Universality of the Hall conductivity in interacting electron systems

Alessandro Giuliani
University of Roma Tre, Department of Mathematics and Physics,
L.go S. L. Murialdo 1, 00146 Roma, Italy

Vieri Mastropietro
University of Milano, Department of Mathematics "F. Enriquez",
Via C. Saldini 50, 20133 Milano, Italy

Marcello Porta
University of Zürich, Mathematics Department,
Winterthurerstrasse 190, 8057 Zürich, Switzerland

We prove the quantization of the Hall conductivity in the presence of weak many-body interactions in a general weakly interacting gapped fermionic systems on two-dimensional periodic lattices and we prove Our result applies, among others, to the interacting Haldane and Hofstadter models. The proof is based on fermionic cluster expansion techniques combined with exact lattice Ward identities.

1. INTRODUCTION

Two-dimensional condensed matter systems often present remarkable transport properties. A famous example is the *Integer Quantum Hall Effect* (IQHE): the Hall conductivity of thin samples at very low temperatures, exposed to strong transverse magnetic fields, is equal to an integer times the von Klitzing constant e^2/h, [**?**]. This measurement is amazingly sharp: the observation of the Hall plateaux is by now used to measure the fine structure constant, at a very high level of accuracy. In view of the complexity of the underlying microscopic Hamiltonians, depending on a number of parameters related to the material details, the universality of the Hall conductance is quite a remarkable phenomenon. The Hall conductivity for non interacting fermions has a beautiful topological interpretation [**?**], and the intrinsic robustness of a topological quantity offers a natural qualitative explanation of the observed universality. The universality of Hall conductivity in presence of disorder has been established with full mathematical rigor in [**? ? ? ?**]. A similar universality property is also expected to be true also in presence of many body interaction. However, while in presence of disorder the properties of the many body problem can be deduced by the single particle Schroedinger equation, in order to take into account the interaction one has to consider the full N-particle Schroedinger equation; this explain way a mathematical proof of the quantization of the Hall conductance for *interacting* electrons remained open [**?**

] for years. Effective field theories [? ? ? ? ? ?], has been used for explaining a possible "topological" mechanisms underlying both the integral and the fractional QHE; however one assumes certain properties, like the *incompressibility* of the "quantum Hall fluid", which may be very hard to check from first principles in concrete models. In recent times, the quantization of the Hall conductivity has been proved using the notion of quasi-adiabatic evolution of the ground state, under changes of the magnetic fluxes acting on the system, [?]; again one needs on the assumption that the existence of a gap, which is unproven in most physically relevant cases (the only case in which is known are perturbations of "topologically trivial" reference states, see [? ?], or [?]).

In this work, we use a quite different approach to prove the quantization of the Hall conductance for general weakly interacting fermionic systems, under the assumption that the reference non-interacting system is gapped. In particular, our result implies the quantization of the Hall conductivity of the interacting version of the Hofstadter and Haldane [?] models. This provides a theoretical justification of the numerical [?] an experimental [?] observations in the interacting Haldane model. Our proof does not require any a priori assumption on the interacting spectrum of the system, but it is based on constructive renormalization group techniques combined with lattice Ward Identities; crucial assumptions are the fact that the interaction has to be weak and short ranged. We use the many body Euclidean formalism and we write a convergent power series expansion for the d.c. Kubo conductivity, showing that higher orders corrections are exactly vanishing. The idea that the universality of Hall conductance follows from Ward Identities is well known in physics , see [?],[?], but its implementation was done in continuum effective quantum field theory models plagued by ultraviolet divergences and by using formal manipulations of non convergent Feynman graph expansions. In our approach we consider lattice and well defined Hamiltonian lattice model, in which the continuity equation implies exact Ward Identities. The convergence of the perturbative expansions is achieved avoiding Feynman graphs expansion and using constructive Renormalization Group methods. Such methods used earlier for constructing the ground state of several low-dimensional interacting Fermi systems, and for proving universality relations among critical exponents, amplitudes and conductivities [? ? ? ? ? ? ? ?]. In this paper we apply these ideas for the first time to the study of the transverse (Hall) conductivity.

An informal statement of our main result is the following.

Consider a fermionic system on a two-dimensional periodic lattice, with grand canonical Hamiltonian $H_0 + UV$, where H_0 is a quadratic gapped Hamiltonian, V is a density-density interaction, decaying faster than any power at large distances, and U is its strength. If U is small enough, then the interacting correlation functions are analytic in U and decay faster than any power at large distances, uniformly in the system size and in the temperature. The conductivity matrix, defined by the Green-Kubo formula, is analytic as well, and its infinite volume and zero temperature limit is independent of U. In particular, the longitudinal conductivity is zero, while the transverse one is quantized.

The rest of the paper is devoted to the proof of the main result. In section 2 we define the general class of Hamiltonians we consider. In section 3 we define the current observable and the conductivity, and state our main result in a mathematically precise way. In section 4 we prove our main result, under the assumption of analyticity and smoothness of the multipoint current correlations, by making use of Ward Identites, which are nothing but the restatement of the continuity equation for the current at the level of correlation functions. In section 5 we prove the analyticity and smoothness of the correlations, by using multiscale fermionic cluster expansion techniques. Strictly speaking, the content of section 5 is a straightforward adaptation of previous results, but we include it here in order to make the paper self-contained. In the appendices we collect some auxiliary results, some of which are already known, but are included here for completeness: in appendix A we show for completeness that in the non interacting case we recover the usual formula for the Chern number; in appendix B we apply our main result to the interacting Haldane model, and show that it displays a non-trivial Hall phase diagram; in appendix **??** we collect a few technical aspects of the derivation of the Ward Identities.

2. THE MODEL

Lattice fermionic operators. Let $L \in \mathbb{N}$, and let Λ_L be a finite Bravais lattice, generated by two linearly independent vectors $\vec{\ell}_1,\ \vec{\ell}_2 \in \mathbb{R}^2$:

$$\Lambda_L = \big\{ \vec{x} \mid \vec{x} = n_1 \vec{\ell}_1 + n_2 \vec{\ell}_2,\ n_i \in \mathbb{Z},\ 0 \le n_i \le L-1 \big\}. \tag{2.1}$$

The number of sites of Λ_L is $|\Lambda_L| = L^2$. With each site $\vec{x} \in \Lambda$, we associate fermionic creation and annihilation operators $\psi^{\pm}_{\vec{x},\sigma}$, with $\sigma \in I$, and I a finite set of indices, which can be thought of as “color” labels, possibly corresponding to the spin, or to different sublattices. In particular, the fermion labeled by σ can be thought of as living on a physical lattice obtained by translating Λ_L by a fixed amount $\vec{r}_\sigma \in \mathbb{R}$ (possibly equal to $\vec{0}$, in the case that, e.g., σ is a spin index).

The fermionic operators satisfy the usual canonical anticommutation relations:

$$\{ \psi^{\varepsilon}_{\vec{x},\sigma}, \psi^{\varepsilon'}_{\vec{y},\sigma'} \} = \delta_{\varepsilon,-\varepsilon'}\, \delta_{\vec{x},\vec{y}}\, \delta_{\sigma,\sigma'}\ , \tag{2.2}$$

where $\varepsilon, \varepsilon' = \pm$, $\vec{x}, \vec{y} \in \Lambda_L$, $\sigma, \sigma' \in I$, and $\delta_{\cdot,\cdot}$ is the Kronecker delta. We impose periodic boundary conditions on Λ_L, that is we identify the fermionic operators obtained by translating $\vec{x}$ by an integer multiple of $L\vec{\ell}_i$. We let $\vec{G}_1$, $\vec{G}_2$ be a basis of the reciprocal lattice Λ_L^* of Λ, i.e., $\vec{G}_i \cdot \vec{\ell}_j = 2\pi\delta_{i,j}$, and we define the finite-volume Brillouin zone as

$$\mathcal{B}_L := \Big\{ \vec{k} \mid \vec{k} = \frac{n_1}{L}\vec{G}_1 + \frac{n_2}{L}\vec{G}_2,\ n_i \in \mathbb{Z},\ 0 \le n_i \le L-1 \Big\} \tag{2.3}$$

We let the Fourier transforms of the fermionic operators be:

$$\psi^{\pm}_{\vec{x},\sigma} = \frac{1}{L^2} \sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\cdot\vec{x}} \hat{\psi}^{\pm}_{\vec{k},\sigma} , \quad \forall \vec{x} \in \Lambda_L , \qquad \Longleftrightarrow \qquad \hat{\psi}^{\pm}_{\vec{k},\sigma} = \sum_{\vec{x}\in\Lambda_L} e^{\mp i\vec{k}\cdot x} \psi^{\pm}_{\vec{x},\sigma} , \quad \forall \vec{k} \in \mathcal{B}_L . \tag{2.4}$$

Note that, with this definition, the fermionic operators in momentum space are periodic over the first Brillouin zone, that is $\hat{\psi}^{\pm}_{\vec{k},\sigma} = \hat{\psi}^{\pm}_{\vec{k}+\vec{G}_i,\sigma}$, $i = 1, 2$. Moreover,

$$\{\hat{\psi}^{\varepsilon}_{\vec{k},\sigma}, \hat{\psi}^{\varepsilon'}_{\vec{k}',\sigma'}\} = L^2 \delta_{\varepsilon,-\varepsilon'} \delta_{\vec{k},\vec{k}'} \delta_{\sigma,\sigma'} . \tag{2.5}$$

The Hamiltonian. The grand-canonical Hamiltonian of the system is assumed to be of the form:

$$\mathcal{H}_L - \mu\mathcal{N}_L = \mathcal{H}^{(0)}_L + U\mathcal{V}_L - \mu\mathcal{N}_L , \tag{2.6}$$

with

$$\begin{aligned}
&\mathcal{H}^{(0)}_L = \sum_{\vec{x},\vec{y}\in\Lambda_L} \sum_{\sigma,\sigma'\in I} \psi^{+}_{\vec{x},\sigma} H^{(0)}_{\sigma\sigma'}(\vec{x}-\vec{y}) \psi^{-}_{\vec{y},\sigma'} , \\
&\mathcal{V}_L = \sum_{\vec{x},\vec{y}\in\Lambda_L} \sum_{\sigma,\sigma'\in I} n^{\sigma}_{\vec{x}}\, v_{\sigma\sigma'}(\vec{x}-\vec{y})\, n^{\sigma'}_{\vec{y}} , \quad \text{where} \quad n^{\sigma}_{\vec{x}} = \psi^{+}_{\vec{x},\sigma}\psi^{-}_{\vec{x},\sigma} , \\
&\text{and} \qquad \mathcal{N}_L = \sum_{\vec{x}\in\Lambda_L} \sum_{\sigma\in I} n^{\sigma}_{\vec{x}} .
\end{aligned} \tag{2.7}$$

The operator $\mathcal{H}^{(0)}_L$ is called the *free Hamiltonian*, while $U\mathcal{V}_L$ is the *many-body interaction*, and U plays the role of the interaction strength. The constant μ is the *chemical potential*, or Fermi level.

We assume the *hopping function* $H^{(0)}_{\sigma\sigma'}(\vec{x})$ to be periodic on Λ_L, and such that $H^{(0)}_{\sigma\sigma}(\vec{0}) = 0$. In order for the free Hamiltonian to be self-adjoint, we require $\big[H^{(0)}_{\sigma\sigma'}(\vec{x})\big]^* = H^{(0)}_{\sigma'\sigma}(-\vec{x})$. Moreover, we assume that it decays faster than any power at large distances:

$$\|H^{(0)}(\vec{x})\| \le \frac{C_N}{1+|\vec{x}|^N} , \quad \forall N \ge 0 . \tag{2.8}$$

As a consequence of these assumptions, we see that the *Bloch Hamiltonian*

$$\hat{H}^{(0)}(\vec{k}) := \sum_{\vec{x}\in\Lambda_L} e^{i\vec{k}\cdot\vec{x}} H^{(0)}(\vec{x}) , \tag{2.9}$$

is a self-adjoint matrix, so that the spectrum $\sigma(\hat{H}^{(0)}(\vec{k})) = \{\varepsilon_\sigma(\vec{k})\}_{\sigma\in I}$ is real. The functions $\vec{k} \mapsto \varepsilon_\sigma(\vec{k})$ are called the *energy bands*. We let

$$\mathfrak{e}_0 = \sup_L \sup_{\vec{k}\in\mathcal{B}_L} ||\hat{H}^{(0)}(\vec{k})||, \tag{2.10}$$

which sets the energy scale. Note also that the infinite volume limit of $\hat{H}^{(0)}(\vec{k})$ is infinitely differentiable in $\vec{k}$.

Concerning the interaction, we assume, similarly, that $v_{\sigma\sigma'}(\vec{x})$ is periodic function on Λ_L, such that $v_{\sigma\sigma}(\vec{0}) = 0$, $v_{\sigma\sigma'}(\vec{x}-\vec{y}) = v_{\sigma'\sigma}(\vec{y}-\vec{x})$ and

$$\|v(\vec{x})\| \le \frac{C_N}{1+|\vec{x}|^N}\,, \quad \forall N \ge 0\,. \tag{2.11}$$

In particular, the infinite volume limit of

$$\hat{v}_{\sigma\sigma'}(\vec{p}) = \sum_{\vec{x}\in\Lambda_L} e^{i\vec{p}\cdot\vec{x}} v_{\sigma\sigma'}(\vec{x}) \tag{2.12}$$

is infinitely differentiable in $\vec{p}$.

Finally, concerning the choice of the Fermi level, we assume the following *gap condition*:

$$\delta_\mu := \lim_{L\to\infty} \delta_{L,\mu} > 0, \quad \text{where} \quad \delta_{L,\mu} := \inf_{\vec{k}\in\mathcal{B}_L} \text{dist}(\mu, \sigma(\hat{H}^{(0)}(\vec{k})))\,. \tag{2.13}$$

... sono arrivato qui ....

Gibbs state and Euclidean correlation functions. The grand-canonical Gibbs state associated to this model is denoted by $\langle\cdot\rangle_{\beta,\mu,L}$. Given a self-adjoint operator O on the fermionic Fock space $\mathcal{F}$, an *observable*, its expectation value is:

$$\langle O\rangle_{\beta,\mu,L} := \frac{\text{Tr}_{\mathcal{F}}\, e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}O}{\text{Tr}_{\mathcal{F}}\, e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}} \tag{2.14}$$

where $\mathcal{F}$ is the fermionic Fock space.

In the following, we will not write explicitly the μ-dependence of the finite volume Gibbs state: $\langle\cdot\rangle_{\beta,\mu,L} \equiv \langle\cdot\rangle_{\beta,L}$ (keeping in mind that μ satisfies Eq. (**??**)). Also, we shall denote by $\langle\cdot\rangle^{(0)}_{\beta,L}$ the non-interacting Gibbs state, correspoding to the choice $U = 0$ in Eq. (2.7).

Let O_{x_0} be the imaginary time evolution of O, namely

$$O_{x_0} := e^{x_0(\mathcal{H}_L-\mu\mathcal{N}_L)} O e^{-x_0(\mathcal{H}_L-\mu\mathcal{N}_L)}\,, \qquad x_0 \in [0,\beta)\,. \tag{2.15}$$

Given n observables $O^{(1)}_{x_{0,1}}, \ldots, O^{(n)}_{x_{0,n}}$, we define their time-ordered average as:

$$\langle \mathbf{T}\, O^{(1)}_{x_{0,1}} \cdots O^{(n)}_{x_{0,n}}\rangle_{\beta,L} := \frac{\text{Tr}_{\mathcal{F}}\, e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}\mathbf{T}\big\{O_{x_{0,1}}\cdots O_{x_{0,n}}\big\}}{\text{Tr}_{\mathcal{F}}\, e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}}\;; \tag{2.16}$$

$\mathbf{T}$ is the usual fermionic time-ordering, acting on a product of fermionic operators as (omitting the $\vec{x}$, σ labels for simplicity):

$$\mathbf{T}\{\psi^{\varepsilon_1}_{x_{0,1}} \cdots \psi^{\varepsilon_n}_{x_{0,n}}\} = \mathrm{sgn}(\pi)\psi^{\varepsilon_{\pi(1)}}_{x_{0,\pi(1)}} \cdots \psi^{\varepsilon_{\pi(n)}}_{x_{0,\pi(n)}} \;, \tag{2.17}$$

where π is a permutation of $\{1,\dots,N\}$ with sign $\mathrm{sgn}(\pi) \in \{-1,+1\}$, such that $x_{0,\pi(1)} \geq \ldots \geq x_{0,\pi(n)}$; if some fields are evaluated at the same time, the ambiguity is solved by normal ordering.

Also, we denote by $\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle_{\beta,L}$ the time-ordered truncated correlation function, or *cumulant*, of $O^{(i)}_{x_{0,i}}$, $i=1,\dots,n$. Given a general state $\langle\cdot\rangle$, the time-ordered cumulant is defined as [**?**] I think this is only valid for even observables, clarify :

$$\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,O^{(2)}_{x_{0,2}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle := \frac{\partial^n}{\partial t_1\cdots\partial t_n}\log\Big\{1+\sum_{I\subseteq\{1,2,\dots,n\}} t(I)\langle\mathbf{T}\,O(I)\rangle\Big\}\Big|_{\underline{t}=\underline{0}} \;, \tag{2.18}$$

where: the sum in the right-hand side is over all subsets $I=\{i_1,\dots,i_k\}$ of $\{1,2,\dots,n\}$, with $i_1 < i_2 < \ldots < i_k$; $t(I) := \prod_{i=1}^k t_i$; and $O(I) := O^{(i_1)}_{x_{0,1}} \ldots O^{(i_k)}_{x_{0,i_k}}$. For $n=1$, this definition reduces to $\langle O^{(1)}_{x_{0,1}}\rangle$. For $n=2$ one gets $\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,O^{(2)}_{x_{0,2}}\rangle = \langle \mathbf{T}\, O^{(1)}_{x_{0,1}} O^{(2)}_{x_{0,2}}\rangle - \langle O^{(1)}_{x_{0,1}}\rangle\langle O^{(2)}_{x_{0,2}}\rangle$, and so on. Again, in case two observables are evaluated at equal times, the ambiguity is solved by putting them into normal order.

Notice that this definition also applies to observables that depend on more than one time variable (*e.g.*, $O^{(i)} = O'_{y_0}O''_{z_0}$). In case all observables depend on just one time variable and all times are different, it is easy to see that Eq. (2.18) reduces to:

$$\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,O^{(2)}_{x_{0,2}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle := \langle O^{(\pi(1))}_{x_{0,\pi(1)}}\,;\,O^{(\pi(2))}_{x_{0,\pi(2)}}\,;\,\cdots\,;\,O^{(\pi(n))}_{x_{0,\pi(n)}}\rangle \;, \tag{2.19}$$

where the permutation π is such that $x_{0,\pi(1)} > \ldots > x_{0,\pi(n)}$.

Finally, we introduce the notion of Fourier transform for the correlations of the Gibbs state $\langle\cdot\rangle_{\beta,L}$, in the imaginary time variables. Let $p_{0,i} \in \frac{2\pi}{\beta}\mathbb{Z}$, $i=1,\dots,n$; that is, $\{p_{0,i}\}$ are (bosonic) *Matsubara frequencies.* The Fourier transform of the correlation $\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle_{\beta,L}$ is defined as:

$$\begin{aligned}\int_{[0,\beta)^n} \Big[\prod_{i=1}^n dx_{0,i}\Big] e^{ip_{0,1}x_{0,1}+\ldots+ip_{0,n}x_{0,n}} \langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle_{\beta,L} \\ = \beta\delta_\beta\Big(\sum_i p_{0,i}\Big)\frac{1}{\beta}\langle \mathbf{T}\, \widehat{O}^{(1)}_{p_{0,1}}\,;\,\cdots\,;\,\widehat{O}^{(n-1)}_{p_{0,n-1}}\,;\,\widehat{O}^{(n)}_{-p_{0,1}-\ldots-p_{0,n-1}}\rangle_{\beta,L}\end{aligned} \tag{2.20}$$

with $\delta_\beta(p_0)$ the Kronecker delta of $p_0 \in \frac{2\pi}{\beta}\mathbb{Z}$ and $\widehat{O}^{(i)}_{p_0} := \int_0^\beta dx_0\, e^{ip_0x_0}\, O^{(i)}_{x_0}$. The second line in Eq. (2.20) is implied by translation invariance in the imaginary-time variable:

$$\langle \mathbf{T}\, O^{(1)}_{x_{0,1}}\,;\,\cdots\,;\,O^{(i)}_{x_{0,i}}\,;\,\cdots\,;\,O^{(n)}_{x_{0,n}}\rangle_{\beta,L} = \langle \mathbf{T}\, O^{(1)}_{x_{0,1}-x_{0,n}}\,;\,\cdots\,;\,O^{(i)}_{x_{0,i}-x_{0,n}}\,;\,\cdots\,;\,O^{(n)}_{0}\rangle_{\beta,L} \;, \tag{2.21}$$

which follows from the ciclicity of the trace. Notice that, as $\beta\to\infty$, the combination $\beta\delta_\beta(\cdot)$ in Eq. (2.20) formally converges to $(2\pi)^{-1}\delta(\cdot)$, where $\delta(\cdot)$ is the Dirac delta function on $\mathbb{R}$.

3. LINEAR RESPONSE THEORY

Here we shall discuss the transport properties of our general interacting gapped systems, in the linear response approximation. In Section 3 A, we define the current operator, and we discuss the associated conservation laws. Then, in Section 4 A, we introduce the current-current correlations, and prove their analyticity for weak enough electron-electron interactions. The transport coefficients we are interested in are defined according to Green-Kubo formula, introduced in Section 3 B; our main result, discussed in Section 3 C and proven in Section 4, is a rigorous statement on the universality properties of the Green-Kubo conductivity matrix.

A. Current operator

In the following, we will be interested in the response of the system to an external time-dependent field, constant in space. Here we define the current operator, and prove a crucial conservation law.

