Dense gaps in the interacting Aubry-André model

Vieri Mastropietro[1]

[1] *Università degli Studi di Milano, Via Saldini, 50, 20133 Milano - Italy*

We consider the interacting Aubry-André model describing fermions on a one dimensional lattice with an incommensurate potential and a short range many-body interaction. The single particle spectrum has infinitely many gaps in the extended phase and at zero temperature is an insulator for almost all the chemical potentials. The many body interaction has the effect that the gaps are strongly decreased or increased depending on the attractive or repulsive nature of the interaction, but even the smallest gaps remain open. The system is a band-insulator for generic chemical potentials even in presence of interaction and a quantum phase transition is excluded at weak coupling.

I. INTRODUCTION

Recent cold atoms experiments, see e.g. [1],[2], [3] have renewed the interest in the *interacting Aubry-André model*, describing fermions on a one dimensional lattice with a quasi-periodic potential $\lambda \cos 2\pi\omega x$ [4] and a short range many body density-density interaction with coupling U. The model was introduced for describing quasi-crystals [5] and is a paradigmatic system for understanding the interplay of disorder and interaction [6]. The Aubry-André model belongs to a class of one dimensional fermionic systems in which the interaction drastically modifies the single body behavior and produces dramatic effect. The one-dimensionality greatly simplifies the analysis and offers a way to understand phenomena which may have a counterpart in higher dimensions, where are much more difficult to analyze. Even for such one dimensional models most of the properties are usually derived with severe approximations and on the other hand such models provide only a qualitative description of real metals; therefore when comparing theoretical predictions with experiments it is often difficult to understand if discrepancies are due to approximations in the theoretical analysis or to the model itself. The realization of optical lattices provides a reasonable clean realization of such systems, to be compared directly with theoretical predictions. As optical lattice systems play the role of "quantum simulators" for such models, precise analytical predictions are necessary as benchmark for more complex systems, whose properties are usually analytically unaccessible.

In the absence of many body interaction the properties of the Aubry-André model are quite well understood. The single particle eigenstates show a transition, when the strength of the potential is increased, between an extended and a localized phase [7], similarly to what happens with three dimensional random disorder. In both regimes the spectrum is a Cantor set [8], a fact which has deep consequences for transport. In particular, in the extended phase there are infinitely many gaps in correspondence of quasi-momenta $2n\pi\omega$ mod. 2π, forming a dense set, and their size is decreasing exponentially with n. Therefore for almost all choices of chemical potentials the system is a band-insulator at zero temperature

Much less is known when a many body interaction is present, in particular in the extended regime; the localized interacting phase is somewhat more accessible numerically [9]-[15] and analytically [16]. In the extended regime any coupling $U \neq 0$ is greater than most gaps and can produce in principle their closure, causing a quantum phase transition between a band insulator to a metallic phase. There is indeed some evidence, based on second order perturbative Renormalization Group (RG), that in the case of a Fibonacci quasi-periodic potential the interaction closes the smallest gaps, see [18],[19], causing a quantum phase transition at $U > 0$. The method for Fibonacci potential, based on second order truncation, cannot be applied to the interacting André-Aubry model as smallest gap are generated at very high orders; in that case one needs non-perturbative methods and by them the persistence of the largest gap was established [17]. Numerical simulations, performed both in the fermionic [9], [10], [11],[12] and bosonic case[13],[14],[15], do not provide much information on the extended regime.

The fate of small gaps and the existence of a quantum phase transition at $U \neq 0$ in the interacting Aubry-André model are therefore open questions to which we provide an answer in this paper, using analytical methods. We show, if the interaction is sufficiently small, that the gaps are strongly suppressed or enlarged by a power law driven by critical exponents, depending on the attractive or repulsive nature of the interaction, but they are nonvanishing. Therefore, there is no quantum phase transition at $U \neq 0$ and the system remains a band-insulator at zero temperature for almost all the chemical potentials. In addition, we show that the exponents appearing in the gaps depend on all the microscopic details but verify exact scaling relations.

The main difficulty of the analysis is related to the combined effect of Umklapp and the incommensurability of potential, which has the effect that a large momentum exchange can connect points arbitrarily close to the Fermi points. This produces small divisors very similar to the ones appearing in quasi-periodic solutions of nearly integrable Hamiltonian systems, a fact making a perturbative approach unreliable. In the RG language, small divisors produce an infinite number of running coupling constants and in order to control their flow one needs to exploit number theoretical properties of irrationals, as is done in the Kolmogorov-Arnold-Moser (KAM) theorem

or in the Harper equation. Contrary to most application of RG, a continuum approximation cannot be performed as it misses the essential qualitative features of the problem, related to the incommensurability of the potential with respect to the lattice; moreover gaps are generated only at high orders so truncation of series is not possible. Therefore, the analysis needs to be performed using exact and non perturbative Renormalization Group methods.

The paper is organized in the following way. In §II we define the model, and we present our main result; in §III we set up our exact Renormalization Group analysis. According to dimensional considerations, there is an infinite number of relevant or marginal interactions; however, in §IV we will show that indeed a huge number of such terms is indeed irrelevant and only a finite number of effective interactions is relevant or marginal. In §V we define a multiscale integration in terms of running coupling constants and in §VI we study their flow. Finally in §VII we prove the persistence of the gaps and the validity of the scaling relations and in §VIII the main conclusions are presented.

II. THE MODEL AND MAIN RESULTS

The Hamiltonian of the fermionic interacting Aubry-André model, is

$$H = \sum_x \frac{1}{2}(a^\dagger_{x+1}a_x + a^\dagger_x a_{x+1}) - \mu \sum_x a^\dagger_x a_x \qquad (1)$$
$$+\lambda \sum_x \cos(2\pi\omega x) a^\dagger_x a_x + U \sum_{x,y} v(x-y) a^\dagger_x a_x a^\dagger_y a_y$$

with $x = 0, \pm 1, \pm 2, ..., > 0$ and $a^\dagger_x, a_x$ fermionic creation or annihilation operators. and $v(x-y)$ is a non local interaction. The irrational frequency ω is assumed *Diophantine*

$$||2n\pi\omega|| \geq C|n|^{-\tau}, \qquad n \neq 0 \qquad (2)$$

$||.||$ being the norm on the one dimensional 2π torus. This is the standard condition usually assumed for the non interacting case $U = 0$ [7] and it is physically not restrictive as Diophantine numbers have full measure.

