

## **$N$ -DIMENSIONAL ELLIPTIC INVARIANT TORI FOR THE PLANAR $(N + 1)$ -BODY PROBLEM\***

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**Abstract.** For any  $N \geq 2$  we prove the existence of quasi-periodic orbits lying on  $N$ -dimensional invariant elliptic tori for the planetary planar  $(N + 1)$ -body problem. For small planetary masses, such orbits are close to the limiting solutions given by the  $N$  planets revolving around the sun on planar circles. The eigenvalues of the linearized secular dynamics are also computed asymptotically. The proof is based on an appropriate averaging and KAM theory which overcomes the difficulties caused by the intrinsic degeneracies of the model. For concreteness, we focus on a caricature of the outer solar system.

**Key words.**  $N$ -body problem, nearly integrable Hamiltonian systems, lower-dimensional elliptic tori

**AMS subject classifications.** 70F10, 34C27, 37J40, 70K43

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### **1. Introduction and results.**

**1.1. Quasi-periodic motions in the many-body problem.** The existence of stable trajectories of the many-body problem viewed as a model for the solar system has been the subject of researches of many distinguished scientists both in the past and in recent years; see, for example, the theoretical work of Poincaré [Poi1905], Arnold [A63], Herman [H95], and the numerical investigations of Laskar [L96]. Only recently, a complete proof, based on [H95], of the existence of quasi-periodic motions (corresponding to maximal invariant tori of dimension  $3N - 1$ ) for the  $(N + 1)$ -body problem for arbitrary  $N$  has been produced in [F04]. We recall that the main difficulties that one encounters in the application of general tools (such as averaging and KAM theory) to particular cases of interest in celestial mechanics, are related to the strong degeneracies of the analytical models.

The scope of this paper is to show the existence of quasi-periodic orbits lying on  $N$ -dimensional invariant elliptic tori for the planar  $(N + 1)$ -body problem. The main difference from [H95] and [F04], besides the dimension of the constructed tori, relies on the explicit evaluations of the eigenvalues of the linearized secular dynamics (which allow us to apply more standard KAM methods).

Though the method exposed here is quite general, for concreteness we will focus our attention on a caricature of the outer solar system. More precisely, our model will be given by a Sun and  $N$  planets with relatively small masses (say, of order  $\varepsilon$ ). All these  $(N + 1)$  bodies are considered as point masses in mutual gravitational interaction. Two planets (such as Jupiter and Saturn in the real world) will be assumed to have mass considerably bigger than the other planets. The bodies lie in a given plane and we assume that the initial configuration is far from collisions. We

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also assume, mimicking the case of the outer solar system, that the two big planets have an orbit which is internal with respect to the orbits of the small planets. We will establish, for a large set of semi-axes, the existence of quasi-periodic orbits with small eccentricities filling up  $N$ -dimensional invariant elliptic tori. Such orbits can be seen as continuations of “limiting” circular trajectories of the system obtained by neglecting the mutual interactions among the planets. A more precise statement is given in Theorem 1.1 below.

The above “outer model,” which roughly mimics some traits of physically relevant cases, has also the nice feature of providing particularly simple expressions in the related perturbing functions, as we will see in section 3 below. We stress, however, that many other situations (such as one large planet plus  $N - 1$  small planets; “inner” or “mixed” models, etc.) may be easily dealt with using the techniques and results presented in this paper.

The proof of our result is based on techniques developed in [BCV03] and on the explicit computation of the eigenvalues of the quadratic part of the so-called principal part of the perturbation for the planar many-body problem.

The first result on quasi-periodic orbits of interest in celestial mechanics goes back to [A63], where quasi-periodic orbits lying on 4-dimensional tori are shown to exist for the planar three-body problem (the general case was discussed there, but no complete proof was given). Related results were given in [JM66], which found linearly unstable quasi-periodic orbits lying on 2-dimensional tori for the nonplanar three-body problem. More recently, [LR95] and [R95] and [BCV03] proved the existence of quasi-periodic orbits for the nonplanar three-body problem, lying on 4-dimensional and linearly stable 2-dimensional tori, respectively. Two-dimensional invariant tori for the planar three-body problem have been found in [F02]. Periodic orbits of the nonplanar three-body problem winding around invariant tori have been constructed in [BBV04]. Finally, the existence of a positive measure set of initial data giving rise to maximal invariant tori for the planetary  $(N + 1)$ -body problem has been established in [F04].

The paper is organized as follows. In section 1.2, we give a more precise statement of our main result. In section 2 we write down the  $(N + 1)$ -body problem Hamiltonian in Delaunay–Poincaré variables. In section 3 (which, in a sense, is the crucial part of the paper) we discuss degeneracies. In section 4 we give the proof of the main result. The scheme of proof is similar to the one presented in [BCV03] (see also [BBV04]) in the three-body case and it is based on a “general” averaging theorem and on KAM theory for lower-dimensional tori (see [P96], [BCV03], [BBV04]). For completeness, we include a classical (but not easy to find) description of analytical properties of the Delaunay–Poincaré variables (see section 2 and Appendix A); in Appendix B we collect some simple linear algebra lemmata that are used in the arguments given in section 3.

**1.2. Statement of results.** We denote the  $N + 1$  massive points (“bodies”) by  $P_0, \dots, P_N$  and let  $m_0, \dots, m_N$  be their masses interacting through gravity (with constant of gravitation 1). Fix  $m_0 > 0$  and assume that

$$(1.1) \quad m_i = \varepsilon \mu_i, \quad i = 1, \dots, N, \quad 0 < \varepsilon < 1.$$

Here,  $\varepsilon$  is regarded as a small parameter and  $\mu_i$  is of order 1 in  $\varepsilon$ . The point  $P_0$  represents the “Sun” and the points  $P_i$ ,  $i = 1, \dots, N$ , the “planets.” We assume that all the bodies lie on a fixed plane, that will be identified with  $\mathbf{R}^2$ . The phase space of this dynamical system—the planetary, planar  $(N + 1)$ -body system—has dimension  $4N$  (after reduction by the symmetries of translations).

We will state the result in terms of orbital elements of the “osculating ellipses” of the two-body problems associated to  $(P_0, P_j)$ . Let  $u^{(0)}$  and  $u^{(j)}$  denote the coordinates of  $P_0$  and  $P_j$  (at a given time) and let  $\dot{u}^{(0)}$  and  $\dot{u}^{(j)}$  denote the corresponding velocities. By definition, the “osculating ellipse” is the ellipse described by the solution of the two-body problem  $(P_0, P_j)$  with initial data given by  $(u^{(0)}, u^{(j)}, \dot{u}^{(0)}, \dot{u}^{(j)})$ . Of course, such ellipses describe the motions of the full  $(N+1)$ -body problem only approximately; nevertheless, they provide a nice set of coordinates allowing, for example, to describe the true motions in terms of the eccentricities  $e_j$  and the major semiaxes  $a_j$  of the osculating ellipses. For further details and pictures of the orbital elements, we refer the reader to [Ch88] and [BCV03].

In this paper we consider a planetary (planar) model with planets evolving from phase points corresponding to well-separated nearly circular ellipses ( $e_i \ll 1$ ); here “well-separated” means that

$$(1.2) \quad 0 < a_i < \theta a_{i+1}, \quad 1 \leq i \leq N-1.$$

for a suitable constant  $0 < \theta < 1$ . For concreteness, we shall focus on a caricature of the outer solar system; i.e., we will assume that, for some  $m_0 < \bar{\mu}_i < 4m_0$ ,

$$(1.3) \quad \begin{aligned} \mu_i &= \bar{\mu}_i & \text{for } i &= 1, 2, \\ \mu_i &= \delta \bar{\mu}_i & \text{for } i &= 3, \dots, N, \quad 0 < \delta < 1. \end{aligned}$$

In this setting,  $P_1$  and  $P_2$  imitate (in a very rough way, of course) the physical<sup>1</sup> features of the giant planets Jupiter and Saturn, while  $P_3$  and  $P_4$  represent Uranus and Neptune.<sup>2</sup>

A rough description of our main result is given in the following theorem; a more precise and quantitative version is given in Theorem 4.2 below.

**THEOREM 1.1.** *Consider a planar, planetary  $(N+1)$ -body system satisfying (if  $N \geq 3$ ) (1.1) and (1.3). Let  $A \subset \mathbf{R}^N$  be a compact set of semiaxes where (1.2) holds for a suitable  $0 < \theta < 1$ . Then, there exists  $\delta^* > 0$  and for any  $0 < \delta < \delta^*$  there exists  $\varepsilon^* > 0$  so that the following holds. For any  $0 < \varepsilon < \varepsilon^*$ , the planetary, planar  $(N+1)$ -body system possesses a family of  $N$ -dimensional elliptic invariant Diophantine quasi-periodic tori; such family is parametrized by the osculating major semiaxes varying in a subset of  $A$  of density<sup>3</sup>  $1 - C_1\varepsilon^{c_1}$ . These motions correspond to orbits with osculating eccentricities bounded by  $C_2\varepsilon^{c_2}$  and the variation in time of the osculating major semiaxes of these orbits is bounded by  $C_3\varepsilon^{c_3}$ .*

We have the following few comments.

- The numbers  $\delta^*$  and  $\theta$  can be easily computed in the course of the proof and are not “very small”; in fact  $\theta$  is a “universal” constant while  $\delta^*$  depends only on  $N$  and  $A$ . On the other hand,  $\varepsilon^*$ , which depends on  $N$ ,  $A$ , and  $\delta$ , is related to a KAM smallness condition and rough estimates lead, as is well known, to ridiculously small quantities (for somewhat more serious KAM estimates, we refer the reader to [CC03]). Finally, the positive constants  $C_i$ ’s depend on  $N$ ,  $A$ , and  $\delta$ , while the  $c_i$ ’s depend only on  $N$  (and could also be easily calculated; see (4.43)).
- The assumptions (1.2) and (1.3) in the theorem are used to check *explicitly* suitable “nondegeneracy” conditions. However, giving explicit constants and

<sup>1</sup>A mathematical motivation for considering *two* dominant planets is given in Remark 3.2(iii).

<sup>2</sup>The Jupiter/Saturn mass ratio is approximately 3.34, while the Neptune/Uranus mass ratio is about 1.18 (to have it all, the Jupiter/Uranus mass ratio is  $\sim 21.78$ ).

<sup>3</sup>Here and in what follows, the “density” is intended with respect to Lebesgue measure.

estimates, one can show that the thesis of the theorem holds, essentially, with no hypotheses on the semiaxes  $a_j$  and the rescaled masses  $\mu_j$  (provided  $a_i \neq a_j > 0$  and  $\mu_j > 0$ ); a rigorous argument, based on analytic continuation of the eigenvalues, could be given along the lines discussed in [F04].

- The invariant tori found in Theorem 1.1 are *lower-dimensional elliptic tori* meaning that the dimension of the tori is strictly smaller than (in fact, half of) the dimension of the Lagrangian (maximal) tori, which have dimension  $2N$ . “Elliptic” means that the tori are linearly stable. It is not difficult to show that such elliptic tori are surrounded by a set of positive measure of maximal tori.
- The proof given below is based on a well-known elliptic KAM theorem, which works under “nondegeneracy” (or Melnikov) conditions. To check these conditions one has to study the eigenvalues of the “secular” (or averaged) quadratic part of the Newtonian many-body interaction, which will be denoted  $\bar{\mathcal{H}}_{1,2}$ ; “quadratic” here refers to the symplectic Cartesian variables measuring the eccentricity and the orientation of the osculating ellipses. The diagonalization of  $\bar{\mathcal{H}}_{1,2}$  is trivial (under the only assumption that  $a_i \neq a_j$ ), while conditions (1.2) and (1.3) will be used to check that the associated eigenvalues are nonzero, simple, and distinct so that Melnikov conditions are satisfied. The proof is noninductive on  $N$ .

**2. Poincaré Hamiltonian setting.** The results described in this section are classical (even if not easy to find) and go back to Delaunay and Poincaré; the reader not familiar with Delaunay and Poincaré variables will find a self-contained exposition in Appendix A.

Consider  $N + 1$  bodies  $P_0, \dots, P_N$ , in a fixed (ecliptic) plane, of masses  $m_0, \dots, m_N$  interacting through gravity (with constant of gravitation 1). We assume that the mass of  $P_0$  (the “star”) is much larger than the mass of the other bodies (the “planets”); i.e., we assume (1.1). In heliocentric planar (suitably rescaled) variables, the dynamics of the planar  $(N + 1)$ -body problem is governed (as explained in Appendix A) by the Hamiltonian

$$(2.1) \quad \mathcal{H}^{(N)}(X, x) := \mathcal{H}_0^{(N)}(X, x) + \varepsilon \mathcal{H}_1^{(N)}(X, x),$$

where  $X := (X^{(1)}, \dots, X^{(N)}) \in \mathbf{R}^{2N}$  and  $x := (x^{(1)}, \dots, x^{(N)}) \in \mathbf{R}^{2N}$  are conjugated Cartesian symplectic variables and

$$(2.2) \quad \begin{aligned} \mathcal{H}_0^{(N)} &:= \sum_{i=1}^N \left( \frac{1}{2m_i} |X^{(i)}|^2 - \frac{m_i M_i}{|x^{(i)}|} \right), \\ \mathcal{H}_1^{(N)} &:= \sum_{1 \leq i < j \leq N} \left( X^{(i)} \cdot X^{(j)} - \frac{\mu_i \mu_j}{m_0^2} \frac{1}{|x^{(i)} - x^{(j)}|} \right); \end{aligned}$$

here we have introduced the dimensionless masses<sup>4</sup>

$$(2.3) \quad M_i := 1 + \varepsilon \frac{\mu_i}{m_0}, \quad m_i := \frac{\mu_i}{m_0 + \varepsilon \mu_i} = \frac{\mu_i}{m_0} \frac{1}{M_i}.$$

The Hamiltonian  $\mathcal{H}_0^{(N)}$  is simply the sum of  $N$  uncoupled planar Kepler problems (formed by the star and the  $i$ th planet). Being interested in phase region where

<sup>4</sup>Beware not to confuse the dimensionless masses  $m_i$  with the real masses  $m_i$  introduced at the beginning of section 1.2.

the uncoupled Kepler problem describes nearly circular orbits, we introduce *planar Poincaré variables*, the construction of which is based on the classical 4-dimensional symplectic map (2.8) below. Let

$$(2.4) \quad F_1(t) := \left(1 - \frac{t}{4}\right)^{\frac{1}{2}}, \quad F_2(t) := \frac{1}{2} \left(1 - \frac{t}{4}\right)^{-1}, \quad \left(|t| < \frac{1}{4}\right);$$

let  $G_0(s, t) = t + st + \dots$  be the function analytic in a neighborhood of  $(0, 0)$  implicitly defined by

$$(2.5) \quad G_0(0, 0) = 0, \quad G_0 = s \sin G_0 + t \cos G_0;$$

define the following four functions of three variables  $(\hat{\eta}, \hat{\xi}, \lambda)$  real-analytic in a neighborhood of the set  $\{(\hat{\eta}, \hat{\xi}) = (0, 0)\} \times \mathbf{T}$ :

$$(2.6) \quad \begin{aligned} G(\hat{\eta}, \hat{\xi}, \lambda) &:= G_0 \left( (\hat{\eta} \cos \lambda - \hat{\xi} \sin \lambda) F_1(t), (\hat{\xi} \cos \lambda - \hat{\eta} \sin \lambda) F_1(t) \right), \\ \mathcal{E}_s(\hat{\eta}, \hat{\xi}, \lambda) &:= \left( \hat{\xi} \cos(\lambda + G) + \hat{\eta} \sin(\lambda + G) \right) F_1(t), \\ \mathcal{C}(\hat{\eta}, \hat{\xi}, \lambda) &:= \cos(\lambda + \mathcal{E}_s) - \hat{\eta} F_1(t) - \hat{\xi} \mathcal{E}_s F_1(t) F_2(t), \\ \mathcal{S}(\hat{\eta}, \hat{\xi}, \lambda) &:= \sin(\lambda + \mathcal{E}_s) + \hat{\xi} F_1(t) - \hat{\eta} \mathcal{E}_s F_1(t) F_2(t), \end{aligned}$$

where  $t$  is short for  $t = \hat{\eta}^2 + \hat{\xi}^2$ ,  $G$  is short for  $G(\hat{\eta}, \hat{\xi}, \lambda)$ , and  $\mathcal{E}_s$  is short for  $\mathcal{E}_s(\hat{\eta}, \hat{\xi}, \lambda)$ .