For $\vec{x} \in \Lambda_L$, let $\vec{x}^{(\sigma)} := \vec{x} + \vec{\delta}^{(\sigma)} \in \Lambda_L^{(\sigma)}$ (see Section **??**). The current operator is defined as:

$$\begin{aligned} \vec{J} &:= i\Big[\mathcal{H}_L, \sum_{\sigma\in I}\sum_{\vec{x}\in\Lambda_L} \vec{x}^{(\sigma)} n_{\vec{x}}^{(\sigma)}\Big] \\ &= i\Big[\mathcal{H}_L^{(0)}, \sum_{\sigma\in I}\sum_{\vec{x}\in\Lambda_L} \vec{x}^{(\sigma)} n_{\vec{x}}^{(\sigma)}\Big]\,, \end{aligned} \tag{3.1}$$

where the second line follows from the fact that the operator $n_{\vec{x}}^{(\sigma)}$ commutes with the interaction term $\mathcal{V}_L$ in Eq. (2.7). Notice that the second argument of the commutator is simply the second quantization of the position operator. More explicitly, one finds:

$$\begin{aligned} \vec{J} &= i\sum_{\vec{x},\vec{y}\in\Lambda_L}\sum_{\sigma,\sigma'\in I} (\vec{y}^{(\sigma')} - \vec{x}^{(\sigma)})\, \psi^+_{\vec{x},\sigma} H^{(0)}_{\sigma\sigma'}(\vec{x}-\vec{y})\psi^-_{\vec{y},\sigma'} \\ &= \frac{i}{2}\sum_{\vec{x},\vec{y}\in\Lambda_L}\sum_{\sigma,\sigma'\in I} (\vec{y}^{(\sigma')} - \vec{x}^{(\sigma)})\big[\psi^+_{\vec{x},\sigma} H^{(0)}_{\sigma\sigma'}(\vec{x}-\vec{y})\psi^-_{\vec{y},\sigma'} - \psi^+_{\vec{y},\sigma'} H^{(0)}_{\sigma'\sigma}(\vec{y}-\vec{x})\psi^-_{\vec{x},\sigma}\big] \end{aligned} \tag{3.2}$$

where in the second line we rewrote the outcome of the commutator in a more symmetric form, using that $\overline{H_{\sigma\sigma'}(\vec{x}-\vec{y})} = H_{\sigma'\sigma}(\vec{y}-\vec{x})$. Let us introduce the shorthand notation:

$$J_{\vec{x}}(\vec{z},\sigma,\sigma') := i\big[\psi^+_{\vec{x},\sigma} H^{(0)}_{\sigma\sigma'}(-\vec{z})\psi^-_{\vec{x}+\vec{z},\sigma'} - \psi^+_{\vec{x}+\vec{z},\sigma'} H^{(0)}_{\sigma'\sigma}(\vec{z})\psi^-_{\vec{x},\sigma}\big]\ ; \tag{3.3}$$

notice that

$$J_{\vec{x}}(\vec{z},\sigma,\sigma') = -J_{\vec{x}+\vec{z}}(-\vec{z},\sigma',\sigma)\ . \tag{3.4}$$

Physically, $J_{\vec{x}}(\vec{z},\sigma,\sigma')$ corresponds to the *bond current* flowing on the bond between $x^{(\sigma)}$ and $(\vec{x}+\vec{z})^{(\sigma')}$. We rewrite Eq. (3.2) as:

$$\vec{J} = \frac{1}{2}\sum_{\vec{z}\in\Lambda_L}\sum_{\sigma,\sigma'\in I}(\vec{z}+\vec{\delta}^{(\sigma)}-\vec{\delta}^{(\sigma')})\sum_{\vec{x}\in\Lambda_L}J_{\vec{x}}(\vec{z},\sigma,\sigma')\;. \tag{3.5}$$

In Eq. (3.5), the factor 1/2 takes into account the fact that we are summing twice over the same bonds.

Let us consider the imaginary time evolution of the σ-density, $n^{(\sigma)}_{(x_0,\vec{x})} := e^{(\mathcal{H}_L-\mu\mathcal{N}_L)x_0}n^{(\sigma)}_{\vec{x}}e^{-(\mathcal{H}_L-\mu\mathcal{N}_L)x_0}$. We define:

$$\tilde{J}_{0,(x_0,\vec{p})} := \sum_{\sigma\in I}\sum_{\vec{x}\in\Lambda_L}e^{i\vec{p}\cdot\vec{x}^{(\sigma)}}n^{(\sigma)}_{(x_0,\vec{x})}\;. \tag{3.6}$$

We have:

$$\begin{aligned}\partial_{x_0}\tilde{J}_{0,(x_0,\vec{p})} &= \big[\mathcal{H}^{(0)}_L,\tilde{J}_{0,(x_0,\vec{p})}\big] \\ &= \sum_{\vec{x},\vec{y}\in\Lambda_L}\sum_{\sigma,\sigma'\in I}\big[e^{i\vec{p}\cdot\vec{y}^{(\sigma')}}-e^{i\vec{p}\cdot\vec{x}^{(\sigma)}}\big]\psi^+_{\vec{x},\sigma}H^{(0)}_{\sigma\sigma'}(\vec{x}-\vec{y})\psi^-_{\vec{y},\sigma'} \\ &= \frac{1}{2}\sum_{\vec{x},\vec{y}\in\Lambda_L}\sum_{\sigma,\sigma'\in I}\big[e^{i\vec{p}\cdot\vec{y}^{(\sigma')}}-e^{i\vec{p}\cdot\vec{x}^{(\sigma)}}\big]\big[\psi^+_{\vec{x},\sigma}H^{(0)}_{\sigma\sigma'}(\vec{x}-\vec{y})\psi^-_{\vec{y},\sigma'}-\psi^+_{\vec{y},\sigma'}H^{(0)}_{\sigma'\sigma}(\vec{y}-\vec{x})\psi^-_{\vec{x},\sigma}\big]\;,\end{aligned} \tag{3.7}$$

where in the last step follows by symmetrization. In terms of the bond currents (3.3), Eq. (3.7) reads:

$$\begin{aligned}\partial_{x_0}\tilde{J}_{0,(x_0,\vec{p})} &= \frac{-i}{2}\sum_{\vec{z}\in\Lambda_L}\sum_{\sigma,\sigma'}\big[e^{i\vec{p}\cdot(\vec{z}+\vec{\delta}^{(\sigma')}-\vec{\delta}^{(\sigma)})}-1\big]e^{i\vec{p}\cdot\vec{\delta}^{(\sigma)}}\sum_{\vec{x}\in\Lambda_L}e^{i\vec{p}\cdot\vec{x}}J_{(x_0,\vec{x})}(\vec{z},\sigma,\sigma') \\ &\equiv \frac{1}{2}\vec{p}\cdot\sum_{\vec{z}\in\Lambda_L}\sum_{\sigma,\sigma'\in I}(\vec{z}+\vec{\delta}^{(\sigma)}-\vec{\delta}^{(\sigma')})\,\eta_{\vec{p}}(\vec{z},\sigma,\sigma')\,\tilde{J}_{(x_0,\vec{p})}(\vec{z},\sigma,\sigma')\;,\end{aligned} \tag{3.8}$$

where $J_{(x_0,\vec{x})}(\cdots)$ is the imaginary time evolution of $J_{\vec{x}}(\cdots)$, and:

$$\eta_{\vec{p}}(\vec{z},\sigma,\sigma') := \frac{\big[e^{i\vec{p}\cdot(\vec{z}+\vec{\delta}^{(\sigma')}-\vec{\delta}^{(\sigma)})}-1\big]}{i\vec{p}\cdot(\vec{z}+\vec{\delta}^{(\sigma')}-\vec{\delta}^{(\sigma)})}e^{i\vec{p}\cdot\vec{\delta}^{(\sigma)}}\;,\qquad \tilde{J}_{(x_0,\vec{p})}(\vec{z},\sigma,\sigma') := \sum_{\vec{x}\in\Lambda_L}e^{i\vec{p}\cdot\vec{x}}J_{(x_0,\vec{x})}(\vec{z},\sigma,\sigma')\;. \tag{3.9}$$

Notice that:

$$\eta_{\vec{p}}(-\vec{z},\sigma',\sigma) = e^{-i\vec{p}\cdot\vec{z}}\eta_{\vec{p}}(\vec{z},\sigma,\sigma')\;,\qquad \tilde{J}_{(x_0,\vec{p})}(-\vec{z},\sigma',\sigma) = -e^{i\vec{p}\cdot\vec{z}}\tilde{J}_{(x_0,\vec{p})}(\vec{z},\sigma,\sigma')\;. \tag{3.10}$$

Again, the factor 1/2 in Eq. (3.7) keeps into account the fact that we are summing twice over the same bonds.

The relation Eq. (3.8) is a conservation law for a space-time lattice current, with components $\tilde{J}_{\mu,(x_0,\vec{p})}$, $\mu=0,1,2$, with $\tilde{J}_{0,(x_0,\vec{p})}$ given by Eq. (3.6), and

$$\tilde{J}_{i,(x_0,\vec{p})} = \frac{1}{2}\sum_{\vec{z}\in\Lambda_L}\sum_{\sigma,\sigma'\in I}(\vec{z}_i+\vec{\delta}^{(\sigma)}_i-\vec{\delta}^{(\sigma')}_i)\,\eta_{\vec{p}}(\vec{z},\sigma,\sigma')\,\tilde{J}_{(x_0,\vec{p})}(\vec{z},\sigma,\sigma')\;,\qquad i=1,2\;; \tag{3.11}$$

with these notations, Eq. (3.8) takes the compact form

$$\partial_{x_0} \tilde{J}_{0,(x_0,\vec{p})} = \vec{p} \cdot \tilde{\vec{J}}_{(x_0,\vec{p})} \ . \tag{3.12}$$

It is important to notice that, being $\eta_{\vec{p}}(\vec{z},\sigma,\sigma')$ analytic in $\vec{p} \in \mathcal{B}$, and using that $\lim_{\vec{p}\to\vec{0}} \eta_{\vec{p}}(\vec{z},\sigma,\sigma') = 1$, we have:

$$\partial_{x_0} \tilde{J}_{0,(x_0,\vec{p})} = \vec{p} \cdot \vec{J}_{x_0} + O(|\vec{p}|^2) \ , \qquad \text{as } \vec{p} \to 0, \tag{3.13}$$

where $\vec{J}_{x_0}$ is the imaginary time evolution of the current operator defined in Eq. (3.1).

B. Green-Kubo formula

The ground-state conductivity matrix of the system is defined according to Green-Kubo formula in the euclidean formalism, see for instance [**?**] or [**?**]

$$\sigma_{ij} := \lim_{p_0 \to 0^+} -\frac{1}{A}\frac{1}{p_0}\big[\widehat{K}_{ij}(p_0,\vec{0}) - \widehat{K}_{ij}(\mathbf{0})\big] \ , \qquad i = 1,2 \ , \tag{3.14}$$

where A is the area of the fundamental cell, $A = |\vec{\ell}_1 \wedge \vec{\ell}_2|$. The labels i, j refer to the basis $\vec{e}_1 = (1,\,0)$, $\vec{e}_2 = (0,\,1)$. *If* the infinite-volume current-current correlation function $\widehat{K}_{ij}(\mathbf{p})$ is differentiable in $p_0 = 0$, this definition reduces to:

$$\sigma_{ij} = -\frac{1}{A}\frac{\partial}{\partial p_0}\widehat{K}_{ij}(p_0,\vec{0})\big|_{p_0=0} \ , \qquad i = 1,2 \ . \tag{3.15}$$

This is the case for the class of systems we are considering, because of the gap in the spectrum of the noninteracting theory and thanks to fermionic cluster expansion; see Proposition 4.1. In App. A we show that the above formula in the non interacting case reduces to the usual formula for Chern numbers; and in the interacting case we prove that all the interaction corrections vanishes. The above formula is taken as our starting point; the problem of a deriving it from linear response theory in our interacting model, or the proof that the limit of zero frequency does not depend from the path (that is , that the limit along the imaginary line is the same as along a line in the complex plane parallel to the real axis), is not addressed here.

C. Main result

For gapped systems it is well-known that, in the absence of interactions, the off-diagonal part of the conductivity matrix, the *Hall conductivity*, has a topological interpretation [**?**]; this remarkable observation implies, in particular, that σ_{12} can only take integer values (in units $e^2 = h = 1$). In Appendix A we review this fact by showing that, for noninteracting systems, the definition (3.14) agrees with the sum of the Chern numbers of the occupied bands.

In the presence of many-body interactions, it has been recently proven that, under suitable assumptions of the spectrum of the interacting system, the Hall conductivity σ_{12} is still quantized, [**?**]. More precisely, in [**?**] the Authors assume that the interacting spectrum is gapped, and that the interacting ground state is nondegenerate. These assumptions can be very hard to check in concrete systems, in the infinite volume limit.

Here we give a new proof of the quantization of σ_{12}, for the class of interacting systems introduced in Section 2. More generally, in our main result, Theorem 3.1, we prove that the ground-state conductivity matrix of the class of systems introduced in Section 2 is *universal*: it does not depend on weak many-body interaction.

Theorem 3.1 [Universality of the conductivity matrix.]
Let σ_{ij} be the conductivity matrix, as defined in Eq. (3.14). Assume that the chemical potential μ is in a gap of the noninteracting Hamiltonian,

$$\mu \notin \sigma(\hat{H}^{(0)}(\vec{k}))\;, \qquad \forall \vec{k} \in \mathcal{B}\;. \tag{3.16}$$

Then, there exists $U_0 > 0$ such that, for $|U| < U_0$:

1. *the zero temperature, infinite-volume conductivity matrix $(\sigma_{ij})_{i,j=1,2}$ is analytic in U;*
2. *the zero temperature, infinite-volume conductivity matrix is given by:*

$$\sigma_{ij} = \sigma_{ij}^{(0)}\;, \qquad \forall i,j = 1,2\;, \tag{3.17}$$

where $\sigma_{ij}^{(0)} = \sigma_{ij}\big|_{U=0}$.

Remark 1 *1. A consequence of this theorem is that, in the analyticity domain:*

$$\sigma_{11} = \sigma_{22} = 0\;, \qquad \sigma_{12} \in \mathbb{Z}\;. \tag{3.18}$$

This result proves the stability of the Integer Quantum Hall effect in presence of weak many-body interactions.

*2. Our methods are different from those of [**?**]; our analysis is based on fermionic cluster expansion, and on Ward identities. In particular, we stress that, with respect to [**?**] we only assume the existence of a gap for the noninteracting theory, and that the many-body interaction is weak (uniformly in temperature and system size).*

The main advantage of our result with respect to [**?**] is that it does not rely on assumptions for the interacting spectrum. This makes the result useful in concrete situations, where the assumptions of [**?**] might be hard to check. The application of Theorem 3.1 to the Haldane model is spelled out explicitly in Appendix ???.

4. PROOF OF THEOREM 3.1

A. Current-current correlation functions

The current-current correlation functions play a crucial role in linear response theory. Here we define them, and prove that they are analytic for weak many-body interactions, see Proposition 4.1. Then, we derive some crucial identities between then, called *Ward identities*, which follow from the conservation law Eq. (3.8). Finally, we prove a suitable decomposition formula for the correlation functions.

Let r, $s \in \mathbb{N}$, $n = r + s$. Specify that $r, s \geq 0$ and $n \geq 1$. If $n = 1$ the correlation is independent of momentum, and it is defined as the average of the corresponding operator at $\mathbf{p} = \mathbf{0}$. Let $\mathbf{p}_i = (p_{i,0}, \vec{p}_i) \in (2\pi/\beta)\mathbb{Z} \times \mathcal{B}_L$, for $i = 1, \ldots, n-1$. We define:

$$\begin{aligned}&\widehat{K}^{\beta,L}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \\ &\quad := \frac{1}{\beta L^2} \langle \mathbf{T}\, \hat{J}_{\mu_1,\mathbf{p}_1} \,;\, \cdots \,;\, \hat{J}_{\mu_s,\mathbf{p}_s} \,;\, \hat{n}^{(\sigma_1)}_{\mathbf{p}_{s+1}} \,;\, \cdots \,;\, \hat{n}^{(\sigma_r)}_{-\mathbf{p}_1 - \ldots - \mathbf{p}_{n-1}} \rangle_{\beta,L} \,, \end{aligned} \tag{4.1}$$

where:

$$\hat{n}^{(\sigma)}_{\mathbf{p}} := \int_0^\beta dx_0\, e^{ip_0 x_0}\, \tilde{n}^{(\sigma)}_{(x_0,\vec{p})} \,, \qquad \hat{J}_{\mu,\mathbf{p}} := \int_0^\beta dx_0\, e^{ip_0 x_0}\, \tilde{J}_{\mu,(x_0,\vec{p})} \,, \tag{4.2}$$

with $\tilde{n}^{(\sigma)}_{(x_0,\vec{p})} = e^{(\mathcal{H}_L - \mu \mathcal{N}_L)x_0}\, \hat{n}^{(\sigma)}_{\vec{p}}\, e^{-(\mathcal{H}_L - \mu \mathcal{N}_L)x_0}$ and $\tilde{J}_{\mu,(x_0,\vec{p})}$ defined in Eq. (3.6), (3.11) (for $\mu = 0$, $\mu = 1, 2$, respectively). Also, we set, for $\mathbf{p}_i \in \mathbb{R} \times \mathcal{B}$:

$$\widehat{K}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) := \lim_{\beta,L\to\infty} \widehat{K}^{\beta,L}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \,. \tag{4.3}$$

Analyticity of the correlations. The current-current correlation functions allow to describe the linear response of the system. In particular, as we shall see, the conductivity matrix can be computed in terms of the two-point current-current correlation function. The following proposition proves some crucial regularity properties of the interacting current-current correlations.

Proposition 4.1 [Existence and regularity of the interacting correlations.] *Let $\beta > 0$, $L \in \mathbb{N}$, $s, r \in \mathbb{N}$, $n = s + r$. Let $\mathbf{p}_i \in (2\pi/\beta)\mathbb{Z} \times \mathcal{B}_L$ for $i = 1, \ldots, n-1$. There exists $U_0 > 0$, independent of β, L, such that, for $|U| < U_0$:*

1. *the current-current correlations $\widehat{K}^{\beta,L}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ are analytic in U;*

2. *the limit $\widehat{K}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = \lim_{\beta\to\infty} \lim_{L\to\infty} \widehat{K}^{\beta,L}_{\mu_1,\ldots,\mu_s,\sigma_1,\ldots,\sigma_r}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ exists, it is C^2 in $\vec{p}_i \in \mathcal{B}$ and C^∞ in $p_{0,i}$, for all $i = 1, \ldots, n-1$.*

Remark 2 *1. The proposition is proven in Appendix 5; its proof follows from standard fermionic cluster expansion methods, and from the Gram-Hadamard inequality. Notice that, in general, the analyticity radius U_0 depends on $\delta_\mu > 0$, (see discussion after Eq.* **??***).*

2. *The $\vec{p}_i$ regularity is constrained by the properties of the interaction potential, (2.11). In general, the correlations cannot be more regular than $v_{\sigma\sigma'}(\vec{p})$.*

Ward Identities. The WI are exact identities for the correlation functions of the model, which ultimately follow from the continuity equation, Eq. (3.12). This is the content of the next proposition, whose proof is deferred to Appendix **??**.

Proposition 4.2 **[Ward Identities.]** *Let $\beta > 0$, $L \in \mathbb{N}$, and let $|U| < U_0$. Let $n \in \mathbb{N}$. Let $\mathbf{p}_i \in (2\pi/\beta)\mathbb{Z} \times \mathcal{B}_L$ for $i = 1, \ldots, n-1$. We have:*

$$\begin{aligned} -ip_{1,0}\widehat{K}^{\beta,L}_{0,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) &= \sum_{i=1,2} p_{1,i}\widehat{K}^{\beta,L}_{i,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \\ -ip_{1,0}\widehat{K}^{\beta,L}_{0,\mu,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) &= \sum_{i=1,2} p_{1,i}\widehat{K}^{\beta,L}_{i,\mu,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) + \widehat{S}^{\beta,L}_{0,\mu,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \end{aligned} \tag{4.4}$$

with

$$\widehat{S}^{\beta,L}_{\mu,\nu,\underline{\sigma}}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1}) := -\frac{1}{L^2}\left\langle \mathbf{T}\left[\tilde{J}_{\mu,(0,\vec{p}_1)}, \tilde{J}_{\nu,(0,\vec{p}_2)}\right] ; \hat{n}^{(\sigma_1)}_{\mathbf{p}_3} ; \cdots ; \hat{n}^{(\sigma_r)}_{-\mathbf{p}_1-\ldots-\mathbf{p}_{n-1}}\right\rangle_{\beta,L} . \tag{4.5}$$

Remark 3 1. *The second term appearing in the right-hand side of Eq. (4.4), called the Schwinger term, it is due to the fact that the theory is defined on a lattice; it would be absent for a continuum quantum field theory.*

2. *Using that $\widehat{K}^{\beta,L}_{\mu_1,\mu_2,\underline{\sigma}}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1}) = \widehat{K}^{\beta,L}_{\mu_2,\mu_1,\underline{\sigma}}(\mathbf{p}_2, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$, we also have:*

$$-ip_{2,0}\widehat{K}^{\beta,L}_{\mu,0,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = \sum_{i=1,2} p_{2,i}\widehat{K}^{\beta,L}_{\mu,i,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) + \widehat{S}^{\beta,L}_{0,\mu,\underline{\sigma}}(\mathbf{p}_2, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) . \tag{4.6}$$

3. *Clearly, $\widehat{S}^{\beta,L}_{\mu,\mu,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = 0$. Also, notice that $\widehat{S}^{\beta,L}_{\mu,\nu,\underline{\sigma}}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1})$ is constant in $p_{1,0}$, $p_{2,0}$.*

4. *Similar identities can be derived for the truncated expectations of an arbitrary number of $\hat{J}_\mu$'s, and gauge-invariant observables (i.e., the many-body interaction).*

The Ward identities have important consequences on the momentum-dependence of the current-current correlations. The following corollary will play a crucial role in the proof of our main result.

