Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$

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Abstract

We study the following singularly perturbed nonlocal Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-2}\Big[\frac{1}{|x|^{\mu}} * F(u)\Big]f(u) \quad \text{in} \quad \mathbb{R}^2,$$

where $V(x)$ is a continuous real function on $\mathbb{R}^2$, $F(s)$ is the primitive of $f(s)$, $0 < \mu < 2$ and ε is a positive parameter. Assuming that the nonlinearity $f(s)$ has critical exponential growth in the sense of Trudinger-Moser, we establish the existence and concentration of solutions by variational methods.

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1 Introduction and main results

The nonlocal elliptic equation

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{\mu-N}\Big[\frac{1}{|x|^{\mu}}*F(u)\Big]f(u)\quad\text{in}\quad\mathbb{R}^N,\qquad (SNS)$$

the so-called Choquard equation when $N=3$, appears in the theory of Bose-Einstein condensation and is used to describe the finite-range many-body interactions between particles. Here $V(x)$ is the external potential, $F(s)$ is the primitive of the nonlinearity $f(s)$ and the parameters $\varepsilon>0$, $0<\mu<N$. For $\mu=1$ and $F(s)=\frac{1}{2}|s|^2$, equation (SNS) was investigated by S.I. Pekar in [42] to study the quantum theory of a polaron at rest. In [28] P. Choquard suggested to use it as an approximation to the Hartree-Fock theory of one-component plasma. This equation was also proposed by R. Penrose in [36] as a model for self-gravitating particles and in that context it is known as the Schrödinger-Newton equation.

Notice that if u is a solution of the nonlocal equation (SNS) and $x_0\in\mathbb{R}^N$, then the function $v=u(x_0+\varepsilon x)$ satisfies

$$-\Delta v+V(x_0+\varepsilon x)v=\Big[\frac{1}{|x|^{\mu}}*F(v)\Big]f(v)\quad\text{in}\quad\mathbb{R}^N.$$

This suggests some convergence, as $\varepsilon\to0$, of the family of solutions of (SNS) to a solution u_0 of the limit problem

$$-\Delta v+V(x_0)v=\Big[\frac{1}{|x|^{\mu}}*F(v)\Big]f(v)\quad\text{in}\quad\mathbb{R}^N.\qquad(1.1)$$

This is known as semi-classical limit for the nonlocal Choquard equation and we refer to [8, 9] for a survey on this topic. The study of semiclassical states for the Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=g(u)\quad\text{in}\quad\mathbb{R}^N,\qquad(1.2)$$

S.S

goes back to the pioneering work [24] by Floer and Weinstein. Since then, it has been studied extensively under various hypotheses on the potential and the nonlinearity, see for example [7, 16, 17, 24, 25, 26, 43, 44, 46, 48] and the references therein. In the study of semiclassical problems for local Schrödinger equations, the Lyapunov-Schmidt reduction method has been proved to be one of the most powerful tools. However, this technique relies greatly on the uniqueness and non-degeneracy of the ground states of the limit problem which is not completely settled for the ground states of the nonlocal Choquard equation

$$-\Delta u+u=\Big[\frac{1}{|x|^{\mu}}*F(u)\Big]f(u)\ \text{ in }\mathbb{R}^N.\qquad(1.3)$$

CC

In [15, 33, 37], the authors investigated the qualitative properties of solutions and established the regularity, positivity, radial symmetry and decaying behavior at infinity. Moroz and Van Schaftingen in [38] established the existence of ground states with Berestycki-Lions type general nonlinearity. For $N=3$, $\mu=1$ and $F(s)=\frac{1}{2}|s|^2$, by proving the uniqueness and non-degeneracy of the ground states, Wei and Winter [47] constructed a family of solutions by a Lyapunov-Schmidt type reduction. In presence of non-constant electric and magnetic potentials, Cingolani et.al. [14] showed the existence of family of solutions concentrating at regions localized by the minima of the potential. Moroz and Van Schaftingen [39] developed a nonlocal penalization technique to show that the equation (SNS) has a family of solutions concentrating at the local minimum of V provided V satisfies some additional assumptions at infinity. In [51], Yang and Ding considered the following equation

$$-\varepsilon^2\Delta u + V(x)u = \Big[\frac{1}{|x|^\mu} * u^p\Big]u^{p-1}, \quad \text{in} \quad \mathbb{R}^3.$$

By using variational methods, for suitable parameters p, μ, the authors obtained the existence of solutions. In [5], Alves and Yang proved the existence, multiplicity and concentration of solutions by penalization methods and Lusternik-Schnirelmann theory.

Let us recall the Hardy-Littlewood-Sobolev inequality, see [27], which will be frequently used throughout this paper:

HLS **Proposition 1.1** (Hardy-Littlewood-Sobolev inequality). *Let $s, r > 1$ and $0 < \mu < N$ with $1/s+\mu/N+1/r = 2$, $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(s, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N} [\frac{1}{|x|^\mu} * f(x)]h(x) \leq C(s, N, \mu, r)|f|_s|h|_r.$$

Applying the Hardy-Littlewood-Sobolev inequality, we know

$$\int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(u)\Big]F(u)$$

is well defined if $F(u) \in L^s(\mathbb{R}^N)$ for $s > 1$ given by

$$\frac{2}{s} + \frac{\mu}{N} = 2.$$

This means that we must require

$$F(u) \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

Assume that $F(u) = |u|^p$ and $N \geq 3$, to preserve the variational structure, the Sobolev Embedding Theorem implies that the exponent p must satisfy

$$\frac{2N-\mu}{N} \leq p \leq \frac{2N-\mu}{N-2}.$$

The confining exponents above play the role of critical exponents for the nonlocal Choquard equation in $\mathbb{R}^N$, $N \geq 3$. To the authors' best knowledge, most of the works afore mentioned are set in $\mathbb{R}^N$, $N \geq 3$ with non-critical growth nonlinearities. Except for the case of the lower-critical exponent considered in [40], there is no results available on the existence and concentration of solutions for the nonlocal Choquard equation with upper-critical exponent $\frac{2N-\mu}{N-2}$.

The case $N = 2$ is very special, as for bounded domains $\Omega \subset \mathbb{R}^2$ the corresponding Sobolev embedding yields $H^1_0(\Omega) \subset L^q(\Omega)$ for all $q \geq 1$, but $H^1_0(\Omega) \nsubseteq L^\infty(\Omega)$. For dimension $N = 2$, the Pohozaev-Trudinger-Moser inequality [34, 45] can be treated as a substitute of the Sobolev inequality as it establishes the following sharp maximal exponential integrability for functions in $H_0^1(\Omega)$:

$$\sup_{u\in H_0^1(\Omega)\,:\,\|\nabla u\|_2\le 1}\int_\Omega e^{\alpha u^2}\le C|\Omega|\quad \text{if } \alpha\le 4\pi,$$

for a positive constant which depends only on α, where $|\Omega|$ denotes Lebesgue measure of Ω. As a consequence we say that a function $f(s)$ has *critical exponential growth* if there exists $\alpha_0>0$ such that

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=0,\quad \forall\alpha>\alpha_0,\quad \text{and}\quad \lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=+\infty,\quad \forall\alpha<\alpha_0. \tag{1.4}$$

ecg

This definition of criticality was introduced by Adimurthi and Yadava [3], see also de Figueiredo, Miyagaki and Ruf [18]. The first version of the Pohozaev-Trundiger-Moser inequality in $\mathbb{R}^2$ was established by Cao in [12], see also [2, 41, 13], and reads as follows

linger-Moser **Lemma 1.2.** *If $\alpha>0$ and $u\in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2}\left[e^{\alpha|u|^2}-1\right]<\infty. \tag{1.5}$$

TM1

Moreover, if $|\nabla u|_2^2\le 1$, $|u|_2\le M<\infty$, and $\alpha<\alpha_0=4\pi$, then there exists a constant C, which depends only on M and α, such that

$$\int_{\mathbb{R}^2}\left[e^{\alpha|u|^2}-1\right]\le C(M,\alpha). \tag{1.6}$$

TM2

We refer the reader to [3, 30] for related problems and [13, 31, 52] for recent advances on this topic. Actually just a few papers deal with semiclassical states for local Schrödinger equations with critical exponential growth. In [19], do Ó and Souto proved the existence of solutions concentrating around local minima of of $V(x)$ which are not necessarily nondegenerate. For N-Laplacian equation in $\mathbb{R}^N$, Alves and Figueiredo [4] studied the multiplicity of semiclassical solutions with Rabinowitz type assumption on the potential. Recently, do Ó and Severo [20] and do Ó, Moameni and Severo [21] also studied a class of quasilinear Schrödinger equations in $\mathbb{R}^2$ with critical exponential growth.

Hence it is quite natural to wonder if the existence and concentration results for local Schrödinger equations still hold for the nonlocal equation with critical growth in the sense of Pohozaev-Trudinger-Moser inequality. The purpose of this paper is two-fold: on the one hand we study the existence of nontrivial solution for the critical nonlocal equation with periodic potential, namely we consider the equation

$$-\Delta u+W(x)u=\Big(\frac{1}{|x|^\mu}*F(u)\Big)f(u),\quad \text{in } \mathbb{R}^2. \tag{1.7}$$

A1

Assume for the potential the following conditions:

(W_1) $W(x)\ge W_0>0$ in $\mathbb{R}^2$ for some $W_0>0$;

(W_2) $W(x)$ is a 1-periodic continuous function.

and for the nonlinearity f the following conditions:

(f_1) (i)$f(s) = 0 \quad \forall s \leq 0,\ 0 \leq f(s) \leq Ce^{4\pi s^2}, \quad s \geq 0;$
(ii) $\exists\, s_0 > 0, M_0 > 0,$ and $q \in (0, 1]$ such that $0 < s^q F(s) \leq M_0 f(s),\ \forall\, |s| \geq s_0$.

(f_2) There exists $p > \dfrac{2-\mu}{2}$ and $C_p > 0$ such that $f(s) \sim C_p s^p$, as $s \to 0$.

(f_3) There exists $K > 1$ such that $f(s)s > KF(s)$ for all $s > 0$, where $F(t) = \int_0^t f(s)ds$.

(f_4) $\displaystyle\lim_{s\to+\infty} \frac{sf(s)F(s)}{e^{8\pi s^2}} \geq \beta$, with $\displaystyle\beta > \inf_{\rho>0} \frac{e^{\frac{4-\mu}{4}V_0\rho^2}}{16\pi^2\rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}$.

Our first main result reads as follows,

m-Existence **Theorem 1.3.** *Assume* $0 < \mu < 2$, *suppose that the potential* V *satisfies* $(W_1) - (W_2)$ *and the nonlinearity* f *satisfies conditions* $(f_1) - (f_4)$. *Then equation* (1.7) *has a ground state solution in* $H^1(\mathbb{R}^2)$.

On the other hand, we establish the existence and concentration of semiclassical ground state solutions of the following equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-2}\Big[\frac{1}{|x|^\mu} * F(u)\Big]f(u) \ \text{ in } \mathbb{R}^2. \tag{1.8}$$ EC

Here we assume the following conditions on V:

(V_1) $V(x) \geq V_0 > 0$ in $\mathbb{R}^2$ for some $V_0 > 0$;

(V_2) $0 < \inf_{x\in\mathbb{R}^2} V(x) = V_0 < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty$.

The condition (V_2) was introduced by Rabinowitz in [46]. Hereafter, we will denote by

$$M = \{x \in \mathbb{R}^2 : V(x) = V_0\},$$

the minimum points set of $V(x)$.

