UNIVERSITÀ DEGLI STUDI DI MILANO

Facoltà di Scienze e Tecnologie
Dipartimento di Matematica
Corso di Dottorato in Scienze Matematiche, Ciclo XXIX

Deformations of nodal surfaces

Elaborato Finale di:
Yan Zhao

Tutor: Prof. Bert van Geemen
Prof. Peter Stevenhagen (Universiteit Leiden)
Prof. Ronald van Luijk (Universiteit Leiden)
Coordinatore di Dottorato: Prof. Vieri Mastropietro

Anno Accademico 2015/2016

This work was funded by Erasmus Mundus Algant-Doc and was carried out at Universiteit Leiden and Università degli studi di Milano.

Contents

Chapter 1

Introduction

1.1 Background

Classical algebraic geometry is the study of geometric objects defined locally by systems of polynomial equations. While this could be done over any base field k, in this thesis, we will work exclusively over the complex numbers $\mathbb{C}$. In addition to the Zariski topology which is defined in relation to the zero sets of polynomial equations, complex algebraic varieties can also be endowed with the Euclidean topology, and can be studied using analytic techniques. Serre showed in his landmark paper *Géométrie Algébrique et Géométrie Analytique* [Ser56] a precise correspondence between a complex algebraic variety and its analytification. This correspondence provides a wealth of tools to study both the local and global structures.

One important tool on the analytic side is Hodge theory. Let X be a smooth projective complex algebraic variety of dimension n, that is, a complex manifold that can be embedded into a projective space $\mathbb{P}^N_{\mathbb{C}}$ for some $N > n$. Algebraic topology provides a set of invariants, namely the cohomology groups $H^k(X, \mathbb{Z})$. They are however too coarse to be useful: many varieties have the same cohomology groups. A Hodge structure is an enhancement on $H^k(X, \mathbb{Z})$. Hodge theory gives a direct sum decomposition

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{p,k-p}(X)$$

with $H^{p,k-p}(X) = \overline{H^{k-p,p}(X)}$. The group $H^{p,q}(X)$ is naturally isomorphic to the cohomology group $H^q(X, \Omega^p_X)$ of the sheaf of differential p-forms on X.

Hodge structures encode many geometric properties of the variety X. One can show that every algebraic subvariety $Z \subset X$ of codimension d defines a class in $H^{d,d}(X) \cap H^{2d}(X, \mathbb{Z})$. Let $A^{2d}(X) \subset H^{2d}(X, \mathbb{Z})$ denote the subgroup generated by all such classes. The Hodge conjecture predicts that there is an equality

$$A^{2d}(X) \otimes \mathbb{Q} = H^{2d}(X, \mathbb{Q}) \cap H^{d,d}(X).$$

The right hand side is a linear algebraic object that can be computed relatively easily. The Hodge conjecture thus relates the set of algebraic subvarieties of X, which is a priori difficult to understand, to an algebraic invariant of X.

A hyperplane section $L \subset X$ defines a class $\eta \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, called a polarization of X. The polarization fixes the embedding of X into a projective space, induces a decomposition of $H^k(X, \mathbb{Q})$ into primitive direct components, called the Lefschetz decomposition. Note that the Lefschetz decomposition does not hold on $H^k(X, \mathbb{Z})$ usually since most polarizations are not principal. The polarization also defines a bilinear form

$$q : H^k(X, \mathbb{C}) \otimes H^k(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$$

such that the primitive components of $H^k(X, \mathbb{C})$ are orthogonal. The bilinear form q restricts to a bilinear form $q_{\mathbb{Q}} : H^k(X, \mathbb{Z}) \times H^k(X, \mathbb{Z}) \to \mathbb{Z}$, which is called a polarization of the Hodge structure on $H^k(X, \mathbb{Z})$. Polarized Hodge structures are much finer algebraic invariants than the cohomology groups. We see in Section 1.1.2 that in some cases they uniquely determine the polarized variety X.

1.1.1 Geometric realization of Hodge structures

Hodge structures can also be defined abstractly. Let $V_{\mathbb{Z}}$ be an abelian group. A Hodge structure of weight k on $V_{\mathbb{Z}}$ is a direct sum decomposition of the vector space

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p=0}^{k} V^{p,k-p}$$

satisfying $V^{p,k-p} = \overline{V^{k-p,p}}$. A polarization on a Hodge structure is a bilinear form $q : V_{\mathbb{Z}} \times V_Z \to \mathbb{Z}$ satisfying certain conditions. The numbers $h^{p,q} = \dim V^{p,q}$ are called the Hodge numbers of $V_{\mathbb{Z}}$.

In a recent paper [Sch15], Schreieder showed that under mild assumptions, almost all symmetric sequences of numbers $(h^{k,0}, \dots, h^{0,k})$ can be obtained as the weight k Hodge numbers of some smooth projective variety X. However, Hodge structures contain more information in the form of the embedding $V_{\mathbb{Z}}$ in $V_{\mathbb{C}}$ and one may ask if all Hodge structures arise geometrically.

To avoid problems with torsion groups, we only consider rational Hodge structures $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$. A polarized Hodge structure is called simple if it does not have any non-trivial polarized sub-Hodge structures. Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space, given a simple polarized rational Hodge structure $(V_{\mathbb{Q}}, V^{p,q}, q)$, we ask if there exists a smooth projective variety X such that $(H^k(X, \mathbb{Q}), H^{p,q}(X), q_H) \supseteq (V_{\mathbb{Q}}, V^{p,q}, q_V)$ with $q_V = q_{H|V_{\mathbb{Q}}}$.

For $k = 0$, the answer is trivially positive. For $k = 1$, given any polarized Hodge structure on a $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$, there exists an abelian variety $A = V^{1,0}/V_{\mathbb{Z}}$, where $V_{\mathbb{Z}}$ is any lattice in $V_{\mathbb{Q}}$ with the required polarization. For $k \geq 2$, there exist Hodge structures which do not arise geometrically, but there are no general results on when a Hodge structure is geometric.

An interesting case is when the weight $k = 2$ and $\dim V^{2,0} = 1$. In this case, Kuga and Satake [KS67] showed that any polarized weight 2 rational Hodge structure $\mathcal{V} = (V_{\mathbb{Q}}, V^{p,q})$ with $\dim V^{2,0} = 1$ is actually geometric. In their construction (cf. [Gee00]), they showed that there exists a polarized weight 1 Hodge structure $C^+(Q)$ with an inclusion of polarized Hodge structures $\mathcal{V} \hookrightarrow C^+(Q) \times C^+(Q)$. Let A be an abelian variety with polarized weight 1 Hodge structure $C^+(Q)$, then $H^2(A \times A, \mathbb{Z})$ contains $\mathcal{V}$ as a sub-Hodge structure. The abelian variety A obtained through the Kuga-Satake construction is of dimension 2^n where $n = \dim V$, so $H^2(A \times A, \mathbb{Q})$ becomes extremely large and intractable as n increases. One can then ask if there exist smaller geometric Hodge structures containing $\mathcal{V}$.

We say that a weight 2 Hodge structure is of type (p, n, p) if $\dim V^{2,0} = p$ and $\dim V^{1,1} = n$. Projective K3 surfaces provide examples of simple Hodge structures of type $(1, n, 1)$ for all $n \leq 19$. On the other hand, by the Enriques-Kodaira classification of minimal surfaces, there does not exist any smooth projective surface with $h^{2,0} = 1$ containing a simple Hodge structure of type $(1, n, 1)$ for $n > 19$. For $n = 20$, it is known that a general deformation of a Hilbert scheme $Z^{[2]}$ of a K3 surface Z contains a simple weight 2 sub-Hodge structure of type $(1, 20, 1)$. There are no known smooth projective varieties of any dimension with $h^{2,0} = 1$, containing simple Hodge structure of type $(1, n, 1)$ for $n > 20$.

For larger n, one should thus look for varieties X with Hodge structures of type (p, m, p) where $p > 1$ and $m > n$ containing a simple sub-Hodge structure of type $(1, n, 1)$. Note that if X is a smooth projective variety, and S is a surface obtained by taking successive hyperplane sections of X, then by the Lefschetz hyperplane theorem, we have $H^2(X, \mathbb{Q}) \hookrightarrow H^2(S, \mathbb{Q})$. Hence, if $H^2(X, \mathbb{Q})$ contains $\mathcal{V}$ as a sub-Hodge structure, then so does $H^2(S, \mathbb{Q})$ and it suffices to look for surfaces containing $\mathcal{V}$.

To find sub-Hodge structures, one can look for quotients by finite groups. Suppose S is a smooth projective surface and G is a finite abelian group acting on S. There is a quotient map $f : S \to F := S/G$ and an eigenspace decomposition

$$H^2(S, \mathbb{C}) = \bigoplus_{\chi \in \hat{G}} H^2(S, \mathbb{C})_\chi$$

where $\hat{G}$ is the character group of G and $H^2(S, \mathbb{C})_\chi$ is the eigenspace of the character χ, that is, $\sigma(s) = \chi(\sigma)(s)$ for all $\sigma \in G$ and $s \in H^2(S, \mathbb{C})_\chi$. Note that the eigenspace $H^2(S,\mathbb{C})_1$ of the trivial character is equal to $H^2(F,\mathbb{C})$.

The eigenspace decomposition is a decomposition of rational Hodge structures if $\chi(\sigma) \in \mathbb{Q}$ for all $\chi \in \hat{G}$ and $\sigma \in G$, that is, if $G = (\mathbb{Z}/2\mathbb{Z})^k$ is a product of involutions. It is thus interesting to seek surfaces S with an involution ι such that the (-1)-eigenspace $H^2(S,\mathbb{Q})_-$ contains a simple Hodge structure of type $(1,n,1)$ with $n > 20$.

In Chapter 4 of this thesis, we study two examples of nodal surfaces in detail. A nodal surface is a surface whose only singularities are ordinary double points. Let $F \subset \mathbb{P}^3$ be a nodal surface. The set of nodes on F is said to form an even set if there exists a double cover $f : S \to F$ which is branched precisely on the set of nodes of F.

Such surfaces have been studied by Casnati, Catanese and Tonoli [CC97; CT07]. They showed that there are very few possibilities for the cardinality of the set of nodes. For F being a sextic surface, an even set of nodes can have cardinality $t \in \{24, 32, 40, 56\}$. We studied the cases where $t = 40$ and $t = 56$.

Of particular interest are nodal sextic surfaces with an even set of 40 nodes (cf. Chapter 4.2). In this case, we showed that $H^2(S,\mathbb{Q})_-$ is of Hodge type $(1, 26, 1)$. However, we constructed the complete family of even 40-nodal sextic surfaces and showed that they can, in general, be obtained as hyperplane sections of EPW sextic fourfolds, which were extensively studied by Kieran O'Grady [OGr06; OGr13]. His work shows that $H^2(S,\mathbb{Q})_-$ has a sub-Hodge structure $\mathcal{V}$ of type $(1, 20, 1)$, and that $\mathcal{V}$ is the weight 2 Hodge structure of a deformation of a Hilbert scheme $Z^{[2]}$ for some K3 surface Z. As mentioned above, such Hodge structures $\mathcal{V}$ are well-understood, and we do not obtain any new interesting simple Hodge structures.

1.1.2 Deformations and Torelli type results

In complex geometry, one often seeks examples of surfaces satisfying certain properties. For example, K3 surfaces are simply connected compact Kähler manifolds with trivial canonical bundles $\omega_X = \mathcal{O}_X$ (cf. [Huy15]). It is possible to find specific examples of K3 surfaces, for example, any smooth projective quartic surface in $\mathbb{P}^3$ is a K3 surface, but when studying such examples, one needs to distinguish between properties specific to these examples and properties that are satisfied by a "general" K3 surface. A smooth projective quartic surface in $\mathbb{P}^3$ has an ample divisor given by a hyperplane section, but a "general" K3 surface is not projective, and hence has no ample divisors. A natural question to ask is: how many K3 surfaces are there, and how many of them contain ample divisors? The answers to both of these questions are known: the moduli space of K3 surfaces is 20 dimensional, in which the moduli space of projective K3 surfaces forms a countable union of 19-dimensional subspaces. It is also not a coincidence that a K3 surface X has $h^{1,1}(X) := \dim H^{1,1}(X) = 20$ and, if X is projective, then the Neron-Severi group $NS(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ has rank ≤ 19. Indeed, in Proposition 3.3.15, we recall that this is the only case in which the deformation of a projective hypersurface may not be projective.

We see that the relevant objects of study should be families of varieties rather than varieties. A moduli space, informally speaking, is the set of isomorphism classes of (polarized) varieties satisfying certain properties, and can be endowed with a natural topology making it an algebraic variety (or scheme or stack). For example, one can talk about the moduli space of algebraic curves of fixed genus g.

Moduli spaces, if they exist, are usually very singular and difficult to describe. One can map them to better understood moduli spaces, and try to understand the image and the fibres of the morphism. One such space is the period domain, which is a moduli space of Hodge structures over a fixed $\mathbb{Z}$-module $V_{\mathbb{Z}}$ (or $\mathbb{Q}$-vectorspace $V_{\mathbb{Q}}$). The map that sends a variety to its Hodge structure is called the period map. Deformation theory and period maps are rich subjects, covered in many books, eg. [CMP03].

A Torelli-type result asks if the period map is injective. It is named after Torelli, who proved that the period map for smooth projective curves of genus g is injective. However, in general, Torelli-type results are difficult to obtain. They are only known to hold for K3 surfaces and most projective hypersurfaces.

An easier question is whether the period map is locally injective. By taking the derivative of the period map at the point corresponding to a variety X, we obtain the infinitesimal period map at X. A variety is said to satisfy the infinitesimal Torelli property if the infinitesimal period map is injective.

An important result of Kodaira and Spencer is that for a smooth projective variety X, the set of isomorphism classes of infinitesimal deformations of X can be parametrized by the cohomology group $H^1(X, T_X)$ where T_X is the tangent sheaf of X. The infinitesimal period map can also be expressed entirely in terms of sheaves on X:

$$d\mathcal{P}^k : H^1(X, T_X) \to \bigoplus_{i=0}^{k-1} \mathrm{Hom}(H^{k-i,i}(X), H^{k-i-1,i+1}(X)).$$

To study families of nodal surfaces, we need to extend these classical results to singular varieties. This forms the bulk of Chapters 2 and 3 of this thesis.

Steenbrink extensively studied the Hodge structure on varieties with quotient singularities, called V-manifolds [Ste77]. Using certain sheaves of differential forms on such varieties, he showed that V-manifolds have pure Hodge structures. Using Steenbrink's definitions, we prove the infinitesimal Torelli theorem for nodal surfaces (Theorem 3.3.16).

1.2 Organization of the thesis

The main goal of this thesis is to study families of nodal surfaces.

Chapters 2 and 3 set the stage by extending general constructions and results for smooth projective varieties to singular varieties. Many results in these two chapters may be familiar to experts but some proofs have been included since appropriate references could not be found.

Chapter 2 focuses on Hodge theoretical aspects. We review classical Hodge theory (Section 2.1) and Steenbrink's construction of sheaves of differential forms on V-manifolds (Section 2.2). We also recall the explicit computation of the cohomology groups as sub-modules of polynomial rings in the case of projective hypersurfaces (Section 2.3).

Extending Steenbrink's definitions, we define tangent sheaves on V-manifolds (Section 2.2.2). We also state the log-cotangent short exact sequence for V-manifolds as divisors on smooth projective varieties (Theorem 2.2.14) but defer the technical proof to Chapter 5. Instead, we prove it directly for nodal surfaces in Section 2.2.4.

In Chapter 3, we recall the definition of the Kodaira-Spencer map (Section 3.1), including that for divisors and for G-equivariant deformations, and the infinitesimal period map (Section 3.2).

The main new result of this chapter is the infinitesimal Torelli theorem for nodal surfaces (Theorem 3.3.16), which is proven in Section 3.3.

Chapter 4 forms the bulk of the thesis. We construct and study two families of nodal surfaces, their deformations and Hodge structures.

Even 56-nodal sextic surfaces are studied in Section 4.1. A family of such surfaces has previously been constructed by Catanese and Tonoli [CT07], but we give a simpler and more geometric construction of even 56-nodal sextic surfaces, starting from a non-hyperelliptic genus 3 curve C and the choice of a divisor $B \in |2K_{S^2C}|$ (Theorem 4.1.1). We show that the 12-dimensional deformation family we obtain is a smooth open dense subset of the family obtained in [CT07, Main Theorem B] (Corollary 4.1.7) and that deformations in the family are unobstructed (4.1.14). We also give an explicit method for constructing numerical examples of such surfaces in Section 4.1.4.

The contents of Chapter 4.1, other than Section 4.1.3, are contained in the preprint [GZ16], which has been accepted for publication by the Journal of Algebraic Geometry.

In Chapter 4.2, we study even 40-nodal sextic surfaces. We recall three constructions of even 40-nodal sextic surfaces, due to Gallarati, Casnati-Catanese and the one arising from EPW sextic fourfolds. We give explicit examples of each construction, and use them to prove numerous results. We prove that all three constructions yield the same universal smooth irreducible 28-dimensional family of even 40-nodal sextic surfaces (Proposition 4.2.11, Corollary 4.2.15 and Theorem 4.2.20). Using the EPW sextic construction, we show that the negative eigenspace $H^2(S,\mathbb{C})_-$ of type $(1,26,1)$ has a sub-Hodge structure of type $(1,20,1)$. In Section 4.2.5, we describe an example of an even 40-nodal sextic surface with additional involutions using the results of Camere [Cam12] for EPW sextic fourfolds. All results in this section, other than the constructions, are original.

Finally, in Chapter 5, we prove two technical results (Remark 2.2.6 and Theorem 2.2.14) from Chapter 2. In Sections 5.1 and 5.2, we give a quick introduction to the theories of perverse sheaves [BBD82] and mixed Hodge modules [Sai88], recalling only the results necessary for our application. The proofs of our results are given in Section 5.3.

Chapter 2

Hodge theory

Throughout this thesis, we shall only work with varieties defined over the complex numbers.

Complex Hodge structures on smooth projective varieties are relatively well-understood, following the classical works of Hodge [Hod41] and Griffiths [Gri68]. While Deligne [Del71a] introduced the notion of mixed Hodge theory to deal with singular varieties, in practice, the mixed Hodge structures can be extremely complicated and difficult to compute.

In this chapter, we give concrete results for computing the complex Hodge structures on certain singular varieties. The singularities on the varieties we will consider are called quotient singularities, which are obtained as quotients of smooth manifolds by finite groups. They were studied extensively by Steenbrink [Ste77]. We shall review complex Hodge theory, the results of Steenbrink and others, as well as give some simple extensions of these results.

2.1 Review

In this section, we state some definitions and results from classical Hodge theory. This is by no means complete, interested readers may refer to the many excellent textbooks on this subject, for example, [Voi02].

Definition 2.1.1. Let H_R be an R-module ($R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$). A pure R-Hodge structure of weight k on H_R is the data of a finite decreasing filtration $F^pH_\mathbb{C}$ on the complexification $H_\mathbb{C} := H_R \otimes \mathbb{C}$, called the Hodge filtration, satisfying the condition that

$$F^pH_\mathbb{C} \cap \overline{F^{k+1-p}H_\mathbb{C}} = 0 \qquad \text{and} \qquad F^pH_\mathbb{C} \oplus \overline{F^{k+1-p}H_\mathbb{C}} = H_\mathbb{C} \qquad \forall\, 0 \leq p \leq k.$$

Let $H^{p,k-p} = F^pH_\mathbb{C} \cap \overline{F^{k-p}H_\mathbb{C}}$, then there is a direct sum decomposition

$$H_\mathbb{C} = \bigoplus_{i\in\mathbb{Z}} H^{i,k-i}.$$

A Hodge structure is a purely algebraic object. The interest in Hodge structures arise from the fact that, for every smooth projective manifold X, it is possible to associate a weight k Hodge structure to its cohomology group $H^k(X, \mathbb{Z})$. This is done as follows: there is a resolution of the constant sheaf $\mathbb{C}_X$ by sheaves of differential forms

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$$

where $\dim X = n$ and Ω^p_X are the sheaves of holomorphic differential p-forms on X. This is called the (holomorphic) de Rham complex. We set $\Omega^p_X = 0$ for $p < 0$ and $p > n$. To this resolution, one can associate the Fröhlicher spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \Longrightarrow H^{p+q}(X, \mathbb{C}).$$

If X is a compact Kähler manifold, this spectral sequence can be shown to degenerate at E_1, that is to say, there is a direct sum decomposition

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X) \quad \text{where} \quad H^{p,q}(X) := H^q(X, \Omega^p_X).$$

Furthermore, there are isomorphisms $H^{p,q}(X) = \overline{H^{q,p}(X)}$, so we have a Hodge structure on $H^k(X, \mathbb{C})$ given by the filtration

$$F^p H^k(X, \mathbb{C}) = \bigoplus_{i \geq p} H^{i,k-i}(X).$$

More intrinsically, there is a filtration on the de Rham complex given by

$$F^p \Omega^\bullet_X = \Omega^{\geq p}_X = (0 \to \Omega^p_X \to \cdots \Omega^n_X \to 0)$$

which induces the isomorphism

$$F^p H^k(X, \mathbb{C}) = \mathbb{H}^k(X, F^p \Omega^\bullet_X).$$

However, if X is not a smooth projective variety, the de Rham resolution may not induce a pure Hodge structure on $H^k(X, \mathbb{Q})$. Deligne [Del71b] introduced the notion of mixed Hodge structures, on which there is a weight filtration in addition to the Hodge filtration, and every graded weight component has a pure Hodge structure. He also showed that, on any variety X, $H^k(X, \mathbb{Q})$ has a mixed Hodge structure. We will not use the weight filtration in this thesis, interested readers can refer to numerous texts such as [Del71b; PS08; Voi03].

Consider the de Rham complex on a smooth open algebraic variety U. If U is affine, then $H^i(U, \Omega^p_U) = 0$ for all $i > 0$, so the standard de Rham complex does not give any interesting structure on $H^k(U, \mathbb{C})$. We instead consider the log-de Rham complex.

Definition 2.1.2. Let Y be smooth projective variety and X be a divisor on Y. The sheaf of log differential forms is the sheaf of forms ω having simple poles along X whose differentials have simple poles as well, i.e.

$$\begin{aligned}\Omega_Y^p(\log X) &= \{\omega \in \Omega_Y^p(X) \mid d\omega \in \Omega_Y^p(X)\}\\ &= \ker(\Omega_Y^p(X) \xrightarrow{d} \Omega_Y^{p+1}(2X)/\Omega_Y^{p+1}(X)).\end{aligned}$$

There is a well-defined complex

$$\Omega_Y^\bullet(\log X) = \big(0 \to \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1(\log X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_Y^{n-1}(\log X) \xrightarrow{d} \Omega_Y^n(X) \to 0\big)$$

of sheaves of log differential forms, where d is the usual differential. This is called the log-de Rham complex.

We also define $T_Y(-\log X) = \mathcal{H}\mathrm{om}(\Omega_Y^1(\log X), \mathcal{O}_Y) =: \Omega_Y^1(\log X)^\vee$.

Example 2.1.3. Suppose $X \subset Y$ is a simple normal crossing divisor. Let $z_1, \ldots, z_n$ be local coordinates on Y, with X being defined by $z_1 \cdots z_k = 0$. Then, $\Omega_Y^p(\log X)$ is locally generated by p-th exterior products of the differential forms $\{\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n\}$.

Let U be a smooth open algebraic variety of dimension n. Then there exists a compactification $Y \supset U$ such that $X = Y \setminus U$ is a simple normal crossing divisor. Let $j : U \to Y$ be the open inclusion. There is a quasi-isomorphism between $Rj_*\mathbb{C}_U$ and the log-de Rham complex:

$$Rj_*\mathbb{C}_U \overset{\text{qis}}{=} \Omega_Y^\bullet(\log X).$$

Define a Hodge filtration on $Rj_*\mathbb{C}_U$ by

$$F^p(Rj_*\mathbb{C}_U) = \Omega_Y^{\geq p}(\log X) = (0 \to \Omega_Y^p(\log X) \to \cdots \to \Omega_Y^n(X) \to 0).$$

In this case, the Leray spectral sequence

$$E_1^{pq} = H^q(Y, \Omega_Y^p(\log X)) \Longrightarrow H^{p+q}(X, Rj_*\mathbb{C}_U) = H^{p+q}(U, \mathbb{C}_U)$$

degenerates at E_1, but there is no Hodge symmetry, i.e. $E_1^{pq} \neq E_1^{qp}$, hence it does not define a pure Hodge structure on $H^k(U, \mathbb{Q}_U)$. One can refer to [Voi02, Section 8.4.1] for the definition of the weight filtration on $H^k(U, \mathbb{Q}_U)$.

A key step in the construction of the weight filtration uses a set of short exact sequences, which are of independent interest. We present them in following proposition.

Proposition 2.1.4 ([EV92, §2.3])**.** *Let Y be a smooth algebraic variety of dimension n and $X \subset Y$ a smooth reduced divisor. Then, there are short exact sequences*

$$0 \to \Omega_Y^p \xrightarrow{i} \Omega_Y^p(\log X) \xrightarrow{r} \Omega_X^{p-1} \to 0, \qquad 1 \leq p \leq n; \tag{2.1}$$

$$0 \to \Omega_Y^p(\log X)(-X) \xrightarrow{i} \Omega_Y^p \xrightarrow{r} \Omega_X^p \to 0, \qquad 0 \leq p \leq n-1. \tag{2.2}$$

More generally, let $X = \bigcup_{i=1}^k X_i$ be a simple normal crossing divisor with irreducible components X_i. Then, there are short exact sequences

$$0 \to \Omega_Y^1 \xrightarrow{i} \Omega_Y^1(\log X) \xrightarrow{r} \bigoplus_{i=1}^k \mathcal{O}_{X_i} \to 0; \tag{2.3}$$

$$0 \to \Omega_Y^{n-1}(\log X)(-X) \xrightarrow{i} \Omega_Y^{n-1} \xrightarrow{r} \bigoplus_{i=1}^k \omega_{X_i} \to 0. \tag{2.4}$$

Proof. These short exact sequences can be defined locally. Let $z_1, \dots, z_n$ be a set of local coordinates around a point $x \in X$ such that the divisor X is given by $z_1 = 0$. The first map i in each sequence is an inclusion of sheaves. The second map r in (2.1) is the residue map defined by taking $\frac{dz_1}{z_1} \mapsto 1$ and killing all terms without the factor $\frac{dz_1}{z_1}$. The second map r in (2.2) is the restriction map that kills z_1. It is an easy exercise to check that the two sequences are exact. With a similar argument, one can check that the short exact sequences (2.1) for $p = 1$ and (2.2) for $p = n - 1$ generalize to simple normal crossing divisors. □

We obtain from the short exact sequence (2.4) a short exact sequence relating the tangent and the log tangent sheaves of Y.

Corollary 2.1.5. *Let Y be a smooth algebraic variety of dimension n and $X = \bigcup_{i=1}^k X_i$ be a simple normal crossing divisor in Y with irreducible components X_i. Then there is a short exact sequence*

$$0 \to T_Y(-\log X) \to T_Y \to \bigoplus_{i=1}^k \mathcal{O}_{X_i}(X_i) \to 0.$$

Proof. This sequence is obtained by tensoring the short exact sequence (2.4) by the locally free sheaf ω_Y^{-1}. The perfect pairings [Voi02]

$$\Omega_Y^p \otimes \Omega_Y^{n-p} \to \omega_Y, \qquad \Omega_Y^p(\log X) \otimes \Omega_Y^{n-p}(\log X) \to \omega_Y(X)$$

induce canonical isomorphisms

$$\begin{aligned} \Omega_Y^p &= \mathcal{H}\mathrm{om}(\Omega_Y^{n-p}, \omega_Y), \\ \Omega_Y^p(\log X) &= \mathcal{H}\mathrm{om}(\Omega_Y^{n-p}(\log X), \omega_Y(X)) = \mathcal{H}\mathrm{om}(\Omega_Y^{n-p}(\log X)(-X), \omega_Y). \end{aligned}$$

Hence, there are isomorphisms

$$\begin{aligned} T_Y &= \Omega_Y^{1\vee} = \mathcal{H}\mathrm{om}(\Omega_Y^{n-1}, \omega_Y)^\vee = \Omega_Y^{n-1} \otimes \omega_Y^{-1} \qquad \text{and} \\ T_Y(-\log X) &= \Omega_Y^1(\log X)^\vee = \mathcal{H}\mathrm{om}(\Omega_Y^{n-1}(\log X)(-X), \omega_Y)^\vee \\ &= \Omega_Y^{n-1}(\log X)(-X) \otimes \omega_Y^{-1}. \end{aligned}$$

For the last term, the adjunction formula gives $\omega_{X_i} = \omega_Y(X_i)_{|X_i}$, so $\Omega^{n-1}_{X_i} \otimes \omega_Y^{-1} = \omega_{X_i} \otimes \omega_{X_i}^{-1}(X_i) = \mathcal{O}_{X_i}(X_i)$. □

Remark 2.1.6. In this section, we only considered complex Hodge structures, where we take $R = \mathbb{C}$. Complex Hodge structures are the easiest to study, and it is sufficient for the purpose of this thesis. However, it is worth noting that the integral and rational Hodge structures contain most of the interesting geometrical information, but they are less well understood. The rational Hodge structure, for example, is the subject of the Hodge conjecture, which asks if, for a smooth projective variety X, all classes in $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ arise from algebraic cycles, in other words, they are of geometric origin.

2.2 V-manifolds

In this section, we study in greater detail the Hodge structure on a special type of singular complex analytic variety known as V-manifolds. The singularities on V-manifolds are by definition quotient singularities and are "mild". We shall show that the Hodge structures of V-manifolds are pure by directly defining sheaves of differential forms $\tilde{\Omega}^\bullet$ on them. We will study these sheaves in greater detail. Most of the results in this section are due to Steenbrink [Ste77, Section 1].

2.2.1 Definition and first properties

Definition 2.2.1. A V-manifold is a complex analytic variety X of dimension n which admits an open covering $X = \bigcup_{i \in I} U_i$ such that for each $i \in I$, there is an analytic isomorphism $U_i = D_i/G_i$ where $D_i \subset \mathbb{C}^n$ is an open ball and $G_i \subset GL(n, \mathbb{C})$ is a finite subgroup.

A V-manifold is normal and hence the singular locus Σ has codimension $\mathrm{codim}_X \Sigma \geq 2$. The singularities of a V-manifold are quotient singularities by definition.

Definition 2.2.2. A finite subgroup G of $GL(n, \mathbb{C})$ is called small if no element of G has 1 as an eigenvalue of multiplicity exactly $n-1$, i.e. G does not contain rotations about hyperplanes. Conversely, a subgroup $G \subset GL(n, \mathbb{C})$ is called big if it is generated by elements of G that have 1 as an eigenvalue of multiplicity exactly $n-1$, that is, it is generated by rotations about hyperplanes.

Every finite subgroup $G \subset GL(n, \mathbb{C})$ admits a unique maximal big normal subgroup G_{big} such that the quotient G/G_{big} is small. The quotient by a big subgroup is smooth, that is, there is an isomorphism $\mathbb{C}^n/G_{\text{big}} \cong \mathbb{C}^n$. If $x \in \Sigma$ is a singular point in X, then there is an open neighbourhood U of x such that $U = D/G$ where $D \subset \mathbb{C}^n$ is an open ball and G is a small subgroup. We will focus on quotients by small subgroups in this section.

Definition 2.2.3. Let X be a V-manifold with singular locus Σ. Let $j : X \setminus \Sigma \to X$ be the open inclusion. Define $\tilde{\Omega}^p_X = j_* \Omega^p_{X\setminus\Sigma}$.

Remark 2.2.4. Any meromorphic function on X that is holomorphic outside a subset Σ of codimension ≥ 2 extends uniquely to a holomorphic function on X. Hence, $\tilde{\Omega}^0_X = j_*\mathcal{O}_{X\setminus\Sigma} = \mathcal{O}_X$. (cf. [Ser66, Proposition 4]).

We summarize some results of Steenbrink regarding the properties of $\tilde{\Omega}^p_X$:

Theorem 2.2.5 ([Ste77, (1.8–1.13)])**.** *Let X be a V-manifold and the sheaves $\tilde{\Omega}^p_X$ be defined as in Definition 2.2.3.*

(i) Let $U \subset X$ be an open subset such that $U = D/G$ where $D \subset \mathbb{C}^n$ is an open ball and G is a small subgroup. Let $f : D \to U$ be the quotient map. Then $\tilde{\Omega}^p_{X|U} = (f_*\Omega^p_D)^G$.

(ii) Let $\pi : \tilde{X} \to X$ be a resolution of singularities of X. Then, $\tilde{\Omega}^p_X = \pi_*\Omega^p_{\tilde{X}}$.

(iii) There is a perfect pairing $\tilde{\Omega}^p_X \otimes \tilde{\Omega}^{n-p}_X \to \tilde{\Omega}^n_X =: \tilde{\omega}_X$ and $\tilde{\omega}_X$ is the dualizing sheaf ([GR70, Section 3.2], cf. [Ste77, Proof of 1.12]), that is, for any coherent sheaf $\mathcal{F}$ on X, there is a canonical isomorphism $\mathrm{Ext}^p(\mathcal{F}, \tilde{\omega}_X)^\vee = H^{n-p}(X, \mathcal{F})$.

Remark 2.2.6. The isomorphism of (ii) induces an injective morphism in cohomology
$$H^k(X, \tilde{\Omega}^p_X) \cong H^k(X, \pi_*\Omega^p_{\tilde{X}}) \hookrightarrow H^k(\tilde{X}, \Omega^p_{\tilde{X}}).$$
This map is usually not surjective since the higher derived images $R^i\pi_*\Omega^p_{\tilde{X}}$ ($i > 0$) do not vanish in general. It is possible to find a sheaf on $\tilde{X}$ whose derived direct image is isomorphic to $\tilde{\Omega}^p_X$. To do so requires some advanced machinery which we will treat in Chapter 5. In Proposition 2.2.21, we shall describe it for the simplest case of nodal surfaces.

A consequence is that $\tilde{\Omega}^p_X$ is coherent for all p and vanishes for $p < 0$ and $p > n$ [Ste77, (1.10)]. The complex
$$\tilde{\Omega}^\bullet_X = (0 \to \mathcal{O}_X \to \tilde{\Omega}^1_X \to \cdots \to \tilde{\Omega}^n_X \to 0)$$
is a resolution of $\mathbb{C}_X$ [Ste77, (1.9)]. Similar to the smooth case in Section 2.1, Peters and Steenbrink showed

Theorem 2.2.7 ([PS08, Theorem 2.43]). *Let X be a projective V-manifold. Then $H^k(X,\mathbb{Q})$ admits a pure Hodge structure of weight k. In particular, there is a Hodge decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad \textit{where} \quad H^{p,q}(X) := H^q(X, \tilde{\Omega}^p_X).$$

2.2.2 Tangent sheaves

As we are interested in studying the deformation theory of V-manifolds, we need a notion of a tangent sheaf.

Definition 2.2.8. We define the tangent sheaf of a V-manifold X to be $\tilde{T}_X = \mathcal{H}\mathrm{om}(\tilde{\Omega}^1_X, \mathcal{O}_X)$.

We claim that, on each open subset $U = D/G \subset X$, the tangent sheaf so defined is precisely the G-invariant part of T_D. This justifies the definition (cf. Theorem 2.2.5(i)). We first need a technical lemma.

Lemma 2.2.9. *Let $D \subset \mathbb{C}^n$ be an open subvariety, endowed with the action of a finite subgroup $G \subset GL(n,\mathbb{C})$. Suppose $f : D \to U = D/G$ is a finite étale covering (i.e. it is unramified). Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on D and U respectively, then there is an isomorphism of $\mathcal{O}_U$-modules*

$$\mathcal{H}\mathrm{om}((f_*\mathcal{E})^G, \mathcal{F}) \cong f_*\mathcal{H}\mathrm{om}(\mathcal{E}, f^*\mathcal{F})^G.$$

Proof. It suffices to check the isomorphism locally around each point $x \in U$. Since f is finite étale, we can choose a sufficiently small open neighbourhood V of x such that $f^{-1}V = \coprod_{i=1}^g V_i$ where $g = |G|$ and V_i are all isomorphic to V. On V, we can evaluate

$$\begin{aligned}\Gamma(V, \mathcal{H}\mathrm{om}((f_*\mathcal{E})^G, \mathcal{F})) &= \mathrm{Hom}((\bigoplus \mathcal{E}(V_i))^G, \mathcal{F}(V)),\\ \Gamma(V, f_*\mathcal{H}\mathrm{om}(\mathcal{E}, f^*\mathcal{F})^G) &= \mathrm{Hom}(\bigoplus \mathcal{E}(V_i), \bigoplus f^*\mathcal{F}(V_j))^G.\end{aligned}$$

Note that $\mathrm{Hom}(\mathcal{E}(V_i), f^*\mathcal{F}(V_j)) = 0$ unless $i = j$, so

$$\mathrm{Hom}(\bigoplus \mathcal{E}(V_i), \bigoplus f^*\mathcal{F}(V_j)) = \bigoplus \mathrm{Hom}(\mathcal{E}(V_i), f^*\mathcal{F}(V_i)).$$

The group G acts by permuting the components of the direct sum, so there are isomorphisms $\mathcal{F}(V) \cong f^*\mathcal{F}(V_i)$ for all i. Therefore, we get

$$\begin{aligned}\Gamma(V, \mathcal{H}\mathrm{om}((f_*\mathcal{E})^G, \mathcal{F})) &= \mathrm{Hom}((\bigoplus \mathcal{E}(V_i))^G, \mathcal{F}(V)) \cong \mathrm{Hom}(\mathcal{E}(V_1), \mathcal{F}(V))\\ &\cong \mathrm{Hom}(\mathcal{E}(V_1), f^*\mathcal{F}(V_1)) \cong \bigoplus \mathrm{Hom}(\mathcal{E}(V_i), f^*\mathcal{F}(V_i))^G\\ &\cong \Gamma(V, f_*\mathcal{H}\mathrm{om}(\mathcal{E}, f^*\mathcal{F})^G).\end{aligned}$$

□

Proposition 2.2.10. *Let $U = D/G$ where $D \subset \mathbb{C}^n$ is an open ball and $G \subset GL(n, \mathbb{C})$ is a small subgroup. Let $f : D \to U$ be the quotient map. Then, $\tilde{T}_U = (f_* T_D)^G$.*

Proof. The proof follows by formal operations using the definition of $\tilde{T}_U$. Let $\Sigma = \operatorname{Sing} U$. Then,

$$\begin{aligned}
\tilde{T}_U &= \mathcal{H}\text{om}(\tilde{\Omega}^1_U, \mathcal{O}_U) \\
&= \mathcal{H}\text{om}(j_* \Omega^1_{U \setminus \Sigma}, j_* \mathcal{O}_{U \setminus \Sigma}) && \text{(Definition 2.2.3)} \\
&= j_* \mathcal{H}\text{om}(j^* j_* \Omega^1_{U \setminus \Sigma}, \mathcal{O}_{U \setminus \Sigma}) && \text{(Projection formula)} \\
&= j_* \mathcal{H}\text{om}(\Omega^1_{U \setminus \Sigma}, \mathcal{O}_{U \setminus \Sigma}) && (j^* j_* = \text{id}) \\
&= j_* \mathcal{H}\text{om}((f_* \Omega^1_{D \setminus \Sigma})^G, \mathcal{O}_{U \setminus \Sigma}) && \text{(G-invariance for finite étale covers)} \\
&= j_* f_* \mathcal{H}\text{om}(\Omega^1_{D \setminus \Sigma}, f^* \mathcal{O}_{U \setminus \Sigma})^G && \text{(Lemma 2.2.9)} \\
&= f_* j_* \mathcal{H}\text{om}(\Omega^1_{D \setminus \Sigma}, \mathcal{O}_{D \setminus \Sigma})^G && (j_* f_* (-)^G = f_* j_* (-)^G) \\
&= (f_* j_* T_{D \setminus \Sigma})^G = (f_* T_D)^G.
\end{aligned}$$

The last equality holds by a similar argument to the first four equalities. □

By Theorem 2.2.5(iii), we have

$$\tilde{T}_X = (\tilde{\Omega}^1_X)^\vee = \mathcal{H}\text{om}(\tilde{\Omega}^{n-1}_X, \tilde{\omega}_X)^\vee = \tilde{\omega}^\vee_X \otimes \tilde{\Omega}^{n-1}_X.$$

This gives us an alternative characterization of the tangent sheaf which will be useful later:

Lemma 2.2.11. *Let X be a V-manifold. Then, $\tilde{T}_X = \tilde{\Omega}^{n-1}_X \otimes \tilde{\omega}^\vee_X$.*

To end the section, we combine these characterizations of the tangent and cotangent sheaves on V-manifolds with classical results for quotients by big subgroups.

Definition 2.2.12. Let Y be a V-manifold with singular locus Σ. Let $j : Y \setminus \Sigma \to Y$ be the open inclusion. For a divisor $X \subset Y$, define $\tilde{\Omega}^p_Y(\log X) = j_* \Omega^p_{Y \setminus \Sigma}(\log X \setminus \Sigma)$ and $\tilde{T}_Y(-\log X) = \tilde{\Omega}^1_Y(\log X)^\vee$.