LEMMA 2.1 (planar Poincaré variables). *Fix  $\varepsilon, \mu, m_0 > 0$  and let*

$$(2.7) \quad \begin{aligned} M &:= 1 + \varepsilon \frac{\mu}{m_0}, & m &:= \frac{\mu}{m_0} \frac{1}{M}, & \bar{m} &:= \frac{\mu}{m_0} \frac{1}{\sqrt{M}}, \\ \sigma &:= \left(\frac{\mu}{m_0}\right)^3 \frac{1}{M}, & a &= a(\Lambda; \mu, \varepsilon) := \frac{\Lambda^2}{\bar{m}^2}. \end{aligned}$$

Then, for any  $\Lambda_+ > \Lambda_- > 0$ , there exists a ball  $B$  around the origin in  $\mathbf{R}^2$  such that the 4-dimensional map

$$\Psi_P : (\Lambda, \lambda, \eta, \xi) \in D := (\Lambda_-, \Lambda_+) \times \mathbf{T} \times B \rightarrow (X, x) \in \mathbf{R}^4,$$

where

$$(2.8) \quad \begin{aligned} x_1 &= x_1(\Lambda, \lambda, \eta, \xi; \mu, \varepsilon) := a(\Lambda; \mu, \varepsilon) \mathcal{C} \left( \frac{\eta}{\sqrt{\Lambda}}, \frac{\xi}{\sqrt{\Lambda}}, \lambda \right), \\ x_2 &= x_2(\Lambda, \lambda, \eta, \xi; \mu, \varepsilon) := a(\Lambda; \mu, \varepsilon) \mathcal{S} \left( \frac{\eta}{\sqrt{\Lambda}}, \frac{\xi}{\sqrt{\Lambda}}, \lambda \right), \\ X &= X(\Lambda, \lambda, \eta, \xi; \mu, \varepsilon) := \frac{\bar{m}^4}{\Lambda^3} \frac{\partial x}{\partial \lambda}(\Lambda, \lambda, \eta, \xi; \mu, \varepsilon) = \frac{\bar{m}}{a(\Lambda; \mu, \varepsilon)^{3/2}} \frac{\partial x}{\partial \lambda}, \end{aligned}$$

is real-analytic in  $D$  and symplectic:

$$d\Lambda \wedge d\lambda + d\eta \wedge d\xi = dX_1 \wedge dx_1 + dX_2 \wedge dx_2.$$

Furthermore, if  $\mathcal{H}_0^{(1)}$  denotes the two-body Hamiltonian

$$\mathcal{H}_0^{(1)}(X, x) := \frac{1}{2m} |X|^2 - \frac{mM}{|x|},$$

then, on the phase region of negative energies  $(\mathcal{H}_0^{(1)})^{-1}(-\frac{\sigma}{2\Lambda_-^2}, -\frac{\sigma}{2\Lambda_+^2})$ , one has

$$\mathcal{H}_0^{(1)} \circ \Psi_P = -\frac{\sigma}{2\Lambda^2};$$

in the planar coordinates  $x \in \mathbf{R}^2$  the corresponding motion describes an ellipse of major semiaxis  $a = a(\Lambda; \mu, \varepsilon)$  and eccentricity

$$(2.9) \quad e = \sqrt{\frac{\eta^2 + \xi^2}{\Lambda}} F_1\left(\frac{\eta^2 + \xi^2}{\Lambda}\right) = \sqrt{\frac{\eta^2 + \xi^2}{\Lambda} \left(1 - \frac{\eta^2 + \xi^2}{4\Lambda}\right)}.$$

The proof of this lemma can be found in Appendix A. Note that

$$(2.10) \quad \mathcal{C}(0, 0, \lambda) = \cos \lambda, \quad \mathcal{S}(0, 0, \lambda) = \sin \lambda,$$

so that the  $(a, \lambda) \rightarrow x$  transformation is, for  $\eta = \xi = 0$ , just polar coordinates. Let, now,  $\Psi_P^{(N)}$  be the  $4N$ -dimensional map, parametrized by  $(\mu_1, \dots, \mu_N, \varepsilon)$ , defined by

$$(2.11) \quad \Psi_P^{(N)} : \left( (\Lambda_1, \lambda_1, \eta_1, \xi_1), \dots, (\Lambda_N, \lambda_N, \eta_N, \xi_N) \right) \in \left( (0, \infty) \times \mathbf{T} \times \mathbf{R}^2 \right)^N \rightarrow (X, x)$$

with

$$(2.12) \quad \begin{aligned} (X, x) &= \left( (X^{(1)}, \dots, X^{(N)}), (x^{(1)}, \dots, x^{(N)}) \right), \\ (X^{(i)}, x^{(i)}) &= \Psi_P(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon). \end{aligned}$$

Then,  $\Psi_P^{(N)}$  is symplectic and

$$(2.13) \quad \mathcal{H}_0^{(N)} \circ \Psi_P^{(N)} = -\frac{1}{2} \sum_{i=1}^N \frac{\sigma_i}{\Lambda_i^2} =: \mathcal{H}_0(\Lambda), \quad \sigma_i := \left(\frac{\mu_i}{m_0}\right)^3 \frac{1}{M_i}.$$

In such Poincaré variables the full planar  $(N + 1)$ -body Hamiltonian  $\mathcal{H}^{(N)}$  becomes

$$(2.14) \quad \mathcal{H}(\Lambda, \lambda, \eta, \xi) = \mathcal{H}_0(\Lambda) + \varepsilon \mathcal{H}_1(\Lambda, \lambda, \eta, \xi), \quad \mathcal{H}_1 := \mathcal{H}_1^{(N)} \circ \Psi_P^{(N)} =: \mathcal{H}_1^{\text{compl}} + \mathcal{H}_1^{\text{princ}},$$

where the so-called “complementary part”  $\mathcal{H}_1^{\text{compl}}$  and the “principal part”  $\mathcal{H}_1^{\text{princ}}$  of the perturbation are, respectively, the functions

$$(2.15) \quad \sum_{1 \leq i < j \leq N} X^{(i)} \cdot X^{(j)} \quad \text{and} \quad \sum_{1 \leq i < j \leq N} \frac{\mu_i \mu_j}{m_0^2} \frac{1}{|x^{(i)} - x^{(j)}|}$$

expressed in Poincaré variables:<sup>5</sup>

$$X^{(i)} = X(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon) \quad \text{and} \quad x^{(i)} = x(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon).$$

<sup>5</sup> $X = (X_1, X_2)$  and  $x = (x_1, x_2)$  denote here the functions defined in (2.8).

Notice that, since  $X^{(i)} = (\bar{m}_i^4/\Lambda_i^3)\partial_{\lambda_i}x^{(i)}$  the  $\lambda$ -average of  $\mathcal{H}_1^{\text{compl}}$  vanishes. Moreover, as it is well known, the  $\lambda$ -average of  $\mathcal{H}_1$  is an even function of  $(\eta, \xi)$ ; see, also, Appendix A. Hence, we may split the perturbation function as

$$(2.16) \quad \mathcal{H}_1 = \bar{\mathcal{H}}_1 + \tilde{\mathcal{H}}_1$$

with

$$(2.17) \quad \bar{\mathcal{H}}_1(\Lambda, \eta, \xi) := \int_{\mathbf{T}^N} \mathcal{H}_1 \frac{d\lambda}{(2\pi)^N}, \quad \int_{\mathbf{T}^N} \tilde{\mathcal{H}}_1 d\lambda = 0.$$

Furthermore,  $\bar{\mathcal{H}}_1$  may be written as

$$(2.18) \quad \bar{\mathcal{H}}_1(\Lambda, \eta, \xi) = \bar{\mathcal{H}}_{1,0}(\Lambda) + \bar{\mathcal{H}}_{1,2}(\Lambda, \eta, \xi) + \bar{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi),$$

where  $\bar{\mathcal{H}}_{1,0} := \bar{\mathcal{H}}_1(\Lambda, 0, 0)$ ,  $\bar{\mathcal{H}}_{1,2}$  is the  $(\eta, \xi)$ -quadratic part of  $\bar{\mathcal{H}}_1$  while  $\bar{\mathcal{H}}_{1,*}$  is the “remainder of order four”:

$$|\bar{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi)| \leq \text{const } |(\eta, \xi)|^4.$$

**3. The averaged quadratic potential  $\bar{\mathcal{H}}_{1,2}$ .** In this section we analyze the function  $\bar{\mathcal{H}}_{1,2}$  (i.e., the  $(\eta, \xi)$ -quadratic part of the  $\lambda$ -average of the perturbation) defined in (2.18), which may be written as

$$(3.1) \quad \bar{\mathcal{H}}_{1,2} = \frac{1}{2} \sum_{1 \leq i, j \leq N} Q_{ij} \begin{pmatrix} \eta_j \\ \xi_j \end{pmatrix} \cdot \begin{pmatrix} \eta_i \\ \xi_i \end{pmatrix},$$

where  $Q_{ij}$  are  $(2 \times 2)$  matrices defined as

$$Q_{ij} := \left( \begin{array}{cc} \partial_{\eta_i, \eta_j}^2 \bar{\mathcal{H}}_{1,2} & \partial_{\eta_i, \xi_j}^2 \bar{\mathcal{H}}_{1,2} \\ \partial_{\xi_i, \eta_j}^2 \bar{\mathcal{H}}_{1,2} & \partial_{\xi_i, \xi_j}^2 \bar{\mathcal{H}}_{1,2} \end{array} \right) \Big|_{(\Lambda, 0, 0)}.$$

The aim of this section is to prove that there exists a symplectic linear change of variables  $(p, q) \rightarrow (\eta, \xi)$  putting the quadratic part (3.1) in the normal form

$$(3.2) \quad \frac{1}{2} \sum_{i=1}^N \bar{\Omega}_i (p_i^2 + q_i^2);$$

see Remark 3.1(i). A crucial fact, in order to apply KAM theory, consists in proving that such  $\bar{\Omega}_i$ 's are nondegenerate<sup>6</sup> in the sense that they are nonvanishing and distinct. Such nondegeneracy is proved in Proposition 3.2 in which we manage to compute explicitly the asymptotics of the  $\bar{\Omega}_i$ 's.

In view of the definition of the Poincaré variables, we look at the rescaled variables  $(\hat{\eta}, \hat{\xi})$  rather than  $(\eta, \xi)$ . Therefore, we define

$$(3.3) \quad \bar{f}_{ij}(\Lambda, \hat{\eta}, \hat{\xi}) := \frac{1}{(2\pi)^N} \int_{\mathbf{T}^N} \frac{d\lambda}{|x^{(i)}(\Lambda_i, \lambda_i, \sqrt{\Lambda_i} \hat{\eta}_i, \sqrt{\Lambda_i} \hat{\xi}_i; \mu_i, \varepsilon) - x^{(j)}(\Lambda_j, \lambda_j, \sqrt{\Lambda_j} \hat{\eta}_j, \sqrt{\Lambda_j} \hat{\xi}_j; \mu_j, \varepsilon)|}.$$

<sup>6</sup>See the “Melnikov condition” (4.33).

Thus, letting<sup>7</sup>

$$(3.4) \quad \begin{aligned} a_i &:= a(\Lambda_i; \mu_i, \varepsilon), & c_{ij} &:= \frac{1}{m_0} \left( \frac{M_i M_j}{a_i a_j} \right)^{1/4}, \\ A_{ij} &:= \left( \begin{array}{cc} \partial_{\hat{\eta}_i \hat{\eta}_j}^2 \bar{f}_{ij} & \partial_{\hat{\eta}_i \hat{\xi}_j}^2 \bar{f}_{ij} \\ \partial_{\hat{\xi}_i \hat{\eta}_j}^2 \bar{f}_{ij} & \partial_{\hat{\xi}_i \hat{\xi}_j}^2 \bar{f}_{ij} \end{array} \right) \Big|_{\hat{\eta}=\hat{\xi}=0}, & B_{ij} &:= \left( \begin{array}{cc} \partial_{\hat{\eta}_j \hat{\eta}_j}^2 \bar{f}_{ij} & \partial_{\hat{\eta}_j \hat{\xi}_j}^2 \bar{f}_{ij} \\ \partial_{\hat{\xi}_j \hat{\eta}_j}^2 \bar{f}_{ij} & \partial_{\hat{\xi}_j \hat{\xi}_j}^2 \bar{f}_{ij} \end{array} \right) \Big|_{\hat{\eta}=\hat{\xi}=0}, \end{aligned}$$

we find

$$Q_{ij} = \begin{cases} \sqrt{\mu_i \mu_j} c_{ij} A_{ij} & \text{if } i \neq j, \\ \sum_{k \neq j} \sqrt{\mu_k \mu_j} c_{kj} B_{kj} & \text{if } i = j. \end{cases}$$

It is a remarkable fact that, for the planar planetary  $(N + 1)$ -body problem the matrices  $A_{ij}$  and  $B_{ij}$  are proportional to the  $(2 \times 2)$  identity matrix  $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and have simple integral representation. In fact, define, for  $a \neq b$ ,

$$\begin{aligned} \mathcal{J}(a, b) &:= \frac{1}{2\pi} \int_0^{2\pi} \frac{-17ab \cos t + 8(a^2 + b^2) \cos(2t) + ab \cos(3t)}{(a^2 + b^2 - 2ab \cos t)^{5/2}} dt, \\ \mathcal{I}(a, b) &:= \frac{1}{2\pi} \int_0^{2\pi} \frac{-7ab + 4(a^2 + b^2) \cos t - ab \cos(2t)}{(a^2 + b^2 - 2ab \cos t)^{5/2}} dt, \end{aligned}$$

and denote, for  $a_i \neq a_j$ ,

$$(3.5) \quad \alpha_{ij} := \frac{a_i a_j}{8} \mathcal{J}(a_i, a_j), \quad \beta_{ij} := \frac{a_i a_j}{4} \mathcal{I}(a_i, a_j).$$

Then, the following ‘‘algebraic’’ result holds.