Periodic boundary conditions are imposed considering a sequence of periodic potentials of period L such that quasi-periodicity is recovered in the thermodynamic limit. In order to do that we start from the continued fraction representation of a number ω

$$\omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \qquad (3)$$

As an example, the golden ratio $\omega = \frac{\sqrt{5}+1}{2}$ has representation $1; 1, ..1, ..$ and it verifies the Diophantine condition (2) with $\tau = 1$ and $C_0 = \frac{3+\sqrt{5}}{2}$. We approximate ω by a sequence of rational numbers (*convergents*) $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$, $\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}$ and so on. For the golden ratio, the sequence is given by the ratio of Fibonacci numbers $\{1, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{13}{8}, ., \frac{p_i}{q_i}, ...\}$. Properties of the convergents imply that if ω verifies the Diophantine condition then $|\pi(n\frac{p_i}{q_i} - k)| \geq \frac{C}{2|n|^\tau}$ if $q_1 \leq n \leq \frac{q_i}{2}$ and any k. Therefore we can impose periodic boundary conditions by considering a sequence of frequencies $\omega_i = \frac{p_i}{q_i}$ and $L_i = q_i$.

We are interested in the thermodynamical correlations at zero temperature, like the 2-point function with imaginary time $< a_{\mathbf{x}} a^\dagger_{\mathbf{y}} >$, $a_{\mathbf{x}} = e^{Hx_0} a_x e^{-Hx_0}$, $\mathbf{x} = (x_0, x)$ and $<>= \frac{\mathrm{Tr} e^{-\beta H}\mathcal{T}.}{\mathrm{Tr} e^{-\beta H}}$ and $\mathcal{T}$ is the time order product. Another important quantity is the density-density correlation $< \rho_{\mathbf{x}}; \rho_{\mathbf{y}} >_T$, with $\rho_{\mathbf{x}} = a^\dagger_{\mathbf{x}} a_{\mathbf{x}}$ and T denotes truncation. In the non interacting $U = \lambda = 0$ limit $< a_{\mathbf{x}} a^\dagger_{\mathbf{y}} > |_{U=u=0} = g(\mathbf{x}, \mathbf{y})$ with

$$g(\mathbf{x}, \mathbf{y}) = \int d\mathbf{k} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + \cos k - \mu} \qquad (4)$$

We call p_F the Fermi momentum defined as $\mu = \cos p_F$; the denominator of (4) is vanishing in correspondence of the two Fermi momenta $\pm p_F$.

Our main result can be summarized by the following theorem.

Theorem *Assume λ, U small, $\sum_x |x||v(x)| < \infty$, ω Diophantine (2) and choose the interacting Fermi momentum equal to $p_F = n\pi\omega$ with n integer. The 2-point function and the density correlations decay exponentially with rate*

$$\Delta_{n,U} \sim [\lambda^{2n}(a_n + F)]^{X_n} \qquad (5)$$

with $F = O(|U| + |\lambda|)$, a_n non vanishing and $X_n = X_n(U)$ is a critical exponent such that $X_n(0) = 1$. Moreover if K_n and η_z are the critical exponents appearing respectively in the 2-point function and in the oscillating part of the density correlation then $\eta_z = \frac{2-K_n-K_n^{-1}}{2}$ and $X_n = \frac{1}{2-K_n}$.

The rate of the exponential decay of the correlations is an estimate of the gap size; for large n the gaps can be much smaller than the many body coupling U but nevertheless they are all non vanishing, so that the system remains a band-insulator and no quantum phase transition is present; there is no transition to a metallic phase with weak interactions, contrary to what happens with other kind of quasi-random disorder like Fibonacci quasi-periodic potential. This is true provided that the quasi-random disorder is weak and the many body interaction is weak and decay for large distances at least as $|x-y|^{-3}$. The interaction strongly modifies the ratio of the interacting and bare gaps; as $K_n < 1$ for repulsive and $K_n > 1$ for attractive interactions, the relative size is strongly enlarged or decreased depending on the sign of U. The critical exponents are non trivial functions of U verifying exact scaling relations.

The spinning version of the Aubry-André model (1) with a local on site interaction has been experimentally realized in [1], considering two incommensurate optical lattices and tuning the interaction via a magnetic Feshbach resonance. Longer ranged interactions with a power law decay with the distance, as the one in (1), are more difficult to realize but can be generated by trapping particles with strong dipolar momentum, i.e. magnetic atoms and polar molecules. In such systems the interaction decays for large distances as r^{-3} and the strength and sign of dipolar interactions can be tuned [22], [23]. An example is in [24], where is described an optical lattice device composed by an array of one dimensional tubes loaded with dipolar fermionic molecules, with dipole moments polarized by an external field in an arbitrary direction and no tunneling between tubes. Varying the polar and azimuthal angle of the external field with respect to the plane, one can obtain in particular that the inter-tube interaction vanishes while the intra-tube can be attractive or repulsive. With another trapping potential incommensurate with the optical lattice one has a realization of the model (1). We predict that if the quasi random random disorder is realized by an Aubry-André potential, and if both the amplitude of the potential and of the interaction are weak with respect to the hopping, the interaction does not alter the insulating behavior due to the Cantor set of gaps which is present in the non interacting case; however the width of the gaps is strongly modified, see (38), depending on the attractive or repulsive nature of the interaction, whose sign can be tuned varying the angle of the external field. On the other hand, according to [18], [19], a transition to a metallic phase should be present with quasi-random superposing lattices realizing a Fibonacci potential; therefore, changing the quasi-random disorder would produce a transition from an insulating to a metallic behavior which should be experimentally visible, for instance using the technique in [1] monitoring the time evolution of local observables following a quench of system parameters. Finally, increasing the width of the potential a transition to a localized phase, persisting even in presence of interaction, is expected [16].