We also assume that the nonlinearity satisfies the following

(f_5) $s \to f(s) \quad$ is strictly increasing on $(0, +\infty)$.

Then we prove our second main result.

T1 **Theorem 1.4.** *Suppose that the nonlinearity* $f(s)$ *satisfies* $(f_1)-(f_5)$ *and the potential function* $V(x)$ *satisfies assumptions* $(V_1) - (V_2)$. *Then, for any* $\varepsilon > 0$ *small, problem* (1.8) *has at least one positive ground state solution. Moreover, let* u_ε *denotes one of these positive solutions with* $\eta_\varepsilon \in \mathbb{R}^2$ *its global maximum, then*

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

Notation:
- C, C_i denote positive constants.
- B_R denotes the open ball centered at the origin with radius $R > 0$.
- $C_0^\infty(\mathbb{R}^2)$ denotes the space of the functions infinitely differentiable with compact support in $\mathbb{R}^2$.
- For a mensurable function u, we denote by u^+ and u^- its positive and negative parts respectively, given by

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

- $\| \ \|$ and $| \ |_s$ denote the usual norms of the spaces $H^1(\mathbb{R}^2)$ and $L^s(\mathbb{R}^2)$ respectively.
- Let E be a real Hilbert space and $I : E \to \mathbb{R}$ a functional of class $\mathcal{C}^1$. We say that $\{u_n\} \subset E$ is a Palais-Smale ((PS) for short) sequence at c for I if $\{u_n\}$ satisfies

$$I(u_n) \to c \text{ and } I'(u_n) \to 0, \text{ as } n \to \infty.$$

Moreover, I satisfies the (PS) condition at level c, if any (PS) sequence $\{u_n\}$ such that $I(u_n) \to c$ possesses a convergent subsequence.

2 A critical nonlocal equation with periodic potential: proof of Theorem 1.3

In [6], Alves and Yang studied equation (1.7) under hypothesis $(W1)$ and (W_2) for the potential and the following conditions on the nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$ of class $\mathcal{C}^1$:

$$f(0) = 0, \quad \lim_{s\to 0} f'(s) = 0. \tag{f_1'}$$

It is of critical growth at infinity with $\alpha_0 = 4\pi$. Moreover, there exists C_0 such that

$$|f'(s)| \leq \ C_0 e^{4\pi s^2}, \quad \forall s > 0. \tag{f_2'}$$

There exists $\theta > 2$ such that

$$0 < \theta F(s) \leq 2f(s)s, \quad \forall s > 0, \tag{f_3'}$$

Furthermore, they suppose that there exists $p > \frac{4-\mu}{2}$ such that

$$F(s) \geq C_p s^p, \quad \forall s > 0 \tag{f_4'}$$

where

$$C_p > \frac{\left[\frac{4\theta(p-1)}{(2-\mu)(\theta-2)}\right]^{\frac{p-1}{2}} S_p^p}{p^{\frac{p}{2}}}.$$

and

$$S_p = \inf_{u\in H^1(\mathbb{R}^2), u\neq 0} \frac{\left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + |W|_\infty |u|^2\right)\right)^{1/2}}{\left(\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * |u|^p\right] |u|^p\right)^{\frac{1}{2p}}}.$$

Combining the above estimates with the Hardy-Littlewood-Sobolev inequality and some results due to P.L. Lions, the following existence result was obtained in [6].

AQ1 **Theorem 2.1.** *Suppose that conditions* $(f_1') - (f_4')$ *hold. Then problem* (1.7) *has at least one ground state solution* w.

A key tool in [6] is assumption (f_4') which enables one to obtain estimates of the Mountain-Pass level for the energy functional related to the nonlocal Choquard equation, for $0 < \mu < 2$,

$$\begin{cases} -\Delta u + W(x)u = \Big(\frac{1}{|x|^\mu} * F(u)\Big) f(u), & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2) \\ u(x) > 0 \text{ for all } x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$ A

Condition (f_4') involves the explicit value of the best constant of the embedding $H^1 \hookrightarrow L^p$, $p \in (2, \infty)$, which is so far unknown and still an open challenging problem. In terms of the nonlinear source, condition (f_4') prescribe a global growth which can not be actually verified. This somehow affects possible further applications. The aim of this section is to overcome condition (f_4') which we replace with the assumption (f_4). For this purpose, we set

$$W_\rho := \sup_{|x| \le \rho} W(x)$$

and

$$\mathcal{W} := \inf_{\rho > 0} \frac{e^{\frac{4-\mu}{4} W_\rho \rho^2}}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.$$

Notice that if $W(x)$ is continuous and (W_2) is satisfied, then W_ρ is a positive continuous function and $\mathcal{W}$ can be attained by some $\rho > 0$. Moreover, it is worth to point out that assumption $(f_1) - (ii)$ implies that for any $\eta > 0$ there exists $C_\eta > 0$ and s_η such that for all $s \ge s_\eta$

$$\eta f(s) \ge F(s) \tag{2.2}$$ ARcond

and as s is large enough

$$F(s) \ge C_\eta e^{s^{q+1}}.$$

On the other hand, $(f_1) - (ii)$ implies, for some $\gamma > 0$,

$$F(s) \le e^{\gamma s^2} - 1, \qquad \text{for any } s > 0$$

which agrees with (f_2). Notice also that assumptions (f_2) and (f_3) yield

$$K > \frac{4-\mu}{2} > 1.$$

Assumption (f_4) is inspired by [18, 52], but here we have the extra difficulty to handle integrals where both the two nonlinearities $F(s)$ and $sf(s)$ appear simultaneously. This situation forces us to assume condition (f_4) which is sharper than the following assumption of [18]

$$\lim_{s \to +\infty} \frac{F(s)}{e^{4\pi s^2}} \ge \gamma. \tag{2.3}$$ f_4 bis

Actually, condition (2.3), combined with (2.2) implies

$$\lim_{s \to +\infty} \frac{sf(s)}{e^{4\pi s^2}} \ge \gamma \eta^{-1} \quad \text{for any } \eta > 0,$$

so that (f_4) is trivially satisfied for any choice of $\gamma > 0$. Finally, note that (f_4) together with (2.2) still imply

$$\lim_{s\to+\infty} \frac{sf(s)}{e^{4\pi s^2}} = +\infty,$$

but it may happen that

$$\lim_{s\to+\infty} \frac{F(s)}{e^{4\pi s^2}} = 0$$

in contrast with (2.3). This is the case, for instance, if

$$F(s) \sim \frac{e^{4\pi s^2}}{s} \text{ and } f(s) \sim 8\pi e^{4\pi s^2}, \quad s \to +\infty.$$

Since we are looking for positive solutions $u \geq 0$, from now on we assume $f(s) = 0$ for $s \leq 0$. The energy functional associated with problem (2.1) is given by

$$\Phi_W(u) = \frac{1}{2}\|u\|_W^2 - \mathfrak{F}(u),$$

where

$$\mathfrak{F}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(u)\Big] F(u)$$

and

$$\|u\|_W := \left(\int_{\mathbb{R}^2} |\nabla u|^2 + W(x)|u|^2\right)^{1/2}$$

Let E denote the space $H^1(\mathbb{R}^2)$ equipped with the norm $\|u\|_W$, which is equivalent to the standard Sobolev norm.

As a consequence of Cao's inequality in Lemma 1.2, (f_2) and Hölder's inequality we have $F(u) \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ (note that (f_2) is weaker then (f_1') of [6]), and the functional $\Phi_W(u)$ is $\mathcal{C}^1(E)$ thanks to a generalization of a Lions' result recently proved in [22]. Then the Mountain Pass geometry can be proved as in [6]. By the Ekeland Variational Principle [23], there exists a (PS) sequence $(u_n) \subset E \subset H^1(\mathbb{R}^2)$ such that

$$\Phi_W'(u_n) \to 0, \quad \Phi_W(u_n) \to m_W,$$

where the Mountain Pass e m_W can be characterized by

$$0 < m_W := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_W(\gamma(t)) \tag{2.4}$$ m

with

$$\Gamma := \left\{\gamma \in \mathcal{C}^1([0,1], E) : \gamma(0) = 0, \Phi_W(\gamma(1)) < 0\right\}.$$

el-estimate **Lemma 2.2.** *The mountain pass level m_W satisfies*

$$m_W < \frac{4-\mu}{8}.$$

Proof. It is enough to prove that there exists s a function $w \in E, \|w\|_W = 1$, such that

$$\max_{t \geq 0} \Phi_W(tw) < \frac{4-\mu}{8}.$$

Let us introduce the following Moser type functions supported in B_ρ by

$$\overline{w}_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq \frac{\rho}{n}, \\ \dfrac{\log(\rho/|x|)}{\sqrt{\log n}}, & \frac{\rho}{n} \leq |x| \leq \rho, \\ 0, & |x| \geq \rho. \end{cases}$$

One has that

$$\begin{aligned} \|\overline{w}_n\|_W^2 &= \int_{B_\rho} |\nabla \overline{w}_n|^2 + \int_{B_\rho} W(x)|\overline{w}_n|^2 \\ &\leq \int_{\rho/n}^{\rho} \frac{dr}{r \log n} dr + W_\rho \int_0^{\rho/n} \log n \, r dr + W_\rho \int_{\rho/n}^{\rho} \frac{\log^2(\rho/r)}{\log n} r dr \\ &= 1 + \delta_n, \end{aligned}$$

where

$$\delta_n = W_\rho \rho^2 \left[\frac{1}{4 \log n} - \frac{1}{4n^2 \log n} - \frac{1}{2n^2} \right] > 0. \tag{2.5}$$

And then, setting $w_n = \overline{w}_n / \sqrt{1+\delta_n}$, we get $\|w_n\|_W = 1$.