**PROPOSITION 3.1.** *Assume  $a_i \neq a_j$  for  $i \neq j$ . Then  $A_{ij} = \alpha_{ij} \mathbf{1}_2$  and  $B_{ij} = \beta_{ij} \mathbf{1}_2$ .*

*Remark 3.1.* (i) An immediate corollary of this result is that, in the collisionless domain  $\{a_i \neq a_j\}$ ,  $\overline{\mathcal{H}}_{1,2}$  has the simple form

$$(3.6) \quad \overline{\mathcal{H}}_{1,2} = \frac{1}{2} (M\eta \cdot \eta + M\xi \cdot \xi),$$

$M$  being the real, symmetric  $(N \times N)$  matrix with entries

$$(3.7) \quad M_{ij} = \begin{cases} \sqrt{\mu_i \mu_j} c_{ij} \alpha_{ij} & \text{if } i \neq j, \\ \sum_{k \neq j} \sqrt{\mu_k \mu_j} c_{kj} \beta_{kj} & \text{if } i = j. \end{cases}$$

The Hamiltonian (3.6) can be immediately put in symplectic normal form: if  $U$  is the real orthogonal matrix ( $U^T = U^{-1}$ ) which diagonalizes  $M$  ( $U^T M U = \text{diag}(\bar{\Omega}_1, \dots, \bar{\Omega}_N)$ ), then the map  $p = U^T \eta$ ,  $q = U^T \xi$  is symplectic and, in such variables, the new Hamiltonian takes the form (3.2).

(ii) The functions  $\mathcal{J}$  and  $\mathcal{I}$  (which admit simple representations in terms of Gauss hypergeometric functions) are symmetric ( $\mathcal{J}(a, b) = \mathcal{J}(b, a)$  and  $\mathcal{I}(a, b) = \mathcal{I}(b, a)$ ) and satisfy

$$\mathcal{J}(a, b) = b^{-3} \mathcal{J}(a/b, 1), \quad \mathcal{I}(a, b) = b^{-3} \mathcal{I}(a/b, 1), \quad a < b.$$

<sup>7</sup>Recall (2.7) and (2.3).



The functions of one real variable  $s \in (-1, 1) \rightarrow \mathcal{J}(s, 1)$  and  $s \in (-1, 1) \rightarrow \mathcal{I}(s, 1)$  are, respectively, even and odd in  $s$ , and satisfy, for small  $s$ , the following asymptotics:

$$(3.8) \quad \mathcal{J}(s, 1) = -\frac{15}{8}s^2 - \frac{105}{8}s^4 + O(s^6), \quad \mathcal{I}(s, 1) = 3s + \frac{45}{8}s^3 + O(s^5).$$

(iii) Proposition 3.1 is a suitable version of a well-known result which can be found, e.g., in [Poi1905]; see also [LR95].

(iv) The asymptotics of the  $\alpha_{ij}$ 's and  $\beta_{ij}$ 's may be also computed in terms of the Laplace coefficients (see, e.g., [LR95]); for our purposes it is simpler to derive the needed asymptotics directly from the integral representations given before (3.5).

*Proof of Proposition 3.1.* The computations we are going to perform are algebraic in character and it is therefore enough to consider real variables. Fix  $i \neq j$  and define

$$(3.9) \quad \mathcal{R}_{ij}(\Lambda, \lambda, \hat{\eta}, \hat{\xi}) := \left| x^{(i)}(\Lambda_i, \lambda_i, \sqrt{\Lambda_i} \hat{\eta}_i, \sqrt{\Lambda_i} \hat{\xi}_i; \mu_i, \varepsilon) - x^{(j)}(\Lambda_j, \lambda_j, \sqrt{\Lambda_j} \hat{\eta}_j, \sqrt{\Lambda_j} \hat{\xi}_j; \mu_j, \varepsilon) \right|^2,$$

so that (recall (3.3))

$$(3.10) \quad \bar{f}_{ij}(\Lambda, \hat{\eta}, \hat{\xi}) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \frac{d\lambda}{\sqrt{\mathcal{R}_{ij}}}.$$

By (2.8) we find

$$(3.11) \quad \mathcal{R}_{ij} = a_i^2 \chi_i^2 + a_j^2 \chi_j^2 - 2a_i a_j (C_i C_j - S_i S_j),$$

where  $C_k$ ,  $S_k$ , and  $\chi_k$  are short for, respectively,

$$C_k = \mathcal{C}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k), \quad S_k = \mathcal{S}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k), \quad \text{and} \quad \chi_k = \sqrt{C_k^2 + S_k^2}.$$

The proof will consist in computing explicitly  $\lambda$ -averages of quantities of the form

$$(3.12) \quad \rho_{\zeta_i, \zeta_j}(\lambda_i, \lambda_j) := \left. \partial_{\zeta_i \zeta_j}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \right|_{\hat{\eta}=\hat{\xi}=0} = \left. \frac{3(\partial_{\zeta_i} \mathcal{R}_{ij})(\partial_{\zeta_j} \mathcal{R}_{ij}) - 2\mathcal{R}_{ij}(\partial_{\zeta_i \zeta_j} \mathcal{R}_{ij})}{4\mathcal{R}_{ij}^{5/2}} \right|_{\hat{\eta}=\hat{\xi}=0},$$

where  $\zeta_k$  denotes either of the variables  $\hat{\eta}_k$  or  $\hat{\xi}_k$ . Thus, what we need to do is to compute suitable orders in the variables  $(\hat{\eta}_k, \hat{\xi}_k)$  of the function  $\mathcal{R}_{ij}$ . For this purpose the following lemma will be useful.

LEMMA 3.1. *Define the following elementary functions:*

$$\begin{aligned} C_+(\lambda) &:= 1 + \cos^2 \lambda = \frac{3 + \cos(2\lambda)}{2}, \\ C_-(\lambda) &:= 1 + \sin^2 \lambda = \frac{3 - \cos(2\lambda)}{2}, \\ S_0(\lambda) &:= \cos \lambda \sin \lambda = \frac{1}{2} \sin(2\lambda), \\ \bar{\chi}(x, y, \lambda) &:= 1 - 2y \cos \lambda + 2x \sin \lambda, \\ S(x, y, \lambda) &:= \sin \lambda + x C_+(\lambda) + y S_0(\lambda), \\ C(x, y, \lambda) &:= \cos \lambda - y C_-(\lambda) - x S_0(\lambda), \end{aligned}$$

and denote by  $O_p(z_1, \dots, z_n)$  a function of the variables  $(z_1, \dots, z_n)$  (depending possibly on other variables) analytic in a neighborhood of  $(0, \dots, 0)$  and starting with a homogeneous polynomial of degree  $p$  in  $(z_1, \dots, z_n)$ . Then,

$$\begin{aligned} \chi_k^2 &= \mathcal{C}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k)^2 + \mathcal{S}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k)^2 = \bar{\chi}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k) + O_2(\hat{\eta}_k, \hat{\xi}_k), \\ \mathcal{C}_k &= \mathcal{C}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k) = C(\hat{\xi}_k, \hat{\eta}_k, \lambda_k) + O_2(\hat{\eta}_k, \hat{\xi}_k), \\ (3.13) \quad \mathcal{S}_k &= \mathcal{S}(\hat{\eta}_k, \hat{\xi}_k, \lambda_k) = S(\hat{\xi}_k, \hat{\eta}_k, \lambda_k) + O_2(\hat{\eta}_k, \hat{\xi}_k). \end{aligned}$$

The proof of this lemma follows at once from the explicit expressions for  $\mathcal{C}$  and  $\mathcal{S}$  given in Lemma 2.1 and is left to the reader.

We consider first the matrices  $A_{ij}$  (which allow to compute  $Q_{ij}$  for  $i \neq j$ ) and then we turn to the matrices  $B_{ij}$  (which allow to compute  $Q_{jj}$ ).

*Computation of the matrices  $A_{ij}$ .* First, observe that the two derivatives involved in the definition of  $A_{ij}$  are *always* mixed in the variables with indexes  $i$  and  $j$ . Thus, we can neglect the terms of third order in  $(\hat{\eta}_i, \hat{\xi}_i, \hat{\eta}_j, \hat{\xi}_j)$  and the terms of *second order* of the type  $O_2(\hat{\eta}_i, \hat{\xi}_i)$  and  $O_2(\hat{\eta}_j, \hat{\xi}_j)$ .

By Lemma 3.1, the function  $\mathcal{R}_{ij}$  in (3.11) has the form

$$\begin{aligned} \mathcal{R}_{ij} &= a_i^2(1 - 2\hat{\eta}_i \cos \lambda_i + 2\hat{\xi}_i \sin \lambda_i) + a_j^2(1 - 2\hat{\eta}_j \cos \lambda_j + 2\hat{\xi}_j \sin \lambda_j) \\ &\quad - 2a_i a_j \left[ (\cos \lambda_i - \hat{\xi}_i S_0(\lambda_i) - \hat{\eta}_i C_-(\lambda_i))(\cos \lambda_j - \hat{\xi}_j S_0(\lambda_j) - \hat{\eta}_j C_-(\lambda_j)) \right. \\ &\quad \left. + (\sin \lambda_i + \hat{\xi}_i C_+(\lambda_i) + \hat{\eta}_i S_0(\lambda_i))(\sin \lambda_j + \hat{\xi}_j C_+(\lambda_j) + \hat{\eta}_j S_0(\lambda_j)) \right] \\ (3.14) \quad &+ O_2(\hat{\eta}_i, \hat{\xi}_i) + O_2(\hat{\eta}_j, \hat{\xi}_j) + O_3(\hat{\eta}_i, \hat{\xi}_i, \hat{\eta}_j, \hat{\xi}_j). \end{aligned}$$

Therefore, letting  $(\cdot)|_0$  be short for  $(\cdot)|_{\hat{\eta}_i=\hat{\xi}_i=\hat{\eta}_j=\hat{\xi}_j=0}$ , one finds

$$\begin{aligned} \mathcal{R}_{ij}|_0 &= a_i^2 + a_j^2 - 2a_i a_j \cos(\lambda_i - \lambda_j), \\ \partial_{\hat{\eta}_i} \mathcal{R}_{ij}|_0 &= -2a_i^2 \cos \lambda_i - 2a_i a_j [-C_-(\lambda_i) \cos \lambda_j + S_0(\lambda_i) \sin \lambda_j], \\ \partial_{\hat{\eta}_j} \mathcal{R}_{ij}|_0 &= -2a_j^2 \cos \lambda_j - 2a_i a_j [-C_-(\lambda_j) \cos \lambda_i + S_0(\lambda_j) \sin \lambda_i], \\ \partial_{\hat{\xi}_i} \mathcal{R}_{ij}|_0 &= 2a_i^2 \sin \lambda_i - 2a_i a_j [C_+(\lambda_i) \sin \lambda_j - S_0(\lambda_i) \cos \lambda_j], \\ \partial_{\hat{\xi}_j} \mathcal{R}_{ij}|_0 &= 2a_j^2 \sin \lambda_j - 2a_i a_j [C_+(\lambda_j) \sin \lambda_i - S_0(\lambda_j) \cos \lambda_i], \\ \partial_{\hat{\eta}_i \hat{\eta}_j}^2 \mathcal{R}_{ij}|_0 &= -2a_i a_j [C_-(\lambda_i) C_-(\lambda_j) + S_0(\lambda_i) S_0(\lambda_j)], \\ \partial_{\hat{\xi}_i \hat{\xi}_j}^2 \mathcal{R}_{ij}|_0 &= -2a_i a_j [C_+(\lambda_i) C_+(\lambda_j) + S_0(\lambda_i) S_0(\lambda_j)], \\ \partial_{\hat{\eta}_i \hat{\xi}_j}^2 \mathcal{R}_{ij}|_0 &= -2a_i a_j [C_-(\lambda_i) S_0(\lambda_j) + S_0(\lambda_i) C_+(\lambda_j)], \\ (3.15) \quad \partial_{\hat{\xi}_i \hat{\eta}_j}^2 \mathcal{R}_{ij}|_0 &= -2a_i a_j [S_0(\lambda_i) C_-(\lambda_j) + C_+(\lambda_i) S_0(\lambda_j)]. \end{aligned}$$

In particular,  $\mathcal{R}_{ij}$  and  $\partial_{\hat{\eta}_i} \mathcal{R}_{ij}$  are even in  $(\lambda_i, \lambda_j) \in \mathbf{T}^2$ , while  $\partial_{\hat{\xi}_j} \mathcal{R}_{ij}$  and  $\partial_{\hat{\eta}_i \hat{\xi}_j}^2 \mathcal{R}_{ij}$  are odd. Thus, recalling the definition of  $\rho_{\zeta_i, \zeta_j}(\lambda_i, \lambda_j)$  in (3.12), we find that  $\rho_{\hat{\eta}_i, \hat{\xi}_j}(\lambda_i, \lambda_j)$  is odd in  $(\lambda_i, \lambda_j)$  and it has therefore zero average. For the same reasons, also  $\rho_{\hat{\xi}_i, \hat{\eta}_j}(\lambda_i, \lambda_j)$  has zero average. Hence, the off-diagonal terms of  $A_{ij}$  are zero. We

now compute the diagonal terms of  $A_{ij}$ . We begin with  $\rho_{\hat{\eta}_i, \hat{\eta}_j}(\lambda_i, \lambda_j)$ . By (3.12) and the list in (3.15), we find

$$(3.16) \quad \rho_{\hat{\eta}_i, \hat{\eta}_j}(\lambda_i, \lambda_j) := \partial_{\hat{\eta}_i \hat{\eta}_j}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_0 = \frac{\rho_1(\lambda_i, \lambda_j)}{\rho_2(\lambda_i, \lambda_j)}$$

with

$$(3.17) \quad \begin{aligned} \rho_1(\lambda_i, \lambda_j) &:= a_i a_j \cdot \left[ -24a_i^2 \cos(2\lambda_i) - 24a_j^2 \cos(2\lambda_j) + 8(a_i^2 + a_j^2) \cos(2(\lambda_i - \lambda_j)) \right. \\ &\quad - 3a_i a_j \cos(\lambda_i - 3\lambda_j) - 17a_i a_j \cos(\lambda_i - \lambda_j) \\ &\quad + a_i a_j \cos(3(\lambda_i - \lambda_j)) - 3a_i a_j \cos(3\lambda_i - \lambda_j) \\ &\quad \left. + 54a_i a_j \cos(\lambda_i + \lambda_j) \right], \\ \rho_2(\lambda_i, \lambda_j) &:= 8 \left( a_i^2 + a_j^2 - 2a_i a_j \cos(\lambda_i - \lambda_j) \right)^{5/2}. \end{aligned}$$