III. RENORMALIZATION GROUP ANALYSIS

The correlations can be obtained by the derivatives of the *generating function*, expressed by the following Grassmann integral

$$e^{W(\phi,J)} = \int P(d\psi) e^{\mathcal{V}(\psi)+B(\phi,J)} \tag{6}$$

where $\psi^{\pm}_{\mathbf{x}}$ are Grassmann variables, $P(d\psi)$ is the Grassmann gaussian integration with propagator (4), with $\mu = \cos n_F \pi\omega$, $\mathcal{V}$ is the effective interaction

$$\mathcal{V} = -U \int d\mathbf{x} d\mathbf{y} v(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} \psi^+_{\mathbf{y}} \psi^-_{\mathbf{y}} + \tag{7}$$
$$\lambda \int d\mathbf{x} \cos(2\pi\omega x) \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} + \nu \int d\mathbf{x} \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}}$$

with $\int d\mathbf{x} = \sum_x \int_{-\beta/2}^{\beta/2} dx_0$, and B is the source term

$$B(\phi, J) = \int d\mathbf{x} [\phi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} + \phi^-_{\mathbf{x}} \psi^+_{\mathbf{x}} + J_{\mathbf{x}} \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}}] \tag{8}$$

The 2-point function is given by $\frac{\partial^2 W}{\partial \phi^-_{\mathbf{x}} \partial \phi^+_{\mathbf{y}}}|_0$ and the density-density correlation is given by $\frac{\partial^2 W}{\partial J_{\mathbf{x}} \partial J_{\mathbf{y}}}|_0$. Note that in the grand-canonical ensamble the chemical potential corresponding to a given density (or Fermi momentum) is a function of U; in order to take this fact into account we choose the chemical potential as $\cos \pi n_F \omega + \nu$ and ν will be properly chosen so that the Fermi momentum is $n_F \pi\omega$.

In order to explain the peculiarities of a quasi-periodic potential, we can consider a class of Feynman graphs, like the chain graphs, as in Fig. 1; their values is given by $\int d\mathbf{k} \phi_{\mathbf{k}} \phi_{\mathbf{k}+\sum_{i\le n} \varepsilon_i 2\pi\omega} H(\mathbf{k})$ with, if $\varepsilon_i = \pm 1$

$$H(\mathbf{k}) = \lambda^n \prod_{j=1}^{n} \widehat{g}(k_0, k + \sum_{i\le j} \varepsilon_i 2\pi\omega) \tag{9}$$

In the periodic case (ω rational) then $H(\mathbf{p}_F) = O(\lambda^n C^n)$ if $\sum_{i\le k} \varepsilon_i 2\pi\omega \neq 0, -2p_F$; that is the contributions involving the exchange of large momenta are small. On the contrary if ω is irrational one gets that even the exchange of large momenta imply dangerous contributions, as , by using (2), $H(\mathbf{p}_F) = O(\lambda^n C^n n!^\tau)$. This non summable behavior (usually called *small divisors* problem) makes a perturbative approach unreliable.

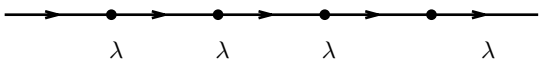

FIG. 1: A chain graph

We evaluate the correlations by an exact Renormalization Group. We introduce a smooth cut-off function $\chi_\rho(\mathbf{k})$, $\mathbf{k} = (k_0, k)$,which is non vanishing for $\sqrt{k_0^2 + ((k - \rho p_F)_{\mathrm{mod}.2\pi})^2)} \le \gamma$, where $\rho = \pm 1$ and $\gamma > 1$ is a suitable constant; therefore we can write the propagator as

$$\widehat{g}(\mathbf{k}) = \widehat{g}^{(u.v.)}(\mathbf{k}) + \sum_{\rho=\pm} \widehat{g}_\rho(\mathbf{k}) \tag{10}$$

where $\widehat{g}_\rho(\mathbf{k}) = \frac{\chi_\rho(\mathbf{k})}{-ik_0+\cos k-\cos p_F}$, and correspondingly $\psi_{\mathbf{k}} = \psi^{(u.v.)}_{\mathbf{k}} + \sum_{\rho=\pm1} \psi_{\mathbf{k}',\rho}$ with $k = k' + \rho p_F$, $\mathbf{k}' = (k_0, k')$. This simply says that we can write the fermionic field as sum of two independent fields living close to one of the Fermi points, up to a regular field. We can further decompose

$$\widehat{g}_\rho(\mathbf{k}) = \sum_{h=-\infty}^{0} \widehat{g}^{(h)}_\rho(\mathbf{k}) \tag{11}$$

with $\widehat{g}^{(h)}_\rho(\mathbf{k})$ similar to $\widehat{g}_\rho(\mathbf{k})$ with χ replaced by f_h with, where $f_h(\mathbf{k})$ is non vanishing in a region $\gamma^{h-1} \leq \sqrt{k_0^2 + v_F^2 k'^2} \leq \gamma^{h+1}$, with $v_F = \sin p_F$. After the integration of $\psi^{(u.v.)}, \psi^{(0)}, .., \psi^{(h+1)}$ the generating function has the form

$$e^{W(\phi,J)} = \int P(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\psi)+B^{(h)}(\psi,\phi,J)} \tag{12}$$

where $P(d\psi^{(\leq h)})$ has propagator $g^{(\leq h)}_\rho = \sum_{k=-\infty}^{h} g^{(k)}_\rho$ and $\mathcal{V}^{(h)}(\psi) =$

$$\sum_{m,n,\underline{\rho}} \int d\mathbf{k}'_1 ... d\mathbf{k}'_m W^{(h)}_{m,n}(\underline{\mathbf{k}}') \psi^{\varepsilon_1(\leq h)}_{\rho_1,\mathbf{k}'_1} ... \psi^{\varepsilon_m(\leq h)}_{\rho_m,\mathbf{k}'_m} \delta_{n,m}(\underline{\mathbf{k}}') \tag{13}$$

where $\delta_{n,m}(\underline{\mathbf{k}}')$ is $L\beta$ times a periodic Kronecker delta non vanishing for

$$\sum_{i=1}^{m} \varepsilon_i \rho_i k'_i = -\sum_{i=1}^{m} \varepsilon_i \rho_i p_F + 2n\pi\omega + 2l\pi \tag{14}$$

with $n = 0, \pm1, ...$ and $l = 0, \pm1, ..$ The kernels $W^{(h)}_{m,n}$ are sum of Feynman diagrams obtained connecting vertices λ, U or ν with propagators $g^{(k)}$ with $k > h$; $B^{(h)}$ is given by a similar expression with the only difference that some of the external lines are associated to ϕ or J external fields. In each of the Feynman diagrams contributing to $W^{(h)}_{m,n}$ there are a set of vertices $\lambda e^{i\sigma_i \pi\omega x}$, $\sigma_i = \pm$ and $n = \sum_i \sigma_i$. The relation (14) is the momentum conservation; when the r.h.s. is vanishing the momentum *measured from the Fermi points* is also conserved. The single scale propagator has the following form

$$g^{(h)}_\rho(\mathbf{k}') = g^{(h)}_{rel,\rho}(\mathbf{k}') + r^{(h)}(\mathbf{k}') \tag{15}$$

where $g^{(h)}_{rel,\rho}(\mathbf{k}') = \frac{f_h(\mathbf{k}')}{-ik_0+\rho v_F k'}$ is the dominant part of the propagator and $r^{(h)}(\mathbf{k}')$; therefore in the above RG procedure naturally emerges a description in terms of massless relativistic chiral fermions with propagator $g^{(h)}_{rel,\rho}(\mathbf{k}')$. Note also that in the effective potential $\mathcal{V}^{(h)}$ appear terms with any n, m, and only a few of them were originally present in the initial potential $\mathcal{V}$.