We claim that there exists n such that

$$\max_{t \geq 0} \Phi_W(tw_n) < \frac{4-\mu}{8}. \tag{2.6}$$ claim

Let us argue by contradiction and suppose this is not the case, so that for all n let $t_n > 0$ be such that

$$\max_{t \geq 0} \Phi_W(tw_n) = \Phi_W(t_n w_n) \geq \frac{4-\mu}{8}, \tag{2.7}$$ bycontr-assump

then t_n satisfies $\frac{d}{dt}\Phi_W(tw_n)|_{t=t_n} = 0$ and

$$t_n^2 = \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(t_n w_n) \right] t_n w_n f(t_n w_n), \tag{2.8}$$ t_n^2=

it follows from (2.7) that

$$t_n^2 \geq \frac{4-\mu}{4}. \tag{2.9}$$ est-t_n^2

Let us estimate from below the quantity t_n^2. Taking advantage of equation (2.8), thanks to (f_4) we have for any $\varepsilon > 0$,

$$sf(s)F(s) \geq (\beta - \varepsilon)e^{8\pi s^2} \quad \text{for all } s \geq s_\varepsilon \tag{2.10}$$ estimate-sfF

and thus

$$
\begin{aligned}
t_n^2 &\geq \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} F(t_n w_n)\, dx \\
&= \int_{B_{\rho/n}} t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}} f\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}}\right) dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} F\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}}\right) dx \\
&\geq (\beta-\varepsilon) e^{4t_n^2(1+\delta_n)^{-1}\log n} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu}\, dx.
\end{aligned}
$$

Notice that $B_{\rho/n-|x|}(0) \subset B_{\rho/n}(x)$ since $|x| \leq \rho/n$, the last integral can be estimated as follows

$$
\begin{aligned}
\int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{dx}{|x-y|^\mu} &= \int_{B_{\rho/n}} dx \int_{B_{\rho/n}(x)} \frac{dz}{|z|^\mu} \\
&\geq \int_{B_{\rho/n}} dx \int_{B_{\rho/n-|x|}} \frac{dz}{|z|^\mu} \\
&= \frac{2\pi}{2-\mu} \int_{B_{\rho/n}} \left(\frac{\rho}{n} - |x|\right)^{2-\mu} \\
&= \frac{4\pi^2}{2-\mu} \int_0^{\rho/n} \left(\frac{\rho}{n} - r\right)^{2-\mu} r dr \\
&= \frac{4\pi^2}{(2-\mu)(3-\mu)(4-\mu)} \left(\frac{\rho}{n}\right)^{4-\mu} \\
&= C_\mu \left(\frac{\rho}{n}\right)^{4-\mu},
\end{aligned}
\tag{2.11}
$$

where

$$
C_\mu = \frac{4\pi^2}{(2-\mu)(3-\mu)(4-\mu)}.
$$

Consequently, we obtain

$$
\begin{aligned}
t_n^2 &\geq \frac{4\pi^2(\beta-\varepsilon)}{(2-\mu)(3-\mu)(4-\mu)} e^{4t_n^2(1+\delta_n)^{-1}\log n} \left(\frac{\rho}{n}\right)^{4-\mu} \\
&= \frac{4\pi^2(\beta-\varepsilon)\rho^{4-\mu}}{(2-\mu)(3-\mu)(4-\mu)} e^{\log n[4(1+\delta_n)^{-1}t_n^2-(4-\mu)]}
\end{aligned}
$$

which, recalling (2.9), means that t_n is bounded and yields

$$
t_n^2 \longrightarrow \left(\frac{4-\mu}{4}\right)^+
$$

as n goes to infinity. Moreover, as a byproduct we also have that for some $C > 0$

$$
\log n[4(1+\delta_n)^{-1}t_n^2 - (4-\mu)] \leq C,
$$

that is

$$
\frac{t_n^2}{1+\delta_n} = \frac{4-\mu}{4} + \mathrm{O}\left(\frac{1}{\log n}\right). \tag{2.12}
$$

est-t_n^2-bis

This estimate will be used to obtain a finer estimate than (2.9). Notice first that by (f_1) and (f_2) we have

$$F(s) \le Cs^{\frac{4-\mu}{2}} + Mf(s) \le Cs^{\frac{4-\mu}{2}} + C(e^{4\pi s^2} - 1). \tag{2.13}$$

estimate-F

Next define

$$A_n = \{y \in B_\rho : t_n w_n(y) > s_\varepsilon\} \quad \text{and} \quad B_n = B_\rho \setminus A_n,$$

where s_ε was introduced in (2.10). By (2.10) we know

$$\begin{aligned} t_n^2 &= \int_{\mathbb{R}^2} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n)\, dy \\ &= \int_{B_\rho} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n)\, dy \\ &= \int_{A_n} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n)\, dy + \int_{B_n} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n). \end{aligned}$$

Combining the Hardy-Littlewood-Sobolev inequality with (2.13) one has

$$\begin{aligned} \int_{B_n} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n) &\le C\|F(t_n w_n)\|_{\frac{4}{4-\mu}} \|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}} \\ &\le \left[C\|t_n w_n\|_2 + C\left\{\int_{\mathbb{R}^2} e^{4\pi\frac{4}{4-\mu}t_n^2 w_n^2} - 1\right\}^{\frac{4-\mu}{4}}\right] \|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}}. \end{aligned} \tag{2.14}$$

finer-est1

By (2.12), since $\|\nabla \overline{w}_n\|_2 = 1$ and $\overline{w}_n^2 \le 2\pi \log n$, we obtain

$$\int_{\mathbb{R}^2} e^{4\pi\frac{4}{4-\mu}t_n^2 w_n^2} - 1 \;\le\; \int_{B_\rho} e^{4\pi\frac{4}{4-\mu}t_n^2 w_n^2} \;\le\; \int_{B_\rho} e^{4\pi(1+\frac{C}{\log n})\overline{w}_n^2} \;\le\; \int_{B_\rho} Ce^{4\pi\overline{w}_n^2} \;\le\; C,$$

due to the Pohozaev-Trudinger-Moser inequality. Since $t_n w_n \to 0$ a.e. and $t_n w_n$ is bounded on B_n, applying the Lebesgue dominated convergence theorem, we obtain

$$\|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}} \to 0,$$

as $n \to \infty$. Consequently,

$$t_n^2 = \int_{A_n} \left(\frac{1}{|x|^\mu} * F(t_n w_n)\right) t_n w_n f(t_n w_n)\, dy + \mathrm{o}(1), \tag{2.15}$$

est-t_n^2-tris

where $\mathrm{o}(1)$ is actually positive.

Buying the same lines we can estimate the convolution term as follows

$$\begin{aligned} t_n^2 &\ge \int_{A_n} t_n w_n f(t_n w_n)\, dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx + \int_{A_n} t_n w_n f(t_n w_n)\, dy \int_{B_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\ &\ge \int_{A_n} t_n w_n f(t_n w_n)\, dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx + \mathrm{o}(1). \end{aligned}$$

By the definition of w_n, we observe that

$$A_n = \{0 < |x| < \rho e^{-s_\varepsilon\sqrt{2\pi(1+\delta_n)}\sqrt{\log n}}\} \supset B_{\frac{\rho}{n}}, \tag{2.16}$$

calculA_n

then

$$
\begin{aligned}
t_n^2 &\geq \int_{A_n} t_n w_n f(t_n w_n)\, dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\
&\geq \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\
&\quad + \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\
&\quad + \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\
&\quad + \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} t_n w_n f(t_n w_n) dy \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} \frac{F(t_n w_n)}{|x-y|^\mu}\, dx \\
&:= I_1 + I_2 + I_3 + I_4 \\
&\geq I_1 \geq (\beta - \varepsilon) e^{8\pi t_n^2 w_n^2} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu}\, dx
\end{aligned} \tag{2.17}
$$

where we have used the fact that w_n is constant on the ball $B_{\rho/n}$. Thanks to (2.11) we have

$$
\begin{aligned}
I_1 &\geq (\beta - \varepsilon) e^{4t_n^2 (1+\delta_n)^{-1} \log n} \int_{|y| \leq \frac{\rho}{n}} dy \int_{|x| \leq \frac{\rho}{n}} \frac{1}{|x-y|^\mu}\, dx \\
&\geq (\beta - \varepsilon) C_\mu e^{4t_n^2 (1+\delta_n)^{-1} \log n} \left(\frac{\rho}{n}\right)^{4-\mu}
\end{aligned} \tag{2.18}
$$

and hence, recalling the definition of δ_n in (2.5), we also have

$$
\begin{aligned}
I_1 &\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{4t_n^2 (1+\delta_n)^{-1} \log n - (4-\mu) \log n} \\
&\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{(4-\mu) \log n [(1+\delta_n)^{-1} - 1]} \\
&\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-(4-\mu)\delta_n \log n} \\
&= (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-(4-\mu) W_\rho \rho^2 \left[\frac{1}{4} - \frac{1}{4n^2} - \frac{\log n}{2n^2}\right]} \\
&\to (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-\frac{4-\mu}{4} W_\rho \rho^2},
\end{aligned}
$$

as $n \to +\infty$. Combining the previous inequality with (2.17) and passing to the limit we get

$$
\frac{4-\mu}{4} \geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-\frac{4-\mu}{4} W_\rho \rho^2},
$$

and since ε is arbitrary, in turn

$$
\beta \leq \frac{4-\mu}{4 C_\mu \rho^{4-\mu}} e^{\frac{4-\mu}{4} W_\rho \rho^2} = \frac{e^{\frac{4-\mu}{4} W_\rho \rho^2}}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.
$$

However, by the definition of $\mathcal{W}$, since $\beta > \mathcal{W}$ and (f_4), there exists $\rho > 0$ such that

$$
\beta > \frac{e^{\frac{4-\mu}{4} W_\rho \rho^2}}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)} \tag{2.19}
$$

beta_below

which is a contradiction and this concludes the proof. □

Remark 2.3. *It is worth to mention that actually estimate* (2.17) *can be improved, in the sense that the constant* $\mathcal{W}$ *can be sharpened by exploiting* I_2*,* I_3 *and* I_4 *and some additional technical growth assumptions on* $f(s)$*, which we omit here since we do not bring to effective advantages in this context.*

In the spirit of [52] we next prove that the limit of a Palais-Smale sequence for Φ_V yields a weak solution to (2.1).

lem-PS **Lemma 2.4.** *Assume* $(W_1)-(W_2), (f_1)-(f_4)$ *and let* $\{u_n\} \subset E$ *be a Palais-Smale sequence for* Φ_W*, i.e.*

$$\Phi_W(u_n) \to c \quad \text{and} \quad \Phi'_W(u_n) \to 0 \quad \text{in } E^*, \quad \text{as } n \to +\infty.$$

Then there exists $u \in E$ *such that, up to subsequence,* $u_n \rightharpoonup u$ *weakly in* E*,*

$$\left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \to \left[\frac{1}{|x|^\mu} * F(u)\right] F(u), \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2) \tag{2.20}$$ convFF

and u *is a weak solution of* (2.1).

Proof. By hypothesis we have

$$\frac{1}{2}\|u_n\|^2_W - \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \to c \tag{2.21}$$ convPhi

as well as

$$\left|\int_{\mathbb{R}^2} \nabla u_n \nabla v + W u_n v - \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)v\right| \le \tau_n \|v\|_W$$

for all $v \in E$, where $\tau_n \to 0$ as $n \to +\infty$. Taking $v = u_n$ in (2.22) we obtain

$$\left|\|u_n\|^2_W - \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n)\right| \le \tau_n \|u_n\|_W. \tag{2.22}$$ convPhi''

By (f_1) that for any $s > 0$ one has $sf(s) \ge KF(s)$. Then,

$$\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n) \ge K \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)$$

and so

$$\frac{1}{2}\left(1 - \frac{1}{K}\right)\|u_n\|^2_W \le \Phi_W(u_n) - \frac{1}{2K}\langle \Phi'_W(u_n), u_n\rangle \le \frac{c}{2} + \frac{\tau_n}{2K}\|u_n\|_W$$

which implies that $\|u_n\|_W$ is bounded. As a consequence we have from (2.21) and (2.22) that

$$\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \le C, \quad \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n) \le C \tag{2.23}$$ bound

with C independent of n. Moreover, $u_n \rightharpoonup u$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^2)$ for any $1 \le q < \infty$ and $u_n \to u$ a.e. in $\mathbb{R}^2$.