Thus, changing the variable of integration, one finds

$$\begin{aligned} \frac{1}{(2\pi)^2} \int_{\mathbf{T}^2} \partial_{\hat{\eta}_i \hat{\eta}_j}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_0 d\lambda_i d\lambda_j &= \frac{1}{2\pi} \int_{\mathbf{T}} a_i a_j \cdot \frac{-17a_i a_j \cos t + 8(a_i^2 + a_j^2) \cos(2t) + a_i a_j \cos(3t)}{8(a_i^2 + a_j^2 - 2a_i a_j \cos t)^{5/2}} dt \\ &= \frac{a_i a_j}{8} \mathcal{J}(a_i, a_j) =: \alpha_{ij}. \end{aligned}$$

The case  $\rho_{\hat{\xi}_i, \hat{\xi}_j}(\lambda_i, \lambda_j)$  is very similar (and will yield the same result). In place of (3.16) one finds

$$(3.18) \quad \rho_{\hat{\xi}_i, \hat{\xi}_j}(\lambda_i, \lambda_j) := \partial_{\hat{\xi}_i \hat{\xi}_j}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_0 = \frac{\rho_3(\lambda_i, \lambda_j)}{\rho_2(\lambda_i, \lambda_j)}$$

with

$$(3.19) \quad \begin{aligned} \rho_3(\lambda_i, \lambda_j) &:= a_i a_j \cdot \left[ 24a_i^2 \cos(2\lambda_i) + 24a_j^2 \cos(2\lambda_j) + 8(a_i^2 + a_j^2) \cos(2(\lambda_i - \lambda_j)) \right. \\ &\quad + 3a_i a_j \cos(\lambda_i - 3\lambda_j) - 17a_i a_j \cos(\lambda_i - \lambda_j) \\ &\quad + a_i a_j \cos(3(\lambda_i - \lambda_j)) + 3a_i a_j \cos(3\lambda_i - \lambda_j) \\ &\quad \left. - 54a_i a_j \cos(\lambda_i + \lambda_j) \right]. \end{aligned}$$

Integrating, one finds again

$$\frac{1}{(2\pi)^2} \int_{\mathbf{T}^2} \partial_{\hat{\xi}_i \hat{\xi}_j}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_0 d\lambda_i d\lambda_j = \alpha_{ij}.$$

This proves Proposition 3.1 in the case of  $Q_{ij}$ , with  $i \neq j$ .

*Computation of the matrices  $B_{ij}$ .* Observe that the derivatives involved in the definition of  $B_{ij}$  are two derivatives with the same index  $j$ . We can, therefore, neglect the third order terms and set  $\hat{\eta}_i = \hat{\xi}_i = 0$ .

Recalling (2.10) we see that  $\chi_i|_{\hat{\eta}_i=\hat{\xi}_i=0} = 1$  and

$$(3.20) \quad \mathcal{R}_{ij} \Big|_{\hat{\eta}_i=\hat{\xi}_i=0} = a_i^2 + a_j^2 \chi_j^2 - 2a_i a_j \chi_j \left( (\cos \lambda_i) \frac{\mathcal{C}_j}{\chi_j} - (\sin \lambda_i) \frac{\mathcal{S}_j}{\chi_j} \right).$$

Defining  $\varphi_j = \varphi_j(\Lambda_j, \lambda_j, \hat{\eta}_j, \hat{\xi}_j)$  through the relations<sup>8</sup>

$$(3.21) \quad \cos \varphi_j = \frac{\mathcal{C}_j}{\chi_j}, \quad \sin \varphi_j = \frac{\mathcal{S}_j}{\chi_j},$$

we find

$$(3.22) \quad \mathcal{R}_{ij} \Big|_{\hat{\eta}_i=\hat{\xi}_i=0} = a_i^2 + a_j^2 \chi_j^2 - 2a_i a_j \chi_j \cos(\varphi_j - \lambda_i).$$

Denote by  $\langle f \rangle_{\theta, \tau}$  the average of a function  $f$  over the angles  $\theta$  and  $\tau$ . Integrating *first* with respect to  $\lambda_i$  and changing variable of integration ( $t = \lambda_i - \varphi_j$ ), one gets

$$(3.23) \quad \left\langle \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_{\hat{\eta}_i=\hat{\xi}_i=0} \right\rangle_{\lambda_i, \lambda_j} = \left\langle \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \right\rangle_{t, \lambda_j}$$

with

$$(3.24) \quad \tilde{\mathcal{R}}_{ij} := a_i^2 + a_j^2 \chi_j^2 - 2a_i a_j \chi_j \cos t.$$

At this point, the argument is completely analogous to that used above. First, we observe that

$$(3.25) \quad \left\langle \partial_{\zeta_h \zeta_k}^2 \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Big|_0 \right\rangle_{\lambda_i, \lambda_j} = \left\langle \frac{3(\partial_{\zeta_h} \tilde{\mathcal{R}}_{ij})(\partial_{\zeta_k} \tilde{\mathcal{R}}_{ij}) - 2\tilde{\mathcal{R}}_{ij}(\partial_{\zeta_h \zeta_k}^2 \tilde{\mathcal{R}}_{ij})}{4\tilde{\mathcal{R}}_{ij}^{5/2}} \Big|_0 \right\rangle_{t, \lambda_j},$$

where  $\zeta_\ell$  denotes here any of the variables  $\hat{\eta}_j, \hat{\xi}_j$ . From Lemma 3.1 it follows that  $\tilde{\mathcal{R}}_{ij}$  can be written as

$$\tilde{\mathcal{R}}_{ij} = f(t) + g(t)(h_1 - h_2) + a_j^2 h_1^2 + O_3(\hat{\eta}_j, \hat{\xi}_j)$$

with

$$\begin{aligned} f(t) &:= a_i^2 + a_j^2 - 2a_i a_j \cos t, & g(t) &:= -2a_j^2 + 2a_i a_j \cos t, \\ h_1 &:= \hat{\eta}_j \cos \lambda_j - \hat{\xi}_j \sin \lambda_j, & h_2 &:= \hat{\xi}_j^2 \cos^2 \lambda_j + \hat{\eta}_j^2 \sin^2 \lambda_j + \hat{\eta}_j \hat{\xi}_j \sin(2\lambda_j). \end{aligned}$$

Thus, since  $h_1$  is of order one in  $(\hat{\eta}_j, \hat{\xi}_j)$  and  $h_2$  is of order two in  $(\hat{\eta}_j, \hat{\xi}_j)$ ,

$$\begin{aligned} \tilde{\mathcal{R}}_{ij} \Big|_0 &= f(t), \\ \partial_{\eta_j} \tilde{\mathcal{R}}_{ij} \Big|_0 &= g(t) \cos \lambda_j, \\ \partial_{\xi_j} \tilde{\mathcal{R}}_{ij} \Big|_0 &= -g(t) \sin \lambda_j, \\ \partial_{\hat{\eta}_j \hat{\eta}_j}^2 \tilde{\mathcal{R}}_{ij} \Big|_0 &= -2g(t) \sin^2 \lambda_j + 2a_j^2 \cos^2 \lambda_j, \\ \partial_{\hat{\eta}_j \hat{\xi}_j}^2 \tilde{\mathcal{R}}_{ij} \Big|_0 &= -(g(t) + a_j^2) \sin(2\lambda_j), \\ \partial_{\hat{\xi}_j \hat{\xi}_j}^2 \tilde{\mathcal{R}}_{ij} \Big|_0 &= -2g(t) \cos^2 \lambda_j + 2a_j^2 \sin^2 \lambda_j. \end{aligned}$$

<sup>8</sup>Physically,  $\varphi_j$  coincides with  $v_j + g_j$  where  $v_j$  and  $g_j$  are, respectively, the true anomaly and the argument of the perihelion of the osculating ellipse associated to the star and the  $j$ th planet; compare to Appendix A.

Therefore, using (3.25), one finds

$$\left\langle \partial_{\hat{\eta}_j \hat{\xi}_j}^2 \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \Big|_0 \right\rangle_{\lambda_i, \lambda_j} = \left\langle \frac{-(\frac{3}{2}g^2 + 2(a_j^2 - g)f) \sin(2\lambda_j)}{4f^{5/2}} \Big|_0 \right\rangle_{t, \lambda_j} = 0$$

(since the integrand is odd in  $\lambda_j$ ), showing that also  $B_{ij}$  is a diagonal matrix. To compute the diagonal elements we calculate

$$(3.26) \quad \partial_{\hat{\eta}_j \hat{\eta}_j}^2 \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \Big|_0 = \frac{\tilde{\rho}_1(\lambda_j, t)}{\tilde{\rho}_2(\lambda_j, t)} \quad \text{and} \quad \partial_{\hat{\xi}_j \hat{\xi}_j}^2 \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \Big|_0 = \frac{\tilde{\rho}_3(\lambda_j, t)}{\tilde{\rho}_2(\lambda_j, t)}$$

with

$$\begin{aligned} \tilde{\rho}_1 &= -7a_i^2 a_j^2 + (9a_i^2 a_j^2 + 8a_j^4) \cos(2\lambda_j) + \frac{7}{2} a_i^2 a_j^2 \cos(2\lambda_j - 2t) \\ &\quad - (2a_i^3 a_j + 10a_i a_j^3) \cos(2\lambda_j - t) + 4(a_i^3 a_j + a_i a_j^3) \cos(t) \\ &\quad - a_i^2 a_j^2 \cos(2t) - (2a_i^3 a_j + 10a_i a_j^3) \cos(2\lambda_j + t) + \frac{7}{2} a_i^2 a_j^2 \cos(2\lambda_j + 2t); \\ \tilde{\rho}_2 &= 4(a_i^2 + a_j^2 - 2a_i a_j \cos t)^{5/2}; \\ \tilde{\rho}_3 &= -7a_i^2 a_j^2 - (9a_i^2 a_j^2 + 8a_j^4) \cos(2\lambda_j) - \frac{7}{2} a_i^2 a_j^2 \cos(2\lambda_j - 2t) \\ &\quad + (2a_i^3 a_j + 10a_i a_j^3) \cos(2\lambda_j - t) + 4(a_i^3 a_j + a_i a_j^3) \cos(t) \\ &\quad - a_i^2 a_j^2 \cos(2t) + (2a_i^3 a_j + 10a_i a_j^3) \cos(2\lambda_j + t) - \frac{7}{2} a_i^2 a_j^2 \cos(2\lambda_j + 2t); \end{aligned}$$

taking the  $\lambda_j$ -average, one finds immediately

$$\left\langle \partial_{\hat{\eta}_j \hat{\eta}_j}^2 \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \Big|_0 \right\rangle_{\lambda_i, \lambda_j} = \left\langle \partial_{\hat{\xi}_j \hat{\xi}_j}^2 \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}} \Big|_0 \right\rangle_{\lambda_i, \lambda_j} = \frac{a_i a_j}{4} \mathcal{I}(a_i, a_j) =: \beta_{ij}. \quad \square$$

The next result shows that, for  $\delta$  and  $\varepsilon$  small, generically the eigenvalues of  $M$  in (3.6)–(3.7) are nonvanishing, simple, and distinct. We formulate the result regarding the semiaxis  $a_j$  as *independent variables*. Recall the definitions of  $\alpha_{ij}$  and  $\beta_{ij}$  in (3.5) and let (if  $N \geq 3$ )

$$(3.27) \quad \beta_j := \sum_{k=1,2} \frac{\sqrt{\mu_k \mu_j}}{m_0} \frac{1}{\sqrt[3]{a_k a_j}} \beta_{kj}, \quad j \geq 3.$$

PROPOSITION 3.2. Assume that  $a_j$  and  $\bar{\mu}_j$  verify<sup>9</sup>

$$(3.28) \quad \alpha_{12} \neq 0 \quad \text{and} \quad \beta_{12} \neq \pm \alpha_{12}, \quad \beta_i \neq 0 \quad \text{and} \quad \beta_i \neq \beta_j \quad \text{for} \quad i \neq j.$$

Then, there exist  $0 < \delta^* < 1$  and  $0 < \varepsilon_0 < 1$  such that for all  $0 < \delta < \delta^*$  and  $0 \leq \varepsilon < \varepsilon_0$  the eigenvalues  $\{\bar{\Omega}_1, \dots, \bar{\Omega}_N\}$  of the matrix  $M$  are nonvanishing, simple,

<sup>9</sup>Clearly, if  $N = 2$ , the statements regarding the  $\beta_j$  and the eigenvalues  $\bar{\Omega}_j$  for  $j \geq 3$  have to be omitted.

and distinct. Furthermore the following asymptotics hold:<sup>10</sup>

$$\begin{aligned}
 \bar{\Omega}_1 &= \frac{\sqrt{\mu_1\mu_2}}{m_0} \frac{\beta_{12} + \alpha_{12}}{\sqrt[4]{a_1a_2}} + O(\sqrt{\delta}, \varepsilon), \\
 \bar{\Omega}_2 &= \frac{\sqrt{\mu_1\mu_2}}{m_0} \frac{\beta_{12} - \alpha_{12}}{\sqrt[4]{a_1a_2}} + O(\sqrt{\delta}, \varepsilon), \\
 (3.29) \quad \bar{\Omega}_j &= \sqrt{\delta} \beta_j + \sqrt{\delta} O(\sqrt{\delta}, \varepsilon), \quad 3 \leq j \leq N.
 \end{aligned}$$

As mentioned above (see Remark 3.1(iv)) the asymptotic of the  $\alpha_{ij}$ 's and  $\beta_{ij}$ 's may be evaluated in terms of the Laplace coefficients (see, e.g., [L91]). For completeness we give a detailed proof.

*Proof.* First of all, from the definition of  $c_{ij}$  (see (3.4) and (2.3)) it follows that

$$(3.30) \quad c_{ij} = \frac{1}{m_0} \frac{1}{\sqrt[4]{a_i a_j}} + O(\varepsilon).$$

Thus, by definition of  $M$ , by definition of  $\beta_j$  and  $\alpha_{ij}$ , and by the hypothesis on the masses  $\mu_i$  (see (1.3)) we find the following asymptotics:

$$\begin{aligned}
 \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} &= M^* + O(\sqrt{\delta}, \varepsilon), \quad \text{where } M^* := \frac{\sqrt{\mu_1\mu_2}}{m_0 \sqrt[4]{a_1a_2}} \begin{pmatrix} \beta_{12} & \alpha_{12} \\ \alpha_{12} & \beta_{12} \end{pmatrix}, \\
 M_{jj} &= \sqrt{\delta} \beta_j + O(\delta, \varepsilon) && \text{for } j \geq 3, \\
 M_{ij} &= O(\sqrt{\delta}) && \text{for } i = 1, 2 \text{ and } j \geq 3, \text{ or } j = 1, 2 \text{ and } i \geq 3, \\
 M_{ij} &= O(\delta) && \text{for } i, j \geq 3 \text{ with } i \neq j.
 \end{aligned}$$

Therefore<sup>11</sup>

$$M = \begin{pmatrix} M^* + O(\sqrt{\delta}, \varepsilon) & O(\sqrt{\delta}) \\ O(\sqrt{\delta}) & \sqrt{\delta} M_* + O(\delta, \varepsilon) \end{pmatrix},$$

where

$$M_* := \text{diag}(\beta_3, \dots, \beta_N) \in \text{Mat}((N - 2) \times (N - 2)).$$

The eigenvalues of  $M^*$  are

$$\frac{\sqrt{\mu_1\mu_2}}{m_0 \sqrt[4]{a_1a_2}} (\beta_{12} + \alpha_{12}) \quad \text{and} \quad \frac{\sqrt{\mu_1\mu_2}}{m_0 \sqrt[4]{a_1a_2}} (\beta_{12} - \alpha_{12}),$$

which, by the first two requirements in (3.28), are nonzero, simple, and distinct. The matrix  $M_*$  is diagonal and its eigenvalues  $\beta_j$  are also nonzero, simple, and distinct by (3.28). The claim now follows by elementary linear algebra (compare, e.g., Lemma B.2 in Appendix B).  $\square$

<sup>10</sup>We use the standard notation  $a = O(\varepsilon) \iff \exists$  a constant  $c > 0$  (independent of  $\varepsilon$ ) and  $0 < \varepsilon_0 < 1$  s.t.  $|a| \leq c|\varepsilon|$  for all  $|\varepsilon| \leq \varepsilon_0$ ;  $O(\sigma, \varepsilon) = O(\sigma) + O(\varepsilon)$ .

<sup>11</sup>The  $O(\sqrt{\delta})$  in the upper right part of  $M$  is a  $(2 \times (N - 2))$  matrix while the  $O(\sqrt{\delta})$  in the lower left part of  $M$  is an  $((N - 2) \times 2)$  matrix.

*Remark 3.2.* (i) The hypotheses (3.28) of Proposition 3.2 are easily checked, for example, if  $a_j$  verifies (1.2) for a suitable  $\theta > 0$ . In fact the asymptotics for  $\mathcal{J}(s, 1)$  and  $\mathcal{I}(s, 1)$  (see (3.8)) yield immediately

$$\begin{aligned}\alpha_{12} &= -\frac{15}{64} \frac{1}{a_2} \left(\frac{a_1}{a_2}\right)^3 \left[1 + O\left(\left(\frac{a_1}{a_2}\right)^2\right)\right], \\ \beta_{12} &= \frac{3}{4} \frac{1}{a_2} \left(\frac{a_1}{a_2}\right)^2 \left[1 + O\left(\left(\frac{a_1}{a_2}\right)^2\right)\right], \\ \beta_j &= \frac{3}{4} \frac{\sqrt{\mu_j \mu_2}}{m_0} \frac{1}{a_j^{3/2}} \left(\frac{a_2}{a_j}\right)^{7/4} \left[1 + O\left(\left(\frac{a_1}{a_2}\right)^{7/4}\right) + O\left(\left(\frac{a_2}{a_j}\right)^2\right)\right], \quad j \geq 3, \\ \beta_{12} \pm \alpha_{12} &= \frac{3}{4} \frac{1}{a_2} \left(\frac{a_1}{a_2}\right)^2 \left[1 \mp \frac{5}{16} \frac{a_1}{a_2} + O\left(\left(\frac{a_1}{a_2}\right)^2\right)\right], \\ \beta_j - \beta_i &= \frac{3}{4} \frac{\sqrt{\mu_j \mu_2}}{m_0} \frac{1}{a_j^{3/2}} \left(\frac{a_2}{a_j}\right)^{7/4} \\ &\quad \times \left[1 + O\left(\left(\frac{a_1}{a_2}\right)^{7/4}\right) + O\left(\left(\frac{a_2}{a_j}\right)^2\right) + O\left(\left(\frac{a_i}{a_j}\right)^{13/4}\right)\right], \quad i > j \geq 3.\end{aligned}$$

Thus, if  $\theta$  is small enough and if (1.2) holds, one sees that

$$\begin{aligned}\alpha_{12} &< 0, \\ \beta_{12} \pm \alpha_{12} &> 0, \\ \beta_j &> 0 \quad \forall j \geq 3, \\ \beta_j - \beta_i &> 0 \quad \forall i > j \geq 3,\end{aligned}$$

and the hypotheses (3.28) are verified as claimed.

(ii) The  $O(\cdot)$ 's appearing in (3.29) (and in the proof of Proposition 3.2) depend on the  $a_j$ 's (and on<sup>12</sup>  $m_0$ ). Thus, the order in fixing the various parameters is important. One way of proceeding is as follows. First determine  $\theta$  as explained in the previous point (i). Then, let  $\bar{a}_i$ ,  $1 \leq i \leq N$ , be positive numbers such that (1.2) holds, i.e.,  $\bar{a}_i/\bar{a}_{i+1} < \theta$  for any  $1 \leq i \leq N-1$ ; (the  $\bar{a}_i$  may be physically interpreted as observed mean major semiaxis). Now, consider a compact order-one neighborhood  $A \subset \{0 < a_1 < \dots < a_N\}$  of  $(\bar{a}_1, \dots, \bar{a}_N)$  for which (1.2) continues to be valid (such neighborhood exists simply by continuity). Finally, fix  $\delta^*$  and  $\varepsilon_0$  so that Proposition 3.2 holds: such numbers will depend only on  $\bar{a}_j$ 's and the (order-one) size of the chosen neighborhood  $A$ .

(iii) In the case of only *one* dominant planet (i.e.,  $\mu_1 = \bar{\mu}_1 = O(1)$ ,  $\mu_i = O(\delta)$  for  $i \geq 2$ ), the first two asymptotics in (3.29) do not give any information: in particular we cannot assure that  $\bar{\Omega}_1$  and  $\bar{\Omega}_2$  are different from zero. On the other hand, one could also consider the case of three or more dominant planets and the choice of focusing on *two* dominant planets has been made for simplicity.

**4. Existence of  $N$ -dimensional elliptic invariant tori.** In this section we prove the existence of  $N$ -dimensional elliptic invariant tori for the  $(N+1)$ -body problem Hamiltonian  $\mathcal{H}$  in (2.14) for any  $N \geq 2$ .

Let  $m_0 < \bar{\mu}_j < 4m_0$ , let  $\theta$ ,  $A$ ,  $\delta^*$ , and  $\varepsilon_0$  be as in Remark 3.2(ii), and fix  $0 < \delta < \delta^*$ , which henceforth will be kept fixed. In the rest of the paper only  $\varepsilon$  is

<sup>12</sup>Recall that  $m_0 < \bar{\mu}_j < 4m_0$ ; compare with the line before (1.3).

regarded as a free parameter: at the moment,  $\varepsilon$  is assumed not to exceed  $\varepsilon_0$  but later will be required to satisfy stronger smallness conditions. The semimajor axis map

$$(4.1) \quad \vec{a} : \Lambda = (\Lambda_1, \dots, \Lambda_N) \mapsto (a(\Lambda_1; \mu_1, \varepsilon), \dots, a(\Lambda_N; \mu_N, \varepsilon))$$

is a real-analytic diffeomorphism and we define

$$\mathfrak{J} = \vec{a}^{-1}(A),$$

then the Hamiltonian  $\mathcal{H}$  is real-analytic (and bounded) on the domain  $\mathfrak{J} \times \mathbf{T}^N \times B_R^{2N}$  for a suitable  $R > 0$  (here  $B_r^n$  denotes the  $n$ -ball of radius  $r$  and center  $0 \in \mathbf{R}^n$ ).

By Proposition 3.1, the quadratic part  $\overline{\mathcal{H}}_{1,2}$  of the averaged Newtonian interaction  $\mathcal{H}_1$  has the simple form (3.6),  $M$  being the symmetric matrix defined in (3.7). As already pointed out in Remark 3.1, the matrix  $M$  can be diagonalized with eigenvalues, which, thanks to our assumptions and to Proposition 3.2, have the form in (3.29) and, therefore, satisfy

$$(4.2) \quad \inf_{\mathfrak{J}} |\overline{\Omega}_j| > \bar{c}, \quad \inf_{\mathfrak{J}} |\overline{\Omega}_i - \overline{\Omega}_j| > \bar{c}$$

for any  $i \neq j = 1, \dots, N$  and for a suitable positive constant  $\bar{c}$  independent of  $\varepsilon$ . If  $U := U(\Lambda)$  is the symmetric matrix which diagonalizes  $M$ ,  $U^T M U = \text{diag}(\overline{\Omega}_1, \dots, \overline{\Omega}_N)$ , then the map

$$(4.3) \quad \Xi : (I, \varphi, p, q) \mapsto (\Lambda, \lambda, \eta, \xi), \quad \text{where} \quad \begin{cases} p = U^T \eta, & q = U^T \xi, \\ I = \Lambda, \\ \varphi = \lambda + \sum_{h,k,\ell} (\partial_\Lambda U_{k\ell}) U_{h\ell} \eta_k \xi_\ell, \end{cases}$$

is symplectic (and real-analytic) and

$$(4.4) \quad \overline{\mathcal{H}}_{1,2} \circ \Xi = \frac{1}{2} \sum_{i=1}^N \overline{\Omega}_i(I) (p_i^2 + q_i^2).$$

Thus, the  $(N + 1)$ -body problem Hamiltonian  $\mathcal{H}$  in (2.14), in the case we are considering, can be written as

$$(4.5) \quad \mathcal{H} \circ \Xi(I, \varphi, p, q; \varepsilon) = h(I) + f(I, \varphi, p, q; \varepsilon)$$

with

$$h := \mathcal{H}_0, \quad f := \varepsilon f_1(I, p, q; \varepsilon) + \varepsilon f_2(I, \varphi, p, q; \varepsilon),$$

$$f_1 := f_{1,0}(I) + \frac{1}{2} \sum_{i=1}^N \overline{\Omega}_i(I) (p_i^2 + q_i^2) + \tilde{f}_1(I, p, q; \varepsilon),$$

$$f_{1,0} := \overline{\mathcal{H}}_{1,0}, \quad \tilde{f}_1 := \overline{\mathcal{H}}_{1,*} \circ \Xi, \quad f_2 := \tilde{\mathcal{H}}_1 \circ \Xi.$$

Here  $h$  is uniformly strictly concave,

$$|\tilde{f}_1| \leq \text{const} |(p, q)|^4, \quad \text{and} \quad \int_{\mathbf{T}^N} f_2 d\varphi = 0.$$

The construction of elliptic invariant tori for the Hamiltonian (4.5) is based on four steps, which we proceed to describe.



**4.1. Averaging.** Fix  $\tau > N - 1 \geq 1$  and pick two numbers  $b_1, b_2$  such that

$$(4.6) \quad 0 < b_1 < \frac{1}{2}, \quad 0 < b_2 < \left(\frac{1}{2} - b_1\right) \frac{1}{\tau + 1}.$$

Since the integrable Hamiltonian  $h$  depends only on the action  $I$ , the conjugated variable  $\varphi$  is a “fast” angle and, in “first approximation,” the  $(h + f)$ -motions are governed by the averaged Hamiltonian  $h + \varepsilon f_1$ , which possesses an elliptic equilibrium at  $p = q = 0$ . As we, now, proceed to describe, one may remove the  $\varphi$ -dependence of the perturbation function  $f$  up to high order in  $\varepsilon$  by using averaging theory; for detailed information on averaging theory in similar situations, see Proposition A.1 of [BCV03] or Proposition 7.1 of [BBV04].