Corollary 4.1 **[Consequences of the Ward identities.]** *Let $|U| < U_0$. Let $j, j' \in \{1, 2\}$. The zero temperature, infinite volume correlations satisfy the following relations:*

$$\begin{aligned} &\widehat{K}_{j,\underline{\sigma}}((p_0, \vec{0}), \mathbf{p}_2, \ldots, \mathbf{p}_{n-1}) = -ip_0\frac{\partial}{\partial p_j}\widehat{K}_{0,\underline{\sigma}}((p_0, \vec{p}), \mathbf{p}_2, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}=\vec{0}} \qquad \text{for } n \geq 2 , \\ &\widehat{K}_{j,j',\underline{\sigma}}((p_0, \vec{0}), (-p_0, \vec{0}), \mathbf{p}_3, \ldots, \mathbf{p}_{n-1}) + \frac{\partial}{\partial p_{1,j}}\widehat{S}_{0,j',\underline{\sigma}}((p_0, \vec{p}_1), (-p_0, \vec{p}_2), \mathbf{p}_3, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}_1=\vec{p}_2=0} \\ &= p_0^2\frac{\partial^2}{\partial p_{1,j}\partial p_{2,j'}}\widehat{K}_{0,0,\underline{\sigma}}((p_0, \vec{p}_1), (-p_0, \vec{p}_2), \mathbf{p}_3, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}_1=\vec{p}_2=\vec{0}} . \qquad \text{for } n \geq 3 . \end{aligned} \tag{4.7}$$

Remark 4 *1. Thus, gauge invariance and differentiability imply nontrivial relations among correlation functions and their derivatives. In particular, as $p_0 \to 0$, we see that the left-hand side of the first of Eq. (4.7) is linearly vanishing, while the left-hand side of the second of Eq. (4.7) is quadratically vanishing.*

2. These relations are proven in a very simple way starting from the WIs (4.4) (see below). These are just two special examples of relations among correlations and derivatives of correlations that can be obtained starting from the WIs; however, these are the only two relations that will play a role in the proof of our main result.

*3. Similar consequences of the Ward identities have been used by Coleman and Hill in [**?**], to prove that all contributions beyond one-loop to the topological mass of QED_{2+1} vanish.*

Proof of Corollary 4.1. Consider the limit $\beta \to \infty$, $L \to \infty$. By Proposition 4.1, the limits of correlations exist and are smooth in their arguments. Let $n \geq 2$. Differentiating the first of Eq. (4.4) with respect to $p_{1,j}$, $j = 1, 2$, we get:

$$\widehat{K}_{j,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = -\sum_{i=1,2} p_{1,i} \frac{\partial}{\partial p_{1,j}} \widehat{K}_{i,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) - ip_{1,0} \frac{\partial}{\partial p_{1,j}} \widehat{K}_{0,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \; ; \tag{4.8}$$

setting $\vec{p}_1 = \vec{0}$, the first of Eq. (4.7) follows. Let now $n \geq 3$, and consider the second of (4.4), with $\mu = j' = 1, 2$. Differentiating with respect to $p_{1,j}$, $j = 1, 2$, we have:

$$\begin{aligned} &\widehat{K}_{j,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) + \frac{\partial}{\partial p_{1,j}} \widehat{S}_{0,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \\ &= -ip_{1,0} \frac{\partial}{\partial p_{1,j}} \widehat{K}_{0,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) - \sum_{i=1,2} p_{1,i} \frac{\partial}{\partial p_{1,j}} \widehat{K}_{i,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \end{aligned} \tag{4.9}$$

Similarly, consider Eq. (4.6), with $\mu = 0$. Differentiating with respect to $p_{2,j'}$, we find:

$$\begin{aligned} \widehat{K}_{0,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \; &= \; -ip_{2,0} \frac{\partial}{\partial p_{2,j'}} \widehat{K}_{0,0,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \\ &\quad - \sum_{i=1,2} p_{2,i} \frac{\partial}{\partial p_{2,j'}} \widehat{K}_{0,i,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \; . \end{aligned} \tag{4.10}$$

Setting $\vec{p}_2 = \vec{p}_1 = 0$, and plugging (4.10) into (4.9) we get:

$$\begin{aligned} &\widehat{K}_{j,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}_1=\vec{p}_2=0} + \frac{\partial}{\partial p_{1,j}} \widehat{S}_{0,j',\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}_1=\vec{p}_2=0} \\ &= -p_{1,0}p_{2,0} \frac{\partial^2}{\partial p_{1,j} \partial p_{2,j'}} \widehat{K}_{0,0,\underline{\sigma}}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})\big|_{\vec{p}_1=\vec{p}_2=\vec{0}} \; . \end{aligned} \tag{4.11}$$

Choosing $p_{1,0} = -p_{2,0} = p_0$, the second of Eq. (4.7) follows. ∎

Decomposition of the correlations. For U in the analyticity domain, we can expand the current-current correlations as follows:

$$\widehat{K}^{\beta,L}_{\mu,\nu,\underline{\sigma}}(\mathbf{p}_1,\ldots,\mathbf{p}_{n-1}) = \sum_{k\geq 0}\frac{U^k}{k!}\widehat{K}^{\beta,L,(k)}_{\mu,\nu,\underline{\sigma}}(\mathbf{p}_1,\ldots,\mathbf{p}_{n-1})\;, \tag{4.12}$$

where

$$\widehat{K}^{\beta,L,(k)}_{\mu,\nu,\underline{\sigma}}(\mathbf{p}_1,\ldots,\mathbf{p}_{n-1}) \;=\; \frac{1}{\beta L^2}\langle \hat{J}_{\mu,\mathbf{p}_1}\,;\,\hat{J}_{\nu,\mathbf{p}_2}\,;\,\hat{n}^{(\sigma_1)}_{\mathbf{p}_3}\,;\,\cdots\,;\,\hat{n}^{(\sigma_r)}_{-\mathbf{p}_1-\ldots-\mathbf{p}_{n-1}}\,;\,\widetilde{\mathcal{V}}_L^{\,;k}\rangle^{(0)}_{\beta,L}\,. \tag{4.13}$$

This formula follows from the well-known cumulant expansion of the interacting Gibbs state with respect to the noninteracting one, that we prove for completeness in Appendix **??**. In Eq. (4.13), $\widetilde{\mathcal{V}}_L^{\,;k}$ is a shorthand notation for $\underbrace{\widetilde{\mathcal{V}}_L\,;\,\widetilde{\mathcal{V}}_L\,;\,\cdots\,;\,\widetilde{\mathcal{V}}_L}_{k\text{ times}}$, with

$$\begin{aligned}\widetilde{\mathcal{V}}_L \;&:=\; \int_{[0,\beta)} dx_0\, e^{(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)x_0}\mathcal{V}_L e^{-(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)x_0} &(4.14)\\ &=\; \int_{[0,\beta)} dx_0 \int_{[0,\beta)} dy_0 \sum_{\vec{x},\vec{y}\in\Lambda_L}\sum_{\sigma,\sigma'\in I} n^{(\sigma)}_{(x_0,\vec{x})}\, v_{\sigma\sigma'}(\vec{x}-\vec{y})\,\delta^{\text{per}}_\beta(x_0-y_0)\, n^{(\sigma')}_{(y_0,\vec{y})}\\ &\equiv\; \frac{1}{\beta L^2}\sum_{\mathbf{q}\in\frac{2\pi}{\beta}\mathbb{Z}\times\mathcal{B}_L}\sum_{\sigma,\sigma'\in I}\hat{n}^{(\sigma)}_{\mathbf{q}}\hat{v}_{\sigma\sigma'}(\vec{q})\,\hat{n}^{(\sigma')}_{-\mathbf{q}}\;; &(4.15)\end{aligned}$$

in the second line we inserted a periodic Dirac delta function:

$$\delta^{\text{per}}_\beta(x_0-y_0) = \begin{cases} 1 \text{ if } x_0 = n\beta + y_0,\, n\in\mathbb{Z}\\ 0 \text{ otherwise,}\end{cases} \tag{4.16}$$

and the last line of Eq. (4.14) is obtained by writing:

$$\delta^{\text{per}}_\beta(x_0-y_0) = \frac{1}{\beta}\sum_{q_0\in\frac{2\pi}{\beta}\mathbb{Z}} e^{iq_0(x_0-y_0)}\;,\qquad \hat{n}^{(\sigma)}_{(q_0,\vec{q})} = \int_{[0,\beta)} dx_0\, e^{iq_0x_0}\tilde{n}^{(\sigma)}_{(x_0,\vec{q})}\;, \tag{4.17}$$

and exchanging the sum over q_0 with the integrals.

With the next proposition, we prove a crucial identity for the k-th order contribution to the interacting current-current correlation.

Proposition 4.3 [Schwinger-Dyson equation.] *Let $\beta>0$, $L\in\mathbb{N}$. Let $k\in\mathbb{N}$, $k\geq 1$. Let $\mathbf{p}\in\frac{2\pi}{\beta}\mathbb{Z}\times\mathcal{B}_L$. We have:*

$$\begin{aligned}\widehat{K}^{\beta,L,(k)}_{\mu,\nu}(\mathbf{p}) \;=\;& \frac{1}{\beta L^2}\sum_{\mathbf{q}\in\frac{2\pi}{\beta}\mathbb{Z}\times\mathcal{B}_L}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{q})\widehat{K}^{\beta,L,(k-1)}_{\mu,\nu,\sigma,\sigma'}(\mathbf{p},-\mathbf{p},\mathbf{q})\\ &+2\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{0})\widehat{K}^{\beta,L,(m)}_{\mu,\nu,\sigma}(\mathbf{p},-\mathbf{p})\widehat{K}^{\beta,L,(k-1-m)}_{\sigma'}\\ &+2\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{p})\widehat{K}^{\beta,L,(m)}_{\mu,\sigma}(\mathbf{p})\widehat{K}^{\beta,L,(k-1-m)}_{\nu,\sigma'}(-\mathbf{p})\,.\end{aligned}$$

The proof of this proposition relies on the following well-known property of the cumulants, whose proof is deferred to Appendix **??**.

Lemma 4.1 [Decomposition lemma.] *Let $n \geq 1$. Given a state $\langle \cdot \rangle$, and a set of $n+1$ time-dependent observables $O^{(1)}, \ldots, O^{(n+1)}$, the following identity holds:*

$$\langle \mathbf{T}\, O^{(1)} ; O^{(2)} ; \cdots ; O^{(n)} O^{(n+1)} \rangle = \langle \mathbf{T}\, O^{(1)} ; O^{(2)} ; \cdots ; O^{(n)} ; O^{(n+1)} \rangle$$
$$+ \sum^{*}_{\substack{\{i_1,\ldots,i_p\} \\ \{j_1,\ldots j_q\}}} \langle \mathbf{T}\, O^{(i_1)} ; O^{(i_2)} ; \cdots ; O^{(i_p)} ; O^{(n)} \rangle \langle \mathbf{T}\, O^{(j_1)} ; O^{(j_2)} ; \cdots ; O^{(j_q)} ; O^{(n+1)} \rangle \quad (4.18)$$

where the sum is over all partitions of $\{1, \ldots, n-1\}$ into two disjoint subsets, $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_q\}$, with $p + q = n - 1$ and $i_1 < \ldots < i_q$, $j_1 < \ldots < j_q$.

Proof of Proposition 4.3. For $k \geq 1$, we rewrite the k-th order contribution to the expansion of $\widehat{K}^{\beta,L}_{\mu,\nu}(\mathbf{p})$ as:

$$\widehat{K}^{\beta,L,(k)}_{\mu,\nu}(\mathbf{p}) = \frac{1}{(\beta L^2)^2} \sum_{\mathbf{q} \in \frac{2\pi}{\beta}\mathbb{Z} \times \mathcal{B}_L} \sum_{\sigma,\sigma' \in I} \hat{v}_{\sigma\sigma'}(\vec{q}) \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;k-1} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \hat{n}^{(\sigma')}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} . \quad (4.19)$$

Using Lemma 4.1, we get:

$$\begin{aligned}
&\langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;k-1} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \hat{n}^{(\sigma')}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} = \\
&= \frac{1}{2} \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;k-1} ; \hat{n}^{(\sigma)}_{\mathbf{q}} ; \hat{n}^{(\sigma')}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} \\
&\quad + \sum_{m=0}^{k-1} \binom{k-1}{m} \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \rangle^{(0)}_{\beta,L} \langle \mathbf{T}\, \widetilde{\mathcal{V}}_L^{;k-1-m} ; \hat{n}^{(\sigma')}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} \\
&\quad + \sum_{m=0}^{k-1} \binom{k-1}{m} \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \rangle^{(0)}_{\beta,L} \langle \mathbf{T}\, \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;k-1-m} ; \hat{n}^{(\sigma')}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} \\
&\quad + \text{terms obtained replacing } \mathbf{q} \to -\mathbf{q},\ \sigma \leftrightarrow \sigma'. \qquad (4.20)
\end{aligned}$$

The translation invariance of the Gibbs state implies that:

$$\begin{aligned}
\langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \rangle^{(0)}_{\beta,L} &= \delta_\beta(q_0) \delta^{\text{per}}_L(\vec{q}) \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \hat{J}_{\nu,-\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{0}} \rangle^{(0)}_{\beta,L} \\
\langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{q}} \rangle^{(0)}_{\beta,L} &= \delta_\beta(q_0 + p_0) \delta^{\text{per}}_L(\vec{q} + \vec{p}) \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}} ; \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{-\mathbf{p}} \rangle^{(0)}_{\beta,L} \\
\langle \mathbf{T}\, \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{-\mathbf{q}} \rangle^{(0)}_{\beta,L} &= \delta_\beta(q_0) \delta^{\text{per}}_L(\vec{q}) \langle \mathbf{T}\, \widetilde{\mathcal{V}}_L^{;m} ; \hat{n}^{(\sigma)}_{\mathbf{0}} \rangle^{(0)}_{\beta,L} , \qquad (4.21)
\end{aligned}$$

where $\delta_\beta(q_0)$ is the Kronecker delta for $q_0 \in \frac{2\pi}{\beta}\mathbb{Z}$, and $\delta^{\text{per}}_L(\vec{q})$ is the *periodic* Kronecker delta:

$$\delta^{\text{per}}_L(\vec{q}) = \begin{cases} 1 & \text{if } \vec{q} \in \Lambda^*_A \\ 0 & \text{otherwise.} \end{cases} \quad (4.22)$$

The claim (4.18) immediately follows after plugging (4.20) into (4.19), and imposing momentum conservation as in (4.21). ■

....

Here we prove our main result, Theorem 3.1. The proof is based on a combination of the three results discussed in Section 4 A, namely: the analyticity of the correlation functions (Proposition 4.1); Ward identities (Proposition 4.2); and the Schwinger-Dyson formula (Proposition 4.3).

Proof of Theorem 3.1. The analyticity of the conductivity matrix immediately follows from Proposition 4.1 and from the definition (3.14). Also, the vanishing of the longitudinal conductivity is an immediate consequence of the differentiability of the current-current correlations. In fact:

$$\begin{aligned}\widehat{K}^{\beta,L}_{ii}(p_0,\vec{0}) &= \frac{1}{\beta L^2}\int_{[0,\beta)} dx_0 \int_{[0,\beta)} dy_0\, e^{ip_0(x_0-y_0)}\langle \mathbf{T}\, J_{i,x_0}\,;\, J_{i,y_0}\rangle_{\beta,L} \\ &= \frac{1}{\beta L^2}\int_{[0,\beta)} dx_0 \int_{[0,\beta)} dy_0\, e^{ip_0(x_0-y_0)}\langle \mathbf{T}\, J_{i,y_0}\,;\, J_{i,x_0}\rangle_{\beta,L} \\ &= \widehat{K}^{\beta,L}_{ii}(-p_0,\vec{0})\;; \end{aligned} \tag{4.23}$$

thus, being $\widehat{K}_{ij}(\mathbf{p}) := \lim_{\beta,L\to\infty}\widehat{K}^{\beta,L}_{ij}(\mathbf{p})$ differentiable in $\mathbf{p}=\mathbf{0}$,

$$\sigma_{ii} = -\frac{1}{A}\frac{\partial}{\partial p_0}\widehat{K}_{ii}(p_0,\vec{0}) = 0\;. \tag{4.24}$$

Suppose now $i\neq j$. Consider the Taylor expansion in U of σ_{ij}, in the analyticity domain $|U|<U_0$. We have:

$$\sigma_{ij} = \sigma^{(0)}_{ij} + \sum_{k\geq 1}\frac{U^k}{k!}\sigma^{(k)}_{ij}\;,\qquad \text{for } |U|<U_0, \tag{4.25}$$

where $\{\sigma^{(k)}_{ij}\}_{k\geq 0}$ are the Taylor coefficients of σ_{ij}. Being the series convergent, to prove Theorem 3.1 it is sufficient to show that:

$$\sigma^{(k)}_{ij} = 0\;,\qquad \text{for all } k\geq 1. \tag{4.26}$$

To prove this, we write explicity the k-th order in the expansion for σ_{ij}, starting from the definition (3.14) and using Proposition 4.3. We have:

$$\sigma^{(k)}_{ij} = -\frac{1}{A}\lim_{p_0\to 0}\frac{\partial}{\partial p_0}\widehat{K}^{(k)}_{ij}(p_0,\vec{0}) \equiv \mathrm{I}+\mathrm{II}+\mathrm{III}\;, \tag{4.27}$$

where:

$$\mathrm{I} \;:=\; -\frac{1}{A}\lim_{p_0\to 0}\int \frac{d\mathbf{q}}{(2\pi)|\mathcal{B}|}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{q})\frac{\partial}{\partial p_0}\widehat{K}^{(k-1)}_{i,j,\sigma,\sigma'}\big((p_0,\vec{0}),(-p_0,\vec{0}),\mathbf{q}\big) \tag{4.28}$$

$$\mathrm{II} \;:=\; -\frac{2}{A}\lim_{p_0\to 0}\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{0})\frac{\partial}{\partial p_0}\widehat{K}^{(m)}_{i,j,\sigma}\big((p_0,\vec{0}),(-p_0,\vec{0})\big)\widehat{K}^{(k-1-m)}_{\sigma'}$$

$$\mathrm{III} \;:=\; -\frac{2}{A}\lim_{p_0\to 0}\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{p})\frac{\partial}{\partial p_0}\Big[\widehat{K}^{(m)}_{i,\sigma}(p_0,\vec{0})\widehat{K}^{(k-1-m)}_{j,\sigma'}(-p_0,\vec{0})\Big]$$

The idea is to use Corollary 4.1 to prove that the three contributions are separately zero. Let us start with I. In order to be in the position to apply Corollary 4.1, the preliminary remark is that:

$$\begin{aligned}&\frac{\partial}{\partial p_0}\widehat{K}^{(k-1)}_{i,j,\sigma,\sigma'}\big((p_0,\vec{0}),(-p_0,\vec{0}),\mathbf{q}\big)\\ &= \frac{\partial}{\partial p_0}\Big[\widehat{K}^{(k-1)}_{i,j,\sigma,\sigma'}\big((p_0,\vec{0}),(-p_0,\vec{0}),\mathbf{q}\big)+\frac{\partial}{\partial p_{1,i}}\widehat{S}^{(k-1)}_{0,j,\sigma,\sigma'}\big((p_0,\vec{p}_1),(-p_0,\vec{p}_2),\mathbf{q}\big)\big|_{\vec{p}_1=\vec{p}_2=\vec{0}}\Big]\end{aligned} \tag{4.29}$$

Eq. (4.29) simply follows from the fact that $\widehat{S}_{\mu,\nu,\sigma,\sigma'}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ is constant in $p_{1,0}$ and $p_{2,0}$ (recall the definition (4.5)). We are now in the position to use the second of Eq. (4.7). We get:

$$\begin{aligned}\mathrm{I} \;&=\; -\frac{1}{A}\lim_{p_0\to 0}\int \frac{d\mathbf{q}}{(2\pi)|\mathcal{B}|}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{q})\frac{\partial}{\partial p_0}\Big[p_0^2\frac{\partial^2}{\partial p_{1,i}\partial p_{2,j}}\widehat{K}^{(k-1)}_{0,0,\sigma,\sigma'}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})\big|_{\mathbf{p}_1=-\mathbf{p}_2=(p_0,\vec{0})}\Big]\\
&= -\frac{1}{A}\lim_{p_0\to 0}\int \frac{d\mathbf{q}}{(2\pi)|\mathcal{B}|}\sum_{\sigma,\sigma'\in I}\hat{v}_{\sigma\sigma'}(\vec{q})\Big[2p_0\frac{\partial^2}{\partial p_{1,i}\partial p_{2,j}}\widehat{K}^{(k-1)}_{0,0,\sigma,\sigma'}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})\big|_{\mathbf{p}_1=-\mathbf{p}_2=(p_0,\vec{0})}\\
&\quad +p_0^2\frac{\partial^3}{\partial p_{1,0}\partial p_{1,i}\partial p_{2,j}}\widehat{K}^{(k-1)}_{0,0,\sigma,\sigma'}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})\big|_{\mathbf{p}_1=-\mathbf{p}_2=(p_0,\vec{0})}\\
&\quad -p_0^2\frac{\partial^3}{\partial p_{2,0}\partial p_{1,i}\partial p_{2,j}}\widehat{K}^{(k-1)}_{0,0,\sigma,\sigma'}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})\big|_{\mathbf{p}_1=-\mathbf{p}_2=(p_0,\vec{0})}\Big] = 0\;,\end{aligned} \tag{4.30}$$

where in the last step we used that

...