Next let us prove (2.20), that is,

$$\left|\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)dx - \int_\Omega \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)dx\right| \to 0, \quad \forall\, \Omega \subset\subset \mathbb{R}^2$$

This can be done as in [18, Lemma 2.1]. Indeed, since $u \in H^1(\mathbb{R}^2)$, then $\left[\frac{1}{|x|^\mu} * F(u)\right] F(u) \in L^1(\mathbb{R}^2)$, so that

$$\lim_{M\to\infty} \int_{\{u\ge M\}} \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)dx = 0.$$

Let C be the constant in (2.23) and M_0 the constant in (f_1): for any $\delta > 0$ we can choose $M > \max\{(CM_0/\delta)^{q+1}, s_0\}$ such that

$$0 \le \int_{\{u\ge M\}} \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)dx < \delta.$$

From (2.23) and $(f_1)(ii)$ we also have

$$0 \le \int_{\{u_n\ge M\}} \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)dx \le \frac{M_0}{M^{q+1}} \int_{\{u_n\ge M\}} \left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n)dx < \delta,$$

then we obtain

$$\left|\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)dx - \int_\Omega \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)dx\right| \le$$
$$2\delta + \left|\int_{\Omega\cap\{u_n\le M\}} \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)dx - \int_{\Omega\cap\{u\le M\}} \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)dx\right|.$$

It remains then to prove that

$$\int_{|u_n|\le M} \left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n)\chi_\Omega dx \to \int_{|u|\le M} \left[\frac{1}{|x|^\mu} * F(u)\right] F(u)\chi_\Omega dx \qquad (2.24)$$ equivts

as $n \to +\infty$, for any fixed $M > \max\{(CM_0/\delta)^{q+1}, s_0\}$. Let us observe that as $K \to +\infty$

$$\int_{|u|\le M} \int_{|u|\le K} \left[\frac{F(u(y))}{|x-y|^\mu}\right] dy F(u(x))\chi_\Omega(x)dx \to \int_{|u|\le M} \left[\frac{1}{|x|^\mu} * F(u)\right] dy F(u)\chi_\Omega dx.$$

Let C be the constant in (2.23) and choose $K \ge \max\{(CM_0/\delta)^{q+1}, s_0\}$ such that

$$\int_{|u|\le M} \int_{|u|\ge K} \left[\frac{F(u(y))}{|x-y|^\mu}\right] dy F(u(x))dx \le \delta.$$

By $(f_1)(ii)$ one has

$$\begin{aligned}
&\int_{|u_n|\le M}\int_{|u_n|\ge K}\left[\frac{F(u_n(y))}{|x-y|^\mu}\right]F(u_n(x))\chi_\Omega(x)dx\\
&\qquad\le \frac{1}{K^{q+1}}\int_{|u_n|\le M}\int_{|u_n|\ge K}\left[\frac{u_n^{q+1}F(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_\Omega dx\\
&\qquad\le \frac{M_0}{K^{q+1}}\int_{|u_n|\le M}\int_{|u_n|\ge K}\left[\frac{u_nf(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_\Omega dx\\
&\qquad\le \frac{M_0}{K^{q+1}}\int_{|u_n|\le M}\int_{|u_n|\ge K}\left[\frac{u_nf(u_n)}{|x-y|^\mu}\right]dyF(u_n)dx\\
&\qquad= \frac{M_0}{K^{q+1}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\frac{F(u_n)}{|x-y|^\mu}\right]dyu_nf(u_n)dx\\
&\qquad\le \delta,
\end{aligned}$$

then we can see that

$$\left|\int_{|u|\le M}\int_{|u|\ge K}\left[\frac{F(u)}{|x-y|^\mu}\right]dyF(u)\chi_\Omega-\int_{|u_n|\le M}\int_{|u_n|\ge K}\left[\frac{F(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_\Omega\right|\le 2\delta.$$

In order to prove (2.24) it remains to verify that as $n\to+\infty$ there holds

$$\left|\int_{|u|\le M}\int_{|u|\le K}\left[\frac{F(u)}{|x-y|^\mu}\right]dyF(u)\chi_\Omega-\int_{|u_n|\le M}\int_{|u_n|\le K}\left[\frac{F(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_\Omega\right|\to 0$$

for any fixed $K,M>0$. This is a consequence of the Lebesgue's dominated convergence theorem: indeed,

$$\int_{|u_n|\le K}\left[\frac{F(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_{\{\Omega\cap|u_n|\le M\}}\to\int_{|u|\le K}\left[\frac{F(u)}{|x-y|^\mu}\right]dyF(u)\chi_{\{\Omega\cap|u|\le M\}}\qquad\text{a.e.}$$

and by (f_2) we know there exists a constant $C_{M,K}$ depends of M,K such that

$$\begin{aligned}
&\int_{|u_n|\le K}\left[\frac{F(u_n)}{|x-y|^\mu}\right]dyF(u_n)\chi_{\{\Omega\cap|u_n|\le M\}}\\
&\qquad\le C_{M,K}\int_{|u_n|\le K}\left[\frac{u_n^{p+1}}{|x-y|^\mu}\right]dyu_n^{p+1}\chi_{\{\Omega\cap|u_n|\le M\}}\\
&\qquad\le C_{M,K}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*u_n^{p+1}\right]u_n^{p+1}\chi_\Omega\to C_{M,K}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*u^{p+1}\right]u^{p+1}\chi_\Omega
\end{aligned}$$

as $n\to\infty$, applying the Hardy-Sobolev-Littlewood inequality, since $u_n\to u$ in L^s_{loc} for all $s\ge 1$. Hence the proof of (2.20) is now complete.

Let us now prove that the weak limit u yields actually a weak solution to (2.1), namely that

$$\int_{\mathbb{R}^2}\nabla u\nabla\varphi+W(x)u\varphi-\left[\frac{1}{|x|^\mu}*F(u)\right]f(u)\varphi=0 \tag{2.25}$$

weaklim

for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Since $\{u_n\}$ is a $(PS)_{m_V}$ sequence, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, we know that

$$\int_{\mathbb{R}^2} \nabla u_n \nabla \varphi + W(x) u_n \varphi - \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)\varphi \to 0,$$

as $n \to \infty$. Since $u_n \rightharpoonup u$ in E, we just need to prove that, as $n \to \infty$

$$\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)\varphi \to \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u)\right] f(u)\varphi \qquad (2.26) \quad \texttt{weak*conv}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$.

Let Ω be any compact subset of $\mathbb{R}^2$, we claim that there exists $C(\Omega)$ such that

$$\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] \frac{f(u_n)}{1+u_n} dx \le C(\Omega). \qquad (2.27) \quad \texttt{boundbis}$$

In fact, let

$$v_n = \frac{\varphi}{1+u_n},$$

where φ is a smooth function compactly supported in $\Omega' \supset \Omega$, Ω' compact, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in Ω. Direct computation shows that

$$\begin{aligned}
\|v_n\|_W^2 &= \int_{\mathbb{R}^2} |\nabla v_n|^2 + W(x) v_n^2 \\
&= \int_{\mathbb{R}^2} \left|\frac{\nabla \varphi}{1+u_n} - \varphi \frac{\nabla u_n}{(1+u_n)^2}\right|^2 + W \frac{\varphi^2}{(1+u_n)^2} \\
&\le \int_{\mathbb{R}^2} \frac{|\nabla \varphi|^2}{(1+u_n)^2} + 2\frac{\nabla \varphi \nabla u_n}{1+u_n} + \varphi^2 \frac{|\nabla u_n|^2}{(1+u_n)^4} + W\varphi^2 \\
&\le 2\|\varphi\|_W^2 + 2\|u_n\|_W^2,
\end{aligned}$$

which means that $v_n \in E$. Choose v_n as test function in (2.22), then

$$\begin{aligned}
\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] \frac{f(u_n)}{1+u_n} dx &\le \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n) \frac{\varphi}{1+u_n} \\
&\le \int_{\mathbb{R}^2} |\nabla u_n|^2 \frac{\varphi}{(1+u_n)^2} + \frac{\nabla u_n \nabla \varphi}{1+u_n} + W u_n \frac{\varphi}{1+u_n} + \tau_n \|v_n\|_W \\
&\le \int_{\mathbb{R}^2} |\nabla u_n|^2 \frac{\varphi}{(1+u_n)^2} + \frac{\nabla u_n \nabla \varphi}{1+u_n} + W u_n \frac{\varphi}{1+u_n} + 2\tau_n \|u_n\|_W + 2\tau_n \|\varphi\|_W \\
&\le \|\nabla u_n\|_2^2 + C_\varphi \|\nabla u_n\|_2 + \int_{\Omega'} W u_n + 2\tau_n \|u_n\|_W + 2\tau_n \|\varphi\|_W.
\end{aligned}$$

Since $W(x)$ is bounded, u_n is bounded in H^1 and $u_n \to u$ in $L^1(\Omega')$ we easily deduce (2.27).

Now define

$$\xi_n := \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n),$$

we can observe that

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n) dx \\
&\quad \le 2\int_{\{u_n<1\}\cap\Omega} \left[\frac{1}{|x|^\mu} * F(u_n)\right] \frac{f(u_n)}{1+u_n} dx + \int_{\{u_n>1\}\cap\Omega} \left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n) dx \\
&\quad \le 2\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] \frac{f(u_n)}{1+u_n} dx + \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] u_n f(u_n) dx.
\end{aligned}$$

Combining (2.27) and (2.23), it is easy to see that ξ_n is uniformly bounded in $L^1(\Omega)$ with

$$\int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)dx \le 2C(\Omega) + C.$$

Finally, consider the sequence of measures μ_n with density $\xi_n = \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)$, that is

$$\mu_n(E) := \int_E \xi_n \, dx = \int_E \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)\, dx \quad \text{for any measurable } E \subset \Omega.$$

Since $\|\xi_n\|_1 \le C(\Omega)$ and Ω is bounded, the measures μ_n have uniformly bounded total variation. Then, by weak*-compactness, up to a subsequence, $\mu_n \rightharpoonup^* \mu$ for some measure μ,

$$\lim_{n\to\infty} \int_\Omega \xi_n \varphi \, dx = \lim_{n\to\infty} \int_\Omega \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)\varphi \, dx = \int_\Omega \varphi d\mu, \quad \forall\, \varphi \in C_c^\infty(\Omega).$$

Now recall that u_n is a (PS) sequence, so that in particular (2.22) holds and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \nabla u_n \nabla \varphi + W(x) u_n \varphi = \int_\Omega \varphi d\mu, \quad \forall\, \varphi \in C_c^\infty(\Omega),$$

which implies that μ is absolutely continuous with respect to the Lebesgue measure. Then, by the Radon-Nicodym theorem, there exists a function $\xi \in L^1(\Omega)$ such that

$$\int_\Omega \varphi d\mu = \int_\Omega \varphi \xi dx, \quad \forall\, \varphi \in C_c^\infty(\Omega).$$

Since this holds for any compact set $\Omega \subset \mathbb{R}^2$, we have that there exists a function $\xi \in L^1_{loc}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \varphi d\,\mu = \lim_n \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)\varphi \, dx = \int_{\mathbb{R}^2} \varphi \xi d\, x, \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^2),$$

where $\xi = \left[\frac{1}{|x|^\mu} * F(u)\right] f(u)$ and the proof is complete.