Corollary 2.2.13. *Suppose $U = D/G$ where $D \subset \mathbb{C}^n$ is an open ball and $G \subset GL(n, \mathbb{C})$ is an abelian subgroup. Let $f : D \to U$ be the quotient map and let B be the union of the codimension 1 components of the branch locus. Then, there are isomorphisms of sheaves $(f_* \Omega^p_D)^G = \Omega^p_U$ and $(f_* T_D)^G = \tilde{T}_U(-\log B)$.*

Proof. The group G has a maximal big normal subgroup G_{big} such that $G_{\text{small}} = G/G_{\text{big}}$ is small. Let $f' : D \to D' = D/G_{\text{big}}$ be the quotient by the abelian group G_{big}. It is branched along a normal crossing divisor $B' \subset D'$, with $(f'_*\Omega^p_D)^{G_{\text{big}}} = \Omega^p_{D'}$ and $(f'_*T_D)^{G_{\text{big}}} = T_{D'}(-\log B')$ [Par91, Proposition 4.1].

The action of G_{small} on D' fixes a locus Σ of codimension ≥ 2. Under the quotient map $f'' : D' \to U$, the image of B' is precisely the union of the codimension 1 components of the branch locus. Hence, we get

$$(f''_*\Omega^p_{D'})^{G_{\text{small}}} = \tilde{\Omega}^p_U \qquad \text{and} \qquad (f''_*T_{D'}(-\log B'))^{G_{\text{small}}} = \tilde{T}_U(-\log B).$$

□

2.2.3 V-manifolds as divisors on smooth varieties

A common technique in the study of singular varieties, especially from an analytical point of view, is to embed them into a smooth manifold. In this section, we suppose Y is a smooth algebraic variety of dimension n and $X \subset Y$ is a divisor such that X is a V-manifold. We want an analogue of Proposition 2.1.4.

The short exact sequences (2.1) and (2.2) do not hold in general for all singular varieties X (assuming we have a reasonable definition for $\tilde{\Omega}^p_X$). In [Ste06], Steenbrink gave some classes of surface singularities for which the short exact sequences hold. His proof (implicitly) uses computations on vanishing cycles around such singularities. We state the theorem for V-manifolds in general:

Theorem 2.2.14. *Let Y be a smooth projective variety of dimension $n+1$ and $X \subset Y$ be a reduced divisor with only quotient singularities. Then, there are exact sequences*

$$0 \to \Omega^p_Y \xrightarrow{i} \Omega^p_Y(\log X) \xrightarrow{r} \tilde{\Omega}^{p-1}_X \to 0, \qquad (1 \leq p \leq n); \tag{2.5}$$

$$0 \to \Omega^p_Y(\log X)(-X) \xrightarrow{i} \Omega^p_Y \xrightarrow{r} \tilde{\Omega}^p_X, \qquad (0 \leq p \leq n-1). \tag{2.6}$$

However, we will defer the proof to Chapter 5, where we show a more general version as an easy consequence of Saito's theory of mixed Hodge modules. It is possible to prove the result directly for isolated quotient singularities, see Proposition 2.2.23 for the case of nodal surfaces.

Remark 2.2.15. In contrast to the case where X is smooth, the left exact sequence (2.6) is almost never right exact when $p > 0$.

Now, we give some easy consequences of Theorem 2.2.14.

Corollary 2.2.16 (Adjunction formula)**.** *Let Y be a smooth projective variety of dimension n and $X \subset Y$ be a reduced divisor with only quotient singularities. Then $\tilde{\omega}_X = \omega_Y(X)_{|X}$.*

Proof. The short exact sequence

$$0 \to \omega_Y \to \omega_Y(X) \to \tilde{\omega}_X \to 0$$

induces a right exact sequence

$$\omega_{Y|X} \to \omega_Y(X)_{|X} \to \tilde{\omega}_X \to 0.$$

Let $z_1, \ldots, z_n$ be local coordinates in Y of a neighbourhood of $x \in X$, such that X is defined by a holomorphic function f. In these coordinates, the map $\omega_Y \to \omega_Y(X)$ is given by $g dz_1 \wedge \cdots \wedge dz_n \mapsto \frac{gf}{f} dz_1 \wedge \cdots \wedge dz_n$.

On restricting to X, $gf \equiv 0$, so the map $\omega_{Y|X} \to \omega_Y(X)_{|X}$ is zero. Hence, $\tilde{\omega}_X \cong \omega_Y(X)_{|X}$. □

Remark 2.2.17. If $Y = \mathbb{P}^n$, then Corollary 2.2.16 is a special case of [Har77, Theorem III.7.11]. In such a case, $\tilde{\omega}_X$ is an invertible sheaf.

However, $\tilde{\omega}_X$ is not invertible for a general V-manifold. For example, let $X = \mathbb{C}^2/\langle\sigma\rangle$ with σ being the action induced by the matrix $\mathrm{diag}(\zeta_3, \zeta_3)$ where ζ_3 is a primitive third root of unity. Then X is not Gorenstein at the origin and $\tilde{\omega}_X$ is not invertible.

Equipped with the adjunction formula, the same proof as that of Corollary 2.1.5 gives the following result:

Corollary 2.2.18. *Let Y be a smooth projective variety of dimension n and $X \subset Y$ be a reduced divisor with only quotient singularities. Then, there is a left exact sequence*

$$0 \to T_Y(-\log X) \to T_Y \to \mathcal{O}_X(X).$$

2.2.4 Example: nodal surfaces

The simplest example of a quotient singularity is an ordinary double point.

Definition 2.2.19. A nodal surface is a 2-dimensional projective V-manifold, with only ordinary double points as singularities. That is, the singularities of X are locally isomorphic to $\mathbb{C}^2/G$ where $G = \langle \mathrm{diag}(-1, -1) \rangle$, or in local coordinates, they are locally isomorphic to $\{z_1^2 - z_2 z_3 = 0\} \subset \mathbb{C}^3$.

In chapter 4, we will study the Hodge theory and deformations of nodal surfaces in details. The goal of this subsection is to give the construction suggested in Remark 2.2.6 and prove Theorem 2.2.14 without appealing to the machinery of Chapter 5.

Note that the sheaf $\Omega^p_{\tilde{X}}$ in Theorem 2.2.5(ii) is not the unique sheaf satisfying $\pi_*\Omega^p_{\tilde{X}} = \tilde{\Omega}^p_X$. The proof of [Ste77, Lemma 1.11] can be applied without change to obtain the following generalization.

Lemma 2.2.20 (cf. [Ste77, Lemma 1.11])**.** *Let $\pi : \tilde{X} \to X$ be a resolution of singularities for a V-manifold X such that the exceptional divisor E is simple normal crossing. Then, any differential form ω that is holomorphic on $\tilde{X} \setminus E$ and meromorphic on E is holomorphic on all of $\tilde{X}$. Hence, the sheaves $\pi_*\Omega^p_{\tilde{X}}(\log E)$ and $\pi_*\Omega^p_{\tilde{X}}(kE)$ for any $k \geq 0$ are isomorphic to $\tilde{\Omega}^p_X$.*

Proposition 2.2.21. *Let X be a nodal surface with a set of nodes $\Sigma = \{p_1, \ldots, p_k\}$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities of X with exceptional divisor $N = \coprod_{i=1}^k N_i$ such that $N_i = \pi^{-1}(p_i)$. There are isomorphisms*

$$R\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X, \qquad R\pi_*\omega_{\tilde{X}} = \tilde{\omega}_X,$$
$$R\pi_*\Omega^1_{\tilde{X}}(\log N) = R\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) = \tilde{\Omega}^1_X.$$

In particular, $R^1\pi_*\Omega^1_{\tilde{X}} = \bigoplus_{i=1}^k \mathbb{C}_{p_i}$ is a skyscraper sheaf.

Proof. By Lemma 2.2.20, the isomorphisms hold if we replace $R\pi_*$ with the underived direct image π_*. It remains to show that $R^i\pi_*$ of the given sheaves vanish for all $i > 0$.

Without loss of generality, we may assume that $k = 1$ and that X has only one node p, so $N = \pi^{-1}p \cong \mathbb{P}^1$. Consider the Cartesian square

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ p \arrow[r, "i"'] & X \end{tikzcd}$$

The derived base change formula gives

$$Li^*R\pi_*\Omega^l_{\tilde{X}} = R\pi_*Li^*\Omega^l_{\tilde{X}} = R\pi_*\Omega^l_{\tilde{X}|N} = R\Gamma(N, \Omega^l_{\tilde{X}|N}). \tag{2.7}$$

The second equality follows since $\Omega^l_{\tilde{X}}$ is a vector bundle on $\tilde{X}$, so it is flat and $Li^* = i^*$. Since the fibres of π have dimension ≤ 1, the higher direct images $R^i\pi_*\mathcal{F}$ vanish for all $i \geq 2$ and coherent sheaves $\mathcal{F}$, and are skyscraper sheaves concentrated on p for $i = 1$. Taking the first cohomology of (2.7) gives the value of $R^1\pi^*\Omega^l_{\tilde{X}}$ on p as

$$i^*R^1\pi_*\Omega^l_{\tilde{X}} = H^1(N, \Omega^l_{\tilde{X}}).$$

For $l = 0$, we have $H^1(N, \mathcal{O}_{\tilde{X}}) = H^1(N, \mathcal{O}_N) = 0$. For $l = 2$, using the adjunction formula $\omega_N = \omega_{\tilde{X}}(N)_{|N}$ gives $H^1(N, \omega_{\tilde{X}}) = H^1(N, \omega_N(-N)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2+2)) = 0$ since the self intersection $N \cdot N = -2$. Thus, we get the derived isomorphisms $R\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R\pi_*\omega_{\tilde{X}} = \tilde{\omega}_X$.

For $l = 1$, we first show that $R^1\pi_*\Omega^1_{\tilde{X}}(\log N) = 0$. We use the short exact sequence for cotangent bundles (cf. [Har77, Chapter II, Theorem 8.17(2)])

$$0 \to \mathcal{O}_N(-N) \to \Omega^1_{\tilde{X}|N} \to \Omega^1_N \to 0.$$

Since $H^1(N, \mathcal{O}_N(-N)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 0$, we get $H^1(N, \Omega^1_{\tilde{X}}) = H^1(\Omega^1_N) = \mathbb{C}$. Applying π_* to the short exact sequence (2.1)

$$0 \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log N) \to \mathcal{O}_N \to 0$$

gives a long exact sequence

$$\begin{aligned} 0 \to \pi_*\Omega^1_{\tilde{X}} \to \pi_*\Omega^1_{\tilde{X}}(\log N) \to \pi_*\mathcal{O}_N = \mathbb{C}_p \to \\ \to R^1\pi_*\Omega^1_{\tilde{X}} = \mathbb{C}_p \to R^1\pi_*\Omega^1_{\tilde{X}}(\log N) \to R^1\pi_*\mathcal{O}_N = 0. \end{aligned}$$

The first two terms are isomorphic by Theorem 2.2.5(ii) and Lemma 2.2.20. Hence, the map $\pi_*\mathcal{O}_N \to R^1\pi_*\Omega^1_{\tilde{X}}$ is an isomorphism and $R^1\pi_*\Omega^1_{\tilde{X}}(\log N) = 0$. Thus, $R\pi_*\Omega^1_{\tilde{X}}(\log N) = \tilde{\Omega}^1_X$.

To show the other equality $R\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) = \tilde{\Omega}^1_X$, we consider the short exact sequence (2.2)

$$0 \to \Omega^1_{\tilde{X}}(\log N)(-N) \to \Omega^1_{\tilde{X}} \to \Omega^1_N \to 0.$$

Applying the functor π_* gives a long exact sequence

$$\begin{aligned} 0 \to \pi_*\Omega^1_{\tilde{X}}(\log N)(-N) \to \pi_*\Omega^1_{\tilde{X}} \to 0 \to \\ \to R^1\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) \to R^1\pi_*\Omega^1_{\tilde{X}} \to \mathbb{C}_p \to 0. \end{aligned}$$

Since $R^1\pi_*\Omega^1_{\tilde{X}} \cong \mathbb{C}_p$, we get $R^1\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) = 0$, thus giving

$$R\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) = \pi_*\Omega^1_{\tilde{X}} = \tilde{\Omega}^1_X.$$

□

We can also give an explicit characterization of the tangent sheaf $\tilde{T}_X$ for a nodal surface.

Proposition 2.2.22. *Let X be a nodal surface with singular locus Σ. Let $\pi : (\tilde{X}, N) \to (X, \Sigma)$ be the resolution of singularities of X. Then, $\tilde{T}_X \cong R\pi_* T_{\tilde{X}}(-\log N)$.*

Proof. A node is a canonical singularity, that is, if $\pi : (\tilde{X}, N) \to (X, \Sigma)$ is a resolution of singularities, then $K_{\tilde{X}} = \pi^* K_X$ where $K_X := \pi_* K_{\tilde{X}}$ is a Weil divisor. Hence, $\omega_{\tilde{X}} = \pi^*\pi_*\omega_{\tilde{X}} = \pi^*\tilde{\omega}_X$. Furthermore, in the proof of Proposition 2.2.23, we will show that $\tilde{\omega}_X$ is generated by $\frac{dz_2 \wedge dz_3}{z_1}$ in a neighbourhood of the double point, so it is locally free and K_X is a Cartier divisor as well.

By Lemma 2.2.11, Proposition 2.2.21 and the projection formula, we obtain

$$\begin{aligned}
\tilde{T}_X &= \tilde{\Omega}^1_X \otimes \tilde{\omega}_X^\vee = R\pi_*\Omega^1_{\tilde{X}}(\log N)(-N) \otimes \tilde{\omega}_X^\vee \\
&= R\pi_*(\Omega^1_{\tilde{X}}(\log N)(-N) \otimes \pi^*\tilde{\omega}_X^\vee) = R\pi_*(\Omega^1_{\tilde{X}}(\log N) \otimes \omega_{\tilde{X}}^\vee(-N)) \\
&= R\pi_* T_{\tilde{X}}(-\log N).
\end{aligned}$$

The last equality follows since there is a perfect pairing

$$\Omega^1_{\tilde{X}}(\log N) \otimes \Omega^1_{\tilde{X}}(\log N) \to \omega_{\tilde{X}}(N).$$

$\square$

Finally, we prove Theorem 2.2.14 directly for nodal surfaces.

Proposition 2.2.23. *Let X be a nodal surface, then there are short exact sequences*

$$0 \to \Omega^p_{\mathbb{P}^3} \xrightarrow{i} \Omega^p_{\mathbb{P}^3}(\log X) \xrightarrow{r} \tilde{\Omega}^{p-1}_X \to 0, \qquad 1 \le p \le 3); \tag{2.8}$$

$$0 \to \Omega^p_{\mathbb{P}^3}(\log X)(-X) \xrightarrow{i} \Omega^p_{\mathbb{P}^3} \xrightarrow{r} \tilde{\Omega}^p_X, \qquad (0 \le p \le 2). \tag{2.9}$$

Proof. The first maps i in both sequences are inclusions of sheaves. Recall that $\tilde{\Omega}^p_X = j_*\Omega^p_{X\setminus\Sigma}$ where Σ is the singular locus of X and $j : X \setminus \Sigma \to X$ is the open inclusion, so the composition $r \circ i$ is determined by that on the smooth locus of X, and is zero by Proposition 2.1.4. It remains to check that $\ker r = \operatorname{im} i$ and r is surjective in (2.8).

By Proposition 2.1.4, both short exact sequences hold at all smooth points $x \in X$, hence it suffices to check them on the singular locus. Let $x \in X$ be a singular point, and $U \subset X$ be an open ball centered at x. In local coordinates $\{z_1, z_2, z_3\}$ of U, we can define $X \cap U$ by $z_1^2 - z_2 z_3 = 0$ with the singular point x being the origin.

First, we show that $\ker r = \operatorname{im} i$ in both complexes. If $\xi \in \Gamma(U, \ker r)$, then $\xi_{|U\setminus 0} \in \Gamma(U \setminus 0, \ker r) = \Gamma(U \setminus 0, \operatorname{im} r)$ by Proposition 2.1.4. Since the point 0 is of codimension ≥ 2 in U, all differential forms on $U \setminus 0$ extend uniquely to differential forms on U, so $\Gamma(U\setminus 0, \operatorname{im} r) = \Gamma(U, \operatorname{im} r)$ and we get $\xi \in \Gamma(U, \operatorname{im} r)$.

Finally, we show that in (2.8), r is surjective. By Theorem 2.2.5(i), we have $\tilde{\Omega}_U^{p-1} = (f_* \Omega_D^{p-1})^G$ where $D \subset \mathbb{C}^2$ is an open ball and $G = \mathbb{Z}/2\mathbb{Z}$. Choose coordinates (u, v) on D such that G acts by sending $(u, v) \mapsto (-u, -v)$. Then, the map f is given by $(u, v) \mapsto (uv, u^2, v^2)$. The map $\Omega^p_{\mathbb{P}^3}(\log X) \to \tilde{\Omega}_X^{p-1}$ is a morphism of $\mathcal{O}_{\mathbb{P}^3}$-modules, so it suffices to prove that it is surjective on the generators of $\tilde{\Omega}_X^{p-1}$.

For $p = 1$, $\tilde{\Omega}^0_X = \mathcal{O}_X$ is locally generated by $1 \in \mathcal{O}_X$. It is the image of the form

$$\frac{2z_1 dz_1 - z_3 dz_2 - z_2 dz_3}{z_1^2 - z_2 z_3} = \frac{d(z_1^2 - z_2 z_3)}{z_1^2 - z_2 z_3} \in \Omega^1_{\mathbb{P}^3}(X).$$

Note that its differential is

$$d\left(\frac{d(z_1^2 - z_2 z_3)}{z_1^2 - z_2 z_3}\right) = \frac{-d(z_1^2 - z_2 z_3) \wedge d(z_1^2 - z_2 z_3)}{(z_1^2 - z_2 z_3)^2} = 0,$$

so $\frac{2z_1 dz_1 - z_3 dz_2 - z_2 dz_3}{z_1^2 - z_2 z_3} \in \Omega^1_{\mathbb{P}^3}(\log X)$ and r is surjective for $p = 1$.

For $p = 2$, $(\Omega^1_D)^G$ is generated by $\{udu, vdv, udv, vdu\}$ as a $\mathbb{C}[uv, u^2, v^2]$-module. We have $udu = \frac{1}{2} f^* dz_2$, $vdv = \frac{1}{2} f^* dz_3$ and

$$f^*\left(\frac{z_1 dz_2}{z_2}\right) = \frac{2u^2 v du}{u^2} = 2vdu, \qquad f^*\left(\frac{z_1 dz_3}{z_3}\right) = \frac{2v^2 u dv}{v^2} = 2udv.$$

Hence, $\tilde{\Omega}^1_X$ is locally generated by $\{dz_2, dz_3, \frac{z_1 dz_2}{z_2}, \frac{z_1 dz_3}{z_3}\}$. The preimages of the first two generators are

$$\frac{d(z_1^2 - z_2 z_3) \wedge dz_2}{z_1^2 - z_2 z_3}, \quad \frac{d(z_1^2 - z_2 z_3) \wedge dz_3}{z_1^2 - z_2 z_3}.$$

It is easy to see that their differentials are zero, so they lie in $\Omega^2_{\mathbb{P}^3}(\log X)$.

Consider the differential forms

$$\xi_j = \frac{2z_k dz_1 \wedge dz_j - z_1 dz_k \wedge dz_j}{z_1^2 - z_2 z_3} \in \Omega^1_{\mathbb{P}^3}(X), \qquad (\{j, k\} = \{2, 3\}).$$

Their differentials are

$$\begin{aligned}
d\xi_j &= \frac{\begin{pmatrix} (z_1^2 - z_2z_3)d(2z_kdz_1 \wedge dz_j - z_1dz_k \wedge dz_j) \\ \qquad - d(z_1^2 - z_2z_3) \wedge (2z_kdz_1 \wedge dz_j - z_1dz_k \wedge dz_j) \end{pmatrix}}{(z_1^2 - z_2z_3)^2} \\
&\equiv \frac{-2z_jz_kdz_k \wedge dz_1 \wedge dz_j - 2z_1^2dz_1 \wedge dz_k \wedge dz_j}{(z_1^2 - z_2z_3)^2} \quad (\mathrm{mod}\ (z_1^2 - z_2z_3)^{-1}) \\
&= \frac{2dz_k \wedge dz_1 \wedge dz_j}{z_1^2 - z_2z_3} \equiv 0 \quad (\mathrm{mod}\ (z_1^2 - z_2z_3)^{-1}),
\end{aligned}$$

so $\xi_j \in \Omega^2_{\mathbb{P}^3}(\log X)$. We have

$$r(\xi_j) = r\left(\frac{z_1(2\frac{z_jz_k}{z_1}dz_1 \wedge dz_j - z_jdz_k \wedge dz_j - z_kdz_j \wedge dz_j)}{z_j(z_1^2 - z_2z_3)}\right) = \frac{z_1dz_j}{z_j}$$

since $\frac{z_jz_k}{z_1} = z_1$ on X and $2z_1dz_1 - z_jdz_k - z_kdz_j = d(z_1^2 - z_2z_3)$. Hence, r is surjective for $p = 2$.

Similarly, for $p = 3$, $(\Omega^2_D)^G$ is generated by $du \wedge dv$. We have $du \wedge dv = \frac{1}{4}f^*(\frac{dz_2 \wedge dz_3}{z_1})$, so $\frac{dz_2 \wedge dz_3}{z_1}$ generates $\tilde{\Omega}^2_X$. We check that

$$r\left(\frac{dz_1 \wedge dz_2 \wedge dz_3}{z_1^2 - z_2z_3}\right) = r\left(\frac{(2z_1dz_1 - z_3dz_2 - z_2dz_3) \wedge dz_2 \wedge dz_3}{z_1(z_1^2 - z_2z_3)}\right) = \frac{dz_2 \wedge dz_3}{z_1}.$$

□

2.3 Hodge structure of singular hypersurfaces

In this section, we study the Hodge structure of another class of singular varieties, singular hypersurfaces, and in particular, try to explicitly compute their cohomology groups in certain cases. Cohomology groups of smooth hypersurfaces were studied by Griffiths [Gri69a; Gri69b] in the context of period mappings. This was extended to some singular hypersurfaces by Steenbrink [Ste06] and was further generalized by Dimca and Saito [DS14] using mixed Hodge modules. Most results in this section are restatements of results from these three papers.

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface defined by a homogeneous polynomial F of degree d. Let $S = \mathbb{C}[X_0, \ldots, X_{n+1}]$ be the graded ring of polynomials and $J = \langle \frac{\partial F}{\partial X_i} | i = 0, \ldots, n+1 \rangle$ be the Jacobian ideal. Let S_k, J_k and $(S/J)_k$ denote the sub-$\mathbb{C}$-vector spaces of homogeneous polynomials of degree k.

If X is smooth, Griffiths [Gri69b] (cf. [Voi03, Corollary 6.12]) showed that there are isomorphisms of vector spaces

$$H^{n-p,p}(X)_{\mathrm{prim}} \cong (S/J)_{(p+1)d-n-2}.$$

We wish to obtain a similar result for certain singular hypersurfaces. The central part of Griffiths' proof is contained in the case of $H^{n-1,1}(X)$. Since this will be the only case that we will be using later, we shall only prove this case.

We start with a few simple technical lemmas.

Lemma 2.3.1 ([Ste06])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a reduced hypersurface. Then, the sheaf of log differential forms $\Omega^p_{\mathbb{P}^{n+1}}(\log X)$ has a $\Gamma(\mathbb{P}^{n+1}, -)$-acyclic resolution by the complex*

$$0 \to \Omega^p_{\mathbb{P}^{n+1}}(X) \xrightarrow{d} \Omega^{p+1}_{\mathbb{P}^{n+1}}(2X)/\Omega^{p+1}_{\mathbb{P}^{n+1}}(X) \to K^2 \to K^3 \to \cdots$$

where the subcomplex $K^\bullet$ is supported on the singular locus.

Proof. The map

$$\Omega^p_{\mathbb{P}^{n+1}}(X) \xrightarrow{d} \Omega^{p+1}_{\mathbb{P}^{n+1}}(2X)/\Omega^{p+1}_{\mathbb{P}^{n+1}}(X)$$

is surjective away from the singular locus of X (the proof is by local computations, see for example [Voi03, Corollary 6.7]). Hence, the cokernel is supported on $\operatorname{Sing} X$. We may choose $K^\bullet$ to be an acyclic resolution of the cokernel, supported on $\operatorname{Sing} X$.

Hence, by the definition of sheaves of log differentials, the complex

$$0 \to \Omega^p_{\mathbb{P}^{n+1}}(X) \xrightarrow{d} \Omega^p_{\mathbb{P}^{n+1}}(2X)/\Omega^p_{\mathbb{P}^{n+1}}(X) \to K^\bullet$$

is a resolution of $\Omega^p_{\mathbb{P}^{n+1}}(\log X)$ that is acyclic in degrees ≥ 2.

By Bott's formula [OSS80], $H^i(\mathbb{P}^{n+1}, \Omega^p_{\mathbb{P}^{n+1}}(l)) = 0$ for all $i > 0$, $0 \leq p \leq n$ and $l > 0$. Hence, the first two terms of the complex are $\Gamma(\mathbb{P}^{n+1}, -)$-acyclic. $\square$

For $p = n$, we can explicitly compute $H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X))$ using the above resolution. Recall that $J = \langle \frac{\partial F}{\partial z_i} \rangle \subset S$ is the Jacobian ideal of X and we let $I = \sqrt{J}$ be the radical ideal of J. The ideal I is the defining ideal of the (reduced) singular locus of X.

Definition 2.3.2. A singularity is reduced if, for any point $x \in X$, $J_{X,x}$ is a radical ideal of $\mathcal{O}_{X,x}$ where J_X is the Jacobian ideal sheaf.

Example 2.3.3. Simple surface singularities, i.e. ADE singularities, are reduced.

Lemma 2.3.4. *Let $X \subset \mathbb{P}^{n+1}$ be a reduced hypersurface defined by a homogeneous polynomial F of degree d. Suppose further that the singular subscheme of X is non-empty and reduced. Let S, J and I be defined as above. Then, $H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X)) = (I/J)_{2d-n-2}$.*

Proof. By the resolution in Lemma 2.3.1, we see that $H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X))$ is the first cohomology of the complex

$$0 \to H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(d)) \to H^0(\mathbb{P}^{n+1}, \omega_{\mathbb{P}^{n+1}}(2d)/\omega_{\mathbb{P}^{n+1}}(d)) \xrightarrow{k} \\ \to H^0(\mathbb{P}^{n+1}, K^2) = H^0(\operatorname{Sing} X, K^2). \qquad (2.10)$$

Dualizing the short exact sequence

$$0 \to \Omega^1_{\mathbb{P}^{n+1}} \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus n+2} \to \mathcal{O}_{\mathbb{P}^{n+1}} \to 0$$

and tensoring with $\omega_{\mathbb{P}^{n+1}}$ gives

$$0 \to \omega_{\mathbb{P}^{n+1}} = \mathcal{O}_{\mathbb{P}^{n+1}}(-n-2) \xrightarrow{E} \\ \to \omega_{\mathbb{P}^{n+1}}(1)^{\oplus n+2} = \mathcal{O}_{\mathbb{P}^{n+1}}(-n-1)^{\oplus n+2} \to \Omega^n_{\mathbb{P}^{n+1}} \to 0.$$

Using the isomorphism $H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(l)) = S_l$, we can rewrite (2.10) as

$$0 \to S_{d-n-2} \xrightarrow{E} S^{\oplus n+2}_{d-n-1} \xrightarrow{h} S_{2d-n-2}/FS_{d-n-2} \xrightarrow{k} H^0(\operatorname{Sing} X, K^2)$$

where

$$E : A \mapsto (\ldots, AX_i, \ldots), \qquad \text{and} \qquad h : (\ldots, G_i, \ldots) \mapsto \sum_{i=0}^{n+1} G_i \frac{\partial F}{\partial X_i}.$$

The image of h is clearly J_{2d-n-2}, which contains the vector space FS_{d-n-2}. Now it suffices to show that the kernel of k on S_{2d-n-2} is precisely I_{2d-n-2}.

By Lemma 2.3.1, there is an inclusion $H^0(\mathbb{P}^{n+1}, \operatorname{coker} d) \subset H^0(\operatorname{Sing} X, K_2)$, so the map k is induced from the map

$$\tilde{k} : \omega_{\mathbb{P}^{n+1}}(2X)/\omega_{\mathbb{P}^{n+1}}(X) \to \operatorname{coker} d.$$

In local coordinates $z_1, \ldots, z_{n+1}$, we see that the differential d sends a meromorphic differential form $\sigma = \frac{g}{f} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots dz_{n+1}$ (where f is the holomorphic function defining X locally) to

$$d\sigma = \frac{f\frac{\partial g}{\partial z_i} - g\frac{\partial f}{\partial z_i}}{f^2} dz_1 \wedge \cdots \wedge dz_{n+1} \equiv -\frac{g}{f^2}\frac{\partial f}{\partial z_i} dz_1 \wedge \cdots \wedge dz_{n+1} \pmod{\omega_{\mathbb{P}^{n+1}}(X)}.$$

The polynomial $g\frac{\partial f}{\partial z_i}$ is zero on the singular locus of X. On the other hand, since the singular locus of X is reduced, the Jacobian ideal sheaf is locally radical, and hence, $\{\frac{\partial f}{\partial z_i}\}$ locally generate all holomorphic functions that are zero on the singular locus.

Thus, the image of d, or equivalently the kernel of $\tilde{k}$, is precisely the sheaf of all differential forms $\frac{g}{f^2} dz_1 \wedge \cdots \wedge dz_n$ where g is a holomorphic function that is zero on the singular locus. On the global level, we obtain that $\ker k = I_{2d-n-2}/FS_{d-n-2}$. □

Remark 2.3.5. Note that Lemma 2.3.4 does not hold for X smooth. If X is smooth, the radical ideal $I = \sqrt{J}$ is the irrelevant ideal. So, if $2d = n + 2$, then $H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X) = (S/J)_0 \cong \mathbb{C} \neq 0 = (I/J)_0$.

Our goal is to compute the cohomology groups on X, so it remains to relate the cohomology groups of X with the log cohomology groups on $\mathbb{P}^{n+1}$.

Proposition 2.3.6. *Suppose $n \geq 2$. Let $X \subset \mathbb{P}^{n+1}$ be a singular projective hypersurface of degree d with only reduced quotient singularities, then $H^0(X, \tilde{\omega}_X) \cong S_{d-n-2}$ and $H^1(X, \tilde{\Omega}^{n-1}_X)_{\mathrm{prim}} \cong (I/J)_{2d-n-2}$ where $I = \sqrt{J}$ is the radical of the Jacobian ideal J of the hypersurface X.*

Proof. By Theorem 2.2.14, there are short exact sequences

$$0 \to \omega_{\mathbb{P}^{n+1}} \to \omega_{\mathbb{P}^{n+1}}(X) \to \tilde{\omega}_X \to 0 \qquad \text{and}$$
$$0 \to \Omega^n_{\mathbb{P}^{n+1}} \to \Omega^n_{\mathbb{P}^{n+1}}(\log X) \to \tilde{\Omega}^{n-1}_X \to 0.$$

Since $H^i(\mathbb{P}^{n+1}, \omega_{\mathbb{P}^{n+1}}) = 0$ for all $i < n+1$, we obtain

$$H^0(X, \tilde{\omega}_X) = H^0(\mathbb{P}^{n+1}, \omega_{\mathbb{P}^{n+1}}(X)) = H^0(\mathbb{P}^{n+1}, \mathcal{O}(d-4)) \cong S_{d-4}.$$

For $n \geq 2$, Bott's formula gives $H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}) = 0$, so we get an isomorphism

$$\begin{aligned}(I/J)_{2d-n-2} = H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X)) = \ker(H^1(X, \Omega^{n-1}_X) \\ \xrightarrow{\sim} H^2(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}})) = H^1(X, \Omega^{n-1}_X)_{\mathrm{prim}}.\end{aligned}$$

□

We can also compute the first cohomology group $H^1(X, \tilde{T}_X)$ of the tangent sheaf for X having reduced quotient singularities.

Proposition 2.3.7. *Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface of degree d with only reduced quotient singularities. Suppose $n \geq 2$ and $(n, d) \neq (2, 4)$, then $H^1(X, \tilde{T}_X) \cong (I/J)_d$.*

Proof. By [Har77, Theorem III.7.11], for a hypersurface $X \subset \mathbb{P}^{n+1}$, the dualizing sheaf $\tilde{\omega}_X$ is invertible, so $\tilde{\omega}_X^\vee = \tilde{\omega}_X^{-1}$. Thus, by Lemma 2.2.11, we have $\tilde{\omega}_X^\vee \cong \omega^{-1}_{\mathbb{P}^{n+1}}(-X)_{|X} = \mathcal{O}_{\mathbb{P}^{n+1}}(n+2-d)_{|X}$. Tensoring the short exact sequence

$$0 \to \Omega^n_{\mathbb{P}^{n+1}} \to \Omega^n_{\mathbb{P}^{n+1}}(\log X) \to \tilde{\Omega}^{n-1}_X \to 0$$

by $\omega^{-1}_{\mathbb{P}^{n+1}}(-X)$ gives

$$\begin{aligned}0 \to \Omega^n_{\mathbb{P}^{n+1}}(n+2-d) \to \\ \to \Omega^n_{\mathbb{P}^{n+1}}(\log X)(n+2-d) \to \tilde{\Omega}^{n-1}_X \otimes \tilde{\omega}^\vee_X = \tilde{T}_X \to 0.\end{aligned} \tag{2.11}$$

For $n \geq 2$ and $(n,d) \neq (2,4)$, we obtain $H^i(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(n+2-d)) = 0$ for $i = 1,2$ by Bott's formula, so there is an isomorphism $H^1(X, \tilde{T}_X) = H^1(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(\log X)(n+2-d))$. The same proof as that of Lemma 2.3.4, with a twist by $\mathcal{O}(n+2-d)$, gives $H^1(X, \tilde{T}_X) \cong (I/J)_d$. $\square$

Remark 2.3.8. The short exact sequence (2.11) can also be written as

$$0 \to T_{\mathbb{P}^{n+1}}(-X) \to T_{\mathbb{P}^{n+1}}(-\log X) \to \tilde{T}_X \to 0$$

so $H^1(X, \tilde{T}_X) = H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log X)) \cong (I/J)_d$.

Remark 2.3.9. For $(n,d) = (2,4)$, there is an isomorphism

$$(I/J)_4 = \ker(H^1(X, \tilde{T}_X) \to H^2(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}) \cong \mathbb{C}).$$

In Chapter 3, we show that $H^1(X, \tilde{T}_X)$ parametrizes the isomorphism classes of infinitesimal deformations of X. The space $(I/J)_4$ parametrizes those deformations which are projective.

Chapter 3

Deformation theory

In this chapter, we review the Kodaira-Spencer map, which parametrizes the infinitesimal deformations of a variety by the first cohomology group of its tangent sheaf. Using the Kodaira-Spencer map, we define the infinitesimal period map which compares the deformation of a variety to the deformation of its Hodge structure. Finally, we show that the infinitesimal Torelli theorem holds for nodal surfaces.

3.1 The Kodaira-Spencer map

In this section, we define the Kodaira-Spencer map. Much of the contents of this section can be found in [Ser06] or [CMP03].

In their papers [KS58], Kodaira and Spencer showed that infinitesimal deformations of a smooth projective manifold M can be expressed entirely in terms of the cohomology group $H^1(M, T_M)$. They gave an analytic construction (cf. [Man05]) but we shall give an algebraic definition of the Kodaira-Spencer map.

Let $\mathbb{C}[\varepsilon] = \mathbb{C}[x]/(x^2)$ be the square-zero extension of $\mathbb{C}$. A *first order infinitesimal deformation* is a pullback square

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d] & \mathcal{M}_\varepsilon \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{C}[\varepsilon] \end{tikzcd}$$

in which f is a flat morphism. A morphism of infinitesimal deformations is a commutative diagram

$$\begin{tikzcd} \mathcal{M}_\varepsilon \arrow[r] \arrow[d, "f"'] & \mathcal{M}_{\varepsilon'} \arrow[d, "f'"] \\ \operatorname{Spec} \mathbb{C}[\varepsilon] \arrow[r] & \operatorname{Spec} \mathbb{C}[\varepsilon'] \end{tikzcd}$$

which restricts to the identity on the central fibre $M \to \operatorname{Spec} \mathbb{C}$. Let Def_M denote the set of isomorphism classes of first order infinitesimal deformations.

Given any representative $\mathcal{M}_\varepsilon \to \operatorname{Spec} \mathbb{C}[\varepsilon]$ of an isomorphism class, there exists an open affine cover $\{U_i\}$ of M such that the family is trivial on each U_i, i.e. there is an isomorphism $\theta_i : U_i \times \operatorname{Spec} \mathbb{C}[\varepsilon] \xrightarrow{\sim} \mathcal{M}_\varepsilon|_{U_i} = \mathcal{M}_\varepsilon \times_M U_i$. The infinitesimal deformation is uniquely determined by the set of transition maps

$$\theta_{ij} = \theta_i^{-1}\theta_j : U_{ij} \times \operatorname{Spec} \mathbb{C}[\varepsilon] \to U_{ij} \times \operatorname{Spec} \mathbb{C}[\varepsilon] \qquad \text{on } U_{ij} = U_i \cap U_j.$$

The maps θ_{ij} define derivations $\mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$. The tangent sheaf is defined as $T_M = \mathcal{H}om(\Omega^1_M, \mathcal{O}_M) = \operatorname{Der}(\mathcal{O}_M, \mathcal{O}_M)$, so each θ_{ij} defines an element $\eta_{ij} \in \Gamma(U_{ij}, T_M)$. The Čech cocycle condition $\eta_{ij} + \eta_{jk} + \eta_{ki} = 0$ holds on the intersection, and $\{\eta_{ij}\}$ gives a well defined class in $H^1(M, T_M)$. This gives us a well-defined map

$$\kappa : \operatorname{Def}_M \to H^1(M, T_M)$$

called the *Kodaira-Spencer correspondence*. Indeed it is a bijection when M is smooth [Ser06, Prop. 1.2.9], and it gives Def_M a vector space structure.

Let $f : \mathcal{M} \to B$ be a smooth family of smooth projective complex varieties. Let $0 \in B$ and $M = \mathcal{M}_0 = f^{-1}(0)$. A deformation family of M is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d] & \mathcal{M} \arrow[d, "f"] \\ \{\mathrm{pt}\} \arrow[r] & B. \end{tikzcd}$$

Where there is no risk of confusion, we shall just refer to a deformation family by the map f.

For a smooth variety B, the algebraic tangent space at 0 is given by

$$T_{B,0} = \operatorname{Hom}_0(\operatorname{Spec} \mathbb{C}[\varepsilon], B) = \{\phi \in \operatorname{Hom}(\operatorname{Spec} \mathbb{C}[\varepsilon], B) \mid f((\varepsilon)) = 0\}.$$

There is a well-defined map $T_{B,0} \to \operatorname{Def}_M$ taking $\phi \in \operatorname{Hom}_0(\operatorname{Spec} \mathbb{C}[\varepsilon], B)$ to the infinitesimal deformation $\mathcal{M}_\varepsilon \to \operatorname{Spec} \mathbb{C}[\varepsilon]$ which is the pullback of the deformation family $f : \mathcal{M} \to B$ along ϕ. Combining the two maps gives the Kodaira-Spencer map

$$KS_f : T_{B,0} \to \operatorname{Def}_M \xrightarrow{\kappa} H^1(M, T_M). \tag{3.1}$$

A family $f : \mathcal{M} \to B$ is said to be *versal* if KS_f is surjective and *universal* if it is an isomorphism. There exists a versal deformation only if, for all classes $\xi \in H^1(M, T_M)$, the Lie bracket $[\xi, \xi] = 0 \in H^2(M, T_M)$ [KS58, §6]. A smooth manifold M admits a universal family if $H^2(M, T_M) = 0$ [KNS58, Theorem, p. 452].

We can extend the Kodaira-Spencer map in two different ways: the first is to consider deformations of pairs (M, D) where M is a smooth variety and $D \subset M$ is a smooth effective divisor, while the second is to consider deformations of singular varieties. In the second case, we shall only consider the simple situation of a quotient variety of the form $X = M/G$ where M is a smooth manifold and G is a finite group.