According to power counting arguments, the quartic terms are *marginal* and the quadratic terms are *relevant*; all other terms are irrelevant. There are then apparently *infinitely* many dimensionally relevant or marginal terms, depending on the value of n in (13). A natural distinction is if the r.h.s. of (14) is vanishing or not. The first case corresponds to processes *exactly connecting* the Fermi points; this happens in the following cases, if $m = 2, 4$: a)$m = 4$, $n = 0$ and $\sum_i \rho_i \varepsilon_i = 0$ (effective interaction of the form $\psi^+_+\psi^-_+\psi^+_-\psi^-_-$); b)$m = 4$, $|n| = 2n_F$ and $|\sum_i \varepsilon_i \rho_i| = 4$ (effective interaction $\psi^+_+\psi^-_-\psi^+_+\psi^-_-$ whose local part is vanishing, so is indeed irrelevant);c) $m = 2$, $|n| = 0$, $\rho_1 = \rho_2$ (chemical potential $\psi^+_\rho\psi^-_\rho$); d) $m = 2$, $|n| = n_F$, $\rho_1 = -\rho_2$ (gap $\psi^+_\rho\psi^-_{-\rho}$).

The other case is when the r.h.s. of (14) in non vanishing and here comes the main difference between the periodic and quasi-periodic case. In the periodic case when ω is rational the r.h.s. of (14) is *large* (if non vanishing); as the fields $\psi^{(\leq h)}$ carry a momentum k' with size smaller that γ^h, the condition (14) cannot be satisfied and such terms are vanishing for large $|h|$; therefore the terms with $m = 2, 4$ such that r.h.s. of (14) is non vanishing are indeed trivially irrelevant in the periodic case. In the quasi-periodic case instead the r.h.s of (14) can be arbitrarily small due to Umklapp (the momentum is defined modulo 2π for the presence of the lattice) , so that terms with large n in (14) persist at any Renormalization Group iteration. It is remarkable that dangerous processes in the infrared behavior are generated by the exchange of large momenta, which is a sort of ultraviolet-infrared mixing problem. We will see in next section that the relevance or irrelevance of such terms depends in a subtle way from number theoretical properties of the frequency ω and the velocity of decay of the Fourier transform of the quasi-periodic potential.

IV. IRRELEVANCE OF THE NON RESONANT TERMS

According to the above analysis, it is natural to distinguish in the effective potential (13) two kind of terms; in the *resonant terms* there is conservation of the momentum measured from the Fermi points, that is the r.h.s. of (14) is vanishing $\sum_i \rho_i p_F + 2n\pi\omega + 2l\pi = 0$; when the above condition is violated the terms are non resonant. We show now that if ω is irrational the non resonant terms are irrelevant, even if dimensionally relevant or marginal. Roughly speaking the reason is that, by the Diophantine condition (2), the r.h.s. of (14) is very small only if n is very large; this can produce a gain factor, provided that the decay of the harmonics of the potential is fast enough.

In order to put on a quantitative basis the above idea it is convenient, given a Feynman graph, to consider a maximally connected subset of lines corresponding to propagators with scale $h \geq h_v$ with at least a scale h_v, and we call it *cluster* v, see Fig.2; the n^e_v lines external to the cluster v have scale smaller then h_v. Given a non maximal cluster v with scale h_v, there is surely a cluster v' with scale $h_{v'} < h_v$ containing it. The clusters are therefore

subgraphs in which the propagators carries a momentum scale larger than the external lines, that is the momentum measured from the Fermi points in the internal lines is larger than in the external; they are a standard tool in renormalization theory to avoid the so called overlapping divergences. We call $\bar{m}_v$ is the number of λ, ν vertices internal to the cluster v and not of any smaller one; in the cluster v there are $k_{2,v}$ vertices $\lambda e^{i\sigma_i \pi\omega x}$, $\sigma_i = \pm$ such that $N_v = \sum_i \sigma_i$, so that $|N_v| \le k_{2,v}$. To each Feynman graph is associated a hierarchy of clusters; inside each cluster v there are S_v maximal clusters, that is clusters contained only in the cluster v and not in any smaller one, or trivial clusters given by a single vertex. Each of such inner clusters are connected by a tree of propagators with scale h_v; by integrating the propagators in the tree and bounding the others, and using that , $\int d\mathbf{x} |g_\rho^{(h)}(\mathbf{x})| \le C\gamma^{-h}$ and that $|g_\rho^{(h)}(\mathbf{x})| \le C\gamma^h$ we get that each graph of order k contributing to $W_{m,n}^{(h)}$ is bounded by the sum over the scales of

$$C^k U^{k_1} \lambda^{k_2} \gamma^{(2-m/2)h} \prod_v \gamma^{(h_v - h_{v'})D_v} \prod_v \gamma^{-h_v \bar{m}_v} \qquad (16)$$

where k is the perturbative order, $k = k_1 + k_2$ and $D_v = 2 - n_v^e/2$, if n_v^e is the number of external lines of cluster (subgraph) v. By summing over the scales h_v of the subgraphs one can read the scaling dimension; therefore the estimate (16) (a versions of Weinberg theorem for this model) says that no infrared divergence are present in the thermodynamic limit *provided that* there are no inner subgraphs v with four or two external lines and there are only quartic interactions. Indeed when the number of external lines of the clusters is greater then 4 then $D_v \le -1$ so that we can sum over h_v, that is $\sum_{h_v \ge h_{v'}} \gamma^{-(h_v - h_{v'})} \le C$. On the contrary if $D_v = 0$ for some v one gets a factor $|h|$ corresponding to a logarithmic divergence and if $D_v = -1$ a factor γ^{-h} summing over the scales; in this way one recovers the expected fact that the quartic terms are marginal and the quadratic are relevant.

We have however to take into account that ω is irrational and verifies the Diophantine condition. Indeed if v is a non resonant cluster with m external lines and verifying $\sum_{i=1}^m \varepsilon_i \rho_i p_F + 2N_v \pi\omega + 2l\pi \neq 0$ then, by using (2)

$$m\gamma^{h_{v'}} \ge ||\sum_{i=1}^m \rho_i k_i'|| \ge$$
$$||\sum_{i=1}^m \rho_i n_F \pi\omega + 2N_v \pi\omega|| \ge C_0(|n_F| + |N_v|)^{-\tau} \qquad (17)$$

which implies that

$$|N_v| \ge C\gamma^{\frac{-h_{v'}}{\tau}} \qquad (18)$$

On the other hand $N_v = \sum_i \sigma_i$ and $|N_v| \le k_{2,v}$ so that in a non resonant cluster there are necessarily a large number of λ vertices $k_{2,v} \ge \widetilde{C}\gamma^{-h_{v'}/\tau}$, and the associated factor $\lambda^{k_{2,v}}$ is therefore very small. By (18) we get, for $c < 1$ and $\gamma^{\frac{1}{\tau}}/2 = \gamma^\eta > 1$

$$c^{-k_2} \le \prod_v c^{-k_{2,v} 2^{h_{v'}-1}} \le \prod_v c^{-C 2^{h_v} \gamma^{-h_v/\tau}} \le \prod_v \gamma^{4h_v S_v^{NR}}$$

where S_v^{NR} are the non resonant clusters contained in v. This extra factor (bounded by an harmless constant) makes the non resonant clusters irrelevant, even when they have 2 or 4 external fields; indeed $\sum_{h_v \ge h_{v'}} \gamma^{(h_v - h_{v'})D_v} \gamma^{2h_{v'}} \le \sum_{h_v \ge h_{v'}} \gamma^{-(h_v - h_{v'})} \le C$. Note also that in the resonant clusters v containing a non resonant cluster there is an extra γ^{h_v}.

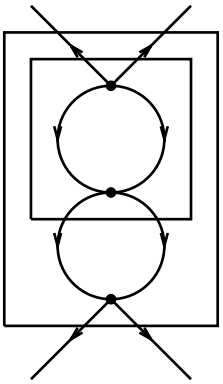

FIG. 2: A graphs and the corresponding clusters

A similar argument could be repeated even if all the harmonics are present in the quasi periodic potential, that is by choosing $\phi_x = \sum_n \widehat{\phi}_n e^{2in\pi\omega x}$, provided that $\widehat{\phi}_n$ decays exponentially fast $|\widehat{\phi}_n| \le e^{-\xi|n|}$, see [17]. If the decay is too slow the above argument does not provide any gain for the non resonant terms. This is the case of Fibonacci potential considered in [18],[19], in which the Fourier coefficients decay only as $O(n^{-1})$.

V. RENORMALIZED EXPANSION

In the previous section we have identified the dangerous terms producing infrared divergences in thermodynamic limit; such divergences has to be removed in order to get physical informations from the expansions. We have to set-up a different integration procedure in which the resonant terms which are dimensionally relevant or marginal are *renormalized*; in this way one produces an expansion in terms of running coupling constants in which no infrared divergences are present. Of

particular importance are the quadratic resonant terms with $\rho_1 = -\rho_2$, corresponding to the generation of a gap. Note that only when $n_F = 1$ the initial interaction V contains such terms, but if $n_F > 1$ they are generated by higher order terms in the RG iterations. It is convenient then to add to the effective action a term of the form

$$\sigma \sum_\rho \int d\mathbf{k}' \sigma \psi^+_{\mathbf{k}',\rho} \psi^-_{\mathbf{k}',-\rho} - \alpha \sum_\rho \int d\mathbf{k}' \psi^+_{\mathbf{k}',\rho} \psi^-_{\mathbf{k}',-\rho} \quad (19)$$

and include the first term in the free integration so that the propagator becomes massive; α is chosen so that the flow of the resonant quadratic terms is bounded. At the end we impose the condition $\alpha(\lambda, U, \sigma) = \sigma$ determining $\sigma(\lambda, U)$, so proving the generation of the gap in the original problem. In the case $n_F = 1$ this is of course not necessary and we consider the $n_F > 1$ case for definiteness. We describe the integration procedure iteratively. Assume that we have integrated $\psi^{(0)}, \psi^{(-1)}, ..., \psi^{(h+1)}$ obtaining

$$e^{\mathcal{W}(0,0)} = \int P_{Z_h,\sigma_h}(d\psi^{(\le h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{\le h})} \quad (20)$$

where

$$g^{(\le h)}_{\rho,\rho'}(\mathbf{x}-\mathbf{y}) = \frac{1}{Z_h} \int d\mathbf{k} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \chi_h(\mathbf{k}) \times \quad (21)$$
$$\begin{pmatrix} -ik_0 + v_F \sin k' + c(k') & \sigma_h \\ \sigma_h & -ik_0 - v_F \sin k' + c(k') \end{pmatrix}^{-1}_{\rho,\rho'}$$

and $\mathcal{V}^{(h)}$ is a sum of monomials with kernels $W^{(h)}_n$ which are expressed as sum of renormalized Feynman diagrams associated to the running coupling constants $U_k, \delta_k, \nu_k, \alpha_k,\ k > h$ or to the non resonant terms λ present in $\mathcal{V}$; they depend also from σ_k, Z_k through the propagators. The single scale propagator is equal to (15) up to terms with the same scaling properties and an extra $\sigma_h \gamma^{-h}$ in the non diagonal component. We have to extract from the effective potential $\mathcal{V}^{(h)}$ the non irrelevant part which is called $\mathcal{L}\mathcal{V}^{(h)}$; that is we write $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$ with $\mathcal{R} = 1 - \mathcal{L}$. $\mathcal{L}$ acts non trivially only on the resonant terms. In particular $\mathcal{R}$ renormalize the non irrelevant subgraphs eliminating the infrared divergences. The presence of a mass has the effect that we integrate up a mass scale h^* defined by the condition $\gamma^{h^*} = \sigma_{h^*}$; we will see that $\sigma_h \sim \sigma_0 \gamma^{\eta_\mu h}$ with $\eta_\mu = O(U)$. The scales $\le h^*$ can be integrated in a single step.

The bilinear terms have scaling dimension 1 and we have to define an $\mathcal{R}$ operation such that their dimension becomes negative. When $m = 2$, $\rho_1 = \rho_2$ we define

$$\mathcal{L} W^{(h)}_{\rho,\rho}(\mathbf{k}') = W^{(h)}_{\rho,\rho}(0) + \mathbf{k}' \partial W^{(h)}_{\rho,\rho}(0) \quad (22)$$

and $\mathbf{k}'^2 \partial^2 W^{(h)}_{\rho,\rho}(0)$ has an extra $\gamma^{2(h_{v'} - h_v)}$ which is sufficient to make the sum over h_v convergent. The term $W^{(h)}_{\rho,\rho}(0)$ contribute to the running coupling constant ν_h, the term $\partial_0 W^{(h)}_{\rho,\rho}(0)$ to the wave function renormalization Z_h and $\partial_1 W^{(h)}_{\rho,\rho}(0) - \partial_0 W^{(h)}_{\rho,\rho}(0)$ to the Fermi velocity renormalization.

Regarding the bilinear terms with $\rho_1 = -\rho_2$ we write $\mathcal{L} W^{(h)}_{\rho,-\rho}(\mathbf{k}') = W^{(h)}_{\rho,-\rho}(0)$ and $W^{(h)}_{\rho,-\rho}(\mathbf{k}') - W^{(h)}_{\rho,-\rho}(0)$ has, in addition to a factor $\gamma^{h_{v'} - h_v}$, an extra $\frac{\sigma_{h_v}}{\gamma^{h_v}} \le \frac{\sigma_{h_v}}{\sigma_{h^*}} \frac{\sigma_{h^*}}{\gamma^{h_v}} \le \gamma^{\frac{1}{2}(h_{v'} - h_v)}$, or an extra $\alpha_h \sim \sigma_0 \le \gamma^{\frac{h^*}{2}} \le \gamma^{\frac{1}{2}(h_{v'} - h_v)}$ or $2n_F$ λ vertex. We write $W^{(h)}_{\rho,-\rho} = W^{(h)}_{a,\rho,-\rho} + W^{(h)}_{b,\rho,-\rho}$, where in $W^{(h)}_{a,\rho,-\rho}$ there are no graphs obtained by contraction of a λ term in $\mathcal{R}\mathcal{V}^0$ (hence there is necessarily at least a σ_k or a α_k) and $W^{(h)}_b$ is the rest and by definition $\mathcal{L} W^{(h)}_{a,\rho,-\rho}(\mathbf{k})$ contribute to σ_h while $\mathcal{L} W^{(h)}_{b,\rho,-\rho}(\mathbf{k})$ to α_h.

Finally for the kernels with $m = 4$ and $\sum_i \rho_i \varepsilon_i = 0$ $\mathcal{L} W^h_4(\underline{\mathbf{k}}) = W^h_4(0)$ which is included in the renormalization of λ_h. In conclusion with the above decomposition of the effective potential and after a redefinition of the effective wave function renormalization and gap , (20) is equal to

$$\int P_{Z_{h-1},\sigma_{h-1}}(d\psi^{\le h}) e^{-\bar{\mathcal{L}}\mathcal{V}^h(\sqrt{Z_{h-1}}\psi^{\le h}) - \mathcal{R}(\sqrt{Z_{h-1}}\psi^{\le h})} \quad (23)$$

with

$$\bar{\mathcal{L}}\mathcal{V}^h = \gamma^h \nu_h F^h_\nu + \gamma^h \alpha_h F^h_\alpha + \delta_h F^h_\delta + U_h F_U \quad (24)$$

where

$$F_U = \int \prod_{i=1}^4 d\mathbf{k}'_i \psi^+_{\mathbf{k}'_1,+} \psi^-_{\mathbf{k}'_2,+} \psi^+_{\mathbf{k}'_3,-} \psi^-_{\mathbf{k}'_4,-} \delta(\sum_i \sigma_i \mathbf{k}'_i) \quad (25)$$

$$F^h_\nu = \sum_\rho \int d\mathbf{k}' \psi^+_{\mathbf{k}',\rho} \psi^-_{\mathbf{k}',\rho} \quad F^h_\alpha = \sum_\rho \int d\mathbf{k}' \psi^+_{\mathbf{k}',\rho} \psi^-_{\mathbf{k}',-\rho}$$

$$F^h_\delta = \sum_\rho \int d\mathbf{k}' \rho v_F \sin k' \psi^+_{\mathbf{k}',\rho} \psi^-_{\mathbf{k}',\rho} \quad (26)$$

The terms $\int d\mathbf{k}' \psi^+_{\mathbf{k}',\rho} \psi^+_{\mathbf{k}',-\rho}$ and $\int d\mathbf{k}' (-ik_0 + v_F \rho \sin k') \psi^+_{\mathbf{k}',\rho} \psi^+_{\mathbf{k}',\rho}$ have been included in the free integration to produce the renormalization of σ_h and Z_h. In conclusion we write $P_{Z_{h-1},\sigma_{h-1}}(d\psi^{\le h-1}) = P_{Z_{h-1},\sigma_{h-1}}(d\psi^{\le h-1}) P_{Z_{h-1},\sigma_{h-1}}(d\psi^{(h)})$ and we can integrate the field $\psi^{(h)}$ obtaining an expression similar to (20) from which the procedure can be iterated. A similar analysis can be repeated in presence of the source term $\int d\mathbf{k}'_1 d\mathbf{k}'_2 W_{2,1}(\mathbf{k}'_1, \mathbf{k}'_2) J \psi^+_{\rho_1,\mathbf{k}'_1} \psi^+_{\rho_1,\mathbf{k}'_2}$; in that case $\mathcal{L} W^{(h)}_{2,1}(\mathbf{k}'_1, \mathbf{k}'_2) = W^{(h)}_{2,1}(0,0)$, which is included in the non oscillating or oscillating renormalization of the density $Z^{(1)}_h$ or Z^+_h depending if $\rho_1 = \rho_2$ or $\rho_1 = -\rho_2$.

The outcome of the above procedure is that the correlations can be expressed by renormalized diagrams functions of the running coupling constants and such that the renormalization $\mathcal{R}$ acts on the resonant clusters eliminating the infrared divergences; the sum over the scales can be done and one gets a finite result provided that the running coupling constants remain inside the convergence

radius; in order to verify this we have to study the flow of the running coupling constants. Note that the renormalization produces extra derivatives in momentum space, corresponding to coordinates to extra factors $(x-y)$ in the contribution to the kernel with four external field or $(x-y)^2$ in the contributions with two lines; note that in the second case one can always find a path of propagators connecting the external lines, avoiding the interaction line; therefore the condition $\sum_x |x||v(x)| < \infty$ is sufficient or having convergence.