□

Proof of Theorem 1.3. As proved in [6, Lemma 2.1], the functional Φ_W satisfies the Mountain Pass geometry, then there exists a $(PS)_{m_W}$ sequence $\{u_n\}$. By Lemma 2.4, up to a subsequence, $\{u_n\}$ weakly converges to a weak solution u of (2.1): it remains only to prove that u is non-trivial. Let us suppose by contradiction that $u \equiv 0$. Since $\{u_n\}$ is bounded, we have either $\{u_n\}$ is vanishing, that is, for any $r > 0$

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |u_n|^2 = 0$$

or it is non-vanishing, i.e. there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^2$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \ge \delta.$$

If $\{u_n\}$ is vanishing, by Lions' concentration-compactness result we have

$$u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}^2) \quad \forall\, s > 2, \tag{2.28}$$ Lions

as $n \to \infty$. In this case we claim that

$$\left[\frac{1}{|x|^\mu} * F(u_n)\right] F(u_n) \to 0 \quad \text{in } L^1(\mathbb{R}^2), \tag{2.29}$$ conv0

as $n \to \infty$. In fact, we need only to repeat the proof of (2.20) in Lemma 2.4 without restricting necessarily to compact sets. Apply the Hardy-Sobolev-Littewood inequality we notice that

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * u_n^{p+1}\right] u_n^{p+1}\right| \le C|u_n|^{2(p+1)}_{\frac{4}{4-\mu}(p+1)} \to 0$$

as $n \to \infty$, since $\frac{4}{4-\mu}(p+1) > 2$ and (2.28) holds. Since $\{u_n\}$ is a $(PS)_{m_W}$ sequence with $m_W < \frac{4-\mu}{8}$, it follows that

$$\lim_{n\to+\infty} \|u_n\|_W^2 = 2m_W < \frac{4-\mu}{4}$$

Then there exist a sufficiently small $\delta > 0$ and $K > 0$ such that

$$\|u_n\|_W^2 \le \frac{4-\mu}{4}(1-\delta), \qquad \forall n > K. \tag{2.30}$$ conv-norm

Using again the Hardy-Sobolev-Littewood inequality we have

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \le C|F(u_n)|_{\frac{4}{4-\mu}}|f(u_n)u_n|_{\frac{4}{4-\mu}}.$$

Combining (f_1) with (f_2), for any $\varepsilon > 0$, $p > 1$ and $\beta > 1$, there exists $C(\varepsilon, p, \beta) > 0$ such that

$$|f(s)| \le \varepsilon|s|^{\frac{2-\mu}{2}} + C(\varepsilon, p, \beta)|s|^{p-1}\left[e^{\beta 4\pi s^2} - 1\right] \ \forall s \in \mathbb{R}.$$

Then,

$$|f(u_n)u_n|_{\frac{4}{4-\mu}} \le \varepsilon|u_n|_2^{\frac{4-\mu}{2}} + C(\varepsilon, p, \beta)|u_n|^{\frac{4-\mu}{4t'}}_{\frac{4pt'}{4-\mu}}\Big(\int_{\mathbb{R}^2}[e^{(\frac{4\beta t}{4-\mu}\|u_n\|_W^2 4\pi\frac{u_n^2}{\|u_n\|_W^2})} - 1]\Big)^{\frac{4-\mu}{4t}}$$

where $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$. In order to conclude by means of [41] by do Ó and Adachi-Tanaka inequality [2] it is enough to choose $\beta, t > 1$ close to 1 such that $\frac{4\beta t}{4-\mu}\|u_n\|_W^2 < 1$, namely

$$1 < \beta t < \frac{1}{1-\delta},$$

we deduce that

$$\Big(\int_{\mathbb{R}^2}[e^{(\frac{4\beta t}{4-\mu}\|u_n\|_W^2 4\pi\frac{u_n^2}{\|u_n\|_W^2})} - 1]\Big)^{\frac{4-\mu}{4t}} \le \Big(\int_{\mathbb{R}^2}[e^{(\frac{4\beta mt}{4-\mu}4\pi\frac{u_n^2}{\|u_n\|_W^2})} - 1]\Big)^{\frac{4-\mu}{4t}} \le C_1 \ \ \forall n > K,$$

for some $C_1 > 0$. Then,

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \le \varepsilon^2|u_n|_2^{4-\mu} + C_2|u_n|^{\frac{4-\mu}{2t'}}_{\frac{4pt'}{4-\mu}}.$$

Since $t > 1$ is close to 1, we have that $\frac{4pt'}{4-\mu} > 2$. By (2.28), we have

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \to 0$$

as $n \to \infty$. Recalling that $\{u_n\}$ is a $(PS)_{m_W}$ sequence, $u_n \to 0$ in E, and so $\Phi_W(u_n) \to 0$ which implies $m_W = 0$, which is a contradiction. Therefore the vanishing case dose not hold.

Let us now consider the non vanishing case and define $v_n := u_n(\cdot - y_n)$, then

$$\int_{B_r(0)} |v_n|^2 \geq \delta \tag{2.31}$$ nonvan

By the periodicity assumption, Φ_W and Φ'_W are both invariant by $\mathbb{Z}^2$ translations, so that $\{v_n\}$ is again a $(PS)_{m_W}$ sequence. Then $v_n \rightharpoonup v$ in E, with $v \neq 0$ by using (2.31), since $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$. Thereby, v is a nontrivial critical point of Φ_W and $\Phi_W(v) = m_W$, which completes the proof of the theorem.

3 Semiclassical states for the nonlocal Schrödinger equation

Performing the scaling $u(x) = v(\epsilon x)$ one easily sees that problem (1.8) is equivalent to

$$-\Delta u + V(\varepsilon x)u = \Big[\frac{1}{|x|^\mu} * F(u)\Big] f(u). \tag{SNS^*}$$

For $\varepsilon > 0$, we define the following Hilbert space

$$E_\varepsilon = \Big\{u \in E : \int_{\mathbb{R}^2} V(\varepsilon x)|u|^2 < \infty\Big\}$$

endowed with the norm

$$\|u\|_\varepsilon := \Big(\int_{\mathbb{R}^2} \big(|\nabla u|^2 + V(\varepsilon x)|u|^2\big)\Big)^{1/2}.$$

The energy functional associated to equation (SNS^*) is given by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \mathfrak{F}(u)$$

and

$$\langle I'_\varepsilon(u), \varphi\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + V(\varepsilon x)u\varphi) - \mathfrak{F}'(u)[\varphi], \ \ \forall u, \varphi \in E.$$

Let $\mathcal{N}_\varepsilon$ be the Nehari manifold associated to I_ε, that is,

$$\mathcal{N}_\varepsilon = \Big\{u \in E_\varepsilon : u \neq 0, \langle I'_\varepsilon(u), u\rangle = 0\Big\}.$$

The following Lemma states that the Nehari manifold $\mathcal{N}_\varepsilon$ is bounded away from 0.

LN **Lemma 3.1.** *Suppose that conditions $(f_1) - (f_3)$ hold. Then there exists $\alpha > 0$, independent of ε, such that*

$$\|u\|_\varepsilon \geq \alpha, \ \ \forall u \in \mathcal{N}_\varepsilon. \tag{3.1}$$ alpha2

Proof. For any $\delta > 0$, $p > 1$ and $\beta > 1$, there exists $C_\delta > 0$ such that

$$F(s) < \frac{1}{K} f(s)s \leq \delta s^{\frac{4-\mu}{2}} + C(\delta, p, \beta) s^p \big[e^{\beta 4\pi s^2} - 1\big], \forall s \in \mathbb{R},$$

it follows

$$|F(u)|_{\frac{4}{4-\mu}} \leq C|f(u)u|_{\frac{4}{4-\mu}} \leq \delta C |u|_2^{\frac{4-\mu}{2}} + C(\delta, p, \beta) \big| u^p \big[e^{\beta 4\pi u^2} - 1\big] \big|_{\frac{4}{4-\mu}}. \tag{3.2}$$ mp1

Since the imbedding $E_\varepsilon \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for any $p \in (2, +\infty)$, we know there exists a constant C_1 such that

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^{\frac{4p}{4-\mu}} \big[e^{\beta 4\pi u^2} - 1\big]^{\frac{4}{4-\mu}} &\leq \Big(\int_{\mathbb{R}^2} |u|^{\frac{8p}{4-\mu}}\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} \big[e^{\beta 4\pi u^2} - 1\big]^{\frac{4}{4-\mu}}\Big)^{\frac{1}{2}} \\ &\leq C_1 \|u\|_\varepsilon^{\frac{4p}{4-\mu}} \Big(\int_{\mathbb{R}^2} \big[e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1\big]\Big)^{\frac{1}{2}}.\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2} \big[e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1\big] = \int_{\mathbb{R}^2} \big[e^{(\frac{4\beta}{4-\mu} \|u\|_\varepsilon^2 4\pi \frac{u^2}{\|u\|_\varepsilon^2})} - 1\big],$$

then, fixing $\xi \in (0,1)$ and making $\frac{4\beta}{4-\mu}\|u\|_\varepsilon^2 = \xi < 1$, Lemma 1.2 implies that there exists a constant C_2 such that

$$\int_{\mathbb{R}^2} \big[e^{(\xi 4\pi \frac{u^2}{\|u\|_\varepsilon^2})} - 1\big] \leq C_2.$$

thus, by (3.2), we know there exists C_3 such that

$$|F(u)|_{\frac{4}{4-\mu}} \leq \delta \|u\|_\varepsilon^{\frac{4-\mu}{2}} + C_3 \|u\|_\varepsilon^p.$$

By the Hardy-Littlewood-Sobolev inequality, if $\|u\|_\varepsilon^2 = \frac{\xi(4-\mu)}{4\beta}$, there holds

$$\mathfrak{F}'(u)[u] \leq \delta^2 C_4 \|u\|_\varepsilon^{4-\mu} + C_4 \|u\|_\varepsilon^{2p}.$$

Since $u \in \mathcal{N}_\varepsilon$, there holds

$$\|u\|_\varepsilon^2 = \mathfrak{F}'(u)[u],$$

and so

$$\|u\|_\varepsilon^2 \leq \delta^2 C_5 \|u\|_\varepsilon^{4-\mu} + C_5 \|u\|_\varepsilon^{2p},$$

then the conclusion follows immediately. □

Next we show that the functional I_ε satisfies the Mountain Pass geometry.

mountain:1 **Lemma 3.2.** *Suppose that conditions $(f_1) - (f_3)$ hold, then*

(i) There exist $\rho, \delta_0 > 0$ such that $I_\varepsilon|_S \geq \delta_0 > 0$ for all $u \in S = \{u \in E_\varepsilon : \|u\|_\varepsilon = \rho\}$;

(ii) There is e with $\|e\|_\varepsilon > \rho$ such that $I_\varepsilon(e) < 0$.