Denote by  $D_R^n$  the complex  $n$ -ball of center zero and radius  $R > 0$  and, for any  $V \subset \mathbf{R}^N$ , denote by  $V_R$  the complex neighborhood of radius  $R > 0$  of the set  $V$  given by  $V_R := \cup_{x \in V} D_R(x)$ . Next, define the set  $\hat{\mathcal{J}}$  as the following “Diophantine subset” of  $\mathcal{J}$ :

$$(4.7) \quad \hat{\mathcal{J}} := \left\{ I \in \mathcal{J} : |\partial_I h(I) \cdot k| \geq \frac{\tilde{\gamma}}{|k|^\tau} \forall k \in \mathbf{Z}^N \setminus \{0\} \right\} \quad \text{with} \quad \tilde{\gamma} := \text{const } \varepsilon^{b_1}.$$

Notice that (as it is standard to prove)

$$(4.8) \quad \text{meas} \left( \mathcal{J} \setminus \hat{\mathcal{J}} \right) \leq \text{const } \tilde{\gamma} = \text{const } \varepsilon^{b_1}.$$

The Hamiltonian  $h + f$  in (4.5) is real-analytic on the complex domain

$$(4.9) \quad \mathfrak{D}_{r,s,\rho} := \hat{\mathcal{J}}_r \times \mathbf{T}_s^N \times D_\rho^{2N} \subset \mathbf{C}^{4N},$$

with

$$(4.10) \quad r := \text{const } \sqrt{\varepsilon}, \quad s := \text{const}, \quad \rho := \text{const } \varepsilon^{b_2}.$$

The definition of  $\hat{\mathcal{J}}$  is motivated by the necessity to have an estimate on small divisors. In fact, let  $I \in \hat{\mathcal{J}}_r$  (and  $\varepsilon$  small enough) and let  $I_0 \in \hat{\mathcal{J}}$  be a point at distance less than  $r$  from  $I$ . Then, for any  $k \in \mathbf{Z}^N \setminus \{0\}$  such that  $|k| \leq K := \text{const } \varepsilon^{-b_2}$ , by the second relation in (4.6), by (4.7), and by Cauchy estimates, one finds

$$(4.11) \quad \begin{aligned} |\partial_I h(I) \cdot k| &\geq |\partial_I h(I_0) \cdot k| - |\partial_I h(I_0) - \partial_I h(I)| |k| \\ &\geq \frac{\tilde{\gamma}}{K^\tau} - \max |\partial_I^2 h| r K \\ &\geq \frac{\tilde{\gamma}}{2K^\tau} =: \alpha = \text{const } \varepsilon^{b_1 + \tau b_2}, \quad \left( 0 < |k| \leq K := \text{const } \varepsilon^{-b_2} \right). \end{aligned}$$

In order to apply averaging theory (see, e.g., [N77]) so as to remove the  $\varepsilon$ -dependence up to order  $\exp(-\text{const } K)$ , one has to verify the following “smallness condition” (compare condition (A.2), p. 110 in [BCV03])

$$\|f\|_{r,s,\rho} \leq \text{const } \frac{\alpha \min\{rs, \rho^2\}}{K},$$

where the norm  $\|\cdot\|_{r,s,\rho}$  is defined as the standard “sup-Fourier norm”

$$(4.12) \quad \|f\|_{r,s,\rho} := \sum_{k \in \mathbf{Z}^N} \left( \sup_{(I,p,q) \in \hat{\mathcal{J}}_r \times D_\rho^{2N}} |f_k(I,p,q)| \right) e^{|k|s},$$

$(f_k(I, p, q)$  denoting Fourier coefficients of the multiperiodic, real-analytic function  $\varphi \mapsto f(I, \varphi, p, q)$ ). Such condition, in view of (4.6), can be achieved by taking  $\varepsilon$  small enough since, by (4.11) and (4.10), one has

$$\|f\|_{r,s,\rho} = O(\varepsilon) \quad \text{and} \quad \frac{\alpha \min\{rs, \rho^2\}}{K} = O(\varepsilon^{b_1+(\tau+1)b_2+1/2}).$$

Hence, there exists a close-to-identity (real-analytic) symplectic change of variables  $(I', \varphi', p', q') \mapsto (I, \varphi, p, q)$  verifying (compare formulae (2.16) and (A.7) of [BCV03])

$$(4.13) \quad |I' - I| \leq \text{const } \varepsilon^{\frac{1}{2}+b_2} \quad \text{and} \quad |p' - p|, |q' - q| \leq \text{const } \sqrt{\varepsilon},$$

and such that the Hamiltonian expressed in the new symplectic variables becomes

$$(4.14) \quad h(I') + \hat{g}(I', p', q') + \hat{f}(I', \varphi', p', q'), \quad \hat{g} := \varepsilon f_1(I', p', q') + \varepsilon \hat{f}_1(I', p', q')$$

with  $\hat{f}_1$  and  $\hat{f}$  real-analytic on the complex domain  $\mathfrak{D}_{r/2,s/6,\rho/2}$  and satisfying

$$(4.15) \quad \|\hat{f}_1\|_{r/2,s/6,\rho/2} \leq \frac{\varepsilon}{\text{or}} = \text{const } \varepsilon^{b_2+b_3} \quad \text{with} \quad b_3 := \frac{1}{2} - b_1 - (\tau + 1)b_2 > 0,$$

$$\|\hat{f}\|_{r/2,s/6,\rho/2} \leq \text{const } e^{-\text{const } K} \ll \text{const } \varepsilon^3.$$

**4.2. New elliptic equilibrium.** Due to the (small) term  $\hat{f}_1$  in (4.14), zero is no longer an elliptic equilibrium for the “averaged” (i.e.,  $\varphi$ -independent) Hamiltonian  $h + \hat{g}$ . Using the implicit function theorem, we can find a new elliptic equilibrium for  $h + \hat{g}$ , which is  $\varepsilon^{b_2+b_3}$  close to zero. Hence we construct a real-analytic symplectic transformation

$$(4.16)$$

$$(J', \psi', v', u') \mapsto (I', \varphi', p', q') \quad \text{with} \quad I' = J' \quad \text{and} \quad \varepsilon^{b_2+b_3}\text{-close-to-the-identity},$$

such that in the new symplectic variables  $(J', \psi', v', u')$  the Hamiltonian takes the form

$$h(J') + \tilde{g}(J', v', u') + \tilde{f}(J', \psi', v', u')$$

with  $\tilde{g}$  having  $v' = u' = 0$  as elliptic equilibrium; the functions  $\tilde{g}$  and  $\tilde{f}$  are real-analytic on a slightly smaller complex domain, say  $\mathfrak{D}_{r/7,s/7,\rho/7}$ , where they satisfy bounds similar to those in (4.15). Furthermore, for  $j = 1, \dots, N$ , the eigenvalues  $\tilde{\Omega}_j(J')$  of the symplectic quadratic part of  $\tilde{g}$  are purely imaginary and  $\varepsilon^{1+b_2+b_3}$ -close to  $\varepsilon\tilde{\Omega}_j(J')$ .

**4.3. Symplectic diagonalization of the quadratic term.** Using a well-known result on the symplectic diagonalization of quadratic Hamiltonians, we can find a real-analytic, symplectic transformation

$$(4.17)$$

$$(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}) \mapsto (J', \psi', v', u') \quad \text{with} \quad J' = \tilde{J} \quad \text{and} \quad \varepsilon^{b_2+b_3}\text{-close-to-the-identity},$$

such that the quadratic part of  $\tilde{g}$  becomes, simply,  $\sum_{i=1}^N \tilde{\Omega}_i(\tilde{J}) (\tilde{u}_i^2 + \tilde{v}_i^2)$ . Whence, the new Hamiltonian becomes (compare formula (2.22) of [BCV03])

$$(4.18) \quad \tilde{\mathcal{H}} := h_0(\tilde{J}) + \sum_{i=1}^N \tilde{\Omega}_i(\tilde{J}) (\tilde{u}_i^2 + \tilde{v}_i^2) + \tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u}) + \tilde{f}_0(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}),$$

where

$$(4.19) \quad h_0(\tilde{J}) := h(\tilde{J}) + \varepsilon \tilde{g}(\tilde{J}, 0, 0),$$

$\tilde{g}_0, \tilde{f}_0, \tilde{\Omega}_j$  are real-analytic, and

$$(4.20)$$

$$|\tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u})| \leq \text{const } \varepsilon |(\tilde{v}, \tilde{u})|^3, \quad |\tilde{\Omega}| \leq \text{const } \varepsilon, \quad \|\tilde{f}\|_{r/8, s/8, \rho/8} \leq \text{const } \varepsilon^3.$$

Finally, because of (4.2),

$$(4.21) \quad \inf |\tilde{\Omega}_i| \geq \text{const } \varepsilon > 0, \quad \inf |\tilde{\Omega}_2 - \tilde{\Omega}_1| \geq \text{const } \varepsilon > 0.$$

**4.4. Applying KAM theory.** We rewrite now the Hamiltonian  $\tilde{\mathcal{H}}$  in (4.18) in a form suitable for applying (elliptic) KAM theory. Introducing translated variables  $y := \tilde{J} - \mathbf{p}$  and complex variables  $z, \bar{z}$ , we define

$$(4.22) \quad H = \tilde{\mathcal{H}}\left(\mathbf{p} + y, \psi, \frac{z + l\bar{z}}{\sqrt{2}}, \frac{z - \bar{z}}{i\sqrt{2}}\right),$$

here  $\mathbf{p}$  is regarded as a parameter and the symplectic form is  $\sum_{j=1}^N dy_j \wedge d\psi_j + i \sum_{j=1}^N dz_j \wedge d\bar{z}_j$  with  $i := \sqrt{-1}$ . The Hamiltonian  $H$  is then seen to have the form

$$H = \mathcal{N} + P$$

with

$$(4.23) \quad \mathcal{N} = e + \omega \cdot y + \sum_{j=1}^N \Omega_j z_j \bar{z}_j, \quad e := h_0(\mathbf{p}), \quad \omega := \partial_{\tilde{J}} h_0(\mathbf{p}), \quad \Omega := \tilde{\Omega}(\mathbf{p}),$$

and  $P$  a perturbation, which can naturally be split into four terms:

$$P = \sum_{1 \leq k \leq 4} P_k$$

with

$$(4.24) \quad \begin{aligned} P_1 &= h_0(\mathbf{p} + y) - h_0(\mathbf{p}) - \partial_{\tilde{J}} h_0(\mathbf{p}) \cdot y \sim y^2, \\ P_2 &= \sum_{j=1}^n \left( \tilde{\Omega}_j(\mathbf{p} + y) - \tilde{\Omega}_j(\mathbf{p}) \right) z_j \bar{z}_j \sim y|z||\bar{z}|, \\ P_3 &= \tilde{g}_0\left(\mathbf{p} + y, \frac{z + \bar{z}}{\sqrt{2}}, \frac{z - \bar{z}}{i\sqrt{2}}\right) \sim \varepsilon(|z| + |\bar{z}|)^3, \quad (\text{by (4.20)}), \\ P_4 &= \tilde{f}_0\left(\mathbf{p} + y, \psi, \frac{z + \bar{z}}{\sqrt{2}}, \frac{z - \bar{z}}{i\sqrt{2}}\right) = O(\varepsilon^3). \end{aligned}$$

The parameter  $\mathbf{p}$  runs over the Diophantine set  $\hat{\mathcal{J}}$  defined in (4.7). Notice that the integrable Hamiltonian  $\mathcal{N}$  affords, for any given value of the parameter  $\mathbf{p}$ , the  $N$ -dimensional elliptic torus

$$(4.25) \quad \{y = 0\} \times \mathbf{T}^N \times \{z = \bar{z} = 0\},$$

which is invariant for the Hamiltonian flow generated by  $\mathcal{N}$ , the flow being, simply, the Diophantine translation  $x \mapsto x + \omega t$ , with  $\omega$  as in (4.23).

Since  $\det \partial_{\bar{j}}^2 h_0 \neq 0$ , we can use the frequencies  $\omega$  as parameters rather than the actions  $\mathbf{p}$ . We, therefore, set

$$(4.26) \quad \mathcal{O} := \partial_{\bar{j}} h_0(\hat{\mathcal{J}}) = \left\{ \omega = \partial_{\bar{j}} h_0(\mathbf{p}) : \mathbf{p} \in \hat{\mathcal{J}} \right\}.$$

Notice that, by (4.7), we have

$$(4.27) \quad \text{meas}(\partial_{\bar{j}} h_0(\hat{\mathcal{J}}) \setminus \mathcal{O}) \leq \text{const } \varepsilon^{b_1}.$$

Now, if we put  $\mathbf{p} = \mathbf{p}(\omega) := (\partial_{\bar{j}} h_0)^{-1}(\omega)$  in (4.22), we can rewrite the  $(N + 1)$ -body Hamiltonian in the form

$$(4.28) \quad H(y, \psi, z, \bar{z}; \omega) := \mathcal{N}(y, z, \bar{z}; \omega) + P(y, \psi, z, \bar{z}; \omega),$$

where

$$(4.29) \quad \mathcal{N}(y, z, \bar{z}; \omega) := e(\omega) + \omega \cdot y + \sum_{j=1}^N \Omega_j(\omega) z_j \bar{z}_j, \quad e(\omega) := h_0(\mathbf{p}(\omega)), \quad \Omega(\omega) := \tilde{\Omega}(\mathbf{p}(\omega)),$$

and the perturbation  $P(y, \psi, z, \bar{z}; \omega)$  is obtained by replacing  $\mathbf{p}$  with  $\mathbf{p}(\omega)$  in (4.24). Recalling (4.10), the Hamiltonian  $H$  in (4.28) is real-analytic in

$$(4.30) \quad (y, \psi, z, \bar{z}; \omega) \in \mathfrak{D}_{\mathbf{r}^2, \mathfrak{s}, \mathbf{r}, \mathfrak{d}} := D_{\mathbf{r}^2}^N \times \mathbf{T}_{\mathfrak{s}}^N \times D_{\mathbf{r}}^{2N} \times \mathcal{O}_{\mathfrak{d}}^N$$

with

$$(4.31) \quad \mathbf{r} := \text{const } , \varepsilon^{3/4}, \quad \mathfrak{s} := \text{const}, \quad \mathfrak{d} := \text{const } \sqrt{\varepsilon}.$$

We recall, now, a well-known KAM result concerning the persistence of lower-dimensional elliptic tori for nearly integrable Hamiltonian systems (see [M65], [E88], [K88]). The version we present here is, essentially, a reformulation of Pöschel's theorem in [P89] (compare, also, with Theorem 5.1 of [BBV04]).