VI. THE FLOW OF THE RUNNING COUPLING CONSTANTS

The expansion described in the previous section is convergent provided that the effective couplings are small. The flow of the relevant couplings α_h, ν_h is controlled by choosing properly α, ν and one gets

$$\alpha_h \sim \lambda^{2n_F}\gamma^h \quad \nu_h \sim (\lambda + |U|)\gamma^h \tag{27}$$

and that U_h, δ_h remain close to their initial value U_0, δ_0. To prove (27) we note that the flow equation for ν_h is

$$\nu_{h-1} = \gamma^{-h}(\nu_0 + \sum_{k=0}^{h} \gamma^k \beta_\nu^{(k)}) \tag{28}$$

and by choosing ν_0 so that $\nu_0 + \sum_{k=h^*}^{0} \gamma^k \beta_\nu^{(k)} = 0$ one gets

$$\nu_{h-1} = -\gamma^{-h} \sum_{k=h^*}^{h} \gamma^k \beta_\nu^{(k)} \tag{29}$$

By construction to $\beta_\nu^{(k)}$ contribute: a)terms depending only from the running coupling constants U, δ; in this case the contributions containing only propagators $g_L^{(k)}$ give a vanishing contributions, in the others there is an extra γ^h; b)terms containing at least α_h, ν_h which are $\sim \gamma^h$. Using (29) we get the second of (27). Similarly we write

$$\alpha_{h-1} = \gamma^{-h}(\alpha_0 + \sum_{k=0}^{h} \gamma^k \beta_\alpha^{(k)}) \tag{30}$$

By choosing $(\alpha_0 - \sum_k \gamma^k \beta_\alpha^{(k)}) = 0$ we get

$$\alpha_{h-1} = -\gamma^{-h} \sum_{k=h^*}^{h} \gamma^k \beta_\alpha^{(k)} \tag{31}$$

By construction $\beta_\alpha^{(k)} = O(\gamma^h)$ as they are obtained at least contracting a λ-term from $\mathcal{RV}^0$ (and the contraction happens at some finite scale by the compact support of propagators) so that $\alpha_h \sim \lambda^{2n_F}\gamma^h$. The lowest order contribution to $\int d\mathbf{k} W^h(\mathbf{k}')\psi^-_{\mathbf{k}',+}\psi^-_{\mathbf{k}',-}$ is obtained by the chain graph with $2n_F$ λ vertices and propagators carrying momentum $k' + (n_F-1)\pi\omega, ..., k' - (n_F+1)\pi\omega$, and the corresponding contribution to α_0, obtained setting $\mathbf{k}' = 0$, is $a_{n_F}\lambda^{2n_F}$ with, $n_F > 1$

$$a_{n_F} = \prod_{k=1}^{2n_F-1} \frac{1}{\cos(\pi\omega(n_F-k)) - \cos n_F\pi\omega} \tag{32}$$

which is non vanishing. Regarding the higher order terms the are at least $O(\lambda^{2n_F+1})$ or $O(U\lambda^{2n_F})$; by imposing $\sigma_0 = \alpha_0$ we get

$$\sigma_0 = \lambda^{2n_F}(a_{n_F} + \lambda F_1(\lambda) + UF_2(U,\lambda)) \tag{33}$$

with F_1, F_2 bounded; therefore the term a_{n_F} dominates if U, λ are small enough. Regarding the flow for U_h, δ_h again we decompose the beta function in a part depending only from U_h, δ_h and propagators $g_L^{(k)}$, see (15), and a rest which is $\sim \gamma^k$ (as there is a $r^{(h)}$ (15) or a α_h, ν_h, or irrelevant terms). This second part is summable while the first coincides with the Luttinger model one and is asymptotically vanishing [21] so that

$$U_h \to_{h\to-\infty} U_{-\infty}(U) = U + O(U^2) \tag{34}$$

and similarly $\delta_h \to_{\to-\infty} \delta_{-\infty}(U)$. The beta function for σ_h can be divided in a part containing only σ_k, U_k, δ_k and a rest which is $\sim \gamma^h$ (as there is a $r^{(h)}$ (15) or a ν_h, α_h, or irrelevant terms), so that

$$\frac{\sigma_{h-1}}{\sigma_h} = 1 - \frac{1}{2\pi v_F} U_h + \widehat{\beta}_\sigma^h \tag{35}$$

with $\widehat{\beta}_\sigma^h = O(U_h^2) + O((\lambda+U)\gamma^h)$ Therefore

$$\sigma_h \sim \sigma_0 \gamma^{\eta_\mu h} \qquad \eta_\mu = -\frac{U_{-\infty}}{2\pi v_F} + ... \tag{36}$$

where the exponent η_μ takes contribution only from g_L depends only from $U_{-\infty}, \delta_{-\infty}$, that is is an universal expression as function of $U_{-\infty}, \delta_{-\infty}$. Similarly the wave function and density renormalizations behaves as

$$Z_h \sim \gamma^{\eta_z h} \quad Z_h^{(1)} \sim \gamma^{\eta_z h} \quad Z_h^{(+)} \sim \gamma^{\eta_+ h} \tag{37}$$

where η_z, η_+ takes no contribution contribution only from g_L and $U_{-\infty}, \delta_{-\infty}$; again the contribution with $\alpha > k, \nu_k$ or from $r^{(h)}$ (15) are $O(\gamma^h)$.

VII. GAP RENORMALIZATION AND SCALING RELATIONS

The Renormalization Group is iterated up to a scale h^* such that $\sigma_{h^*} \sim \gamma^{h^*}$; as $\sigma_h = \sigma_0\gamma^{\eta_\mu h}$ then $\gamma^{h^*} = (\sigma_0)^{\frac{1}{1-\eta_\mu}}$. All the scales $< h^*$ can be integrated in a single step, as the scaling properties of the propagator $g^{<h^*}(\mathbf{x})$ are the same as $g^{(k)}(\mathbf{x})$ for $k \geq h^*$. As a consequence, the 2-point propagator decays faster that any power with rate $\gamma^{h^*} = (\sigma_0)^{\frac{1}{1-\eta_\mu}}$; this provides an estimate of the gap

$$\Delta_{n,U} \sim (\Delta_{n,0} + F_{n,U})^{X_n} \tag{38}$$

with $X_n = \frac{1}{1-\eta_\mu}$, $\Delta_{n,U}$ is the gap in the non interacting case and $F_{n,u,U}$ is of order $U\lambda^{2n_F}$. In conclusion when $p_F = n\pi\omega$ the 2-point function decays for large distances as