Proof. The proof of (i) easily follows the line of Lemma 3.1, and so we only prove (ii). Fixed $u_0 \in E_\varepsilon$ with $u_0^+(x) = \max\{u_0(x), 0\}$, we set

$$w(t) = \mathfrak{F}(\frac{tu_0}{\|u_0\|_\varepsilon}) > 0, \quad \text{for } t > 0.$$

By the Ambrosetti-Rabinowitz condition (f_3) we know

$$\frac{w'(t)}{w(t)} \geq \frac{2K}{t} \quad \text{for } t > 0.$$

Integrate this over $[1, s\|u_0\|_\varepsilon]$ with $s > \frac{1}{\|u_0\|_\varepsilon}$ to get

$$\mathfrak{F}(su_0) \geq \mathfrak{F}(\frac{u_0}{\|u_0\|_\varepsilon})\|u_0\|_\varepsilon^{2K} s^{2K}.$$

Therefore

$$I_\varepsilon(su_0) \leq C_1 s^2 - C_2 s^{2K} \quad \text{for} \quad s > \frac{1}{\|u_0\|_\varepsilon}.$$

Since $K > 1$, (ii) follows taking $e = su_0$ and s large enough. □

By the Ekeland Variational Principle [23] we know there is a $(PS)_{c_\varepsilon}$ sequence $(u_n) \subset E$, i.e.

$$I'_\varepsilon(u_n) \to 0, \quad I_\varepsilon(u_n) \to c_\varepsilon,$$

where c_ε is defined by

$$0 < c_\varepsilon := \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tu) \tag{3.3}$$ m1

and moreover there is a constant $c > 0$ independent of ε such that $c_\varepsilon > c > 0$. Using assumption (f_5), for each $u \in E_\varepsilon \setminus \{0\}$, there is an unique $t = t(u)$ such that

$$I_\varepsilon(t(u)u) = \max_{s \geq 0} I_\varepsilon(su) \quad \text{and} \quad t(u)u \in \mathcal{N}_\varepsilon.$$

Then it is standard to see (see [50]) that the minimax value c_ε can be characterized by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u). \tag{3.4}$$ m2

EML **Lemma 3.3.** *Suppose that assumptions $(f_1)-(f_5)$, (V_1) and (V_2) hold. Let c_ε be the minimax value defined in* (3.3)*, then there holds*

$$\lim_{\varepsilon \to 0} c_\varepsilon = m_{V_0},$$

where m_{V_0} is the minimax value defined in (2.4) *with $W(x) \equiv V_0$. Hence, by Lemma 2.2, there is $\varepsilon_0 > 0$ such that*

$$c_\varepsilon < \frac{4-\mu}{8}, \quad \forall \varepsilon \in [0, \varepsilon_0).$$

Moreover, since $m_{V_0} < m_{V_\infty}$, we also have

$$\lim_{\varepsilon \to 0} c_\varepsilon \leq m_{V_\infty}.$$

Proof. Let $w \in E$ be the ground state solution obtained in Theorem 1.3, then there holds

$$\int_{\mathbb{R}^2} \big(|\nabla w|^2 + V_0|w|^2\big) = \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(w)\Big] f(w)w.$$

In what follows, given $\delta > 0$, we fix $w_\delta \in C_0^\infty(\mathbb{R}^2)$ verifying

$$w_\delta \in \mathcal{N}_{V_0}, \ w_\delta \to w \ \text{ in } \ E \ \text{ and } \ \Phi_{V_0}(w_\delta) < m_{V_0} + \delta. \tag{3.5}$$ ESc1

Now, choose $\eta \in C_0^\infty(\mathbb{R}^2, [0,1])$ be such that $\eta = 1$ on $B_1(0)$ and $\eta = 0$ on $\mathbb{R}^2 \backslash B_2(0)$, let us define $v_n(x) = \eta(\varepsilon_n x) w_\delta(x)$, where $\varepsilon_n \to 0$. Clearly

$$v_n \to w_\delta \ \text{ in } \ E, \ \text{ as } \ n \to +\infty.$$

From the definition of $\mathcal{N}_\varepsilon$, we know that there exists unique t_n such that $t_n v_n \in \mathcal{N}_{\varepsilon_n}$. Consequently,

$$c_{\varepsilon_n} \le I_{\varepsilon_n}(t_n v_n) = \frac{t_n^2}{2} \int_{\mathbb{R}^2} \big(|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2\big) - \frac{1}{2}\int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(t_n v_n)\Big] F(t_n v_n).$$

Observe that

$$\langle I'_{\varepsilon_n}(t_n v_n), t_n v_n \rangle = 0,$$

or equivalently,

$$\begin{aligned} t_n^2 \int_{\mathbb{R}^2} \big(|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2\big) &= \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(t_n v_n)\Big] f(t_n v_n) t_n v_n \\ &\ge C t_n^{2K} \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * |v_n|^K\Big] |v_n|^K \end{aligned} \tag{3.6}$$ ESc

which means $\{t_n\}$ is bounded and thus, up to subsequence, we may assume that $t_n \to t_0 \ge 0$. Notice that there is a constant $c > 0$ independent of ε such that $c_{\varepsilon_n} > c > 0$. Then, this information implies that $t_0 > 0$. Take a limit in the equality in (3.6) to find

$$\int_{\mathbb{R}^2} \big(|\nabla w_\delta|^2 + V_0|w_\delta|^2\big) = t_0^{-2} \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(t_0 w_\delta)\Big] f(t_0 w_\delta) t_0 w_\delta. \tag{3.7}$$ ESc2

Hence, from (3.5) and (3.7),

$$t_0^{-2} \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(t_0 w)\Big] f(t_0 w) t_0 w - \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^\mu} * F(w)\Big] f(w) w = 0.$$

Thereby, by monotone assumption (f_5), we derive that

$$t_0 = 1.$$

Since

$$\int_{\mathbb{R}^2} \big(V(\varepsilon_n x) - V_0\big)|v_n|^2 \to 0 \ \text{ and } \ \Phi_{V_0}(t_n v_n) \to \Phi_{V_0}(w_\delta),$$

the following inequality

$$c_{\varepsilon_n} \le I_{\varepsilon_n}(t_n v_n) = \Phi_{V_0}(t_n v_n) + \frac{t_n^2}{2} \int_{\mathbb{R}^2} \big(V(\varepsilon_n x) - V_0\big)|v_n|^2,$$

gives

$$\limsup_{n\to+\infty} c_{\varepsilon_n} \leq \Phi_{V_0}(w_\delta) \leq m_{V_0} + \delta.$$

As δ is arbitrary, we deduce that

$$\limsup_{n\to+\infty} c_{\varepsilon_n} \leq m_{V_0}.$$

As ε_n is also arbitrary, it follows that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq m_{V_0}. \tag{3.8}$$ PASSO1

On the other hand, we already know that

$$c_\varepsilon \geq m_{V_0}, \quad \forall \varepsilon > 0,$$

which implies

$$\liminf_{\varepsilon\to 0} c_\varepsilon \geq m_{V_0}. \tag{3.9}$$ PASSO2

From (3.8) and (3.9) we get

$$\lim_{\varepsilon\to 0} c_\varepsilon \geq m_{V_0}.$$

and the proof follows by using Lemma 2.2.

□

PS **Lemma 3.4.** *Suppose that assumptions $(f_1)-(f_5)$, (V_1) and (V_2) hold. Let $\{u_n\}$ be a $(PS)_{c_\varepsilon}$ sequence with $\varepsilon \in [0, \varepsilon_0)$. Let u_ε be the weak limit of u_n, then $\{u_n\}$ converges strongly to u_ε in E_ε, i.e. I_ε satisfies $(PS)_{c_\varepsilon}$ condition for $\varepsilon \in [0, \varepsilon_0)$.*

Proof. First recall that

$$c_\varepsilon < \frac{4-\mu}{8}, \quad \forall \varepsilon \in [0, \varepsilon_0) \tag{3.10}$$

$$m_{V_0} < m_{V_\infty}. \tag{3.11}$$

and there are positive constants a_1, a_2 such that

$$a_1 < \|u_n\|_\varepsilon < a_2, \quad \forall n \in \mathbb{N} \quad \text{(for some subsequence)}. \tag{3.12}$$ EST3

In the sequel, our first goal is to prove that $u_\varepsilon \neq 0$. To do that, we will argue by contradiction, assuming that $u_\varepsilon = 0$.

Claim: There exist $\beta, \tilde{R} > 0$ and $\{y_n\} \subset \mathbb{R}^2$ such that

$$\int_{B_{\tilde{R}}(y_n)} |u_n|^2 \geq \beta.$$

Otherwise, by applying a result due to Lions, we obtain

$$u_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^2) \quad \forall q \in (2, +\infty).$$

Following line by line the argument of Section 2, we have

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*F(u_n)\right]F(u_n)\right|\to 0,\quad n\to\infty.$$

Since (u_n) be a $(PS)_{c_\varepsilon}$ sequence with $c_\varepsilon<\frac{4-\mu}{8}$, we know that

$$\limsup_{n\to\infty}\|u_n\|_\varepsilon^2=2c_\varepsilon<\frac{4-\mu}{4}. \tag{3.13}$$ EST4

As in the proof of Theorem 1.3, we can conclude that

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*F(u_n)\right]f(u_n)u_n\right|\to 0,\quad n\to\infty.$$

This together with $\langle I'_\varepsilon(u_n),u_n\rangle=o_n(1)$ implies that

$$\lim_{n\to+\infty}\|u_n\|_\varepsilon^2=0$$

which contradicts with (3.13), proving the claim.

Next, we fix $t_n>0$ such that $t_nu_n\in\mathcal{N}_{V_\infty}$. We claim that $\{t_n\}$ is bounded. In fact, setting $v_n=u_n(x+y_n)$, by Claim 1, we may assume that, up to a subsequence, $v_n\rightharpoonup v$ in E_ε. Moreover, using the fact that $u_n\geq 0$ for all $n\in\mathbb{N}$, there exists $a_3>0$ and a subset $\Omega\subset\mathbb{R}^2$ with positive measure such that $v(x)>a_3$ for all $x\in\Omega$. We have

$$\int_{\mathbb{R}^2}(|\nabla u_n|^2+V_\infty|u_n|^2)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\frac{F(t_nu_n(y))f(t_nu_n(x))t_nu_n(x)}{t_n^2|x-y|^\mu}\Big)$$

and so,

$$\int_{\mathbb{R}^2}(|\nabla u_n|^2+V_\infty|u_n|^2)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\frac{F(t_nv_n(y))f(t_nv_n(x))t_nv_n(x)}{t_n^2|x-y|^\mu}\Big)$$

from which

$$\int_{\mathbb{R}^2}(|\nabla u_n|^2+V_\infty|u_n|^2)\geq\int_{\Omega}\int_{\Omega}\Big(\frac{F(t_nv_n(y))f(t_nv_n(x))t_nv_n(x)}{t_n^2|x-y|^\mu}\Big)$$

Since

$$\liminf_{n\to\infty}\frac{F(t_nv_n(y))f(t_nv_n(x))t_nv_n(x)}{t_n^2|x-y|^\mu}=+\infty\quad\text{a.e.}$$

Fatou's lemma gives

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^2}(|\nabla u_n|^2+V_\infty|u_n|^2)=+\infty,$$

which is a contradiction since $\{u_n\}$ is bounded in E_ε. Thus, without loss of generality, we may assume

$$\lim_{n\to+\infty}t_n=t_0>0.$$

In what follows, we divide the remaining part of the proof into three steps.

Step 1. The number t_0 is less or equal to 1.