3.1.1 Kodaira-Spencer map for divisors on varieties

Let M be a smooth algebraic variety and $D \subset M$ an effective divisor. Let $\mathcal{L} = \mathcal{O}_M(D)$ be the line bundle associated to D and $\Sigma_{\mathcal{L}}$ be the sheaf of differential operators of degree ≤ 1 on M. Let $s \in H^0(\mathcal{L})$ be the section defining D, then s defines a morphism

$$d_1 s : \Sigma_{\mathcal{L}} \to \mathcal{L} \ : \ \partial \mapsto \partial s.$$

An infinitesimal deformation of the triple $(M, \mathcal{L}, s)$ is defined to be a triple $(M_\varepsilon, \mathcal{L}_\varepsilon, s_\varepsilon)$ where M_ε is a flat $\mathbb{C}[\varepsilon]$-scheme ($\varepsilon^2 = 0$), $\mathcal{L}_\varepsilon$ is a line bundle on M_ε and $s_\varepsilon \in H^0(M_\varepsilon, \mathcal{L}_\varepsilon)$, satisfying isomorphisms $M_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} \cong M$ and $\mathcal{L}_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} \cong \mathcal{L}$ which send $s_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}$ to s. Two infinitesimal deformations $(M_\varepsilon, \mathcal{L}_\varepsilon, s_\varepsilon)$ and $(M'_\varepsilon, \mathcal{L}'_\varepsilon, s'_\varepsilon)$ are isomorphic if there are $\mathbb{C}[\varepsilon]$-isomorphisms $M_\varepsilon \xrightarrow{\sim} M'_\varepsilon$ and $L_\varepsilon \xrightarrow{\sim} L'_\varepsilon$ sending s_ε to s'_ε, restricting to the identity on $(M, \mathcal{L}, s)$ (cf. [Wel83, Section 1]).

We shall call an infinitesimal deformation of a triple $(M, \mathcal{L}, s)$ satisfying the assumptions in the first paragraph an infinitesimal deformation of the pair (M, D), and denote the vector space of isomorphism classes of infinitesimal deformations of (M, D) by $\mathrm{Def}_{M,D}$.

Welters [Wel83, Prop. 1.2] proved that the set of isomorphism classes of infinitesimal deformations of the triple $(M, \mathcal{L}, s)$ is given by the first hypercohomology group $\mathbb{H}^1(M, d_1 s)$ of the complex

$$0 \to \Sigma_{\mathcal{L}} \xrightarrow{d_1 s} \mathcal{L} \to 0.$$

A simple manipulation gives us the following proposition.

Proposition 3.1.1. *Let M be a smooth algebraic variety and $D \subset M$ an effective divisor. Then,* $\mathrm{Def}_{M,D} = \mathbb{H}^1(M, \overline{d}_1 s)$ *where $\overline{d}_1 s$ is the complex*

$$0 \to T_M \xrightarrow{\overline{d}_1 s} \mathcal{O}_D(D) \to 0.$$

In particular, if D is smooth, $\mathrm{Def}_{M,D} = H^1(M, T_M(-\log D))$.

Proof. The sheaf $\Sigma_{\mathcal{L}}$ lies in a short exact sequence [Wel83, p. 178, (1.10)]

$$0 \to \mathcal{O}_M \to \Sigma_{\mathcal{L}} \to T_M \to 0.$$

Since the composition $\mathcal{O}_M \to \Sigma_{\mathcal{L}} \xrightarrow{d_1 s} \mathcal{L}$ given by $f \mapsto fs$ is injective, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_M \arrow[r] \arrow[d, equal] & \Sigma_{\mathcal{L}} \arrow[r] \arrow[d, "d_1 s"] & T_M \arrow[r] \arrow[d, "\overline{d}_1 s"] & 0 \\
0 \arrow[r] & \mathcal{O}_M \arrow[r] & \mathcal{L} = \mathcal{O}_M(D) \arrow[r] & \mathcal{O}_D(D) \arrow[r] & 0.
\end{tikzcd}$$

This gives a quasi-isomorphism of the latter two vertical complexes in the derived category $D^b(\mathcal{O}_M)$, hence $\mathrm{Def}_{M,D} = \mathbb{H}^1(M, d_1 s) = \mathbb{H}^1(M, \overline{d}_1 s)$.

If D is a smooth, the short exact sequence (Corollary 2.1.5)

$$0 \to T_M(-\log D) \to T_M \xrightarrow{\overline{d}_1 s} \mathcal{O}_D(D) \to 0$$

implies that there a quasi-isomorphism between $T_M(-\log D)$ seen as a complex concentrated in degree 0 and $T_M \xrightarrow{\overline{d}_1 s} \mathcal{O}_D(D)$. This gives an isomorphism of cohomology groups

$$H^1(M, T_M(-\log D)) = \mathbb{H}^1(M, \overline{d}_1 s).$$

$\square$

If D is not smooth, but is a V-manifold instead, the situation is more complicated. By Corollary 2.2.18, the sequence

$$0 \to T_M(-\log D) \to T_M \to \mathcal{O}_D(D)$$

is usually only left exact. Let $\mathcal{C}$ be the cokernel of the map $T_M(-\log D) \to T_M$ and $\mathcal{C}'$ be the cokernel of the inclusion $\mathcal{C} \to \mathcal{O}_D(D)$. We then get a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_M \arrow[r, equal] \arrow[d] & T_M \arrow[r] \arrow[d, "\overline{d}_1 s"] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{C} \arrow[r] & \mathcal{O}_D(D) \arrow[r] & \mathcal{C}' \arrow[r] & 0.
\end{tikzcd}$$

This induces a long exact sequence in hypercohomology

$$0 \to H^1(M, T_M(-\log D)) \to \mathbb{H}^1(M, \overline{d}_1 s) \to H^0(M, \mathcal{C}').$$

Hence, we can conclude:

Corollary 3.1.2. *Let M be a smooth algebraic variety and $D \subset M$ an effective divisor. Suppose that D is a V-manifold. Then, $H^1(M, T_M(-\log D)) \subseteq$* $\mathrm{Def}_{M,D}$.

Example 3.1.3. Suppose $M \cong \mathbb{P}^{n+1}$ and D is a hypersurface defined by a homogeneous polynomial F of degree d. Let $J = \langle \frac{\partial F}{\partial X_i} \rangle \subset S = \mathbb{C}[X_0, \ldots, X_{n+1}]$ be the Jacobian ideal and $I = \sqrt{J}$ be the radical of J. In this case, we can evaluate $\mathrm{Def}_{M,D}$: there is a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^{n+1}} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus(n+2)} \arrow[r] \arrow[d, "h"] & T_{\mathbb{P}^{n+1}} \arrow[r] \arrow[d, "\overline{d}_1 s"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^{n+1}} \arrow[r] & \mathcal{O}_{\mathbb{P}^{n+1}}(d) \arrow[r] & \mathcal{O}_D(D) \arrow[r] & 0
\end{tikzcd}$$

where h is given by $(G_i) \mapsto \sum_{i=0}^{n+1} G_i \frac{\partial F}{\partial X_i}$. Furthermore, $\mathcal{O}(1)^{n+2}$ and $\mathcal{O}(d)$ are $\Gamma(\mathbb{P}^{n+1}, -)$-acyclic, so h is an acyclic resolution $\overline{d}_1 s$. Hence,

$$\mathrm{Def}_{\mathbb{P}^{n+1},D} = \mathbb{H}^1(\mathbb{P}^{n+1}, \overline{d}_1 s) = \mathbb{H}^1(\mathbb{P}^{n+1}, h) = \mathrm{coker}\,(H^0(h)) = (S/J)_d.$$

Recall from Remark 2.3.8 that $H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D)) \cong (I/J)_d$. There is a short exact sequence

$$\begin{aligned}0 \to H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D)) \cong (I/J)_d \to& \\ \to \mathrm{Def}_{\mathbb{P}^{n+1},D} \cong (S/J)_d \to (S/I)_d \to 0.&\end{aligned}$$

Since $\mathbb{P}^{n+1}$ has no non-trivial deformations, $\mathrm{Def}_{\mathbb{P}^{n+1},D}$ parametrizes the deformations of D in $\mathbb{P}^{n+1}$. The kernel of the map $(S/J)_D \to (S/I)_D$ is precisely the deformations whose defining polynomials remain in I, thus fixing the singular locus. Hence, $H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D))$ parametrizes the deformations of D in $\mathbb{P}^{n+1}$ that preserve the singular locus.

3.1.2 Kodaira-Spencer map for quotient varieties

Let M be a smooth projective complex algebraic variety, G a finite group that acts on M and $X = M/G$ be the quotient variety.

Definition 3.1.4. Let X be a quotient variety. A deformation of X as a quotient variety over a smooth base B is defined to be a deformation $f : \mathcal{M} \to B$ of M such that the action of G on M extends to a (holomorphic or algebraic) action on $\mathcal{M}$ such that the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & \mathcal{M} \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

commutes for all $\sigma \in G$. Such a deformation of M is also called a G-equivariant deformation. Let the space of isomorphism classes of G-equivariant first order infinitesimal deformations of M be denoted by Def_M^G or Def_X where $X = M/G$.

We shall construct the Kodaira-Spencer map for G-equivariant deformations.

There is a natural G-action on the category of deformation families of M [Rim80] which acts by sending a deformation family $f : \mathcal{M} \to B$ to the deformation family

$$\begin{tikzcd} M \arrow[r, "i\sigma^{-1}"] \arrow[d] & \mathcal{M} \arrow[d, "f"] \\ \mathrm{Spec}\ \mathbb{C} \arrow[r] & B \end{tikzcd}$$

for each $\sigma \in G$. This action preserves isomorphism classes, so it induces an action on Def_M. It is clear that $\mathrm{Def}_M^G = (\mathrm{Def}_M)^G$ is the G-invariant subspace of first order infinitesimal deformations.

Remark 3.1.5. There is a natural induced G-action on the tangent bundle T_M. It is defined as follows. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of M by open balls such that G acts as a permutation on the indices in I, i.e., for each $\sigma \in G$, there is an isomorphism $\sigma_{|U_i} : U_i \xrightarrow{\sim} U_{\sigma(i)}$. Let $\{y_k\}$ and $\{x_k\}$ be local coordinates of $U_{\sigma^{-1}(i)}$ and U_i respectively such that $\sigma \in G$ sends y_k to x_k. The induced G-action on T_M is by sending the basis $\{\frac{\partial}{\partial y_k}\}$ to $\{\frac{\partial}{\partial x_k}\}$.

Proposition 3.1.6. *The Kodaira-Spencer correspondence*

$$\kappa : \mathrm{Def}_M \to H^1(M, T_M)$$

is an isomorphism of G-modules. Hence, it induces an isomorphism

$$\kappa : \mathrm{Def}_M^G \to H^1(M, T_M)^G.$$

Proof. Let $\mathcal{U}$ be an open covering of M and $f : \mathcal{M}_\varepsilon \to B$ be a representative of a class $[f] \in \mathrm{Def}_M$. The Kodaira-Spencer correspondence κ is defined by sending the transition functions $U_{ij} \times \mathrm{Spec}\ \mathbb{C}[\varepsilon] \xrightarrow{\theta_{ij}} U_{ij} \times \mathrm{Spec}\ \mathbb{C}[\varepsilon]$ to the Čech 1-cycles $\{\eta_{ij}\}$, which defines a class in $H^1(M, T_M)$.

Choose the open covering $\mathcal{U}$ defined in Remark 3.1.5. Then, $\sigma \in G$ acts by sending

$$(\theta_{ij}) \mapsto (\sigma \theta_{\sigma^{-1}(i)\sigma^{-1}(j)} \sigma^{-1}) \quad \text{and} \quad (\eta_{ij}) \mapsto (\sigma \eta_{\sigma^{-1}(i)\sigma^{-1}(j)} \sigma^{-1}),$$

hence the action of G commutes with the Kodaira-Spencer correspondence κ. □

Combining Corollary 2.2.13 and Proposition 3.1.6, we get

Corollary 3.1.7. *Let M be a smooth manifold endowed with the action of a finite group G. Let B be the union of the codimension 1 components of the branch locus of $X = M/G$. Then, the vector space Def_X of isomorphism classes of the infinitesimal deformations of X as a quotient variety is parametrized by $H^1(M, T_M)^G = H^1(X, \tilde{T}_X(-\log B))$.*

Remark 3.1.8. We contrast our result with that of Pardini [Par91] for big subgroups of $GL(n, \mathbb{C})$. In [Par91, Proposition 4.1], Pardini showed that for a quotient map $f : M \to X$ branched over a divisor B on X, the infinitesimal G-equivariant deformations are parametrized by $H^1(M, T_M)^G = H^1(X, T_X(-\log B))$.

We also understand the deformation of a divisor on a quotient variety.

Corollary 3.1.9. *Let M be a smooth manifold endowed with the action of a finite group G and $f : M \to X = M/G$ be the quotient map. Let $D \subset X$ be a divisor and $\tilde{D} = f^{-1}D$ be the preimage of D on M. Denote the space of isomorphism classes of the infinitesimal deformations of the pair (X, D) obtained as a quotient of $(M, \tilde{D})$ by $\mathrm{Def}^G_{M,\tilde{D}} = \mathrm{Def}_{X,D}$. Then, $\mathrm{Def}_{X,D} = \mathbb{H}^1(M, \overline{d}_1 s)^G$ where $\overline{d}_1 s$ is the complex*

$$0 \to T_M \xrightarrow{\overline{d}_1 s} \mathcal{O}_{\tilde{D}}(\tilde{D}) \to 0$$

and $s \in \mathcal{O}_M(\tilde{D})$ is the section defining D. Furthermore, the G-invariant cohomology group $H^1(M, T_M(-\log \tilde{D}))^G$ is a subspace of $\mathrm{Def}_{X,D}$ and equality holds if $\tilde{D}$ is smooth.

Proof. Let $\mathcal{L} = \mathcal{O}_M(\tilde{D})$ and $s \in H^0(M, \mathcal{L})$ be the section defining $\tilde{D}$. The section s is invariant under G.

Fixing an open cover $\mathcal{U}$ of M, a deformation in $\mathrm{Def}_{M,\tilde{D}}$ can be represented by a triple $(\theta_{ij}, l_{ij}, \tilde{s}_i = s_i + b_i\varepsilon)$ where $U_{ij} \times \mathrm{Spec}\ \mathbb{C}[\varepsilon] \xrightarrow{\theta_{ij}} U_{ij} \times \mathrm{Spec}\ \mathbb{C}[\varepsilon]$ are transition functions,

$$l_{ij} = \begin{pmatrix} 1 & 0 \\ \eta_{ij} & 1 \end{pmatrix} \in \mathrm{End}(H^0(U_{ij}, \mathcal{L})[\varepsilon]) \quad \text{with } \eta_{ij} \in \mathrm{End}(H^0(U_{ij}, \mathcal{L})),$$

and $\tilde{s}_i \in H^0(U_i, \mathcal{L})[\varepsilon]$ are infinitesimal deformations of $s_i = s_{|U_i}$. By the proof of [Wel83, Proposition 1.2], the Kodaira-Spencer correspondence for manifold-divisor pairs sends a deformation (θ_{ij}, l_{ij}) to a pair $(b_i, \eta_{ij}) \in C^0(\mathcal{U}, \mathcal{L}) \oplus C^1(\mathcal{U}, \Sigma_{\mathcal{L}})$ which is a 1-cocycle in $\mathbb{H}^1(d_1 s)$.

As in the proof of Proposition 3.1.6, using the basis given in Remark 3.1.5, it is clear that the G-action commutes with the Kodaira-Spencer correspondence. The conclusion then follows from Proposition 3.1.1 and Corollary 3.1.2. $\square$

3.2 Infinitesimal period map

In this section, we briefly recall the definition of a period map, and generalize it to define the infinitesimal period map for V-manifolds. The results for smooth projective varieties in this section are due to Griffiths [Gri68] and can be found in many standard texts, for example, [Voi02, Chapter 10] and [CMP03, Chapter 5].

Let $B \ni 0$ be an open ball, and $f : \mathcal{M} \to B$ be a family of smooth projective varieties. We wish to understand how the Hodge structure of $M_b = f^{-1}(b)$ varies across the family. The period map is a holomorphic map

$$\mathcal{P}^k : B \to \mathcal{D}_k \ : \ [M_b] \mapsto (F^p H^k(M_b, \mathbb{C}))$$

where $\mathcal{D}_k$ is the period domain, which is the moduli space of pure Hodge structures of weight k. Fix $M_0 = f^{-1}(0)$ and the vector space $V = H^k(M_0, \mathbb{Q})$. Since B is contractible, Ehresmann's lemma gives canonical isomorphisms $\phi_b : H^k(M_b, \mathbb{C}) \xrightarrow{\sim} V_{\mathbb{C}}$ for all $b \in B$, so $F^p H^k(M_b, \mathbb{C})$ can be canonically identified with subspaces of $V_{\mathbb{C}}$. Furthermore, the Hodge numbers are constant in the family (cf. [Voi02, Proposition 9.20]), so $\mathcal{D}_k$ is in fact a subspace of a product of Grassmannians

$$\prod_{p=1}^{k} \mathrm{Gr}(b_{p,k}, V_{\mathbb{C}}) \qquad \text{where} \quad b_{p,k} = \dim\,(F^p H^k(M_0, \mathbb{C}).$$

By Lefschetz's hyperplane theorem, the Hodge structure on $H^k(M_b, \mathbb{Q})$ is determined by that on a general hyperplane section of M_b for all $k \neq n = \dim M_b$. Hence, the most interesting case of the period map is when $k = n$.

If the deformation is trivial, the Hodge structure is constant over the family and the period map is trivial, so we suppose that all deformations in the family $f : \mathcal{M} \to B$ are non-trivial. If the period map $\mathcal{P}^n$ is injective, then the family can be identified with a subspace of the period domain. A universal family $f : \mathcal{M} \to B$ is said to satisfy the Torelli property if the period map $\mathcal{P}^n$ is injective.

Checking whether a family satisfies the Torelli property is difficult. A significantly easier question is to ask if the period map $\mathcal{P}^n$ is locally injective, that is, if, for any $[M] \in B$, its differential

$$d\mathcal{P}^n : T_{[M]}B \to T_{\mathcal{P}^n[M]}\mathcal{D}$$

is injective.

By the Kodaira-Spencer isomorphism for universal families, we know that $T_{[M]}B \cong H^1(M, T_M)$. The codomain of $d\mathcal{P}^k$ can be expressed in terms of the Hodge structure of M:

Proposition/Definition 3.2.1 ([Voi02, Theorem 10.21])**.** *Let M be a smooth projective variety. The infinitesimal period map is the morphism*

$$d\mathcal{P}^k : H^1(M, T_M) \to \bigoplus_{p=1}^{k} \mathrm{Hom}(H^{k-p}(M, \Omega_M^p), H^{k-p+1}(M, \Omega_M^{p-1})) \tag{3.2}$$

defined by sending $\eta \in H^1(M, T_M)$ to the map $\eta \cup - : \omega \mapsto \eta \cup \omega$. In local coordinates $z_1, \ldots, z_n$, we can write η as $\sum f_i \frac{\partial}{\partial z_i}$, so $\eta \cup -$ is given by contracting the differential forms, with the action on each one form given by $\frac{\partial}{\partial z_i}(f dz_i) = \frac{\partial f}{\partial z_i}$.

Definition 3.2.2. Let M be a smooth projective variety of dimension n. M is said to satisfy the infinitesimal Torelli property if the infinitesimal period map $d\mathcal{P}^n$ is injective.

Let $X = M/G$ be a quotient variety where G is an abelian group. Taking the G-invariant components on both sides of the map (3.2) gives a map

$$\begin{aligned} d\mathcal{P}^k : H^1(M, T_M)^G \to & \bigoplus_{p=1}^{k} \mathrm{Hom}(H^{p,k-p}(M), H^{p-1,k-p+1}(M))^G \\ &= \bigoplus_{p=1}^{k} \bigoplus_{\chi \in \hat{G}} \mathrm{Hom}(H^{p,k-p}(M)_\chi, H^{p-1,k-p+1}(M)_{\chi^{-1}}) \end{aligned}$$

where $\hat{G}$ is the character group of G and $H^{p,q}(M)_\chi$ is the eigenspace of $H^{p,q}(M)$ corresponding to the character χ.

Recall from Theorem 2.2.5(i) that the eigenspace corresponding to the trivial character $H^{p,q}(M)_1 = H^{p,q}(M)^G$ is precisely equal to $H^{p,q}(X)$, which defines a pure Hodge structure on $H^{p+q}(X, \mathbb{Q})$ by Theorem 2.2.7. Thus, one can define the infinitesimal period map for X by projecting onto the components with trivial characters.

However, from a geometrical perspective, the period map $\mathcal{P}^k$ is only well-defined if the filtration $(F^p H^k(-, \mathbb{C}))$ is constant dimensional, at least in an open neighbourhood of $[X]$. We thus need to impose an additional condition in the definition.

Definition 3.2.3. Let $X = M/G$ be a quotient variety of dimension n and $f : \mathcal{M} \to B$ a versal G-equivariant deformation family with $M = M_0 = f^{-1}(0)$. Let $M_b = f^{-1}(b)$ for any $b \in B$. Suppose $H^{p,k-p}(M_b)^G$ is constant dimensional for all b in an open neigbourhood of 0. Then, the infinitesimal period map is defined to be

$$d\mathcal{P}^k : H^1(M, T_M)^G = H^1(X, \tilde{T}_X(-\log D))$$
$$\longrightarrow \bigoplus_{p=1}^{k} \mathrm{Hom}(H^{k-p}(X, \tilde{\Omega}^p_X), H^{k-p+1}(X, \tilde{\Omega}^{p-1}_X))$$

where D is the union of the codimension 1 components of the branch divisor. X is said to satisfy the infinitesimal Torelli property if the $d\mathcal{P}^n$ is injective.

Remark 3.2.4. The domain of the infinitesimal period map in Definition 3.2.3 is restricted to locally-trivial or G-equivariant infinitesimal deformations of X (cf. Section 3.1.2). If we embed a general singular variety X as a divisor in a smooth projective variety Y, we see from Corollary 3.1.2 that a general infinitesimal deformation of X in Y is not equisingular. A general deformation will cause a jump in Hodge numbers, and as such the period map is not well-defined.

In this thesis, we will only be consider G-equivariant deformations of V-manifolds (cf. Chapter 4). The infinitesimal Torelli property determines if the Hodge structure on the middle cohomology distinguishes all non-trivial deformations of the V-manifold X.

3.3 Infinitesimal Torelli theorem for nodal surfaces

After the proof of the original Torelli theorem for smooth projective curves, the next major result is Griffiths' proof of the Torelli theorem for most smooth projective hypersurfaces [Gri69a; Gri69b]. The first step in Griffiths' proof is to prove the infinitesimal Torelli theorem.

Theorem 3.3.1 (Infinitesimal Torelli theorem for smooth hypersurfaces [Gri69a, Theorem 9.8(b)])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and suppose $n \geq 3$ and $d \geq 3$ or $n = 2$ and $d > 3$. Then, the infinitesimal period map*

$$d\mathcal{P}^n : H^1(X, T_X) \to \bigoplus_{p=1}^{n} \mathrm{Hom}(H^{p,n-p}(X), H^{p-1,n-p+1}(X))$$

is injective.

The proof uses the description of cohomology groups using polynomial rings given in Section 2.3.

Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface defined by a homogeneous polynomial equation $F(z_0, \dots, z_{n+1}) = 0$ of degree d. Let $S = \mathbb{C}[z_0, \dots, z_{n+1}]$ be the ring of polynomials and $J = \langle \frac{\partial F}{\partial z_i} \rangle$ be the Jacobian ideal. We denote by S_d, J_d and $(S/J)_d$ the homogeneous parts of degree d.

In Section 2.3, we showed that there are isomorphisms $H^{p,n-p}(X)_{\text{prim}} = (S/J)_{(n-p+1)d-n-2}$ and $H^1(X, T_X) = (S/J)_d$. The infinitesimal period map factors through the map

$$
\begin{array}{rcc}
\Pi : (S/J)_d & \to & \bigoplus_{p=1}^{n} \operatorname{Hom}((S/J)_{(n-p+1)d-n-2}, (S/J)_{(n-p+2)d-n-2}) \\
{[P]} & \mapsto & ([Q] \mapsto [P \cdot Q]).
\end{array}
$$

The proof of the injectivity of Π (and hence Theorem 3.3.1) relies on a key lemma.

Lemma 3.3.2 (Macaulay's theorem [Mac16, §86], cf. [CMP03, Theorem 7.4.1])**.** *Let $(P_0, \dots, P_{n+1})$ be a regular sequence of homogeneous polynomials of degrees $d_0, \dots, d_{n+1}$ in S and $\rho = \sum_{i=0}^{n+1} d_i - (n+2)$. Then $(S/J)_l = 0$ for all $l > \rho$ and there is a perfect pairing*

$$(S/J)_l \otimes (S/J)_{\rho-l} \to (S/J)_\rho \cong \mathbb{C}$$

induced by multiplication in S.

To prove Theorem 3.3.1, the lemma is applied to $P_i = \frac{\partial F}{\partial z_i}$. The sequence (P_i) is regular if and only if the hypersurface $X \subset \mathbb{P}^n$ is smooth. The ideal $\langle P_0, \dots, P_n \rangle$ is precisely the Jacobian ideal J and $\rho = (n+2)(d-2)$.

However, for singular hypersurfaces, the radical ideal $I = \operatorname{rad}(J)$ of the Jacobian is not the irrelevant ideal $\mathfrak{m}$, so the sequence $(\frac{\partial F}{\partial z_i})$ is not regular and Macaulay's theorem cannot be applied. Indeed, the radical ideal $I = I(\operatorname{Sing} X)$ is the ideal defining the singular locus of X.

To prove the infinitesimal Torelli theorem for certain singular hypersurfaces, we need an analogue of Macaulay's theorem.

From now on, we shall assume that $n = 2$, so X is a surface of degree d in $\mathbb{P}^3$. We further assume that X is a nodal surface.

Let $X \subset \mathbb{P}^3$ be a nodal surface defined by a homogeneous polynomial $F \in \mathbb{C}[z_0, \dots, z_3]$ of degree d. Let $Z = \{p_1, \dots, p_k\}$ be the set of nodes of X and $F_i = \frac{\partial F}{\partial z_i}$ $(0 \leq i \leq 3)$ be the partial derivatives of F. So, Z is the zero locus of the set of partial derivatives. Let $J = \langle F_0, \dots, F_3 \rangle$ be the Jacobian ideal of F and $I = \sqrt{J}$ be its radical.

Let $\mathcal{E} = \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(d-1)$ and s be the section of $\mathcal{E}$ defined by $(F_0, \dots, F_3)$. Consider the Koszul complex

$$\mathcal{K}^\bullet = \left(0 \to \bigwedge^4 \mathcal{E}^\vee \to \bigwedge^3 \mathcal{E}^\vee \to \bigwedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}^3} \to 0\right)$$

where the differential maps are contractions by the section s. We place $\mathcal{O}_{\mathbb{P}^3}$ in degree 0.

Since the zero locus of the section s is the finite set Z, which is of codimension 3 in $\mathbb{P}^3$, by [CMP03, Problem 7.2.2(b)], the Koszul complex $\mathcal{K}^\bullet$ is exact in degrees $< -(4-3) = -1$. Note that the map $\mathcal{E}^\vee \xrightarrow{(F_0,\dots,F_3)} \mathcal{O}_{\mathbb{P}^3}$ is surjective away from Z, and the cokernel at each point of Z is isomorphic to $\mathbb{C}$, so there is an exact sequence

$$\mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0.$$

Thus, the extended complex, which we shall also call $\mathcal{K}^\bullet$ by abuse of notation,

$$\mathcal{K}^\bullet = \left(0 \to \bigwedge^4 \mathcal{E}^\vee \to \bigwedge^3 \mathcal{E}^\vee \to \bigwedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0\right)$$

is exact everywhere except in degree -1. Indeed, there is a quasi-isomorphism

$$\mathcal{K}^\bullet \overset{\text{qis}}{\cong} \frac{\ker(\mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}^3})}{\operatorname{im}(\bigwedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee)}[1].$$

The righthand side is supported on Z since the complex $\mathcal{K}^\bullet$ is exact away from Z by [CMP03, Problem 7.2.2(b) or Theorem 7.4.1]. We define K to be the sheaf

$$K := \frac{\ker(\mathcal{E}^\vee \to \mathcal{O}_{\mathbb{P}^3})}{\operatorname{im}(\bigwedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee)}.$$

Consider the twisted complex $\mathcal{K}^\bullet \otimes \mathcal{O}(l)$ for some integer l. We can associate to it the spectral sequence

$$\begin{aligned} E_1^{p,q} = H^q(\mathbb{P}^3, \mathcal{K}^p(l)) &\implies \mathbb{H}^{p+q}(\mathbb{P}^3, \mathcal{K}^\bullet \otimes \mathcal{O}(l)) = H^{p+q}(Z, K[1]), \\ d^r : E_r^{p,q} &\to E_r^{p+r,q-r+1}. \end{aligned} \tag{3.3}$$

Note that $\mathbb{H}^{p+q}(\mathbb{P}^3, \mathcal{K}^\bullet \otimes \mathcal{O}(l)) = H^{p+q+1}(Z, K)$ is independent of the twist since it is supported on a finite set.

The cohomology group $E_1^{p,q} = H^q(\mathbb{P}^3, \mathcal{K}^p(l))$ is zero except where $q = 0$ or $q = 3$, so we have $E_2 = E_3 = E_4$ and $E_5 = E_6 = \cdots = E_\infty$. Indeed, $E_r^{p,q} \neq 0$ only if $q = 0, 3$ and $-4 \leq p \leq 1$.

Consider the map $d_4 : E_4^{-4,3} \to E_4^{0,0}$. We have

$$\begin{aligned}
E_4^{-4,3} &= \frac{\ker(E_1^{-4,3} \xrightarrow{d_1} E_1^{-3,3})}{\operatorname{im}(E_1^{-5,3} = 0 \xrightarrow{d_1} E_1^{-4,3})} \\
&= \ker\Big(H^3(\mathbb{P}^3, \bigwedge^4 \mathcal{E}^\vee(l)) \xrightarrow{\alpha_l} H^3(\mathbb{P}^3, \bigwedge^3 \mathcal{E}^\vee(l))\Big) \\
&= \ker\Big(H^3(\mathbb{P}^3, \mathcal{O}(-\rho-4+l)) \xrightarrow{\alpha_l} H^3(\mathbb{P}^3, \bigoplus_{i=0}^{3} \mathcal{O}(-\rho-5+l+d))\Big) \\
&\cong \operatorname{coker}\Big(H^0(\mathbb{P}^3, \bigoplus_{i=0}^{3} \mathcal{O}(\rho-l-d+1)) \xrightarrow{\alpha_l^*} H^0(\mathbb{P}^3, \mathcal{O}(\rho-l))\Big)^\vee \\
&= \operatorname{coker}\Big(\bigoplus_{i=0}^{3} S_{\rho-l-d+1} \xrightarrow{\alpha_l^*} S_{\rho-l}\Big)^\vee = (S/J)^\vee_{\rho-l}
\end{aligned}$$

where $\rho = 4d-8$ and the fourth isomorphism is given by Serre duality. α_l is multiplication by the polynomials $(F_0, \ldots, F_3)$ and α_l^* is its dual. Similarly,

$$\begin{aligned}
E_4^{0,0} &= \frac{\ker(E_1^{0,0} \xrightarrow{d_1} E_1^{1,0})}{\operatorname{im}(E_1^{-1,0} = 0 \xrightarrow{d_1} E_1^{0,0})} \\
&= \frac{\ker\big(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(\mathbb{P}^3, \mathcal{O}_Z(l))\big)}{\operatorname{im}\big(H^0(\mathbb{P}^3, \mathcal{E}^\vee(l)) \to H^0(Proj^3, \mathcal{O}_{\mathbb{P}^3}(l))\big)} \\
&= \frac{\ker\big(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(\mathbb{P}^3, \mathcal{O}_Z)\big)}{\operatorname{im}\big(H^0(\mathbb{P}^3, \bigoplus_{i=0}^3 \mathcal{O}(l-d_i)) \xrightarrow{\beta_l} H^0(Proj^3, \mathcal{O}(l))\big)} \\
&= \frac{I_l}{\operatorname{im}\big(\bigoplus_{i=0}^3 S_{l-d_i} \xrightarrow{\beta_l} S_l\big)} = (I/J)_l
\end{aligned}$$

where β_l is multiplication by $(F_0, \ldots, F_3)$. Note that $\alpha_l^* = \beta_{\rho-l}$.

Thus, d_4 induces a morphism

$$d_4 : (S/J)^\vee_{\rho-l} \to (I/J)_l. \tag{3.4}$$

Lemma 3.3.3. *The morphism d_4, restricted to $(I/J)^\vee_{\rho-l}$ induces a duality $(I/J)^\vee_{\rho-l} \xrightarrow{\sim} (I/J)_l$. In particular, the morphism $d_4 : (S/J)^\vee_{\rho-l} \to (I/J)_l$ is surjective.*

Proof. The composition of the map (3.4) for l and $\rho - l$ gives

$$(S/J)^\vee_{\rho-l} \to (I/J)_l \to (I/J)^\vee_{\rho-l} \qquad \text{and} \qquad (S/J)^\vee_l \to (I/J)_{\rho-l} \to (I/J)^\vee_l$$

which are dual to the inclusion $I/J \to S/J$. Hence, there is a natural duality $(I/J)_l \cong (I/J)^{\vee}_{\rho-l}$.

We can also prove directly that d_4 is surjective. Since $E_4^{4,-3} = 0$, we have $E_5^{0,0} = \operatorname{coker}(E_4^{-4,3} \xrightarrow{d_4} E_4^{0,0})$. The spectral sequence of (3.3) converges to $H^{p+q}(Z, K[1])$ and $H^0(Z, K[1]) = H^1(Z, K) = 0$ since Z is a finite set, so $E_5^{0,0} = E_\infty^{0,0} = 0$ and d_4 is surjective. $\square$

Remark 3.3.4. Unlike the regular case (Macaulay's theorem, Lemma 3.3.2), the perfect pairing

$$(I/J)_{\rho-l} \otimes (I/J)_l \to \mathbb{C}$$

is not induced by multiplication of polynomials in I. This is clear since $(I/J)_\rho = 0$.

We can however show a weaker form of the multiplication property (Lemma 3.3.11). To do so we need to relate the algebraic independence of the partial derivatives F_i and the independence of the set of nodes.

Definition 3.3.5. Let $Z = \{p_1, \ldots, p_k\}$ be a set of points on a hypersurface X and $\{e_i\}$ be a basis for S_l, the vector space of homogeneous polynomials of degree l. We say that the set Z imposes independent conditions in degree l (or simply, is independent in degree l) if the rank of the matrix $(e_i(p_j))_{i,j}$ is k.

It is clear from the definition that if Z is independent in degree l, then it is independent in all degrees $\geq l$.

Remark 3.3.6. The kernel of the matrix $(e_i(p_j))_{i,j}$ is the vector space I_l. So, if Z is a set of k points, then Z is independent in degree l if and only if $\dim S_l - \dim I_l = k$.

There are numerous results regarding independence of nodes. The main result that we shall use in this thesis is that of Severi. He showed that the set of nodes Z on a nodal surface $X \subset \mathbb{P}^3$ is independent in degree $2d-5$ [Sev46, §14].

A recent work of Mustaţă and Popa uses mixed Hodge modules and Hodge ideals to give a vast generalization of this lemma. They showed that for any reduced hypersurface $D \subset \mathbb{P}^{n+1}$ of degree d, with isolated singularities of multiplicity m, the set Z of singular points imposes independent conditions in degree $(\lfloor \frac{n+1}{m} \rfloor + 1)d - n - 2$ [MP16, Corollary H]. However, in the case of nodal surfaces, their result is slightly weaker, it only yields independence in degree $2d-4$, which is insufficient for proving the infinitesimal Torelli theorem for nodal surfaces.

To prove the infinitesimal Torelli theorem, we need conditions on the algebraic independence of the partial derivatives F_i.

Lemma 3.3.7. *Let $X \subset \mathbb{P}^3$ be a nodal surface defined by a homogeneous polynomial F of degree d. Then, the partial derivatives $F_i = \frac{\partial F}{\partial z_i}$ are algebraically independent in degrees $\leq 2d-4$, that is, if $\sum_{i=0}^{3} H_i F_i = 0$ with $\deg H_i F_i \leq 2d-4$, then $H_i = 0$ for all i.*

Proof. Since the Jacobian ideal J is generated by the four partial derivatives F_i, which are of degree $d-1$, the homogeneous module J_{2d-4} is generated by the products $F_i z_{j_d} \cdots z_{j_{2d-4}}$ where $0 \leq j_d \leq \cdots \leq j_{2d-4} \leq 3$. Hence, $\dim J_{2d-4} \leq 4 \dim S_{d-3} = 4\binom{d}{3}$. The partial derivatives are algebraically independent in degrees $\leq 2d-4$ if and only if the given set of generators is linearly independent, i.e. $\dim J_{2d-4} = 4\binom{d}{3}$.

Let $Z = \{p_1, \ldots, p_k\}$ be the set of nodes. Since $I = I(Z)$ is the ideal of polynomials that vanish on Z, the homogeneous part I_{2d-4} is generated by the kernel of the matrix $(e_i(p_j))_{i,j}$, which has dimension at least $\dim S_{2d-4} - k$.

By Proposition 2.3.6, we have $\dim (I/J)_{2d-4} = h^1(\tilde{\Omega}^1_X)_{\text{prim}} = h^1(\tilde{\Omega}^1_X) - 1$. Let $\pi : \tilde{X} \to X$ be a resolution of singularities of X. By Proposition 2.2.21, there is an isomorphism $\tilde{\Omega}^1_X = R\pi_* \Omega^1_{\tilde{X}}(\log E)$. The distinguished triangle

$$R\pi_*\Omega^1_{\tilde{X}} \to R\pi_*\Omega^1_{\tilde{X}}(\log E) = \tilde{\Omega}^1_X \to R\pi_*\mathcal{O}_E \xrightarrow{+1}$$

gives a long exact sequence

$$0 \to H^{1,0}(\tilde{X}) \xrightarrow{\sim} H^{1,0}(X) \to H^0(E, \mathcal{O}_E) \cong \mathbb{C}^k \to H^{1,1}(\tilde{X}) \to H^{1,1}(X) \to 0.$$

So, $h^{1,1}(\tilde{X}) = h^{1,1}(X) + k$, and by Noether's formula, for a smooth hypersurface,

$$h^{1,1}(\tilde{X}) = 10\chi(\mathcal{O}_{\tilde{X}}) - K^2_{\tilde{X}} = 10\left(1 + \binom{d-1}{3}\right) - d(d-4)^2.$$

We can thus compute the dimension of the vector space J_{2d-4} to be

$$\begin{aligned} 4\binom{d}{3} \geq \dim J_{2d-4} &= \dim I_{2d-4} - \dim (I/J)_{2d-4} \\ &\geq \dim S_{2d-4} - k - h^1(\tilde{\Omega}^1_X) + 1 \\ &= \binom{2d-1}{3} - k - h^{1,1}(\tilde{X}) + k + 1 = 4\binom{d}{3}. \end{aligned} \tag{3.5}$$

Hence, equality holds throughout, and we conclude that the partial dervatives are algebraically independent in degree $\leq 2d-4$. □

Note the fact that equality holds in (3.5) implies that $\dim S_{2d-4} - \dim I_{2d-4} = k$, so by Remark 3.3.6, the nodes are independent in degree $2d-4$. Thus, we have given an alternate proof of the result.

We now want to further show that the partial derivatives are algebraically independent in degrees up to $2d-3$.

Recall the surjective map $d_4 : (S/J)^{\vee}_{\rho-l} \to (I/J)_l$ (cf. (3.4)) induced from the spectral sequence (3.3). We try to understand the kernel of the map d_4. Note that $E_5^{-4,3} = \ker(E_4^{-4,3} \xrightarrow{d_4} E_4^{0,0})$ and

$$H^0(Z,K) = H^{-1}(X, K[1]) = E_5^{-4,3} \oplus E_5^{-1,0}. \tag{3.6}$$

We can compute

$$\begin{aligned} E_5^{-1,0} = E_2^{-1,0} &= \frac{\ker\left(H^0(\mathbb{P}^3, \mathcal{E}^{\vee}(l)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l))\right)}{\operatorname{im}\left(H^0(\mathbb{P}^3, \bigwedge^2 \mathcal{E}^{\vee}(l)) \to H^0(\mathbb{P}^3, \mathcal{E}^{\vee}(l))\right)} \\ &= \frac{\ker\left(\bigoplus_{i=0}^3 S_{l-d+1} \xrightarrow{\beta_l} S_l\right)}{\operatorname{im}\left(\bigoplus_{0\le i<j\le 3} S_{l-2d+2} \to \bigoplus_{i=0}^3 S_{l-d+1}\right)}. \end{aligned}$$

Note that $J_l = \operatorname{im}\left(\bigoplus_{i=0}^3 S_{l-d+1} \xrightarrow{\beta_l} S_l\right)$.

Definition 3.3.8. We define the defect $\operatorname{Dft}(J_l)$ of J_l be the kernel of the map $\bigoplus_{i=0}^3 S_{l-d+1} \xrightarrow{\beta_l} S_l$ and let $\operatorname{dft}(J_l) = \dim \operatorname{Dft}(J_l)$. Then $\operatorname{dft}(J_l) + \dim J_l = 4\dim S_{l-d+1}$.

Proposition 3.3.9. *The partial derivatives F_i are algebraically independent in degree $2d-3$, i.e.* $\operatorname{dft}(J_{2d-3}) = 0$.