Hence, (**??**) is given by a sum of contributions proportional to, using the shorthand notations $\int_{(\beta,L)} d\mathbf{x} := \int_0^\beta \sum_{\vec x\in\Lambda_L}$, $v_{\sigma\sigma'}(\mathbf{x}) := \delta_\beta^{\rm per}(x_0) v_{\sigma\sigma'}(\vec x)$:

$$\begin{aligned}
&\frac{1}{(\beta L^2)^2}\sum_{\mathbf{q}\in\frac{2\pi}{\beta}\mathbb{Z}\times\mathcal{B}_L}\hat v_{\sigma\sigma'}(\vec q)\int_{(\beta,L)} d\mathbf{x}_1 d\mathbf{x}_2\int_{(\beta,L)} d\mathbf{y}_1\cdots d\mathbf{y}_{2k-2}\int_{(\beta,L)} d\mathbf{w}_1 d\mathbf{w}_2 \qquad (4.31)\\
&\quad\cdot e^{i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)} e^{i\mathbf{q}(\mathbf{w}_1-\mathbf{w}_2)} v_{\sigma_1\sigma_2}(\mathbf{y}_1-\mathbf{y}_2)\cdots v_{\sigma_{2k-3}\sigma_{2k-2}}(\mathbf{y}_{2k-3}-\mathbf{y}_{2k-2})\\
&\quad\cdot\langle \mathbf{T}\, J_{\mathbf{x}_1}(\vec z_1,\tilde\sigma_1,\tilde\sigma_2)\,;\, J_{\mathbf{x}_2}(\vec z_2,\tilde\sigma_3,\tilde\sigma_4)\,;\, n^{(\sigma_1)}_{\mathbf{y}_1} n^{(\sigma_2)}_{\mathbf{y}_2}\,;\,\cdots\,;\, n^{(\sigma_{2k-3})}_{\mathbf{y}_{2k-3}} n^{(\sigma_{2k-2})}_{\mathbf{y}_{2k-2}}\,;\, n^{(\sigma)}_{\mathbf{w}_1}\,;\, n^{(\sigma')}_{\mathbf{w}_2}\rangle^{(0)}_{\beta,L}\\
&=\frac{1}{\beta L^2}\int_{(\beta,L)} d\mathbf{x}_1 d\mathbf{x}_2\int_{(\beta,L)} d\mathbf{y}_1\cdots d\mathbf{y}_{2k-2}\int_{(\beta,L)} d\mathbf{w}_1 d\mathbf{w}_2\\
&\quad\cdot e^{i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)} v_{\sigma\sigma'}(\mathbf{w}_1-\mathbf{w}_2) v_{\sigma_1\sigma_2}(\mathbf{y}_1-\mathbf{y}_2)\cdots v_{\sigma_{2k-3}\sigma_{2k-2}}(\mathbf{y}_{2k-3}-\mathbf{y}_{2k-2})\\
&\quad\cdot\langle \mathbf{T}\, J_{\mathbf{x}_1}(\vec z_1,\tilde\sigma_1,\tilde\sigma_2)\,;\, J_{\mathbf{x}_2}(\vec z_2,\tilde\sigma_3,\tilde\sigma_4)\,;\, n^{(\sigma_1)}_{\mathbf{y}_1} n^{(\sigma_2)}_{\mathbf{y}_2}\,;\,\cdots\,;\, n^{(\sigma_{2k-3})}_{\mathbf{y}_{2k-3}} n^{(\sigma_{2k-2})}_{\mathbf{y}_{2k-2}}\,;\, n^{(\sigma)}_{\mathbf{w}_1}\,;\, n^{(\sigma')}_{\mathbf{w}_2}\rangle^{(0)}_{\beta,L}\,.
\end{aligned}$$

Expanding the expectation in connected Feynman diagrams, and using the estimate (**??**) together with $|H_{\sigma\sigma'}(\vec z)|\le C$ and the decay properties (2.11) of $v_{\sigma\sigma'}(\vec x)$, it is easy to see that the last integral in (4.31) is absolutely convergent.

...

Consider now II. Proceeding as before, and using again the second of Eq. (4.7), we get:

$$\begin{aligned}
{\rm II} &= -\frac{2}{A}\lim_{p_0\to 0}\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{\sigma,\sigma'}\hat v_{\sigma\sigma'}(\vec 0)\frac{\partial}{\partial p_0}\Big[p_0^2\frac{\partial^2}{\partial p_{1,i}\partial p_{2,j}}\widehat K^{(m)}_{0,0,\sigma}(\mathbf{p}_1,\mathbf{p}_2)\big|_{\mathbf{p}_1=-\mathbf{p}_2=(p_0,\vec 0)}\Big]\widehat K^{(k-1-m)}_{\sigma'}\\
&= 0\,. \qquad (4.32)
\end{aligned}$$

Finally, consider III. From the first of Eq. (4.7), we have that:

$$\begin{aligned}
\widehat K^{(m)}_{i,\sigma}(p_0,\vec 0) &= -ip_0\frac{\partial}{\partial p_i}\widehat K^{(m)}_{0,\sigma}(\mathbf{p})\big|_{\vec p=\vec 0}\\
\widehat K^{(k-1-m)}_{j,\sigma'}(-p_0,\vec 0) &= ip_0\frac{\partial}{\partial p_i}\widehat K^{(k-1-m)}_{j,\sigma'}(-p_0,\vec p)\big|_{\vec p=\vec 0}\,; \qquad (4.33)
\end{aligned}$$

plugging these two identities in III, we get

$$\text{III} = 0\,. \qquad (4.34)$$

This conclues the proof of (4.26), and of Theorem 3.1. ∎

5. ANALYTICITY

In this section we prove Proposition 4.1, concerning analyticity in U and smoothness in $\mathbf{p}$ of the multi-point current/density correlation functions. Roughly, the strategy will consist in: (i) reformulating the correlation functions in terms of a Grassmann integral, in the limit where a suitable cutoff function is removed; (ii) proving analyticity of the Grassmann integral, uniformly in the cutoff parameter; (iii) using Vitali's uniform convergence theorem for analytic functions, to conclude that the correlations themselves are analytic.

A. Grassmann representation

Let us preliminarily recall a few known facts about perturbation theory for the free energy and correlations of interacting fermionic systems, which we need for justifying their Grassmann representation. We first discuss the free energy, which is simpler. Using Duhamel's expansion, we can rewrite the (a priori formal) series expansion of the interacting partition function in the parameter U as:

$$\frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}} = 1+\sum_{n\geq 1}(-U)^n\int_0^\beta dt_1\cdots\int_0^{t_{n-1}} dt_n \frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}} \tag{5.1}$$

where $\mathcal{V}_L(t) = e^{t(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}\mathcal{V}_L e^{-t(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}$ is the non-interacting ($U=0$) version of the imaginary time evolution of $\mathcal{V}_L$, cfr. Eq.(2.15). Symmetrizing over the permutations of $t_1,\ldots,t_n$, this can be rewritten as

$$\frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L)}} = 1+\sum_{n\geq 1}\frac{(-U)^n}{n!}\int_0^\beta dt_1\cdots\int_0^{\beta} dt_n \langle \mathbf{T}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)\rangle^0_{\beta,L}, \tag{5.2}$$

where $\langle \mathbf{T}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)\rangle^0_{\beta,L}$ is defined by the analogue of Eq.(2.16) with $U=0$. Since $\mathcal{H}_L^{(0)}-\mu\mathcal{N}_L$ is quadratic in the fermionic creation/annihilation operators, $\langle\cdot\rangle^0_{\beta,L}$ can be computed via the fermionic Wick rule, which is the following. In order to evaluate $\langle \mathbf{T}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)\rangle^0_{\beta,L}$ (where the times t_i are all different from each other, as we can suppose with no loss of generality), recall that each $\mathcal{V}_L(t)$ is a linear combination of quartic monomials in the (imaginary time evolution of the) creation/annihilation operators, so that the product $\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)$ itself is a linear combination of monomials, all of order $4n$. For each such monomial, consider all possible pairings of the creation/annihilation operators such that each annihilation operator $\psi^-_{\vec{x},\sigma}(t)$ is paired with a creation operator $\psi^+_{\vec{x}',\sigma'}(t')$. Then associate each pairing with a value, given by the sign of the permutation required to move every creation operator to the immediate right of the annihilation operator it is paired with, times the product over the pairs of the corresponding propagators, where the propagator corresponding to the pair $(\psi^-_{\vec{x},\sigma}(t),\psi^+_{\vec{x}',\sigma'}(t'))$ is

$$\begin{aligned} g^{\beta,L}_{\sigma,\sigma'}(t-t',\vec{x}-\vec{x}') &= \langle \mathbf{T}\psi^-_{\vec{x},\sigma}(t)\psi^+_{\vec{x}',\sigma'}(t')\rangle^0_{\beta,L} \\ &= \frac{1}{L^2}\sum_{\vec{k}\in\mathcal{B}_L} e^{-i\vec{k}(\vec{x}-\vec{x}')}\Big[e^{-(t-t')(\hat{H}^{(0)}(\vec{k})-\mu)}\Big(\frac{1(t>t')}{1+e^{-\beta(\hat{H}^{(0)}(\vec{k})-\mu)}}-\frac{1(t\leq t')\,e^{-\beta(\hat{H}^{(0)}(\vec{k})-\mu)}}{1+e^{-\beta(\hat{H}^{(0)}(\vec{k})-\mu)}}\Big)\Big]_{\sigma,\sigma'}. \end{aligned} \tag{5.3}$$

In the following, we denote by $g^{\beta,L}(t,\vec{x})$ the matrix whose elements are $g^{\beta,L}_{\sigma,\sigma'}(t,\vec{x})$. Note that, if $0<t<\beta$, then $g^{\beta,L}(t-\beta,\vec{x}) = -g^{\beta,L}(t,\vec{x})$. Therefore, it is natural to extend $g^{\beta,L}(t,\vec{x})$, which is a priori defined only on the time interval $(-\beta,\beta)$, to the whole real line, by anti-periodicity in the imaginary time, i.e., via the rule $g^{\beta,L}(t+n\beta,\vec{x}) = (-1)^n g^{\beta,L}(t,\vec{x})$.

The resulting extension can be expanded in Fourier series w.r.t. t, so that, for all $t \neq n\beta$,

$$g^{\beta,L}(t,\vec{x}) = \frac{1}{\beta L^2} \sum_{\substack{k_0 \in \mathcal{B}_\beta \\ \vec{k} \in \mathcal{B}_L}} e^{-i\vec{k}\cdot\vec{x} - ik_0 t}\, \hat{g}^{\beta,L}(k_0,\vec{k}) \tag{5.4}$$

with $\mathcal{B}_\beta = \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$ and

$$\hat{g}^{\beta,L}(k_0,\vec{k}) := \frac{1}{-ik_0 + \hat{H}^{(0)}(\vec{k}) - \mu}\ . \tag{5.5}$$

If, instead, $t = n\beta$, then $g^{\beta,L}(n\beta,\vec{x}) = (-1)^n \lim_{t\to 0^-} g^{\beta,L}(t,\vec{x})$. Note that, by the very definition of the propagator and the canonical anti-commutation relations, $g^{\beta,L}_{\sigma,\sigma'}(0^+,\vec{x}) - g^{\beta,L}_{\sigma,\sigma'}(0^-,\vec{x}) = \delta_{\vec{x},\vec{0}}\delta_{\sigma,\sigma'}$, so that the only discontinuity points of $g^{\beta,L}(t,\vec{x})$ are $(n\beta,\vec{0})$.

In the following we will also need a variant of $g^{\beta,L}(t,\vec{x})$, to be denoted by $\bar{g}^{\beta,L}(t,\vec{x})$, which coincides with $g^{\beta,L}(t,\vec{x})$, $\forall (t,\vec{x}) \neq (n\beta,\vec{0})$, and with the arithmetic mean of $g^{\beta,L}(0^+,\vec{0})$ and $g^{\beta,L}(0^-,\vec{0})$ at the discontinuity points:

$$\bar{g}^{\beta,L}(\mathbf{x})\big|_{\mathbf{x}=(n\beta,\vec{x})} = \frac{g^{\beta,L}(0^+,\vec{0}) + g^{\beta,L}(0^-,\vec{0})}{2}. \tag{5.6}$$

The function $\bar{g}^{\beta,L}(\mathbf{x})$ is a natural object to introduce, in that it is the limit as $M\to\infty$ of a regularization of $g^{\beta,L}(\mathbf{x})$ obtained by cutting off the ultraviolet modes $|k_0| > 2^M$ in the right side of (5.4). More specifically, if we take a smooth even compact support function $\chi_0(t)$, equal to 1 for $|t| < 1$ and equal to 0 for $|t| > 2$, and we define

$$\bar{g}^{\beta,L,M}(\mathbf{x}) = \frac{1}{\beta L^2} \sum_{\mathbf{k}\in\mathcal{B}_\beta\times\mathcal{B}_L} e^{-i\mathbf{k}\cdot\mathbf{x}} \chi_0(2^{-M}k_0/\delta_\mu)\hat{g}^{\beta,L}(\mathbf{k}), \tag{5.7}$$

then

$$\bar{g}^{\beta,L}(\mathbf{x}) = \lim_{M\to\infty} \bar{g}^{\beta,L,M}(\mathbf{x}). \tag{5.8}$$

These propagators can be used to re-express the formal perturbation theory in (5.2) in terms of the limit of a regularized theory with finitely many degrees of freedom, which is advantageous for performing rigorous bounds on the convergence of the series. More precisely, we note that (5.2), as an identity between (a priori formal) power series, can be equivalently rewritten as

$$\frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L - \mu\mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L^{(0)} - \mu\mathcal{N}_L)}} = \lim_{M\to\infty} \Big[1 + \sum_{n\geq 1} \frac{(-U)^n}{n!} \int_0^\beta dt_1 \cdots \int_0^\beta dt_n\, \bar{\mathbb{E}}^{(M)}_{\beta,L}\Big(\bar{\mathcal{V}}_L(t_1)\cdots\bar{\mathcal{V}}_L(t_n)\Big)\Big], \tag{5.9}$$

where

$$\bar{\mathcal{V}}_L(t) = \sum_{\vec{x},\vec{y}\in\Lambda_L} \sum_{\sigma,\sigma'\in I} \Big(\psi^+_{(t,\vec{x}),\sigma}\psi^-_{(t,\vec{x}),\sigma} + \frac{1}{2}\Big) v_{\sigma\sigma'}(\vec{x}-\vec{y}) \Big(\psi^+_{(t,\vec{y}),\sigma'}\psi^-_{(t,\vec{y}),\sigma'} + \frac{1}{2}\Big) \tag{5.10}$$

and $\bar{\mathbb{E}}^{(M)}_{\beta,L}(\cdot)$ acts linearly on normal-ordered polynomials in $\psi^{\pm}_{(t,\vec{x}),\sigma}$, the action on a normal-ordered monomial being defined by the fermionic Wick rule with propagator

$$\bar{\mathbb{E}}^{(M)}_{\beta,L}(\psi^-_{(t,\vec{x}),\sigma}\psi^+_{(t',\vec{x}'),\sigma'}) = \bar{g}^{\beta,L,M}_{\sigma,\sigma'}(t-t',\vec{x}-\vec{x}').$$

In order to check that the right side of (5.9) coincides order by order with the right side of (5.2), it is enough to note the following (assume, again without loss of generality, that the times $t_1,\ldots,t_n$ are all distinct):

- all the pairings contributing to $\langle \mathbf{T}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)\rangle^0_{\beta,L}$ without *tadpoles* (i.e., without contractions of two fields at the same space-time point) give the same contribution as the corresponding pairing in $\lim_{M\to\infty}\bar{\mathbb{E}}^{(M)}_{\beta,L}\Big(\bar{\mathcal{V}}_L(t_1)\cdots\bar{\mathcal{V}}_L(t_n)\Big)$, simply because $g^{\beta,L}(\mathbf{x}) = \bar{g}^{\beta,L}(\mathbf{x})$, $\forall \mathbf{x}\neq(\beta n,\vec{0})$;
- in the pairings contributing to $\langle \mathbf{T}\mathcal{V}_L(t_1)\cdots\mathcal{V}_L(t_n)\rangle^0_{\beta,L}$ that contain tadpoles, every tadpole corresponds to a factor $\langle\psi^+_{(t,\vec{x}),\sigma}\psi^-_{(t,\vec{x}),\sigma}\rangle^0_{\beta,L} = -g^{\beta,L}_{\sigma,\sigma}(0^-,\vec{0})$, while the corresponding tadpole in $\lim_{M\to\infty}\bar{\mathbb{E}}^{(M)}_{\beta,L}\Big(\bar{\mathcal{V}}_L(t_1)\cdots\bar{\mathcal{V}}_L(t_n)\Big)$ contributes a factor

  $$\lim_{M\to\infty}\bar{\mathbb{E}}^{(M)}_{\beta,L}\Big(\psi^+_{(t,\vec{x}),\sigma}\psi^-_{(t,\vec{x}),\sigma}\Big) = -\bar{g}^{\beta,L}_{\sigma,\sigma}(0,\vec{0}) = -\frac{1}{2}\big[g^{\beta,L}_{\sigma,\sigma}(0^+,\vec{0})) + g^{\beta,L}_{\sigma,\sigma}(0^-,\vec{0})\big].$$

  The difference between the two is

  $$-\bar{g}^{\beta,L}_{\sigma,\sigma}(0,\vec{0}) + g^{\beta,L}_{\sigma,\sigma}(0,\vec{0}) = -\frac{1}{2}\big[g^{\beta,L}_{\sigma,\sigma}(0^+,\vec{0})) - g^{\beta,L}_{\sigma,\sigma}(0^-,\vec{0})\big] = -\frac{1}{2},$$

  which is compensated exactly by the $+\frac{1}{2}$'s appearing in the definition (5.10).

A concise way of rewriting the series in brackets in (5.9) is in terms of Grassmann integrals:

$$1+\sum_{n\geq 1}\frac{(-U)^n}{n!}\int_0^\beta dt_1\cdots\int_0^\beta dt_n\,\bar{\mathbb{E}}^{(M)}_{\beta,L}\Big(\bar{\mathcal{V}}_L(t_1)\cdots\bar{\mathcal{V}}_L(t_n)\Big) = \int P_{\leq M}(d\Psi)e^{-UV_{\beta,L}(\Psi)}, \quad (5.11)$$

where $V_{\beta,L}(\Psi)$ and $\int P_{\leq M}(d\Psi)$ are, respectively, an element of a finite Grassmann algebra, and a linear map from the even part of the same algebra to the real numbers, defined as follows. Let $\mathcal{B}^*_\beta = \mathcal{B}_\beta\cap\{k_0\,:\,\chi_0(2^{-M}k_0)>0\}$, with $\mathcal{B}_\beta$ defined after (5.4), and $\mathcal{B}^*_{\beta,L} = \mathcal{B}^*_\beta\times\mathcal{B}_L$. We consider the finite Grassmann algebra generated by the Grassmann variables $\{\hat{\Psi}^{\pm}_{\mathbf{k},\sigma}\}^{\sigma\in I}_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}$ and we let

$$V_{L,\beta}(\Psi) = \sum_{\substack{\vec{x},\vec{y}\in\Lambda_L\\ \sigma,\sigma'\in I}}\int_0^\beta dt\,\Big(\Psi^+_{(t,\vec{x}),\sigma}\Psi^-_{(t,\vec{x}),\sigma}+\frac{1}{2}\Big)v_{\sigma\sigma'}(\vec{x}-\vec{y})\Big(\Psi^+_{(t,\vec{y}),\sigma'}\Psi^-_{(t,\vec{y}),\sigma'}+\frac{1}{2}\Big), \quad (5.12)$$

where

$$\Psi^{\pm}_{\mathbf{x},\sigma} = \frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} e^{\pm i\mathbf{k}\mathbf{x}}\hat{\Psi}^{\pm}_{\mathbf{k},\sigma}\,. \quad (5.13)$$

Moreover, $\int P_{\le M}(d\Psi)$ acts on a generic even monomial in the Grassmann variables as follows: it gives non zero only if the number of $\hat\Psi^+_{\mathbf{k},\sigma}$ variables is the same as the number of $\hat\Psi^-_{\mathbf{k},\sigma}$ variables, in which case

$$\int P_{\le M}(d\Psi)\hat\Psi^-_{\mathbf{k}_1,\sigma_1}\hat\Psi^+_{\mathbf{p}_1,\sigma'_1}\cdots\hat\Psi^-_{\mathbf{k}_m,\sigma_m}\hat\Psi^+_{\mathbf{p}_m,\sigma'_m} = \det[C(\mathbf{k}_i,\sigma_i;\mathbf{p}_j,\sigma'_j)]_{i,j=1,\dots,m}, \tag{5.14}$$

where $C(\mathbf{k},\sigma;\mathbf{p},\sigma') = \beta L^2\delta_{\mathbf{k},\mathbf{p}}\chi_0(2^{-M}k_0/\delta_\mu)\hat g^{\beta,L}_{\sigma,\sigma'}(\mathbf{k})$. In particular,

$$\int P_{\le M}(d\Psi)\Psi^-_{\mathbf{x}}\Psi^+_{\mathbf{y}} = \bar g^{\beta,L,M}(\mathbf{x}-\mathbf{y}). \tag{5.15}$$

If needed, $\int P_{\le M}(d\Psi)$ can be written explicitly in terms of the usual Berezin integral $\int d\Psi$, which is the linear functional on the Grassmann algebra acting non trivially on a monomial only if the monomial is of maximal degree, in which case

$$\int d\Psi \prod_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\prod_{\sigma\in I}\hat\Psi^-_{\mathbf{k},\sigma}\hat\Psi^+_{\mathbf{k},\sigma} = 1.$$

The explicit expression of $\int P_{\le M}(d\Psi)$ in terms of $\int d\Psi$ is

$$\int P_{\le M}(d\Psi)(\,\cdot\,) = \frac{1}{N_{\beta,L,M}}\int d\Psi \exp\Big\{-\frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\chi_0^{-1}(2^{-M}k_0)\hat\Psi^+_{\mathbf{k},\cdot}\big[\hat g^{\beta,L}_{\mathbf{k}}\big]^{-1}\hat\Psi^-_{\mathbf{k},\cdot}\Big\}(\,\cdot\,),$$
$$\text{with}\qquad N_{\beta,L,M} = \prod_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}[\beta L^2\chi_0(2^{-M}k_0/\delta_\mu)]^{|I|}\det\hat g^{\beta,L}_{\mathbf{k}}\ , \tag{5.16}$$

which motivates the appellation "Gaussian integration" that is usually given to the reference "measure" $P_{\le M}(d\Psi)$. Because of (5.15), $P_{\le M}(d\Psi)$ is also called the Gaussian integration with propagator $\bar g^{\beta,L,M}$.

It is straightforward to check that the definitions above are given in such a way that the two sides of (5.11) coincide, order by order in U. Note, by the way, that (5.11) is a (finite) polynomial in U, for every finite β, L, M, simply because the Grassmann algebra entering the definition of the right side of (5.11) is finite.