In fact, suppose by contradiction that the above claim does not hold. Then, there exist $\delta>0$ and a subsequence of (t_n), still denoted by itself, such that

$$t_n\geq 1+\delta\ \text{ for all }\ n\in\mathbb{N}.$$

Since $\langle I'_\varepsilon(u_n), u_n\rangle = o_n(1)$ and $(t_n u_n) \subset \mathcal{N}_{V_\infty}$, we have

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(\varepsilon x)|u_n|^2) = \mathfrak{F}'(u_n)[u_n] + o_n(1)$$

and

$$t_n^2 \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) = \mathfrak{F}'(t_n u_n)[t_n u_n].$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^2} (V_\infty - V(\varepsilon x))|u_n|^2 + o_n(1) \\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(\frac{F(t_n u_n(y)) f(t_n u_n(x)) t_n u_n(x)}{t_n^2 |x-y|^\mu} - \frac{F(u_n(y)) f(u_n(x)) u_n(x)}{|x-y|^\mu}\Big).
\end{aligned}$$

Given $\zeta > 0$, from assumptions (V_1) and (V_2), there exists $R = R(\zeta) > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \zeta, \text{ for any } |x| \geq R. \tag{3.14}$$ V1

Using the fact that $u_n \to 0$ in $L^2(B_R(0))$, we conclude that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(\frac{F(t_n u_n(y)) f(t_n u_n(x)) t_n u_n(x)}{t_n^2 |x-y|^\mu} - \frac{F(u_n(y)) f(u_n(x)) u_n(x)}{|x-y|^\mu}\Big) \leq \zeta C + o_n(1),$$

where $C = \sup_{n\in\mathbb{N}} |u_n|_2^2$. Using the sequence $v_n = u_n(x + y_n)$ again, we find the inequality

$$\begin{aligned}
0 &< \int_\Omega\int_\Omega \frac{|v_n(y)||v_n(x)|}{|x-y|^\mu} \Big[\frac{F((1+\delta)v_n(y)) f((1+\delta)v_n(x))(1+\delta)v_n(x)}{(1+\delta)|v_n(y)|(1+\delta)|v_n(x)|} \\
&\quad - \frac{F(v_n(y)) f(v_n(x)) v_n(x)}{|v_n(y)||v_n(x)|}\Big] \\
&= \int_\Omega\int_\Omega \Big[\frac{F((1+\delta)v_n(y)) f((1+\delta)v_n(x))(1+\delta)v_n(x)}{(1+\delta)^2|x-y|^\mu} - \frac{F(v_n(y)) f(v_n(x)) v_n(x)}{|x-y|^\mu}\Big] \\
&\leq \zeta C + o_n(1)
\end{aligned}$$

Letting $n \to \infty$ in the last inequality and applying Fatou's lemma, it follows that

$$0 < \int_\Omega\int_\Omega \frac{F((1+\delta)v(y)) f((1+\delta)v(x))(1+\delta)v(x)}{(1+\delta)^2|x-y|^\mu} - \frac{F(v(y)) f(v(x)) v(x)}{|x-y|^\mu} \leq \zeta C$$

which is absurd, since the arbitrariness of ζ.

Step 2. $t_0 = 1$.

In this case, we begin with recalling that $m_{V_\infty} \leq \Phi_{V_\infty}(t_n u_n)$. Therefore,

$$c_\varepsilon + o_n(1) = I_\varepsilon(u_n) \geq I_\varepsilon(u_n) + m_{V_\infty} - \Phi_{V_\infty}(t_n u_n)$$

and from

$$\begin{aligned}
I_\varepsilon(u_n) - \Phi_{V_\infty}(t_n u_n) &= \frac{(1-t_n^2)}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(\varepsilon x)|u_n|^2 \\
&\quad - \frac{t_n^2}{2}\int_{\mathbb{R}^2} V_\infty |u_n|^2 + \mathfrak{F}(t_n u_n) - \mathfrak{F}(u_n),
\end{aligned}$$

and the fact that $\{u_n\}$ is bounded in E_ε as well as $u_n \rightharpoonup 0$, we derive from (3.14)

$$c_\varepsilon + o_n(1) \geq m_{V_\infty} - \zeta C + o_n(1),$$

and since ζ is arbitrary we obtain

$$\limsup_{\varepsilon \to 0} c_\varepsilon \geq m_{V_\infty},$$

which contradicts Lemma 3.3.

Step 3. $t_0 < 1$.
In this case, we may assume that $t_n < 1$ for all $n \in \mathbb{N}$. Since $m_{V_\infty} \leq \Phi_{V_\infty}(t_n u_n)$ and $\langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \rangle = 0$, we have

$$\begin{aligned}
m_{V_\infty} &\leq \Phi_{V_\infty}(t_n u_n) - \frac{1}{2}\langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \rangle \\
&= \frac{1}{2}\mathfrak{F}'(t_n u_n)[t_n u_n] - \mathfrak{F}(t_n u_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(t_n u_n(y)) f(t_n u_n(x)) t_n u_n(x)}{|x-y|^\mu} - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(t_n u_n(y)) F(t_n u_n(x))}{|x-y|^\mu} \\
&< \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(u_n(y)) f(u_n(x)) u_n(x)}{|x-y|^\mu} - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(u_n(y)) F(u_n(x))}{|x-y|^\mu} \\
&= I_\varepsilon(u_n) - \frac{1}{2}\langle I'_\varepsilon(u_n), u_n \rangle \\
&= c_\varepsilon + o_n(1),
\end{aligned}$$

which also yields a contradiction. From Steps 1, 2 and 3, we deduce that $u_\varepsilon \neq 0$. Hence, by Fatou's Lemma and using the characterization of c_ε, it follows that

$$\begin{aligned}
c_\varepsilon \leq I_\varepsilon(u_\varepsilon) &= I_\varepsilon(u_\varepsilon) - \frac{1}{2}\langle I'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(u_\varepsilon(y))[f(u_\varepsilon(x))u_\varepsilon(x) - F(u_\varepsilon(x)]}{|x-y|^\mu} \\
&= \liminf_{n \to +\infty} \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{F(u_n(y))[f(u_n(x))u_n(x) - F(u_n(x)]}{|x-y|^\mu} \\
&\leq \limsup_{n \to +\infty} (I_\varepsilon(u_n) - \frac{1}{2}\langle I'_\varepsilon(u_n), u_n \rangle) = c_\varepsilon,
\end{aligned}$$

thus

$$I_\varepsilon(u_\varepsilon) = c_\varepsilon.$$

Now, using the following inequalities

$$\begin{aligned}
c_\varepsilon &= I_\varepsilon(u_\varepsilon) - \frac{1}{2K}\langle I'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle \\
&\leq \liminf_{n \to +\infty} (I_\varepsilon(u_n) - \frac{1}{2K}\langle I'_\varepsilon(u_n), u_n \rangle) \\
&\leq \limsup_{n \to +\infty} (I_\varepsilon(u_n) - \frac{1}{2K}\langle I'_\varepsilon(u_n), u_n \rangle) \\
&= c_\varepsilon
\end{aligned}$$

we actually have

$$u_n \to u_\varepsilon \quad \text{in} \quad E_\varepsilon,$$

showing that I_ε verifies the $(PS)_{c_\varepsilon}$ condition. □

As an immediate consequence of Lemma 3.4, we have

Existence **Corollary 3.5.** *The minimax value c_ε is achieved if ε is small enough and hence problem (SNS^*) has a solution of least energy if ε is small enough.*

4 Concentration phenomena: proof of Theorem 1.4 completed

In this section our goal is to establish the concentration phenomenon fro ground state solutions of the singularly perturbed equation (SNS^*) . For this purpose, the following technical lemma will play a fundamental role.

BNT1 **Lemma 4.1.** *Suppose that assumptions (f_1) and (f_2) hold. If $h \in H^1(\mathbb{R}^2)$, then the function $\frac{1}{|x|^\mu} * F(h)$ belongs to $L^\infty(\mathbb{R}^2)$.*

Proof. For $\beta > 1$, there exists $C_0 > 0$ such that

$$F(s) \le C_0\Big(|s|^{\frac{4-\mu}{2}} + |s|\big[e^{\beta 4\pi s^2} - 1\big]\Big), \forall s \in \mathbb{R}.$$

Then,

$$\begin{aligned}
\big|\frac{1}{|x|^\mu} * F(h)\big| &= \Big|\int_{\mathbb{R}^2} \frac{F(h)}{|x-y|^\mu}\Big| \\
&= \Big|\int_{|x-y|\le 1} \frac{F(h)}{|x-y|^\mu}\Big| + C\Big|\int_{|x-y|\ge 1} \frac{F(h)}{|x-y|^\mu}\Big| \\
&\le \int_{|x-y|\le 1} \frac{|h|^{\frac{4-\mu}{2}} + |h|\big[e^{\beta 4\pi |h|^2} - 1\big]}{|x-y|^\mu} \\
&\quad + C\int_{|x-y|\ge 1} \Big(\frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} + |h|\big[e^{\beta 4\pi |h|^2} - 1\big]\Big).
\end{aligned}$$

Since

$$\frac{1}{|y|^\mu} \in L^{\frac{2+\delta}{\mu}}(B_1^c(0)), \quad \forall \;\; \delta > 0,$$

take $\delta \approx 0^+$ such that

$$q_{1,\delta} = \frac{(4-\mu)}{2}\frac{(2+\delta)}{(2+\delta)-\mu} > 2.$$

Using Hölder inequality, we get

$$\int_{|x-y|\ge 1} \frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} \le C_0 \left(\int_{|x-y|\ge 1} |h|^{q_{1,\delta}}\right)^{\frac{(2+\delta)-\mu}{2+\delta}} = C_1.$$

On the other hand, by Lemma 1.2

$$e^{2\beta 4 s\pi |h|^2} - 1 \in L^1(\mathbb{R}^2), \quad \forall s \ge 1,$$

Again by Hölder's inequality

$$\int_{|x-y|\geq 1} |h|\big[e^{\beta 4\pi |h|^2}-1\big] \leq |h|_2 \int_{\mathbb{R}^2} \Big(\big[e^{2\beta 4\pi \frac{|h|^2}{\|h\|_\varepsilon^2}}-1\big]\Big)^{\frac{1}{2}} \leq C_2.$$

for some positive constant C_2.
Choosing $t \in (\frac{2}{2-\mu}, +\infty)$, we have that $\frac{(4-\mu)t}{2} > 2$ and $1 - \frac{t\mu}{t-1} > -1$. Then, from Hölder's inequality

$$\begin{aligned}\int_{|x-y|\leq 1} \frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} &\leq \Big(\int_{|x-y|\leq 1} |h|^{\frac{(4-\mu)t}{2}}\Big)^{\frac{1}{t}} \Big(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}}\Big)^{\frac{t-1}{t}} \\ &\leq C_2 \Big(\int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr\Big)^{\frac{t-1}{t}} = C_3.\end{aligned}$$

Furthermore, using again Lemma 1.2, we get

$$\begin{aligned}&\int_{|x-y|\leq 1} \frac{|h|\big[e^{\beta 4\pi |h|^2}-1\big]}{|x-y|^\mu} \\ &\leq \Big(\int_{|x-y|\leq 1} |h|\big[e^{\beta 4\pi |h|^2}-1\big]|^t\Big)^{\frac{1}{t}} \Big(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}}\Big)^{\frac{t-1}{t}} \\ &\leq \Big(\int_{|x-y|\leq 1} |h|^{2t}\Big)^{\frac{1}{2t}} \Big(\int_{|x-y|\leq 1} \big[e^{2\beta t 4\pi |h|^2}-1\big]\Big)^{\frac{1}{2t}} \Big(\int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr\Big)^{\frac{t-1}{t}} \\ &\leq C_4.\end{aligned}$$

The lemma thus follows from the above estimates. □

Seq **Proposition 4.2.** *Let $\varepsilon_n \to 0$ and $\{u_n\}$ be the sequence of solutions obtained in Corollary 3.5. Then, there exists a sequence $\{y_n\} \subset \mathbb{R}^2$, such that $v_n = u_n(x + y_n)$ has a convergent subsequence in E. Moreover, up to a subsequence, $y_n \to y \in M$.*

Proof. Let $\{u_n\}$ be the sequence of solutions obtained in Corollary 3.5, it is easy to see $c_{\varepsilon_n} = I_{\varepsilon_n}(u_n) \to m_{V_0}$, $\{u_n\}$ is bounded in E and

$$0 < m_{V_0} = \limsup_{n\to\infty} c_{\varepsilon_n} < \frac{(4-\mu)}{8}.$$

By following the argument in the proof of Theorem 1.3 in Section 2, there exist $r, \delta > 0$ and $\tilde{y}_n \in \mathbb{R}^2$ such that

$$\liminf_{n\to\infty} \int_{B_r(\tilde{y}_n)} |u_n|^2 \geq \delta. \tag{4.1}$$ B1'

Setting $v_n(x) = u_n(x + \tilde{y}_n)$, up to a subsequence, if necessary, we may assume $v_n \rightharpoonup v \not\equiv 0$ in E. Let $t_n > 0$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Then,

$$m_{V_0} \leq \Phi_{V_0}(\tilde{v}_n) = \Phi_{V_0}(t_n u_n) \leq I_\varepsilon(t_n u_n) \leq I_\varepsilon(u_n) \to m_{V_0}$$

and so,

$$\Phi_{V_0}(\tilde{v}_n) \to m_{V_0} \text{ and } (\tilde{v}_n) \subset \mathcal{N}_{V_0}.$$

Then the sequence $\{\tilde{v}_n\}$ is a minimizing sequence, and by the Ekeland Variational Principle [23], we may also assume it is a bounded (PS) sequence at m_{V_0}. Thus, for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in E with $\tilde{v} \neq 0$ and $\Phi'_{V_0}(\tilde{v}) = 0$. Repeating the same arguments used in the proof of Lemma 3.4, we have that $\tilde{v}_n \to \tilde{v}$ in E. Since (t_n) is bounded, we can assume that for some subsequence $t_n \to t_0 > 0$, and so $v_n \to v$ in E.