Proof. Note that for $l < 2d-2$, we have $S_{l-2d+2} = 0$, so $E_5^{-1,0} = \operatorname{Dft}(J_l)$. Thus, by the equation (3.6), for $l < 2d-2$, there are isomorphisms

$$H^0(Z,K) = \ker\left((S/J)^{\vee}_{\rho-l} \xrightarrow{d_4} (I/J)_l\right) \oplus \operatorname{Dft}(J_l).$$

By Lemma 3.3.3, the map d_4 is surjective, so we can compute the dimension of $H^0(Z,K)$ as

$$h^0(K) = \dim S_{\rho-l} - \dim J_{\rho-l} - \dim I_l + \dim J_l + \operatorname{dft}(J_l).$$

Now we restrict to the cases where $2d-5 \le l \le 2d-3$. Severi [Sev46, §14] showed that the set of nodes on a nodal surface is independent in degrees $\ge 2d-5$, so $\dim S_l - \dim I_l = k$.

Recall that $\operatorname{dft}(J_l) + \dim J_l = 4\dim S_{l-d+1}$. Thus, for $2d-5 \le l \le 2d-3$, we have

$$h^0(K) = \dim S_{\rho-l} - 4\dim J_{\rho-l-d+1} + \operatorname{dft}(J_{\rho-l}) - \dim S_l + k + 4\dim J_{l-d+1}.$$

Since $h^0(K)$ is independent of l, we can compare the dimensions for $l = 2d-5$ and $l = 2d-3$ to obtain $\mathrm{dft}(J_{2d-3}) = \mathrm{dft}(J_{2d-5}) = 0$ by Lemma 3.3.7. $\square$

Remark 3.3.10. Proposition 3.3.9 gives the largest possible degree for algebraic independence. One can verify on Explicit examples 4.1.15 and 4.2.7 that the partial derivatives F_i are no longer algebraically independent in degree $2d-2$.

Lemma 3.3.11. *Let $l < 2d-4$ and $G \in I_l$. Suppose $Gz_j \in J_{l+1}$ for all $j = 0, \ldots, 3$, then $G \in J_l$.*

Proof. Let $F_k = \frac{\partial F}{\partial z_k}$ and $Gz_j = \sum_k H_{jk}F_k$ for each j with H_{jk} of degree $l-d+2 \leq l$. Then, for any $i \neq j$,

$$0 = G(z_j z_i - z_i z_j) = \sum_{k=0}^{3}(z_i H_{jk} - z_j H_{ik})F_k.$$

By Proposition 3.3.9, the F_i's are algebraically independent in degree $l+2 \leq 2d-3$, so the coefficients $z_i H_{jk} - z_j H_{ik} = 0$ for all i, j, k. In particular, z_j divides H_{jk} for each j. Let $H_{jk} = H'_{jk}z_j$, then $G = \sum_k H'_{jk}F_k$ lies in J_l. $\square$

We can now conclude:

Proposition 3.3.12. *Let $X \subset \mathbb{P}^3$ be a nodal surface of degree $d \geq 4$. Then, the map*

$$d\mathcal{P}^2 : H^1(X, \tilde{T}_X) \to \mathrm{Hom}(H^0(X, \tilde{\omega}_X), H^1(X, \tilde{\Omega}^1_X))$$

is injective.

Proof. The map $d\mathcal{P}^2$ factors through

$$\begin{aligned} \Pi : & H^1(X, \tilde{T}_X) = (I/J)_d \\ & \longrightarrow \mathrm{Hom}(H^0(X, \tilde{\omega}_X), H^1(X, \tilde{\Omega}^1_X)_{\mathrm{prim}}) = \mathrm{Hom}(S_{d-4}, (I/J)_{2d-4}) \end{aligned}$$

which is given by multiplication of polynomials. Suppose $[P] \in (I/J)_d$ is in the kernel of Π, that is, $P \cdot Q \in J_{2d-4}$ for all $Q \in S_{d-4}$. We shall prove that $P \in J_d$, so $[P] = 0 \in (I/J)_d$.

Suppose, to the contrary, that $P \notin J_d$, then there exists a homogeneous polynomial $R \in S_{<d-4}$ of maximal degree such that $P \cdot R \notin J$. By hypothesis, $P \cdot R \cdot z_i \in J$ for all $i = 0, \ldots, 3$. Lemma 3.3.11 then implies that $P \cdot R \in J$, giving the contradiction. $\square$

Remark 3.3.13. In [EM15, Corollaire 5.2.2], Eyssidieux and Mégy proved a similar result, that

$$\Pi : (I/J)_d \to \mathrm{Hom}((S/J)_{d-n-2}, (I/J)_{2d-n-2})$$

is injective. However, they needed to impose the condition that $(I/J)_{\geq d-n-2}$ is generated by $(I/J)_{d-n-2}$. Our result, though much more restricted, avoids this condition. In fact, for all our examples of nodal sextic surfaces in Chapter 4, $(I/J)_{d-n-2} = (I/J)_2$ are zero.

Remark 3.3.14. The proof of Proposition 3.3.9 suggests that there is some form of duality between the algebraic independence of F_i in degree l and the independence of the nodes in degree $\rho - l$, for example, an equality $\mathrm{dft}(J_{\rho-l}) = k'_l$ where $k'_l \leq k$ is the rank of the matrix $(e_i(p_j))_{i,j}$ determining the dependence of the nodes.

However, such an equality cannot hold for $l \geq 2d - 2$ because the term $\mathrm{im}\left(\bigoplus_{0\leq i<j\leq 3} S_{l-2d+2} \to \bigoplus_{i=0}^{3} S_{l-d+1}\right)$ in $E_5^{-1,0}$ is non-zero. If we can understand this term better, it may be possible to use Mustaţă and Popa results [MP16, Corollary H] on independence of nodes to prove the injectivity of the infinitesimal period map for hypersurfaces in higher dimensions with isolated singularities.

To prove the infinitesimal Torelli theorem, we further assume that $X = M/G$ is a quotient surface and G is small in the neighbourhoods of all points on the singular locus of X. This assumption is not essential, but we have only defined the infinitesimal period map for quotient surfaces, and not for general V-manifolds. We need to show that the Hodge numbers are constant in an open neighbourhood of X on a versal family of G-equivariant deformations of M.

Proposition 3.3.15. *Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface of degree d such that X is a V-manifold. Suppose $n \geq 2$ and $(n, d) \neq (2, 4)$. Then all small deformations of X parametrized by $H^1(X, \tilde{T}_X)$ are projective hypersurfaces of degree d, preserving the singularities of X.*

Proof. By Proposition 2.3.7 and Remark 2.3.8, there is an isomorphism

$$H^1(X, \tilde{T}_X) = H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log X))$$

under the hypothesis $n \geq 2$ and $(n, d) \neq (2, 4)$. By Example 3.1.3, the latter parametrizes all infinitesimal deformations of X in $\mathbb{P}^{n+1}$ that preserve the singularities. Since there is a unique linear system $\mathcal{O}_{\mathbb{P}^{n+1}}(d) = \mathcal{O}_{\mathbb{P}^{n+1}}(X)$ in $\mathbb{P}^{n+1}$ of degree d, all deformations of X are projective hypersurfaces of degree d. □

Theorem 3.3.16. *Let $X \subset \mathbb{P}^3$ be a nodal surface of degree $d \geq 5$. Then X satisfies the infinitesimal Torelli property.*

Proof. The injectivity of the infinitesimal period map was already proven in Proposition 3.3.12. It remains to show that the period domain is well-defined in a small neighbourhood $\mathcal{U}$ of X in the moduli family parametrized by $H^1(X, \tilde{T}_X)$.

Suppose X has k nodes. By Proposition 3.3.15, any $X' \in \mathcal{U}$ is also a nodal surface of degree d with k nodes. The Hodge numbers of a nodal surface of degree d is only dependent on the number of nodes, hence the Hodge numbers are constant on $\mathcal{U}$ and the period domain is well-defined. $\square$

Remark 3.3.17. Deformations of nodal surfaces have been studied in greater depth by Burns and Wahl [BW74]. They showed that the formal moduli space of surfaces of degree d with at most nodal singularities extends that of smooth surfaces of degree d, and is a reduced complete intersection of dimension $\binom{d+3}{3} - 16$ for $d \geq 5$ (Corollary 2.11).

They further showed that G-equivariant deformations of a nodal surface X of degree d is unobstructed if and only if the set Z of nodes is independent in degree d [BW74, Corollary 4.3]. Thus, if $d \leq 5$, all G-equivariant deformations are unobstructed by Severi's result on independence of nodes. For the two examples we give in Chapter 4 with $d = 6$, we show directly in both cases that the infinitesimal deformations are unobstructed (Propositions 4.1.14 and 4.2.12), and hence the nodes are independent in degree 6. This gives an improvement over Severi's result.

Chapter 4

Nodal sextic surfaces

Recall from Definition 2.2.19 that a nodal surface is a projective surface with only ordinary double points as singularities.

Definition 4.0.1. A set of nodes of a surface F is said to be even if there exists a double cover $S \to F$ branched exactly on the nodes from that set. An even k-nodal surface is a surface whose singular locus is exactly k nodes forming an even set.

The involution defining the double cover $f : S \to F$ of an even nodal surface induces a decomposition of the Hodge structure of S. In particular, the negative eigenspace $H^2(S, \mathbb{Q})_-$ of the weight 2 Hodge structure often has small $h^{2,0}$ and large $h^{1,1}$.

Hence, the double covers of even nodal surfaces provide a source of potentially interesting simple weight 2 Hodge structures of geometric origin. In this chapter, we study some families of such examples with detailed discussion of their Hodge structures.

We shall focus on a set of examples previously considered by Casnati, Catanese and Tonoli [CC97; CT07]. In [CT07], Catanese and Tonoli showed that an even set of nodes on a sextic surface has cardinality in $\{24, 32, 40, 56\}$. They provided a construction of these nodal surfaces in the paper [CC97], where it is shown that even sets of nodes correspond to certain symmetric maps between vector bundles. In this way, one can find explicit examples of such surfaces, but the equations tend to be rather complicated and it is not easy to understand the geometry of these surfaces. In this thesis we present more explicit constructions of these nodal surfaces, and study their Hodge structures through additional involutions and deformations.

Let F be an even k-nodal sextic surface and $f : S \to F$ be the double cover of F ramified on the nodes $p_1, \dots, p_k$. There is a decomposition

$$f_* \mathcal{O}_S = \mathcal{O}_F \oplus \mathcal{F}.$$

Casnati and Catanese [CC97, Proposition 3.1] showed that $\mathcal{F}$ is a quadratic coherent sheaf, that is, there is an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee$.

Let $\pi_F : \tilde{F} \to F$ be the desingularization of F. There is a Cartesian diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\pi_S"] \arrow[d, "\tilde{f}"'] & S \arrow[d, "f"] \\ \tilde{F} \arrow[r, "\pi_F"] & F. \end{tikzcd}$$

Let $N_i \subset \tilde{F}$ be the inverse images of the nodes $p_i \in F$. The double cover $\tilde{f}$ is branched over the exceptional divisor $\tilde{\Delta} = \sum N_i$. Since p_i are nodes, the exceptional curves N_i are (-2)-curves. Let $E_i = \tilde{f}^{-1}N_i$, so

$$E_i \cdot E_i = \frac{1}{4}\tilde{f}^*N_i \cdot \tilde{f}^*N_i = \frac{1}{2}(N_i \cdot N_i) = -1$$

and E_i are (-1)-curves. Blowing down the exceptional curves E_i give the surface S, hence $\tilde{S}$ is the blow up of S at $f^{-1}\Delta$ where $\Delta = \bigcup p_i$ is the set of nodes of F.

There exists a divisor L on $\tilde{F}$ such that $2L \equiv \tilde{\Delta}$. Let $\mathcal{L} = \mathcal{O}_{\tilde{F}}(L)$, then there is a decomposition (cf. [Par91, Proposition 4.1])

$$\tilde{f}_*\mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{F}} \oplus \mathcal{L}^{-1}.$$

The $\mathcal{O}_{\tilde{F}}$-linear involution on $\tilde{f}_*\mathcal{O}_{\tilde{S}}$ decomposes it into the eigensheaves $\mathcal{O}_{\tilde{F}}$ and $\mathcal{L}^{-1}$ with eigenvalues $+1$ and -1 respectively. We denote the eigenspaces on the cohomology groups by $H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})_+ = H^i(\tilde{F}, \mathcal{O}_{\tilde{F}})$ and $H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})_- = H^i(\tilde{F}, \mathcal{L}^{-1})$. Since $H^i(S, \mathcal{O}_S) = H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$ and $H^i(F, \mathcal{O}_F) = H^i(\tilde{F}, \mathcal{O}_{\tilde{F}})$, we have $H^i(F, \mathcal{F}) = H^i(\tilde{F}, \mathcal{L}^{-1})$.

Lemma 4.0.2. *Let $F \subset \mathbb{P}^3$ be an even k-nodal sextic surface and use the notations as above. Then,*

$$h^i(\mathcal{O}_{\tilde{S}})_+ = h^i(\mathcal{O}_{\tilde{F}}) = h^i(\mathcal{O}_F) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 10 & i = 2 \end{cases} \qquad \textit{and}$$

$$\chi(O_{\tilde{S}})_- = \chi(\mathcal{L}^{-1}) = \chi(\mathcal{F}) = 11 - \frac{k}{4}.$$

Furthermore, $K_F^2 = K_{\tilde{F}}^2 = 24$, $K_{\tilde{S}}^2 = 48 - k$ and $K_S^2 = 48$.

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-6) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_F \to 0$$

and its associated long exact sequence. Since $h^i(\mathcal{O}_{\mathbb{P}^3}) = 0$ for all $i > 0$ and $h^i(\mathcal{O}_{\mathbb{P}^3}(-6)) = 0$ for all $i < 3$, we have $h^0(\mathcal{O}_F) = h^0(\mathcal{O}_{\mathbb{P}^3}) = 1$ and $h^2(\mathcal{O}_F) = h^3(\mathcal{O}_{\mathbb{P}^3}(-6)) = h^0(\omega_{\mathbb{P}^3}(6)) = h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \dim \mathbb{C}[x_0, \ldots, x_3]_2 = 10.$

By the adjunction formula, the canonical divisor $K_{\tilde{F}} = \pi_F^* K_F = \pi_F^*(K_{\mathbb{P}^3} + F)_{|F} = \pi_F^* \mathcal{O}(2)_{|F}$ is twice the general hyperplane section H. Since the intersection product $H \cdot \tilde{\Delta}$ is zero, the Riemann-Roch theorem for surfaces gives

$$\chi(\mathcal{L}^{-1}) = \chi(\mathcal{O}_{\tilde{F}}) + \frac{(-L)(-L - K_{\tilde{F}})}{2} = 11 + \frac{\tilde{\Delta}^2}{8} = 11 - \frac{k}{4}.$$

The adjunction formula gives $K_{\tilde{F}}^2 = (\mathcal{O}(2) \cdot \mathcal{O}(2))|_F = 2 \cdot 2 \cdot 6 = 24$. There is an isomorphism $\tilde{f}^*(K_{\tilde{F}} + L) = K_{\tilde{S}}$, so $K_{\tilde{S}}^2 = \tilde{f}^*(K_{\tilde{F}} + L)^2 = 2((2H)^2 + \frac{1}{4}\tilde{\Delta}^2) = 2(24 - \frac{1}{4}(2k)) = 48 - k$. Since $\tilde{\Delta}_S = \tilde{f}^{-1}(\tilde{\Delta}) \subset \tilde{S}$ is the blowup of smooth points on S, we have $K_{\tilde{S}} = \pi_S^* K_S + \tilde{\Delta}_S = \pi_S^* K_S + \tilde{f}^* L$. Hence, $\pi_S^* K_S = K_{\tilde{S}} - \tilde{f}^* L = \tilde{f}^* K_{\tilde{F}}$ and $K_S^2 = 2K_{\tilde{F}}^2 = 48$. $\square$

4.1 56-nodal sextic surfaces

In [CT07], Catanese and Tonoli constructed a family of even 56-nodal sextic surfaces as the degeneracy locus of a symmetric map. In this section, we give a more direct construction starting from the theta divisor Θ of a principally polarized abelian threefold. We will construct a family of even 56-nodal sextic surfaces F with double cover S equipped with another double cover $S \to \Theta$ induced by the Albanese map of S. Any such surface F lies in a commutative diagram of covering maps

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "f"'] & \Theta \arrow[d, "\phi"] \\ F \arrow[r, "\overline{p}"'] & \overline{\Theta} \end{tikzcd}$$

The double cover ϕ is induced by the involution $[-1]$ on the abelian threefold $\mathrm{Alb}(S)$, and it fixes 28 points on Θ. The nodal surface $\overline{\Theta}$ has been studied before, cf. [DO88, Chapter IX.6, Theorem 4 and Remark 6].

In Section 4.1.3, we shall show that a general member of the family of Catanese and Tonoli lies in the family we constructed. By construction, F has an involution with quotient $\overline{\Theta}$.

The contents of Sections 4.1.1, 4.1.2 and 4.1.4 are contained in the preprint [GZ16].

4.1.1 Construction of a family of even 56-nodal surfaces

Let C be a smooth non-hyperelliptic curve of genus 3 and consider its Jacobian $A = \mathrm{Jac}(C)$. The abelian variety A admits a principal polarization defined by a theta divisor Θ and we will identify $\Theta = S^2C$. We can choose Θ to be a symmetric divisor on A, i.e. $[-1]^*\Theta = \Theta$. The involution $[-1]$ on Θ corresponds to the involution $D \mapsto K_C - D$ on S^2C, where K_C is the canonical divisor on C.

The linear system $|2\Theta|$ is totally symmetric, that is, $[-1]^*D = D$ for all $D \in |2\Theta|$. It defines a morphism

$$\varphi_{2\Theta} : A \to \mathbb{P}^7$$

which is the quotient map by the involution $[-1]$. Let $\overline{A} = A/[-1]$, the Kummer variety of A, be the image of $\varphi_{2\Theta}$. The singular locus of $\overline{A}$ consists of 64 nodes, which are the images of the two-torsion points of A.

Consider the hyperplane $H_{2\Theta}$ of $\mathbb{P}^7$ corresponding to the divisor 2Θ. The intersection of $H_{2\Theta}$ with $\overline{A}$ is the image $\overline{\Theta} = \Theta/[-1]$ of Θ, with multiplicity two. As Θ contains 28 of the two-torsion points of A, the surface $\overline{\Theta}$ has 28 nodes. Equivalently, these are the images of the 28 odd theta characteristics in S^2C.

To describe this map $\varphi_{2\Theta|\Theta} : \Theta \to \mathbb{P}^6$, notice that the adjunction formula on A shows that the canonical class of Θ is $K_\Theta = \Theta_{|\Theta}$. Thus $\mathcal{O}_\Theta(2\Theta) \cong \omega_\Theta^{\otimes 2}$. Moreover, the cohomology of the restriction sequence

$$0 \to \mathcal{O}_A(\Theta) \to \mathcal{O}_A(2\Theta) \to \mathcal{O}_\Theta(2\Theta) \to 0$$

combined with $H^i(A, \mathcal{O}_A(\Theta)) = 0$ for $i > 0$ (Kodaira vanishing or Riemann-Roch on A), shows that $h^0(\omega_\Theta^{\otimes 2}) = h^0(\mathcal{O}_\Theta(2\Theta)) = 7$. Hence, when restricted to Θ, the morphism $\varphi_{2\Theta}|_\Theta = \varphi_{2K_\Theta}$ is given by the complete linear system $|2K_\Theta|$.

To understand this morphism better, we first consider the map φ_{K_Θ}. From the restriction sequence above, twisted by $\mathcal{O}_A(-\Theta)$, one deduces that $H^0(\Theta, \omega_\Theta) \cong H^1(A, \mathcal{O}_A)$ is three dimensional. The map $\varphi_{K_\Theta} : \Theta \to \mathbb{P}^2$ is the Gauss map, which is a morphism of degree $(\Theta_{|\Theta})^2 = \Theta^3 = 6$ factoring through $\overline{\Theta}$. As φ_{K_Θ} is surjective, the natural map $S^2H^0(\Theta, \omega_\Theta) \to H^0(\Theta, \omega_\Theta^{\otimes 2})$ is injective, thus the image has codimension one.

Let $t \in H^0(\Theta, \omega_\Theta^{\otimes 2})$ be a general section in the complement of the image of $S^2H^0(\Theta, \omega_\Theta)$. Since $|2K_\Theta|$ is basepoint free, we may assume that the divisor B in Θ defined by $t = 0$ is smooth and does not pass through any two-torsion points. Since $|2K_\Theta|$ is totally symmetric, we have $[-1]^*B = B$. Let s_0, s_1, s_2 be a basis of $H^0(\Theta, \omega_\Theta)$. Then we have:

$$\begin{aligned} \varphi_{2K_\Theta} : \Theta &\to \mathbb{P}H^0(\Theta, \omega_\Theta^{\otimes 2}) \cong H_{2\Theta} \cong \mathbb{P}^6, \\ x &\mapsto (\ldots : s_i(x)s_j(x) : \ldots : t(x))_{0 \le i \le j \le 2}. \end{aligned}$$

The image $\overline{\Theta}$ of Θ thus lies in a cone over the Veronese surface of $\mathbb{P}^2$. This cone is the image Y of the weighted projective 3-space $\mathbb{P}(1,1,1,2)$, which is embedded into $\mathbb{P}^6$ by the (very) ample generator $\mathcal{O}_Y(2)$ of its Picard group:

$$\begin{aligned} \mathbb{P}(1,1,1,2) \quad &\rightarrow \quad Y \subset \mathbb{P}^6, \\ (y_0 : y_1 : y_2 : y_3) \ &\mapsto \ (\ldots : y_i y_j : \ldots : y_3)_{0 \leq i \leq j \leq 2}. \end{aligned}$$

As φ_{K_Θ} has no base points, the surface $\overline{\Theta} \subset Y$ does not contain the singular point $v = (0 : \ldots : 0 : 1)$ of Y, the vertex of the cone over the Veronese surface. Hence, $\overline{\Theta}$ is a Cartier divisor on Y. The projection of $\overline{\Theta}$ from v onto the Veronese surface is the Gauss map φ_{K_Θ}, which has degree $6/2 = 3$ on $\overline{\Theta}$. This implies that $\overline{\Theta}$ lies in the linear system on Y defined by three times the ample generator. Since the map $S^3 H^0(Y, \mathcal{O}_Y(2)) \rightarrow H^0(Y, \mathcal{O}_Y(6))$ is surjective, we conclude that $\overline{\Theta}$ is defined by a weighted homogeneous polynomial p of degree six in $Y = \mathbb{P}(1,1,1,2)$:

$$\overline{\Theta} = \Bigg\{ (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}(1,1,1,2) : \\ p(y_0, \ldots, y_3) = \sum_{i=0}^{3} p_{2i}(y_0, y_1, y_2) y_3^{3-i} = 0 \Bigg\},$$

where each p_{2i} is homogeneous of degree $2i$ in y_0, y_1, y_2. Since $v \notin \overline{\Theta}$, we may and will assume that $p_0 = 1$.

The weighted projective space $\mathbb{P}(1,1,1,2)$ is also the quotient of $\mathbb{P}^3$ by the involution $i_3 : (x_0 : \ldots : x_3) \mapsto (x_0 : x_1 : x_2 : -x_3)$, the quotient map is explicitly given by:

$$\overline{p} : \mathbb{P}^3 \rightarrow \mathbb{P}(1,1,1,2), \qquad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2 : x_3^2).$$

Now we define a surface F in $\mathbb{P}^3$ as $F := \overline{p}^{-1}(\overline{\Theta})$, thus F is defined by the sextic equation $P = 0$ where

$$P := p_6(x_0, x_1, x_2) + p_4(x_0, x_1, x_2) x_3^2 + p_2(x_0, x_1, x_2) x_3^4 + x_3^6.$$

The double cover $\overline{p} : F \rightarrow \overline{\Theta}$ is branched over the points where $x_3 = 0$, so the branch locus is the divisor $\overline{B} \subset \overline{\Theta}$ defined by $t = 0$. Here $\overline{B} = B/[-1]$, which is a smooth curve by the assumption that B is smooth and does not pass through the 28 fixed points of $[-1]$ in Θ. The same assumption implies that the singular locus of F consists of 56 nodes. The 28 nodes of $\overline{\Theta}$ form an even set since the double cover $\Theta \rightarrow \overline{\Theta}$ is branched only over the nodes. Hence, the preimage $\Delta \subset F$ of these nodes is also an even set. In fact, F has a double cover S branched only over the nodes by pulling back the double cover $\Theta \rightarrow \overline{\Theta}$ along $\overline{p} : F \rightarrow \overline{\Theta}$, cf. diagram (4.1).

We summarize the construction as follows:

Theorem 4.1.1. *There exists a family of even 56-nodal sextic surfaces, which is parametrized by pairs (C, B) where C is a non-hyperelliptic curve of genus 3 and $B \in |2K_{S^2C}|$ is a general divisor. Each surface F is obtained as the double cover of $\overline{\Theta}$ branched over $\overline{B}$ where $\overline{\Theta}$ and $\overline{B}$ are images of $\Theta = S^2C$ and $B \subset \Theta$ under the quotient map $A = J(C) \to A/[-1]$.*

In particular, we have a $6 + 6 = 12$ dimensional family of such surfaces. Moreover, each surface in the family has an automorphism of order two.

4.1.2 Coverings of $\overline{\Theta}$

From the definition of S as the base change along $\overline{p} : F \to \overline{\Theta}$ of the double cover $\Theta \to \overline{\Theta}$, it follows that the covering $S \to \overline{\Theta}$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-covering. Let ι_1 and ι_2 be involutions on S with quotient surfaces F and Θ respectively. Let $\iota_3 = \iota_1\iota_2$, then ι_3 is an involution and we define $T := S/\iota_3$.

This gives a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "p"] \arrow[dd, "f"'] \arrow[dr] & & \Theta \arrow[dd, "\phi"] \\ & T \arrow[dr] & \\ F \arrow[rr, "\overline{p}"'] & & \overline{\Theta} \end{tikzcd} \tag{4.1}$$

Proposition 4.1.2. *The double cover $S \to T$ is unramified. In particular, T is smooth and $T \to \overline{\Theta}$ is branched along $\overline{B}$ and the 28 nodes.*

Proof. The ramification locus of $S \to T$ is the fixed locus R_3 of ι_3, which is precisely the points $s \in S$ such that $\iota_3 \in \mathrm{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s)$. The fixed loci of ι_1 and ι_2 are $f^{-1}\Delta$ and $p^{-1}B$ respectively. Since the branch curve B does not contain any of the 28 two-torsion points on Θ, the intersection of the fixed loci $f^{-1}\Delta \cap p^{-1}B = \emptyset$. Hence, there are no points $s \in S$ such that $\mathrm{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s) = (\mathbb{Z}/2\mathbb{Z})^2$. In particular, $R_3 = \{s \in S | \mathrm{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s) = \langle\iota_3\rangle\}$ is disjoint from $f^{-1}\Delta \cup p^{-1}B$. Since the ramification locus of $S \to \overline{\Theta}$ is precisely the union of that of f and p, we conclude that $R_3 = \emptyset$ and $S \to T$ is unramified. □

We now consider the Hodge numbers of the surfaces in diagram 4.1.

Proposition 4.1.3. *The smooth surfaces Θ, S, T and $\tilde{F}$ have Hodge numbers:*

	$h^{1,0}$	$h^{2,0}$	$h^{1,1}$
Θ	3	3	10
S	3	10	38
$\tilde{S}$	3	10	94
$\tilde{F}$	0	10	86
T	0	3	16

Proof. The cohomology groups of $\mathcal{O}_\Theta$ are computed using the short exact sequence

$$0 \to \mathcal{O}_A(-\Theta) \to \mathcal{O}_A \to \mathcal{O}_\Theta \to 0.$$

Since Θ is ample on A and $K_A = 0$, $h^i(\mathcal{O}_A(-\Theta)) = 0$ for $i < 3$ by Kodaira vanishing and, using Serre duality, $h^3(\mathcal{O}_A(-\Theta)) = h^0(\mathcal{O}_A(\Theta)) = 1$ since Θ is a principal polarization. Moreover, $h^i(\mathcal{O}_A) = \binom{3}{i}$, hence $h^{1,0}(\Theta) = h^1(\mathcal{O}_\Theta) = 3$ and $h^{2,0}(\Theta) = 3$. As

$$\chi_{\text{top}}(\Theta) = 2-4h^{1,0}(\Theta)+2h^{2,0}(\Theta)+h^{1,1}(\Theta) = 2-12+6+h^{1,1}(\Theta) = h^{1,1}(\Theta)-4,$$

we can compute $h^{1,1}(\Theta)$ from Noether's formula:

$$\chi(\mathcal{O}_\Theta) = \frac{\chi_{\text{top}}(\Theta) + K_\Theta^2}{12} \;\Rightarrow\; h^{1,1}(\Theta) = 12\chi(\mathcal{O}_\Theta) - K_\Theta^2 + 4 = 12 - 6 + 4 = 10.$$

For the double cover $p : S \to \Theta$, branched over the divisor B, there is an isomorphism

$$p_*\mathcal{O}_S = \mathcal{O}_\Theta \oplus \mathcal{L}^{-1} \qquad \text{with} \quad \mathcal{L} \cong \omega_\Theta,$$

so $\mathcal{L}^{\otimes 2} = \mathcal{O}_\Theta(B)$. Thus $h^{i,0}(S) = h^i(\mathcal{O}_\Theta) + h^i(\mathcal{L}^{-1})$. As $\mathcal{L} = \omega_\Theta$ is ample, by Kodaira vanishing we get $h^i(\mathcal{L}^{-1}) = 0$ for $i < 2$. Hence, by Riemann-Roch

$$h^2(\mathcal{L}^{-1}) = \chi(\omega_\Theta^{-1}) = \chi(\mathcal{O}_\Theta) + \frac{K_\Theta \cdot (K_\Theta + K_\Theta)}{2} = 7.$$

By Lemma 4.0.2, $K_S^2 = 48$, so we obtain $h^{1,1}(S) = 38$ by Noether's formula. The blowup $\pi_S : \tilde{S} \to S$ at 56 points does not change $h^{1,0}$ and $h^{2,0}$, and $h^{1,1}(\tilde{S}) = h^{1,1}(S) + 56$.

Lemma 4.0.2 give the Hodge numbers $h^{i,0}(F)$ and $h^{i,0}(\tilde{F})$, and also gives $K_{\tilde{F}}^2 = 24$. By Noether's formula, we obtain $h^{1,1}(\tilde{F}) = 86$.

Since $S \to T$ is an unramified double cover, we have $\chi_{\text{top}}(S) = 2\chi_{\text{top}}(T)$ and $K_S^2 = 2K_T^2$, so Noether's formula implies that $2\chi(\mathcal{O}_T) = \chi(\mathcal{O}_S) = 8$. As $h^1(\mathcal{O}_\Theta) = h^1(\mathcal{O}_S)$ and $h^1(\mathcal{O}_F) = 0$, the covering involutions of p and f act as multiplication by $+1$ and -1 on $H^1(\mathcal{O}_S)$ respectively. The covering involution of $S \to T$ is their product, thus it acts as -1 and we get $h^1(\mathcal{O}_T) = 0$. Hence, $h^2(\mathcal{O}_T) = \chi(\mathcal{O}_T) - 1 = 3$. We have $\chi_{\text{top}}(T) = 2 - 0 + 2h^{2,0}(T) + h^{1,1}(T) = 8 + h^{1,1}(T)$, so from $2\chi_{\text{top}}(T) = \chi_{\text{top}}(S)$ we obtain $h^{1,1}(T) = 16$. □

A consequence of the fact that we have a morphism $p : S \to \Theta$ and $h^{1,0}(S) = h^{1,0}(\Theta)$, is that the Albanese map of S factors over the Albanese map for Θ, which is just the inclusion $\Theta \hookrightarrow A$, hence $A \cong \mathrm{Alb}(S)$.

4.1.3 Deformations of even 56-nodal surfaces

We want to understand the family of even 56-nodal sextic surfaces we constructed better. Recall that the 12 dimensional family $\mathcal{M}$ we constructed in Theorem 4.1.1 is fibred over the moduli space $\mathcal{M}_3$ of smooth non-hyperelliptic curves of genus 3, which is 6-dimensional. The fibre over $C \in \mathcal{M}_3$ is an open subset of the linear system $|2K_{S^2C}| \cong \mathbb{P}^6$.

From our construction, we can obtain explicit polynomials describing the deformation of an even 56-nodal sextic surface in each fibre.

Recall from Proposition 2.3.7 that $\mathrm{Def}_F = H^1(F, \tilde{T}_F) \cong (I/J)_6$ where $J \subset S = \mathbb{C}[x_0, \dots, x_3]$ is the Jacobian ideal and $I = \sqrt{J}$ is its radical.

First, we give an explicit description of the generators of $H^1(F, \tilde{T}_F)$ which parametrize deformations of F fixing the curve C.

The short exact sequence

$$0 \to T_\Theta(-\log B) \to T_\Theta \to \mathcal{O}_B(B) \to 0$$

of Corollary 2.1.5 induces a long exact sequence in cohomology. By the computations of Lemma 4.1.13, we see that $H^0(\Theta, T_\Theta) = 0$, so there is a left exact sequence

$$\begin{aligned} 0 \to H^0(B, \mathcal{O}_B(B))^G = H^0(\overline{B}, \mathcal{O}_{\overline{B}}(\overline{B})) \to \\ \to H^1(\Theta, T_\Theta(-\log B))^G = H^1(F, \tilde{T}_F) \to H^1(\Theta, T_\Theta)^G. \end{aligned}$$

Thus, the subspace $H^0(\overline{B}, \mathcal{O}_{\overline{B}}(\overline{B})) \subset H^1(\Theta, T_\Theta(-\log B))^G$ parametrizes the infinitesimal deformations of F that fix the curve C (cf. Section 4.1.1) but vary the divisor $B \in |2K_\Theta|$. We give a basis of the subspace $H^0(\overline{B}, \mathcal{O}_{\overline{B}}(\overline{B}))$ in terms of polynomials in I.

Proposition 4.1.4. *Let F be an even 56-nodal sextic surface defined by the polynomial*

$$P = p_6(x_0, x_1, x_2) + p_4(x_0, x_1, x_2)x_3^2 + p_2(x_0, x_1, x_2)x_3^4 + x_3^6$$

where $p_i \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous polynomials of degree i. Then

$$\langle P_{ij} = x_i x_j (p_4 + 2p_2 x_3^2 + 3x_3^4) \mid 0 \le i \le j \le 2 \rangle$$

is a 6-dimensional subspace of $(I/J)_6$*, which parametrizes* $H^0(\overline{B}, \mathcal{O}_{\overline{B}}(\overline{B})) \subset H^1(F, \tilde{T}_F)$*. Hence, the polynomials* $P + \sum_{i,j} \varepsilon_{ij} P_{ij}$ $(\varepsilon_{ij}^2 = 0)$ *define infinitesimal deformations of* F *which fix* C *but vary the divisor* $B \in |2K_\Theta|$*.*

Proof. The surface F is a double cover of $\overline{\Theta} \subset \mathbb{P}(1,1,1,2)$, the latter being defined by the polynomial $Q(u) = p_6 + p_4 u + p_2 u^2 + u^3 \in \mathbb{C}[x_0, x_1, x_2][u]$ where u is of weight 2. In Section 4.1.1, we showed that there is an embedding

$$\mathbb{P}(1,1,1,2) \to \mathbb{P}^6 \ , \ (x_0 : x_1 : x_2 : u) \mapsto (\ldots : x_i x_j : \ldots : u)_{0 \le i \le j \le 2}$$

and the double cover $\phi : \Theta \to \overline{\Theta}$ is induced by

$$\phi_{2K_\Theta} : \Theta \to \mathbb{P}^6 \ , \ x \mapsto (\ldots : s_i(x) s_j(x) : \ldots : t(x))_{0 \le i \le j \le 2}$$

where $s_i \in H^0(\Theta, \omega_\Theta)$ and $t \in H^0(\Theta, \omega_\Theta^{\otimes 2})$ is the zero section defined by the divisor B.

An infinitesimal deformation of B is given by a substitution $t \mapsto t + \sum_{i,j} \varepsilon_{ij} s_i s_j$ where $\varepsilon_{ij}^2 = 0$. This induces a substitution $u \mapsto u + \sum \varepsilon_{ij} x_i x_j$. Applying the substitution to the polynomial Q gives

$$\begin{aligned} Q(u + \sum \varepsilon_{ij} x_i x_j) &= Q(u) + \sum \varepsilon_{ij} x_i x_j \frac{\partial Q}{\partial u} \\ &= Q(u) + \sum \varepsilon_{ij} x_i x_j (p_4 + 2p_2 u + 3u^2). \end{aligned}$$

Hence, $\{x_i x_j (p_4 + 2p_2 u + 3u^2)\}$ is a basis of $H^0(\overline{B}, \mathcal{O}_{\overline{B}}(\overline{B}))$ and pulling back along $\mathbb{P}^3 \to \mathbb{P}(1,1,1,2)$ gives the basis $\{P_{ij}\}$. □

Remark 4.1.5. Observe that $\frac{\partial P}{\partial x_3} = 2x_3(p_4 + 2p_2 x_3^2 + 3x_3^4)$ and since the nodes do not lie on the ramification locus $x_3 = 0$, we have $p_4 + 2p_2 x_3^2 + 3x_3^4 \in I$. Indeed, it is the unique generator of I_4. Note that $\dim S_4 = 35$ is much smaller than the number of nodes, so it is a surprise that $I_4 \neq 0$ for a general even 56-nodal surface.

Using Proposition 4.1.4, we show that the family obtained in Theorem 4.1.1 is locally a family of non-trivial deformations.

Proposition 4.1.6. *Let* F *be an even 56-nodal sextic surface in the family* $\mathcal{M}$ *of Theorem 4.1.1. Then, there is a small neighbourhood* $\mathcal{U}$ *of* F *in* $\mathcal{M}$ *such that any deformation of* F *in* $\mathcal{U}$ *is non-trivial.*

Proof. Suppose there exists a one-parameter family F_ε $(0 \le \varepsilon \ll 1)$ of small deformations of F in $\mathcal{U}$ with $F_0 = F$ such that there is a family of isomorphisms $\sigma_\varepsilon : F \xrightarrow{\sim} F_\varepsilon$. There is a family of involutions ι_ε on F_ε such that $F_\varepsilon / \langle \iota_\varepsilon \rangle = \overline{\Theta}_\varepsilon$. By Proposition 3.3.15, the automorphism group of F_ε is a subgroup of

$\mathrm{Aut}(\mathbb{P}^3) = PGL(3)$. It permutes the 56 nodes, so it is a finite subgroup. In particular, it is discrete, so the automorphisms σ_ε are constant on the family. Since $\sigma_0 = \mathrm{id}$, we have $\sigma_\varepsilon^*(\iota_\varepsilon) = \iota$ for all ε. Hence, the isomorphisms σ_ε induce isomorphisms on the quotient spaces $\sigma_\varepsilon : \overline{\Theta} \to \overline{\Theta}_\varepsilon$.

For each ε, there is a unique double cover Θ_ε branched over the 28 nodes of $\overline{\Theta}_\varepsilon$. By definition of the family $\mathcal{F}$, Θ_ε is the theta divisor associated to some non-hyperelliptic genus 3 curve C_ε. By [Kem81, Thm. 3.6], there is a natural isomorphism of the infinitesimal deformations of C and Θ:

$$H^1(C, T_C) \cong H^1(\Theta, T_\Theta).$$

In particular, we see that σ_ε induces an isomorphism $\sigma_\varepsilon : C \to C_\varepsilon$. Hence, in a sufficiently small open neighbourhood $\mathcal{U} \subset \mathcal{M}$ around F, any trivial deformation of F in $\mathcal{U}$ must lie in the fibre of the curve C in the fibration $\mathcal{M} \to \mathcal{M}_3$.

Note that in the proof of Proposition 4.1.4, the substitution $u \mapsto u + \sum \varepsilon_{ij} x_i x_j$ produce all small deformations of F on the fibre, and all such deformations are non-trivial by the same proposition. Hence, shrinking the neighbourhood $\mathcal{U}$ if necessary, all small deformations of F in $\mathcal{U}$ are non-trivial. □

We now deduce that the 12 dimensional family of 56-nodal sextics we constructed is contained in the family constructed by Catanese and Tonoli in [CT07, Main Theorem B]. Notice that they obtained a 27 dimensional subvariety of the space of sextic surfaces parametrizing 56-nodal sextics, but modulo the action of $\mathrm{Aut}(\mathbb{P}^3)$ one again finds a $27 - 15 = 12$ dimensional family. When using their Macaulay scripts (which can be found in the eprint arXiv:math/0510499), we noticed that it produces sextics which are invariant under the involution $x_0 \mapsto -x_0$ in $\mathbb{P}^3$.