Summarizing,

$$\frac{\mathrm{Tr}_{\mathcal{F}}e^{-\beta(\mathcal{H}_L-\mu\mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}}e^{-\beta(\mathcal{H}^{(0)}_L-\mu\mathcal{N}_L)}} = \lim_{M\to\infty}\int P_{\le M}(d\Psi)e^{-UV_{\beta,L}(\Psi)}, \tag{5.17}$$

as an identity between (a priori formal) power series in U. In a similar way, one can show (details left to the reader) that the power series expansion for the truncated multipoint current-density correlations can be rewritten as

$$\langle \mathbf{T}J_{\mathbf{x}_1}(\vec z_1,\sigma_1,\sigma'_1);\cdots;J_{\mathbf{x}_m}(\vec z_m,\sigma_m,\sigma'_m);n^{(\sigma_{m+1})}_{\mathbf{x}_{m+1}};\cdots;n^{(\sigma_{m+n})}_{\mathbf{x}_{m+n}}\rangle_{\beta,L} = \tag{5.18}$$
$$= \lim_{M\to\infty}\frac{\partial^{m+n}}{\partial A^{\sigma_1\sigma'_1}_{\mathbf{x}_1}(\vec z_1)\cdots\partial\phi^{\sigma_{m+n}}_{\mathbf{x}_{m+n}}}\log\int P_{\le M}(d\Psi)e^{-UV_{\beta,L}(\Psi)+(\phi,n)+(A,J)}\Big|_{A=\phi=0}\ ,$$

where

$$(\phi, n) = \int_0^\beta dt \sum_{\vec{x},\sigma} \phi^\sigma_{(t,\vec{x})} \big(\Psi^+_{(t,\vec{x}),\sigma} \Psi^-_{(t,\vec{x}),\sigma} + \frac{1}{2}\big) , \tag{5.19}$$

$$(A, J) = \int_0^\beta dt \sum_{\vec{x},\vec{z}} \sum_{\sigma,\sigma'} A^{\sigma\sigma'}_{(t,\vec{x})}(\vec{z}) \big[i\Psi^+_{(t,\vec{x}),\sigma} H^{(0)}_{\sigma\sigma'}(-\vec{z}) \Psi^-_{(t,\vec{x}+\vec{z}),\sigma'} - i\Psi^+_{(t,\vec{x}+\vec{z}),\sigma'} H^{(0)}_{\sigma'\sigma}(\vec{z}) \Psi^-_{(t,\vec{x}),\sigma} \big] .$$

The goal of the incoming discussion is to show that (5.17) and (5.18) are not just identities between formal power series, but rather between analytic functions of U. In order to prove this, it suffices to prove the uniform analyticity in M, as $M \to \infty$, and the existence of the limit as $M \to \infty$ of the regularized free energy per site and correlations, as the following elementary lemma shows.

Lemma 5.1 *Assume that, for any finite β and L, there exists $\varepsilon_{\beta,L} > 0$ such that the regularized free energy per site*

$$f_{\beta,L,M} = -\frac{1}{\beta L^2} \log \int P_{\le M}(d\Psi) e^{-UV_{\beta,L}(\Psi)} \tag{5.20}$$

and the regularized truncated correlations

$$K^{\beta,L,M}(\mathbf{x}_1, \vec{z}_1, \sigma_1, \sigma'_1; \ldots ; \mathbf{x}_{m+n}, \sigma_{m+n}) = \tag{5.21}$$
$$= \frac{\partial^{m+n}}{\partial A^{\sigma_1\sigma'_1}_{\mathbf{x}_1}(\vec{z}_1) \cdots \partial \phi^{\sigma_{m+n}}_{\mathbf{x}_{m+n}}} \log \int P_{\le M}(d\Psi) e^{-UV_{\beta,L}(\Psi)+(\phi,n)+(A,J)} \Big|_{A=\phi=0}$$

are analytic functions of U in the domain $D_{\beta,L} = \{U : \; |U| < \varepsilon_{\beta,L}\}$, uniformly in M as $M \to \infty$. Moreover, assume that in any compact subset of $D_{\beta,L}$ the sequences $\{f_{\beta,L,M}\}_{M\ge1}$ and $\{K^{\beta,L,M}(\mathbf{x}_1, \vec{z}_1, \sigma_1, \sigma'_1; \ldots ; \mathbf{x}_{m+n}, \sigma_{m+n})\}_{M\ge1}$ converge uniformly as $M \to \infty$. Then (5.17) *and* (5.18) *are valid as identities between analytic functions of U in $D_{\beta,L}$.*

Remark 5 *In the following we will prove the assumption of this lemma, and actually much more: namely, we will prove the analyticity of $f_{\beta,L,M}$ and $K^{\beta,L,M}(\mathbf{x}_1, \vec{z}_1, \sigma_1, \sigma'_1; \ldots ; \mathbf{y}_n, \sigma''_n)$, uniformly in β, L, M (not just in M). We will also prove that these functions converge not only as $M \to \infty$, but also as $L \to \infty$ and $\beta \to \infty$, which in turn implies that the limiting correlations in the thermodynamic and zero temperature limits are analytic as well, as claimed in Proposition 4.1.*

Proof of Lemma 5.1. Let us start by proving (5.17), which is equivalent to

$$\frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L - \mu\mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}^{(0)}_L - \mu\mathcal{N}_L)}} = \lim_{M\to\infty} e^{-\beta L^2 f_{\beta,L,M}} . \tag{5.22}$$

The first key remark is that, if β, L are finite, the left side of this equation is an entire function of U, as it follows from the fact that the Fock space generated by the fermion operators $\psi^{\pm}_{\vec{x},\sigma}$, with $\vec{x} \in \Lambda_L$, $\sigma \in I$, is finite dimensional. On the other hand, by assumption, $f_{\beta,L,M}$ is analytic in $D_{\beta,L}$ and uniformly convergent as $M \to \infty$ in every compact subset of $D_{\beta,L}$. Hence, by Weierstrass' convergence theorem for analytic functions, the limit $f_{\beta,L} = \lim_{M\to\infty} f_{\beta,L,M}$ is analytic in $D_{\beta,L}$ and its Taylor coefficients coincide with the limits as $M \to \infty$ of the Taylor coefficients of $f_{\beta,L,M}$. Moreover, by construction, as discussed after (5.9), the Taylor coefficients of $e^{-\beta L^2 f_{\beta,L}}$ coincide with the Taylor coefficients of the left side of (5.22), which implies the validity of (5.22) as an identity between analytic functions in $D_{\beta,L}$, simply because the left side is entire in U, the right side is analytic in $D_{\beta,L}$ and the Taylor coefficients at the origin of the two sides are the same. By taking the logarithm at both sides, we also find that

$$f_{\beta,L} = -\frac{1}{\beta L^2} \log \frac{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}_L - \mu \mathcal{N}_L)}}{\mathrm{Tr}_{\mathcal{F}} e^{-\beta(\mathcal{H}^{(0)}_L - \mu \mathcal{N}_L)}}$$

as an identity between analytic functions in $D_{\beta,L}$. In particular, the left side of (5.22) does not vanish on $D_{\beta,L}$.

In order to prove the analogous claim for the correlation functions, we note that the truncated correlations $\langle \mathbf{T} J_{\mathbf{x}_1}(\vec{z}_1, \sigma_1, \sigma'_1); \cdots ; J_{\mathbf{x}_m}(\vec{z}_m, \sigma_m, \sigma'_m); n^{(\sigma_{m+1})}_{\mathbf{x}_{m+1}}; \cdots ; n^{(\sigma_{m+n})}_{\mathbf{x}_{m+n}} \rangle_{\beta,L}$ are linear combination of ratios of entire functions, simply because they are linear combinations of products of non-truncated functions, each of which is a ratio of entire functions. The denominator in these ratios is proportional to a power of the left side of (5.22) that, as observed earlier, does not vanish on $D_{\beta,L}$. Therefore, the truncated correlations are analytic in $D_{\beta,L}$, which allow us to repeat the same argument used above for the free energy, to conclude the validity of (5.18) as well, as an identity between analytic functions in $D_{\beta,L}$.

B. Uniform analyticity of the regularized correlation functions

In this section, we prove the uniform analyticity of the regularized free energy per site and regularized correlations, in a domain D independent not only of M, but also of β, L. Later, we will discuss the existence of the limit as $M, L, \beta \to \infty$ of the regularized functions, thus proving the assumptions of Lemma 5.1, as well as the existence and analyticity of the infinite volume and zero temperature limits. Throughout the proof, $C, C_i, c, c_i, \ldots,$ stand for unspecified constants, independent of β, L, M and of δ_μ, unless specified otherwise. The key result proved in this section is the following.

Lemma 5.2 *There exists $\varepsilon_0 = \varepsilon_0(\delta_\mu) > 0$ such that the regularized free energy $f_{\beta,L,M}$ and correlations $K^{\beta,L,M}(\mathbf{x}_1, \vec{z}_1, \sigma_1, \sigma'_1; \ldots; \mathbf{x}_{m+n}, \sigma_{m+n})$ are analytic in the common analyticity domain $D_0 = \{U : |U| \le \varepsilon_0\}$. Moreover, the regularized correlations are translation invariant and they satisfy the cluster property with faster-than-any-power decay rate, i.e., for any collection of integers $\underline{m} = \{m_{i,j}, m_k\}^{i,j=1,\ldots,m+n}_{k=1,\ldots,m} \ge 0$, there exists a constant $C_{\underline{m}} =$*

$C_{\underline{m}}(\delta_\mu)$ such that

$$\frac{1}{\beta L^2}\int_{\Lambda^{m+n}_{\beta,L}} d\underline{\mathbf{x}} \sum_{\underline{\vec z}\in\Lambda_L^m} \left|K^{\beta,L,M}(\mathbf{x}_1,\vec z_1,\sigma_1,\sigma_1';\dots;\mathbf{x}_{m+n},\sigma_{m+n})\right| d_{\underline{m}}(\underline{\mathbf{x}},\underline{\vec z}) \le C_{\underline{m}}. \tag{5.23}$$

Here $\underline{\mathbf{x}}=\{\mathbf{x}_1,\dots,\mathbf{x}_{m+n}\}$, $\underline{\vec z}=\{\vec z_1,\dots,\vec z_m\}$, $\Lambda_{\beta,L}=(0,\beta)\times\Lambda_L$, $\int_{\Lambda_{\beta,L}} d\mathbf{x}$ is a shorthand for $\int_0^\beta dx_0\sum_{\vec x\in\Lambda_L}$, and $d_{\underline{m},\underline{m}'}(\underline{\mathbf{x}},\underline{\vec z})=|\mathbf{x}_i-\mathbf{x}_j|^{m_{i,j}}|\vec z_i|_L^{m_i}$, where, if $|x_0|_\beta=\min_{n\mathbb{Z}}|x_0+n\beta|$ is the distance on the one-dimensional torus of size β and $|\vec x|_L=\min_{\vec n\in\mathbb{Z}^2}|\vec x+\vec nL|$ is the distance on the periodic lattice of size L, we denoted $|\mathbf{x}|=\mathfrak{e}_0|x_0|_\beta+|\vec x|_L$, with $\mathfrak{e}_0$ the energy scale defined in (2.10).

Proof of Lemma 5.2. The proof is long and, therefore, we split it into three main steps: we first define the multiscale decomposition of the Grassmann integral, which we intend to perform in an iterative fashion; next, we explain in detail how to integrate the first scale; finally, we explain the iterative procedure, whose output is conveniently organized in the form of a tree expansion.

Multiscale decomposition. In order to prove the analyticity of the regularized free energy and correlations, we perform the Grassmann integration in a multiscale fashion, by rewriting the propagator $\bar g^{\beta,L,M}$ as a sum of smooth "single scale" propagators $g^{(h)}$, $h=0,1,\dots,M$, each decaying faster than any power on a specific time scale $\sim 2^h$:

$$\bar g^{\beta,L,M}(\mathbf{x})=\sum_{h=0}^M g^{(h)}(\mathbf{x}),\qquad g^{(h)}(\mathbf{x})=\frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{f_h(k_0)}{-ik_0+\hat H^{(0)}(\vec k)-\mu}. \tag{5.24}$$

Here $f_h(k_0)=\chi_0(2^{-h}k_0/\delta_\mu)-\chi_0(2^{-h+1}k_0/\delta_\mu)$ for $h\ge1$ and $f_0(k_0)=\chi_0(k_0/\delta_\mu)$. For later use, note that the single scale propagator $g^{(h)}(\mathbf{x})$ satisfies the bound

$$|g^{(h)}(\mathbf{x})|\le\frac{C_K}{1+(2^h\delta_\mu|x_0|_\beta+(\delta_\mu/\mathfrak{e}_0)|\vec x|_L)^K}\ ,\qquad \forall 0\le h\le M,\quad \forall K\ge0\,. \tag{5.25}$$

In particular,

$$\|g^{(h)}\|_{1,n}:=\int d\mathbf{x}\,\|g^{(h)}(\mathbf{x})\|\cdot|\mathbf{x}|^n\le C_n\delta_\mu^{-3-n}2^{-h}. \tag{5.26}$$

where $\int d\mathbf{x}\equiv\int_{\Lambda_{\beta,L}}d\mathbf{x}$ is a shorthand for $\int_0^\beta dx_0\sum_{\vec x\in\Lambda_L}$. If $n=0$, we shall denote $\|g^{(h)}\|_1=\|g^{(h)}\|_{1,0}$. Moreover, $g^{(h)}(\mathbf{x})$ admits a Gram decomposition, which will be useful in deriving combinatorially optimal bounds on the generic order of perturbation theory:

$$g^{(h)}(\mathbf{x}-\mathbf{y})=\big(A_{h,\mathbf{x}},B_{h,\mathbf{y}}\big)\equiv\int d\mathbf{z}\,A^*_{h,\mathbf{x}}(\mathbf{z})\cdot B_{h,\mathbf{y}}(\mathbf{z})\ , \tag{5.27}$$

with

$$A_{h,\mathbf{x}}(\mathbf{z})=\frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\frac{\sqrt{f_h(k_0)}}{k_0^2+(\hat H_0(\vec k)-\mu)^2}e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})}\,,$$

$$B_{h,\mathbf{x}}(\mathbf{z})=\frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\sqrt{f_h(k_0)}\,(ik_0+\hat H_0(\vec k)-\mu)e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})}\ ,$$

and

$$||A_{h,\mathbf{x}}||^2 := (A_{h,\mathbf{x}}, A_{h,\mathbf{x}}) \leq C(\delta_\mu 2^h)^{-3}\;, \qquad ||B_h||^2 \leq C(\delta_\mu 2^h)^3\;. \tag{5.28}$$

The decomposition (5.24) of the propagator allows us to compute the regularized Grassmann generating function,

$$\mathcal{W}_M(\phi, A) = \log \int P_{\leq M}(d\Psi) e^{-UV_{\beta,L}(\Psi)+(\phi,n)+(A,J)}\;, \tag{5.29}$$

in an iterative way, by first integrating the degrees of freedom corresponding to $g^{(M)}$, then those corresponding to $g^{(M-1)}$, and so on. Technically, we make use of the so-called addition formula for Grassmann Gaussian integrations: if g_1, g_2 are two propagators and $g := g_1 + g_2$, then the Gaussian integration $P_g(d\psi)$ with propagator g can be rewritten as $P_g(d\psi) = P_{g_1}(d\psi_1)P_{g_2}(d\psi_2)$, in the sense that for every polynomial f

$$\int P_g(d\psi) f(\psi) = \int P_{g_1}(d\psi_1) \int P_{g_2}(d\psi_2) f(\psi_1 + \psi_2)\;. \tag{5.30}$$

In our context, we rewrite $P_{\leq M}(d\Psi) = \prod_{h=0}^M P_h(d\Psi^{(h)})$, where $P_h(d\Psi^{(h)})$ is the Gaussian integration with propagator $g^{(h)}$, so that

$$e^{\mathcal{W}_M(\phi,A)} = \int P_0(d\Psi^{(0)}) \cdots P_h(\Psi^{(h)}) e^{-\mathcal{V}^{(h)}(\Psi^{(\leq h)},\phi,A)}, \tag{5.31}$$

where $\Psi^{(\leq h)} := \sum_{j=0}^h \Psi^{(j)}$, so that

$$\mathcal{V}^{(h)}(\Psi,\phi,A) = -\log \int P_{h+1}(d\Psi^{(h+1)}) \cdots P_M(\Psi^{(M)}) e^{-UV_{\beta,L}(\Psi+\Psi^{(h+1)}+\cdots+\Psi^{(M)})+(\phi,n)+(A,J)}. \tag{5.32}$$

and $\mathcal{V}^{(M)}(\Psi,\phi,A) = UV_{\beta,L}(\Psi) - (\phi,n) - (A,J)$.

The first integration step. In order to compute the sequence $\mathcal{V}^{(h)}$ iteratively, let us start by explaining in detail the first step:

$$\mathcal{V}^{(M-1)}(\Psi,\phi,A) = -\log \int P_M(d\Psi^{(M)}) e^{-\mathcal{V}^{(M)}(\Psi+\Psi^{(M)},\phi,A)}\;. \tag{5.33}$$

The logarithm in the right side can be expressed as a series of truncated expectations:

$$\log \int P_M(d\Psi^{(M)}) e^{-\mathcal{V}^{(M)}(\Psi+\Psi^{(M)},\phi,A)} = \tag{5.34}$$

$$= \sum_{s\geq 1} \frac{(-1)^s}{s!} \mathcal{E}^T_M\big(\underbrace{\mathcal{V}^{(M)}(\Psi+\Psi^{(M)},\phi,A);\cdots;\mathcal{V}^{(M)}(\Psi+\Psi^{(M)},\phi,A)}_{s\text{ times}}\big)\;, \tag{5.35}$$

where

$$\mathcal{E}^T_M(X_1(\Psi^{(M)});\cdots;X_s(\Psi^{(M)})) = \frac{\partial^s}{\partial\lambda_1\cdots\partial\lambda_s} \log \int P_M(d\Psi^{(M)}) e^{\lambda_1 X_1(\Psi^{(M)})+\cdots+\lambda_s X_s(\Psi^{(M)})}\Big|_{\lambda_i=0}\;, \tag{5.36}$$

and the X_i's are all even elements of the Grassmann algebra generated by the field $\Psi^{(M)}$ we are integrating over and by the "external" Grassmann field Ψ. The functional $\mathcal{E}_M^T$ is multilinear in its arguments, the action on a collection of monomials being defined by the *truncated* Wick rule with propagator $g^{(M)}$, which is similar to the usual fermionic Wick rule, modulo the extra condition that, if the number s of monomials involved is ≥ 2, then the pairings one has to sum over are only those for which the collection of monomials $X_1, \ldots,$ X_s is *connected* (this means that for all $\mathcal{I} \subsetneq \{1, \ldots, s\}$, there exists at least one contracted pair involving one variable in the group $\{X_i\}_{i\in\mathcal{I}}$ and one in $\{X_i\}_{i\in\mathcal{I}^c}$).

A convenient representation of the truncated expectation, due to Battle, Brydges and Federbush [**? ? ?**], is the following (for a proof, see, e.g., [**? ?**]). For a given (ordered) set of indices $P = (f_1, \ldots, f_p)$, with $f_i = (\mathbf{x}_i, \sigma_i, \varepsilon_i)$, let

$$\Psi_P := \Psi^{\varepsilon(f_1)}_{\mathbf{x}(f_1),\sigma(f_1)} \cdots \Psi^{\varepsilon(f_p)}_{\mathbf{x}(f_p),\sigma(f_p)} \, , \tag{5.37}$$

where $\mathbf{x}(f_i) = \mathbf{x}_i$, etc. It is customary to represent each variable $\Psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f)}$ as an oriented half-line, emerging from the point $\mathbf{x}(f)$ and carrying an arrow, pointing in the direction entering or exiting the point, depending on whether $\varepsilon(f)$ is equal to $-$ or $+$, respectively; moreover, the half-line carries the labels $\sigma(f) \in I$. Given n sets of indices $P_1, \ldots, P_n$, we can enclose the points $\mathbf{x}(f)$ belonging to the set P_j in a box: in this way, assuming that all the points $\mathbf{x}(f)$, $f \in \cup_i P_i$, are distinct, we obtain n disjoint boxes. Given these definitions, if $\sum_{i=1}^s |P_i|$ is even we can write

$$\mathcal{E}_M^T(\Psi_{P_1}; \ldots; \Psi_{P_s}) = \sum_{T\in\mathbf{T}_M} \alpha_T \prod_{\ell\in T} g_\ell^{(M)} \int dP_T(\mathbf{t}) \det G_T^{(M)}(\mathbf{t}) \, , \tag{5.38}$$

where:

- any element T of the set $\mathbf{T}_M = \mathbf{T}_M(P_1, \ldots, P_s)$ is a set of lines forming an *anchored tree* between the boxes $P_1, \ldots, P_s$, i.e., T is a set of lines that becomes a tree if one identifies all the points in the same box; each line ℓ corresponds to a pair of half-lines indexed by two distinct variables $f, f' \in \cup_i P_i$ such that $\varepsilon(f) = -\varepsilon(f')$ (i.e., the directions of the two half-lines have to be compatible); if ℓ is obtained by contracting f and f', we shall write $\ell = (f, f')$, with the convention that $\varepsilon(f') = -\varepsilon(f) = +$.

- α_T is a sign (irrelevant for the subsequent bounds), which depends on the choice of the anchored tree T;

- if $\ell = (f, f')$, then $g_\ell^{(M)}$ stands for $g^{(M)}_{\sigma(f),\sigma(f')}(\mathbf{x}(f) - \mathbf{x}(f'))$;

- if $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \leq i, i' \leq n\}$, then $dP_T(\mathbf{t})$ is a probability measure (depending on the anchored tree T) with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm;

- if $2N = \sum_{i=1}^s |P_i|$, then $G_T^{(M)}(\mathbf{t})$ is a $(N-s+1)\times(N-s+1)$ matrix (depending both on the sets P_i and on the anchored tree T), whose elements are given by $[G_T^{(M)}(\mathbf{t})]_{f,f'} = t_{i(f),i(f')} g^{(M)}_{(f,f')}$, where $f, f' \in \cup_i P_i \setminus \cup_{\ell\in T}\{f_\ell^-, f_\ell^+\}$ (with $\ell = (f_\ell^-, f_\ell^+)$), and $i(f) \in \{1,\dots,s\}$ is the index such that $f \in P_{i(f)}$.