**THEOREM 4.1.** *Let  $H$  have the form in (4.28), (4.29) and let it be real-analytic on a domain  $\mathfrak{D}_{\mathbf{r}^2, \mathfrak{s}, \mathbf{r}, \mathfrak{d}}$  of the form (4.30) for some  $\mathbf{r}$ ,  $\mathfrak{s}$ , and  $\mathfrak{d}$  positive. Assume that*

$$(4.32) \quad \sup_{\omega \in \mathcal{O}_{\mathfrak{d}}} |\partial_{\omega} \Omega(\omega)| \leq \frac{1}{4}$$

*and that the nonresonance (or Melnikov) condition*

$$(4.33) \quad |\Omega(\omega) \cdot k| \geq \gamma_0 \quad \forall 1 \leq |k| \leq 2, \quad k \in \mathbf{Z}^N, \forall \omega \in \mathcal{O},$$

*is satisfied for some  $\gamma_0 > 0$ . Then, if  $\mathfrak{d} \geq \gamma_0$  and  $P$  is sufficiently small, i.e.,*

$$(4.34) \quad \|P\|_{\mathbf{r}, \mathfrak{s}, \mathfrak{d}} := \sup_{\omega \in \mathcal{O}_{\mathfrak{d}}} \|P(\cdot; \omega)\|_{\mathbf{r}^2, \mathfrak{s}, \mathbf{r}} \leq \text{const } \gamma_0 \mathbf{r}^2,$$

*then there exist a normal form  $\mathcal{N}_* := e_*(\omega) + \omega \cdot y_* + \Omega_*(\omega) z_* \bar{z}_*$ , a Cantor set  $\mathcal{O}(\gamma_0) \subset \mathcal{O}$  with*

$$(4.35) \quad \text{meas}(\mathcal{O} \setminus \mathcal{O}(\gamma_0)) \leq \text{const } \gamma_0,$$

and a transformation

$$\mathcal{F} : D_{\tau^2/4}^N \times \mathbf{T}_{s/2}^N \times D_{\tau/2}^{2N} \times \mathcal{O}(\gamma_0) \longrightarrow D_{\tau^2}^N \times \mathbf{T}_s^N \times D_{\tau}^{2N} \times \mathcal{O}_\delta$$

$$(y_*, \psi_*, z_*, \bar{z}_*; \omega) \longmapsto (y, \psi, z, \bar{z}; \omega)$$

real-analytic and symplectic for each  $\omega$  and Whitney smooth in  $\omega$ , such that

$$(4.36) \quad H \circ \mathcal{F} = \mathcal{N}_* + R_* \quad \text{with} \quad \partial_{y_*}^j \partial_{z_*}^h \partial_{\bar{z}_*}^k R_* = 0 \quad \text{if} \quad 2|j| + |h+k| \leq 2.$$

In particular, for each  $\omega \in \mathcal{O}(\gamma_0)$ , the torus  $\{y_* = 0\} \times \mathbf{T}^N \times \{z_* = \bar{z}_* = 0\}$  is an  $N$ -dimensional, linearly elliptic, invariant torus run by the flow  $\psi_* \rightarrow \psi_* + \omega t$ . Finally

$$(4.37) \quad |y_* - y|, \quad \tau|z_* - z|, \quad \tau|\bar{z}_* - \bar{z}| \leq \text{const} \frac{\|P\|_{\tau, s, \delta}}{\gamma_0}.$$

In this section we have shown that the many-body Hamiltonian (2.14) (under the hypotheses spelled out at the beginning of the section) has indeed the form assumed in the KAM theorem (Theorem 4.1). Furthermore, by (4.21), the elliptic frequencies  $\Omega_i$  verify the Melnikov conditions (4.33) with

$$(4.38) \quad \gamma_0 = \text{const} \varepsilon,$$

and, by (4.24) and (4.31), the perturbation  $P$  verifies, for small  $\varepsilon$ , the KAM condition (4.34), since

$$(4.39) \quad \|P\|_{\tau, s, \delta} = O(\tau^4 + \varepsilon\tau^3 + \varepsilon^3) = O(\varepsilon^3) \leq \text{const} \gamma_0 \tau^2 = O(\varepsilon^{5/2}).$$

Thus, the existence of the desired quasi-periodic orbits follows at once from Theorem 4.1. We may summarize the final result as follows.

**THEOREM 4.2.** *Let  $N \geq 2$  and let  $\mathcal{H}$  be the  $(N + 1)$ -body problem Hamiltonian in Poincaré variables defined in (2.14). Let  $m_0 < \bar{\mu}_j < 4m_0$ , let  $\theta, A, \delta^*$ , and  $\varepsilon_0$  be as in Remark 3.2(ii). Fix  $0 < \delta < \delta^*$  and let  $\mathfrak{J} = \bar{a}^{-1}(A)$  where  $\bar{a}$  is the semimajor axis map defined in (4.1). Let  $\tau > N - 1$  and pick  $b_1, b_2$  as in (4.6). Finally, let  $0 < \varepsilon^* < \varepsilon_0$  be such that (4.39) holds for any  $\varepsilon \leq \varepsilon^*$  and such that all conditions on  $\varepsilon$  required for constructing the symplectic transformations introduced in sections 4.1–4.3 are satisfied for  $\varepsilon < \varepsilon^*$ . Then, for any  $\varepsilon < \varepsilon^*$ , there exist a Cantor set  $\mathfrak{J}_* \subset \mathfrak{J}$ , with*

$$(4.40) \quad \text{meas}(\mathfrak{J} \setminus \mathfrak{J}_*) \leq \text{const} \varepsilon^{b_1},$$

and a Lipschitz continuous family of tori embedding

$$\phi : (\vartheta, \mathbf{p}) \in \mathbf{T}^N \times \mathfrak{J}_* \mapsto \left( \Lambda(\vartheta; \mathbf{p}), \lambda(\vartheta; \mathbf{p}), \eta(\vartheta; \mathbf{p}), \xi(\vartheta; \mathbf{p}) \right) \in \mathfrak{J} \times \mathbf{T}^N \times B_{\rho_*}^{2N}$$

with  $\rho_* := \text{const} \varepsilon^{b_2}$  such that, for any  $\mathbf{p} \in \mathfrak{J}_*$ ,  $\phi(\mathbf{T}^N; \mathbf{p})$  is a real-analytic elliptic  $\mathcal{H}$ -invariant torus, on which the  $\mathcal{H}$ -flow is analytically conjugated to the linear flow  $\vartheta \rightarrow \vartheta + \omega_* t$ ,  $\omega_*$  being  $(\gamma, \tau)$ -diophantine with  $\gamma = O(\varepsilon^{b_1})$ . Furthermore, the following bounds hold uniformly on  $\mathbf{T}^N \times \mathfrak{J}_*$ :

$$(4.41) \quad |\Lambda(\vartheta; \mathbf{p}) - \mathbf{p}| \leq \text{const} \varepsilon^{\frac{1}{2} + b_2},$$

$$(4.42) \quad |\eta(\vartheta; \mathbf{p})| + |\xi(\vartheta; \mathbf{p})| \leq \text{const} \varepsilon^{b_2}.$$

Theorem 1.1 follows, now, by taking (recall the definitions of  $b_k$  in (4.6))

$$(4.43) \quad c_1 := b_1, \quad c_2 := b_2, \quad c_3 := b_2 + \frac{1}{2}.$$

In particular the statements on the density of the set of the osculating major semiaxes, on the bound on the osculating eccentricities and on the variation of the osculating major semiaxes, follows from (4.40), (2.9), (4.42), (2.7), and (4.41).

**Appendix A. Poincaré variables for the planar  $(N + 1)$ -body problem.**

We briefly recall in this appendix the classical derivation of the Poincaré variables for the planar  $N$ -body problem,<sup>13</sup> showing, in particular, the validity of Lemma 2.1, which is proven in subsections A.1 and A.2; subsections A.3 and A.4 are included for completeness.

**A.1. Canonical variables for the two-body problem.** Consider two bodies  $P_0, P_1$  of masses  $m_0, m_1$  and position  $u^{(0)}, u^{(1)} \in \mathbf{R}^2$  (with respect to an inertial frame). We assume that  $P_0$  and  $P_1$  interact through gravity, with gravitational constant 1. By Newton’s laws, the equations of motion for such two-body problem are

$$\begin{aligned} \ddot{u}^{(0)} &= m_1 \frac{(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3}, \\ \ddot{u}^{(1)} &= m_0 \frac{(u^{(0)} - u^{(1)})}{|u^{(0)} - u^{(1)}|^3}. \end{aligned}$$

Let

$$(A.1) \quad \mathfrak{M} := m_0 + m_1, \quad m := \frac{m_0 m_1}{\mathfrak{M}}, \quad x := u^{(1)} - u^{(0)}, \quad X := m\dot{x}.$$

Then, the above equations of motion become

$$\ddot{x} = \frac{\mathfrak{M}x}{|x|^3},$$

and the motion of the two bodies is governed by the Hamiltonian

$$(A.2) \quad \mathcal{K}(X, x) = \frac{1}{2m}|X|^2 - \frac{m\mathfrak{M}}{|x|},$$

with  $(X, x) \in \mathbf{R}^2 \times \mathbf{R}^2$  conjugate variables; i.e., the equations of motion are  $\dot{x} = \partial_X \mathcal{K}$ ,  $\dot{X} = -\partial_x \mathcal{K}$ .

As well known, such system is integrable and for  $\mathcal{K} < 0$  the orbits are ellipses. More precisely, one has the following proposition.

PROPOSITION A.1. Fix  $\Lambda_- > 0 > \mathcal{K}_0$  and let  $\Lambda_+ := \left(\frac{m^3 \mathfrak{M}^2}{-2\mathcal{K}_0}\right)^{\frac{1}{2}} > \Lambda_-$ . Then, there exist  $\hat{\rho} > 0$  and a real-analytic symplectic transformation

$$\Psi_{\text{DP}} : \left( (\Lambda, \eta), (\lambda, \xi) \right) \in \left( [\Lambda_-, \Lambda_+] \times [-\hat{\rho}, \hat{\rho}] \right) \times \left( \mathbf{T} \times [-\hat{\rho}, \hat{\rho}] \right) \mapsto (X, x) \in \left\{ |x| \geq \frac{\hat{\rho}^2}{m^2 \mathfrak{M}} \right\},$$

casting (A.2) into the integrable Hamiltonian  $(-m^3 \mathfrak{M}^2)/(2\Lambda^2)$ .

<sup>13</sup>For a review of the Poincaré variables in the nonplanar case, see, for instance, [Ch88] and [BCV03].

This classical proposition is a planar version of the classical one, due to Poincaré (see [Poi1905, Chapter III]) and the variables  $(\Lambda, \eta, \lambda, \xi)$  are, usually, called (planar) Poincaré variables. The proof of Proposition A.1 is particularly interesting from the physical point of view and rests upon the introduction of three different (famous) changes of variables, which we, now, proceed to describe briefly.

Let  $\ell$  and  $g$  denote, respectively, the mean anomaly and the argument of the perihelion.

*Step 1.* The system is set in “*symplectic*” *polar variables*; namely, we consider the symplectic map  $\Psi_{\text{spc}} : ((R, \Phi), (r, \varphi)) \mapsto (X, x)$  (where  $r > 0$  and  $\varphi \in \mathbf{T}$ ) given by

$$(A.3) \quad \Psi_{\text{spc}} : \begin{cases} x_1 = r \cos \varphi, \\ x_2 = r \sin \varphi, \end{cases} \quad X = \begin{pmatrix} \cos \varphi & -\frac{\sin \varphi}{r} \\ \sin \varphi & \frac{\cos \varphi}{r} \end{pmatrix} \begin{pmatrix} R \\ \Phi \end{pmatrix}$$

and consider the new Hamiltonian  $\mathcal{K}_{\text{spc}} := \mathcal{K} \circ \Psi_{\text{spc}}$ .

*Step 2.* There is a symplectic map  $\Psi_{\text{D}} : ((L, G), (\ell, g)) \mapsto ((R, \Phi), (r, \varphi))$  that integrates the system:  $\Psi_{\text{D}}$  is obtained via the generating function

$$(A.4) \quad S(L, G, r, \varphi) = \int \sqrt{-\frac{\mathfrak{m}^4 \mathfrak{M}^2}{L^2} + \frac{2\mathfrak{m}^2 \mathfrak{M}}{r} - \frac{G^2}{r^2}} dr + G\varphi.$$

The variables  $((L, G), (\ell, g))$  are known as (planar) *Delaunay variables*. In such variables, the new Hamiltonian becomes

$$\mathcal{K}_{\text{D}} := \mathcal{K}_{\text{spc}} \circ \Psi_{\text{D}} = -\frac{\mathfrak{m}^3 \mathfrak{M}^2}{2L^2}.$$

Also, if  $C$  is the angular momentum of the planet and  $a$  is the major semiaxis, by construction, one has that

$$G = |C| \quad \text{and} \quad L = \mathfrak{m}\sqrt{\mathfrak{M}a}.$$

*Step 3.* We need now to remove singularities, which appear for small eccentricity. To this aim, we first introduce (planar) *Poincaré action-angle variables* by means of the linear symplectic transformation

$$\Psi_{\text{Paa}} : ((\Lambda, H), (\lambda, h)) \mapsto ((L, G), (\ell, g))$$

given by

$$(A.5) \quad \Psi_{\text{Paa}} : \begin{cases} \Lambda = L, & H = L - G, \\ \lambda = \ell + g, & h = -g. \end{cases}$$

Then, we let  $\Psi_{\text{P}} : ((\Lambda, \eta), (\lambda, \xi)) \mapsto ((\Lambda, H), (\lambda, h))$  be the symplectic map defined by

$$(A.6) \quad \sqrt{2H} \cos h = \eta, \quad \sqrt{2H} \sin h = \xi.$$

As Poincaré showed (see [Poi1905], [Ch88], [BCV03]), the symplectic map

$$\Psi_{\text{DP}} : ((\Lambda, \eta), (\lambda, \xi)) \mapsto (X, x)$$

with

$$(A.7) \quad \Psi_{\text{DP}} := \Psi_{\text{spc}} \circ \Psi_{\text{D}} \circ \Psi_{\text{Paa}} \circ \Psi_{\text{P}}$$

is real-analytic in a (complex) neighborhood of

$$\Lambda \in [\Lambda_-, \Lambda_+], \quad |\eta|, |\xi| \leq \text{const} \sqrt{\Lambda_-}, \quad \lambda \in \mathbf{T}.$$

Also, the two-body Hamiltonian, in Poincaré variables, is  $\mathcal{K} \circ \Psi = -\frac{m^3 \mathfrak{M}^2}{2\Lambda^2}$ .

*Remark A.1.* (i) If we denote  $(X, x) = \Phi_{\text{DP}}((\Lambda, \eta, p), (\lambda, \xi, q))$ , then

$$(A.8) \quad X = \frac{m^4 \mathfrak{M}^2}{\Lambda^3} \frac{\partial x}{\partial \lambda}.$$

Indeed, from the Hamilton equations one sees that:  $\dot{\lambda} = \partial_{\Lambda}(-\frac{m^3 \mathfrak{M}^2}{2\Lambda^2}) = \frac{m^3 \mathfrak{M}^2}{\Lambda^3}$ , and  $\dot{\Lambda} = \dot{\xi} = \dot{\eta} = \dot{p} = \dot{q} = 0$ . Thus, by the chain rule,  $X = m\dot{x} = m(\partial_{\lambda} x) \dot{\lambda} = \frac{m^4 \mathfrak{M}^2}{\Lambda^3} \frac{\partial x}{\partial \lambda}$ , proving (A.8).

(ii) We collect some useful relations among the above-introduced quantities. Let, as usual,  $e$  denote the eccentricity of the Keplerian ellipse and let  $a$  denote the major semiaxis. Then, by construction, one sees that

$$(A.9) \quad \Lambda = m\sqrt{\mathfrak{M}a}, \quad \sqrt{\xi^2 + \eta^2} = \sqrt{\Lambda} e (1 + O(e^2)).$$

Also, if  $C$  is the angular momentum of the system, one infers that

$$(A.10) \quad |C| = \Lambda \sqrt{1 - e^2} = \Lambda(1 + O(e^2)).$$

(iii) A proof of the analyticity of Poincaré variables will also follow by directly inspecting the formulae given in Lemma 2.1, which is proved in the coming section.