Corollary 4.1.7. *The* 12 *dimensional family of even 56-nodal sextic surfaces constructed in Theorem 4.1.1 is an open dense sub-family of the* 12 *dimensional family from [CT07, Main Theorem B].*

Proof. For a double cover $f : S \to F$ of a 56-nodal sextic surface F, branched exactly over the nodes of F, the quadratic sheaf $\mathcal{F}$ on F defined by $f_*\mathcal{O}_S = \mathcal{O}_F \oplus \mathcal{F}$ must satisfy $(\tau, a) = (3, 3)$ or $(\tau, a) = (3, 4)$, where $2\tau = h^1(F, \mathcal{F}(1))$ and $a = h^1(F, \mathcal{F})$, cf. [CT07, Theorem 2.5]. The family constructed in [CT07] is the one with invariants $(\tau, a) = (3, 3)$. For our surfaces, we have $h^{1,0}(S) = h^1(F, f_*\mathcal{O}_S) = h^1(F, \mathcal{O}_F) + h^1(F, \mathcal{F})$ so we get $h^1(F, \mathcal{F}) = 3$, which shows that they lie in the same family. Furthermore, the family constructed by Catanese and Tonoli is irreducible, so since the two families are of the same dimension, the family $\mathcal{M}$ of Proposition 4.1.6 is an open dense subvariety of that of [CT07]. □

We can further show that for any $F \in \mathcal{M}$, the space of isomorphism classes of infinitesimal deformations $\mathrm{Def}_F = H^1(F, \tilde{T}_F)$ is 12-dimensional. Def_F is isomorphic to the tangent space of $\mathcal{M}$ at F, so this implies that the family is reduced and smooth and all deformations are unobstructed.

By definition, $\mathrm{Def}_F = \mathrm{Def}_S^G = H^1(S, T_S)^G$ where $G \cong \mathbb{Z}/2\mathbb{Z}$ is the group acting on S with quotient F. Let $H \cong \mathbb{Z}/2\mathbb{Z}$ be the other group action inducing $S \to \Theta$. By [Par91, Proposition 4.1], there is a decomposition

$$H^1(S, T_S) = H^1(\Theta, T_\Theta(-\log B)) \oplus H^1(\Theta, T_S \otimes \omega_\Theta^{-1}).$$

Hence,

$$\mathrm{Def}_F = H^1(\Theta, T_\Theta(-\log B))^G \oplus H^1(\Theta, T_S \otimes \omega_\Theta^{-1})^G. \tag{4.2}$$

Computing the cohomology groups requires careful consideration of the induced action of G on the sheaves.

On Θ, the action of G is given by the involution $[-1] : \Theta \to \Theta$ which is the restriction of the involution on the abelian variety A. The involution $[-1]$ induces (non-unique) involutions on coherent sheaves on A, Θ and B (since B is a symmetric divisor, $[-1]$ acts on B as well). On sheaves such as Ω_A^p and T_A, we can define a standard action of G as the action given by pulling back the differentials. We denote the $(+1)$- and (-1)-eigenspaces with respect to this standard basis by $H_+^*(X, \mathcal{F}) = H^*(X, \mathcal{F})^G$ and $H_-^*(X, \mathcal{F})$ respectively for $X = A, \Theta, B$ and $\mathcal{F}$ a coherent sheaf on X endowed with a G-action.

To compute the cohomology groups, we use the Atiyah-Bott fixed point formula to compare the relative dimensions of H_+^* and H_-^*. We recall the theorem here in the form that we need:

Theorem 4.1.8 ([AB68, Thm. 4.12])**.** *Let X be a smooth algebraic variety, $\tau : X \to X$ an automorphism with only a set P of isolated fixed points. Let $\mathcal{F}$ be any vector bundle on X and $\varphi : \tau^{-1}\mathcal{F} \to \mathcal{F}$ be an isomorphism. Then, there is an equality*

$$\sum_{i \geq 0} (-1)^i \mathrm{tr}(\varphi, H^i(X, \mathcal{F})) = \sum_{p \in P} \frac{\mathrm{tr}(\varphi_p, \mathcal{F}_p)}{\det(1 - d\tau_p, T_{X,p})}. \tag{4.3}$$

Remark 4.1.9. For the computations, we need the following short exact sequences:

$$0 \to T_\Theta \to i^*T_A \cong (\mathcal{O}_\Theta)^{\oplus 3} \to \mathcal{O}_\Theta(\Theta) \cong \omega_\Theta \to 0 \tag{4.4}$$
$$0 \to \mathcal{O}_\Theta \to \mathcal{O}_\Theta(B) \to \mathcal{O}_B(B) \to 0 \tag{4.5}$$
$$0 \to T_\Theta(-\log B) \to T_\Theta \to \mathcal{O}_B(B) \to 0 \tag{4.6}$$

It is easy to check that all three exact sequences commute with φ for the standard action of G. Note however that the isomorphisms in the first sequence requires choosing the action $-\varphi$ on $\mathcal{O}_\Theta$ and ω_Θ, where φ is the standard action.

Let $\chi_\pm(\mathcal{F}) = \sum_{i\geq 0}(-1)^i h^i(\mathcal{F})_\pm$ denote the Euler characteristics on the positive and negative eigenspaces respectively, then the left hand side of (4.3) is equal to

$$\chi_+(\mathcal{F}) - \chi_-(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \mathrm{tr}(\varphi, H^i(X, \mathcal{F})).$$

Corollary 4.1.10.

$$\chi_+(\mathcal{F}) = \chi_-(\mathcal{F}) \quad \forall\ \mathcal{F} \textit{ on } B,$$
$$\chi_+(\mathcal{F}) - \chi_-(\mathcal{F}) = \begin{cases} 7 & \mathcal{F} = \mathcal{O}_\Theta, \omega_\Theta, \mathcal{O}_\Theta(B) \\ -14 & \mathcal{F} = T_\Theta, T_\Theta(-\log B) \end{cases}.$$

Proof. Since B does not pass through any fixed points of the symmetric involution τ, the right hand side of the Lefschetz fixed point formula is 0, giving the first equality.

The divisor Θ passes through 28 fixed points, and since B does not pass through them, locally around each $p \in P$, $T_{\Theta,p} = T_\Theta(-\log B)_p$ and $\mathcal{O}_{\Theta,p} = \omega_{\Theta,p} = \mathcal{O}_\Theta(B)_p$, so it suffices to show the formula for T_Θ and $\mathcal{O}_\Theta$. Let z_1, z_2 be the local coordinates around $p \in P$. φ induces the identity map on $\mathcal{O}_{\Theta,p}$ while it sends $\frac{\partial}{\partial z_i} \mapsto -\frac{\partial}{\partial z_i}$ on the tangent space. Hence, for each $p \in P$,

$$\frac{\mathrm{tr}_{\mathbb{C}}(\varphi_p)}{\det(1 - d\tau_p)} = \begin{cases} \frac{1}{\det(2I)} = \frac{1}{4} & \mathcal{F} = \mathcal{O}_\Theta \\ \frac{\mathrm{tr}(-I)}{\det(2I)} = -\frac{2}{4} & \mathcal{F} = T_\Theta \end{cases}$$

Summing over the fixed points gives the desired results. $\square$

We state, without proof, the following obvious counting result which we will use multiple times:

Lemma 4.1.11. *Suppose* $\chi_+(\mathcal{F}) - \chi_-(\mathcal{F}) = \sum_{i\geq 0} h^i(\mathcal{F})$, *then*

$$h^i_+(\mathcal{F}) = \begin{cases} h^i(\mathcal{F}) & i \textit{ even} \\ 0 & i \textit{ odd}, \end{cases} \qquad h^i_-(\mathcal{F}) = \begin{cases} 0 & i \textit{ even} \\ h^i(\mathcal{F}) & i \textit{ odd}. \end{cases}$$

Similarly for $\chi_+(\mathcal{F}) - \chi_-(\mathcal{F}) = -\sum_{i\geq 0} h^i(\mathcal{F})$.

Lemma 4.1.12. $H^1_+(\Theta, T_S \otimes \omega_\Theta^{-1}) = 0$.

Proof. Tensoring the short exact sequence (4.4) with ω_Θ^{-1} gives

$$0 \to T_\Theta \otimes \omega_\Theta^{-1} \to (\omega_\Theta^{-1})^{\oplus 3} \to \mathcal{O}_\Theta \to 0.$$

Since ω_Θ is ample, by Serre duality and Kodaira vanishing, we have $h^i(\omega_\Theta^{-1}) = h^{2-i}(\omega_\Theta^{\otimes 2}) = 0$ for $i < 2$. This implies that $H^1(\Theta, T_\Theta \otimes \omega_\Theta^{-1}) \cong H^0(\Theta, \mathcal{O}_\Theta) \cong \mathbb{C}$. By Remark 4.1.9, the standard isomorphism φ on $H^1(\Theta, T_\Theta \otimes \omega_\Theta^{-1})$ induces $-\varphi$ on $H^0(\Theta, \mathcal{O}_\Theta)$, so $H^1_+(\Theta, T_\Theta \otimes \omega_\Theta^{-1}) \cong H^0_-(\Theta, \mathcal{O}_\Theta)$, the latter being 0-dimensional since $H^0(\Theta, \mathcal{O}_\Theta)_+ = H^0(\overline{\Theta}, \mathcal{O}_{\overline{\Theta}}) = H^0(\Theta, \mathcal{O}_\Theta) = \mathbb{C}$. $\square$

Lemma 4.1.13. $9 \leq h^1_+(T_\Theta(-\log B)) \leq 12$.

Proof. The proof is a straightforward computation of the cohomologies of the short exact sequences (4.4)-(4.6) using Corollary 4.1.10 and Lemma 4.1.11.

Short exact sequence (4.4) gives the cohomologies of T_Θ

	T_Θ	i^*T_A	$\mathcal{O}_\Theta(\Theta)$
h^0_+	0	0	0
h^1_+	6	9	3
h^2_+	0	0	0

	T_Θ	i^*T_A	$\mathcal{O}_\Theta(\Theta)$
h^0_-	0	3	3
h^1_-	0	0	0
h^2_-	8	9	1

while short exact sequence (4.5) gives that of $\mathcal{O}_B(B)$

	$\mathcal{O}_\Theta$	$\mathcal{O}_\Theta(B)$	$\mathcal{O}_B(B)$
h^0_+	1	7	6
h^1_+	0	0	3
h^2_+	3	0	0

	$\mathcal{O}_\Theta$	$\mathcal{O}_\Theta(B)$	$\mathcal{O}_B(B)$
h^0_-	0	0	3
h^1_-	3	0	0
h^2_-	0	0	0

Finally, computing the cohomologies of short exact sequence (4.6) gives

	$T_\Theta(-\log B)$	T_Θ	$\mathcal{O}_B(B)$
h^0_+	0	0	6
h^1_+	$x+9$	6	3
h^2_+	x	0	0

	$T_\Theta(-\log B)$	T_Θ	$\mathcal{O}_B(B)$
h^0_-	0	0	3
h^1_-	3	0	0
h^2_-	8	8	0

where $0 \leq x \leq 3$. $\square$

From these computations, we conclude

Proposition 4.1.14. *Let F be an even 56-nodal sextic surface obtained from the construction in Section 4.1.1. Then, $h^1(F, \tilde{T}_F) = 12$, the deformations of F are unobstructed and the family $\mathcal{M}$ is reduced and smooth at F.*

Proof. Applying Lemmas 4.1.12 and 4.1.13 to (4.2), we have

$$h^1(F, \tilde{T}_F) = h^1_+(\Theta, T_\Theta(-\log B)) + h^1_+(\Theta, T_\Theta \otimes \omega_\Theta^{-1}) \leq 12.$$

Theorem 4.1.1 shows that the deformation family is at least 12 dimensional, and since Def_F is the tangent space to the family at F, we have $\dim \mathrm{Def}_F = h^1(F, \tilde{T}_F) = 12$. $\square$

4.1.4 Construction of explicit examples

Let C be a non-hyperelliptic genus three curve, we will also denote the canonical model of C, a quartic curve in $\mathbb{P}^2$, by C. Recall that $\Theta = S^2C$, the symmetric product of C.

We show how to find the global sections in $H^0(\Theta, \omega_\Theta^{\otimes 2})$ in terms of the geometry of C, following [BV96]. Note that if we map $S^2C \to \mathrm{Jac}(C)$ by $p+q \mapsto p+q-t$ where $t \in S^2C$ is an odd theta characteristic (so $2t \equiv K_C$), then the image of S^2C is a symmetric theta divisor. Let $d = z_1 + \ldots + z_4$ be an effective canonical divisor on C, $D = \sum(z_i + C)$ be the corresponding divisor on S^2C and Δ be the diagonal in S^2C. Then, $2K_{S^2C} = 2D - \Delta$. By [BV96, Lemma 4.7], we have the restriction sequence

$$0 \to \mathcal{O}_{S^2C}(2K_{S^2C}) \to \mathcal{O}_{S^2C}(2D) \to \mathcal{O}_\Delta(2D) \cong \mathcal{O}_C(4d) \to 0.$$

and

$$H^0(S^2C, \omega_{S^2C}^{\otimes 2}) \cong \ker\left(S^2H^0(C, \omega_C^{\otimes 2}) \xrightarrow{\mu} H^0(C, \omega_C^{\otimes 4})\right).$$

where μ is the multiplication map. As $h^0(C, \omega_C^{\otimes 2}) = 6$, $\dim S^2H^0(C, \omega_C^{\otimes 2}) = 21$ and $h^0(C, \omega_C^{\otimes 4}) = 14$. By the same lemma, μ is surjective so indeed $h^0(S^2C, \omega_{S^2C}^{\otimes 2}) = 7$.

Let $\sigma_0, \sigma_1, \sigma_2$ be a basis of $H^0(C, \omega_C)$. It induces a basis $\sigma_i \otimes \sigma_j$ of the tensor product $H^0(C^2, \omega_{C^2}) = H^0(C, \omega_C)^{\otimes 2}$. The sections of $H^0(\Theta, \omega_\Theta) \cong \wedge^2 H^0(C, \omega_C)$ define the Gauss map $S^2C \cong \Theta \to \mathbb{P}^2$. Explicitly, the Gauss map is induced by the map

$$C \times C \to \mathbb{P}^2, \quad (x, y) \mapsto (p_{12} : p_{13} : p_{23}), \quad p_{ij}(x, y) := \sigma_i(x)\sigma_j(y) - \sigma_j(x)\sigma_i(y).$$

The six products $p_{ij}p_{kl}$ span a six dimensional subspace of $\ker(\mu)$ which is the image of $S^2H^0(\Theta, \omega_\Theta)$ in $H^0(\Theta, \omega_\Theta^{\otimes 2})$.

Let $f(z)$ be a homogeneous quartic polynomial in $\mathbb{C}[z_0, z_1, z_2]$ such that $f(\sigma_0(x), \sigma_1(x), \sigma_2(x)) = 0$ for all $x \in C$, that is, f defines the curve $C \subset \mathbb{P}^2$. Choose any polynomial $g(u, v)$ of bidegree $(2, 2)$ in $\mathbb{C}[u_0, u_1, u_2, v_0, v_1, v_2]$ such that $g(z, z) = f(z)$ and let $g_s(u, v) := g(u, v) + g(v, u)$, then $\tilde{g}(x, y) := g_s(\sigma_0(x), \ldots, \sigma_2(y)) \in S^2H^0(C, \omega_C^{\otimes 2})$ and lies in $\ker(\mu)$. Thus the choice of $\tilde{g}$ provides the section t used to construct the map φ_{2K_Θ}, any other choice of $\tilde{g}$ is of the form $\lambda\tilde{g} + \sum \lambda_{ij}p_{ij}$ for complex numbers λ, λ_{ij} with $\lambda \neq 0$.

The map $\varphi_{2K_\Theta} : \Theta \to \mathbb{P}^6$ is therefore induced by the map

$$C \times C \to \mathbb{P}^6, \qquad (x, y) \mapsto \big(\ldots : p_{ij}(x, y)p_{kl}(x, y) : \ldots : \tilde{g}(x, y)\big).$$

A homogeneous polynomial P in seven variables gives an equation for the image of this map if $P(\ldots, p_{ij}(u, v)p_{kl}(u, v), \ldots, \tilde{g}(u, v))$ lies in the ideal of $\mathbb{C}[u_0, \ldots, v_2]$ generated by $f(u)$ and $f(v)$.

Explicit example 4.1.15. An explicit example, worked out using the computer program Magma [BCP97], is provided by the choice $f = z_0z_1^3 + z_1z_2^3 + z_2z_0^3$, which defines the Klein curve in $\mathbb{P}^2$. We will take $g = u_0u_1v_1^2 + u_1u_2v_2^2 + u_2u_0v_0^2$ and the map φ_{2K_Θ} is given by:

$$(y_{00} : y_{01} : \ldots : y_{22} : y_g) = \big(p_{01}^2 : p_{01}p_{02} : p_{01}p_{12} : p_{02}^2 : p_{02}p_{12} : p_{12}^2 : \tilde{g}\big).$$

One of the equations for the image is

$$y_{00}^2y_{02} - y_{12}y_{22}^2 - y_{01}y_{11}^2 - 5y_{01}^2y_{22} + (-y_{00}y_{01} + y_{02}y_{22} - y_{11}y_{12})y_g - y_g^3 = 0$$

(this equation thus defines the image in $\mathbb{P}(1, 1, 1, 2) \subset \mathbb{P}^6$). Next we pull this equation back to $\mathbb{P}^3$ along the map $\overline{p}$ by substituting $y_{ij} = x_ix_j$ and $y_g = x_3^2$, moreover we change the sign of x_1 in order to simplify the equation and we obtain

$$Q := x_0^5x_2 + x_0x_1^5 + x_1x_2^5 - 5x_0^2x_1^2x_2^2 \; + \; (x_0^3x_1 + x_0x_2^3 + x_1^3x_2)x_3^2 - x_3^6 = 0.$$

The singular locus of the surface F defined by $Q = 0$ consists of 56 nodes and these are thus an even set of nodes. To find all the nodes, we observe that $\mathrm{Aut}(F)$ contains a subgroup G_{336} of order 336 with generators

$$g_7 := \mathrm{diag}(\omega, \omega^4, \omega^2, 1),$$
$$g_2 := \frac{1}{\sqrt{-7}}\begin{pmatrix} a & c & b & 0 \\ c & b & a & 0 \\ b & a & c & 0 \\ 0 & 0 & 0 & \sqrt{-7} \end{pmatrix}, \qquad \begin{cases} a = \omega^2 - \omega^5, \\ b = \omega - \omega^6, \\ c = \omega^4 - \omega^3, \end{cases}$$

where ω is a primitive seventh root of unity. One of the nodes is $(1 : 1 : 1 : 1)$ and G_{336} acts transitively on the 56 nodes, the stabilizer of a node is isomorphic to the symmetric group S_3. The covering involution $\mathrm{diag}(1, 1, 1, -1)$ generates the center of G_{336} and $G_{336} \cong \{\pm 1\} \times G_{168}$ where $G_{168} \cong SL(3, \mathbb{F}_2)$ is the automorphism group of the Klein curve. The equation of F can be written as $p_6 + p_4x_3^2 - x_3^6$, the discriminant of the cubic polynomial $p_6 + p_4T - T^3$ has degree 12 in $\mathbb{C}[x_0, x_1, x_2]$ and the curve it defines is the dual of the Klein curve. This dual curve appears since the composition of the bicanonical map φ_{2K_Θ} with the projection from the vertex of the cone Y over the Veronese surface is the canonical map $\varphi_{K_\Theta} : \Theta \to \mathbb{P}^2$ which, under the identification $\Theta = S^2C$, is the map sending $p + q$ to the line in (the dual) $\mathbb{P}^2$ spanned by p and q on the canonical curve.

4.2 40-nodal sextic surfaces

We now turn our attention to even 40-nodal sextic surfaces. By Lemma 4.0.2, for a 40-nodal sextic, $\chi(\mathcal{L}^{-1}) = 1$. Since $h^0(\mathcal{L}^{\otimes 2}) > 1$, $\mathcal{L}$ is non-trivial and we have $h^0(\mathcal{L}^{-1}) = 0$ and so $h^2(\mathcal{L}^{-1}) - h^1(\mathcal{L}^{-1}) = 1$. Recall that $h^i(\mathcal{L}^{-1}) = h^{0,i}(S)_-$.

Hence, we have $h^{0,2}(S)_- \geq 1$. In Subsection 4.2.1, we present a construction, due to Gallarati, of a family of even 40-nodal sextic surfaces with double cover S such that $h^{0,2}(S)_- = 1$ (Theorem 4.2.3, Proposition 4.2.4). The construction realises an even 40-nodal sextic surface F as a sextic surface tangent to a Kummer surface K along a curve C of genus 15. We then show that a general even 40-nodal sextic surface is tangent to some Kummer surface along such a curve, so this family forms a dense open subset of all even 40-nodal sextic surfaces (Corollary 4.2.15).

We can compute the Hodge numbers of the weight 2 Hodge structure of surfaces in this family.

Lemma 4.2.1. *Let F be an even 40-nodal sextic surface and $f : S \to F$ the double cover branched over the nodes. Suppose $h^{0,2}(S)_- = 1$. Then, $h^{1,1}(S)_- = 26$ and $H^2(S, \mathbb{Q})$ has a sub-Hodge structure $H^2(S, \mathbb{Q})_-$ of type $(1, 26, 1)$.*

Proof. Recall that $\omega_{\tilde{F}} = \pi_F^* \mathcal{O}_F(2H)$ for a general hyperplane section H. Hence, Noether's formula gives

$$\chi(\mathcal{O}_{\tilde{F}}) = \frac{K_{\tilde{F}}^2 + \chi_{\mathrm{top}}(\tilde{F})}{12} = \frac{4H^2 + 2\chi(\mathcal{O}_{\tilde{F}}) - 2h^{1,0}(\tilde{F}) + h^{1,1}(\tilde{F})}{12}$$
$$\Rightarrow \; h^{1,1}(\tilde{F}) = 10\chi(\mathcal{O}_{\tilde{F}}) + 2h^{1,0}(\tilde{F}) - 4H^2 = 10 \cdot 11 - 24 = 86$$

since $h^{0,1}(F) = 0$ by Lemma 4.0.2. Thus, $h^{1,1}(F) = h^1(\tilde{\Omega}_F^1) = h^1(\Omega_{\tilde{F}}^1) - 40 = 46$.

The canonical divisors of $\tilde{S}$, S and $\tilde{F}$ are related by the formula $K_{\tilde{S}} = \tilde{f}^* K_{\tilde{F}} + \tilde{\Delta}$ and $K_{\tilde{S}} = \pi_S^* K_S + \tilde{\Delta}$, so $K_S^2 = \pi_S^* K_S^2 = \tilde{f}^* K_{\tilde{F}}^2 = 48$. We also have $h^{0,1}(S) = h^{0,1}(F) + h^{0,1}(S)_- = 0$. Noether's formula again gives us

$$h^{1,1}(S) = 10\chi(\mathcal{O}_S) - K_S^2 = 10 \cdot 12 - 48 = 72.$$

Hence, $h^{1,1}(S)_- = h^{1,1}(S) - h^{1,1}(F) = 26$. $\square$

By Lemma 4.2.1, $H^2(S, \mathbb{Q})$ contains a sub-Hodge structure of type $(1, 26, 1)$. The question is whether there exists a double cover S of a 40-nodal sextic for which the sub-Hodge structure of type $(1, 26, 1)$ is simple. Unfortunately, the answer is negative. In addition to Gallarati's construction, we study 2 other constructions of 40-nodal sextic surfaces, and show that a general even 40-nodal sextic surface can be obtained by any of these 3 constructions. In particular, using the EPW sextic construction (Subsection 4.2.4), we show that $H^2(S, \mathbb{Q})_-$ contains a sub-Hodge structure of type $(1, 20, 1)$, so one cannot produce a simple $(1, 26, 1)$ Hodge structure using even 40-nodal sextic surfaces (Corollary 4.2.21).

4.2.1 Gallarati's construction

In the paper [Cat81], Catanese gave a construction, due to Gallarati, of a family of 40-nodal sextic surfaces starting from Kummer surfaces. We recall the construction in detail since many proofs in later sections rely heavily on this construction.

A Kummer surface K is the quotient of a principally polarized abelian surface (A, Θ) under the involution $[-1] : a \mapsto -a$. Similar to the previous section, we may choose Θ to be a symmetric divisor on A, so 2Θ is totally symmetric. It induces a morphism

$$\phi = \phi_{2\Theta} : A \to \mathbb{P}^3$$

which is the quotient map by $[-1]$. This gives an explicit embedding of the image $K = \phi_{2\Theta}(A)$ as a 16-nodal quartic surface in $\mathbb{P}^3$. By an abuse of notation, we denote by ϕ the double cover $A \to K$ as well. It is ramified at precisely the even set Δ_{16} of 16 nodes. We see that $2\Theta \equiv \phi^* H$ where H is the generic hyperplane section on K.

Let $\pi_A : \tilde{A} \to A$ and $\pi_K : \tilde{K} \to K$ be the blowups of A and K at the 16 2-torsion points respectively, and let $\tilde{\Delta}_{16} = \pi_K^{-1}\Delta_{16}$ and $L_{16} = \frac{1}{2}\tilde{\Delta}_{16}$. Where there is no risk of confusion, we denote the hyperplane section on both a nodal hypersurface and its blowup by H. We consider linear systems of the type $|kH - L_{16}|$ on $\tilde{K}$.

Lemma 4.2.2. *Let $k > 0$ be an integer. Then, $H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(kH - L_{16}))$ is isomorphic to the negative eigenspace of $H^0(A, \mathcal{O}_A(2k\Theta))$ under the involution induced by $[-1]$. Hence, the (projective) dimension of the linear system $|kH - L_{16}|$ is*

$$\dim|kH - L_{16}| = h^0(\mathcal{O}_{\tilde{K}}(kH - L_{16})) - 1 = 2k^2 - 3.$$

Proof. Let $\tilde{\Theta}$ be the strict transform of Θ under the blowup $\pi_A : \tilde{A} \to A$. Applying the projection formula to the double cover $\tilde{\phi} : \tilde{A} \to \tilde{K}$ on the blowups, we get

$$\begin{aligned}\tilde{\phi}_*\mathcal{O}_{\tilde{A}}(2k\tilde{\Theta}) &= \tilde{\phi}_*(\mathcal{O}_{\tilde{A}} \otimes \tilde{\phi}^*\mathcal{O}_{\tilde{K}}(kH)) = \tilde{\phi}_*\mathcal{O}_{\tilde{A}} \otimes \mathcal{O}_{\tilde{K}}(kH) \\ &= \mathcal{O}_{\tilde{K}}(kH) \oplus \mathcal{O}_{\tilde{K}}(kH - L_{16}).\end{aligned}$$

There is an isomorphism $R\pi_{A*}\mathcal{O}_{\tilde{A}}(2k\tilde{\Theta}) = R\pi_{A*}\mathcal{O}_{\tilde{A}} \otimes \mathcal{O}_A(2k\Theta) = \mathcal{O}_A(2k\Theta)$, so the negative eigenspace of $H^0(A, \mathcal{O}_A(2k\Theta)) = H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(2k\tilde{\Theta}))$ is precisely the group $H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(kH - L_{16}))$.

Since the divisor $2k\Theta$ is ample, we have $h^0(\mathcal{O}_A(2k\Theta))_\pm = \chi(\mathcal{O}_A(2k\Theta))_\pm$ by Kodaira vanishing. By Riemann-Roch,

$$\chi(\mathcal{O}_A(2k\Theta)) = \frac{(2k)^2\Theta^2}{2} = 4k^2 \qquad \text{and}$$
$$\chi(\mathcal{O}_A(2k\Theta))_+ = \chi(\mathcal{O}_{\tilde{A}}(2k\tilde{\Theta}))_+ = \chi(\mathcal{O}_{\tilde{K}}(kH)) = \chi(\mathcal{O}_{\tilde{K}}) + \frac{k^2H^2}{2} = 2 + 2k^2.$$

Hence, we get

$$h^0(\mathcal{O}_{\tilde{K}}(kH - L_{16})) = h^0(\mathcal{O}_A(2k\Theta))_- = \chi(\mathcal{O}_A(2k\Theta)) - \chi(\mathcal{O}_A(2k\Theta))_+ = 2k^2 - 2.$$

□

By the lemma above, the linear system $|3H - L_{16}|$ on $\tilde{K}$ is 15 dimensional. We can choose a curve $\tilde{C}_K \in |3H - L_{16}|$ such that it is the strict transform of some curve C on K. Indeed, a general choice of the curve C is smooth (cf. proof of [Cat81, Proposition 2.24]).

The curve C is a Weil divisor but not a Cartier divisor on K. Suppose the node is defined locally by the equation $\{x^2 + yz = 0\} \subset \mathbb{C}^3$, then the curve $2C$ can be defined locally, for example, by the intersection with the hyperplane $y = 0$. In this case, we say that the curve C is *tangent to* the surface K at the node.

We can compute the arithmetic genus of the curve C using the adjunction formula:

$$\begin{aligned}&K_C = K_{\tilde{C}_K} = (K_{\tilde{K}} + \tilde{C}_K)_{|\tilde{C}_K} = \tilde{C}_{K|\tilde{C}_K} \\ &\Rightarrow \deg K_C = (\tilde{C}_K)^2 = (3H - L_{16})^2 = 9H^2 + \frac{1}{4}\tilde{\Delta}_{16}^2 = 9 \cdot 4 + \frac{1}{4}(-32) = 28.\end{aligned}$$

Therefore, $p_a(C) = 15$.

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)) = 0$ for all l, the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(l-4) \to \mathcal{O}_{\mathbb{P}^3}(l) \to \mathcal{O}_K(l) \to 0$$

induces a short exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(K, \mathcal{O}_K(lH)) \to 0. \tag{4.7}$$

The surjectivity of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(K, \mathcal{O}_K(lH))$ implies that any section $s \in |lH|$ is defined by a homogeneous polynomial of degree l in $\mathbb{C}[z_0, \dots, z_3]$.

The divisor $2\tilde{C}_K + \tilde{\Delta}_{16}$ lies in the linear system $|6H_{\tilde{K}}| = \pi_K^*|6H|$, so $2C$ lies in the linear system $|6H|$. Hence, there exists a sextic surface F tangent to K such that $\mathrm{div}_K(F) = 2C$. By the short exact sequence (4.7), this surface F is well-defined up to the choice of a quadratic section in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. More precisely, if f_4 and f_6 are homogeneous polynomials of degrees 4 and 6 respectively defining the surfaces K and F, then for any quadratic polynomial $q \in \mathbb{C}[z_0, \dots, z_3]$, the polynomial $f_6' = f_6 + qf_4$ defines another sextic surface tangent to K along C.

Catanese [Cat81, Proposition 2.24] showed that a general choice of F is smooth outside C and on Δ_{16}. By [Cat81, Lemma 2.3], the number of nodes t in an even set on F, the degrees m and n of K and F respectively, and the arithmetic genus p of C are related by

$$p = 1 - nm + \frac{mn(m+2n) - 2t}{8}.$$

Hence, a general F will have exactly 40 simple nodes lying on C which forms an even set $\Delta = \Delta_{40}$.

We can summarize the above construction in the following result.

Theorem 4.2.3 ([Cat81, Proposition 2.24]). *For each principally polarized abelian surface (A, Θ) and symmetric curve $D \in |6\Theta|_-$, there exists a 10-dimensional family of even 40-nodal sextic surfaces tangent to the Kummer surface $K = A/[-1]$ along the curve $C = D/[-1]$, parametrized by homogeneous quadratic polynomials in $\mathbb{C}[z_0, \dots, z_3]$.*

Now, consider the curve C as a Weil divisor on F, so $2C \in |\mathcal{O}_F(K)| = |\mathcal{O}_F(4H)|$. The curve C is tangent to the even set of 40 nodes, so $2\tilde{C} \in |\pi_F^* 4H - \tilde{\Delta}|$ and $\tilde{C} \in |\pi_F^* 2H - L|$.

By [Par91, Proposition 4.1], $(\tilde{f}_* \omega_{\tilde{S}})_- = \omega_{\tilde{F}} \otimes \mathcal{L} = \mathcal{O}_{\tilde{F}}(2H + L)$. Since $H \cdot \tilde{\Delta} = 0$, $H^0(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}(2H + L)) = H^0(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}(\frac{1}{2}\tilde{\Delta})) = H^0(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}(-1)) = 0$, so from the short exact sequence

$$0 \to \mathcal{O}_{\tilde{F}}(2H - L) \to \mathcal{O}_{\tilde{F}}(2H + L) \to \mathcal{O}_{\tilde{\Delta}}(2H + L) \to 0,$$

we obtain an isomorphism $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H - L)) \cong H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H + L)) \cong H^0(\tilde{S}, \omega_{\tilde{S}})_- = H^{2,0}(S)_-$. We have previously shown that $\chi(\mathcal{O}_S)_- = 1$ and we know that $h^{0,0}(S)_- = 0$, so $h^{2,0}(S)_- = h^{0,2}(S)_- \geq 1$. The curve C defines a section $s \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H - L)) \cong H^{2,0}(S)_-$ and we shall show that s is indeed the unique generator of the cohomology group.

Proposition 4.2.4. *Let F be an even 40-nodal sextic surface which is a deformation of one that is obtained from Gallarati's construction. Let $f : S \to F$ be the double cover ramified at the 40 nodes. Then, $h^{0,1}(S) = 0$ and $h^{0,2}(S)_- = 1$. In particular, $h^0(\mathcal{O}_{\tilde{F}}(2H - L)) = h^{0,2}(S)_- = 1$ and the linear system $|2H - L|$ is generated by a unique curve $\tilde{C}$.*

Proof. Consider the family of even 40-nodal sextic surfaces obtained from the Gallarati construction. We give a member of this family in Explicit example 4.2.7. This particular 40-nodal sextic surface F has $h^1(\mathcal{L}^{-1}) = 0$ and $h^2(\mathcal{L}^{-1}) = 1$. Since the Hodge numbers in a family are upper semi-continuous, any general member of the deformation family of even 40-nodal sextic surfaces must have $h^1(\mathcal{L}^{-1}) = 0$ and $h^2(\mathcal{L}^{-1}) = 1$. Hence, for a general double cover $f : S \to F$, we have $h^{0,1}(S) = h^{0,1}(F) + h^1(\mathcal{L}^{-1}) = 0$ and $h^{0,2}(S)_- = h^2(\mathcal{L}^{-1}) = 1$. Since the Hodge numbers remain constant in a deformation family of smooth projective varieties, we have $h^{0,1}(S)_- = h^{0,1}(S) = 0$ and $h^0(\mathcal{O}_{\tilde{F}}(2H - L)) = h^{0,2}(S)_- = 1$ for all S in the deformation family. ☐

Corollary 4.2.5. *The Hodge numbers of F and S are*

	$h^{1,0}$	$h^{2,0}$	$h^{1,1}$
S	0	11	72
F	0	10	46

As shown in Lemma 4.2.1, $H^2(S, \mathbb{Q})_-$ has a Hodge structure of type $(1, 26, 1)$.

Proof. By the previous corollary, $h^{1,0}(S) = 0$ and $h^{2,0}(S) = h^{2,0}(S)_+ + h^{2,0}(S)_- = 10 + 1 = 11$. Noether's formula gives

$$\chi(\mathcal{O}_S) = \frac{\chi_{\text{top}}(S) + K_S^2}{12} \quad \Rightarrow \quad h^{1,1}(S) = 10(1+h^{2,0}(S)) - K_S^2 = 120 - 48 = 72.$$

Hence, $h^{1,1}(F) = h^{1,1}(S)_+ = 72 - 26 = 46$. ☐

We now give a method to explicitly construct examples of even 40-nodal sextic surfaces and provide an explicit example on which we can compute the Hodge numbers.

Example 4.2.6. Let (A, Θ) be a principally polarized abelian surface, $K = A/[-1]$ be a Kummer surface and C be a curve on K whose strict transform on $\tilde{K}$ lies in the linear system $|3H - L_{16}|$. Recall that the non-reduced divisor $2C$ is defined by a section in $H^0(\tilde{K}, 6H - \tilde{\Delta}_{16})$, i.e. a sextic polynomial f_6

which vanishes at all the nodes. However, a general sextic polynomial which vanishes at all the nodes defines an irreducible curve C' of degree 24 on K, so a priori, it is difficult to find a polynomial defining the divisor $2C$. In this example, we give an explicit procedure to construct such a polynomial f_6 given f_4.

By Lemma 4.2.2, the linear system $|2H - L_{16}|$ is non-empty. Consider the map

$$H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16})) \otimes H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(H)) \to H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(3H - L_{16})). \quad (4.8)$$

Let $0 \neq s \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16}))$ be any fixed section. Since $\tilde{K}$ is irreducible, s is not a zero divisor and the map (4.8) is injective for any fixed $s \neq 0$. Let $g_4 \in \mathbb{C}[z_0, \ldots, z_3]$ be a general quartic polynomial in the inverse image of $s^2 \in H^0(K, \mathcal{O}_K(4H))$ to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. By the exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \cong \mathbb{C} \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(K, \mathcal{O}_K(4H)) \to 0,$$

any quartic polynomial in the inverse image of s^2 is of the form $g_4 + cf_4$ for some constant $c \in \mathbb{C}$. Each such quartic polynomial defines a quartic surface tangent to K along a curve D. Let $h \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(H)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \cong \mathbb{C}[z_0, \ldots, z_3]_1$ be any linear polynomial. Then, $hs \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(3H - L_{16}))$ and $(hs)^2 + qf_4 = h^2 g_4 + qf_4 \in \mathbb{C}[z_0, \ldots, z_3]$ defines an even 40-nodal sextic surface for a general $q \in \mathbb{C}[z_0, \ldots, z_3]_2$. Hence, it suffices to give a construction of g_4.

The quartic polynomial g_4 is the square of a section $s \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16})) \cong H^0(A, \mathcal{O}_A(4\Theta))_-$. $H^0(A, \mathcal{O}_A(4\Theta))_-$ is 6-dimensional by Lemma 4.2.2, but it is difficult to describe a general section of the cohomology group. Nevertheless, we can find 6 special sections, which are inverse images of the six odd theta divisors under the multiplication-by-two map $[2] : A \to A$.

The Kummer surface K is the image of A under the map $\phi_{|2\Theta|} : A \to \mathbb{P}^3$. The 2-torsion subgroup $A[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ acts on A and on the totally symmetric linear system $|2\Theta|$. This gives an action of $A[2]$ on K under the morphism $\phi : A \to K \subset \mathbb{P}(|2\Theta|)$. Thus, the isogeny $[2] : A \to A$ induces a covering map $\psi : K \to K$. To avoid confusion, we denote the images of A, Θ and K under these covering maps by $\overline{A}$, $\overline{\Theta}$ and $\overline{K}$ respectively. So, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "{[2]}"] \arrow[d, "p:=\phi_{|2\Theta|}"'] & \overline{A} := A \arrow[d, "\overline{p}:=p"] \\ K \arrow[r, "\psi"'] & \overline{K} \cong K. \end{tikzcd}$$

The multiplication-by-two map $[2]$ sends the set of two torsion points $A[2] \subset A$ to the origin $0 \in \overline{A}$. Hence, ψ maps all 16 nodes of K to a single node $x_0 \in \overline{K}$.

Since $\overline{p} = \phi_{|2\overline{\Theta}|}$, any divisor in the linear system $|2\overline{\Theta}|$ on $\overline{A}$ is the inverse image of a hyperplane section $\overline{H}$ on $\overline{K}$. By [BL92, Corollary 2.3.6], $[2]^*\overline{p}^*\overline{H}$ lies in the linear system $[2]^*|2\overline{\Theta}| = |8\Theta|$, and is thus the inverse image of a quartic polynomial on K. We seek a divisor $\overline{D}$ of $\overline{K}$ such that $\overline{H} := 2\overline{D}$ is a hyperplane section passing through the node $x_0 \in \overline{K}$, then $D := \psi^{-1}(\overline{D})$ passes through all the nodes of K and is the required curve.

Consider the theta divisors of the abelian surface $\overline{A}$. Any theta divisor on $\overline{A}$ is a translation of $\overline{\Theta}$, so it can be written in the form $\overline{\Theta} + a$ for some point $a \in \overline{A}$. If $2(\overline{\Theta} + a) \in |2\overline{\Theta}|$, then $2a = 0$, so a is a two-torsion point. There are 16 such theta divisors. In particular, 6 of them are odd theta divisors, that is, they correspond to the zero locus of odd theta functions ϑ which satisfy $\vartheta(x) = -\vartheta(-x)$ for all $x \in \overline{A}$. Therefore, each of the odd theta divisors $\overline{\Theta}_i$ ($i = 1, \ldots, 6$) passes through the origin $0 \in \overline{A}$, and each also passes through 5 other two-torsion points of $\overline{A}$ by [BL92, Remark 4.7.7(a)].