If $s = 1$ the sum over T is empty, but we can still use the Eq.(5.38) by interpreting the r.h.s. as equal to 1 if P_1 is empty and equal to $\det G^T(\mathbf{1})$ otherwise.

In order to use (5.38) in (5.33)-(5.34), we first rewrite $\mathcal{V}^{(M)}$ as

$$\mathcal{V}^{(M)}(\Psi,\phi,A) = E_M(\phi) + \sum_{\rho=1}^4 \sum_{\sigma,\sigma'\in I} \int d\mathbf{x} d\mathbf{y} K^\rho_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) \big[\phi^\sigma_{\mathbf{x}}\big]^{\delta_{\rho,1}} \big[A^{\sigma\sigma'}_{\mathbf{x},\mathbf{y}}\big]^{\delta_{\rho,2}} \Psi_{P^\rho} \; , \tag{5.39}$$

where $E_M(\phi) = \frac{\beta L^2}{4} U \sum_\sigma \nu_\sigma - \frac12 \sum_\sigma \int d\mathbf{x}\, \phi^\sigma_{\mathbf{x}}$, with $\nu_\sigma = \sum_{\vec x\in\Lambda_L}\sum_{\sigma'\in I} v_{\sigma\sigma'}(\vec x)$. Moreover, $A^{\sigma\sigma'}_{\mathbf{x},\mathbf{y}} = A^{\sigma\sigma'}_{\mathbf{x}}(\vec y - \vec x) - A^{\sigma'\sigma}_{\mathbf{y}}(\vec x - \vec y)$,

$$K^1_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) = -\delta_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{y}) \; , \qquad K^2_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) = -i\delta(x_0-y_0) H^{(0)}_{\sigma\sigma'}(\vec x - \vec y) \; , \tag{5.40}$$

$$K^3_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) = U\nu_\sigma\delta_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{y}) \; , \qquad K^4_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) = U\delta(x_0-y_0) v_{\sigma\sigma'}(\vec x - \vec y) \; , \tag{5.41}$$

and

$$P^1 = P^2 = P^3 = \big((\mathbf{x},\sigma,+),(\mathbf{y},\sigma',-)\big) \; , \tag{5.42}$$

$$P^4 = \big((\mathbf{x},\sigma,+),(\mathbf{x},\sigma,-),(\mathbf{y},\sigma',+),(\mathbf{y},\sigma',-)\big) \; . \tag{5.43}$$

Plugging (5.39) into (5.33)-(5.34), we obtain

$$\mathcal{V}^{(M-1)}(\Psi,\phi,A) = E_M(\phi) - \sum_{s\ge1}\frac{(-1)^s}{s!} \sum_{\substack{\rho_1,\dots,\rho_s\\ \sigma_1,\sigma_1',\dots,\sigma_s,\sigma_s'}} \int d\mathbf{x}_1 d\mathbf{y}_1 \cdots d\mathbf{x}_s\, d\mathbf{y}_s \times$$

$$\times \Big[\prod_{i:\rho_i=1} \phi^{\sigma_i}_{\mathbf{x}_i}\Big] \Big[\prod_{i:\rho_i=2} A^{\sigma_i\sigma_i'}_{\mathbf{x}_i,\mathbf{y}_i}\Big] \Big[\prod_{i=1}^s K^{\rho_i}_{\sigma_i\sigma_i'}(\mathbf{x}_i,\mathbf{y}_i)\Big] \mathcal{E}^T_M\big((\Psi+\Psi^{(M)})_{P_1^{\rho_1}};\cdots;(\Psi+\Psi^{(M)})_{P_s^{\rho_s}}\big) \; .$$

The truncated expectation in the right side can be further rewritten as

$$\mathcal{E}^T_M\big((\Psi+\Psi^{(M)})_{P_1^{\rho_1}};\cdots;(\Psi+\Psi^{(M)})_{P_s^{\rho_s}}\big) = \sum_{P\subseteq\cup_i P_i^{\rho_i}} \alpha_P \Psi_P \mathcal{E}^T_M\big(\Psi^{(M)}_{P_1^{\rho_1}\setminus Q_1};\cdots;\Psi^{(M)}_{P_s^{\rho_s}\setminus Q_s}\big) \; , \tag{5.44}$$

where α_P is a sign, and $Q_i = P \cap P_i^{\rho_i}$, so that, applying (5.38), we find

$$\mathcal{V}^{(M-1)}(\Psi,\phi,A) = E_M(\phi) - \sum_{s\ge1}\frac{(-1)^s}{s!} \sum_{\underline\rho,\underline\sigma} \int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}} \Big[\prod_{i:\rho_i=1}\phi^{\sigma_i}_{\mathbf{x}_i}\Big]\Big[\prod_{i:\rho_i=2} A^{\sigma_i\sigma_i'}_{\mathbf{x}_i,\mathbf{y}_i}\Big] \times$$

$$\times \Big[\prod_{i=1}^s K^{\rho_i}_{\sigma_i\sigma_i'}(\mathbf{x}_i,\mathbf{y}_i)\Big] \sum_{P\subseteq\cup_i P_i^{\rho_i}} \Psi_P \sum_{T\in\mathbf{T}_M} \alpha_{P,T} \prod_{\ell\in T} g^{(M)}_\ell \int dP_T(\mathbf{t}) \det G^{(M)}_T(\mathbf{t}) \; , \tag{5.45}$$

where $\underline{\rho}$, $\underline{\sigma}$, $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are shorthands for $(\rho_1, \ldots, \rho_s)$, $(\sigma_1, \sigma_1', \ldots, \sigma_s, \sigma_s')$, $(\mathbf{x}_1, \ldots, \mathbf{x}_s)$ and $(\mathbf{y}_1, \ldots, \mathbf{y}_s)$, respectively, and $\alpha_{P,T} = \alpha_P \alpha_T$. Eq.(5.45) can be equivalently rewritten as

$$\begin{aligned}
&\mathcal{V}^{(M-1)}(\Psi, \phi, A) = E_M(\phi) + \qquad (5.46)\\
&+\sum_{n\ge 0}\sum_{s_1,s_2\ge 0}\sum_{\underline{\sigma},\underline{\varepsilon}}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}d\underline{\mathbf{z}}W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})\Big[\prod_{i=1}^{s_1}\phi^{\sigma_i}_{\mathbf{x}_i}\Big]\,\Big[\prod_{i=s_1+1}^{s_1+s_2}A^{\sigma_i\sigma_i'}_{\mathbf{x}_i,\mathbf{y}_i}\Big]\Big[\prod_{i=1}^{2n}\Psi^{\varepsilon_i}_{\mathbf{z}_i,\sigma_i''}\Big]\ ,
\end{aligned}$$

with

$$\begin{aligned}
&W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}}) = \sum^{*}_{\substack{s_3\ge 0\\ s_4\ge n-1}}\frac{(-1)^{s-1}}{s_1!s_2!s_3!s_4!}\sum_{\substack{\sigma_i,\sigma_i':\\ i>s_1+s_2}}\int\Big[\prod_{i>s_1+s_2}d\mathbf{x}_i d\mathbf{y}_i\Big]\times \qquad (5.47)\\
&\times\Big[\prod_{i=1}^{s}K^{\bar\rho_i}_{\sigma_i\sigma_i'}(\mathbf{x}_i,\mathbf{y}_i)\Big]\sum_{\substack{P\subseteq\cup_i P_i^{\bar\rho_i}:\\ |P|=2n}}\delta(P-P_{ext})\sum_{T\in\mathbf{T}_M}\alpha_{P,T}\prod_{\ell\in T}g^{(M)}_\ell\int dP_T(\mathbf{t})\det G^{(M)}_T(\mathbf{t})\ ,
\end{aligned}$$

where $s = s_1 + s_2 + s_3 + s_4$, the $*$ on the sum indicates the constraint that $s \ge 1$, and $\bar\rho_i$ is equal to 1 if $i \le s_1$, is equal to 2 if $0 < i - s_1 \le s_2$, is equal to 3 if $0 < i - s_1 - s_2 \le s_3$, and is equal to 4 otherwise. Moreover, $P_{ext} = ((\mathbf{z}_1, \sigma_1'', \varepsilon_1), \ldots, (\mathbf{z}_{2n}, \sigma_{2n}'', \varepsilon_{2n}))$, and $\delta\big(P - P_{ext}\big)$ is a shorthand for the product of delta functions $\prod_{f_i\in P}\delta(\mathbf{x}(f_i) - \mathbf{z}_i)\delta_{\sigma(f_i),\sigma_i''}\delta_{\varepsilon(f_i),\varepsilon_i}$, where the labeling $P = (f_1, \ldots, f_{2n})$ is understood. Note that, in the case that $n = s_1 = s_2 = 0$, in the right side of (5.46) there are neither sums over $\underline{\sigma}, \underline{\varepsilon}$ nor integrals over $\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}$, and $W^{(M-1)}_{0,0,0}$ is a constant, given by (5.46), with the understanding that the meaningless factors or sums or integrals should be replaced by one.

We are finally in the position of proving the analyticity of the integral kernels of $\mathcal{V}^{(M-1)}$. By using (5.46) we obtain

$$\begin{aligned}
&\frac{1}{\beta L^2}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\big|W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})\big| \le \qquad (5.48)\\
&\qquad\le\sum^{*}_{\substack{s_3\ge 0\\ s_4\ge n-1}}\frac{|I|^{2s_3+2s_4}}{s_1!s_2!s_3!s_4!}\Big[\prod_{j=1}^{4}||K^j||_1^{s_j}\Big]\binom{2s+2s_4}{2n}(C^s s!)||g^{(M)}||_1\cdot||\det G^{(M)}_T||_\infty\ ,
\end{aligned}$$

where: $|I|^{2s_3+2s_4}$ bounds the number of terms in the sum over σ_i, σ_i'; $||K^j||_1 = \sup_{\sigma,\sigma'}\int d\mathbf{x}|K^j_{\sigma\sigma'}(\mathbf{x}, \mathbf{0})|$; $\binom{2s+2s_4}{2n}$ bounds the number of terms in the sum over P; $(C^s s!)$ bounds the number of terms in the sum over T. Recalling (5.26) for $n = 0$ and the definitions (5.40)-(5.41), from which $||K^j||_1 \le C|U|^{\delta_{j,3}+\delta_{j,4}}$, we find that (5.48) implies

$$\frac{1}{\beta L^2}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\big|W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})\big| \le \sum^{*}_{\substack{s_3\ge 0\\ s_4\ge n-1}}C^s|U|^{s_3+s_4}(\delta_\mu^{-3}2^{-M})^{s-1}||\det G^{(M)}_T||_\infty\ . \qquad (5.49)$$

In order to bound $\det G^{(M)}_T$, we use the *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} = (A_i, B_j)$, where A_i, B_j are vectors in a

Hilbert space with scalar product $(\cdot,\cdot)$, then

$$|\det M| \le \prod_i ||A_i||\cdot||B_i||\,. \tag{5.50}$$

where $||\cdot||$ is the norm induced by the scalar product. In our case, $[G_T^{(M)}(\mathbf{t})]_{f,f'} = \mathbf{u}_{i(f)}\cdot\mathbf{u}_{i(f')}(A_{M,\mathbf{x}(f)},B_{M,\mathbf{x}(f')})$, so that, using (5.28) and recalling that $G_T^{(M)}$ is a $(s_4-n+1)\times(s_4-n+1)$ matrix,

$$||\det G_T^{(M)}||_\infty \le C^{s_4-n+1}. \tag{5.51}$$

Plugging this last ingredient into (5.49), we finally obtain

$$\begin{aligned}\frac{1}{\beta L^2}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\big|W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})\big| \;&\le\; \sum^{*}_{\substack{s_3\ge0\\ s_4\ge n-1}} C^s|U|^{s_3+s_4}(\delta_\mu^{-3}2^{-M})^{s-1}\\ &\le\; C^n|U|^{[n-1]_+}(\delta_\mu^{-3}2^{-M})^{[s_1+s_2+n-2]_+}\,,\end{aligned}\tag{5.52}$$

where $[\cdot]_+=\max\{\cdot,0\}$ denotes the positive part. Eq.(5.52) proves the analyticity of the kernels of $\mathcal{V}^{(M)}$ for U small enough, uniformly in M (but not in δ_μ, in general). Moreover, the kernels $W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})$ decay faster than any power, on scale δ_μ^{-1}, in the relative distances between the coordinates $\mathbf{x}_i,\mathbf{y}_i,\mathbf{z}_i$. In order to prove this, we multiply the argument of the integral in the left side of (5.48) by a product of factors of the form $|\mathbf{x}_i-\mathbf{x}_j|^{m_{i,j}}$, or $|\mathbf{x}_i-\mathbf{y}_j|^{m'_{i,j}}$, etc. We denote by $m=\sum_{i,j}(m_{i,j}+m'_{i,j}+\cdots)$ the sum of these exponents. Again, we use the representation (5.46), and we decompose each factor "along the anchored tree T", that is we bound it by using

$$|\mathbf{x}_i-\mathbf{x}_j| \le \sum_{\ell\in T}|\mathbf{x}(f_\ell^-)-\mathbf{x}(f_\ell^+)| + \sum_{i=1}^{s} d_i, \tag{5.53}$$

where $d_i=\max_{f,f'\in P_{T,i}}|\mathbf{x}(f)-\mathbf{x}(f')|$ and $P_{T,i}=\cup_{\ell\in T}\{f_\ell^-,f_\ell^+\}\cap P_i^{\bar\rho_i}$. In this way, the right side of (5.48) is replaced by a sum of terms, each of which is obtained by replacing some of the factors $||K^j||_1$ and $||g^{(M)}||_1$ by $||K^j||_{1,n_i}=\sup_{\sigma,\sigma'}\int d\mathbf{x}|K^j_{\sigma\sigma'}(\mathbf{x},\mathbf{0})|\,|\mathbf{x}|^{n_i}\le C_{n_i}$ and by $||g^{(M)}||_{1,n'_i}$, respectively. Recall that, by (5.26), the dimensional estimate of $||g^{(M)}||_{1,n'_i}$ differs from that of $||g^{(M)}||_1$ just by a factor $\delta_\mu^{-n'_i}$. Moreover, the total sum of the exponents n_i,n'_i, etc., equals the exponent m introduced earlier. Therefore, the product of the extra factors $\delta_\mu^{-n'_i}$ is smaller than δ_μ^{-m}. All in all, the dimensional estimate on the kernels $W^{(M-1)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})$, multiplied by the extra factors $|\mathbf{x}_i-\mathbf{x}_i|^{m_{i,j}}$, etc, is the same as (5.52), up to an extra factor $C_m\delta_\mu^{-m}$, for all $m\ge0$.

The iterative integration procedure and the tree expansion.

We are now in the position of iterating the procedure used above for computing the integral over the scale M. By using (5.32) and the definition of truncated expectation $\mathcal{E}_h^T$

(which is the same as (5.36), with M replaced by h), we obtain

$$\mathcal{V}^{(h-1)}(\Psi,\phi,A) = -\log\int P_h(d\Psi^{(h)})e^{-\mathcal{V}^{(h)}(\Psi+\Psi^{(h)},\phi,A)} = \tag{5.54}$$

$$= \sum_{s\geq 1}\frac{(-1)^s}{s!}\mathcal{E}_h^T\big(\underbrace{\mathcal{V}^{(h)}(\Psi+\Psi^{(h)},\phi,A);\cdots;\mathcal{V}^{(h)}(\Psi+\Psi^{(h)},\phi,A)}_{s\ \text{times}}\big)\,.$$

Eq.(5.54) can be graphically represented as in Fig.1. The tree in the left side, consisting

FIG. 1: The graphical representation of $\mathcal{V}^{(h-1)}$.

of a single horizontal branch, connecting the left node (called the *root* and associated with the *scale label* $h-1$) with a big black dot on scale h, represents $\mathcal{V}^{(h-1)}$. In the right side, the term with s final points represents the corresponding term in the right side of (5.54): a scale label $h-1$ is attached to the leftmost node (the root); a scale label h is attached to the central node (corresponding to the action of $\mathcal{E}_h^T$); a scale label $h+1$ is attached to the s rightmost nodes with the big black dots (representing $\mathcal{V}^{(h)}$).

Iterating the graphical equation in Fig.1 up to scale M, and representing the endpoints on scale $M+1$ as simple dots (rather than big black dots), we end up with a graphical representation of $\mathcal{V}^{(h)}$ in terms of *Gallavotti-Nicolò* trees, see Fig.2, defined in terms of the following features.

1. Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $N\geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. N will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with N end-points is bounded by 4^N (see, e.g., [**?** , Appendix A.1.2] for a proof of this fact). We shall also consider the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2. We associate a label $0\leq h\leq M-1$ with the root and we denote by $\widetilde{\mathcal{T}}_{M;h,N}$ the corresponding set of labeled trees with N endpoints. Moreover, we introduce a family

FIG. 2: A tree $\tau \in \widetilde{\mathcal{T}}_{M;h,N}$ with $N = 9$: the root is on scale h and the endpoints are on scale $M+1$.

of vertical lines, labeled by an integer taking values in $[h, M+1]$, and we represent any tree $\tau \in \widetilde{\mathcal{T}}_{M;h,N}$ so that, if v is an endpoint, it is contained in the vertical line with index $h_v = M+1$, while if it is a non trivial vertex, it is contained in a vertical line with index $h < h_v \le M$, to be called the *scale* of v; the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

3. There is only one vertex immediately following the root, called v_0 and with scale label equal to $h+1$.

4. Given a vertex v of $\tau \in \widetilde{\mathcal{T}}_{M;h,N}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \ge v$. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it is called a *trivial subtree*.

5. With each endpoint v we associate one of the terms contributing to $\mathcal{V}^{(M)}(\Psi, \phi, A)$, see (5.39). In order to distinguish between the various terms in the right side of (5.39), we introduce a *type label* $\rho_v \in \{0, 1, 2, 3, 4\}$. If $\rho_v = 0$, then we associate the endpoint with a contribution $E_M(\phi)$, while, if $1 \le \rho_v \le 3$, then we associate the endpoint with

a contribution $K^{\rho_v}_{\sigma_v\sigma'_v}(\mathbf{x}_v,\mathbf{y}_v)\big[\phi^{\sigma_v}_{\mathbf{x}_v}\big]^{\delta_{\rho_v,1}}\big[A^{\sigma_v\sigma'_v}_{\mathbf{x}_v,\mathbf{y}_v}\big]^{\delta_{\rho_v,2}}\Psi_{I_v}$.

The field labels attached to the endpoints v of τ are denoted by I_v. If $\rho_v = 0$, then $I_v = \emptyset$; if $\rho_v = 1,2,3$, then $I_v = \big((\mathbf{x}_v,\sigma_v,+),(\mathbf{y}_v,\sigma'_v,-)\big)$; if $\rho_v = 4$, then $I_v = \big((\mathbf{x}_v,\sigma_v,+),(\mathbf{x}_v,\sigma_v,-),(\mathbf{y}_v,\sigma'_v,+),(\mathbf{y}_v,\sigma'_v,-)\big)$. Moreover, given any vertex $v\in\tau$, we denote by I_v the set of field labels associated with the endpoints following the vertex v; given $f\in I_v$, $\mathbf{x}(f)$, $\sigma(f)$ and $\varepsilon(f)$ denote the space-time point, the σ index and the ε index of the Grassmann variable with label f. In the following, the "sum" over the field labels associated with the endpoints should be understood as $\sum_{\underline{\sigma}_{v_0}}\int d\underline{\mathbf{x}}_{v_0}$, where v_0 is the leftmost vertex of τ, $\underline{\sigma}_v = \cup_{f\in I_v}\{\sigma(f)\}$ and $\underline{\mathbf{x}}_v = \cup_{f\in I_v}\{\mathbf{x}(f)\}$.

In terms of trees, the effective potential $\mathcal{V}^{(h)}$, $-1\le h\le M$ (with $\mathcal{V}^{(-1)}$ identified with $\mathcal{W}_M$), can be written as

$$\mathcal{V}^{(h)}(\Psi^{(\le h)}) = \sum_{N=1}^{\infty}\sum_{\tau\in\widetilde{\mathcal{T}}_{M;h,N}}\widetilde{\mathcal{V}}^{(h)}(\tau,\Psi^{(\le h)})\ , \tag{5.55}$$

where, if v_0 is the first vertex of τ and $\tau_1,\dots,\tau_s$ ($s=s_{v_0}$) are the subtrees of τ with root v_0, $\widetilde{\mathcal{V}}^{(h)}(\tau,\Psi^{(\le h)})$ is defined inductively as:

$$\widetilde{\mathcal{V}}^{(h)}(\tau,\Psi^{(\le h)}) = \frac{(-1)^{s-1}}{s!}\mathcal{E}^T_{h+1}\big[\widetilde{\mathcal{V}}^{(h+1)}(\tau_1,\Psi^{(\le h+1)});\dots;\widetilde{\mathcal{V}}^{(h+1)}(\tau_s,\Psi^{(\le h+1)})\big]\ . \tag{5.56}$$

where, if τ is a trivial subtree with root on scale M, then $\widetilde{\mathcal{V}}^{(M)}(\tau,\Psi^{(\le M)}) = \mathcal{V}^{(M)}(\Psi^{(\le M)})$ (for lightness of notation, we are dropping the arguments (ϕ,A), which are implicitly understood here and in the following).