**A.2. Orbital elements.** We now sketch a way to explicitly represent some quantities in terms of Poincaré variables. This will also lead to the proof of Lemma 2.1. Let  $u$  and  $v$  denote the eccentric anomaly and the true anomaly, respectively. By geometric considerations,

$$(A.11) \quad u = \ell + e \sin u$$

and<sup>14</sup>

$$(A.12) \quad \cos v = \frac{\cos u - e}{1 - e \cos u},$$

where

$$(A.13) \quad \ell = \lambda + h.$$

Also, by (A.6),

$$(A.14) \quad H = \frac{\eta^2 + \xi^2}{2}.$$

An explicit expression taking into account  $H$ , the eccentricity, and the major semiaxis is given by

$$(A.15) \quad H = \Lambda (1 - \sqrt{1 - e^2}) = \Lambda \frac{e^2}{2} (1 + O(e^2)),$$

$$(A.16) \quad e(H, \Lambda) = \sqrt{\frac{H}{\Lambda} \left( 2 - \frac{H}{\Lambda} \right)}.$$

<sup>14</sup>Such relations are classical and we refer the reader to [Ch88] and [BCV03] for a geometric interpretation of these anomalies.



In light of (A.12),

$$(A.17) \quad \sin v = \frac{\sqrt{1-e^2} \sin u}{1-e \cos u}.$$

By means of (A.11), we have

$$u - \ell = e \sin(u - \ell + \ell) = e \cos \ell \sin(u - \ell) + e \sin \ell \cos(u - \ell).$$

Thus, in the notation of Lemma 2.1, if  $G_0$  is implicitly defined by

$$G_0(x, y) = x \sin G_0(x, y) + y \cos G_0(x, y),$$

with  $G_0(0, 0) = 0$ , we have that  $G_0$  is real-analytic,  $G_0(x, y) = y + xy + O_3(x, y)$  and

$$(A.18) \quad u - \ell = G_0(e \cos \ell, e \sin \ell).$$

Therefore, we deduce from (A.18) and (A.13) that

$$(A.19) \quad u = \lambda + h + G_0(e \cos h \cos \lambda - e \sin h \sin \lambda, e \sin h \cos \lambda + e \cos h \sin \lambda).$$

Moreover, denoting

$$(A.20) \quad \hat{\eta} = \eta/\sqrt{\Lambda}, \quad \hat{\xi} = \xi/\sqrt{\Lambda},$$

we deduce from (A.6) and (A.16) that

$$(A.21) \quad e \sin h = \sqrt{\frac{2H}{\Lambda}} \cdot \sqrt{1 - \frac{H}{2\Lambda}} \sin h = \hat{\xi} F_1(\hat{\eta}^2 + \hat{\xi}^2),$$

where  $F_1(t) = \sqrt{1 - (t/4)}$  is real-analytic for  $|t| < 4$  (and agrees with the one introduced in Lemma 2.1). Analogously,

$$(A.22) \quad e \cos h = \hat{\eta} F_1(\hat{\eta}^2 + \hat{\xi}^2).$$

Therefore, substituting (A.21) and (A.22) in (A.19), we can write  $G_0$  as an analytic expression of  $(\hat{\eta}, \hat{\xi}, \lambda)$ : more formally, there exists a real-analytic  $(\hat{\eta}, \hat{\xi}, \lambda) \mapsto G(\hat{\eta}, \hat{\xi}, \lambda)$  (which agrees with the one introduced in (2.6) by (A.21) and (A.22)), so that

$$G_0(e \cos h \cos \lambda - e \sin h \sin \lambda, e \sin h \cos \lambda + e \cos h \sin \lambda) = G(\hat{\eta}, \hat{\xi}, \lambda).$$

Hence, from (A.19),

$$(A.23) \quad \begin{aligned} e \cos u &= e \cos h \cos(\lambda + G) - e \sin h \sin(\lambda + G), \\ e \sin u &= e \sin h \cos(\lambda + G) + e \cos h \sin(\lambda + G) \end{aligned}$$

with  $G = G(\hat{\eta}, \hat{\xi}, \lambda)$ . Notice also that, from the formulae in (A.16) and (A.14),

$$\frac{1 - \sqrt{1 - e^2}}{e^2} = F_2(\hat{\eta}^2 + \hat{\xi}^2)$$

for a suitable real-analytic function  $F_2$  (actually,  $F_2(t) = \frac{1}{2}(1 - \frac{t}{4})^{-1}$ , which agrees with the notation in Lemma 2.1). Thus, if we set  $\varphi = \lambda + v - \ell = v - h$ , recalling also

(A.12) and (A.17), we have

$$\begin{aligned}
 \sin \varphi &= \sin v \cos h - \cos v \sin h \\
 &= \frac{1}{1 - e \cos u} [\sqrt{1 - e^2} \sin u \cos h - \cos u \sin h + e \sin h] \\
 &= \frac{1}{1 - e \cos u} [\sin(u - h) + e \sin h - F_2 \cdot (e \sin u) \cdot (e \cos h)] \\
 \text{(A.24)} \quad &= \frac{1}{1 - e \cos u} [\sin(\lambda + e \sin u) + e \sin h - F_2 \cdot (e \sin u) \cdot (e \cos h)]
 \end{aligned}$$

for  $F_2 = F_2(\hat{\eta}^2 + \hat{\xi}^2)$  and analogously

$$\text{(A.25)} \quad \cos \varphi = \frac{1}{1 - e \cos u} [\cos(\lambda + e \sin u) - e \cos h - F_2 \cdot (e \sin u) \cdot (e \sin h)].$$

Hence, from (A.21), (A.22), (A.23), (A.24), and (A.25), it follows that  $\sin \varphi$  and  $\cos \varphi$  are real-analytic functions in  $\lambda, \hat{\eta}, \hat{\xi}$ , for  $\lambda \in \mathbf{T}$  and small  $\hat{\xi}, \hat{\eta}$ . In particular, if  $\mathcal{C}, \mathcal{S}$ , and  $\mathcal{E}_s$  are as defined in Lemma 2.1, we deduce from (A.23), (A.21), and (A.22) that

$$\text{(A.26)} \quad e \sin u = \mathcal{E}_s,$$

and then from (A.25) and (A.24) that

$$\text{(A.27)} \quad (1 - e \cos u) \cos \varphi = \mathcal{C} \quad \text{and} \quad (1 - e \cos u) \sin \varphi = \mathcal{S}.$$

Finally, by geometric considerations, we have

$$\text{(A.28)} \quad r = a(1 - e \cos u),$$

where  $r$  is the distance between the planet and the sun. Thus, the formulae in Lemma 2.1 follow at once by (A.26), (A.27), (A.3), and (A.8).

**A.3. Hamiltonian setting for the planar many-body problem.** Consider  $(N + 1)$  bodies  $P_0, \dots, P_N$  of masses  $m_0, \dots, m_N$ , all lying in the same plane, interacting through gravity (with constant of gravitation 1). Denote by  $u^{(i)}$  the position of  $P_i$  in a given inertial frame of  $\mathbf{R}^2$ , with origin in the center of mass of the system. By Newton’s laws, we have that

$$\text{(A.29)} \quad \ddot{u}^{(i)} = \sum_{0 \leq j \neq i \leq N} \frac{m_j(u^{(j)} - u^{(i)})}{|u^{(j)} - u^{(i)}|^3}.$$

Thus, if  $U^{(i)} := m_i \dot{u}^{(i)}$  denotes the momentum of  $P_i$ , we see that the equations of motion (A.29) come from the Hamiltonian

$$\sum_{i=0}^N \frac{1}{2m_i} |U^{(i)}|^2 - \sum_{0 \leq i < j \leq N} \frac{m_i m_j}{|u^{(i)} - u^{(j)}|},$$

where  $U = (U^{(0)}, \dots, U^{(N)}) \in \mathbf{R}^{2(N+1)}$  and  $u = (u^{(0)}, \dots, u^{(N)}) \in \mathbf{R}^{2(N+1)}$  are conjugate symplectic variables.

We now consider  $P_0$  as the “sun” and introduce canonical heliocentric variables via the linear symplectic transformation

$$\text{(A.30)} \quad \begin{aligned}
 u^{(0)} &= r^{(0)}, & u^{(i)} &= r^{(0)} + r^{(i)}, \\
 U^{(0)} &= R^{(0)} - R^{(1)} - \dots - R^{(N)}, & U^{(i)} &= R^{(i)}, \quad \text{for } i = 1, \dots, N.
 \end{aligned}$$

Notice that, by our choice of coordinates,  $R^{(0)} = 0$ . Thus, the planar many-body problem is governed by the  $(2N)$ -degree-of-freedom Hamiltonian

$$\sum_{i=1}^N \left( \frac{m_0 + m_i}{2m_0 m_i} |R^{(i)}|^2 - \frac{m_0 m_i}{|r^{(i)}|} \right) + \sum_{1 \leq i < j \leq N} \left( \frac{R^{(i)} \cdot R^{(j)}}{m_0} - \frac{m_i m_j}{|r^{(i)} - r^{(j)}|} \right).$$

If  $m_i = \varepsilon \mu_i$  for  $i = 1, \dots, N$ , i.e., if the “planets” are very much smaller than the “sun,” the momenta  $R^{(i)}$  are of order  $\varepsilon$ . Therefore, it is convenient to introduce the following rescaled symplectic variables:

$$(A.31) \quad X^{(i)} = \frac{R^{(i)}}{\varepsilon m_0^{5/3}}, \quad x^{(i)} = \frac{r^{(i)}}{m_0^{2/3}}, \quad i = 1, \dots, N.$$

In such variables, after a time scale of factor  $\varepsilon m_0^{7/3}$  we obtain the Hamiltonian in (2.1). Notice that, in that setting, the Hamiltonian  $\mathcal{H}_0^{(N)}$  corresponds to the sum of  $N$  integrable Hamiltonians of the form (A.2), with  $\mathbf{m}$  and  $\mathfrak{M}$  replaced by  $m_i$  and  $M_i$ , respectively.

**A.4. A parity property.** We recall here the well-known fact that the  $\lambda$ -average of  $\mathcal{H}_1$  (as in (2.14)) is even in  $(\eta, \xi)$ . The proof of this will be accomplished by a 180-degree rotation of the perihelia.

PROPOSITION A.2. *Let*

$$f_1(\Lambda, \eta, \xi) := \frac{1}{\varepsilon} \int_{\mathbf{T}^2} \mathcal{H}_1(\Lambda, \eta, \lambda, \xi) \, d\lambda.$$

Then,  $f_1(\Lambda, -\eta, -\xi) = f_1(\Lambda, \eta, \xi)$ .

The rescaling by  $\frac{1}{\varepsilon}$  is made so that  $f_1$  is a (real-analytic) uniformly bounded (by an order-one constant) function.

*Proof.* The eccentricity  $e_i$ , the semiaxis  $a_i$ , and the mean anomaly  $\ell_i$  of the osculating ellipse of  $P_i$  are invariant under the map  $(\Lambda_i, \lambda_i, H_i, h_i) \mapsto (\Lambda_i, \lambda_i - \pi, H_i, h_i + \pi)$ , while the argument of the perihelion  $g_i$  changes of  $\pi$ . Let us denote  $\vec{\pi} := (\pi, \dots, \pi) \in \mathbf{R}^N$ . In light of the consideration above we have that the map

$$(A.32) \quad (\Lambda, \lambda, H, h) \mapsto (\Lambda, \lambda - \vec{\pi}, H, h + \vec{\pi})$$

leaves  $|r^{(i)} - r^{(j)}|$  invariant, for any  $i, j = 1, \dots, N$ . The map in (A.32) corresponds to

$$(\Lambda, \lambda, \xi, \eta) \mapsto (\Lambda, \lambda - \pi, -\xi, -\eta),$$

usually referred to as “space inversion.”  $\square$

### Appendix B. Simple eigenvalue perturbations.

LEMMA B.1. *Let  $M^* \in \text{Mat}(m \times m)$ ,  $M_* \in \text{Mat}(k \times k)$ , and  $M^\sharp \in \text{Mat}(m \times k)$ . Then,*

$$\det \begin{pmatrix} M^* & M^\sharp \\ \mathbf{0}_{k \times m} & M_* \end{pmatrix} = \det(M^*) \det(M_*).$$

*Proof.* The proof is obvious.  $\square$

LEMMA B.2. Let  $M^* \in \text{Mat}(m \times m)$ ,  $M_* \in \text{Mat}(k \times k)$ ,  $M^\# \in \text{Mat}(m \times k)$ , and  $M_\# \in \text{Mat}(k \times m)$ . Let

$$M_\epsilon := \begin{pmatrix} M^* + O(\epsilon) & \epsilon M^\# + O(\epsilon^2) \\ \epsilon M_\# + O(\epsilon^2) & \epsilon M_* + O(\epsilon^2) \end{pmatrix}.$$

Then

- if  $\bar{\lambda} \neq 0$  is a simple eigenvalue of  $M^*$ , then there exists  $\bar{\lambda}_\epsilon = \bar{\lambda} + O(\epsilon)$  which is an eigenvalue of  $M_\epsilon$ , provided  $|\epsilon|$  is suitably small;
- if  $\tilde{\lambda}$  is a simple eigenvalue of  $M_*$  and  $\det(M^*) \neq 0$ , then there exists  $\tilde{\lambda}_\epsilon = \tilde{\lambda} + O(\epsilon)$  so that  $\epsilon \tilde{\lambda}_\epsilon$  is an eigenvalue of  $M_\epsilon$ , provided  $|\epsilon|$  is suitably small.

*Proof.* For the first claim, apply the implicit function theorem to

$$\mathcal{F}_1(t, \epsilon) := \det(M_\epsilon - t\mathbf{1}_{(m+k)}),$$

noticing that  $\mathcal{F}_1(t, 0) = (-1)^{k_t k} \det(M^* - t\mathbf{1}_m)$ . For the second claim, apply the implicit function theorem to

$$\mathcal{F}_2(t, \epsilon) := \det \begin{pmatrix} M^* - \epsilon t\mathbf{1}_m & M^\# \\ \epsilon M_\# & M_* - t\mathbf{1}_k \end{pmatrix},$$

noticing that  $\epsilon^k \mathcal{F}_2(t, \epsilon) = \det(M_\epsilon - \epsilon t\mathbf{1}_{(m+k)})$  and that, by Lemma B.1,  $\mathcal{F}_2(t, 0) = \det(M^*) \det(M_* - t\mathbf{1}_k)$ .  $\square$

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