Hence, for each odd theta divisor $\overline{\Theta}_i = \overline{\Theta} + a$ with $i = 1, \ldots, 6$, we have $2\overline{\Theta}_i = \overline{p}^*\overline{H}_i$ for some hyperplane $\overline{H}_i \subset \overline{K}$ passing through 6 nodes of $\overline{K}$, including x_0. Thus, there exists a divisor $\overline{D}_i$ on $\overline{K}$ such that $2\overline{D}_i = \overline{H}_i$ and $\overline{p}^*\overline{D}_i = \overline{\Theta}_i$. Let $D_i = \psi^*\overline{D}_i$, so $\psi^*\overline{H}_i = 2D_i$ is a non-reduced curve on K defined by a quartic polynomial $g_{4,i}$ vanishing on the nodes of K. We thus obtain examples of even 40-nodal sextic surfaces defined by polynomials $f_6 = h^2 g_{4,i} + q f_4$ for general linear polynomials h and quadratic polynomials q.

Note that, in fact, the six sections $s_1, \ldots, s_6$ corresponding to the curves $D_1, \ldots, D_6$ form a basis of $H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16}))$.

We can compute an example by choosing explicit equations for the Kummer surfaces K and $\overline{K}$.

The group $A[2]$ acts as the Heisenberg group on $\mathbb{P}^3$ and the images of a point $(z_0 : \ldots : z_3)$ under the generators of this group are

$$(z_0 : z_1 : -z_2 : -z_3),\ (z_0 : -z_1 : z_2 : -z_3),\ (z_1 : z_0 : z_3 : z_2),\ (z_2 : z_3 : z_0 : z_1).$$

Under this action, the Heisenberg invariant polynomials of degree 4 are

$$\begin{gathered} P_0 = z_0^4 + \cdots + z_3^4, \\ P_1 = 2(z_0^2 z_1^2 + z_2^2 z_3^2),\ P_2 = 2(z_0^2 z_2^2 + z_1^2 z_3^2),\ P_3 = 2(z_0^2 z_3^2 + z_1^2 z_2^2), \\ P_4 = 4 z_0 \cdots z_3. \end{gathered}$$

We can choose coordinates on $\mathbb{P}^3$ such that the Kummer surface K is defined by the quartic polynomial $f_4 = \sum_{i=0}^4 a_i P_i$. Indeed, f_4 defines a Kummer surface if the coefficients $a_0, \ldots, a_4$ satisfy the Segre cubic relation

$$a_0^3 - a_0 a_1^2 - a_0 a_2^2 - a_0 a_3^2 + a_0 a_4^2 + 2 a_1 a_2 a_3 = 0.$$

Conversely, a general solution of the equation above gives a Kummer surface.

The Heisenberg polynomials P_i induce a morphism

$$\psi : \mathbb{P}^3 \to Y \subset \mathbb{P}^4, \qquad z \mapsto (P_0(z) : \ldots : P_4(z)) = (w_0 : \cdots : w_4)$$

where Y is the image of the morphism. Y is a quartic threefold in $\mathbb{P}^4$, known as the Igusa quartic, and is defined by the equation

$$Y : \quad w_4^2(w_0^2 - w_1^2 - w_2^2 - w_3^2) + w_1^2 w_2^2 + w_2^2 w_3^2 + w_3^2 w_1^2 + w_4^4 - 2w_0 w_1 w_2 w_3 = 0.$$

From the definition of ψ, we see that the image $\overline{K} = \psi(K)$ of the Kummer surface is the intersection of the hyperplane $H_K = \{h_0 := \sum a_i w_i = 0\} \subset \mathbb{P}^4$ with Y.

Using the generators of the action of $A[2]$ given above, we find that the image of the fixed locus of any non-trivial involution in $A[2]$ is a line in Y. The quartic threefold Y is singular along 15 lines, each defined by the fixed locus of a non-trivial element of $A[2]$. Any general hyperplane on $\mathbb{P}^4$ intersects the singular locus of Y transversely at 15 points. Hence, for a general Kummer surface K, the hyperplane H_K intersects the singular lines transversely, giving rise to 15 singular points on $\overline{K}$. It is easy to check that these singularities are indeed nodes on $\overline{K}$. Together with x_0, the image of the 16 nodes of K, these $15 + 1$ nodes are the images of the two-torsion points of $\overline{A}$.

Computationally, it is easy to find the 6 planes $2\overline{D}_i = \overline{H}_i \subset H_K$ $(i = 1, \ldots, 6)$ passing through 6 nodes of $\overline{K}$, including x_0. Each $\overline{H}_i$ is the intersection of H_K with a hyperplane in $\mathbb{P}^4$ given by a linear polynomial $h_i \in \mathbb{C}[w_0, \ldots, w_4]$. So, the quartic polynomial $g_{4,i}$ is given by $h_i(P_0, \ldots, P_4)$. Since h_i is well-defined modulo h_0, $g_{4,i}$ is well-defined modulo f_4.

Explicit example 4.2.7. We found a specific even 40-nodal sextic surface using Magma [BCP97] as follows.

Let K be a Kummer surface defined by the equation $f_4 := P_0 - iP_4 = 0$ where $i^2 = -1$. The polynomial f_4 is invariant under the action of the subgroup $G \subset SL(4, \mathbb{C})$ generated by $\mathrm{diag}(-1, -1, 1, 1)$ and the symmetric group S_4 given by permutations of coordinates. The nodes of K are generated by the point $(1 : 1 : 1 : -i)$ under the action of the automorphism group G.

The image of the nodes of K in $\overline{K} \subset \mathbb{P}^4$ under the map ψ is $x_0 = (i : 0 : 0 : 0 : 1)$. Each of the six hyperplanes

$$w_i \pm i w_j = 0 \qquad (1 \leq i < j \leq 3)$$

passes through 6 nodes of $\overline{K}$, including x_0.

For example, we may take $g_4 = P_1 + iP_2$, $h = z_1 + 2z_3$ and $q = z_0 z_1 + z_1 z_2 + z_2 z_3$. Then, using Magma, we checked that

$$f_6 = h^2 g_4 + q f_4$$

is a 40-nodal surface, and the set of nodes is even by the Gallarati construction. In this example, the curve C has 2 smooth irreducible components, intersecting at 8 points. It has arithmetic genus 15.

Furthermore, we checked in Magma that the ideal $I = \sqrt{J}$ where J is the Jacobian ideal of f_6 is generated by a single polynomial f_4 in degree 4. So, $H^0(\tilde{F}, 4H - \tilde{\Delta}) \cong I_4 \cong \mathbb{C}$. The kernel of the quadratic map

$$H^0(\tilde{F}, 2H - L) \to H^0(\tilde{F}, 4H - \tilde{\Delta}) \,:\, s \mapsto s^2$$

is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, thus $h^0(\tilde{F}, 2H - L) = 1$. This is used in the proof of Proposition 4.2.4.

We can also check that $H^0(F, \tilde{T}_F) \cong (I/J)_6$ is 28-dimensional (cf. Proposition 4.2.12).

Remark 4.2.8. Any curve C obtained by the method in Example 4.2.6 will have 2 irreducible components intersecting at 8 points. This is because in the construction of the example, we have $f_6 = h^2 g_4 + q f_4$, so the curve $2C$ is defined by the ideal

$$(f_6, f_4) = (h^2 g_4, f_4) = (h^2, f_4) \cdot (g_4, f_4).$$

Hence, C in general will have two irreducible components of degrees 4 and 8, intersecting at 8 points on K. In the next example, we show how to obtain a smooth curve C in the Gallarati construction.

Example 4.2.9. We found a basis $\{s_1, \dots, s_6\}$ of $H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16}))$ in Example 4.2.6. A general section in $H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(2H - L_{16})) \otimes H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(H))$ is given by $\sum_{i=1}^{6} \lambda_i s_i h_i$ where $\lambda_i \in \mathbb{C}$ and $h_i \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(H))$. We shall demonstrate in Explicit example 4.2.10 that a general section gives a smooth curve C under the map (4.8). We will discuss in this example how to construct such a general section.

In Example 4.2.6, we see that there is no explicit polynomial description of s_i, but there is a quartic polynomial g_i (modulo f_4) that defines s_i^2. Now, we seek a quartic polynomial g defining $(\sum_{i=1}^{6} \lambda_i s_i)^2$ for general λ_i. Expanding the square gives

$$(\sum_{i=1}^{6} \lambda_i s_i)^2 = \sum_{i=1}^{6} \lambda_i^2 s_i^2 + 2 \sum_{1 \le i < j \le 6} \lambda_i \lambda_j s_i s_j \in H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(4H - \tilde{\Delta}_{16})). \quad (4.9)$$

Let D_i be the divisor defined by the section s_i. The section $s_i s_j$ corresponds to the reduced divisor $D_i + D_j$, which is determined by a quartic polynomial g_{ij} such that there is an equality of ideals

$$(g_{ij}, f_4) = \sqrt{(g_i, f_4) \cdot (g_j, f_4)}.$$

Note that the divisors, $2D_i$ and $D_i \cup D_j$ only determine the polynomials g_i and g_{ij} (mod f_4) up to constants μ_i and μ_{ij} respectively. Hence, the section in (4.9) is given by the polynomial

$$\sum_{i=1}^{6} \lambda_i^2 \mu_i g_i + 2 \sum_{1 \leq i < j \leq 6} \lambda_i \lambda_j \mu_{ij} g_{ij}.$$

By adjusting λ_i, we may assume without loss of generality that $\mu_i = 1$ for all i. The polynomial

$$\sum_{i=1}^{6} \lambda_i^2 g_i h_i^2 + 2 \sum_{1 \leq i < j \leq 6} \lambda_i \lambda_j \mu_{ij} g_{ij} h_i h_j$$

where h_i are linear polynomials then determines a divisor $2C$ for some curve $C = K \cap F$ which is smooth for a general choice of λ_i and h_i.

It remains to determine the coefficients μ_{ij}. We do so by restricting the sections s_i to a sufficiently small subvariety Z such that there is a polynomial expression for $s_{i|Z}$.

Choose a hyperplane section H_0 of K passing through 4 nodes, so H_0 is a curve of degree 4 in $\mathbb{P}^2$. Then, H_0 has two irreducible components Z_1 and Z_2 of degree 2 in $\mathbb{P}^2$. The intersection $Z_1 \cap Z_2$ is precisely the 4 nodes.

The two irreducible components are smooth, and hence isomorphic to $\mathbb{P}^1$. Let Z be one of the components. Fixing a point on Z gives a standard parametrization

$$\phi : \mathbb{P}^1 \xrightarrow{\sim} Z \subset K \subset \mathbb{P}^3 : (u, v) \mapsto (r_0(u, v) : \cdots : r_3(u, v))$$

where $r_0, \ldots, r_3 \in \mathbb{C}[u, v]$ are quadratic polynomials. Hence, the polynomials $\phi^* g_i$ and $\phi^* g_{ij}$ are homogeneous of degree 8 in $\mathbb{C}[u, v]$.

The divisors $2D_{i|H_0}$ are of degree $4 \cdot 4 = 16$, they are each a sum of 8 points, including the 4 nodes, each with multiplicity 2. Hence, the divisors $2D_{i|Z}$ are of degree 8, more precisely, they are each a sum of the 4 nodes with multiplicities 1 and 2 other points with multiplicities 2. Thus,

$$\phi^* g_i = \eta(u, v) g_{i,0}^2$$

where $\eta(u, v)$ is a quartic polynomial, zero on the 4 nodes, and $g_{i,0}$ are quadratic polynomials, zero on the other two points. The constants μ_{ij} are then the unique constants such that

$$\mu_{ij} \phi^* g_{ij} = \eta(u, v) g_{i,0} g_{j,0}.$$

Explicit example 4.2.10. We demonstrate the above construction by extending the computations in Explicit example 4.2.7. Recall that the six sections s_i^2 are given by the polynomials $P_i \pm i P_j$ $(1 \leq i < j \leq 3)$.

For example, let $g_1 = P_1 + iP_2$ and $g_2 = P_1 - iP_2$, then, using Magma, we obtain a polynomial

$$g_{12} = z_0z_1z_2z_3 + \frac{i}{2}(z_1^4 + z_2^4)$$

which satisfies $(g_{12}, f_4) = \sqrt{(g_1, f_4) \cdot (g_2, f_4)}$.

Consider the plane $z_1 - z_2 = 0$ which passes through the nodes $(\pm 1 : 1 : 1 : \mp i)$, $(\pm i : 1 : 1 : \mp 1)$. The hyperplane section H_0 defined by this plane is the union of two smooth conics, one of which is given by

$$Z = K \cap \left\{ z_0^2 + \sqrt{2}z_0z_2 + i\sqrt{2}z_1^2 + z_2^2 \right\}.$$

The map $\phi : \mathbb{P}^1 \to Z \subset \mathbb{P}^3$ is parametrized by

$$\begin{aligned}(u, v) \mapsto \Big(&(i + \sqrt{2})u^2 - 2\sqrt{2}uv + \sqrt{2}v^2 : -iu^2 + (2i + \sqrt{2})uv - \sqrt{2}v^2 : \\ &-iu^2 + (2i + \sqrt{2})uv - \sqrt{2}v^2 : -u^2 - 2i\sqrt{2}v^2 \Big).\end{aligned}$$

On this particular conic, the divisors D_1 and D_2 are actually supported on the same points, so factoring $\phi^*g_1 = c_1\eta(u, v)g_{1,0}^2$, $\phi^*g_2 = c_2\eta(u, v)g_{2,0}^2$ and $\phi^*g_{12} = c_{12}g_{1,0}g_{2,0}$ gives

$$\begin{aligned}\eta(u, v) &= u(u - (1 + i)v)(u - (1 - i)v)(u - \frac{2(1 - i\sqrt{2})}{3}v) \\ g_{1,0} = g_{2,0} &= (u - (1 - \zeta)v)(u - (1 - \zeta^3)v)\end{aligned}$$

where $\zeta^2 = i$, with coefficients

$$c_1 = -4(1 + i + 2\zeta^3), \qquad c_2 = -4(1 - i + 2\zeta), \qquad c_{12} = 2i + \sqrt{2}.$$

So, we get

$$\mu_{12} = \frac{\sqrt{c_1c_2}}{c_{12}} = 4.$$

Taking $\lambda_1 = 1$, $\lambda_2 = 3$, $h_1 = z_0$, $h_2 = z_1$, so

$$g_6 = z_0^2g_1 + 6z_0z_1g_{12} + 9z_1^2g_2,$$

we get a 40-nodal sextic surface F defined by the polynomial

$$f_6 = g_6 + (z_0z_1 + z_1z_2 + z_2z_3)f_4.$$

We verified with Magma (with computations done over finite fields) that the intersection $K \cap F = 2C$ is given by a smooth curve C of genus 15.

4.2.2 Universality of Gallarati's construction

We shall show that Gallarati's construction gives an irreducible 28-dimensional family of even 40-nodal sextic surfaces. We will further show that a general even 40-nodal sextic surface can be obtained using Gallarati's construction.

Proposition 4.2.11. *There is an irreducible 28-dimensional family $\mathfrak{M}_6$ of isomorphism classes of even 40-nodal sextic surfaces given by Gallarati's construction.*

Proof. Let $\mathcal{F}_6$ be the space of polynomials $f_6 = qf_4 + g_6$ defining even 40-nodal sextic surfaces given by Gallarati's construction (cf. Section 4.2.1). Recall that every such 40-nodal sextic surface F yields a unique Kummer surface $K \in |4H - \tilde{\Delta}|$ which is tangent to F along a curve $2C$. Hence there is a projection from $\mathcal{F}_6$ to the space $\mathcal{P}_{4,6}$ of pairs $(f_4, \langle f_4, g_6 \rangle)$ where $\langle f_4, g_6 \rangle \subset \mathbb{C}[z_0, \ldots, z_3]$ is the ideal generated by f_4 and g_6. There is a natural projection from $\mathcal{P}_{4,6}$ to $\mathcal{F}_4$, the space of polynomials $f_4 \in \mathbb{C}[z_0, \ldots, z_3]$ defining Kummer surfaces.

The spaces $\mathcal{F}_6$, $\mathcal{P}_{4,6}$ and $\mathcal{F}_4$ can be projectivized and the projections commute with the projectivizations to give

$$\mathbb{P}(\mathcal{F}_6) \xrightarrow{\pi_1} \mathbb{P}(\mathcal{P}_{4,6}) \xrightarrow{\pi_2} \mathbb{P}(\mathcal{F}_4).$$

Let $p = ([f_4], \langle f_4, g_6 \rangle)$ be a point in $\mathbb{P}(\mathcal{P}_{4,6})$ where $[f_4]$ is the isomorphism class of f_4 under the equivalence relation $\lambda f_4 \sim f_4$ for all $\lambda \in \mathbb{C}^*$. Fixing a choice of $g_6 \in \langle f_4, g_6 \rangle$, for any $[f_6] \in \pi_1^{-1}(p)$, there is a unique $\lambda \in \mathbb{C}^*$ such that $f_4 | (\lambda f_6 - g_6)$. Hence, $\pi_1^{-1}(p)$ is the affine space

$$\{[f_6] \mid \exists \lambda \in \mathbb{C}^* \in \text{ s.t. } f_4 | \lambda f_6 - g_6\} = \{qf_4 + g_6 \mid q \in \mathbb{C}[z_0, \ldots, z_3]\}.$$

and the fibre of π_1 is a 10-dimensional affine space.

For each Kummer surface K, the curve C of Gallarati's construction lies in the linear system $|3H - L_{16}|$. By Lemma 4.2.2, $\dim |3H - L_{16}| = 15$. The fibre of π_2 is the image of $|3H - L_{16}|$ under the quadratic map

$$|3H - L_{16}| \to |6H - \tilde{\Delta}_{16}| \ : \ C \mapsto 2C = \operatorname{div}(f_4, g_6).$$

This is the projectivization of the quadratic map on the global sections

$$H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(3H - L_{16})) \to H^0(\tilde{K}, \mathcal{O}_{\tilde{K}}(6H - \tilde{\Delta}_{16})) \ : \ s \mapsto s^2.$$

Note that $s^2 = t^2$ if and only if $s = \pm t$, so the quadratic map on the projectivized linear systems is injective. Hence, the fibre of π_2 is 15 dimensional.

By Lemma 3.3.7, the nodes of K are independent in degree 4, so $\dim \mathcal{F}_4 = \dim \mathbb{C}[z_0, \ldots, z_3]_4 - 16 = \binom{4+3}{3} - 16 = 19$ and $\dim \mathbb{P}(\mathcal{F}_4) = 18$. Hence, $\dim \mathbb{P}(\mathcal{F}_6) = 18 + 15 + 10 = 43$.

There is an action of $\mathrm{Aut}(\mathbb{P}^3) = PGL(3, \mathbb{C})$ on $\mathbb{P}(\mathcal{F}_6)$ which fixes the isomorphism class of the sextic surfaces, so the deformation family of even 40-nodal sextic surfaces obtained from Gallarati's construction is at most

$$\dim \mathbb{P}(\mathcal{F}_6) - \dim PGL(3, \mathbb{C}) = 43 - 15 = 28.$$

We shall show that any isotrivial deformation in $\mathbb{P}(\mathcal{F}_6)$ lies in $PGL(3, \mathbb{C})$. By Proposition 3.3.15, all small deformations of F are projective. In particular, if F' is a deformation of F such that there is an isomorphism $\sigma : F \xrightarrow{\sim} F'$, then σ lifts to an automorphism of $\mathbb{P}^3$, so by definition, $\sigma \in PGL(3, \mathbb{C})$.

Therefore, there exists a family $\mathfrak{M}_6$ of even 40-nodal sextic surfaces with no trivial deformations, obtained as the quotient of a dense open subset of $\mathbb{P}(\mathcal{F}_6)$ by $PGL(3, \mathbb{C})$ (or we can see it as the stack quotient of $\mathbb{P}(\mathcal{F}_6)$ by $PGL(3, \mathbb{C})$). This family $\mathfrak{M}_6$ is 28-dimensional. Similarly, we can define $\mathfrak{M}_{4,6}$ and $\mathfrak{M}_4$ from $\mathbb{P}(\mathcal{P}_{4,6})$ and $\mathbb{P}(\mathcal{F}_4)$ and there is a projection

$$\mathfrak{M}_6 \to \mathfrak{M}_{4,6} \to \mathfrak{M}_4.$$

Since projective Kummer surfaces are quotients of principally polarized abelian surfaces induced by the involution $[-1]$, they form an irreducible family of dimension 3. In particular, $\mathfrak{M}_4$ is irreducible. As $\mathfrak{M}_6$ is fibred over $\mathfrak{M}_4$ with smooth equidimensional fibres, we conclude that $\mathfrak{M}_6$ is irreducible. $\square$

We shall show that in fact, the tangent space to any point in $\mathfrak{M}_6$ is also 28-dimensional.

Proposition 4.2.12. *Let F be an even 40-nodal sextic surface obtained using the Gallarati construction. Then $H^1(F, \tilde{T}_F)$ is 28-dimensional and the deformations of F are unobstructed.*

Proof. By Proposition 2.3.7, we have $H^1(F, \tilde{T}_F) \cong (I/J)_6$ and $\dim (I/J)_6 = 68 - k$ where k is the rank of the matrix $(e_j(p_i))_{i,j}$ where $\{e_j\}$ is a basis of the module S_6 of degree 6 polynomials in 4 variables and $\{p_i\}$ is the set of nodes. Hence, $h^1(F, \tilde{T}_F) \geq 28$. Indeed, equality holds if and only if the nodes are independent in degree 6.

Recall that $\mathrm{Def}_F = H^1(F, \tilde{T}_F)$ parametrizes the infinitesimal deformations of 40-nodal sextics as quotient surfaces and $\mathrm{Def}_{K,C}$ parametrizes the infinitesimal deformations of the pair (K, C). Suppose F and K are defined by polynomials f_6 and f_4 respectively, then the spaces Def_F and $\mathrm{Def}_{K,C}$ are the tangent spaces to $\mathfrak{M}_6$ and $\mathfrak{M}_{4,6}$ at $[f_6]$ and $([f_4], \langle f_4, f_6\rangle)$ respectively (cf. proof of Proposition 4.2.11). Since $\pi_1 : \mathbb{P}(\mathcal{F}_6) \to \mathbb{P}(\mathcal{P}_{4,6})$ is fibred over a 10-dimensional vector space, there is a well-defined projection $p : \mathrm{Def}_F \to \mathrm{Def}_{K,C}$ whose kernel is 10-dimensional.

Similarly, the fibre of the map $\pi_2 : \mathbb{P}(\mathcal{P}_{4,6}) \to \mathbb{P}(\mathcal{F}_4)$ is a smooth quadric, so there is a projection on the tangent space $\mathrm{Def}_{K,C} \to \mathrm{Def}_K$ whose kernel is 15-dimensional.

Since C is an ample divisor, all deformations in $\mathrm{Def}_{K,C}$ are ample, so the image of $\mathrm{Def}_{K,C} \to \mathrm{Def}_K = H^1(K, \tilde{T}_K)$ is the sub-vector space of projective deformations of K. In particular, there is an inclusion

$$\mathrm{im}\,(\mathrm{Def}_{K,C} \to \mathrm{Def}_K) \subset \mathrm{im}\,(H^1(\mathbb{P}^3, T_{\mathbb{P}^3}(-\log K)) \to H^1(K, \tilde{T}_K)).$$

where the second map is that given in Remark 2.3.8. By Remark 2.3.9, the second image is isomorphic to $(I/J)_4$ where J is the Jacobian ideal of the quartic polynomial defining K and $I = \sqrt{J}$. By Lemma 3.3.7, the 16 nodes of K are independent in degree 4, so $\dim\,(I/J)_4 = \dim S_4 - \dim J_4 - 16 = \binom{4+3}{3} - 16 - 16 = 3$.

We thus obtain $\dim \mathrm{Def}_{K,C} \le 15 + 3 = 18$. Hence, $h^1(\tilde{T}_F) = \dim\ker p + \dim \mathrm{Def}_{K,C} = 28$ and the nodes are independent in degree 6. $\square$

Remark 4.2.13. We can also compute the dimension of the kernel of the projection $\mathrm{Def}_{K,C} \to \mathrm{Def}_K$ directly using the methods of Section 3.1.1. By Proposition 3.1.9, there is an isomorphism

$$\mathrm{Def}_{K,C} \cong \mathbb{H}^1(A, \overline{d}_1\tilde{s})^G$$

where $\overline{d}_1\tilde{s}$ is the complex

$$0 \to T_A \xrightarrow{\overline{d}_1\tilde{s}} \mathcal{O}_D(D) \to 0,$$

$D = \phi^{-1}C \in |4\Theta|$ is the pullback of C under the double cover $\phi : A \to K$ and $\tilde{s} \in H^0(A, \mathcal{O}_A(4\Theta))$ is the section defining D. Applying the 5 term exact sequence to the spectral sequence

$$E_1^{p,q} = H^q(A, (\overline{d}_1\tilde{s})^p) \Longrightarrow \mathbb{H}^{p+q}(A, \overline{d}_1\tilde{s})$$

gives

$$\begin{aligned} 0 \to & H^0(D, \mathcal{O}_D(D)) \to \mathbb{H}^1(A, \overline{d}_1\tilde{s}) \to \\ & \to \ker(H^1(A, T_A) \to H^1(D, \mathcal{O}_D(D))) \to E_2^{2,0} = 0. \end{aligned}$$

Taking the G-invariant part gives the left exact sequence

$$\begin{aligned} 0 \to & H^0(D, \mathcal{O}_D(D))^G = H^0(C, \mathcal{O}_C(C)) \to \mathrm{Def}_{K,C} = \mathbb{H}^1(A, \overline{d}_1\tilde{s})^G \to \\ & \to H^1(A, T_A)^G = H^1(K, \tilde{T}_K). \end{aligned}$$

Since the Kummer surface K has trivial canonical divisor, we get $K_C = C_{|C}$ by the adjunction formula. So, the first term has dimension $h^0(C, \mathcal{O}_C(C)) = h^0(C, \tilde{\omega}_C) = p_g(C) = 15$.

Remark 4.2.14. Propositions 4.2.11 and 4.2.12 show that the moduli stack of 40-nodal sextic surfaces generated by Gallarati's construction is 28-dimensional, and the tangent space at each point is also 28-dimensional, so it is smooth. Furthermore, the set of 40 nodes are independent in degree 6 since $\dim (S/J)_6 - \dim (I/J)_6 = 40$.

Corollary 4.2.15. *A general even 40-nodal sextic surface can be obtained by Gallarati's construction. Hence, $\mathfrak{M}_6$ is open and dense in the family of all even 40-nodal sextic surfaces.*

Proof. Let F be any sextic surface with an even set of 40 nodes. Then, since $h^0(\mathcal{O}_{\tilde{F}}(2H - L)) = h^{0,2}(\tilde{F}) > 0$, we can find a curve $C \subset F$ such that the strict transform $\tilde{C}$ of C lies in $|2H - L|$. We thus have $\pi_F^*(2C) \in |4H - \tilde{\Delta}|$, so the divisor $2C$ is defined by the ideal $(f_6, f_4) \subset \mathbb{C}[z_0, \ldots, z_3]$ where f_6 is the polynomial defining F and f_4 is a quartic polynomial. By Gallarati's construction, we see that a general f_4 defines a quartic surface which is smooth outside the curve C and has only nodes as singularities on C. By [Cat81, Lemma 2.3], such an f_4 defines a Kummer surface in $\mathbb{P}^3$. Thus, F lies in the family obtained from Gallarati's construction. $\square$

4.2.3 The Casnati-Catanese construction

In [CC97], Casnati and Catanese gave another construction for nodal surfaces with an even set of nodes. Let $F \subset \mathbb{P}^3$ be a nodal surface with an even set of nodes and $f : S \to F$ be a double cover branched at the nodes. The involution on S corresponding to this double cover induces an eigenspace decomposition $f_*\mathcal{O}_S = \mathcal{O}_F \oplus \mathcal{F}$, where $\mathcal{O}_F$ and $\mathcal{F}$ are the $+1$ and -1 eigenspaces respectively of the $\mathcal{O}_F$-linear involution. Since F is singular, $\mathcal{F}$ is not necessarily a line bundle.

The involution on $f_*\mathcal{O}_S$ induces a non-degenerate pairing $\mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_F$, hence there is an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}, \mathcal{O}_F)$. We call such a coherent sheaf $\mathcal{F}$ quadratic.

Theorem 4.2.16 ([CC97, Theorem 0.3])**.** *Let $F \subset \mathbb{P}^3$ be a surface, and let $\mathcal{F}$ be a quadratic sheaf on F. Then $\mathcal{F}$ fits into an exact sequence of the form*

$$0 \to \mathcal{E}^\vee(-d) \xrightarrow{\phi} \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathbb{P}^3}$-module and ϕ is a symmetric map.

Since $\mathcal{F}$ is supported on F, the surface F is obtained as the degeneracy locus of the symmetric map ϕ, i.e., the locus where $\operatorname{rank} \phi < \operatorname{rank} \mathcal{E}$.

For an even 40-nodal sextic F, Casnati and Catanese explicitly gave $\mathcal{E} = \Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$ [CC97, after Theorem 3.8.2]. There is a perfect pairing $\Omega^1_{\mathbb{P}^3} \otimes \Omega^2_{\mathbb{P}^3} \to \omega_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$, so $(\Omega^1_{\mathbb{P}^3})^\vee \cong \Omega^2_{\mathbb{P}^3}(4)$. Theorem 4.2.16 gives us the short exact sequence

$$0 \to \Omega^2_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{\phi} \Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{F} \to 0 \tag{4.10}$$

where

$$\phi = \begin{pmatrix} \phi_0 & \beta \\ \beta^t & q \end{pmatrix}$$

is a symmetric matrix.

We shall show how one could obtain the equation of a 40-nodal sextic surface from the above short exact sequence.

The Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3} \to 0$$

and its dual, tensored by $\mathcal{O}_{\mathbb{P}^3}(-4)$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 4} \to \Omega^2_{\mathbb{P}^3} \to 0$$

gives a diagram

$$\begin{tikzcd}
\Omega^2_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \arrow[r, "\phi"] & \Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[d] \\
\mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \arrow[u] \arrow[r, "\tilde{\phi}"] & \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)
\end{tikzcd}$$

Recall that there is an isomorphism $\mathcal{O}_{\mathbb{P}^3}(l) \cong \tilde{S}(l)$ where $S = \mathbb{C}[z_0, \ldots, z_3]$ is the graded polynomial ring in 4 variables. Under this isomorphism, the morphisms in the diagram above are given by multiplication of matrices of homogeneous polynomials. In particular, $\tilde{\phi}$ is given by a 5×5 matrix

$$\tilde{\phi} = \begin{pmatrix} \tilde{\phi}_0 & \tilde{\beta} \\ \tilde{\beta}^t & q \end{pmatrix}$$

of homogeneous quadratic polynomials, where $\tilde{\phi}_0$, $\tilde{\beta}$ and q are sub-matrices of dimensions 4×4, 4×1 and 1×1 respectively.

The first vertical map is the quotient map to the cokernel of $G \mapsto (Gz_i)$ while the second vertical map is the inclusion of the kernel of $G_i \mapsto \sum_{i=0}^{3} G_i z_i$. Thus, a non-zero element of $\Omega^1_{\mathbb{P}^3}(-1)$ can be represented by polynomials $G_i \in \mathcal{S}$ such that $\sum G_i z_i = 0$. This is equivalent to imposing the condition on $\tilde{\beta}$ that $\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} \tilde{\beta} = 0$. By symmetry, this imposes a necessary and sufficient condition for $\Omega^2_{\mathbb{P}^3}(-1)$ as well.

Such a resolution preserves the rank of the matrix, so at any point $p \in \mathbb{P}^3$, we have $\operatorname{rank} \phi(p) = \operatorname{rank} \tilde{\phi}(p)$. Hence, the surface F is the locus where $\operatorname{rank} \tilde{\phi}(p) \leq 3$, i.e. where any 4×4 minor of $\tilde{\phi}$ is zero. A 4×4 minor is a homogeneous polynomial of degree 8, hence F is defined by a sextic polynomial f_6 which is the greatest common divisor of all such minors.

The Casnati-Catanese construction is related to Gallarati's construction in the following way.

For a 4×4 symmetric matrix $A = (a_{ij})$, there is a determinantal identity [Cat81, (1.3)]

$$(a_{ii}a_{jj} - a_{ij}^2) \det A = \det A_{kk} \det A_{44} - (\det A_{k4})^2$$

where $\{i, j, k\} = \{1, 2, 3\}$ and A_{ij} is the (i, j)-th minor. Applying the identity to $A = \phi(p)$ for any point $p \in F$ and noting that $\det \phi(p) = 0$, we see that

$$\det \phi(p)_{kk} \det \phi_0(p) = (\det \phi(p)_{k4})^2.$$

The nodes of F lie in the locus where $\operatorname{rank} \tilde{\phi}(p) \leq 2$. In particular, $\phi_0(p) = 0$ for any node p. Hence, ϕ_0 is a quartic polynomial passing through the nodes of F. Furthermore, since the righthand side of the equation is a square, we have $\operatorname{div}_F(K) = 2C$ for some curve C on F. Hence, by the same proof as Corollary 4.2.15, a general such K is a Kummer surface.

4.2.4 EPW sextics

The next construction arises from taking hyperplane sections of EPW sextic fourfolds $Y_A \subset \mathbb{P}^5$.

EPW sextic fourfolds were first introduced by Eisenbud, Popescu and Walter in [EPW01] and were extensively studied by O'Grady [OGr06; OGr13; OGr15; OGr16], in particular, with respect to their double covers and the moduli space of EPW sextics. They are related to an important class of algebraic varieties, the Gushel-Mukai (GM) varieties [IM11]. Debarre and Kuznetsov understood the moduli space and periods of GM varieties by relating them to the results of O'Grady on EPW sextics. Correspondences between double EPW sextics X_A and GM varieties X of dimensions $n = 4$ or 6 induce isomorphisms of the primitive Hodge structures $H^2(X_A, \mathbb{Z})_0 \cong H^n(X, \mathbb{Z})_0$ of type $(1, 20, 1)$. Interested readers may refer to [DK15; DK16] and also to [KP16] for a study of the derived categories of GM varieties.

In [OGr06], O'Grady showed that a general EPW sextic admits a smooth double cover X_A, ramified precisely along its singular locus, which is a smooth surface of degree 40. Cutting Y_A by two generic hyperplanes gives a 40-nodal surface F with a double cover $S \subset X_A$ ramified at the nodes. We show that a general even 40-nodal sextic surface can be obtained through this construction and $H^2(S, \mathbb{Q})_-$ contains a sub-Hodge structure $H^2(X_A, \mathbb{Q})_-$ of type $(1, 20, 1)$.

EPW sextics are defined as follows (cf. [OGr06]). Let V be a 6 dimensional vector space and fix a volume form $\Lambda^6 V \xrightarrow{\sim} \mathbb{C}$. This allows us to define a symplectic form on $\Lambda^3 V$ by $(\alpha, \beta) = \mathrm{vol}(\alpha \wedge \beta)$.

There is a subbundle $E \subset \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ defined, on each point $v \in \mathbb{P}(V)$, by the fibre

$$E_v = \{\alpha \in \Lambda^3 V | \alpha \wedge v = 0\} = v \wedge \Lambda^2 V'$$

where $\dim V' = 5$ and $V = V' \oplus \mathbb{C}v$. Hence, $\dim E_v = \binom{5}{2} = 10 = \frac{1}{2}\dim \Lambda^3 V$. It is clear that E_v is a Lagrangian subspace of $\Lambda^3 V$, so E is a Lagrangian subbundle of $\Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$.

Let $A \subset \Lambda^3 V$ be any Lagrangian subspace and λ_A be the composition of the inclusion and the projection

$$\lambda_A : E \hookrightarrow \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \twoheadrightarrow (\Lambda^3 V / A) \otimes \mathcal{O}_{\mathbb{P}(V)}$$

An EPW sextic Y_A is defined to be the degeneracy locus $V(\det \lambda_A)$ of the map of vector bundles λ_A. The singular locus of Y_A is the locus where $\operatorname{rank} \lambda_A \leq 8$. It is in general a smooth surface of degree 40 in $\mathbb{P}^5$.

There exists a unique surface X_A which is a double cover of Y_A branched along the singular locus of Y_A. It is called a double EPW sextic. In [OGr06], O'Grady showed that the pair (X_A, Y_A) is a deformation of $(Z^{[2]}, \phi_H(Z^{[2]}))$ where Z is a K3 surface, $Z^{[2]}$ is the Hilbert scheme of Z, H is an ample divisor on $Z^{[2]}$ with $\dim |H| = 5$ and the morphism generated by the global sections of $\mathcal{O}_{Z^{[2]}}(H)$

$$\phi_H : Z^{[2]} \to \mathbb{P}^5$$

is a double cover on the image, which is a nodal sextic fourfold.

It is well-known that the Beauville-Bogomolov quadratic form on $H^2(Z^{[2]}, \mathbb{Z})$ defines a polarized Hodge structure of type $(1, 21, 1)$ [Bea83]. Since $\phi_H(Z^{[2]})$ is a hypersurface in $\mathbb{P}^5$, the Lefschetz hyperplane theorem implies that the positive eigenspace $H^2(Z^{[2]}, \mathbb{Q})_+ = H^2(\phi_H(Z^{[2]}), \mathbb{Q}) = H^2(\mathbb{P}^5, \mathbb{Q})$ has Hodge type $(0, 1, 0)$. Hence, the negative eigenspace $H^2(Z^{[2]}, \mathbb{Q})_-$ is of type $(1, 20, 1)$. Since the Hodge numbers are invariant in a deformation family, $H^2(X_A, \mathbb{Q})_-$ has Hodge type $(1, 20, 1)$ for all double EPW sextics X_A, and is simple in general.

By the Lefschetz hyperplane theorem, there is an embedding of Hodge structures $H^2(X_A, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{Z})$ which respects the involution on S and X_A. Hence, $H^2(S, \mathbb{Z})_-$ contains a sub-Hodge structure of type $(1, 20, 1)$.

We now study the deformations of even 40-nodal sextic surfaces obtained from EPW sextic fourfolds. We show that there is a 28-dimensional family of such surfaces where all deformations are non-trivial.

EPW sextic fourfolds are parametrized by $\mathbb{LG}(\Lambda^3 V)$, the space of Lagrangian subspaces $A \subset \Lambda^3 V$. In [OGr15], O'Grady showed that there is a well-defined GIT-quotient $\mathfrak{M} = \mathbb{LG}(\Lambda^3 V) /\!\!/ PGL(V)$ where $PGL(V)$ is the group of automorphisms of $\mathbb{P}(V)$. In the proof of [OGr13, Theorem 4.25], it was shown that the family has dimension

$$\dim \mathfrak{M} = \dim \mathbb{LG}(\Lambda^3 V) - \dim PGL(V) = 55 - 35 = 20.$$

Furthermore, there is a rational map [OGr15, (0.0.6)]

$$\mathfrak{p} : \mathfrak{M} \dashrightarrow \mathbb{D}_\Lambda^{BB}$$

where $\mathbb{D}_\Lambda^{BB}$ is the Baily-Borel compactification of the period domain of polarized weight 2 Hodge structures of type $(1, 20, 1)$. It is defined by sending the class of Y_A to the polarized weight 2 Hodge structure $H^2(X_A, \mathbb{C})_-$ when the double EPW-sextic X_A is smooth.

The map $\mathfrak{p}$ is regular and injective when restricted to the dense open subspace $\mathfrak{M}^0 = \mathbb{LG}(\Lambda^3 V)^0 /\!\!/ PGL(V)$ where $\mathbb{LG}(\Lambda^3 V)^0 \subset \mathbb{LG}(\Lambda^3 V)$ is the dense open subset parametrizing smooth double EPW-sextics X_A.

Lemma 4.2.17. *Suppose F is an even 40-nodal sextic surface obtained from some EPW sextic fourfold. Then there is a unique class $[Y_A] \in \mathfrak{M}^0$ such that F is the intersection of Y_A with two hyperplane sections.*

Proof. Let $f : S \to F$ be the double cover branched over the 40 nodes of F, and suppose F is obtained by the intersection of an EPW sextic fourfold Y_A with two hyperplane sections. The Lefschetz hyperplane theorem gives an embedding of Hodge structures $H^2(X_A, \mathbb{C}) \hookrightarrow H^2(S, \mathbb{C})$. In particular, $H^2(S, \mathbb{C})_-$ has a sub-Hodge structure $H^2(X_A, \mathbb{C})_-$ of type $(1, 20, 1)$. By the injectivity of the map $\mathfrak{p} : \mathfrak{M}^0 \to \mathbb{D}_\Lambda^{BB}$, $H^2(X_A, \mathbb{C})_-$ determines a unique class $[Y_A]$. Hence, the surface F is determined by a unique isomorphism class of EPW sextic fourfolds. $\square$

An immediate consequence of the lemma is that if F' is a deformation of F such that $F \cong F'$, then there is a unique (up to isomorphism) EPW sextic fourfold $Y_A \subset \mathbb{P}^5$ such that F and F' are intersections of Y_A with subspaces $W, W' \subset \mathbb{P}^5$ respectively where $W \cong W' \cong \mathbb{P}^3$.

Now, fix an EPW sextic fourfold $Y_A \subset \mathbb{P}^5$, an even 40-nodal sextic surface F is obtained by taking the intersection of Y_A with two (general) hyperplane sections. Deformations of F fixing Y_A are obtained by moving the hyperplane sections, i.e. choosing a subspace $\mathbb{P}^3 \subset \mathbb{P}^5$. They are parametrized by the Grassmannian $\mathrm{Gr}(4,6)$, which is $(6-4)\cdot 4 = 8$-dimensional.

Explicitly, choose coordinates $z_0, \dots, z_5$ on Y_A such that the two hyperplane sections are defined by $z_0 = 0$ and $z_1 = 0$. Then, a deformation F_t of F fixing Y_A can be given by the hyperplane sections

$$z_0 + t_{02}z_2 + t_{03}z_3 + t_{04}z_4 + t_{05}z_5 = 0, \quad z_1 + t_{12}z_2 + t_{13}z_3 + t_{14}z_4 + t_{15}z_5 = 0$$

where $t = (t_{ij}) \in \mathbb{C}^8$ corresponds to a point in the 8-dimensional Grassmannian $\mathrm{Gr}(4,6)$.

Let $h_6(z_0, \dots, z_5) \in \mathbb{C}[z_0, \dots, z_5]$ be the polynomial defining the EPW sextic fourfold, so the surface F is defined by the polynomial $f_6 = h_6(0, 0, y_0, \dots, y_3) \in \mathbb{C}[y_0, \dots, y_3]$. As a polynomial in $\mathbb{C}[z_2, \dots, z_5][z_0, z_1]$, we can write h_6 as

$$\begin{aligned} h_6 = f_6(z_2, \dots, z_5) + z_0 f_{50}(z_2, \dots, z_5) + z_1 f_{51}(z_2, \dots, z_5) \\ + \text{higher order terms in } z_0,\ z_1 \end{aligned}$$

where $f_{5j} = \frac{\partial h_6}{\partial z_j}_{|z_0 = z_1 = 0}$ with $j = 0, 1$. Note that the polynomial f_{5j} vanishes at all the nodes of F, so $f_{5j} \in I = \sqrt{J}$ where J is the Jacobian ideal of f_6.

After the change of coordinates

$$\Big\{ w_0 = z_0 + t_{02}z_2 + \ldots + t_{05}z_5\,,\ w_1 = z_1 + t_{12}z_2 + \ldots + t_{15}z_5\,, \\ w_2 = z_2\,,\ \ldots\,,\ w_5 = z_5 \Big\}\,,$$

the surface F_t is defined by the hyperplane sections $w_0 = w_1 = 0$. Let $h_{6,t}(w_0, \dots, w_5)$ be the polynomial obtained from h_6 under the change of coordinates, so $f_{6,t} = h_{6,t}(0, 0, y_0, \dots, y_3)$ defines the surface F_t. We have

$$\begin{aligned} h_{6,t} &= h_6(w_0 - t_{02}w_2 - \ldots - t_{05}w_5, w_1 - t_{12}w_2 - \ldots - t_{15}w_5, w_2, \dots, w_5) \\ &= f_6(w_2, \dots, w_5) - \sum_{\substack{i \in \{2, \dots, 5\} \\ j \in \{0,1\}}} t_{ij} z_i f_{5j}(w_2, \dots, w_5) + \text{higher order terms in } t_{ij}. \end{aligned}$$

We obtain a set of 8 polynomials $\{z_i f_{5j} | i \in \{2, \dots, 5\}, j \in \{0, 1\}\}$ in $(I/J)_6 \cong H^1(F, \tilde{T}_F)$ generating the infinitesimal deformations parametrized by the Grassmannian $\mathrm{Gr}(4,6)$. Hence, we have the following criterion

Lemma 4.2.18. *Let $Y_A \subset \mathbb{P}^5$ be an EPW sextic fourfold and F be an even 40-nodal sextic surface defined by intersecting Y_A with the hyperplanes $z_0 = z_1 =$*

0. *Let* $h_6 \in \mathbb{C}[z_0, \ldots, z_5]$ *be the polynomial defining* Y_A *and* $f_{5j} = \frac{\partial h_6}{\partial z_j}_{|z_0=z_1=0}$ *for* $j = 0, 1$. *The infinitesimal deformations of* F *fixing* Y_A *are parametrized by the Grassmannian* $\mathrm{Gr}(4, 6)$, *and all infinitesimal deformations of* F *in* $\mathrm{Gr}(4, 6)$ *are non-trivial if and only if the set*

$$\{z_i f_{5j} | i \in \{2, \ldots, 5\}, j \in \{0, 1\}\} \subset (I/J)_6$$

is linearly independent.

Let $\mathfrak{M}_2$ be the family of even 40-nodal sextic surfaces obtained from EPW sextics, fibred over the family $\mathfrak{M}$ of EPW sextic fourfolds Y_A, with fibres isomorphic to $\mathrm{Gr}(4, 6)$. Hence, the dimension of $\mathfrak{M}_2$ is equal to

$$\dim \mathfrak{M} + \dim \mathrm{Gr}(4, 6) = 20 + 8 = 28.$$

Lemma 4.2.19. *Let* F *be an even 40-nodal sextic surface obtained from an EPW sextic fourfold* Y_A. *Suppose all infinitesimal deformations of* F *in the fibre of* Y_A *under the fibration* $\mathfrak{M}_2 \to \mathfrak{M}$ *are non-trivial. Let* $F' \subset Y_{A'}$ *be a sufficiently small deformation of* F *in* $\mathfrak{M}_2$, *then all infinitesimal deformations of* F' *in the fibre of* $Y_{A'}$ *are non-trivial as well. Hence, all small deformations of* F *in* $\mathfrak{M}_2$ *are non-trivial.*

Proof. Let $h_6 \in \mathbb{C}[z_0, \ldots, z_5]$ be the polynomial defining Y_A. Since $\dim \mathfrak{M}_2 = 28$, there exists a polynomial $\tilde{h}_6 \in \mathbb{C}[z_0, \ldots, z_5][u_1, \ldots, u_{28}]$, homogeneous of degree 6 in the variables $z_0, \ldots, z_5$, such that $h_6 = \tilde{h}_6(0, \ldots, 0)$ and, for any $u = (u_1, \ldots, u_{28})$ with $|u| \ll 1$, the polynomial $h_{6,u} = \tilde{h}_6(u_1, \ldots, u_{28}) \in \mathbb{C}[z_0, \ldots, z_5]$ defines a small deformation $Y_{A,u}$ of Y_A. Let $\tilde{f}_{5j} = \frac{\partial \tilde{h}_6}{\partial z_j}_{|z_0=z_1=0}$ and $f_{5j,u} = \frac{\partial h_{6,u}}{\partial z_j}_{|z_0=z_1=0}$ for $j = 0, 1$.

By Lemma 4.2.18, all small deformations of F_u on the fibre are non-trivial if and only if the set $\{z_i f_{5j,u} | i \in \{2, \ldots, 5\}, j \in \{0, 1\}\}$ is linearly independent. Let $\{e_k\}$ be the set of degree 6 monomials in $\mathbb{C}[z_0, \ldots, z_5]$ and M be the matrix of the coefficients of each e_k in $z_i \tilde{f}_{5j}$. Thus, M is of size 8×84 with terms in $\mathbb{C}[u_1, \ldots, u_{28}]$.

There exist trivial deformations of F_u on the fibre if and only if $\operatorname{rank} M(u) < 8$. This is a closed algebraic condition on $\mathbb{C}^{28}$. By assumption, F_0 has no trivial deformations on the fibre, so there is an open neighbourhood $\mathcal{U} \subset \mathbb{C}^{28}$ containing the origin such that for any $u \in \mathcal{U}$, all deformations of F_u on the fibre are non-trivial.

Indeed, if $u, u' \in \mathcal{U}$ such that $Y_{A,u} \ncong Y_{A,u'}$, then by Lemma 4.2.17, $F_u \ncong F_{u'}$, so all deformations of F_0 parametrized by $\mathcal{U}$ are non-trivial. Hence, $\mathcal{U}$ parametrizes an open neighbourhood of F in $\mathfrak{M}_2$ on which all deformations are non-trivial. $\square$

Theorem 4.2.20. *A general even 40-nodal sextic surface $F \subset \mathbb{P}^3$ is the intersection of an EPW sextic fourfold with two hyperplanes in $\mathbb{P}^5$.*

Proof. In Proposition 4.2.11 and Corollary 4.2.15, we showed that a general even 40-nodal sextic surfaces can be obtained from Gallarati's construction, and that yields an irreducible 28-dimensional family. The rational map $\mathfrak{M}_2 \dashrightarrow \mathfrak{M}_6$ from the family $\mathfrak{M}_2$ of 40-nodal surfaces obtained from EPW sextic fourfolds to the family $\mathfrak{M}_6$ of even 40-nodal sextic surfaces is algebraic, so the image is a Zariski-constructible subset of $\mathfrak{M}_6$. Any constructible subset of maximal dimension in an irreducible variety is in fact open and dense, so it suffices to show that the image of $\mathfrak{M}_2$ in $\mathfrak{M}_6$ is 28-dimensional.

Since $\dim \mathfrak{M}_2 = 28$, we just need to show that there exists an open neighbourhood $\mathcal{U} \subset \mathfrak{M}_2$ of some even 40-nodal sextic surface F on which all small deformations are non-trivial. Let Y_A be the image of F under the fibration $\mathfrak{M}_2 \to \mathfrak{M}$. By Lemma 4.2.19, it suffices to find a surface F such that all infinitesimal deformations of F in the fibre of Y_A are non-trivial.

Consider the EPW sextic Y_A and surface F constructed in Explicit example 4.2.24. Applying a change of coordinates

$$(z_0, \ldots, z_5) \mapsto (z_2, 2z_2 - z_0, z_3, z_4, 2z_5 - z_1, z_5),$$

the surface F is defined by the hyperplane sections $z_0 = z_1 = 0$. Using Magma, we can check that the set of polynomials $\{z_i f_{5j} | i \in \{2, \ldots, 5\}, j \in \{0, 1\}\}$ are linearly independent in $(I/J)_6$, so by Lemma 4.2.18, all infinitesimal deformations of F in the fibre of Y_A are non-trivial. □

We obtain the following corollary from the characterization of the Hodge structure of EPW sextics.

Corollary 4.2.21. *Let $f : S \to F$ be the double cover of an even 40-nodal sextic surface, branched at the 40 nodes. Then, $H^2(S, \mathbb{Q})_-$ contains a simple sub-Hodge structure of type $(1, n, 1)$ where $n \leq 20$. In particular, $H^2(S, \mathbb{Q})_-$ does not contain a simple sub-Hodge structure of type $(1, 26, 1)$.*

Proof. By Theorem 4.2.20, a general 40-nodal sextic surface F is obtained from an EPW sextic Y_A, so Lefschetz's hyperplane theorem gives an inclusion of Hodge structures $H^2(X_A, \mathbb{Q})_- \hookrightarrow H^2(S, \mathbb{Q})_-$, so $H^2(S, \mathbb{Q})_-$ contains a simple sub-Hodge structure of type $(1, 20, 1)$. Thus, $\dim NS(S)_- = h^{1,1}(S) - 20 = 6$. Since the dimension of the Neron-Severi group in a family is upper semi-continuous, $\dim NS(S)_- \geq 6$ for any double cover S of a 40-nodal sextic surface. Hence, $H^2(S)_-$ contains a simple sub-Hodge structure of type $(1, n, 1)$ where $n \leq 20$. □

4.2.5 Involutions on certain EPW sextic surfaces

In this final subsection, we study involutions on a family of even 40-nodal sextic surfaces obtained using the EPW construction, and compute the induced decompositions of Hodge structures on these surfaces. These involutions arise as restrictions of symplectic involutions on EPW sextic fourfolds, first studied by Camere in [Cam12].

Recall that the construction of an EPW sextic fourfold requires the choice of a Lagrangian subspace $A \subset \Lambda^3 V$ where V is a 6-dimensional vector space equipped with a symplectic form. Let ι be an involution of V. This induces a decomposition of V into $(+1)$- and (-1)-eigenspaces V_+ and V_-. Assume that $\dim V_+ = 4$ and $A \subset \Lambda^3 V$ is invariant under ι. Then, A decomposes into eigenspaces $A = A_+ \oplus A_-$. Let f_1, f_2 be a basis of V_-. Further assume that the eigenspaces of A take the precise forms

$$\begin{aligned} A_+ &= \big\{ f_1 \wedge f_2 \wedge v + \phi(v) \mid v \in V_+,\, \phi : V_+ \xrightarrow{\sim} \Lambda^3 V_+ \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{s.t. } v \wedge \phi(w) = w \wedge \phi(v)\ \forall\, v, w \big\} \\ A_- &= \big\{ f_1 \wedge x + f_2 \wedge u(x) \mid x \in \Lambda^2 V_+,\, u : \Lambda^2 V_+ \to \Lambda^2 V_+ \text{ is self-adjoint.} \big\} \end{aligned}$$

In [Cam12, Section 8], Camere showed that such an EPW sextic Y_A admits a symplectic involution $\bar{\iota}$ which lifts to a symplectic involution ι_0 on X_A.

The fixed locus of the involution $\bar{\iota}$ on Y_A is the union of $Y_A \cap \mathbb{P}(V_+)$ and $Y_A \cap \mathbb{P}(V_-)$. By [Cam12, Proposition 17], the first is the union of a smooth quadric surface and a quartic Kummer surface, while the latter is a set of 6 isolated points. In [Cam12, Proposition 19], Camere showed that the induced symplectic involution ι_0 on X_A fixes a K3 surface, which is the preimage of the quadric surface under f_A, and a set of 28 isolated points. 12 of the points are the preimages of the 6 isolated points while the remaining 16 are preimages of the nodes of the Kummer surface, which are ramified under $f_A : X_A \to Y_A$.

There is another anti-symplectic involution $\iota_1 = \iota_A \iota_0$ on X_A. Camere showed that the branch locus of f_A intersects the fixed locus of the involution $\bar{\iota}$ along an octic curve which is the intersection of the quadric and the Kummer surfaces and at the 16 nodes of the Kummer surface. Hence, ι_1 fixes precisely the preimage of the Kummer surface.

Recall that a 40-nodal surface F can be obtained by taking the intersection of a general $\mathbb{P}^3 \subset \mathbb{P}(V)$ with Y_A. By our choice of A, the fourfold Y_A is invariant under the action of ι on $\mathbb{P}(V)$. Let $W \subset V$ be a 4-dimensional sub-vector space. $\mathbb{P}(W)$ is invariant under the action of ι if and only if $W = W_+ \oplus W_-$ where $W_+ \subset V_+$ and $W_- \subset V_-$.

In this section, we shall only consider W such that $\dim (W_+, W_-) = (3, 1)$. Let $F = \mathbb{P}(W) \cap Y_A$ and $S = f_A^{-1} F$. Let $q : F \to Z$ and $p_i : S \to V_i$ $(i = 1, 2)$

be the quotient of F and S by $\bar{\iota}$ and ι_i respectively. We obtain a commutative diagram

$$\begin{tikzcd}
S \arrow[rr, "p_0"] \arrow[dr, "p_1"] \arrow[dd, "f"'] & & V_0 \arrow[dd, "g_0"] \\
& V_1 \arrow[dr, "g_1"] & \\
F \arrow[rr, "q"'] & & Z
\end{tikzcd}$$

where $S \to Z$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover.

By our assumption that $\dim W \cap V_+ = 3$, the intersection $\mathbb{P}(W) \cap \mathbb{P}(V_+)$ is a plane in $\mathbb{P}(V)$, hence for a general choice of W, the fixed locus of $\bar{\iota}$ on F is the union of a smooth quadric curve $\overline{C}_0$ and a smooth quartic curve $\overline{C}_1$ intersecting at 8 points (it does not pass through the 6 isolated fixed points and the 16 nodes of the Kummer surface). The 8 intersection points lie on the branch locus of f_A, so they are nodes on F. The curves $\overline{C}_0$ and $\overline{C}_1$ do not contain any other nodes of F.

The induced involution ι_0 on S fixes the curve C_0 and defines the quotient map $p_0 : S \to V_0 = S/\langle \hat{\imath} \rangle$ while $\iota_1 = \iota_A \iota_0$ fixes the curve C_1 and defines the quotient map p_1.

The surface Z is singular with 16 nodes, whose preimages are the 32 nodes of F not contained in $\overline{C}_0 \cup \overline{C}_1$. The 8 nodes in $\overline{C}_0 \cap \overline{C}_1$ map to 8 smooth points, which form the intersection of $\overline{C}_0$ and $\overline{C}_1$ on Z. The maps g_i are ramified at the union of the 16 nodes and $\overline{C}_{1-i}$.

We can obtain the Hodge numbers of the quotients of S by a rather long and uninspiring computation which we shall omit.

Proposition 4.2.22. *The Hodge numbers of V_0, V_1 and Z are*

	$h^{1,0}$	$h^{2,0}$	$h^{1,1}$
V_0	0	4	36
V_1	0	3	32
Z	0	3	21

It is more instructive to write the Hodge numbers in terms of the eigenspace decomposition of $H^{p,q}(S)$:

$$h^{2,0}: \quad \begin{array}{r|rr} 11 & 4 & 7 \\ \hline 10 & 3 & 7 \\ 1 & 1 & 0 \end{array} \qquad\qquad h^{1,1}: \quad \begin{array}{r|rr} 72 & 36 & 36 \\ \hline 46 & 21 & 25 \\ 26 & 15 & 11 \end{array}$$

The top left corner gives $h^{p,q}(S)$ and the columns and rows are decompositions of $H^{p,q}(S)$ with respect to ι_0 and ι_A respectively.

To end off the section, we will give an example of a 40-nodal sextic surface containing 4 such involutions.

Example 4.2.23. First, we describe the construction for EPW sextics. Let V, $A \subset \Lambda^3 V$ and $\lambda_A : E \to (\Lambda^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)}$ be defined as in Subsection 4.2.4. For any point $v \in V$, it is easy to see that

$$\det \lambda_{A,v} = 0 \quad \Longleftrightarrow \quad (v \wedge \Lambda^2 V) \cap A \neq 0 \quad \Longleftrightarrow \quad \dim (v \wedge A) < \dim A.$$

Let $v_0, \ldots, v_5$ be a basis for V, then $v_{ijk} = v_i \wedge v_j \wedge v_k$ $(i < j < k)$ is a basis for $\Lambda^3 V$. We can write a basis for A as

$$\left\{ a_m = \sum_{0 \leq i < j < k \leq 5} \alpha_{ijk,m} v_{ijk} \right\}_{1 \leq m \leq 10}.$$

Let $v = (z_0, \ldots, z_5)$. In the basis $v_{ijkl} = v_i \wedge v_j \wedge v_k \wedge v_l$ of $\Lambda^4 V$, the image of $v \wedge A$ is given by the 15×10 matrix of linear polynomials

$$M(z) = (z_i \alpha_{jkl,m} - z_j \alpha_{ikl,m} + z_k \alpha_{ijl,m} - z_l \alpha_{ijk,m})_{0 \leq i < j < k < l \leq 5,\, m}.$$

The locus of $v \in V$ where $\dim (v \wedge A) < \dim A$ is precisely the locus on $\mathbb{P}(V)$ where the rank of $M(z)$ is less than 10. Eisenbud, Popescu and Welter [EPW01] showed that, for a general choice of A, the greatest common divisor of all the 10×10 minors of $M(z)$ is an irreducible sextic polynomial defining an EPW sextic.

Explicit example 4.2.24. In our example, we take the Lagrangian subspace

$$\begin{aligned} A = \langle & v_{014} + 3v_{235},\ 3v_{134} + v_{025},\ v_{234} + 3v_{015},\ 3v_{012} - v_{345},\ v_{013} - 3v_{245}, \\ & 3v_{023} - v_{145},\ v_{024} - 3v_{135},\ 3v_{034} - v_{125},\ v_{123} - 3v_{045},\ 3v_{124} - v_{035} \rangle. \end{aligned}$$

Using Magma [BCP97], we find the EPW sextic fourfold Y_A using the above algorithm. Consider the surface F defined by intersecting Y_A with the hyperplanes defined by $z_1 = 2z_0$ and $z_5 = 2z_4$. The surface F is given by the sextic polynomial

$$\begin{aligned} f_6 = {} & 23625 z_0^6 - 47925 z_0^4 z_1^2 + 48075 z_0^4 z_2^2 + 144375 z_0^4 z_3^2 + 9855 z_0^2 z_1^4 \\ & - 5790 z_0^2 z_1^2 z_2^2 - 292950 z_0^2 z_1^2 z_3^2 - 440000 z_0^2 z_1 z_2 z_3^2 + 9855 z_0^2 z_2^4 \\ & - 484950 z_0^2 z_2^2 z_3^2 + 144375 z_0^2 z_3^4 - 243 z_1^6 + 1971 z_1^4 z_2^2 + 9855 z_1^4 z_3^2 \\ & + 1971 z_1^2 z_2^4 - 5790 z_1^2 z_2^2 z_3^2 - 47925 z_1^2 z_3^4 - 243 z_2^6 + 9855 z_2^4 z_3^2 \\ & + 48075 z_2^2 z_3^4 + 23625 z_3^6. \end{aligned}$$

We checked that the surface F is invariant under the four involutions given by

$$(z_0, z_1, z_2, z_3) \mapsto \begin{cases} (-z_0, z_1, z_2, z_3) \\ (z_0, z_1, z_2, -z_3) \\ (z_3, z_1, z_2, z_0) \\ (z_3, z_1, z_2, -z_0). \end{cases}$$

As in the discussion above, each such involution fixes a reducible sextic curve which is the union of a smooth conic and a smooth quartic curve, which intersect at 8 nodes of F. We verified that the sets of nodes fixed by the involutions are pairwise disjoint. An interesting observation is that the remaining set of 8 nodes also lies on a plane, defined by $12z_1 + 41z_2 = 0$. The intersection of F with this plane is also the union of a smooth conic and a smooth quartic curve. However, there is no involution of F around this plane.

We computed the associated Kummer surface K and the curve C such that $\mathrm{div}_F(K) = 2C$. It is interesting to note that, in this example, the curve C is the union of 5 irreducible components, namely the smooth quartic curve in the plane $12z_1 + 41z_2 = 0$ and the 4 conics fixed by each of the 4 involutions.

This example was used in the proof of Theorem 4.2.20. We showed that all infinitesimal deformations of this particular surface F fixing Y_A are non-trivial and used it to deduce that all small deformations of F are non-trivial.

Chapter 5

Generalizations using mixed Hodge modules

In this chapter, we give the first step in generalizing the Hodge theoretical results in Chapter 2 to more general singular varieties. To do so, we make use of mixed Hodge modules, a technique developed by Morihiko Saito to generalize variations of Hodge structures to singular varieties. We will give a very brief introduction to mixed Hodge modules in the first two sections. In the third section, we will apply these theories and demonstrate how they give a coherent picture for understanding the behaviour of singularities on algebraic varieties.

5.1 Perverse sheaves

We will begin by studying the underlying topological obstructions to variations of Hodge structures. This is best presented in the theory of perverse sheaves, which can be seen as an extension of local systems to singular varieties. Perverse sheaves were first introduced by Beilinson, Bernstein and Deligne in [BBD82]. There are many other excellent exposés on perverse sheaves, eg. [CM09; Moz08]. Over here, we shall just describe some basic properties and let interested readers refer to the earlier texts for further details.

Let X be a complex algebraic variety of dimension n. A stratification $\mathcal{S}$ of X is a sequence of Zariski-open subsets

$$\emptyset = U_{n+1} \subset U_n \subset \cdots \subset U_1 \subset U_0 = X$$

such that $S_i = U_i \setminus U_{i+1}$ is either smooth of dimension i or empty. A constructible sheaf on X is a sheaf $\mathcal{F}$ of k-vector spaces ($k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) such that there exists a stratification $\mathcal{S}$ where $\mathcal{F}|_{S_i}$ is locally constant for each i.

Let $\mathbf{Mod}_c k_X$ be the category of constructible sheaves and denote by $D^b_c(k_X)$ the bounded derived category of complexes with constructible cohomology, i.e. for each $\mathcal{F}^\bullet \in D^b_c(k_X)$, the sheaf $\mathcal{H}^i(\mathcal{F}^\bullet)$ is constructible for each i and zero for $|i|$ sufficiently large.

Let $f : X \to Y$ be a morphism of varieties. We can define six basic geometric functors (known as Grothendieck's six functors) between the derived categories:

$$\begin{array}{rl} f^* : D^b_c(k_Y) \to D^b_c(k_X) & f_* : D^b_c(k_X) \to D^b_c(k_Y) \\ f_! : D^b_c(k_X) \to D^b_c(k_Y) & f^! : D^b_c(k_Y) \to D^b_c(k_X) \\ -\otimes- : D^b_c(k_X) \times D^b_c(k_X) \to D^b_c(k_X) & \mathcal{H}\mathrm{om}(-,-) : D^b_c(k_X) \times D^b_c(k_X) \to D^b_c(k_X) \end{array}$$

These are simply the derived versions of our usual functors: $f^* = f^{-1}$, $f_* = Rf_*$, $f_! = Rf_!$, $\otimes = \otimes^L$ and $\mathcal{H}\mathrm{om} = R\mathcal{H}\mathrm{om}$. To simplify notations, in this chapter, we shall denote all derived functors as above and write, for example, $R^0 f_*$ or $\mathcal{H}^0(f_*)$ for the usual underived functors when needed.

The six functors form adjoint pairs (f^*, f_*), $(f_!, f^!)$ and $(- \otimes \mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{F}, -))$ for any $\mathcal{F} \in D^b_c(k_X)$. There is also a notion of duality, called the Verdier duality. The duality functor

$$\mathbb{D} : D^b_c(k_X) \to D^b_c(k_X)$$

satisfies the identities $\mathbb{D} f_* \mathbb{D} = f_!$ and $\mathbb{D} f^* \mathbb{D} = f^!$. The dualizing sheaf (or complex) is the object $IC_X = a^!_X k \in D^b_c(k_X)$ where $a_X : X \to \mathrm{Spec}\ \mathbb{C}$ is the structure morphism, and we have the identity $\mathbb{D}\mathcal{F} = \mathcal{H}\mathrm{om}(\mathcal{F}, IC_X)$.

If f is proper, then $f_* = f_!$. If f is finite étale, then $f^* = f^!$.

From another perspective, we may start with the triangulated category $D^b_c(k_X)$. On a triangulated category, there is a notion of t-structures (see [BBD82] or any other text on triangulated categories) which is a $\mathbb{Z}$-partitioning of the category. The 0-th partition is called the heart of the t-structure. The heart is a sub-abelian category of the triangulated category and each other partition is a translation of the heart by the shift functor $[i]$. For the derived category $D^b(A)$ of an abelian category A, there is a standard t-structure where the heart is the abelian category A viewed as complexes concentrated in the 0-th degree while the i-th partition is the category of complexes concentrated in the i-th degree.

However, we can also choose a different t-structure on $D^b_c(k_X)$, one of which is the perverse t-structure. It is possible to explicitly define this t-structure, but we shall only describe its heart. A perverse sheaf K is an object in $D^b_c(k_X)$, a complex concentrated in degrees $-n$ to 0, such that $\dim \mathrm{Supp}(\mathcal{H}^{-i}(\mathcal{P})) \leq i$ and $\dim \mathrm{Supp}(\mathcal{H}^{-i}(\mathbb{D}\mathcal{P})) \leq i$. Note that a perverse sheaf is a complex of sheaves rather than a sheaf. The category $\mathrm{Perv}(k_X)$ of perverse sheaves is abelian and is the heart of the perverse t-structure on $D^b_c(k_X)$.

Rather than trying to understand the definition of perverse sheaves, we shall give a few examples. Most importantly, the dualizing complex IC_X is perverse. When X is smooth, we have $IC_X = k_X[n]$ and Verdier duality is just the

Poincaré duality. More generally, given a local system L on U_n, there exists a unique simple perverse sheaf $IC_X(L)$ such that $IC_X(L)|_{U_n} = L$. This is called the intersection complex of L. We have $IC_X = IC_X(k_X)$.

We say that a functor F is left (or right, respectively) t-exact if $\mathcal{H}^0(F)$ is left (right, resp.) exact on the abelian heart of the t-structure. On the standard t-structure, f_*, $f_!$ and $\mathcal{H}om$ are left t-exact, $\otimes$ is right t-exact and f^* is t-exact. It is different with respect to the perverse t-structure. We have the following proposition:

Proposition 5.1.1 ([Moz08, Prop 3.29]). *Let $f : X \to Y$ be an algebraic morphism. Then, with respect to the perverse t-structure,*

1. *If f is quasi-finite, then $f_!$ and f^* are right t-exact while $f^!$ and f_* are left t-exact.*
2. *if f is affine, then f_* is right t-exact while $f_!$ is left t-exact.*
3. *If f is finite, then $f_* = f_!$ are t-exact.*

Let ${}^p\mathcal{H}^i(\mathcal{F})$ be the i-th cohomology of a complex $\mathcal{F} \in D^b_c(k_X)$ with respect to the perverse t-structure. A key result of Beilinson, Bernstein and Deligne [BBD82, Théorème 6.2.10] can be restated using [Del68, Théorème 1.5]

Theorem 5.1.2. *Let $f : X \to Y$ be a proper morphism of algebraic varieties, and $\mathcal{F} \in D^b_c(k_X)$ be of geometric origin. Then, there exists a non-canonical quasi-isomorphism*

$$f_*\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(f_*\mathcal{F})[-i].$$

In later works by Saito [Sai88] and de Cataldo-Migliorini [CM05], they showed, using different methods, that for $\mathcal{F} = IC_X$ the quasi-isomorphism above can be chosen to compatible with Hodge theory. In the next section, we will present Saito's enhancement of this theorem.

Let $Z \subset X$ be a Zariski-closed subset and consider the morphisms

$$Z \xrightarrow{i} X \xleftarrow{j} U = X \setminus Z.$$

We have that $i_* = i_!$ and $j^* = j^!$ and the adjunctions induce natural morphisms

$$j_!j^! \to \mathrm{id} \to i_*i^*, \qquad i_!i^! \to \mathrm{id} \to j_*j^*.$$

It is a theorem that these two sequences are distinguished triangles in $D^b_c(k_X)$.

5.2 Mixed Hodge Modules

Mixed Hodge modules were first introduced by Morihiko Saito [Sai88] as a generalization of the variation of Hodge structures to singular varieties. The definition of mixed Hodge modules is extremely technical and beyond the scope of this thesis. We will instead focus on the important properties of mixed Hodge modules and specific examples. Interested readers may refer to Saito's original papers [Sai88; Sai90] or Schnell's excellent exposé [Sch14].

Let X be a smooth algebraic variety of dimension n. Consider the data of a quadruple $(\mathcal{M}, F, W, K)$ where $\mathcal{M}$ is a regular holonomic right D_X-module (see, for example, [HTT08] for the definition), F and W are filtrations on $\mathcal{M}$ (the Hodge and weight filtrations respectively) and $K \in \mathrm{Perv}(\mathbb{Q}_X)$ is a $\mathbb{Q}$-perverse sheaf with a fixed isomorphism $K \otimes_{\mathbb{Q}} \mathbb{C} \cong DR(\mathcal{M})$. Here, DR is the de Rham functor defined by

$$DR : D^b(D_X) \to D^b_c(\mathbb{C}_X) \;:\; \mathcal{M} \mapsto (\mathcal{M} \otimes \wedge^n T_X \to \cdots \to \mathcal{M} \otimes T_X \to \mathcal{M})[n]$$

where $D^b(D_X)$ is the derived category of D_X-modules and the map sends $\mathcal{M} \in \mathbf{Mod}(D_X)$ to a complex lying in degrees $-n$ to 0.

Remark 5.2.1. Note that in this thesis, we adopt the convention of Saito and Schnell and define Hodge modules using right D_X-modules (cf. [Sch14, Section A.3] for a discussion of the difference). Other texts such as [PS08, Chapters 13 and 14] use left D_X-modules. The two conventions are equivalent, but lead to different notations (cf. [PS08, Section 13.3.2]). For example, over a smooth projective variety X, the equivalence sends the left D_X-module $\mathcal{O}_X$ to the right D_X-module ω_X. Using left D_X-modules, the de Rham functor is given by the usual de Rham complex.

A mixed Hodge module is such a quadruple $(\mathcal{M}, F, W, K)$ where $\mathcal{M}$ satisfying certain conditions (the conditions imposed ensure that they are of "geometric" origin). Denote the abelian category of mixed Hodge modules by $MHM(X)$ and its bounded derived category by $D^b(MHM(X))$.

In this chapter, almost all mixed Hodge modules we consider are actually pure, that is, the weight filtration is trivial. Hence, we shall forget the weight filtration W. For examples with standard choices of Hodge filtration, we will often denote the Hodge module $(\mathcal{M}, F, W, K)$ simply by K^H. If we are not concerned about the rational Hodge structure, we may simply denote a Hodge module by $K^H_{\mathbb{C}}$.

Note that F also induces a filtration on $DR\mathcal{M}$ given by

$$F_p DR(\mathcal{M}) = (F_{p-n}\mathcal{M} \otimes \wedge^n T_X \to \cdots \to F_{p-1}\mathcal{M} \otimes T_X \to F_p\mathcal{M})[n].$$

When we talk about the Hodge filtration on $K^H_{\mathbb{C}}$, we refer to the induced Hodge filtration on $DR\mathcal{M} = K_{\mathbb{C}}$.

Example 5.2.2. Let X be a smooth projective algebraic variety of dimension n. The "structure" Hodge module is $\mathbb{Q}_X[n]^H = (\omega_X, F_\bullet, \mathbb{Q}_X[n])$ where the Hodge filtration on ω_X is defined by

$$F_p\omega_X = \begin{cases} \omega_X, & p \geq -n; \\ 0, & p < -n. \end{cases}$$

We see that the induced filtration on $DR\omega_X = \mathbb{C}_X$ is the stupid filtration

$$F_p\mathbb{C}_X = (0 \to \cdots \to 0 \to \Omega^p_X \to \cdots \to \Omega^{n-1}_X \to \omega_X)[n]$$

and classical Hodge theory shows that $\mathrm{Gr}^F_p\mathbb{C}_X = \Omega^{p+n}_X$.

The forgetful functor

$$\mathrm{rat} : D^b(MHM(X)) \to D^b_c(\mathbb{Q}_X) \ : \ (\mathcal{M}, F, W, K) \mapsto K$$

is exact since the de Rham functor is. The Hodge filtration is strict [Sai16, (2.3.3)], that is,

$$\mathrm{Gr}^F_\cdot : D^b(MHM(X)) \to D^b(\mathcal{O}_X)$$

is an exact functor, where $D^b(\mathcal{O}_X)$ is the derived category of $\mathcal{O}_X$-modules on X. Hence, given any distinguished triangle

$$A \to B \to C \xrightarrow{+1}$$

in $D^b(MHM(X))$, we have a distinguished triangle of D-modules

$$\mathrm{Gr}^F_p A \to \mathrm{Gr}^F_p B \to \mathrm{Gr}^F_p C \xrightarrow{+1} \qquad \forall\, p \in \mathbb{Z}.$$

Saito also defined the six functors $f_*, f^*, f_!, f^!, \otimes, \mathcal{H}\mathrm{om}$ on mixed Hodge modules and showed that they commute with the forgetful functor rat [Sai90]. Note that Gr^F does not commute with the six functors in general. If $f : X \to Y$ is a projective morphism, then Gr^F commutes with the direct image functor $f_* = f_!$ up to a shift, more precisely $\mathrm{Gr}^F_p f_* = f_*\mathrm{Gr}^F_p[\dim X - \dim Y]$ (see the definition of f_* in [Sai90, p. 2.13]).

There exists a Hodge module enhancement for most geometric results on perverse sheaves.

Theorem 5.2.3 ([Sai88, Théorème 1])**.** *Let $f : X \to Y$ be a proper algebraic morphism. Then, the functor $f_* : D^b(MHM(X)) \to D^b(MHM(Y))$ is strict. Let $(\mathcal{M}, F, W, K) \in D^b(MHM(X))$ be any mixed Hodge module of geometric origin. There is an isomorphism*

$$f_*(\mathcal{M}, F, W, K) \cong \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(f_*(\mathcal{M}, F, W, K))[-i].$$

We now give a few more examples of Hodge modules.

Example 5.2.4. Let $D \subset X$ be a reduced closed subscheme and $j : U = X \setminus D \to X$ be the open immersion. Consider the sheaf

$$\omega_X(*D) = \bigcup_{k \geq 0} \omega_X(kD)$$

of meromorphic differential forms with arbitary poles on D. Then, its image under the de Rham functor is $DR(\omega_X(*D)) = j_*\mathbb{C}_U[n]$.

There is a Hodge module

$$(j_*\mathbb{Q}_U[n])^H = (\omega_X(*D), F_\bullet, j_*\mathbb{Q}_U[n])$$

where the Hodge filtration $F_\bullet$ is dependent on the singularities of D. The precise definition of the Hodge filtration is beyond the scope of this introduction (see, for example, [Sai07] or [MP16]), we shall just state some properties.

The Hodge filtration $F_{k-n}\omega_X(*D)$ is contained in the pole order filtration $P_{k-n}\omega_X(*D) = \omega_X((k+1)D)$, more precisely, there is an ideal $I_k(D) \subset \mathcal{O}_X$ such that $F_{k-n}\omega_X(*D) = \omega_X((k+1)D) \otimes I_k(D)$. These are called Hodge ideals, and are invariants of the types of singularities on D [MP16]. In general, the worse the singularities of D, the smaller the $I_k(D)$. If D is smooth, then $I_k(D) = \mathcal{O}_X$ for all k. Mustaţă and Popa showed that $I_0(D) = \mathcal{O}_X$ if and only if (X, D) is log-canonical [MP16, Corollary 10.3].

Saito ([Sai07, Theorem 1] or [MP16, Theorem 6.1]) gave another equivalent Hodge filtration on $(j_*\mathbb{Q}_U[n])^H$ which is often easier to work with. We may also take this as the definition. Let $\pi : (\tilde{X}, E) \to (X, D)$ be a log resolution, that is, $\pi : \tilde{X} \to X$ is a resolution such that $E = \pi^*D$ and E is a normal crossing divisor in $\tilde{X}$. There is an isomorphism of filtered complexes

$$\pi_*(\Omega^\bullet_{\tilde{X}}(\log E), F_\bullet) \cong DR(\mathcal{O}_X(*D), F_\bullet) \tag{5.1}$$

where the filtration on the left hand side is given by the stupid truncation

$$F_p(\Omega^\bullet_{\tilde{X}}(\log E)) = \\ (0 \to \cdots \to 0 \to \Omega^p_{\tilde{X}}(\log E) \to \Omega^{k+1}_{\tilde{X}}(\log E) \to \cdots \to \Omega^n_{\tilde{X}}(\log E))[n]$$

with $\mathrm{Gr}^F_p(j_*\mathbb{C}_U[n]) = \Omega^p_{\tilde{X}}(\log E)$.

Example 5.2.5. Let X, D and U be defined as in the previous example. Recall that Verdier duality gives an equivalence $\mathbb{D}j_! = j_*\mathbb{D}$. Verdier duality is compatible with the mixed Hodge module structures, so it induces a duality on the Hodge filtration

$$\mathrm{Gr}^F_{p-n}(j_!\mathbb{C}_U[n])^H \otimes \mathrm{Gr}^F_{-p}(j_*\mathbb{C}_U[n])^H \to \mathrm{Gr}^F_0 \mathbb{C}_X[n]^H.$$

From the previous examples, we have $\mathrm{Gr}^F_{-p}(j_*\mathbb{C}_U[n])^H = \pi_*\Omega^{n-p}_{\tilde{X}}(\log E)$ and $\mathrm{Gr}^F_0\mathbb{C}_X[n]^H = \omega_X$. There is a classical duality for Hodge structures on $(\tilde{X}, E)$ given by the perfect pairing

$$\Omega^p_{\tilde{X}}(\log E) \otimes \Omega^{n-p}_{\tilde{X}}(\log E) \to \omega_{\tilde{X}}(E).$$

Hence, we conclude that $\mathrm{Gr}^F_{p-n}(j_!\mathbb{C}_U[n])^H = \pi_*\Omega^p_{\tilde{X}}(\log E)(-E)$.

The reason for the convoluted definition of $(j_*\mathbb{Q}_U[n])^H$ is to ensure compatibilty with the Hodge modules on D.

So far, we have only discussed Hodge modules on smooth varieties. This is because D-modules are, a priori, only well-defined for smooth varieties. To extend the definition to a singular variety X, we can embed X as a closed subvariety of some smooth variety Y and define a D-module on X to be one on Y supported on X. We can define

$$D^b(MHM(X)) = D^b_X(MHM(Y))$$

where $D^b_X(MHM(Y))$ is the full subcategory of Hodge modules on Y supported on X. Saito showed that this definition is independent of the choice of embedding $X \subset Y$ [Sai90, p. 223].

For a variety X of pure dimension n embedded as a divisor in Y, the Hodge module IC^H_X is the cocone of

$$\mathbb{C}_Y[n+1]^H \to (j_*\mathbb{C}_U[n+1])^H$$

where $j : U = Y \setminus X \to Y$ is the open embedding. One can show that the Hodge filtration on IC^H_X gives precisely the weight k part of the mixed Hodge structure on $H^k(X, \mathbb{C})$ for each k. We define $\tilde{\Omega}^p_X := \mathrm{Gr}^F_{p-n} IC^H_X$.

Dually, the Hodge module $\mathbb{C}_X[n]^H$ is the cone of

$$(j_!\mathbb{C}_U[n+1])^H[-1] \to \mathbb{C}_Y[n+1]^H[-1]$$

and we define $\Omega^p_X := \mathrm{Gr}^F_{p-n}\mathbb{C}_X[n]^H$. In [du 81], du Bois defined a resolution $\Omega^\bullet_X$ of the constant sheaf $\mathbb{C}_X$ for any variety X. In Lemma 5.3.3, we will show that the complexes Ω^p_X obtained using mixed Hodge modules gives precisely the same resolution.

Remark 5.2.6. Note that in [Ste06], Steenbrink used the notation $\tilde{\Omega}^\bullet_X$ for du Bois' resolution. The author apologizes for the clash in notation. The Hodge structure of a V-manifold X is pure and we have $IC_X = \mathbb{C}_X[n]$, so $\tilde{\Omega}^\bullet_X = \Omega^\bullet_X$ and in that case, the notation is consistent.

More generally, the distinguished triangles

$$j_!j^! \to \mathrm{id} \to i_*i^* \xrightarrow{+1}, \qquad i_!i^! \to \mathrm{id} \to j_*j^* \xrightarrow{+1}$$

hold in $D^b(MHM(Y))$ as well [Sai90, p. 2.24].

5.3 Hodge theory of singular varieties

In general, the complex $\tilde{\Omega}_X^p$ is rather obscure and difficult to understand. In this section, we prove a few results that allow us to compute its cohomologies.

First, we prove that in the case of V-manifolds, the complex $\tilde{\Omega}_X^p$ we defined is indeed a sheaf and coincides with Steenbrink's definition (see Section 2.2).

Proposition 5.3.1. *Let $X = M/G$ where $M \subset \mathbb{C}^n$ is an open ball and $G \subset GL(n, \mathbb{C})$ is a small subgroup and let $f : M \to X$ be the quotient map. Then, there is an isomorphism $(f_*\Omega_M^p)^G = \tilde{\Omega}_X^p$. Hence, for a V-manifold X, the Hodge module definition of $\tilde{\Omega}_X^p$ coincides with Steenbrink's (Definition 2.1.2) by Theorem 2.2.5(i).*

Proof. Let $\Sigma \subset X$ be the singular locus. Since G is small, the singular locus coincides with the branch locus. Let $U = M \setminus f^{-1}\Sigma$ and $V = X \setminus \Sigma$, and consider the diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ V \arrow[r, "j"'] & X \end{tikzcd}$$

The morphism $f : U \to V$ is a finite étale map of smooth varieties, so there is a decomposition

$$f_*\mathbb{C}_U[n] = \bigoplus_{\chi \in G^*} L_\chi[n]$$

where G^* is the group of characters of G and L_χ are local systems on V. Note that if G is non-abelian, the rank of L_χ may be larger than one. Nevertheless, the trivial character gives a trivial local system $(f_*\mathbb{C}_U[n])^G = L_1[n] = \mathbb{C}_V[n]$.

By Theorem 5.2.3, the eigenspace decomposition of $f_*\mathbb{C}_U[n]$ lifts to a decomposition on the level of Hodge modules, so $(f_*\mathbb{C}_U[n]^H)^G = \mathbb{C}_V[n]^H$.

Note that $\tilde{j}_{!*}\mathbb{C}_U[n]^H = \mathbb{C}_M[n]^H$ and $\tilde{j}_{!*}\mathbb{C}_V[n]^H = IC_X^H$. Since f is proper, we have $f_* = f_! = f_{!*}$. Hence, we obtain

$$(f_*\mathbb{C}_M[n]^H)^G = (f_*\tilde{j}_{!*}\mathbb{C}_U[n]^H)^G = j_{!*}(f_*\mathbb{C}_U[n]^H)^G = j_{!*}\mathbb{C}_V[n]^H = IC_X^H.$$

Taking the $(p-n)$-th graded component of the Hodge filtration and noting that f_* commutes with Gr_p^F, we get $(f_*\Omega_M^p)^G = \tilde{\Omega}_X^p$. $\square$

Next, we show how we can, in some cases, compute the sheaves $\tilde{\Omega}_X^p$ in terms of a desingularization of X.

Lemma 5.3.2. *Let X be a projective variety of dimension n and let $\Sigma \subset X$ be the singular locus of X. Suppose $\mathrm{codim}_X \Sigma = d$. Let $\pi : (\tilde{X}, E) \to (X, \Sigma)$ be a log-resolution of (X, Σ). Then,*

- *(i) there are isomorphisms $\tilde{\Omega}_X^p = \pi_*\Omega_{\tilde{X}}^p(\log E)$ for $p \leq d$;*
- *(ii) for any $0 \leq p \leq n$, the cohomology of the complex $\tilde{\Omega}_X^p$ vanishes in negative degrees, i.e. $\mathcal{H}^i(\tilde{\Omega}_X^p) = 0$ for all $i < 0$;*
- *(iii) for any $0 \leq p \leq n$, the map $\mathcal{H}^0(\tilde{\Omega}_X^p) \to R^0\pi_*\Omega_{\tilde{X}}^p(\log E)$ is injective, and it is an isomorphism if $d \geq 2$;*
- *(iv) there are isomorphisms $\tilde{\Omega}_X^p = \pi_*\Omega_{\tilde{X}}^p(\log E)(-E)$ for $p \geq n-d$.*

Proof. There is a diagram of morphisms

$$\begin{tikzcd} \tilde{U} = \tilde{X} \setminus E \arrow[r, "\tilde{j}"] \arrow[d, "\pi", "\wr"'] & \tilde{X} \arrow[d, "\pi"] & E \arrow[l, "\tilde{i}"'] \arrow[d, "\pi"] \\ U = X \setminus \Sigma \arrow[r, "j"'] & X & \Sigma \arrow[l, "i"] \end{tikzcd} \tag{5.2}$$

We consider the distinguished triangle

$$i_! i^! IC_X^H = i_* IC_\Sigma^H \to IC_X^H \to j_*j^* IC_X^H = (j_*\mathbb{C}_U[n])^H \xrightarrow{+1}.$$

We take the $(p-n)$-th graded component of the aboved distinguished triangle. Note that $\mathrm{Gr}_{p-n}^F i_* IC_\Sigma^H = i_*\mathrm{Gr}_{(p-d)-(n-d)}^F IC_\Sigma^H[-d] = \tilde{\Omega}_\Sigma^{p-d}[-d]$. We can also write $\mathrm{Gr}_{p-n}^F(j_*\mathbb{C}_U[n])^H$ in terms of the log sheaf on $(\tilde{X}, E)$ using (5.1). Thus, we get a distiguished triangle

$$i_*\tilde{\Omega}_\Sigma^{p-d}[-d] \to \tilde{\Omega}_X^p \to \pi_*\Omega_{\tilde{X}}^p(\log E) \xrightarrow{+1}. \tag{5.3}$$

(i) If $p < d$, then $\tilde{\Omega}_\Sigma^{p-d} = 0$, giving the isomorphism $\tilde{\Omega}_X^p = \pi_*\tilde{\Omega}_{\tilde{X}}^p(\log E)$.

(ii) We prove by induction on the dimension of X. Suppose it is true for all varieties of dimension $k \leq n-1$, in particular, it is true on Σ. So, taking the cohomologies of the distinguished triangle (5.3) gives exact sequences

$$\mathcal{H}^{i-d}(\tilde{\Omega}_\Sigma^{p-d}) \to \mathcal{H}^i(\tilde{\Omega}_X^p) \to \mathcal{H}^i(\pi_*\Omega_{\tilde{X}}^p(\log E)).$$

The first and last terms are zero when $i < 0$ (by induction hypothesis and since π_* is left exact), so $\mathcal{H}^i(\tilde{\Omega}^p_X) = 0$ for all $i < 0$.

(iii) follows from taking the cohomologies of the distinguished triangle (5.3) in degree 0 and using (ii).

(iv) Consider the dual sequence

$$j_!j^!IC^H_X = (j_!\mathbb{C}_U[n])^H \to IC^H_X \to i_*i^*IC^H_X \xrightarrow{+1} .$$

Taking the $(p-n)$-th graded component gives the distinguished triangle

$$\pi_*\Omega^p_{\tilde{X}}(\log E)(-E) \to \tilde{\Omega}^p_X \to$$
$$\to \mathrm{Gr}^F_{p-n}i_*i^*IC^H_X = i_*\mathrm{Gr}^F_{p-n}i^*IC_X[-d]^H = i_*\mathrm{Gr}^F_{p-n+d}i^*IC^H_X[-d] \xrightarrow{+1} .$$

Since i^* is right exact, we have ${}^p\mathcal{H}^{>0}(i^*IC_X) = 0$, so $\mathrm{Gr}^F_{p-n+d}i^*IC^H_X = 0$ whenever $p > n-d$. This gives us the required isomorphism. □

Lemma 5.3.3. Ω^p_X *are precisely the sheaves defined by du Bois in [du 81].*

Proof. Using the same setup as in the diagram (5.2), we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
(j_!\mathbb{C}_U[n])^H \arrow[r] \arrow[d, equal] & \mathbb{C}_X[n]^H \arrow[r] \arrow[d] & i_*\mathbb{C}_\Sigma[n]^H \arrow[r, "+1"] \arrow[d] & {} \\
\pi_*(j_!\mathbb{C}_{\tilde{U}}[n])^H \arrow[r] & \pi_*\mathbb{C}_{\tilde{X}}[n]^H \arrow[r] & \pi_*i_*\mathbb{C}_E[n]^H \arrow[r, "+1"] & {} .
\end{tikzcd}$$

Since the first terms are isomorphic, we obtain a new distinguished triangle

$$\mathbb{C}_X[n]^H \to \mathbb{C}_\Sigma[n]^H \oplus \pi_*\mathbb{C}_{\tilde{X}}[n]^H \to \pi_*i_*\mathbb{C}_E[n]^H \xrightarrow{+1} .$$

Taking the $(p-n)$-th graded component of the Hodge filtration gives a distinguished triangle

$$\Omega^p_X \to \Omega^p_\Sigma \bigoplus \pi_*\Omega^p_{\tilde{X}} \to \pi_*\Omega^p_E \xrightarrow{+1} .$$

This is precisely the characterization given by du Bois in [du 81, Proposition 3.9]. □

We shall now prove a generalization of Theorem 2.2.14.

Proposition 5.3.4. *Let Y be a smooth algebraic variety of dimension n and $X \subset Y$ be a divisor. Then, there are short exact sequences*

$$0 \to \Omega^p_Y \to \Omega^p_Y(\log X) \to \mathcal{H}^0(\tilde{\Omega}^{p-1}_X) \to 0$$

for $1 \le p \le n$.

Proof. Let $\pi : (\tilde{Y}, \tilde{X} + E) \to (Y, X)$ be a log-resolution of (Y, X) such that $\tilde{X}$ is the strict transform of X and $E = \pi^{-1}\Sigma$ is the total transform of the singular locus Σ of X in Y. We have a diagram

$$\begin{tikzcd}
\tilde{U} = \tilde{Y} \setminus (\tilde{X} \cup E) \arrow[r, "\tilde{j}"] \arrow[d, "\pi", "\wr"'] & \tilde{Y} \arrow[d, "\pi"] & \tilde{X} \cup E \arrow[l, "\tilde{i}"'] \arrow[d, "\pi"] \\
U = Y \setminus X \arrow[r, "j"'] & Y & X \arrow[l, "i"]
\end{tikzcd}$$

Consider the distinguished triangle

$$i_! i^! I\mathbb{C}_Y[n]^H = i_* IC_X^H \to \mathbb{C}_Y[n]^H \to j_* j^* \mathbb{C}_Y[n]^H = (j_* \mathbb{C}_U[n])^H \xrightarrow{+1} .$$

Similar to the proof of Lemma 5.3.2, the $(p-n)$-th graded component of the above distinguished triangle gives another distinguished triangle

$$i_* \tilde{\Omega}_X^{p-1}[-1] \to \Omega_Y^p \to \pi_* \Omega_{\tilde{Y}}^p(\log(\tilde{X} + E)) \xrightarrow{+1} .$$

Taking the cohomology gives an exact sequence

$$\mathcal{H}^{-1}(\tilde{\Omega}_X^{p-1}) = 0 \to \Omega_Y^p \to R^0\pi_* \Omega_{\tilde{Y}}^p(\log(\tilde{X} + E)) \to \mathcal{H}^0(\tilde{\Omega}_X^{p-1}) \to 0.$$

It remains to show that $\Omega_Y^p(\log X) = R^0\pi_*\Omega_{\tilde{Y}}^p(\log(\tilde{X} + E))$.

There is a resolution of $\Omega_{\tilde{Y}}^p(\log(\tilde{X} + E))$ as

$$0 \to \Omega_{\tilde{Y}}^p(\tilde{X} + E) \xrightarrow{d} \Omega_{\tilde{Y}}^{p+1}(2(\tilde{X} + E))/\Omega_{\tilde{Y}}^{p+1}(\tilde{X} + E) \to 0.$$

By Lemma 2.2.20, $R^0\pi_*\Omega_{\tilde{Y}}^p = \Omega_Y^p$. Since $\pi^*\mathcal{O}_Y(X) = \mathcal{O}_{\tilde{Y}}(\tilde{X} + E)$, by the projection formula, we get $R^0\pi_*\Omega_{\tilde{Y}}^p(k(\tilde{X} + E)) = \Omega_{\tilde{Y}}^p(kX)$. Hence,

$$\begin{aligned}
&R^0\pi_*\Omega_{\tilde{Y}}^p(\log(\tilde{X} + E)) \\
&= \ker\left(R^0\pi_*\Omega_{\tilde{Y}}^p(\tilde{X} + E) \xrightarrow{d} R^0\pi_*(\Omega_{\tilde{Y}}^{p+1}(2(\tilde{X} + E))/\Omega_{\tilde{Y}}^{p+1}(\tilde{X} + E))\right) \\
&= \ker\left(\Omega_Y^p(X) \to \Omega_Y^{p+1}(2X)/\Omega_Y^{p+1}(X)\right) \\
&= \Omega_Y^p(\log X).
\end{aligned}$$

□

Proposition 5.3.5. *Let Y be a smooth algebraic variety of dimension n and $X \subset Y$ be a divisor. Then, there are exact sequences*

$$0 \to \Omega_Y^p(\log X)(-X) \to \Omega_Y^p \to \Omega_X^p$$

for $0 \le p \le n-1$.

Proof. Let $\tilde{Y}$, $\tilde{X}$, E and all the morphisms be as defined in the proof of Proposition 5.3.4.

Consider the distinguished triangle

$$j_! j^! \mathbb{C}_Y[n]^H = (j_! \mathbb{C}_U[n])^H = \pi_*(\tilde{j}_! \mathbb{C}_{\tilde{U}}[n])^H \to \mathbb{C}_Y[n]^H \to \\ \to i_* i^* \mathbb{C}_Y[n]^H = i_* \mathbb{C}_X[n]^H \xrightarrow{+1} . \tag{5.4}$$

The $(p-n)$-th graded component of the Hodge filtration on the distinguished triangle 5.4 gives

$$\pi_* \Omega^p_{\tilde{Y}}(\log(\tilde{X}+E))(-\tilde{X}-E) \to \Omega^p_Y \to \mathrm{Gr}^F_{p-n} i_* \mathbb{C}_X[n]^H \xrightarrow{+1} . \tag{5.5}$$

The last term is isomorphic to

$$\mathrm{Gr}^F_{p-n} i_* \mathbb{C}_X[n]^H = i_* \mathrm{Gr}^F_{p-n} \mathbb{C}_X[n]^H[-1] = i_* \mathrm{Gr}^F_{p-n+1} \mathbb{C}_X[n-1]^H = i_* \Omega^p_X$$

which is a sheaf by Lemma 5.3.3.

Similar to the last part of the proof of Proposition 5.3.4, we obtain that $R^0 \pi_* \Omega^p_{\tilde{Y}}(\log(\tilde{X}+E))(-\tilde{X}-E) = \Omega^p_Y(\log X)(-X)$. Hence, the degree 0 cohomology of the distinguished triangle (5.5) gives the required left exact sequence. □

Theorem 2.2.14 follows immediately from the preceeding propositions.

Proof of Theorem 2.2.14. The theorem follows from Propositions 5.3.4 and 5.3.5 by noting that $\Omega^p_X = \tilde{\Omega}^p_X$ for V-manifolds. □

Bibliography

[AB68] M.F. Atiyah and R. Bott. “A Lefschetz fixed point formula for elliptic complexes: II. Applications”. In: *Annals of Mathematics* 88.3 (1968), pp. 451–491.

[BBD82] A.A. Beilinson, J.N. Bernstein, and P. Deligne. *Faisceaux Pervers*. Astérisque 100. Paris: Société Mathématique de France, 1982.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. “The Magma algebra system. I. The user language”. In: *J. Symbolic Comput.* 24.3-4 (1997), pp. 235–265.

[Bea83] A. Beauville. “Variétes Kähleriennes dont la premiére classe de Chern est nulle”. In: *J. Differential Geometry* 18 (1983), pp. 755–782.

[BL92] C. Birkenhake and H. Lange. *Complex Abelian Varieties*. Grundlehren der mathematischen Wissenschaften 302. Springer-Verlag, 1992.

[BV96] S. Brivio and A. Verra. “The theta divisor of $SU_C(2,2d)^s$ is very ample if C is not hyperelliptic”. In: *Duke Math. J.* 82.3 (1996), pp. 503–552.

[BW74] D.M. Burns Jr. and J.M. Wahl. “Local contributions to global deformations of surfaces”. In: *Inventiones Mathematicae* 26 (1974), pp. 67–88.

[Cam12] C. Camere. “Symplectic involutions of holomorphic symplectic fourfolds”. In: *Bull. London Math. Soc.* 44.4 (2012), pp. 687–702.

[Cat81] F. Catanese. “Babbage’s conjecture, contact of surfaces, symmetric determinantal varieties and applications”. In: *Inventiones Mathematicae* 63 (1981), pp. 433–465.

[CC97] G. Casnati and F. Catanese. “Even sets of nodes are bundle symmetric”. In: *J. Differential Geometry* 47 (1997), pp. 237–256.

[CM05] M.A.A. de Cataldo and L. Migliorini. “The Hodge theory of algebraic maps”. In: *Ann. Sci. de l’E.N.S.* 4.38 (2005), pp. 693–750.

[CM09] M.A.A. de Cataldo and L. Migliorini. “The decomposition theorem, perverse sheaves, and the topology of algebraic maps”. In: *Bulletin of the A.M.S.* 46.4 (2009), pp. 535–633.

[CMP03] J. Carlson, S. Müller-Stach, and C.A.M. Peters. *Period Mappings and Period Domains*. Cambridge studies in advanced mathematics 85. Cambridge University Press, 2003.

[CT07] F. Catanese and F. Tonoli. "Even sets of nodes on sextic surfaces". In: *J. Eur. Math. Soc.* 9 (2007), pp. 705–737.

[Del68] P. Deligne. "Théorème de Lefschetz et critères de dégénérescence de suites spectrales". In: *Publications Mathématiques de l'I.H.É.S.* 35 (1968), pp. 107–126.

[Del71a] P. Deligne. "Théorie de Hodge: I". In: *Actes du Congrès internationale des Mathématiciens (Nice, 1970)*. Gauthier-Villars, 1971, pp. 425–430.

[Del71b] P. Deligne. "Théorie de Hodge: II". In: *Publications Mathématiques de l'I.H.É.S.* 40 (1971), pp. 5–58.

[DK15] O. Debarre and A. Kuznetsov. *Gushel-Mukai varieties: classification and birationalities*. 2015. URL: `http://arxiv.org/abs/1510.05448`.

[DK16] O. Debarre and A. Kuznetsov. *Gushel-Mukai varieties: linear spaces and periods*. 2016. URL: `http://arxiv.org/abs/1605.05648`.

[DO88] I. Dolgachev and D. Ortland. *Point Sets in Projective Spaces and Theta Functions*. Astérisque 165. Paris: Société Mathématique de France, 1988.

[DS14] A. Dimca and M. Saito. *Generalization of theorems of Griffiths and Steenbrink to hypersurfaces with ordinary double points*. 2014. URL: `http://arxiv.org/abs/1403.4563`.

[du 81] P. du Bois. "Complexe de de Rham filtré d'une variété singulière". In: *Bulletin de la S. M. F.* 109 (1981), pp. 41–81.

[EM15] P. Eyssidieux and D. Mégy. "Sur les applications des périodes d'une variation de structure de Hodge attachée aux familles d'hypersurfaces à singularités simples". In: *Comment. Math. Helv.* 90 (2015), pp. 731–759.

[EPW01] D. Eisenbud, S. Popescu, and C. Walter. "Lagrangian subbundles and codimension 3 subcanonical subschemes". In: *Duke Math. J.* 107 (2001), pp. 427–467.

[EV92] H. Esnault and E. Viehweg. *Lectures on vanishing theorems*. Basel: Birkhäuser-Verlag, 1992.

[Gee00] B. van Geemen. "Kuga-Satake varieties and the Hodge conjecture". In: *The Arithmetic and Geometry of Algebraic Cycles*. Ed. by B.B. Gordon et al. Series C: Mathematical and Physical Sciences, Vol. 548. Springer, 2000, pp. 51–82. URL: `https://arxiv.org/abs/math/9903146`.

[GR70] H. Grauert and O. Riemenschneider. “Verschwindingssätze für analytische kohomlogiegruppen auf komplexen Räumen”. In: *Inventiones Mathematicae* 11 (1970), pp. 263–292.

[Gri68] P.A. Griffiths. “Periods of integrals on algebraic manifolds, I, II”. In: *Amer. J. Math.* 90 (1968), pp. 568–626, 805–865.

[Gri69a] P.A. Griffiths. “On the periods of certain rational integrals: I”. In: *Annals of Mathematics* 90.3 (1969), pp. 460–495.

[Gri69b] P.A. Griffiths. “On the periods of certain rational integrals: II”. In: *Annals of Mathematics* 90.3 (1969), pp. 496–541.

[GZ16] B. van Geemen and Y. Zhao. *Genus three curves and 56 nodal sextic surfaces*. To appear in J. Algebraic Geometry. 2016. URL: http://arxiv.org/abs/1603.06247.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics Vol. 52. Springer, 1977.

[Hod41] W.V.D. Hodge. *"The Theory and Applications of Harmonic Theories"*. Cambridge University Press, 1941.

[HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*. Progress in Mathematics 236. Birkaüser, 2008.

[Huy15] D. Huybrechts. *Lectures on K3 Surfaces*. Cambridge University Press, 2015.

[IM11] A. Iliev and L. Manivel. “Fano manifolds of degree 10 and EPW sextics”. In: *Ann. Sci. de l'E.N.S.* 44 (2011), pp. 393–426.

[Kem81] G.R. Kempf. “Deformation of symmetric products”. In: *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*. Ed. by I. Kra and B. Maskit. Annals of Mathmatics Studies 97. Princeton University Press and University of Tokyo Press, 1981, pp. 319–341.

[KNS58] K. Kodaira, L. Nirenberg, and D.C. Spencer. “On the existence of deformations of complex analytic structures”. In: *Annals of Mathematics* 68.2 (1958), pp. 450–459.

[KP16] A. Kuznetsov and A. Perry. *Derived categories of Gushel-Mukai varieties*. 2016. URL: http://arxiv.org/abs/1605.06568.

[KS58] K. Kodaira and D.C. Spencer. “On deformations of complex analytic structures I,II”. In: *Annals of Mathematics* 67 (1958), pp. 328–466.

[KS67] M. Kuga and I. Satake. “Abelian varieties associated to polarized K3 surfaces”. In: *Math. Ann.* 169 (1967), pp. 239–242.

[Mac16] F.S. Macaulay. *The Algebraic Theory of Modular Systems*. Cambridge University Press, 1916.

[Man05] M. Manetti. *Lectures on deformations of complex manifolds.* 2005. URL: `http://arxiv.org/abs/math/0507286`.

[Moz08] S. Mozgovoy. *Introduction to perverse sheaves.* 2008. URL: `http://www.maths.tcd.ie/~mozgovoy/data/ps_lecture.pdf`.

[MP16] M. Mustaţă and M. Popa. *Hodge ideals.* 2016. URL: `https://arxiv.org/abs/1605.08088`.

[OGr06] K.G. O'Grady. "Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics ". In: *Duke Math. J.* 134.1 (2006), pp. 99–137.

[OGr13] K.G. O'Grady. "Double covers of EPW-sextics". In: *Michigan Math. J.* 62.1 (2013), pp. 143–184.

[OGr15] K.G. O'Grady. "Periods of double EPW-sextics". In: *Mathematische Zeitschrift* 280.1 (2015), pp. 485–524.

[OGr16] K.G. O'Grady. "Moduli of double EPW-sextics". In: *Memoirs of the AMS* 240 (2016).

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector Bundles on Complex Projective Spaces.* Progress in Mathematics 3. Springer, 1980.

[Par91] R. Pardini. "Abelian covers of algebraic varieties". In: *J. reine angew. Math.* 417 (1991), pp. 191–213.

[PS08] C.A.M. Peters and J.H.M. Steenbrink. *Mixed Hodge Structures.* Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge Vol 52. Berlin: Springer, 2008.

[Rim80] D.S. Rim. "Equivariant G-structure on versal deformations". In: *Trans. Amer. Math. Soc.* 257.1 (1980), pp. 217–226.

[Sai07] M. Saito. "Direct image of logarithmic complexes and infinitesimal invariants of cycles". In: *Algebraic cycles and motives Vol. 2.* LMS Lecture Notes Series, Vol. 344. Cambridge University Press, 2007, pp. 304–318.

[Sai16] M. Saito. *A young person's guide to mixed Hodge modules.* 2016. URL: `https://arxiv.org/abs/1605.00435`.

[Sai88] M. Saito. "Modules de Hodge polarisables". In: *Publ. RIMS, Kyoto Univ.* 24 (1988), pp. 849–995.

[Sai90] M. Saito. "Mixed Hodge modules". In: *Publ. RIMS, Kyoto Univ.* 26 (1990), pp. 221–333.

[Sch14] C. Schnell. *An overview of Morihiko Saito's theory of mixed Hodge modules.* 2014. URL: `http://arxiv.org/abs/1405.3096`.

[Sch15] S. Schreieder. "On the construction problem for Hodge numbers". In: *Geometry and Topology* 19 (2015), pp. 295–342.

[Ser06] E. Sernesi. *Deformations of Algebraic Schemes.* Grundlehren der mathematischen Wissenschaften 334. Springer, 2006.

[Ser56] J.-P. Serre. "Géométrie algébrique et géométrie analytique". In: *Annales de l'institut Fourier* 6 (1956), pp. 1–42.

[Ser66] J.-P. Serre. "Prolongement de faisceaux analytiques cohérents ". In: *Ann. de l'institut Fourier* 16.1 (1966), pp. 363–374.

[Sev46] F. Severi. "Sul massimo numero di nodi di una superficie di dato ordine dello spazio ordinario o di una forma di un iperspazio". In: *Ann. Mat. Pura Appl.* 25 (1946), pp. 1–41.

[Ste06] J.H.M. Steenbrink. "Adjunction conditions for one-forms on surfaces in projective three-space". In: *Singularities and Computer Algebra.* London Math. Soc. Lecture Note Ser. 324. Cambridge: Cambridge University Press, 2006, pp. 301–314.

[Ste77] J.H.M. Steenbrink. "Mixed Hodge Structure on the vanishing cohomology". In: *Real and complex singularities, Oslo 1976.* Ed. by P. Holm. Alphen a/d Rijn: Sijthoff–Noordhoff, 1977, pp. 525–563.

[Voi02] C. Voisin. *Hodge Theory and Complex Algebraic Geometry I.* Cambridge studies in advanced mathematics 76. Cambridge: Cambridge University Press, 2002.

[Voi03] C. Voisin. *Hodge Theory and Complex Algebraic Geometry II.* Cambridge studies in advanced mathematics 77. Cambridge: Cambridge University Press, 2003.

[Wel83] G. Welters. "Polarized abelian varieties and the heat equations". In: *Compositio Mathematica* 49 (1983), pp. 173–194.

Acknowledgements

I am very grateful to my advisors, Bert van Geemen, Ronald van Luijk, and Peter Stevenhagen, for their guidance and support throughout my PhD studies.

Bert has been instrumental in the course of my research, sharing with me his immense knowledge of the subject. While granting me a large degree of freedom to pursue my topic of interest, he provided ideas when I was stuck and invested much time and effort to guide me through my thesis. He was always available when I needed help, and was able to give useful insights even if my questions were not in his area of expertise.

Although I did not work much with Ronald and Peter, they encouraged me and provided important feedback on my work and progress, ensuring that I was on track and had the resources that I needed.

Many people have inspired me in my career so far. I would like to thank Luca Migliorini, whose lectures in Rennes in 2014 gave me the first (albeit unsuccessful) idea for my PhD project, for taking time to discuss with me and answer my silly questions; Gabriele Vezzosi, Pieter Belmans, Mauro Porta, Paolo Stellari, Mattia Ornaghi, and Bas Edixhoven for the many insightful discussions, broadening my perspective on mathematics; Paolo and Mattia, again, for organizing the seminar on non-commutative motives in Milan, which was a great learning experience; and many others whom I've talked to and worked with over the last three years.

I would also like to thank the reading committee and the anonymous referees, for taking the time to read my thesis, providing valuable comments and pointing out major mistakes.

Special thanks go to Swantje for proofreading my thesis and designing the cover as well as Bert, Ronald, Leonardo, Mima, and Rosa for the translation of the summary. Cover image is courtesy of David Madore who kindly supplied the source code.

I am grateful towards Erasmus Mundus and the ALGANT program for funding both my masters and doctorate studies, providing me the opportunity to study in Europe and to meet many new people.

My deepest appreciation goes to my flatmate in Milan for two years, Qijun, for all that we have done together, for encouraging me when I was down and

for bearing with me when I was being intolerable. Thank you for being a great friend, flatmate and colleague!

The same appreciation also goes to Djordjo, my flatmate in Leiden for the first three months, and to Giulio, who hosted me for way longer than expected.

My PhD experience would have been totally different, and nowhere near as enriching, without my friends and colleagues at the department: in particular, Chloe, Mima, Djordjo, Giulio, Rosa, Raymond, Erik, Carlo, David, Andrea, Leonardo, Marta, Michiel in Leiden; Qijun, Francesca, Carolina, Matteo, Laura, Mattia, Simone, Martino in Milan. Thank you Kathelijne for always being so cheery and making everything run smoothly.

Thank you to the friends who have stuck with me over many years: Yuxin, Aylwin, Esther, Rong, Stefano, Danilo; as well as the new friends I have made: Swantje, Jit, Liuba, Aleks, Catalina, Velten, my hiking buddies in Milan, and many others.

Last but not least, a big thank you to my parents, for their never-ending support, despite being so far away, over the last 28 years of my life and for the years to come.

Summary

Ordinary double points are the simplest type of singularity on an algebraic surface, and can be defined locally by the polynomial $x^2 - yz \in \mathbb{C}[x, y, z]$. Nodal surfaces are projective surfaces with only ordinary double points as singularities.

Hodge theory and deformation theory of nodal surfaces are similar to those of smooth surfaces in many respects, for example, nodal surfaces also have pure Hodge structures. The similarities and differences are described in detail in Chapters 2 and 3 of the thesis. Our first main result is the infinitesimal Torelli theorem for nodal surfaces in $\mathbb{P}^3$, which states that all small non-trivial deformations of nodal surfaces induce non-trivial variations of Hodge structures.

A nodal surface F is said to have an even set of nodes if there exists a double cover $f : S \to F$ branched precisely over the nodes. We studied two families of such surfaces in Chapter 4, namely sextic surfaces in $\mathbb{P}^3$ with even sets of 56 and 40 nodes respectively.

We found a new geometric construction for a universal family of even 56-nodal surfaces whose double covers S satisfy $h^{1,0}(S) = 3$, by showing that any such surface is in turn the double cover of an even 28-nodal surface $\Theta/[-1]$ where $\Theta = S^2C$ is a symmetric theta divisor on the Jacobian of a non-hyperelliptic curve C of genus 3 and $[-1]$ is induced by the involution on the Jacobian.

The involution inducing the double cover $f : S \to F$ gives a decomposition of Hodge structures into eigenspaces

$$H^2(S, \mathbb{Q}) = H^2(S, \mathbb{Q})_+ \oplus H^2(S, \mathbb{Q})_-, \qquad \text{with} \quad H^2(S, \mathbb{Q})_+ = H^2(F, \mathbb{Q}).$$

The double cover S of an even 40-nodal surface has $H^2(S, \mathbb{Q})_-$ of Hodge type $(1, 26, 1)$. We seek simple sub-Hodge structures of types $(1, n, 1)$ with $n > 20$, since we know very few examples of these. However, by comparing various constructions of families of even 40-nodal surfaces, we showed that $H^2(S, \mathbb{Q})_-$ always contains a sub-Hodge structure, of type $(1, 20, 1)$, arising from some deformation of a Hilbert scheme of a K3 surface. Thus, this family of examples failed to provide new geometric Hodge structures of interest.

In the final chapter, we extended some constructions from Chapter 2 to more general singularities, using Saito's theory of mixed Hodge modules. These results provide ways to compute the Hodge decompositions of singular varieties.

Samenvatting

Normale dubbelpunten zijn de eenvoudigste singulariteiten op een algebraïsch oppervlak, lokaal zijn ze gedefinieerd door het polynoom $x^2 - yz \in \mathbb{C}[x, y, z]$. Nodale oppervlakken zijn projectieve oppervlakken met als enige singulariteiten normale dubbelpunten.

De Hodge theorie en de deformaties van nodale oppervlakken lijken erg veel op die van gladde oppervlakken, nodale oppervlakken hebben bijvoorbeeld ook pure Hodge structuren. De overeenkomsten en verschillen worden in detail beschreven in Hoofdstukken 2 en 3 van dit proefschrift. Het eerste belangrijke resultaat is de infinitesimale Torelli stelling voor nodale oppervlakken in $\mathbb{P}^3$. Deze stelling zegt dat een niet-triviale deformatie van een nodaal oppervlak een niet-triviale variatie van de Hodge structuur induceert.

We zeggen dat een nodaal oppervlak F een even verzameling van dubbelpunten heeft als er een overdekking $f : S \to F$ van graad twee is die precies vertakt over de dubbelpunten. In Hoofdstuk 4 worden twee families van zulke oppervlakken bestudeerd, namelijk zesdegraads oppervlakken met even verzamelingen van respectivelijk 56 en 40 dubbelpunten.

We hebben een nieuwe meetkundige constructie gevonden voor een universele familie van nodale oppervlakken met een even verzameling van 56 dubbelpunten en met een dubbele overdekking S zodat $h^{1,0}(S) = 3$, door te laten zien dat deze oppervlakken zelf ook een dubbele overdekking van een oppervlak $\Theta/[-1]$ met 28 dubbelpunten zijn, waarbij $\Theta = S^2C$ de symmetrische theta divisor is op de Jacobiaan van een niet-hyperelliptische kromme C van geslacht 3 en $[-1]$ is geïnduceerd door de involutie op de Jacobiaan.

De involutie op S geïnduceerd door de overdekking $f : S \to F$ geeft een decompositie van Hodge structuren in eigenruimtes

$$H^2(S, \mathbb{Q}) = H^2(S, \mathbb{Q})_+ \oplus H^2(S, \mathbb{Q})_-, \text{ met } H^2(S, \mathbb{Q})_+ = H^2(F, \mathbb{Q}).$$

Voor een oppervlak met een even verzameling van 40 dubbelpunten heeft de overdekking S de sub-Hodge structuur $H^2(S, \mathbb{Q})_-$ met Hodge getallen (1, 26, 1). We zijn geïnteresseerd in irreducibile sub-Hodge structuren met Hodge getallen $(1, n, 1)$ en $n > 20$ omdat er weinig voorbeelden bekend zijn. Echter, nadat we meerdere constructies van oppervlakken met 40 dubbelpunten bestudeerd hebben, zijn we tot de conclusie gekomen dat $H^2(S, \mathbb{Q})_-$ altijd reducibel is, omdat er een sub-Hodge structuur met Hodge getallen $(1, 20, 1)$ is die afkomstig is van een deformatie van het tweede Hilbert schema van een

K3 oppervlak. Helaas geeft deze familie dus geen meetkundige constructie van interessante nieuwe Hodge structuren.

In het laatste hoofdstuk geven we met behulp van Saito's theorie van gemengde Hodge modulen generalisaties van enkele constructies uit Hoodstuk 2 voor algemenere singulariteiten. Deze resultaten zijn van belang voor het berekenen van de Hodge decompositie van singuliere variëteiten.

Sommario

I nodi sono le singolarità le più semplici di una superficie algebrica, definite localmente dal polinomio $x^2 - yz \in \mathbb{C}[x, y, z]$. Le superfici nodali sono superfici proiettive con soltanto nodi come punti singolari.

Le teorie di Hodge e della deformazione delle superfici nodali sono simili a quelle delle superfici lisce, ad esempio anche le superfici nodali hanno strutture di Hodge pure. Le similarità e le differenze sono studiate in dettaglio nei capitoli 2 e 3 della tesi. Il primo, fra i risultati principali di questa tesi, è il teorema di Torelli infinitesimale per superfici nodali in $\mathbb{P}^3$ che afferma che ogni deformazione non-banale di una superficie nodale induce una variazione non-banale di strutture di Hodge.

Una superficie nodale F ha un insieme di nodi pari se esiste un rivestimento doppio $f : S \to F$ ramificato esattamente sopra i nodi di F. Nel capitolo 4 si studiano due famiglie di tali superfici di grado sei in $\mathbb{P}^3$ con insiemi pari di 56 e 40 nodi rispettivamente.

Si è trovata una nuova costruzione geometrica di una famiglia universale di superfici con 56 nodi pari, il cui rivestimento doppio S soddisfa $h^{1,0}(S) = 3$, usando che una qualsiasi tale superficie è a sua volta il rivestimento doppio di una superficie $\Theta/[-1]$ 28-nodale pari dove $\Theta = S^2C$ è un divisore theta simmetrico della jacobiana di una curva non-iperellittica C di genere 3 e $[-1]$ è indotta dall'involuzione della jacobiana.

L'involuzione che induce il rivestimento doppio $f : S \to F$ dà una decomposizione della struttura di Hodge in autospazi:

$$H^2(S, \mathbb{Q}) = H^2(S, \mathbb{Q})_+ \oplus H^2(S, \mathbb{Q})_-, \qquad \text{con} \quad H^2(S, \mathbb{Q})_+ = H^2(F, \mathbb{Q}).$$

La sottostruttura di Hodge $H^2(S, \mathbb{Q})_-$ del rivestimento doppio S di una superficie pari 40-nodale ha numeri di Hodge $(1, 26, 1)$. Si è interessati alle sottostrutture irriducibili di Hodge con numeri di Hodge $(1, n, 1)$ con $n > 20$ poiché soltanto pochi esempi sono noti. A seguito dello studio di varie costruzioni di tali superfici, si è mostrato che, tuttavia, $H^2(S, \mathbb{Q})_-$ ha sempre una sottostruttura con numeri di Hodge $(1, 20, 1)$ che è indotta dalla deformazione di uno schema di Hilbert associato ad una superficie K3. Questi esempi, quindi, non producono nuove strutture di Hodge geometriche di interesse.

Nel capitolo finale è stata usata la teoria dei moduli misti di Hodge, introdotta

da Saito, per estendere alcune costruzioni del Capitolo 2 a singolarità più generali. Questi risultati permettono di determinare la decomposizione di Hodge di varietà singolari.

Curriculum vitae

Yan Zhao was born on September 27, 1988, in Fuzhou, China. When he was four, he moved with his family to Singapore where he attended Raffles Institution and Raffles Junior College, graduating in 2006. He represented Singapore at the International Mathematical Olympiad in 2005 and 2006.

After finishing high school, Yan took a gap year travelling through China before moving to Canberra, Australia, in 2008 for his undergraduate studies at the Australian National University. His broad-based study program allowed him to pursue numerous research internships in geology, seismology and quantum chemistry. During the academic year of 2009/2010, he went on an exchange program to the University of Cambridge, where he read the Mathematics Tripos Part II. Returning to Australia, he completed his studies and graduated in July 2011 with a Bachelor of Science (double major in mathematics).

In 2011, Yan was awarded the Erasmus Mundus Masters Scholarship to pursue his masters degree under the ALGANT (Algebra, Geometry and Number Theory) program at the University of Padova and University of Paris XI. He wrote his masters thesis, entitled "Extended topological field theories and the cobordism hypothesis", under the supervision of Dr. Grégory Ginot at the University of Paris VI.

Continuing in the same program, Yan commenced his PhD studies in October 2013. He switched to the field of complex geometry and worked under the joint supervision of Prof. Bert van Geemen at the University of Milan and Dr. Ronald van Luijk at Leiden University. He hopes to graduate in December 2016.

Yan is an avid hiker and enjoys exploring remote corners of the world.