For what follows, it is important to specify the action of the truncated expectations on the branches connecting any endpoint v to the closest *non-trivial* vertex v' preceding it. In fact, if τ has only one end-point, it is convenient to rewrite $\widetilde{\mathcal{V}}^{(h)}(\tau,\Psi^{(\le h)}) = \mathcal{E}^T_{h+1}\mathcal{E}^T_{h+2}\cdots\mathcal{E}^T_M(\mathcal{V}(\Psi^{(\le M)}))\equiv\widetilde{\mathcal{V}}^{(h)}(\Psi^{(\le h)})$ as:

$$\widetilde{\mathcal{V}}^{(h)}(\Psi^{(\le h)}) = \mathcal{V}^{(M)}(\Psi^{(\le h)}) + \mathcal{E}^T_{h+1}\cdots\mathcal{E}^T_M\big(\mathcal{V}^{(M)}(\Psi^{(\le M)}) - \mathcal{V}^{(M)}(\Psi^{(\le h)})\big)\ . \tag{5.57}$$

The second term in the right side can be evaluated explicitly and gives:

$$\mathcal{E}^T_{h+1}\cdots\mathcal{E}^T_M\big(\mathcal{V}^{(M)}(\Psi^{(\le M)}) - \mathcal{V}^{(M)}(\Psi^{(\le h)})\big) = e_{[h+1,M]} + \sum_{\sigma,\sigma'}\int d\mathbf{x}d\mathbf{y}\, k^{[h+1,M]}_{\sigma\sigma'}(\mathbf{x},\mathbf{y})\Psi^+_{\mathbf{x},\sigma}\Psi^-_{\mathbf{y},\sigma'}\ , \tag{5.58}$$

where, denoting $g^{[h+1,M]}(\mathbf{x}) = \sum_{h'=h+1}^{M} g^{(h')}(\mathbf{x})$,

$$e_{[h+1,M]} = e_{[h+1,M]}(\phi,A) = -\sum_{\sigma,\sigma'}\int d\mathbf{x}d\mathbf{y}\,\Big\{\big[K^1_{\sigma\sigma'}(\mathbf{x},\mathbf{y})\phi^\sigma_{\mathbf{x}} + K^2_{\sigma\sigma'}(\mathbf{x},\mathbf{y})A^{\sigma\sigma'}_{\mathbf{x},\mathbf{y}} + K^3_{\sigma\sigma'}(\mathbf{x},\mathbf{y})\big]\cdot$$
$$\cdot g^{[h+1,M]}_{\sigma'\sigma}(\mathbf{0}) + K^4_{\sigma\sigma'}(\mathbf{x},\mathbf{y})\big[g^{[h+1,M]}_{\sigma\sigma'}(\mathbf{x}-\mathbf{y})g^{[h+1,M]}_{\sigma'\sigma}(\mathbf{y}-\mathbf{x}) - g^{[h+1,M]}_{\sigma\sigma}(\mathbf{0})g^{[h+1,M]}_{\sigma'\sigma'}(\mathbf{0})\big]\Big\}\ ,$$

and

$$k_{\sigma\sigma'}^{[h+1,M]}(\mathbf{x},\mathbf{y}) = 2U g_{\sigma\sigma'}^{[h+1,M]}((0,\vec{x}-\vec{y}))\delta(x_0-y_0)\big[v_{\sigma\sigma'}(\vec{x}-\vec{y}) - \nu_\sigma \delta_{\sigma\sigma'}\delta(\vec{x}-\vec{y})\big]\,. \tag{5.59}$$

Therefore, it is natural to shrink all the branches of $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ consisting of a subtree $\tau' \subseteq \tau$, having root r' on scale $h' \in [h,M]$ and only one endpoint on scale $M+1$, into a trivial subtree, rooted in r' and associated with a factor $\widetilde{\mathcal{V}}^{(h')}(\Psi^{(\le h')})$, which has the same structure as the right side of (5.39), with $E_M(\phi)$ replaced by $E_{h'}(\phi,A) = E_M(\phi) + e_{[h'+1,M]}(\phi,A)$, $K^3_{\sigma\sigma'}(\mathbf{x},\mathbf{y})$ replaced by $K^3_{h'+1;\sigma\sigma'}(\mathbf{x},\mathbf{y}) := K^3_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) + k_{\sigma\sigma'}^{[h'+1,M]}(\mathbf{x},\mathbf{y})$, and Ψ replaced by $\Psi^{(\le h')}$. Note that $k_{\sigma\sigma'}^{[h+1,M]}(\mathbf{x},\mathbf{y})$ is bounded proportionally to U, and decays faster than any power, *uniformly in* M, in the sense that

$$\|k^{[h+1,M]}\|_{1,n} = \sup_{\sigma,\sigma'} \int d\mathbf{x} |k_{\sigma\sigma'}^{[h+1,M]}(\mathbf{x},\mathbf{0})| \cdot |\mathbf{x}|^n \le C_n 2^{-h}|U|, \qquad \forall n \ge 0\,. \tag{5.60}$$

In particular, the $(1,n)$-norm of $K^3_{h'}$ is bounded uniformly in h' and M, proportionally to $|U|$. By shrinking all the linear subtrees in the way explained above, we end up with an alternative representation of the effective potentials, which is based on a slightly modified tree expansion. The set of modified trees with N endpoints contributing to $\mathcal{V}^{(h)}$ will be denoted by $\mathcal{T}_{M;h,N}$; every $\tau \in \mathcal{T}_{M;h,N}$ is characterized in the same way as the elements of $\widetilde{\mathcal{T}}_{M;h,N}$, but for two features: (i) the endpoints of $\tau \in \mathcal{T}_{M;h,N}$ are not necessarily on scale $M+1$; (ii) every endpoint v of τ is attached to a non-trivial vertex on scale $h_v - 1$ and is associated with the factor $\widetilde{\mathcal{V}}^{(h_v-1)}(\Psi^{(\le h_v-1)})$. See Fig.3. In terms of these modified trees,

FIG. 3: A tree $\tau \in \mathcal{T}_{M;h,N}$ with $N=9$: the root is on scale h and the endpoints are on scales $\le M+1$.

(5.56) is changed into

$$\mathcal{V}^{(h)}(\Psi^{(\le h)}) = \sum_{N=1}^{\infty} \sum_{\tau\in\mathcal{T}_{M;h,N}} \mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}) \;, \tag{5.61}$$

where

$$\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}) = \frac{(-1)^{s-1}}{s!} \mathcal{E}^T_{h+1}\big[\mathcal{V}^{(h+1)}(\tau_1, \Psi^{(\le h+1)}); \ldots ; \mathcal{V}^{(h+1)}(\tau_s, \psi^{(\le h+1)})\big] \tag{5.62}$$

and, if τ is a trivial subtree with root on scale $k \in [h, M]$, then $\mathcal{V}^{(k)}(\tau, \Psi^{(\le k)}) = \widetilde{\mathcal{V}}(\Psi^{(\le k)})$.

Using its inductive definition Eq.(5.62), the right hand side of Eq.(5.61) can be further expanded (it is a sum of several contributions, differing for the choices of the field labels contracted under the action of the truncated expectations $\mathcal{E}^T_{h_v}$ associated with the vertices v that are not endpoints), and in order to describe the resulting expansion we need some more definitions (allowing us to distinguish the fields that are contracted or not "inside the vertex v").

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v \ge 1$ vertices immediately following it (such that, in particular, $h_{v_i} = h_v + 1$), then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $s_v > 1$. Given $\tau \in \mathcal{T}_{M;h,N}$ and the set of field labels I_v associated with the endpoints v of τ, there are many possible choices of the subsets P_v associated with the vertices that are not endpoints, which are compatible with all the constraints. We shall denote by $\mathcal{P}_\tau$ the family of all these choices and by $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)})$ as:

$$\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}) = \sum_{\underline{\sigma}_{v_0}} \int d\underline{\mathbf{x}}_{v_0} \sum_{\mathbf{P}\in\mathcal{P}_\tau} K^{(h+1)}_{\tau,\mathbf{P}} \Psi^{(\le h)}_{P_{v_0}} \;, \tag{5.63}$$

where $K^{(h+1)}_{\tau,\mathbf{P}}$ is defined inductively by the following equation, which is valid for any $v \in \tau$ that is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P}} = \frac{1}{s_v!} \prod_{i=1}^{s_v} [K^{(h_{v_i})}_{\tau_i,\mathbf{P}_i}] \;\; \mathcal{E}^T_{h_v}[\Psi^{(h_v)}_{P_{v_1}\setminus Q_{v_1}}, \ldots, \Psi^{(h_v)}_{P_{v_{s_v}}\setminus Q_{v_{s_v}}}] \;. \tag{5.64}$$

Here $\tau_1, \ldots, \tau_{s_v}$ are the subtrees with root v, v_i are their leftmost vertices (such that, in particular, $h_{v_i} = h_v + 1$), and $\mathbf{P}_i = \{P_w, w \in \tau_i\}$. Moreover, if v_i is an endpoint, then $K^{(h_{v_i})}_{\tau_i,\mathbf{P}_i} = K_{v_i}$, with

$$K_v = \begin{cases} E_{h_v-1}(\phi, A) & \text{if} \quad \rho_v = 0 \;, \\ K^{\rho_v}_{h_v;\sigma_v\sigma'_v}(\mathbf{x}_v, \mathbf{y}_v) \big[\phi^{\sigma_v}_{\mathbf{x}_v}\big]^{\delta_{\rho_v,1}} \big[A^{\sigma_v\sigma'_v}_{\mathbf{x}_v,\mathbf{y}_v}\big]^{\delta_{\rho_v,2}} & \text{if} \quad \rho_v > 0 \;, \end{cases} \tag{5.65}$$

where $K^{\rho_v}_{h_v;\sigma_v\sigma'_v}$ should be identified with $K^{\rho_v}_{\sigma_v\sigma'_v}$ in the case that $\rho_v = 1, 2, 4$. Combining (5.61) with (5.63) and (5.64), and using the determinant representation of the truncated expectation, see (5.38), we finally get:

$$\mathcal{V}^{(h)}(\Psi^{(\le h)}) = E_h(\phi, A) + \sum_{N=1}^{\infty} \sum^{*}_{\tau\in\mathcal{T}_{M;h,N}} \sum_{\underline{\sigma}_{v_0}} \int d\underline{\mathbf{x}}_{v_0} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} W^{(h)}_{\tau,\mathbf{P},T}(\underline{\mathbf{x}}_{v_0}, \underline{\sigma}_{v_0}) \Psi^{(\le h)}_{P_{v_0}} , \qquad (5.66)$$

where the $*$ on the sum over τ indicates the constrain that there are no endpoints of type 0, and $\mathbf{T}$ is the set of the tree graphs on $\underline{\mathbf{x}}_{v_0}$ obtained by putting together an anchored tree graph T_v for each non-trivial vertex v and by adding a line (which is by definition the only element of T_v) for the couple of space-time points belonging to the set $\underline{\mathbf{x}}_v$ for each endpoint v. Moreover,

$$W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}, \underline{\sigma}_{v_0}) = \alpha_T \Big[\prod_{v \text{ e.p.}} K_v \Big] \prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{(h_v)}_{T_v}(\mathbf{t}_v) \prod_{\ell\in T_v} g^{(h_v)}_\ell , \qquad (5.67)$$

where α_T is a sign and $G^{(h_v)}_{T_v}(\mathbf{t}_v)$ is a matrix analogous to the one defined after (5.38), with $g^{(M)}$ replaced by $g^{(h_v)}$. Note that $W_{\tau,\mathbf{P},T}$ depends on M only through: (i) the choice of the scale labels, and (ii) the (weak) M-dependence of the endpoints v of type $\rho_v = 3$, whose value is $K^3_{h_v;\sigma_v\sigma'_v} = K^3_{\sigma_v\sigma'_v} + k^{[h_v,M]}_{\sigma_v\sigma'_v}$, with $k^{[h_v,M]}_{\sigma_v\sigma'_v}$ as in (5.59). From (5.66) and (5.67) we see that $\mathcal{V}^{(h)}(\Psi)$ can be rewritten as in (5.46), with $M-1$ replaced by h, and

$$W^{(h)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}) = \sum_{N\ge 1} \sum^{**}_{\tau\in\mathcal{T}_{M;h,N}} \sum_{\underline{\sigma}_{v_0}} \int d\underline{\mathbf{x}}_{v_0} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2n}} \delta(I^1_{v_0} - I^1_{ext}) \delta(I^2_{v_0} - I^2_{ext}) \delta\big(P_{v_0} - P_{ext}\big) \times$$

$$\times \Big[\prod_{v \text{ e.p.}} K^{\rho_v}_{h_v;\sigma_v\sigma'_v}(\mathbf{x}_v, \mathbf{y}_v) \Big] \sum_{T\in\mathbf{T}} \alpha_T \prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{(h_v)}_{T_v}(\mathbf{t}_v) \prod_{\ell\in T_v} g^{(h_v)}_\ell , \qquad (5.68)$$

where the $**$ on the sum over τ indicates the constraint that τ has s_1 endpoints of type 1, s_2 of type 2, and no endpoints of type 0. Note also that, in order for $|P_{v_0}|$ to be equal to $2n$, the number of endpoints of type 3 and 4 must be $\ge n-1$, that is $N \ge s_1 + s_2 + n - 1$. Moreover, $I^1_{ext} = \big((\mathbf{x}_1, \sigma_1), \dots, (\mathbf{x}_{s_1}, \sigma_{s_1})\big)$, $I^2_{ext} = \big((\mathbf{x}_{s_1+1}, \mathbf{y}_{s_1+1}, \sigma_{s_1+1}, \sigma'_{s_1+1}), \dots, (\mathbf{x}_{s_2}, \mathbf{y}_{s_2}, \sigma_{s_2}, \sigma'_{s_2})\big)$, $P_{ext} = \big((\mathbf{z}_1, \sigma''_1, \varepsilon_1), \dots, (\mathbf{z}_{2n}, \sigma''_{2n}, \varepsilon_{2n})\big)$, and the functions $\delta(I^1_{v_0} - I^1_{ext})$, etc, are shorthands of products of delta functions, in the same sense as $\delta(P - P_{ext})$ in (5.47). Using the explicit expression (5.68), we obtain a bound analogous to (5.48):

$$\frac{1}{\beta L^2} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} \big| W^{(h)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}) \big| \le \qquad (5.69)$$

$$\le \sum_{\substack{N\ge 1:\\ N\ge s_1+s_2+n-1}} C^N \sum^{**}_{\tau\in\mathcal{T}_{M;h,N}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2n}} \Big[\prod_{v \text{ e.p.}} \|K^{\rho_v}\|_1 \Big] \sum_{T\in\mathbf{T}} \Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} \big\| \det G^{(h_v)}_{T_v} \big\|_\infty \prod_{\ell\in T_v} \|g^{(h_v)}_\ell\|_1 \Big] .$$

Now: (i) the contribution of the endpoints is bounded as $\|K^{\rho_v}\|_1 \le C|U|^{\delta_{\rho_v,3}+\delta_{\rho_v,4}}$, (ii) the 1-norm of the propagators is bounded as in (5.26), that is $\|g^{(h_v)}_\ell\|_1 \le C\delta_\mu^{-3} 2^{-h}$, and (iii) the determinant, recalling the Gram representation of the propagator (5.27), can be bounded by using the Gram–Hadamard inequality (5.50) in a way analogous to (5.51), that is

$$\|\det G_{T_v}^{(h_v)}\|_\infty \leq C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\ , \tag{5.70}$$

where $v_1,\ldots,v_{s_v}$ are the vertices immediately following v on τ. Plugging these bounds into (5.69), and using the fact that $\sum_{v\text{ not. e.p.}}(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|)\leq 4(N-s_1-s_2)$, we obtain

$$\sum_{\substack{N\geq 1:\\ N\geq s_1+s_2+n-1}} C^N|U|^{N-s_1-s_2}\sum^{**}_{\tau\in\mathcal{T}_{M;h,N}}\ \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2n}}\ \sum_{T\in\mathbf{T}}\Big[\prod_{\substack{v\text{ not}\\ \text{e.p.}}}\frac{1}{s_v!}(C\delta_\mu^{-3}2^{-h_v})^{s_v-1}\Big]\ . \tag{5.71}$$

Using the following relation, which can be easily proved by induction,

$$\sum_{\substack{v\text{ not}\\ \text{e.p.}}} h_v(s_v-1)=h(N-1)+\sum_{\substack{v\text{ not}\\ \text{e.p.}}}(h_v-h_{v'})(n(v)-1)\ , \tag{5.72}$$

where v' is the vertex immediately preceding v on τ and $n(v)$ the number of endpoints following v on τ, we find that Eq.(5.71) can be rewritten as

$$\sum_{\substack{N\geq 1:\\ N\geq s_1+s_2+n-1}}\ \sum^{**}_{\tau\in\mathcal{T}_{M;h,N}}\ \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2n}}\ \sum_{T\in\mathbf{T}} C^N\delta_\mu^{-3(N-1)}|U|^{N-s_1-s_2}2^{-h(N-1)}\Big[\prod_{\substack{v\text{ not}\\ \text{e.p.}}}\frac{1}{s_v!}2^{(h_v-h_{v'})(n(v)-1)}\Big]\ . \tag{5.73}$$

where, by construction, if $N>1$, then $n(v)>1$ for any vertex v of $\tau\in\mathcal{T}_{M;h,N}$ that is not an endpoint (simply because every endpoint v of τ is attached to a non-trivial vertex on scale h_v-1, see the discussion after (5.59) and item (ii) after (5.60)). If $N=1$, the only tree contributing to the sum in (5.73) is trivial, with four possible type labels attached to the endpoint. The corresponding contribution to (5.73) is (const.)$|U|^{\delta_{s_1+s_2,0}}$. The contribution to (5.73) from the terms with $N\geq 2$ can be bounded as follows: first of all, the number of terms in $\sum_{T\in\mathbf{T}}$ is bounded by $C^N\prod_{v\text{ not e.p.}}s_v!$ (see, e.g., [? , Appendix A.3.3]); moreover, $|P_v|\leq 4n(v)$ and $n(v)-1\geq\max\{1,\frac{n(v)}{2}\}$, so that $n(v)-1\geq\frac{1}{2}+\frac{|P_v|}{16}$, and, therefore,

$$\frac{1}{\beta L^2}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\big|W^{(h)}_{2n,s_1,s_2,\underline{\sigma},\underline{\varepsilon}}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}})\big|\leq \sum_{\substack{N\geq 1:\\ N\geq s_1+s_2+n-1}} C^N\delta_\mu^{-3(N-1)}|U|^{N-s_1-s_2}2^{-h(N-1)}\times$$
$$\times\sum^{**}_{\tau\in\mathcal{T}_{M;h,N}}\Big(\prod_{v\text{ not e.p.}}2^{-\frac{1}{2}(h_v-h_{v'})}\Big)\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2n}}\Big(\prod_{\substack{v\text{ not}\\ \text{e.p.}}}2^{-|P_v|/16}\Big)\ . \tag{5.74}$$

Now, the sums over τ and $\mathbf{P}$ in the second line can be both bounded by (const.)^N, see [? , Lemma A.2 in Appendix A.1 and Appendix A.6.1], which implies the uniform analyticity of the kernels of the effective potentials on scale h, for all $-1\leq h<M$, provided U is small enough, namely $|U|\leq$ (const.)δ_μ^3. Note that the regularized free energy and correlation

functions are nothing but the constant part and the kernels of the effective potential with $h = -1$. Therefore, the regularized free energy is analytic in $|U|$, uniformly in β, L, M. Similarly, the regularized correlation functions are uniformly analytic and satisfy (5.23), uniformly in β, L, M, for $\underline{m} = \underline{0}$ and $|U|$ small enough. The proof of (5.23) for general choices of $\underline{m}$ follows similarly, by combining the previous strategy with the idea of decomposing the factors $|\mathbf{x}_i - \mathbf{x}_j|$ along the tree T, as in (5.53) and following discussion. *This concludes the proof of (5.23).* ∎

C. Proof of Proposition 4.1

We are left with proving the existence of the limit as $\beta, L, M \to \infty$ of the regularized free energy and correlation functions. In order to prove it, we show that these regularized functions form a Cauchy sequence. Let us start by showing that, for fixed β, L, and $M' > M$, for all $0 < \theta < 1$, there exists $C_\theta > 0$ such that

$$\|K^{\beta,L,M}_{m,n} - K^{\beta,L,M'}_{m,n}\|_{1,r} \le C_\theta 2^{-\theta M} , \tag{5.75}$$

where

$$\|K^{\beta,L,M}_{m,n}\|_{1,r} = \frac{1}{\beta L^2} \sup_{\underline{\sigma}} \sup_{\substack{\underline{m}: \\ |\underline{m}|=r}} \int_{\Lambda^{m+n}_{\beta,L}} d\underline{\mathbf{x}} \sum_{\underline{\vec{z}} \in \Lambda^m_L} \left| K^{\beta,L,M}(\mathbf{x}_1, \vec{z}_1, \sigma_1, \sigma'_1; \ldots; \mathbf{x}_{m+n}, \sigma_{m+n}) \right| d_{\underline{m}}(\underline{\mathbf{x}}, \underline{\vec{z}}) . \tag{5.76}$$

As already remarked above, the regularized correlation function are the kernels of the effective potential on scale -1. Therefore, both $K^{\beta,L,M}$ and $K^{\beta,L,M'}$ can be expressed in terms of the tree expansion described above. As already remarked after (5.67), the expansions for $K^{\beta,L,M}$ and $K^{\beta,L,M'}$ differ among each other only because of: (i) the choice of the scale labels (the trees contributing to $K^{\beta,L,M}$, resp. $K^{\beta,L,M'}$, have endpoints on scales $\le M+1$, resp. $\le M'+1$); (ii) the dependence on the ultraviolet cutoff of the endpoints of type 3, whose value is $K^3_{h_v;\sigma_v\sigma'_v} = K^3_{\sigma_v\sigma'_v} + k^{[h_v,M]}_{\sigma_v\sigma'_v}$ in the trees contributing to $K^{\beta,L,M}$, and similarly for $K^{\beta,L,M'}$. This means that the difference $K^{\beta,L,M} - K^{\beta,L,M'}$ can be expressed as a sum over trees whose root is on scale -1 and: (A) either there is at least one endpoint on scale $> M+1$, or (B) there is one endpoint of type 3 associated with a difference $k^{[h_v,M]}_{\sigma_v\sigma'_v} - k^{[h_v,M']}_{\sigma_v\sigma'_v} = k^{[M+1,M']}_{\sigma_v\sigma'_v}$.