Next we will show that $\{y_n\} = \{\varepsilon_n \tilde{y}_n\}$ has a subsequence satisfying $y_n \to y \in M$. We begin with proving that $\{y_n\}$ is bounded in $\mathbb{R}^2$. Indeed, if not there would exist a subsequence, which we still denote by $\{y_n\}$, such that $|y_n| \to \infty$. Since $\tilde{v}_n \to \tilde{v}$ in E and $V_0 < V_\infty$, we have

$$\begin{aligned}
m_{V_0} &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 + \frac{1}{2}\int_{\mathbb{R}^2} V_0 |\tilde{v}|^2 - \mathfrak{F}(\tilde{v}) \\
&< \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 + \frac{1}{2}\int_{\mathbb{R}^2} V_\infty |\tilde{v}|^2 - \mathfrak{F}(\tilde{v}) \\
&\leq \liminf_{n\to\infty} \left[\frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{v}_n|^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(\epsilon_n x + y_n)|\tilde{v}_n|^2 - \mathfrak{F}(\tilde{v}_n)\right] \\
&= \liminf_{n\to\infty} \left[\frac{t_n^2}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V(\epsilon_n x)|u_n|^2 - \mathfrak{F}(t_n^2 u_n)\right] \\
&= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \\
&\leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) \\
&= m_{V_0}
\end{aligned}$$

hence the absurd which shows that $\{y_n\}$ stays bounded and up to a subsequence, $y_n \to y \in \mathbb{R}^2$. Then, necessarily $y \in M$ otherwise we would get again a contradiction as above. □

Let $\varepsilon_n \to 0$ as $n \to \infty$, u_n be the ground state solution of

$$-\Delta u + V(\varepsilon_n x)u = \Big[\frac{1}{|x|^\mu} * F(u)\Big] f(u) \quad \text{in} \quad \mathbb{R}^2.$$

From Lemma 3.3 we know

$$I_{\varepsilon_n}(u_n) \to m_{V_0}.$$

Then, there exists a sequence $\tilde{y}_n \in \mathbb{R}^2$, such that $v_n = u_n(x + \tilde{y}_n)$ is a solution of

$$-\Delta v_n + V_n(x) v_n = \Big[\frac{1}{|x|^\mu} * F(v_n)\Big] f(v_n), \quad \text{in} \quad \mathbb{R}^2,$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$. Moreover, (v_n) has a convergent subsequence in E and $y_n \to y \in M$, up to a subsequence, where $y_n = \varepsilon_n \tilde{y}_n$. Hence, there exists $h \in H^1(\mathbb{R}^2)$ such that

$$|v_n(x)| \leq h(x) \quad \text{a.e in} \quad \mathbb{R}^2 \quad \forall n \in \mathbb{N}. \tag{4.2}$$ h

Lemma 4.3. *Suppose that conditions $(f_1)-(f_5)$, (V_1) and (V_2) hold. Then there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^2)} \leq C$ for all $n \in \mathbb{N}$. Furthermore*

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof. Let us first show that the sequence

$$W_n(x) := \Big[\frac{1}{|x|^\mu} * F(v_n)\Big],$$

stays bounded in $L^\infty(\mathbb{R}^2)$. Indeed, as F is an increasing function, by (4.2) we know that

$$0 \le W_n(x) := \Big[\frac{1}{|x|^\mu} * F(v_n)\Big] \le \Big[\frac{1}{|x|^\mu} * F(h)\Big]$$

Hence claim will hold provided the function

$$W(x) = \Big[\frac{1}{|x|^\mu} * F(h)\Big]$$

belongs to $L^\infty(\mathbb{R}^2)$ and this is an immediate consequence of Lemma 4.1.
For any $R > 0$, $0 < r \le \frac{R}{2}$, let $\eta \in C^\infty(\mathbb{R}^2)$, $0 \le \eta \le 1$ with $\eta(x) = 1$ if $|x| \ge R$ and $\eta(x) = 0$ if $|x| \le R - r$ and $|\nabla \eta| \le \frac{2}{r}$. For $L > 0$, let

$$v_{L,n} = \begin{cases} v_n(x), & v(x) \le L \\ L, & v_n(x) \ge L, \end{cases}$$

and

$$z_{L,n} = \eta^2 v_{L,n}^{2(\gamma-1)} v_n \quad \textit{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\gamma-1}$$

with $\gamma > 1$ to be determined later. Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)} |\nabla v_n|^2 + \int_{\mathbb{R}^2} \tilde{V}_{\varepsilon_n}(x) |v_n|^2 \eta^2 v_{L,n}^{2(\gamma-1)} \\ &= -2(\gamma-1) \int_{\mathbb{R}^2} v_n v_{L,n}^{2\gamma-3} \eta^2 \nabla v_n \nabla v_{L,n} + \int_{\mathbb{R}^2} W_n(x) f(v_n) \eta^2 v_n v_{L,n}^{2(\gamma-1)} \\ &\quad - 2 \int_{\mathbb{R}^2} \eta v_{L,n}^{2(\gamma-1)} v_n \nabla v_n \nabla \eta. \end{aligned} \tag{4.3}$$ E1

Using Lemma 1.2, for all $\beta, s > 1$, we know that

$$\int_{\mathbb{R}^2} \big[e^{\beta 4\pi v_n^2} - 1\big]^s \le \int_{\mathbb{R}^2} \big[e^{\beta 4\pi |h|^2} - 1\big]^s = C < \infty \quad \forall n \in \mathbb{N}. \tag{4.4}$$ E2

Let $t = \sqrt{s}$, $p > \frac{2t}{t-1} > 2$ and $\gamma = \frac{p(t-1)}{2t}$, for any $\delta > 0$, there exists $C(\delta, p, \beta) > 0$ such that

$$F(u) \le \delta u^2 + C(\delta, p, \beta) u^{p-1} \big[e^{\beta 4\pi |u|^2} - 1\big], \ \ \forall u \in \mathbb{R}.$$

Thus for δ sufficiently small, as (W_n) is bounded in $L^\infty(\mathbb{R}^2)$, gathering (4.3) and Young's inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)} |\nabla v_n|^2 + V_0 \int_{\mathbb{R}^2} |v_n|^2 \eta^2 v_{L,n}^{2(\gamma-1)} \\ &\quad \le C \int_{\mathbb{R}^2} v_n^p \eta^2 v_{L,n}^{2(\gamma-1)} \big[e^{\beta 4\pi |h|^2} - 1\big] + C \int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)} |\nabla \eta|^2. \end{aligned} \tag{4.5}$$ E3

Using this fact, from [4] we have

$$|w_{L,n}|_p^2 \leq C\gamma^2\Big(C' + \Big[\int_{|x|\geq R-r} v_n{}^{(p-2)t}\big[e^{\beta 4\pi |h|^2} - 1\big]^t\Big]^{\frac{1}{t}}\Big)\Big[\int_{|x|\geq R-r} v_n{}^{\frac{2\gamma t}{t-1}}\Big]^{\frac{t-1}{t}}.$$

By (4.4) and Hölder's inequality, we know

$$|w_{L,n}|_p^2 \leq C\gamma^2\Big[\int_{|x|\geq R-r} v_n{}^{\frac{2\gamma t}{t-1}}\Big]^{\frac{t-1}{t}}.$$

Now, following the same iteration arguments explored in [4], we find

$$|v_n|_{L^\infty(|x|\geq R)} \leq C|v_n|_{p(|x|\geq R/2)}. \tag{4.6}$$ BD1

For $x_0 \in B_R$, we can use the same argument taking $\eta \in C_0^\infty(\mathbb{R}^2, [0,1])$ with $\eta(x) = 1$ if $|x - x_0| \leq \rho'$ and $\eta(x) = 0$ if $|x - x_0| > 2\rho'$ and $|\nabla\eta| \leq \frac{2}{\rho'}$, to prove that

$$|v_n|_{L^\infty(|x-x_0|\leq\rho')} \leq C|v_n|_{p(|x|\leq 2\rho')}. \tag{4.7}$$ BD2

With (4.6) and (4.7), by a standard covering argument it follows that

$$|v_n|_\infty < C$$

for some positive constant C. Then, using again the convergence of (v_n) to v in E in the right side of (4.6), for each $\delta > 0$ fixed, there exists $R > 0$ such that $|v_n|_{L^\infty(|x|\geq R)} < \delta, \forall n \in N$. Thus,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in} \quad n \in \mathbb{N},$$

and the proof is complete. □

The last lemma establishes an estimate from below in terms of the L^∞-norm of $\{v_n\}$.

MP **Lemma 4.4.** *There exists $\delta_0 > 0$ such that $|v_n|_\infty \geq \delta_0$ for all $n \in \mathbb{N}$.*

Proof. Recall that,

$$\delta \leq \int_{B_r(\tilde{y}_n)} |u_n|^2,$$

then

$$\delta \leq \int_{B_r(0)} |v_n|^2 \leq |B_r||v_n|_\infty^2,$$

from where it follows

$$|v_n|_\infty \geq \delta_0,$$

showing the lemma. □

Concentration around maxima. Let b_n denote a maximum point of v_n, we know it is a bounded sequence in $\mathbb{R}^2$. Thus, there is $R > 0$ such that $b_n \in B_R(0)$. Thus the global maximum of u_{ε_n} is attained at $z_n = b_n + \tilde{y}_n$ and

$$\varepsilon_n z_n = \varepsilon_n b_n + \varepsilon_n \tilde{y}_n = \varepsilon_n b_n + y_n.$$

From the boundedness of $\{b_n\}$ we have

$$\lim_{n\to\infty} z_n = y,$$

which together with the continuity of V yields

$$\lim_{n\to\infty} V(\varepsilon_n z_n) = V_0.$$

If u_ε is a positive solution of (SNS^*) the function $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of (1.8). Thus, the maxima points η_ε and z_ε of respectively w_ε and u_ε, satisfy the equality $\eta_\varepsilon = \varepsilon z_\varepsilon$ and in turn

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

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