$$| < \psi^-_{\mathbf{x}} \psi^+_{\mathbf{y}} > | \leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{1+\eta_z}} \frac{C_N}{(\Delta_{n,U}|\mathbf{x}-\mathbf{y}|)^N} \tag{39}$$

for any integer N. Similarly the density-density correlations can be written as

$$< \rho_{\mathbf{x}} \rho_{\mathbf{y}} >= G_1(\mathbf{x},\mathbf{y}) + \sin p_F(x-y) G_2(\mathbf{x},\mathbf{y}) + G_3(\mathbf{x},\mathbf{y}) \tag{40}$$

where

$$|G_2(\mathbf{x},\mathbf{y})| \leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{2K_n}} \frac{1}{(\Delta_{n,U}|\mathbf{x}-\mathbf{y}|)^N} \tag{41}$$

with $2K_n = 2(1+\eta_+ - \eta_z)$; G_1 and G_3 verify similar bounds with K_n replaced by 1 and 3/2 respectively. By construction $\eta_\mu = \eta_+ - \eta_z$ and using that $X_n = \frac{1}{1-\eta_\mu}$ we get finally

$$X_n = \frac{1}{2-K_n} \tag{42}$$

and $\eta_z = \frac{2-K_n-K_n^{-1}}{2}$ follows from the fact that η_z, K_n are λ-independent functions of $U_{-\infty}, \delta_{-\infty}$ [21].

VIII. CONCLUSIONS

We have considered the Aubry-André model for spinless fermions with a non local many body interaction. In the non interacting case there is an insulating behavior at zero temperature; with certain quasi-random disorders is believed that the interaction can generate a metallic behavior, but our analysis excludes this in the case of Aubry-André disorder; insulating behavior persists even in presence of interaction. We have shown that, even if most of the infinitely many gaps in the single particle spectrum are infinitesimal, the interaction do not close any of them and there is no quantum phase transition from an insulating to a metallic phase; the interaction however has the effect of strongly decreasing or increasing the gap amplitude depending on its attractive or repulsive nature. Such behavior follows crucially from the irrelevance of all the processes involving an high momentum exchange, which is consequence of number theoretical properties of the frequency appearing in the Aubry-André potential. There is only a small number of running coupling constants, describing marginal or relevant terms, and their flow implies that the gaps are renormalized through critical exponents but cannot be closed. This result is consistent with numerical simulations on the interacting Aubry-André model, in which results are in agreement with a flow equation , see eq 3 of [13] , which essentially coincides with (35) truncated at second order; in particular the beta function for the effective many body interaction is essentially vanishing and the quasi-random disorder produces only a single running coupling constant. In the case of other quasi-periodic potentials a rather different behavior has been proposed [18],[19]: namely a quantum phase transition to a metal is expected for any repulsive U, as a consequence of the relevance of all the process involving large momentum exchange. This indicates that the interplay of interaction with quasi-random disorder depends critically on the decay properties of the Fourier transform of the noise.

The system (1) can experimentally realized in cold atoms experiments by trapping particles with strong dipolar momentum [22], [23], [24], and the amplitude and the form of quasi-random disorder together with the sign of the interaction can be tuned. In such systems a different behavior is therefore predicted depending on the nature of the quasi-random noise; the system is insulating for Aubry-André disorder (and the gaps are increased or decreased depending on the sign of the interaction) while a metallic behavior is expected with disorder with a slow decaying Fourier transform. At strong disorder, a localized behavior is instead expected.

[1] M. Schreiber, S. Hodgman, P. Bordia, H. Luschen, M. Fischer, R. Vosk, E. Altman, U. Schneider, I. Bloch, Science 349, 842 (2015)

[2] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, M. Inguscio, Nature 453, 895 (2008).

[3] C. D Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio, G. Modugno, Phys. Rev. Lett. 113, 095301 (2014)

[4] S. Aubry, G. André, Ann. Israel Phys. Soc. 3, 133 (1980).

[5] D. Schechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. 53, 1951 (1984).

[6] V. Oganesyan, D. A. Huse, Phys. Rev. B 75, 155111 (2007)

[7] J. Fröhlich, T. Spencer, T. Wittwer, Comm. Math. Phys. 88, 151 (1983)

[8] A. Avila, S. Jitomirskaya, Ann. of. Math. 170, 303 (2009)

[9] K. He, I. I. Satija, C. W. Clark, A. M. Rey, and M. Rigol, Physical Review A 85, 013617 (2012)

[10] S. Iyer, V. Oganesyan, G. Refael, D. A. Huse, Phys. Rev. B 87, 134202 (2013)

[11] V.P. Michal, B.L. Altshuler, G.V. Shlyapnikov Phys. Rev. Lett. 113, 045304 (2014)

[12] Y. E. Kraus, O. Zilberberg and R. Berkovits, Phys. Rev. B 89, 161106(R) (2014).

[13] C. Schuster, R. A. Römer, M. Schreiber, Phys. Rev. B, 65, 115114 (2002).

[14] T. Roscilde, Phys. Rev. A 77, 063605 (2008)

[15] G. Roux, T. Barthel, I. P. McCulloch, C. Kollath, U. Schollwoeck, T. Giamarchi Phys. Rev. A 78, 023628 (2008)
[16] V. Mastropietro Phys. Rev. Lett. 115, 180401 (2015); Comm. Math. Phys. 342, 1, 217 (2016)
[17] V. Mastropietro, Comm. Math. Phys. 201, 81 (1999)
[18] G. Vidal, D. Mouhanna, T. Giamarchi, Phys, Rev. Lett. 83, 3908 (1999)
[19] G. Vidal, D. Mouhanna, T. Giamarchi, Physical Review B, vol. 65, Issue 1, id. 014201 (2002)
[20] C. Sire and R. Mosseri, J. Phys. France, 50 (1989), 34473461
[21] G. Benfatto, P. Falco, V Mastropietro Phys. Rev. Lett. 104. 075701 (2010); Comm. Math. Phys. 330, 1, 217282 (2014)
[22] M. A. Baranov, Phys. Rep. 464, 71 (2008).
[23] L.D. Carr, M. DeMille, R.V. Krems, J. Ye New J. of Physics, 11, 5, 055049 (2009)
[24] Yi-Ping Huang, Daw-Wei Wang, Phys. Rev. A 80, 053610 (2009).