The contributions from the case (A) can be bounded as in (5.73), with $h = -1$ and the extra constraint that there is at least one endpoint on scale $> M+1$. This means that the factor $\prod_{v \text{ not e.p.}} 2^{(h_v - h_{v'})(n(v)-1)}$ is smaller than 2^{-M}. The idea is then to split this term into two factors, in the form $\left[\prod_{v \text{ not e.p.}} 2^{\theta(h_v - h_{v'})(n(v)-1)}\right] \times \left[\prod_{v \text{ not e.p.}} 2^{(1-\theta)(h_v - h_{v'})(n(v)-1)}\right]$. The first factor is smaller than $2^{-\theta M}$, while the sum over the scale and field labels of the second factor can be bounded exactly in the same away as it was explained after (5.74).

Concerning case (B), it is enough to note that the norm of $k^{[M+1,M']}_{\sigma_v\sigma'_v}$ is proportional to 2^{-M}, see (5.60), which implies that the overall contribution from these trees is smaller than the norm of $K^{\beta,L,M}$ by a factor 2^{-M}.

In conclusion, we obtain (5.75). By Vitali's uniform convergence theorem for analytic functions, we conclude that the limit as $M \to \infty$ of the regularized correlations is analytic, and its Taylor coefficients are the $M \to \infty$ limit of the coefficients of the regularized correlations. The same argument is valid for the limit as $\beta, L \to \infty$, see [**?** , Appendix D] for a thorough discussion of this limit. Of course, the same claims are valid for the regularized free energy, too.

Finally, the statement of Proposition 4.1 follows from the remark that that the correlation functions in momentum space can be expressed as the Fourier transforms of their space-time counterparts, and that their derivatives of order r are controlled by the $(1, r)$ norms (5.76) of the space-time correlation functions, which are finite and bounded uniformly in β, L, M, as we just proved.

APPENDIX A: EQUIVALENCE OF DEFINITIONS OF CONDUCTIVITY

In this appendix we show that, in the absence of interactions, our definition of conductivity Eq. (3.14) is equivalent to the Kubo formula (in units $e = \hbar = 1$):

$$\sigma_{ij}^{(0)} = i \sum_{\substack{\alpha \\ \varepsilon_\alpha(\vec{k})<\mu}} \int_{\mathcal{B}} \frac{d\vec{k}}{(2\pi)^2} \,\mathrm{Tr}\, P_\alpha(\vec{k})[\partial_i P_\alpha(\vec{k}), \partial_j P_\alpha(\vec{k})] \,, \tag{A.1}$$

where μ is the Fermi energy, and $P_\alpha(\vec{k}) = |v_\alpha(\vec{k})\rangle\langle v_\alpha(\vec{k})|$ is the projector over the α-th Bloch band; the Bloch function $v_\alpha(\vec{k})$ satisfies $\hat{H}^{(0)}(\vec{k})v_\alpha(\vec{k}) = \varepsilon_\alpha(\vec{k})v_\alpha(\vec{k})$, with $\varepsilon_\alpha(\vec{k})$ the α-th energy band. Of course, $v_\alpha(\vec{k})$ is defined only up to a phase, while $P_\alpha(\vec{k})$ is free from this ambiguity. It is well-known that the integral in (A.1) can only take integer values, and that the corresponding integer has a topological meaning [**?**]; it is the *Chern number* of the Bloch bundle associated to the α-th band. The equivalence of (3.14) with (A.1) is a well-known fact, and we review it for completeness.

The starting point is to rewrite the current operator in Fourer space as follows:

$$\begin{aligned} \vec{J} \,&=\, i\big[\mathcal{H}_L, \sum_{\sigma\in I}\sum_{\vec{x}\in\Lambda_L} \vec{x}^{(\sigma)}\, n_{\vec{x}}^{(\sigma)}\big] \,, \qquad \vec{x}^{(\sigma)} = \vec{x} + \vec{\delta}^{(\sigma)} \,, \\ &=\, \frac{i}{L^2} \sum_{\vec{k}\in\mathcal{B}_L} \psi^+_{\vec{k},\sigma} \big[(-i)\nabla_{\vec{k}} H^{(0)}_{\sigma\sigma'}(\vec{k}) + \vec{\delta}^{(\sigma)} - \vec{\delta}^{(\sigma')}\big] \psi^-_{\vec{k},\sigma'} \end{aligned} \tag{A.2}$$

It turns out that the $\vec{\delta}^{(\sigma)}$ factors in (A.2) can be reabsorbed by conjugating the Bloch Hamiltonian and the fermionic fields with a suitable unitary transformation. Let us define $U(\vec{k}) = \mathrm{diag}\big(e^{-i\vec{k}\cdot\vec{\delta}^{(1)}}\ ,\ \cdots\ ,\ e^{-i\vec{k}\cdot\vec{\delta}^{(N)}}\big)$; then, we can rewrite the current as:

$$\vec{J} = \frac{1}{L^2} \sum_{\vec{k}\in\mathcal{B}_L} \widetilde{\psi}^+_{\vec{k},\sigma} \nabla_{\vec{k}} \widetilde{H}^{(0)}_{\sigma\sigma'}(\vec{k}) \widetilde{\psi}^-_{\vec{k},\sigma'} \tag{A.3}$$

where $\widetilde{H}^{(0)}(\vec{k}) = U(\vec{k})H^{(0)}(\vec{k})U(\vec{k})^*$, $\widetilde{\psi}^- = U(\vec{k})\psi^-_{\vec{k}}$, $\widetilde{\psi}^+ = \psi^+_{\vec{k}}U(\vec{k})^*$. Of course, $\sigma\big(H^{(0)}(\vec{k})\big) = \sigma\big(\widetilde{H}^{(0)}(\vec{k})\big)$. Instead, the eigenvectors of $\widetilde{H}^{(0)}(\vec{k})$ are $\widetilde{v}_\alpha(\vec{k}) = U(\vec{k})v_\alpha(\vec{k})$.

In the absence of interactions, the Green-Kubo conductivity matrix is:

$$\sigma^{(0)}_{ij} = -\frac{1}{A}\frac{\partial}{\partial p_0}\lim_{\beta,L\to\infty}\frac{1}{\beta L^2}\int_0^\beta dx_0 \int_0^\beta dy_0\, e^{ip_0(x_0-y_0)}\big\langle \mathbf{T}\, J_{x_0,i}\,;\, J_{y_0,j}\big\rangle^{(0)}_{\beta,L} . \tag{A.4}$$

We write:

$$\begin{aligned}\frac{1}{\beta L^2}\int_0^\beta dx_0 \int_0^\beta dy_0\, e^{ip_0(x_0-y_0)}\big\langle \mathbf{T}\, J_{x_0,i}\,;\, J_{y_0,j}\big\rangle^{(0)}_{\beta,L} &= \frac{1}{2L^2}\int_0^\beta dx_0\, e^{ip_0x_0}\big\langle J_{x_0,i}J_j\big\rangle^{(0)}_{\beta,L} \\ &\quad + \frac{1}{2L^2}\int_{-\beta}^0 dx_0\, e^{ip_0x_0}\big\langle J_j J_{x_0,i}\big\rangle^{(0)}_{\beta,L} ,\end{aligned} \tag{A.5}$$

where we used that, by the ciclicity of the trace, $\langle J_i\rangle^{(0)}_{\beta,L} = 0$. Let us consider the first term in Eq. (A.5). We get, by Wick's rule:

$$\begin{aligned}&\frac{1}{2L^2}\int_0^\beta dx_0\, e^{ip_0x_0}\big\langle J_{x_0,i}J_j\big\rangle^{(0)}_{\beta,L} = \\ &\quad \frac{-1}{2L^2}\sum_{\vec{k}\in\mathcal{B}_L}\int_0^\beta dx_0\, e^{ip_0x_0}\mathrm{tr}\,\big\{\widetilde{g}^{\beta,L}(x_0,\vec{k})\,\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{g}^{\beta,L}(-x_0,\vec{k})\,\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\big\}\end{aligned}$$

where $\widetilde{g}^{\beta,L}(x_0,\vec{k})$ is the fermionic propagator of the transformed operators $\widetilde{\psi}^\pm$ (see Eq. (**??**)):

$$\begin{aligned}\widetilde{g}^{\beta,L}(x_0,\vec{k}) &= U(\vec{k})g^{\beta,L}(x_0,\vec{k})U(\vec{k})^* \\ &= e^{-x_0(\widetilde{H}^{(0)}(\vec{k})-\mu)}\Big[\frac{1(x_0>0)}{1+e^{-\beta(\widetilde{H}^{(0)}(\vec{k})-\mu)}} - 1(x_0\le 0)\frac{e^{-\beta(\widetilde{H}^{(0)}(\vec{k})-\mu)}}{1+e^{-\beta(\widetilde{H}^{(0)}(\vec{k})-\mu)}}\Big] .\end{aligned} \tag{A.6}$$

Plugging (A.6) into (A.5) and integrating over x_0 we find:

$$\begin{aligned}&\frac{1}{2L^2}\int_0^\beta dx_0\, e^{ip_0x_0}\big\langle J_{x_0,i}J_j\big\rangle^{(0)}_{\beta,L} \\ &= \frac{1}{2L^2}\sum_{\vec{k}\in\mathcal{B}}\sum_{\alpha,\gamma}\frac{\big[e^{\beta(ip_0-\varepsilon_\alpha(\vec{k})+\varepsilon_\gamma(\vec{k}))}-1\big]}{ip_0-\varepsilon_\alpha(\vec{k})+\varepsilon_\gamma(\vec{k})}\cdot\frac{1}{1+e^{-\beta(\varepsilon_\alpha(\vec{k})-\mu)}}\frac{1}{1+e^{\beta(\varepsilon_\gamma(\vec{k})-\mu)}} \\ &\qquad \cdot\langle\widetilde{v}_\alpha(\vec{k}),\,\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{v}_\gamma(\vec{k})\rangle\langle\widetilde{v}_\gamma(\vec{k}),\,\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{v}_\alpha(\vec{k})\rangle .\end{aligned} \tag{A.7}$$

In the same way, the second term in the right-hand side of Eq. (A.5) is:

$$\frac{1}{2L^2}\int_{-\beta}^0 dx_0\, e^{ip_0x_0}\big\langle J_jJ_{x_0,i}\big\rangle^{(0)}_{\beta,L} \tag{A.8}$$

$$\begin{aligned}&= \frac{1}{2L^2}\sum_{\vec{k}\in\mathcal{B}}\sum_{\alpha,\gamma}\frac{\big[e^{\beta(-ip_0-\varepsilon_\alpha(\vec{k})+\varepsilon_\gamma(\vec{k}))}-1\big]}{-ip_0-\varepsilon_\alpha(\vec{k})+\varepsilon_\gamma(\vec{k})}\cdot\frac{1}{1+e^{-\beta(\varepsilon_\alpha(\vec{k})-\mu)}}\frac{1}{1+e^{\beta(\varepsilon_\gamma(\vec{k})-\mu)}} \\ &\qquad \cdot\langle\widetilde{v}_\alpha(\vec{k}),\,\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{v}_\gamma(\vec{k})\rangle\langle\widetilde{v}_\gamma(\vec{k}),\,\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{v}_\alpha(\vec{k})\rangle .\end{aligned} \tag{A.9}$$

Therefore, introducing the shorthand notation $\big[\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma} := \langle\widetilde{v}_\alpha(\vec{k}),\, \partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\,\widetilde{v}_\gamma(\vec{k})\rangle$, we get:

$$\begin{aligned}
&\lim_{\beta\to\infty}\frac{1}{2L^2}\int_0^\beta dx_0\, e^{ip_0x_0}\big\langle J_{x_0,i}J_j\big\rangle^{(0)}_{\beta,L} + \lim_{\beta\to\infty}\frac{1}{2L^2}\int_{-\beta}^0 dx_0\, e^{ip_0x_0}\big\langle J_j J_{x_0,i}\big\rangle^{(0)}_{\beta,L} \\
&= \frac{1}{L^2}\sum_{\vec{k}\in\mathcal{B}_L}\sum_{\substack{\alpha,\gamma \\ \varepsilon_\alpha(\vec{k})>\mu \\ \varepsilon_\gamma(\vec{k})<\mu}} \frac{1}{ip_0-\varepsilon_\gamma(\vec{k})+\varepsilon_\alpha(\vec{k})}\big[\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma}\big[\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\big]_{\gamma\alpha} + \\
&\quad + \frac{1}{L^2}\sum_{\vec{k}\in\mathcal{B}_L}\sum_{\substack{\alpha,\gamma \\ \varepsilon_\alpha(\vec{k})>\mu \\ \varepsilon_\gamma(\vec{k})<\mu}} \frac{1}{-ip_0-\varepsilon_\gamma(\vec{k})+\varepsilon_\alpha(\vec{k})}\big[\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma}\big[\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\big]_{\gamma\alpha}\,. \qquad \text{(A.10)}
\end{aligned}$$

Plugging this computation in (A.4), we obtain:

$$\begin{aligned}
\sigma^{(0)}_{ij} &= \int\frac{d\vec{k}}{(2\pi)^2}\sum_{\substack{\alpha,\gamma \\ \varepsilon_\alpha(\vec{k})>\mu \\ \varepsilon_\gamma(\vec{k})<\mu}} \frac{i}{(\varepsilon_\gamma(\vec{k})-e_\alpha(\vec{k}))^2} \\
&\quad \cdot\Big\{\big[\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma}\big[\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\big]_{\gamma\alpha} - \big[\partial_{k_j}\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma}\big[\partial_{k_i}\widetilde{H}^{(0)}(\vec{k})\big]_{\gamma\alpha}\Big\}\,, \qquad \text{(A.11)}
\end{aligned}$$

where we used that $A|\mathcal{B}| = (2\pi)^2$. Then, since

$$\big[\partial_i\widetilde{H}^{(0)}(\vec{k})\big]_{\alpha\gamma} = (\varepsilon_\gamma(\vec{k})-\varepsilon_\alpha(\vec{k}))\langle\widetilde{v}_\alpha(\vec{k}),\partial_{k_i}\widetilde{v}_\gamma(\vec{k})\rangle \qquad \text{for } \alpha\neq\gamma, \qquad \text{(A.12)}$$

we can rewrite the integrand in Eq. (A.11) as:

$$\begin{aligned}
& i\sum_{\substack{\alpha,\gamma \\ \varepsilon_\alpha(\vec{k})>\mu \\ \varepsilon_\gamma(\vec{k})<\mu}} \langle\widetilde{v}_\alpha(\vec{k}),\partial_{k_i}\widetilde{v}_\gamma(\vec{k})\rangle\langle\widetilde{v}_\gamma(\vec{k}),\partial_{k_j}\widetilde{v}_\alpha(\vec{k})\rangle - \langle\widetilde{v}_\alpha(\vec{k}),\partial_{k_j}\widetilde{v}_\gamma(\vec{k})\rangle\langle\widetilde{v}_\gamma(\vec{k}),\partial_{k_i}\widetilde{v}_\alpha(\vec{k})\rangle \\
&= i\sum_{\gamma:\,\varepsilon_\gamma(\vec{k})<\mu}\Big[-\langle\partial_{k_j}\widetilde{v}_\alpha(\vec{k}),\partial_{k_i}\widetilde{v}_\gamma(\vec{k})\rangle + \langle\partial_{k_j}\widetilde{v}_\gamma(\vec{k}),\widetilde{P}_{\le\mu}(\vec{k})\partial_{k_i}\widetilde{v}_\gamma\rangle\Big] - (i\leftrightarrow j) \\
&= i\sum_{\gamma:\,\varepsilon_\gamma(\vec{k})<\mu}\Big[\langle\partial_{k_i}\widetilde{v}_\gamma(\vec{k}),\partial_{k_j}\widetilde{v}_\gamma(\vec{k})\rangle - \langle\partial_{k_j}\widetilde{v}_\gamma(\vec{k}),\partial_{k_i}\widetilde{v}_\gamma(\vec{k})\rangle\Big]\,; \qquad \text{(A.13)}
\end{aligned}$$

using that $P_\gamma(\partial P_\gamma)P_\gamma = 0$, we get:

$$\begin{aligned}
\langle\partial_{k_j}\widetilde{v}_\gamma(\vec{k}),\partial_{k_i}\widetilde{v}_\gamma(\vec{k})\rangle &= \langle\partial_{k_j}\widetilde{P}_\gamma(\vec{k})\widetilde{v}_\gamma(\vec{k}),\partial_{k_i}\widetilde{P}_\gamma(\vec{k})\widetilde{v}_\gamma(\vec{k})\rangle \\
&= \mathrm{Tr}\widetilde{P}_\gamma(\vec{k})\partial_{k_j}\widetilde{P}_\gamma(\vec{k})\partial_{k_i}\widetilde{P}_\gamma(\vec{k}) + \langle\partial_{k_j}v_\gamma(\vec{k}),\widetilde{P}_\gamma(\vec{k})\partial_{k_i}v_\gamma(\vec{k})\rangle\,. \qquad \text{(A.14)}
\end{aligned}$$

The last term is symmetric with respect to $i\leftrightarrow j$ and hence it does not contribute to $\sigma^{(0)}_{ij}$. Thus, we found (A.1) with $\widetilde{P}_\gamma(\vec{k})$ instead of $P_\gamma(\vec{k})$. To drop the tilde, notice first that

$$\partial_{k_i}\widetilde{P}_\gamma(\vec{k}) = U(\vec{k})\partial_{k_i}P_\gamma(\vec{k})U(\vec{k})^* + U(\vec{k})[A_i,P_\gamma(\vec{k})]U(\vec{k})^* \qquad \text{(A.15)}$$

where $A_i = U(\vec{k})^* \partial_{k_i} U(\vec{k})$ is a diagonal matrix, independent of $\vec{k}$. Therefore,

$$\begin{aligned}&\mathrm{Tr}\,\widetilde{P}_\gamma(\vec{k})[\partial_{k_i}\widetilde{P}_\gamma(\vec{k}), \partial_{k_j}\widetilde{P}_\gamma(\vec{k})] = \mathrm{Tr}\,P_\gamma(\vec{k})[\partial_{k_i}P_\gamma(\vec{k}), \partial_{k_j}P_\gamma(\vec{k})]\\ &\quad + \mathrm{Tr}\,P_\gamma(\vec{k})\big[[A_i, P_\gamma(\vec{k})], \partial_{k_j}P_\gamma(\vec{k})\big] + \mathrm{Tr}\,P_\gamma(\vec{k})\big[\partial_{k_i}P_\gamma(\vec{k}), [A_j, P_\gamma(\vec{k})],\big]\\ &\quad + \mathrm{Tr}\,P_\gamma(\vec{k})\big[[A_i, P_\gamma(\vec{k})], [A_j, P_\gamma(\vec{k})]\big]\ .\end{aligned} \tag{A.16}$$

It is easy to see that the last term in (A.16) is zero for all $\vec{k}$. Consider the second term in (A.16). We have:

$$\begin{aligned}&\mathrm{Tr}\,P_\gamma(\vec{k})\big[[A_i, P_\gamma(\vec{k})], \partial_{k_j}P_\gamma(\vec{k})\big]\\ &= -\mathrm{Tr}\,P_\gamma(\vec{k})A_i(1 - P_\gamma(\vec{k}))\partial_{k_j}P_\gamma(\vec{k}) - \mathrm{Tr}\,P_\gamma(\vec{k})\partial_{k_j}P_\gamma(\vec{k})(1 - P_\gamma(\vec{k}))A_i\\ &= -\partial_{k_j}\mathrm{Tr}\,A_i P_\gamma(\vec{k})\ ,\end{aligned} \tag{A.17}$$

where we used again that $P_\gamma(\partial P_\gamma)P_\gamma = 0$. Being $P_\gamma(\vec{k})$ periodic over $\mathcal{B}$ and A_i constant, the integral of (A.17) vanishes. The same is true for the third term in (A.16). Therefore, the only nontrivial contribution to $\sigma^{(0)}_{ij}$ comes from the first term of (A.16); this concludes the check of (A.1).

APPENDIX B: THE HALDANE MODEL

An interesting model that falls into the general class of two-dimensional systems discussed here is the *Haldane model*, [**?**]; as we shall see later, this model has remarkable transport properties.

The model. The Haldane model describes fermions hopping on the honeycomb lattice, exposed to a suitable external magnetic field. Let Λ_L be the triangular lattice, generated by the basis vectors

$$\vec{\ell}_1 = \frac{1}{2}(3, -\sqrt{3})\,, \qquad \vec{\ell}_2 = \frac{1}{2}(3, \sqrt{3})\ . \tag{B.1}$$

The reciprocal lattice Λ^*_L of Λ_L is the triangular lattice generated by the vectors

$$\vec{G}_1 = \frac{2\pi}{3}(1, -\sqrt{3})\ , \qquad \vec{G}_2 = \frac{2\pi}{3}(1, \sqrt{3})\ . \tag{B.2}$$

The physical lattice of the Haldane model is an hexagonal lattice, which can be obtained as the superposition of two triangular lattices. Thus, the internal degrees of freedom are labelled by $\sigma \in \{A, B\}$, where A, B label the sublattices $\Lambda^{(\sigma)}_L = \Lambda_L + \vec{\delta}^{(\sigma)}$, with $\vec{\delta}^{(A)} = \vec{0}$, $\vec{\delta}^{(B)} = \vec{\delta}_1 = (1,\ 0)$ (we neglect the spin for simplicity). The full honeycomb lattice is $\Lambda^{(A)}_L \cup \Lambda^{(B)}_L$. In other words, we can think of the honeycomb lattice as a triangular lattice, where the sites corresponding to "dimers", given by the pairs $(\vec{x},\ \vec{x} + \vec{\delta}_1)$, with $\vec{x} \in \Lambda^{(A)}_L$. To each dimer, we associate the pair of fermionic operators $(\psi^{\pm}_{\vec{x},A}\ , \psi^{\pm}_{\vec{x},B})$. Each site $\vec{x} \in \Lambda^{(A)}_L$ has three nearest-neighbours $\vec{x} + \vec{\delta}_j$, $j = 1, 2, 3$, with: