EXPLICIT SMOOTHED PRIME IDEALS THEOREMS UNDER GRH

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ABSTRACT. Let $\psi_{\mathbb{K}}$ be the Chebyshev function of a number field $\mathbb{K}$. Let $\psi^{(1)}_{\mathbb{K}}(x) := \int_0^x \psi_{\mathbb{K}}(t)\,\mathrm{d}t$ and $\psi^{(2)}_{\mathbb{K}}(x) := 2\int_0^x \psi^{(1)}_{\mathbb{K}}(t)\,\mathrm{d}t$. We prove under GRH (Generalized Riemann Hypothesis) explicit inequalities for the differences $|\psi^{(1)}_{\mathbb{K}}(x) - \frac{x^2}{2}|$ and $|\psi^{(2)}_{\mathbb{K}}(x) - \frac{x^3}{3}|$. We deduce an efficient algorithm for the computation of the residue of the Dedekind zeta function and a bound on small-norm prime ideals.

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1. Introduction

For a number field $\mathbb{K}$ we denote

$n_{\mathbb{K}}$ its dimension,
$\Delta_{\mathbb{K}}$ the absolute value of its discriminant,
r_1 the number of its real places,
r_2 the number of its imaginary places,
$d_{\mathbb{K}} := r_1 + r_2 - 1$.

Moreover, throughout this paper $\mathfrak{p}$ denotes a maximal ideal of the integer ring $\mathcal{O}_{\mathbb{K}}$ and $\mathrm{N}\mathfrak{p}$ its absolute norm. The von Mangoldt function $\Lambda_{\mathbb{K}}$ is defined on the set of ideals of $\mathcal{O}_{\mathbb{K}}$ as $\Lambda_{\mathbb{K}}(\mathfrak{I}) = \log \mathrm{N}\mathfrak{p}$ if $\mathfrak{I} = \mathfrak{p}^m$ for some $\mathfrak{p}$ and $m \geq 1$, and is zero otherwise. Moreover, the Chebyshev function $\psi_{\mathbb{K}}$ and the arithmetical function $\tilde{\Lambda}_{\mathbb{K}}$ are defined via the equalities

$$\psi_{\mathbb{K}}(x) := \sum_{\substack{\mathfrak{I}\subset\mathcal{O}_{\mathbb{K}}\\ \mathrm{N}\mathfrak{I}\leq x}} \Lambda_{\mathbb{K}}(\mathfrak{I}) =: \sum_{n\leq x} \tilde{\Lambda}_{\mathbb{K}}(n).$$

In 1979, Oesterlé announced [19] a general result implying under the Generalized Riemann Hypothesis that

$$|\psi_{\mathbb{K}}(x) - x| \leq \sqrt{x}\Big[\Big(\frac{\log x}{\pi} + 2\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{\log^2 x}{2\pi} + 2\Big)n_{\mathbb{K}}\Big] \qquad \forall\, x \geq 1, \tag{1.1}$$

but its proof has never appeared. The stronger bound with $\log x$ substituted by $\frac{1}{2}\log x$ has been proved by the authors [8] for $x \geq 100$.

The function $\psi_{\mathbb{K}}(x)$ is the first member of a sequence of similar sums $\psi^{(m)}_{\mathbb{K}}(x)$ which are defined for every $m \in \mathbb{N}$ as

$$\psi^{(0)}_{\mathbb{K}}(x) := \psi_{\mathbb{K}}(x) \qquad \psi^{(m)}_{\mathbb{K}}(x) := m\int_0^x \psi^{(m-1)}_{\mathbb{K}}(u)\,\mathrm{d}u = \sum_{n\leq x} \tilde{\Lambda}_{\mathbb{K}}(n)(x-n)^m$$

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and are smoothed versions of $\psi_{\mathbb{K}}(x)$. They could be studied using (1.1) via a partial summation formula, but a direct attack via the integral identities

$$\psi^{(m)}_{\mathbb{K}}(x) = -\frac{m!}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)\frac{x^{s+m}}{s(s+1)\cdots(s+m)}\,\mathrm{d}s \qquad \forall x\geq 1,\quad \forall m\geq 0 \tag{1.2}$$

(see Section 4) produces better results, as a consequence of the better decay that the kernel in the integral has for $m\geq 1$ with respect to the case $m=0$. In fact, the absolute integrability of the kernel allows us to apply the Cauchy integral formula to quickly obtain that

$$\psi^{(1)}_{\mathbb{K}}(x) = \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - xr_{\mathbb{K}} + r'_{\mathbb{K}} + R^{(1)}_{r_1,r_2}(x), \tag{1.3a}$$

$$\psi^{(2)}_{\mathbb{K}}(x) = \frac{x^3}{3} - \sum_{\rho}\frac{2x^{\rho+2}}{\rho(\rho+1)(\rho+2)} - x^2 r_{\mathbb{K}} + 2xr'_{\mathbb{K}} - r''_{\mathbb{K}} + R^{(2)}_{r_1,r_2}(x), \tag{1.3b}$$

and analogous formulas for every $m\geq 3$, where ρ runs on the set of nontrivial zeros for $\zeta_{\mathbb{K}}$, the constants $r_{\mathbb{K}}$, $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ are defined in (3.8) below and the functions $R^{(m)}_{r_1,r_2}(x)$ in Lemma 3.3. These representations show that the main term for the difference $\psi^{m}_{\mathbb{K}}(x) - \frac{x^m}{m}$ comes from the sum on nontrivial zeros.
Assuming the Generalized Riemann Hypothesis we have the strongest horizontal localization on zeros but we lack any sharp vertical information. Thus we are in some sense forced to estimate the sum with $x^{m+1/2}\sum_{\rho}|\rho(\rho+1)\cdots(\rho+m)|^{-1}$, and the problem here is essentially producing good bounds for this sum. To estimate this type of sums, we use the following method. Let Z be the set of imaginary parts of the nontrivial zeros of $\zeta_{\mathbb{K}}$, counted with their multiplicities, and let

$$\begin{aligned} f(s,\gamma) &:= \operatorname{Re}\Big(\frac{2}{s-(\frac12+i\gamma)}\Big),\\ f_{\mathbb{K}}(s) &:= \sum_{\gamma\in Z} f(s,\gamma). \end{aligned}$$

The sum converges to the real part of a meromorphic function with poles at the zeros of $\zeta_{\mathbb{K}}$. Let g be a non-negative function. Suppose we have a real measure μ supported on a subset $D\subseteq\mathbb{C}$ such that

$$g(\gamma) \leq \int_D f(s,\gamma)\,\mathrm{d}\mu(s), \qquad \forall\gamma\in\mathbb{R}, \tag{1.4}$$

then under moderate conditions on D and μ we have

$$\sum_{\gamma\in Z} g(\gamma) \leq \sum_{\gamma\in Z}\int_D f(s,\gamma)\,\mathrm{d}\mu(s) = \int_D\sum_{\gamma\in Z} f(s,\gamma)\,\mathrm{d}\mu(s) = \int_D f_{\mathbb{K}}(s)\,\mathrm{d}\mu(s). \tag{1.5}$$

To ensure the validity of the estimate it is sufficient to have D on the right of the line $\operatorname{Re}(s) = \frac12 + \varepsilon$ for some $\varepsilon > 0$ and μ of bounded variation. The interest of the method comes from the fact that, using the functional equation of $\zeta_{\mathbb{K}}$, one can produce a formula for $f_{\mathbb{K}}$ independent of the zeros (see (3.7)).

The aforementioned idea works very well for certain g corresponding to $m=1$ and 2 above, allowing us to prove the explicit formulas for $\psi^{(1)}_{\mathbb{K}}(x)$ and $\psi^{(2)}_{\mathbb{K}}(x)$ given in Theorem 1.1. Other applications of this idea can be found in [8] and [9].

Theorem 1.1. *(GRH) For every $x \geq 3$, when $\mathbb{K} \neq \mathbb{Q}$ we have*

$$\begin{aligned}
\Big|\psi^{(1)}_{\mathbb{K}}(x) - \frac{x^2}{2}\Big| \leq{}& x^{3/2}(0.5375 \log \Delta_{\mathbb{K}} - 1.0355 n_{\mathbb{K}} + 5.3879) + (n_{\mathbb{K}} - 1)x\log x \\
&+ x(1.0155 \log \Delta_{\mathbb{K}} - 2.1041 n_{\mathbb{K}} + 8.3419) + \log \Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4, \\
\Big|\psi^{(2)}_{\mathbb{K}}(x) - \frac{x^3}{3}\Big| \leq{}& x^{5/2}(0.3526 \log \Delta_{\mathbb{K}} - 0.8212 n_{\mathbb{K}} + 4.4992) + (n_{\mathbb{K}} - 1)x^2(\log x - \tfrac{1}{2}) \\
&+ x^2(1.0155 \log \Delta_{\mathbb{K}} - 2.1041 n_{\mathbb{K}} + 8.3419) + 2x(\log \Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4) \\
&+ \log \Delta_{\mathbb{K}} - 0.9151 n_{\mathbb{K}} + 2,
\end{aligned}$$

while for $\mathbb{Q}$ the bounds become

$$\begin{aligned}
\Big|\psi^{(1)}_{\mathbb{Q}}(x) - \frac{x^2}{2}\Big| &\leq 0.0462 x^{3/2} + 1.838x, \\
\Big|\psi^{(2)}_{\mathbb{Q}}(x) - \frac{x^3}{3}\Big| &\leq 0.0015 x^{5/2} + 1.838x^2.
\end{aligned}$$

The method can be easily adapted to every $m \geq 3$, but depends on several parameters that we have to set in a proper way to get an interesting result, and whose dependence on m is not clear. As a consequence it is not evident that the bounds for each $m \geq 3$ will be as good as the cases $m = 1$ and 2, despite the fact that our computations for $m = 3$ and 4 show that it should be possible. Moreover, the applications we will show in the next section essentially do not benefit from any such extension, the cases $m = 1$ and 2 giving already the best conclusions (see Remark 4.3 below). Thus we have decided not to include the cases $m = 3$ and 4 in the paper.

Remark 1.2. Integrating (1.4) for $\gamma \in \mathbb{R}$ we find that, if D is in the $\operatorname{Re} s > \frac{1}{2}$ half of the plane, then

$$\mu(D) \geq \frac{1}{2\pi}\int_{\mathbb{R}} g(\gamma)\,\mathrm{d}\gamma.$$

The measure $\mu(D)$ will end up as the main coefficient of $\log \Delta_{\mathbb{K}}$ in our inequalities. This means that the coefficient of $\log \Delta_{\mathbb{K}}$ that we can obtain with this method is necessarily greater than $\frac{1}{2\pi}\int_{\mathbb{R}} g(\gamma)\,\mathrm{d}\gamma$.

Finally, we notice that our method is not limited to upper-bounds, since if we change $\leq$ to $\geq$ in Inequality (1.4), then Inequality (1.5) gives a lower bound. For an application see Remark 4.4 below.

A file containing the PARI/GP [21] code we have used for a set of computations is available at the following address:
`http://users.mat.unimi.it/users/molteni/research/psi_m_GRH/psi_m_GRH_data.gp`.

Notation. $\lfloor x \rfloor$ denotes the integral part of x; γ denotes the imaginary part of the nontrivial zeros, but in some places it will denote also the Euler-Mascheroni constant, the actual meaning being clear from the context.

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2. Applications

Small prime ideals. The bound in (1.1) can be used to prove that $\psi_{\mathbb{K}}(x) > 0$ when $x \geq 4(\log\Delta_{\mathbb{K}}\log^2\log\Delta_{\mathbb{K}}+5n_{\mathbb{K}}+10)^2$. This fact, without explicit constants, was already mentioned by Lagarias and Odlyzko [10] who also gave an argument to remove the double logarithm of the discriminant and hence proving the existence of an absolute constant c such that $\psi_{\mathbb{K}}(x) > 0$ whenever $x \geq c\log^2\Delta_{\mathbb{K}}$. Later Oesterlé [19] announced that $c = 70$ works conditionally (see also [23, Th. 5]). More recently, Bach [1, Th. 4] proved (assuming GRH, again) that the class group of $\mathbb{K}$ is generated by ideals whose norm is bounded by $12\log^2\Delta_{\mathbb{K}}$ and by $(4+o(1))\log^2\Delta_{\mathbb{K}}$ when $\Delta_{\mathbb{K}}$ tends to infinity (see also [4]). This proves the claim with $c = 12$, and $c = 4$ asymptotically. A different approach of Bach and Sorenson [3] proves that for any abelian extension of number fields $\mathbb{E}/\mathbb{K}$ with $\mathbb{E} \neq \mathbb{Q}$ and every $\sigma \in \mathrm{Gal}(\mathbb{E},\mathbb{K})$ there are degree-one primes $\mathfrak{p}$ in $\mathbb{K}$ such that $\big[\frac{\mathbb{E}/\mathbb{K}}{\mathfrak{p}}\big] = \sigma$ with $\mathrm{N}\mathfrak{p} \leq (1+o(1))\log^2\Delta_{\mathbb{E}}$, where the "little-$o$" function is explicit but decays very slowly. As a consequence of the work of Lamzouri, Li and Soundararajan [11, Cor. 1.2] one can take $1+o(1) = \big(\frac{\varphi(q)\log q}{\log\Delta_{\mathbb{K}}}\big)^2$ in the case of the cyclotomic extension $\mathbb{K} = \mathbb{Q}[q]$ of q-th roots of unity.
The case $\mathbb{E} = \mathbb{K}$ of the aforementioned result of Bach and Sorenson implies that there exists a degree-one prime below $(1+o(1))\log^2\Delta_{\mathbb{K}}$. Using the bounds for $\psi^{(1)}_{\mathbb{K}}(x)$ and $\psi^{(2)}_{\mathbb{K}}(x)$ in Theorem 1.1 we reach a similar conclusion with the "little-o" function substituted by an explicit and quite small constant.

Corollary 2.1. *(GRH) For every $\kappa \geq 0$, there are more than κ degree-one prime ideals $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \leq \big(\mathcal{L}_{\mathbb{K}} + \sqrt{8\kappa\log(\mathcal{L}_{\mathbb{K}} + \sqrt[3]{\kappa}\log\kappa)}\big)^2$, where $\mathcal{L}_{\mathbb{K}} := 1.075(\log\Delta_{\mathbb{K}} + 13)$ (with $\sqrt[3]{\kappa}\log\kappa = 0$ for $\kappa = 0$).*

Remark. The same argument, but this time based on bounds for $\psi^{(2)}_{\mathbb{K}}(x)$, $\psi^{(3)}_{\mathbb{K}}(x)$ and $\psi^{(4)}_{\mathbb{K}}(x)$, produces a small improvement on the previous corollary, giving the same conclusion but with $\mathcal{L}_{\mathbb{K}} := 1.0578(\log\Delta_{\mathbb{K}} + c)$ for a suitable constant c which can be explicitly computed. The improvement is due to the fact that the main constants 0.3526 and 0.5375 appearing in Theorem 1.1 satisfy $1.0578 = 3\cdot 0.3526 < 2\cdot 0.5375 = 1.075$. Actually, no further improvement is possible with our technique (see Remark 4.3). In our opinion this very small improvement is unworthy of a detailed exposition: the interested reader will be able to prove it following the proof of Corollary 2.1 in Section 5.

Let $\partial_{\mathbb{K}} = \prod_{\mathfrak{p}}\mathfrak{p}^{c_{\mathfrak{p}}}$ be the decomposition of the different ideal of $\mathbb{K}$. We have $c_{\mathfrak{p}} = e(\mathfrak{p}) - 1$ when $\mathfrak{p}$ is tamely ramified and $c_{\mathfrak{p}} \geq e(\mathfrak{p})$ when $\mathfrak{p}$ is wildly ramified. If $\mathfrak{p}$ is above an odd prime then $\log\mathrm{N}\mathfrak{p} \geq \log 3$ hence $c_{\mathfrak{p}}\log\mathrm{N}\mathfrak{p} \geq \log 3$. If $\mathfrak{p}$ is above 2, then either it is wildly ramified and $c_{\mathfrak{p}} \geq e(\mathfrak{p}) \geq 2$ or it is tamely ramified and $c_{\mathfrak{p}} = e(\mathfrak{p}) - 1 \geq 2$ (by definition of tame ramification). We thus have $c_{\mathfrak{p}}\log\mathrm{N}\mathfrak{p} \geq \log 3$ in all cases. This in turn means that the number of ramifying ideals is at most $\frac{\log\mathrm{N}\partial_{\mathbb{K}}}{\log 3} \leq \log\Delta_{\mathbb{K}}$. We deduce immediately the following

Corollary 2.2. *(GRH) For every $\kappa \geq 0$, there are more than κ unramified degree-one prime ideals $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \leq \big(\mathcal{L}_{\mathbb{K}} + \sqrt{8\kappa'\log(\mathcal{L}_{\mathbb{K}} + \sqrt[3]{\kappa'}\log\kappa')}\big)^2$, where $\kappa' = \kappa + \log\Delta_{\mathbb{K}}$ and $\mathcal{L}_{\mathbb{K}} :=$* $1.075(\log\Delta_{\mathbb{K}} + 13)$.

Remark. If $\mathbb{K}/\mathbb{Q}$ is a Galois extension, then the prime ideals in Corollary 2.2 are totally split, i.e. $\big[\frac{\mathbb{K}/\mathbb{Q}}{\mathfrak{p}}\big] = \mathrm{id}$.

Let $\mathbb{K} := \mathbb{Q}[q]$ be the cyclotomic field of q-th roots of unity. Let p be the largest prime divisor of q and write $q =: p^\nu q'$ with p and q' coprime. There is a ramified prime ideal of degree one if and only if $p \equiv 1 \pmod{q'}$, this condition being trivially true when $q' = 1$, i.e. when q is a prime power. In that case there are $\varphi(q')$ ramified primes of degree one and their norm is p. Therefore, there is necessarily a prime congruent to $1 \pmod q$ below the bound of Corollary 2.1 with $\kappa = \varphi(q')$. A second prime congruent to 1 modulo q is produced setting $\kappa = \varphi(q') + \varphi(q)$. Comparing $\mathcal{L}_\mathbb{K}$ and $\varphi(q)\log q$ we get the following explicit result.

Corollary 2.3. *(GRH) For every $q \geq 5$ there are at least two primes which are congruent to 1 modulo q and $\leq 1.2(\varphi(q)\log q)^2$.*

Proof. We know that $\log\Delta_\mathbb{K} = \varphi(q)\log q - \varphi(q)\sum_{p|q}\frac{\log p}{p-1}$ (see [26, Prop. 2.17]), so that $\mathcal{L}_\mathbb{K} \leq 1.075\varphi(q)\log q$ for every $q > e^{13}$ (and when $q > 32$ if q is not a prime). Define $q =: p^\nu q'$ as above. As observed, we take $\kappa = \varphi(q') + \varphi(q)$ in Corollary 2.1.
Notice that, if $q' \neq 1$, then $p \geq 3$ thus $\varphi(q') = \frac{\varphi(q)}{(p-1)p^{\nu-1}} \leq \frac{1}{2}\varphi(q)$, while if $q' = 1$ the same inequality holds as soon as $q \geq 3$. This proves that $\kappa \leq \frac{3}{2}\varphi(q)$ holds for every $q \geq 3$.
Since $\log\Delta_\mathbb{K} \geq \frac{1}{2}\varphi(q)\log q$ for $q \geq 7$, one has $\varphi(q) \leq 4\mathcal{L}_\mathbb{K}/\log(2\mathcal{L}_\mathbb{K})$. Thus $\kappa \leq 6\mathcal{L}_\mathbb{K}/\log(2\mathcal{L}_\mathbb{K})$ when $q \geq 7$. With this upper bound, for $\mathcal{L}_\mathbb{K} \geq 1.3\cdot 10^5$, we get

$$1.075^2 \cdot \Big(1 + \frac{1}{\mathcal{L}_\mathbb{K}}\sqrt{8\kappa\log(\mathcal{L}_\mathbb{K} + \sqrt[3]{\kappa}\log\kappa)}\Big)^2 \leq 1.2.$$

If $\varphi(q) \geq 24000$, we have $\mathcal{L}_\mathbb{K} \geq 1.075(\frac{1}{2}\varphi(q)\log\varphi(q) + 13) \geq 1.3\cdot 10^5$. For $q \geq 510510 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17$, looking separately the cases where q as at least 7 or less than 7 distinct prime factors, we see that $\varphi(q) \geq 92160 \geq 24000$. This proves the claim for $q \geq 510510$. Then, the explicit computation for $q < 510510$ of the bound in Corollary 2.1 shows that it is $\leq 1.2(\varphi(q)\log q)^2$ for every $q > 4373$; this proves the claim for $4373 < q < 510510$. A direct search shows that two primes $p = 1 \pmod q$ and $p \leq 1.2(\varphi(q)\log q)^2$ exist also in the range $5 \leq q \leq 4373$. ☐

Remark. We can repeat the proof of the previous corollary in a more general setting. Letting $\kappa = \varphi(q') + (k-1)\varphi(q)$ one can prove that, when $q > e^{13}$, there are at least k primes congruent to 1 modulo q and smaller than

$$\big((1.075 + 0.02\sqrt{k\log k})\varphi(q)\log q\big)^2.$$

Computing the residue of $\zeta_\mathbb{K}$. An explicit form for the remainder of the formula for any $\psi_\mathbb{K}^{(m)}$ gives a way to compute within a prefixed error any quantity which can be written as a Dirichlet series in the von Mangoldt function of the field. Among these, the computation of the logarithm of the residue of $\zeta_\mathbb{K}$ with an error lower than $\frac{1}{2}\log 2$ is a particularly important problem, being an essential step of Buchmann's algorithm [6] for the computation of the class group and the regulator of the ring of integral elements in $\mathbb{K}$. The representation

$$\log\zeta_\mathbb{K}(s) - \log\zeta(s) = \sum_{n=2}^\infty \frac{\tilde\Lambda_\mathbb{K}(n) - \Lambda_\mathbb{Q}(n)}{n^s\log n}$$

holds true uniformly in $\mathrm{Re}(s) \geq 1$ by Landau's and de la Vallée–Poussin's estimates for the remainder terms of $\psi_\mathbb{K}(x)$ and $\psi_\mathbb{Q}(x)$. Hence, a simple way to compute the residue is

$$\log\operatorname*{res}_{s=1}\zeta_\mathbb{K}(s) = \lim_{s\to 1}[\log\zeta_\mathbb{K}(s) - \log\zeta(s)] = \sum_{n=2}^\infty \frac{\tilde\Lambda_\mathbb{K}(n) - \Lambda_\mathbb{Q}(n)}{n\log n}.$$

Here, truncating the series at a level N and using the partial summation formula one gets

$$\log \operatorname*{res}_{s=1} \zeta_{\mathbb{K}}(s) = \sum_{n\le N} \big(\tilde{\Lambda}_{\mathbb{K}}(n) - \Lambda_{\mathbb{Q}}(n)\big)\big(f(n) - f(N)\big) + \mathcal{R}(N) \tag{2.1}$$

with $f(x) := (x\log x)^{-1}$ and

$$\mathcal{R}(N) := -\int_N^{+\infty} (\psi_{\mathbb{K}}(x) - \psi_{\mathbb{Q}}(x)) f'(x)\,\mathrm{d}x.$$

Moving the absolute value into the integral and using (1.1) yields

$$|\mathcal{R}(N)| \le \frac{c}{\sqrt{N}}(\log\Delta_{\mathbb{K}} + n_{\mathbb{K}}\log N)$$

for an explicit constant c. This procedure can already be used to compute the residue, but a substantial improvement has been obtained by Bach [2] and very recently published by Belabas and Friedman [5]. They propose different approximations to $\log\operatorname{res}_{s=1}\zeta_{\mathbb{K}}(s)$ with a remainder term which is essentially estimated by $c\frac{\log\Delta_{\mathbb{K}}}{\sqrt{N}\log N}$, with $c = 8.33$ in Bach's work and $c = 2.33$ in the one of Belabas and Friedman. The presence of the extra $\log N$ in the denominator and the small multiplicative constant in their formulas represent a strong boost to the computation, but this is achieved at the cost of some complexities in the proofs and in the implementation of the algorithm.
Using Theorem 1.1 after a further integration by parts of Equation (2.1) we get the same result with a simpler approach and already smaller constants. Even stronger results are available in Section 6. The following corollary is a part of Corollary 6.1.

Corollary 2.4. *(GRH) For $N \ge 3$, we have*

$$\log \operatorname*{res}_{s=1} \zeta_{\mathbb{K}}(s) = \sum_{n\le N} \big(\tilde{\Lambda}_{\mathbb{K}}(n) - \Lambda_{\mathbb{Q}}(n)\big)\big(f(n) - f(N) - (n-N)f'(N)\big) + \mathcal{R}^{(1)}(N)$$

with

$$|\mathcal{R}^{(1)}(N)| \le \alpha_{\mathbb{K}}^{(1)}\Big(\frac{\frac{5}{2}+y}{\sqrt{N}\log N} + \frac{3}{4}\operatorname{E}_1\Big(\frac{1}{2}\log N\Big)\Big) + \beta_{\mathbb{K}}^{(1)}\frac{2+3y}{N} + \gamma_{\mathbb{K}}^{(1)}\frac{2y+y^2}{N} + \delta_{\mathbb{K}}^{(1)}\frac{y+y^2}{N^2},$$

$f(x) := (x\log x)^{-1}$, $y := (\log N)^{-1}$, $E_1(x) := \int_1^{+\infty} e^{-xt}t^{-1}\,\mathrm{d}t$ and

$$\alpha_{\mathbb{K}}^{(1)} = 0.5375\log\Delta_{\mathbb{K}} - 1.0355 n_{\mathbb{K}} + 5.4341, \qquad \beta_{\mathbb{K}}^{(1)} = n_{\mathbb{K}} - 1,$$
$$\gamma_{\mathbb{K}}^{(1)} = 1.0155\log\Delta_{\mathbb{K}} - 2.1041 n_{\mathbb{K}} + 10.1799, \qquad \delta_{\mathbb{K}}^{(1)} = \log\Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4.$$

The E_1 function satisfies the double inequality $1 - 1/x \le xe^x\operatorname{E}_1(x) \le 1$ for every $x > 0$. Thus this strategy produces an error bounded essentially by $2.15\frac{\log\Delta_{\mathbb{K}}}{\sqrt{N}\log N}$: this means that our algorithm is in N of the same order of Bach's and Belabas–Friedman's results with a smaller constant. Moreover, the negative coefficient for the contribution of the degree has the interesting side effect that, for fixed discriminant, the complexity actually decreases for increasing degree.

As shown in Tables 4 and 5 below, in practice Corollary 2.4 improves on Belabas and Friedman's procedure by a factor of about 3, and in some ranges even by a factor of 10.

3. Preliminary inequalities

For $\mathrm{Re}(s) > 1$ we have

$$-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \log(\mathrm{N}\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-ms},$$

which in terms of standard Dirichlet series reads

$$-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{n=1}^{\infty} \tilde{\Lambda}_{\mathbb{K}}(n) n^{-s} \quad \text{observing that} \quad \tilde{\Lambda}_{\mathbb{K}}(n) = \begin{cases} \displaystyle\sum_{\mathfrak{p}|p,\, f_{\mathfrak{p}}|k} \log \mathrm{N}\mathfrak{p} & \text{if } n = p^k \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{\mathfrak{p}}$ is the residual degree of $\mathfrak{p}$. The formula for $\tilde{\Lambda}_{\mathbb{K}}$ shows that $\tilde{\Lambda}_{\mathbb{K}}(n) \leq n_{\mathbb{K}}\Lambda(n)$ for every integer n, so that immediately we get

$$0 < -\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(\sigma) \leq -n_{\mathbb{K}} \frac{\zeta'}{\zeta}(\sigma) \qquad \forall \sigma > 1. \tag{3.1}$$

Let

$$\Gamma_{\mathbb{K}}(s) := \Big[\pi^{-\frac{s+1}{2}} \Gamma\Big(\frac{s+1}{2}\Big)\Big]^{r_2} \Big[\pi^{-\frac{s}{2}} \Gamma\Big(\frac{s}{2}\Big)\Big]^{r_1+r_2} \tag{3.2}$$

and

$$\xi_{\mathbb{K}}(s) := s(s-1)\Delta_{\mathbb{K}}^{s/2} \Gamma_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s). \tag{3.3}$$

The functional equation for $\zeta_{\mathbb{K}}$ then reads

$$\xi_{\mathbb{K}}(1-s) = \xi_{\mathbb{K}}(s). \tag{3.4}$$

Since $\xi_{\mathbb{K}}(s)$ is an entire function of order 1 and does not vanish at $s = 0$, one has

$$\xi_{\mathbb{K}}(s) = e^{A_{\mathbb{K}} + B_{\mathbb{K}} s} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho} \tag{3.5}$$

for some constants $A_{\mathbb{K}}$ and $B_{\mathbb{K}}$, where ρ runs through all the zeros of $\xi_{\mathbb{K}}(s)$. These are precisely the zeros $\rho = \beta + i\gamma$ of $\zeta_{\mathbb{K}}(s)$ for which $0 < \beta < 1$ and are the so-called "nontrivial zeros" of $\zeta_{\mathbb{K}}(s)$. From now on ρ will denote a nontrivial zero of $\zeta_{\mathbb{K}}(s)$. We recall that the zeros are symmetric with respect to the real axis, as a consequence of the fact that $\zeta_{\mathbb{K}}(s)$ is real for $s \in \mathbb{R}$.
Differentiating (3.3) and (3.5) logarithmically we obtain the identity

$$\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = B_{\mathbb{K}} + \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \frac{1}{2}\log \Delta_{\mathbb{K}} - \Big(\frac{1}{s} + \frac{1}{s-1}\Big) - \frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}(s). \tag{3.6}$$

Stark [24, Lemma 1] proved that the functional equation (3.4) implies that $B_{\mathbb{K}} = -\sum_{\rho} \mathrm{Re}(\rho^{-1})$ (see also [17] and [12, Ch. XVII, Th. 3.2]), and that once this information is available one can use (3.6) and the definition of the gamma factor in (3.2) to prove that the function $f_{\mathbb{K}}(s) = \sum_{\rho} \mathrm{Re}\left(\frac{2}{s-\rho}\right)$ can be computed via the alternative representation

$$f_{\mathbb{K}}(s) = 2\,\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) + \log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}} + \mathrm{Re}\Big(\frac{2}{s} + \frac{2}{s-1}\Big) + (r_1+r_2)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + r_2\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big). \tag{3.7}$$

Using (3.3), (3.4) and (3.6) one sees that

$$\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \begin{cases} \dfrac{r_1+r_2-1}{s} + r_{\mathbb{K}} + O(s) & \text{as } s\to 0 \\ \dfrac{r_2}{s+1} + r'_{\mathbb{K}} + O(s+1) & \text{as } s\to -1 \\ \dfrac{r_1+r_2}{s+2} + r''_{\mathbb{K}} + O(s+2) & \text{as } s\to -2, \end{cases} \tag{3.8}$$

where

$$r_{\mathbb{K}} = B_{\mathbb{K}} + 1 - \frac{1}{2}\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}} - \frac{r_1+r_2}{2}\frac{\Gamma'}{\Gamma}(1) - \frac{r_2}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big), \tag{3.9a}$$

$$r'_{\mathbb{K}} = -\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(2) - \log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}} - \frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}\Big) - \frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}(1), \tag{3.9b}$$

$$r''_{\mathbb{K}} = -\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(3) - \log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}} - \frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}(2) - \frac{n_{\mathbb{K}}}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}\Big). \tag{3.9c}$$

In order to prove our results we need explicit bounds for $B_{\mathbb{K}}$, $r_{\mathbb{K}}$, $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ and for some auxiliary functions.

Lemma 3.1. *$B_{\mathbb{K}}$ is real, negative, and under GRH we have*

$$|B_{\mathbb{K}}| \le 0.5155\log\Delta_{\mathbb{K}} - 1.2432 n_{\mathbb{K}} + 9.3419.$$

Proof. We know that $-B_{\mathbb{K}} = \sum_\rho \operatorname{Re}\big(\frac{1}{\rho}\big) = \sum_\rho \frac{\operatorname{Re}(\rho)}{|\rho|^2}$, which is positive. The upper bound will be proved in next section. ☐

Lemma 3.2. *(GRH) We have*

$$\begin{aligned} |r_{\mathbb{K}}| &\le 1.0155\log\Delta_{\mathbb{K}} - 2.1042 n_{\mathbb{K}} + 8.3423,\\ |r'_{\mathbb{K}}| &\le \log\Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4,\\ |r''_{\mathbb{K}}| &\le \log\Delta_{\mathbb{K}} - 0.9151 n_{\mathbb{K}} + 2. \end{aligned}$$

Proof. Substituting the values $-\frac{\Gamma'}{\Gamma}(\frac{1}{2}) = \gamma + 2\log 2$, $-\frac{\Gamma'}{\Gamma}(1) = \gamma$ in (3.9a) we get

$$r_{\mathbb{K}} = B_{\mathbb{K}} - \frac{1}{2}\log\Delta_{\mathbb{K}} + (\log\pi+\gamma)\frac{n_{\mathbb{K}}}{2} + r_2\log 2 + 1. \tag{3.10}$$

By Lemma 3.1 we get

$$r_{\mathbb{K}} \le -\frac{1}{2}\log\Delta_{\mathbb{K}} + (\gamma+\log 2\pi)\frac{n_{\mathbb{K}}}{2} + 1 \le -\frac{1}{2}\log\Delta_{\mathbb{K}} + 1.2076 n_{\mathbb{K}} + 1$$

and

$$\begin{aligned} r_{\mathbb{K}} &\ge -(0.5155\log\Delta_{\mathbb{K}} - 1.2432 n_{\mathbb{K}} + 9.3423) - \frac{1}{2}\log\Delta_{\mathbb{K}} + (\log\pi+\gamma)\frac{n_{\mathbb{K}}}{2} + 1\\ &\ge -1.0155\log\Delta_{\mathbb{K}} + 2.1042 n_{\mathbb{K}} - 8.3423. \end{aligned}$$

The (opposite of the) lower bound for $r_{\mathbb{K}}$ gives the upper bound for $|r_{\mathbb{K}}|$, since the explicit bounds for the discriminant in terms of the degree proved by Odlyzko (see [13, 15, 16, 18] and Table 3 in [14]) show that the difference

$$\begin{aligned} 1.0155\log\Delta_{\mathbb{K}} - 2.1042 n_{\mathbb{K}} + 8.3423 - \big(-\tfrac{1}{2}\log\Delta_{\mathbb{K}} + 1.2076 n_{\mathbb{K}} + 1\big)&\\ = 1.5155\log\Delta_{\mathbb{K}} - 3.3118 n_{\mathbb{K}} + 7.3423& \end{aligned} \tag{3.11}$$

is always positive (use the entry $b = 1.3$ in [14, Tab. 3]).
The bounds for $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ are proved with a similar argument. By (3.9b) and the identities $-\frac{\Gamma'}{\Gamma}(\frac{3}{2}) = \gamma + 2\log 2 - 2$, $-\frac{\Gamma'}{\Gamma}(1) = \gamma$ we have

$$r'_{\mathbb{K}} = -\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(2) - \log\Delta_{\mathbb{K}} + (\log 2\pi + \gamma - 1)n_{\mathbb{K}}.$$

By (3.1) we have

$$r'_{\mathbb{K}} \le -\log\Delta_{\mathbb{K}} + \Big(-\frac{\zeta'}{\zeta}(2) + \log 2\pi + \gamma - 1\Big)n_{\mathbb{K}} \le -\log\Delta_{\mathbb{K}} + 1.9851 n_{\mathbb{K}}$$

and

$$r'_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + (\log 2\pi + \gamma - 1)n_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + 1.415 n_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + 1.415 n_{\mathbb{K}} - 4.$$

The lower bounds for the discriminant prove that the inequality

$$\log\Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4 - \big(-\log\Delta_{\mathbb{K}} + 1.9851 n_{\mathbb{K}}\big) = 2\log\Delta_{\mathbb{K}} - 3.4001 n_{\mathbb{K}} + 4 \ge 0 \tag{3.12}$$

is true for $n_{\mathbb{K}} \ge 5$ (entry $b = 1$ in [14, Tab. 3]). Using the "megrez" number field tables [20] we find that (3.12) has only two exceptions for fields of equation x^2+x+1 and $x^4-x^3-x^2+x+1$. We numerically compute the value of $r'_{\mathbb{K}}$ for these two fields and we find that indeed $|r'_{\mathbb{K}}| \le \log\Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4$.

Finally, by (3.9c)

$$r''_{\mathbb{K}} = -\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(3) - \log\Delta_{\mathbb{K}} + \Big(\log 2\pi + \gamma - \frac{3}{2}\Big)n_{\mathbb{K}}$$

and thus

$$r''_{\mathbb{K}} \le -\log\Delta_{\mathbb{K}} + \Big(-\frac{\zeta'}{\zeta}(3) + \log 2\pi + \gamma - \frac{3}{2}\Big)n_{\mathbb{K}} \le -\log\Delta_{\mathbb{K}} + 1.08 n_{\mathbb{K}}$$

and

$$r''_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + (\log 2\pi + \gamma - \tfrac{3}{2})n_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + 0.9151 n_{\mathbb{K}} \ge -\log\Delta_{\mathbb{K}} + 0.9151 n_{\mathbb{K}} - 2.$$

The lower bounds for the discriminant prove that the inequality

$$\log\Delta_{\mathbb{K}} - 0.9151 n_{\mathbb{K}} + 2 - \big(-\log\Delta_{\mathbb{K}} + 1.08 n_{\mathbb{K}}\big) = 2\log\Delta_{\mathbb{K}} - 1.9951 n_{\mathbb{K}} + 2 \ge 0 \tag{3.13}$$

is true for all $n_{\mathbb{K}}$ (entry $b = 0.6$ in [14, Tab. 3]). □

Lemma 3.3. *For $x \ge 1$ let*

$$f_1^{(1)}(x) := \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)}, \qquad f_2^{(1)}(x) := \sum_{r=2}^{\infty} \frac{x^{2-2r}}{(2r-1)(2r-2)},$$
$$f_1^{(2)}(x) := \sum_{r=2}^{\infty} \frac{x^{2-2r}}{r(2r-1)(2r-2)}, \quad f_2^{(2)}(x) := \sum_{r=1}^{\infty} \frac{x^{1-2r}}{(2r+1)r(2r-1)},$$

and

$$\begin{aligned} R^{(1)}_{r_1,r_2}(x) &:= -d_{\mathbb{K}} x(\log x - 1) + r_2(\log x + 1) - (r_1+r_2)f_1^{(1)}(x) - r_2 f_2^{(1)}(x), \\ R^{(2)}_{r_1,r_2}(x) &:= -d_{\mathbb{K}} x^2\Big(\log x - \frac{3}{2}\Big) + 2r_2 x\log x - (r_1+r_2)\Big(\log x + \frac{3}{2}\Big) \\ &\qquad + (r_1+r_2)f_1^{(2)}(x) + r_2 f_2^{(2)}(x). \end{aligned}$$

If $x \geq 3$ *then*

$$|R^{(1)}_{r_1,r_2}(x)| \leq (n_{\mathbb{K}}-1)x\log x + \delta_{n_{\mathbb{K}},1}\frac{0.5097}{x},$$
$$|R^{(2)}_{r_1,r_2}(x)| \leq (n_{\mathbb{K}}-1)x^2(\log x - \tfrac{1}{2}) + \delta_{n_{\mathbb{K}},1}(\log x + 2)$$

where $\delta_{n_{\mathbb{K}},1}$ *is* 1 *if* $n_{\mathbb{K}} = 1$ *and* 0 *otherwise.*

Proof. We have

$$\begin{aligned}
f^{(1)}_1(x) &= \frac{1}{2}\Big[x\log(1-x^{-2}) + \log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big],\\
f^{(1)}_2(x) &= 1 - \frac{1}{2}\Big[\log(1-x^{-2}) + x\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big],\\
f^{(2)}_1(x) &= \frac{3}{2} - \frac{1}{2}(x^2+1)\log(1-x^{-2}) - x\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big),\\
f^{(2)}_2(x) &= -x + x\log(1-x^{-2}) + \frac{1}{2}(x^2+1)\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big),
\end{aligned}$$

and the claims follow with elementary arguments. □

4. PROOF OF THE THEOREM

When $m \geq 1$ the equality in (1.2) follows by the Dirichlet series representation of $\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)$ and the special integrals

$$\frac{m!}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{y^{s+m}}{\prod_{u=0}^{m}(s+u)}\,\mathrm{d}s = \begin{cases}(y-1)^m & \text{if } y > 1\\ 0 & \text{if } 0 < y \leq 1\end{cases} \qquad \forall m \geq 1.$$

The case $m = 0$ is more complicated but well known (see [10]). Equalities (1.3a–1.3b) come from the Cauchy residue theorem, using the identities

$$\frac{x^{s+1}}{s(s+1)} = \begin{cases}\frac{x}{s} + x\log x - x + O(s) & \text{as } s \to 0\\ -\frac{x}{s+1} - \log x - 1 + O(s+1) & \text{as } s \to -1,\end{cases}$$

$$\frac{x^{s+2}}{s(s+1)(s+2)} = \begin{cases}\frac{x^2}{2s} + \frac{x^2}{2}\log x - \frac{3}{4}x^2 + O(s) & \text{as } s \to 0\\ -\frac{x}{s+1} - x\log x + O(s+1) & \text{as } s \to -1\\ \frac{1}{2(s+2)} + \frac{1}{2}\log x + \frac{3}{4} + O(s+2) & \text{as } s \to -2,\end{cases}$$

and the definitions of $r_{\mathbb{K}}$, $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ in (3.8) and of $R^{(m)}_{r_1,r_2}(x)$ in Lemma 3.3. They show that

$$\Big|\psi^{(1)}_{\mathbb{K}}(x) - \frac{x^2}{2}\Big| \leq x^{3/2}\sum_{\rho}\frac{1}{|\rho(\rho+1)|} + |xr_{\mathbb{K}} - r'_{\mathbb{K}} - R^{(1)}_{r_1,r_2}(x)|,$$

$$\Big|\psi^{(2)}_{\mathbb{K}}(x) - \frac{x^3}{3}\Big| \leq x^{5/2}\sum_{\rho}\frac{2}{|\rho(\rho+1)(\rho+2)|} + |x^2 r_{\mathbb{K}} - 2xr'_{\mathbb{K}} + r''_{\mathbb{K}} - R^{(2)}_{r_1,r_2}(x)|.$$

For $\mathbb{Q}$, we observe that $|xr_{\mathbb{Q}} - r'_{\mathbb{Q}} - R^{(1)}_{1,0}(x)| \leq x\log 2\pi$ and $|x^2 r_{\mathbb{Q}} - 2xr'_{\mathbb{Q}} + r''_{\mathbb{Q}} - R^{(2)}_{1,0}(x)| \leq x^2\log 2\pi$. For generic $\mathbb{K}$ these terms are estimated with the sums of the absolute values, and $|r_{\mathbb{K}}|$, $|r'_{\mathbb{K}}|$, $|r''_{\mathbb{K}}|$ and $|R^{(j)}_{r_1,r_2}(x)|$ have already been estimated in Lemmas 3.2 and 3.3. We thus only need a bound for $\sum_\rho |\rho(\rho+1)|^{-1}$ and $\sum_\rho |\rho(\rho+1)(\rho+2)|^{-1}$. It is easy to check that

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq \frac{2}{3} f_{\mathbb{K}}\Big(\frac{3}{2}\Big) \qquad \text{and} \qquad \sum_\rho \frac{1}{|\rho(\rho+1)(\rho+2)|} \leq \frac{4}{15} f_{\mathbb{K}}\Big(\frac{3}{2}\Big). \tag{4.1}$$

A bound comes from the estimation $f_{\mathbb{K}}\big(\frac{3}{2}\big) \leq \log\Delta_{\mathbb{K}} - (\gamma + \log 8\pi - 2)n_{\mathbb{K}} + \frac{16}{3}$, which is the case $a = 1/2$ of Lemma 5.6 in [1] and of Lemma 4.6 in [3], but we can do better.

Lemma 4.1. *(GRH) We have*

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq 0.5375 \log\Delta_{\mathbb{K}} - 1.0355 n_{\mathbb{K}} + 5.3879,$$

$$\sum_\rho \frac{1}{|\rho(\rho+1)(\rho+2)|} \leq 0.1763 \log\Delta_{\mathbb{K}} - 0.4106 n_{\mathbb{K}} + 2.2496.$$

For the Riemann zeta function the conclusions improve to

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq 0.0462, \qquad \sum_\rho \frac{1}{|\rho(\rho+1)(\rho+2)|} \leq 0.00146.$$

Proof. We apply the method we have described in the introduction with real s, so $f(s,\gamma) = 4(2s-1)/((2s-1)^2+4\gamma^2)$. We choose $D = \{s_j : j = 1,2,\dots\}$ with $s_j := 1 + j/2$, and μ compactly supported on D. For the first claim let $g(\gamma) := 4/((1+4\gamma^2)(9+4\gamma^2))^{1/2}$, so that $\sum_\rho |\rho(\rho+1)|^{-1} = \sum_\gamma g(\gamma)$. Condition (1.4) indicates that we must prove

$$g(\gamma) \leq F(\gamma) := \sum_j a_j f(s_j,\gamma) \qquad \forall \gamma \in \mathbb{R} \tag{4.2}$$

for suitable a_j. Recalling that $f_{\mathbb{K}}(s) = \sum_\gamma f(s,\gamma)$, Inequality (1.5) gives

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq \sum_j a_j f_{\mathbb{K}}(s_j), \tag{4.3}$$

which generalizes (4.1). From (4.3) and (3.7), and once (4.2) is proved, we obtain a bound for $\sum_\rho |\rho(\rho+1)|^{-1}$. The final coefficient of $\log\Delta_{\mathbb{K}}$ will then be the sum of all a_j, thus we are interested in linear combinations for which this sum is as small as possible. We choose the support of μ such that the s_j appearing in (4.2) are those with $1 \leq j \leq 2q$ for a suitable integer q. Let $\Upsilon \subset (0,\infty)$ be a set with $q-1$ numbers. We require:

(1) $g(\gamma) = F(\gamma)$ for all $\gamma \in \{0\} \cup \Upsilon$,
(2) $g'(\gamma) = F'(\gamma)$ for all $\gamma \in \Upsilon$,
(3) $\lim_{\gamma\to\infty} \gamma^2 g(\gamma) = \lim_{\gamma\to\infty} \gamma^2 F(\gamma)$.

This produces a set of $2q$ linear equations for the $2q$ constants a_j. The first conditions impose a double contact between g and F in all the points of Υ. This means that g will almost certainly not cross F at these points. With a little bit of luck, F will be always above g ensuring (4.2). We chose $q := 40$ and $\Upsilon := \{v^i - v + 1 : 1 \leq i \leq q-1\}$ for $v := 1.21$. Finally, with an abuse of notation we took for a_j the solution of the system, rounded above to 10^{-7}: this produces the numbers in Table 6. Then, using Sturm's algorithm, we prove that the values found actually give an upper bound for g, so that (4.3) holds with such a_j's. These constants verify

$$\begin{aligned}
&\sum_j a_j = 0.53747\ldots, && \sum_j a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big) \leq 5.3879, \\
&\sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \leq -0.6838, && \sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \leq -0.1567.
\end{aligned} \tag{4.4}$$

Moreover, the sum $\sum_j a_j \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s_j)$ is negative. Indeed we write it as

$$-\sum_n \tilde{\Lambda}_{\mathbb{K}}(n)S(n) \quad \text{with} \quad S(n) := \sum_j \frac{a_j}{n^{s_j}}$$

and, since the signs of the a_j's alternate, we can easily prove that the sum in pairs $\frac{a_1}{n^{s_1}} + \frac{a_2}{n^{s_2}}$, $\ldots$, $\frac{a_{2q-1}}{n^{s_{2q-1}}} + \frac{a_{2q}}{n^{s_{2q}}}$ are positive for $n \geq 26500$. Then we check numerically that $S(n) > 0$ also for $n \leq 26500$. The result now follows from (3.7), (4.3) and (4.4).

For the second inequality, let $g(\gamma) := 8/((1+4\gamma^2)(9+4\gamma^2)(25+4\gamma^2))^{1/2}$, so that $\sum_\rho |\rho(\rho+1)(\rho+2)|^{-1} = \sum_\gamma g(\gamma)$. We use s_j with $1 \leq j \leq 2q-1$, $q := 20$, $\Upsilon := \{v^i - v + 0.75 \colon 1 \leq i \leq q-1\}$, keeping $v = 1.21$, and the conditions

(1) $g(\gamma) = F(\gamma)$ for all $\gamma \in \{0\} \cup \Upsilon$,
(2) $g'(\gamma) = F'(\gamma)$ for all $\gamma \in \Upsilon$.

We take for a_j the solution of the system, rounded above to 10^{-7}: this produces the numbers in Table 7. We check their validity using Sturm's algorithm as before. We then have

$$\sum_\rho \frac{1}{|\rho(\rho+1)(\rho+2)|} \leq \sum_j a_j f_{\mathbb{K}}(s_j) \tag{4.5}$$

where the constants a_j verify

$$\begin{aligned}
&\sum_j a_j = 0.17629\ldots, && \sum_j a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big) \leq 2.2496, \\
&\sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \leq -0.3130, && \sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \leq -0.1047, \\
&\sum_j a_j \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s_j) \leq 0.
\end{aligned} \tag{4.6}$$

As before, we prove the last inequality noticing that it is $-\sum_n \tilde{\Lambda}_{\mathbb{K}}(n)S(n)$ with $S(n) := \sum_j \frac{a_j}{n^{s_j}}$, and that each $S(n)$ is positive since this is true for $n \leq 16800$ (numerical test) and since the sums in pairs $\frac{a_1}{n^{s_1}} + \frac{a_2}{n^{s_2}}$, $\ldots$, $\frac{a_{2q-3}}{n^{s_{2q-3}}} + \frac{a_{2q-2}}{n^{s_{2q-2}}}$ and the last summand $\frac{a_{2q-1}}{n^{s_{2q-1}}}$ are positive for $n \geq 16800$. The result now follows from (3.7), (4.5) and (4.6).
For the Riemann zeta function we proceed as in the general case, but now using the numerical value of $\sum_j a_j f_{\mathbb{Q}}(s_j)$. □

Remark 4.2. For the Riemann zeta function one has $\sum_{|\gamma|\geq T} |\rho|^{-2} \leq 10^{-5}$ when $T \geq 400000$ (by partial summation, using [22, Th. 19] or [25, Cor. 1]), thus the value of $\sum_\rho |\rho(\rho+1)|^{-1}$ correct up to the fifth digit can be obtained summing the first $7 \cdot 10^5$ zeros. The computation produces the number 0.0461(1). In a similar way, $\frac{1}{T}\sum_{|\gamma|\leq T}|\rho|^{-2}\leq 10^{-10}$ when $T\geq 200000$, thus the value of $\sum_\rho |\rho(\rho+1)(\rho+2)|^{-1}$ correct up to the tenth digit can be obtained summing the first $3\cdot 10^5$ zeros. The computation produces the number 0.001439963(2). In both cases the bounds in Lemma 4.1 essentially agree with the actual values.

Remark 4.3. Let $g_m(\gamma) := \prod_{n=0}^m |n+\frac12+i\gamma|^{-1}$. As observed in Remark 1.2, $\sum_j a_j \geq \frac{1}{2\pi}\int_{\mathbb{R}} g_1(\gamma)\,\mathrm{d}\gamma \geq 0.53659$ in the first case, and $\sum_j a_j \geq \frac{1}{2\pi}\int_{\mathbb{R}} g_2(\gamma)\,\mathrm{d}\gamma \geq 0.1759$ in the second case are the best coefficients of $\log\Delta_{\mathbb{K}}$ we can get from our method. Thus, what we got in Lemma 4.1 are close to the best. Moreover, for a generic $m\geq 1$ one gets

$$\Big|\psi_{\mathbb{K}}^{(m)}(x) - \frac{x^{m+1}}{m+1}\Big| \leq m!\,x^{m+1/2}\sum_\rho g_m(\gamma) + \text{lower order terms}$$

and we need an upper bound of $\sum_\rho g_m(\gamma)$. If we could follow the argument proving Lemma 4.1 for general m we would get a sequence a_j (a different sequence for every m) necessarily satisfying the lower bound $\sum_j a_j \geq \frac{1}{2\pi}\int_{\mathbb{R}} g_m(\gamma)\,\mathrm{d}\gamma$. Since $\int_{\mathbb{R}} g_m(\gamma)\,\mathrm{d}\gamma \sim \frac{\sqrt{m}}{(m+1)!}\int_{\mathbb{R}}|\Gamma(\frac12+i\gamma)|\,\mathrm{d}\gamma$ when m tends to infinity, in this way we cannot produce an upper-bound for $|\psi_{\mathbb{K}}^{(m)}(x) - \frac{x^{m+1}}{m+1}|$ with a coefficient for $\log\Delta_{\mathbb{K}}$ better than $x^{m+1/2}(\frac{1}{2\pi\sqrt{m}}+o(1))\int_{\mathbb{R}}|\Gamma(\frac12+i\gamma)|\,\mathrm{d}\gamma$. Iterating m times the partial summation for the logarithm of the residue of $\zeta_{\mathbb{K}}$ we get a remainder term which, in its main part, is controlled by $2(m+1)!\sum_j a_j$, so that it tends to infinity as $\frac{\sqrt{m}}{\pi}\int_{\mathbb{R}}|\Gamma(\frac12+i\gamma)|\,\mathrm{d}\gamma$: this proves that one cannot expect to improve the algorithm for the residue simply increasing m. A closer look at the sequence $(m+1)!\int_{\mathbb{R}} g_m(\gamma)\,\mathrm{d}\gamma$ shows that it attains its minimum exactly when $m=2$, so that our formulas are already the best we can produce.

Proof of Lemma 3.1. We still follow the method described in the introduction. We use $s_j = 1+j/2$ as in Lemma 4.1. Let $g(\gamma) := 2/(1+4\gamma^2)$, so that $|B_{\mathbb{K}}| = \sum_\gamma g(\gamma)$. Then using Sturm's algorithm we see that $g(\gamma)\leq\sum_{j=1}^{10} a_j f(s_j,\gamma)$ for every $\gamma\in\mathbb{R}$, when the constants a_j have the values in Table 8. As for Lemma 4.1 the numbers a_j have been generated imposing a double contact at the points in $\Upsilon := \{0.84, 2.04, 4.01, 9.61\}$, the equality at $\gamma=0$ and the asymptotic equality for $\gamma\to\infty$. With these constants we have

$$\begin{aligned}
&\sum_j a_j = 0.51543\ldots, &&\sum_j a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big)\leq 9.3419,\\
&\sum_j a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big)\leq -1.0094, &&\sum_j a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big)\leq -0.297,\\
&\sum_j a_j\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s_j)\leq 0,
\end{aligned}\tag{4.7}$$

where the last inequality follows by noticing once again that it is $-\sum_n\tilde{\Lambda}_{\mathbb{K}}(n)S(n)$ with $S(n) := \sum_j\frac{a_j}{n^{s_j}}$, and that each $S(n)$ is positive (for $n<150$ by numerical test, and for every $n\geq 150$ because the sums in pairs $\frac{a_1}{n^{s_1}}+\frac{a_2}{n^{s_2}}$, …, $\frac{a_9}{n^{s_9}}+\frac{a_{10}}{n^{s_{10}}}$ are positive). The result now follows from (3.7) and (4.7). □

Remark 4.4. The best coefficient of $\log\Delta_{\mathbb{K}}$ we can get from our argument is $\frac{1}{2}$. Moreover, trying to find a lower bound, we can prove $|B_{\mathbb{K}}| \geq 0.4512\log\Delta_{\mathbb{K}} - 5.2554n_{\mathbb{K}} + 5.2784$. Unfortunately this bound is not sufficiently strong to produce anything useful for our purposes, thus we do not include its proof.

5. Proof of Corollary 2.1

Proof of the case $\kappa = 0$. We write

$$\psi_{\mathbb{K}}^{(1)}(x) = \sum_{\substack{\mathfrak{p},\, m\\ \mathrm{N}\mathfrak{p}^m \leq x}} \log(\mathrm{N}\mathfrak{p})(x - \mathrm{N}\mathfrak{p}^m)$$

as $S_1 + S_2$, where S_1 is the contribution to $\psi_{\mathbb{K}}^{(1)}(x)$ coming from the primes in the statement, and S_2 is the complementary term. Thus

$$S_1 := \sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\text{ prime}}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m \leq x}} (x - \mathrm{N}\mathfrak{p}^m) = \sum_{p\leq x}\Big(\sum_{\substack{\mathfrak{p}|p\\ \mathrm{N}\mathfrak{p}=p}} 1\Big)\log p \sum_{\substack{m\\ p^m\leq x}}(x - p^m)$$

and

$$S_2 := \sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\text{ not prime}}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m \leq x}} (x - \mathrm{N}\mathfrak{p}^m) = \sum_{p\leq x}\sum_{\substack{\mathfrak{p}|p\\ \mathrm{N}\mathfrak{p}=p^{f_\mathfrak{p}},f_\mathfrak{p}\geq 2}} f_\mathfrak{p}\log p \sum_{\substack{m\\ p^{mf_\mathfrak{p}}\leq x}}(x - p^{mf_\mathfrak{p}}).$$

The definition of S_2 shows that

$$\begin{aligned}
S_2 &\leq \sum_{p\leq x}\sum_{\substack{\mathfrak{p}|p\\ \mathrm{N}\mathfrak{p}=p^{f_\mathfrak{p}},f_\mathfrak{p}\geq 2}} f_\mathfrak{p}\log p \sum_{\substack{m\\ p^m\leq\sqrt{x}}}(x - p^{2m}) \leq n_{\mathbb{K}}\sum_{p}\sum_{\substack{m\\ p^m\leq\sqrt{x}}}\log p(x - p^{2m})\\
&= n_{\mathbb{K}}\sum_{n\leq\sqrt{x}}\Lambda(n)(x-n^2) = n_{\mathbb{K}}(2\sqrt{x}\psi_{\mathbb{Q}}^{(1)}(\sqrt{x}) - \psi_{\mathbb{Q}}^{(2)}(\sqrt{x}))\\
&\leq n_{\mathbb{K}}\Big(\frac{2}{3}x^{3/2} + 2\sqrt{x}\Big|\psi_{\mathbb{Q}}^{(1)}(\sqrt{x}) - \frac{x}{2}\Big| + \Big|\psi_{\mathbb{Q}}^{(2)}(\sqrt{x}) - \frac{x^{3/2}}{3}\Big|\Big). \qquad (5.1)
\end{aligned}$$

Thus, in order to prove that S_1 is positive it is sufficient to verify that $\psi_{\mathbb{K}}^{(1)}(x)$ is larger than the function appearing on the right in (5.1), which can be estimated using the upper bounds for $\mathbb{Q}$ and the lower bound for $\psi_{\mathbb{K}}^{(1)}(x)$ in Theorem 1.1. After some simplifications the inequality is reduced to

$$\sqrt{x} \geq \mathcal{L}_{\mathbb{K}} = 1.075(\log\Delta_{\mathbb{K}} + 13) > A$$

where

$$\begin{aligned}
A :=& 2(0.5375\log\Delta_{\mathbb{K}} - 1.0355n_{\mathbb{K}} + 5.3879) + 2(n_{\mathbb{K}} - 1)\frac{\log x}{\sqrt{x}}\\
&+ \frac{2}{\sqrt{x}}(1.0155\log\Delta_{\mathbb{K}} - 2.1041n_{\mathbb{K}} + 8.3419) + \frac{2}{x^{3/2}}(\log\Delta_{\mathbb{K}} - 1.415n_{\mathbb{K}} + 4)\\
&+ 2n_{\mathbb{K}}\Big(\frac{2}{3} + \frac{0.0939}{x^{1/4}} + \frac{5.514}{x^{1/2}}\Big). \qquad (5.2)
\end{aligned}$$

After some rearrangements the inequality $\mathcal{L}_{\mathbb{K}} > A$ becomes

$$1.5996 + \frac{\log x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}(1.0155\log\Delta_{\mathbb{K}} + 8.3419) + \frac{1}{x^{3/2}}(\log\Delta_{\mathbb{K}} + 4)$$

$$+ n_{\mathbb{K}}\Big(-0.3688 + \frac{\log x}{\sqrt{x}} + \frac{0.0939}{x^{1/4}} + \frac{3.4099}{x^{1/2}} - \frac{1.415}{x^{3/2}}\Big)$$

which is implied by the simpler

$$0.6546 + \frac{\log x}{\sqrt{x}} \geq n_{\mathbb{K}}\Big(-0.3688 + \frac{\log x}{\sqrt{x}} + \frac{0.0939}{x^{1/4}} + \frac{3.4099}{x^{1/2}} - \frac{1.415}{x^{3/2}}\Big) \tag{5.3}$$

because

$$\frac{1.0155\log\Delta_{\mathbb{K}} + 8.3419}{\sqrt{x}} + \frac{\log\Delta_{\mathbb{K}} + 4}{x^{3/2}} \leq 0.945$$

under the assumption $\sqrt{x} \geq 1.075(\log\Delta_{\mathbb{K}}+13)$. The function appearing on the right-hand side of (5.3) is negative for $\sqrt{x} \geq 30$ and this is enough to prove the inequality when $\log\Delta_{\mathbb{K}} \geq 15$. If $\log\Delta_{\mathbb{K}} \leq 15$, Odlyzko's Table 3 [14] of inequalities for the discriminant shows that this may happen only for $n_{\mathbb{K}} \leq 8$. For every $n_{\mathbb{K}} \leq 8$ Inequality (5.3) holds when $x \geq \bar{x}$ for a suitable constant $\bar{x}$ depending on $n_{\mathbb{K}}$. However, for each $n_{\mathbb{K}}$ there is a minimal value $\bar{x}_{\min}$ for x, coming from the minimal discriminant for that degree (estimated again using Odlyzko's table). Values for $\bar{x}$ and $\bar{x}_{\min}$ are shown in Table 1: in every case $\bar{x} < \bar{x}_{\min}$, thus proving (5.3) also for $n_{\mathbb{K}} \leq 8$. □

Proof of the general case. Let $\mathcal{A}$ be the set of all degree-one prime ideals in $\mathcal{O}_{\mathbb{K}}$. Thus the term S_1 appearing in the decomposition of $\psi_{\mathbb{K}}^{(1)}(x)$ as $S_1 + S_2$ in the proof of the case $\kappa = 0$ reads

$$S_1 = \sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\leq x}} \delta_{\mathfrak{p}\in\mathcal{A}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m\leq x}} (x - \mathrm{N}\mathfrak{p}^m)$$

where $\delta_{\mathfrak{p}\in\mathcal{A}}$ is 1 if $\mathfrak{p}\in\mathcal{A}$ and 0 otherwise. With two applications of the Cauchy–Schwarz inequality we get

$$\begin{aligned}
S_1 &\leq \Big(\sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\leq x}} \delta_{\mathfrak{p}\in\mathcal{A}}\Big)^{1/2} \cdot \Big(\sum_{\mathfrak{p}} \log^2 \mathrm{N}\mathfrak{p}\Big(\sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m\leq x}} (x - \mathrm{N}\mathfrak{p}^m)\Big)^2\Big)^{1/2}\\
&\leq \Big(\sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\leq x}} \delta_{\mathfrak{p}\in\mathcal{A}}\Big)^{1/2} \cdot \Big(\sum_{\mathfrak{p}} \log^2 \mathrm{N}\mathfrak{p} \Big\lfloor \frac{\log x}{\log \mathrm{N}\mathfrak{p}}\Big\rfloor \sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m\leq x}} (x - \mathrm{N}\mathfrak{p}^m)^2\Big)^{1/2}\\
&\leq \Big(\sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\leq x}} \delta_{\mathfrak{p}\in\mathcal{A}}\Big)^{1/2} \cdot \sqrt{\log x}\Big(\sum_{\mathfrak{p}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m\\ \mathrm{N}\mathfrak{p}^m\leq x}} (x - \mathrm{N}\mathfrak{p}^m)^2\Big)^{1/2}\\
&= \Big(\sum_{\substack{\mathfrak{p}\\ \mathrm{N}\mathfrak{p}\leq x}} \delta_{\mathfrak{p}\in\mathcal{A}}\Big)^{1/2} \cdot \sqrt{\log x\psi_{\mathbb{K}}^{(2)}(x)}.
\end{aligned}$$

Thus, in order to have $\sum_{\mathrm{N}\mathfrak{p}\leq x} \delta_{\mathfrak{p}\in\mathcal{A}} > \kappa$ it is sufficient to have $S_1 > \sqrt{\kappa\log x\,\psi_{\mathbb{K}}^{(2)}(x)}$, i.e.

$$\psi_{\mathbb{K}}^{(1)}(x) > S_2 + \sqrt{\kappa\log x\ \psi_{\mathbb{K}}^{(2)}(x)}.$$

Recalling the upper bound (5.1) for S_2 and Theorem 1.1 (with $\mathbb{K}\neq\mathbb{Q}$), for the previous inequality it is sufficient to have

$$\sqrt{x} > A + 2\sqrt{\kappa B\log x}$$

where A is given in (5.2) and

$$\begin{aligned}
B :=& \frac{1}{3} + \frac{1}{\sqrt{x}}(0.3526 \log \Delta_{\mathbb{K}} - 0.8212 n_{\mathbb{K}} + 4.4992) + (n_{\mathbb{K}} - 1)\frac{1}{x}\Big(\log x - \frac{1}{2}\Big) \\
&+ \frac{1}{x}(1.0155 \log \Delta_{\mathbb{K}} - 2.1041 n_{\mathbb{K}} + 8.3419) + \frac{2}{x^2}(\log \Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4) \\
&+ \frac{1}{x^3}(\log \Delta_{\mathbb{K}} - 0.9151 n_{\mathbb{K}} + 2).
\end{aligned}$$

We can take $\sqrt{x} = \mathcal{L}_{\mathbb{K}} + \sqrt{8\kappa \log(\mathcal{L}_{\mathbb{K}} + \sqrt[3]{\kappa}\log\kappa)}$ with $\mathcal{L}_{\mathbb{K}} = 1.075(\log \Delta_{\mathbb{K}} + 13)$, and under this hypothesis function B is bounded by $2/3$. To prove it we notice that

$$\frac{1}{x}\Big(\log x - \frac{1}{2}\Big) \leq \frac{0.33}{\sqrt{x}}$$

because $\sqrt{x} \geq \mathcal{L}_{\mathbb{K}} \geq 15$. This remark and the assumption $n_{\mathbb{K}} \geq 2$ show that B is smaller than

$$\begin{aligned}
B \leq& \frac{1}{3} + \frac{1}{\sqrt{x}}(0.3526 \log \Delta_{\mathbb{K}} + 3.6) + \frac{1}{x}(1.0155 \log \Delta_{\mathbb{K}} + 4.2) + \frac{2}{x^2}(\log \Delta_{\mathbb{K}} + 1.2) \\
&+ \frac{1}{x^3}(\log \Delta_{\mathbb{K}} + 0.2).
\end{aligned}$$

It is now easy to prove that this is smaller than $2/3$ for $\sqrt{x} \geq \mathcal{L}_{\mathbb{K}}$.
Since $B \leq \frac{2}{3}$ we only need to prove that

$$\sqrt{x} > A + 2\sqrt{\frac{2}{3}}\sqrt{\kappa \log x}.$$

From the proof of Corollary 2.1 we already know that $\mathcal{L}_{\mathbb{K}} > A$. Thus the inequality holds when $\kappa = 0$ and for $\kappa > 0$ it is sufficient to verify that

$$(\mathcal{L}_{\mathbb{K}} + \sqrt[3]{\kappa}\log\kappa)^{3/2} \geq \mathcal{L}_{\mathbb{K}} + (8\kappa \log(\mathcal{L}_{\mathbb{K}} + \sqrt[3]{\kappa}\log\kappa))^{1/2}$$

which holds true for every $\mathcal{L}_{\mathbb{K}} \geq 15$ and every $\kappa > 0$. □

6. Proof of Corollary 2.4 and improvements

Starting with (2.1) and with, respectively, one and two further integrations by parts one gets

$$\log \operatorname*{res}_{s=1} \zeta_{\mathbb{K}}(s) = \sum_{n \leq N} \big(\tilde{\Lambda}_{\mathbb{K}}(n) - \Lambda_{\mathbb{Q}}(n)\big) W^{(1)}(n, N) + \mathcal{R}^{(1)}(N), \tag{6.1a}$$

$$\log \operatorname*{res}_{s=1} \zeta_{\mathbb{K}}(s) = \sum_{n \leq N} \big(\tilde{\Lambda}_{\mathbb{K}}(n) - \Lambda_{\mathbb{Q}}(n)\big) W^{(2)}(n, N) + \mathcal{R}^{(2)}(N) \tag{6.1b}$$

with the weights

$$\begin{aligned}
W^{(1)}(n, N) &:= f(n) - f(N) - (n - N) f'(N), \\
W^{(2)}(n, N) &:= f(n) - f(N) - (n - N) f'(N) - \frac{1}{2}(n - N)^2 f''(N)
\end{aligned}$$

and the remainders

$$\mathcal{R}^{(1)}(N) := \int_N^{+\infty} (\psi_{\mathbb{K}}^{(1)}(x) - \psi_{\mathbb{Q}}^{(1)}(x)) f''(x) \,\mathrm{d}x, \tag{6.2a}$$

$$\mathcal{R}^{(2)}(N) := -\frac{1}{2}\int_N^{+\infty} (\psi_{\mathbb{K}}^{(2)}(x) - \psi_{\mathbb{Q}}^{(2)}(x)) f'''(x)\,\mathrm{d}x, \tag{6.2b}$$

giving immediately the bounds

$$\big|\mathcal{R}^{(1)}(N)\big| \le \int_N^{+\infty} |\psi_{\mathbb{K}}^{(1)}(x) - \psi_{\mathbb{Q}}^{(1)}(x)| \cdot |f''(x)|\,\mathrm{d}x, \tag{6.3a}$$

$$\big|\mathcal{R}^{(2)}(N)\big| \le \frac{1}{2}\int_N^{+\infty} |\psi_{\mathbb{K}}^{(2)}(x) - \psi_{\mathbb{Q}}^{(2)}(x)| \cdot |f'''(x)|\,\mathrm{d}x. \tag{6.3b}$$

We can now prove

Corollary 6.1. *(GRH) In Equations* (6.1a) *and* (6.1b) *the remainders satisfy*

$$|\mathcal{R}^{(1)}(N)| \le \mathcal{R}_{\mathrm{bas}}^{(1)}(N) \quad \textit{and} \quad |\mathcal{R}^{(2)}(N)| \le \mathcal{R}_{\mathrm{bas}}^{(2)}(N) \qquad \forall N \ge 3, \tag{6.4}$$

with

$$\begin{aligned}\mathcal{R}_{\mathrm{bas}}^{(1)}(N) := {}&\alpha_{\mathbb{K}}^{(1)}\Big(\frac{\frac{5}{2}+y}{\sqrt{N}\log N} + \frac{3}{4}\,\mathrm{E}_1\Big(\frac{1}{2}\log N\Big)\Big) + \beta_{\mathbb{K}}^{(1)}\frac{2+3y}{N} \\ &+ \gamma_{\mathbb{K}}^{(1)}\frac{2y+y^2}{N} + \delta_{\mathbb{K}}^{(1)}\frac{y+y^2}{N^2},\end{aligned} \tag{6.5a}$$

$$\begin{aligned}\mathcal{R}_{\mathrm{bas}}^{(2)}(N) := {}&\alpha_{\mathbb{K}}^{(2)}\Big(\frac{\frac{33}{8}+\frac{11}{4}y+y^2}{\sqrt{N}\log N} + \frac{15}{16}\,\mathrm{E}_1\Big(\frac{1}{2}\log N\Big)\Big) + \beta_{\mathbb{K}}^{(2)}\frac{3+\frac{11}{2}y+\frac{3}{2}y^2}{N} \\ &+ \gamma_{\mathbb{K}}^{(2)}\frac{3y+\frac{5}{2}y^2+y^3}{N} + \delta_{\mathbb{K}}^{(2)}\frac{\frac{3}{2}y+2y^2+y^3}{N^2} + \eta_{\mathbb{K}}^{(2)}\frac{y+\frac{3}{2}y^2+y^3}{N^3}\end{aligned} \tag{6.5b}$$

where $\mathrm{E}_1(x) := \int_1^{+\infty} e^{-xt}t^{-1}\,\mathrm{d}t$ *is the exponential integral,* $y := (\log N)^{-1}$ *and*

$$\begin{aligned}\alpha_{\mathbb{K}}^{(1)} &= 0.5375\log\Delta_{\mathbb{K}} - 1.0355 n_{\mathbb{K}} + 5.4341, & \beta_{\mathbb{K}}^{(1)} &= n_{\mathbb{K}} - 1,\\ \gamma_{\mathbb{K}}^{(1)} &= 1.0155\log\Delta_{\mathbb{K}} - 2.1041 n_{\mathbb{K}} + 10.1799, & \delta_{\mathbb{K}}^{(1)} &= \log\Delta_{\mathbb{K}} - 1.415 n_{\mathbb{K}} + 4,\end{aligned}$$

$$\begin{aligned}\alpha_{\mathbb{K}}^{(2)} &= 0.3526\log\Delta_{\mathbb{K}} - 0.8212 n_{\mathbb{K}} + 4.5007, & \beta_{\mathbb{K}}^{(2)} &= n_{\mathbb{K}} - 1,\\ \gamma_{\mathbb{K}}^{(2)} &= 1.0155\log\Delta_{\mathbb{K}} - 2.6041 n_{\mathbb{K}} + 10.6799, & \delta_{\mathbb{K}}^{(2)} &= 2\log\Delta_{\mathbb{K}} - 2.83 n_{\mathbb{K}} + 8,\\ \eta_{\mathbb{K}}^{(2)} &= \log\Delta_{\mathbb{K}} - 0.9151 n_{\mathbb{K}} + 2.\end{aligned}$$

Proof. Suppose we have found constants $\alpha_{\mathbb{K}}^{(1)},\dots,\delta_{\mathbb{K}}^{(1)}$ and $\alpha_{\mathbb{K}}^{(2)},\dots,\eta_{\mathbb{K}}^{(2)}$ such that

$$|\psi_{\mathbb{K}}^{(1)}(x) - \psi_{\mathbb{Q}}^{(1)}(x)| \le \alpha_{\mathbb{K}}^{(1)}x^{3/2} + \beta_{\mathbb{K}}^{(1)}x\log x + \gamma_{\mathbb{K}}^{(1)}x + \delta_{\mathbb{K}}^{(1)}, \tag{6.6a}$$

$$|\psi_{\mathbb{K}}^{(2)}(x) - \psi_{\mathbb{Q}}^{(2)}(x)| \le \alpha_{\mathbb{K}}^{(2)}x^{5/2} + \beta_{\mathbb{K}}^{(2)}x^2\log x + \gamma_{\mathbb{K}}^{(2)}x^2 + \delta_{\mathbb{K}}^{(2)}x + \eta_{\mathbb{K}}^{(2)}. \tag{6.6b}$$

For (6.5a) we plug (6.6a) into (6.3a) and we use (A.1a–A.1d): the integrals apply here because $f(x) = (x\log x)^{-1}$ is a completely monotone function, i.e. satisfies $(-1)^k f^{(k)}(x) > 0$ for every $x > 1$ and for every order k.
For (6.5b) we plug (6.6b) into (6.3b) and we use (A.1f–A.1j).
The existence and the values of the constants $\alpha_{\mathbb{K}}^{(j)},\dots$ are an immediate consequence of Theorem 1.1. □

Coming back to the remark below Corollary 2.4 this strategy produces algorithms where the errors $|\mathcal{R}^{(1)}(N)|$ and $|\mathcal{R}^{(2)}(N)|$ are bounded essentially by $2.15\frac{\log\Delta_{\mathbb{K}}}{\sqrt{N}\log N}$, and $2.116\frac{\log\Delta_{\mathbb{K}}}{\sqrt{N}\log N}$, respectively. The minimal N needed for Buchmann's algorithm using Belabas and Friedman's result and ours are compared in Table 4.

The terms $-xr_{\mathbb{K}}$ and $R^{(1)}_{r_1,r_2}(x)$ in (1.3a) and $-x^2r_{\mathbb{K}}$ and $R^{(2)}_{r_1,r_2}(x)$ in (1.3b) are generally of comparable size and opposite in sign for the typical values of x which are needed in this application; thus it is possible to improve the result by estimating the remainders in such a way as to keep these terms together. This remark produces the following corollary.

Corollary 6.2. *(GRH) In Equations* (6.1a) *and* (6.1b) *the remainders satisfy*

$$|\mathcal{R}^{(1)}(N)| \le \mathcal{R}^{(1)}_{\mathrm{imp}}(N) \quad \textit{and} \quad |\mathcal{R}^{(2)}(N)| \le \mathcal{R}^{(2)}_{\mathrm{imp}}(N) \qquad \forall N\ge 3, \tag{6.7}$$

where

(6.8)

$$\begin{aligned}\mathcal{R}^{(1)}_{\mathrm{imp}}(N) := \alpha^{(1)}_{\mathbb{K}}\Big(\frac{\frac52+y}{\sqrt{N}\log N}+\frac34\,\mathrm{E}_1\Big(\frac12\log N\Big)\Big)+\Big(d_{\mathbb{K}}+\frac{r_2}{4N}\Big)\frac{y^2}{N}\\ +\Big|d_{\mathbb{K}}\frac{2+y-y^2}{N}+(r_{\mathbb{K}}-r_{\mathbb{Q}})\frac{2y+y^2}{N}-r_2\frac{1+\frac52y+y^2}{N^2}-(r'_{\mathbb{K}}-r'_{\mathbb{Q}})\frac{y+y^2}{N^2}\Big|,\end{aligned}$$

(6.9)

$$\begin{aligned}\mathcal{R}^{(2)}_{\mathrm{imp}}(N) := \alpha^{(2)}_{\mathbb{K}}\Big(\frac{\frac{33}{8}+\frac{11}{4}y+y^2}{\sqrt{N}\log N}+\frac{15}{16}\,\mathrm{E}_1\Big(\frac12\log N\Big)\Big)+\Big(d_{\mathbb{K}}+\frac{r_2}{4N}\Big)\frac{y^2}{N}\Big(1+\frac{5}{yN^2}\Big)\\ +\frac12\Big|d_{\mathbb{K}}\frac{6+2y-\frac92y^2-3y^3}{N}+(r_{\mathbb{K}}-r_{\mathbb{Q}})\frac{6y+5y^2+2y^3}{N}-2r_2\frac{3+\frac{11}{2}y+3y^2}{N^2}\\ -2(r'_{\mathbb{K}}-r'_{\mathbb{Q}})\frac{3y+4y^2+2y^3}{N^2}+d_{\mathbb{K}}\frac{2+\frac{20}{3}y+\frac{15}{2}y^2+3y^3}{N^3}+(r''_{\mathbb{K}}-r''_{\mathbb{Q}})\frac{2y+3y^2+2y^3}{N^3}\Big|,\end{aligned}$$

and $\alpha^{(1)}_{\mathbb{K}}$ and $\alpha^{(2)}_{\mathbb{K}}$ are as in Corollary 6.1.

Proof. By (6.2a) and the explicit formula (1.3a) we get

$$\begin{aligned}|\mathcal{R}^{(1)}(N)| = \Big|\int_N^{+\infty}\Big(\sum_{\substack{\rho\\ \zeta_{\mathbb{Q}}(\rho)=0}}\frac{x^{\rho+1}}{\rho(\rho+1)}-\sum_{\substack{\rho\\ \zeta_{\mathbb{K}}(\rho)=0}}\frac{x^{\rho+1}}{\rho(\rho+1)}\\ -(r_{\mathbb{K}}-r_{\mathbb{Q}})x+(r'_{\mathbb{K}}-r'_{\mathbb{Q}})+R^{(1)}_{r_1,r_2}(x)-R^{(1)}_{1,0}(x)\Big)f''(x)\,\mathrm{d}x\Big|.\end{aligned}$$

Here we isolate the part depending on the zeros. We estimate it by moving the absolute value in the inner part both of the integral and of the sum, and then applying the upper bound in Lemma 4.1. In this way we get

$$\begin{aligned}|\mathcal{R}^{(1)}(N)| \le \alpha^{(1)}_{\mathbb{K}}\int_N^{+\infty}x^{3/2}|f''(x)|\,\mathrm{d}x\\ +\Big|\int_N^{+\infty}\big(-(r_{\mathbb{K}}-r_{\mathbb{Q}})x+(r'_{\mathbb{K}}-r'_{\mathbb{Q}})+R^{(1)}_{r_1,r_2}(x)-R^{(1)}_{1,0}(x)\big)f''(x)\,\mathrm{d}x\Big|\end{aligned}$$

where $\alpha^{(1)}_{\mathbb{K}}$ is the constant of Corollary 6.1. We apply then Equalities (A.1a–A.1c), thus getting

$$
\begin{aligned}
|\mathcal{R}^{(1)}(N)| \leq{}& \frac{\alpha_{\mathbb{K}}^{(1)}}{\sqrt{N}}(4y - 2y^2 + 12y^3) \\
&+ \Big| -\frac{r_{\mathbb{K}} - r_{\mathbb{Q}}}{N}(2y + y^2) + \frac{r'_{\mathbb{K}} - r'_{\mathbb{Q}}}{N^2}(y + y^2) + \int_N^{+\infty} \big(R^{(1)}_{r_1,r_2}(x) - R^{(1)}_{1,0}(x)\big) f''(x)\,\mathrm{d}x \Big|.
\end{aligned}
$$

Recalling the definition of functions $f_j^{(1)}(x)$ and $R^{(1)}_{r_1,r_2}(x)$ in Lemma 3.3 we have

$$
\begin{aligned}
|\mathcal{R}^{(1)}(N)| \leq{}& \frac{\alpha_{\mathbb{K}}^{(1)}}{\sqrt{N}}(4y - 2y^2 + 12y^3) + \Big| \int_N^{+\infty} (d_{\mathbb{K}} f_1^{(1)}(x) + r_2 f_2^{(1)}(x)) f''(x)\,\mathrm{d}x \Big| \\
&+ \Big| \frac{r_{\mathbb{K}} - r_{\mathbb{Q}}}{N}(2y + y^2) - \frac{r'_{\mathbb{K}} - r'_{\mathbb{Q}}}{N^2}(y + y^2) + \int_N^{+\infty} \big(d_{\mathbb{K}} x(\log x - 1) - r_2(\log x + 1)\big) f''(x)\,\mathrm{d}x \Big|.
\end{aligned}
$$

The part depending on $f_j^{(1)}$ functions is estimated using the inequalities $0 < f_1^{(1)}(x) \leq 0.6x^{-1}$ and $0 < f_2^{(1)}(x) \leq 0.2x^{-2}$ for $x \geq 3$, the other integrals are computed via (A.1b–A.1j). After some computations one gets the bound $|\mathcal{R}^{(1)}(N)| \leq \mathcal{R}^{(1)}_{\mathrm{imp}}(N)$ with $\mathcal{R}^{(1)}_{\mathrm{imp}}(N)$ given in (6.8). The proof of (6.9) is similar using $0 < f_1^{(2)}(x) \leq 0.1x^{-2}$ and $0 < f_2^{(2)}(x) \leq 0.4x^{-1}$ for $x \geq 3$, and (A.1g–A.1l). ☐

In order to apply the formulas in Corollary 6.2 we recall that

$$
r_{\mathbb{Q}} = \log 2\pi \qquad r'_{\mathbb{Q}} = -\frac{\zeta'}{\zeta}(2) + \gamma + \log 2\pi - 1 \qquad r''_{\mathbb{Q}} = -\frac{\zeta'}{\zeta}(3) + \gamma + \log 2\pi - \frac{3}{2}
$$

(for $r_{\mathbb{Q}}$ see [7, Ch. 12], the other two are immediate consequence of (3.9b–3.9c)) but we need also the parameters $r_{\mathbb{K}}$, $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$. They can be estimated as (see the proof of Lemma 3.2)

$$
-1.0155 \log \Delta_{\mathbb{K}} + 2.1042 n_{\mathbb{K}} - 8.3419 \leq r_{\mathbb{K}} \leq -\tfrac{1}{2}\log \Delta_{\mathbb{K}} + 1.2076 n_{\mathbb{K}} + 1 \tag{6.10a}
$$

$$
-\log \Delta_{\mathbb{K}} + 1.415 n_{\mathbb{K}} \leq r'_{\mathbb{K}} \leq -\log \Delta_{\mathbb{K}} + 1.9851 n_{\mathbb{K}} \tag{6.10b}
$$

$$
-\log \Delta_{\mathbb{K}} + 0.9151 n_{\mathbb{K}} \leq r''_{\mathbb{K}} \leq -\log \Delta_{\mathbb{K}} + 1.08 n_{\mathbb{K}}. \tag{6.10c}
$$

Thus we can take the largest value that $\mathcal{R}^{(m)}_{\mathrm{imp}}$ assumes when the parameters run in those ranges. To that effect, it is sufficient to consider the values of the term in the absolute value where $r_{\mathbb{K}}$, $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ are replaced by the maximum and the minimum of their range. The results are summarized in Tables 2–5. Tables 2 and 3 show that in any case the improved estimate beats the plain bound by a quantity which largely depends on the quotient $n_{\mathbb{K}}/\log \Delta_{\mathbb{K}}$, reaching a gain greater than 10% for $\mathcal{R}^{(1)}$ and 16% for $\mathcal{R}^{(2)}$ for some combinations. This behavior agrees with our motivations for the improved formulas: keeping together the quantities $d_{\mathbb{K}} + r_{\mathbb{K}} y$, $r_2 + r'_{\mathbb{K}} y$ (for non-totally real fields) and $d_{\mathbb{K}} + r''_{\mathbb{K}} y$, which are $\approx n_{\mathbb{K}} - \frac{\log \Delta_{\mathbb{K}}}{\log N}$ (times suitable multiple of N^{-1}), we take advantage of their cancellations which can be quite large for suitable values of $n_{\mathbb{K}}/\log \Delta_{\mathbb{K}}$. Tables 4 and 5 show that the new algorithms improve Belabas–Friedman's bound by a factor which is at least 3 and sometimes 10. Finally, Tables 4–5 show that in that range of discriminants and for degrees larger than 10 it is convenient to use $\mathcal{R}^{(2)}_{\mathrm{imp}}$ instead of $\mathcal{R}^{(1)}_{\mathrm{imp}}$.

We could improve the algorithm a bit further by using the relation

$$
r_{\mathbb{K}} = \sum_{n=1}^{+\infty} \frac{\tilde{\Lambda}_{\mathbb{K}}(n) - \Lambda(n)}{n} - \log \Delta_{\mathbb{K}} + (\gamma + \log 2\pi) n_{\mathbb{K}} - \gamma, \tag{6.11}
$$

which follows combining the functional equations for $\zeta_{\mathbb{K}}$ and $\zeta_{\mathbb{Q}}$. In fact, truncating the series at a new level N' and estimating the remainder as in (6.1) via Theorem 1.1 we get an explicit formula which already for $N' \approx 100$ gives for $r_{\mathbb{K}}$ a range shorter than (6.10a). This computation takes only a small fraction of the total time needed for Buchmann's algorithm, and the new range allows us to improve the N computed via $\mathcal{R}_{\mathrm{imp}}^{(m)}$ by a quantity which in our tests has been generally around 1–2%, and occasionally large as 5%.

We can also compute $r'_{\mathbb{K}}$ and $r''_{\mathbb{K}}$ via (3.9b) and (3.9c), but their ranges (6.10b) and (6.10c) are already tight and in the formulas for $\mathcal{R}_{\mathrm{imp}}^{(m)}$ these parameters appear only in terms which are several orders lower than the principal one, and no improvement comes from their computation.

Appendix A. Some integrals

We collect here a lot of computations and approximations of integrals that are used in Section 6; they can easily be proved by integration by parts. Recall that $f(x) = (x \log x)^{-1}$, $N \geq 3$ and $y = (\log N)^{-1}$. Thus

$$f(N) = \frac{y}{N} \qquad f'(N) = -\frac{y+y^2}{N^2} \qquad f''(N) = \frac{2y+3y^2+2y^3}{N^3}.$$

In the following θ is a constant in $(0,1)$, with possibly different values in each occurrence. We have

$$\int_N^{+\infty} x^{3/2} f''(x)\,\mathrm{d}x = \frac{1}{\sqrt{N}}(\tfrac{5}{2}y + y^2) + \tfrac{3}{4}\,\mathrm{E}_1\left(\tfrac{1}{2}\log N\right) \tag{A.1a}$$

$$\int_N^{+\infty} x f''(x)\,\mathrm{d}x = \frac{1}{N}(2y + y^2) \tag{A.1b}$$

$$\int_N^{+\infty} f''(x)\,\mathrm{d}x = \frac{1}{N^2}(y + y^2) \tag{A.1c}$$

$$\int_N^{+\infty} x\log x f''(x)\,\mathrm{d}x = \frac{1}{N}(2 + 3y - \theta y^2) \tag{A.1d}$$

$$\int_N^{+\infty} \log x f''(x)\,\mathrm{d}x = \frac{1}{N^2}(1 + \tfrac{3}{2}y + \tfrac{\theta}{4}y^2) \tag{A.1e}$$

$$\int_N^{+\infty} x^{5/2} f'''(x)\,\mathrm{d}x = -\frac{1}{\sqrt{N}}(\tfrac{33}{4}y + \tfrac{11}{2}y^2 + 2y^3) - \tfrac{15}{8}\,\mathrm{E}_1\left(\tfrac{1}{2}\log N\right) \tag{A.1f}$$

$$\int_N^{+\infty} x^2 f'''(x)\,\mathrm{d}x = -\frac{1}{N}(6y + 5y^2 + 2y^3) \tag{A.1g}$$

$$\int_N^{+\infty} x f'''(x)\,\mathrm{d}x = -\frac{1}{N^2}(3y + 4y^2 + 2y^3) \tag{A.1h}$$

$$\int_N^{+\infty} f'''(x)\,\mathrm{d}x = -\frac{1}{N^3}(2y + 3y^2 + 2y^3) \tag{A.1i}$$

$$\int_N^{+\infty} x^2\log x f'''(x)\,\mathrm{d}x = \frac{1}{N}(-6 - 11y - 3y^2 + 2\theta y^2) \tag{A.1j}$$

$$\int_N^{+\infty} x\log x f'''(x)\,\mathrm{d}x = \frac{1}{N^2}(-3 - \tfrac{11}{2}y - 3y^2 - \tfrac{\theta}{4}y^2) \tag{A.1k}$$

$$\int_N^{+\infty} \log x f'''(x)\,\mathrm{d}x = \frac{1}{N^3}(-2 - \tfrac{11}{3}y - 3y^2 + \tfrac{2\theta}{9}y^2). \tag{A.1l}$$

Table 1. Parameters for (5.3).

$n_{\mathbb{K}}$	2	3	4	5	6	7	8
$\bar{x}$ $\leq$	97	179	253	316	369	414	452
$\bar{x}_{\min}$ $\geq$	229	287	363	456	566	694	840

TABLE 2. Least N for Buchmann's algorithm: $\mathcal{R}^{(1)}_{\text{bas}}$ against $\mathcal{R}^{(1)}_{\text{imp}}$.

	$n=2$		$n=6$		$n=10$		$n=20$		$n=50$	
Δ	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(1)}_{\text{imp}}$
10^5	371	361	211	190	—	—	—	—	—	—
10^{10}	763	752	529	485	341	310	—	—	—	—
10^{20}	1835	1824	1478	1406	1159	1085	—	—	—	—
10^{50}	6961	6950	6305	6231	5678	5541	4248	4088	—	—
10^{100}	20776	20765	19709	19634	18668	18529	16177	15879	9704	9446
10^{200}	64950	64939	63189	63114	61451	61310	57198	56897	45269	44710

TABLE 3. Least N for Buchmann's algorithm: $\mathcal{R}^{(2)}_{\text{bas}}$ against $\mathcal{R}^{(2)}_{\text{imp}}$.

	$n=2$		$n=6$		$n=10$		$n=20$		$n=50$	
Δ	$\mathcal{R}^{(2)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{imp}}$
10^5	466	451	256	221	—	—	—	—	—	—
10^{10}	899	884	601	531	369	317	—	—	—	—
10^{20}	2054	2039	1607	1504	1216	1097	—	—	—	—
10^{50}	7444	7429	6631	6524	5862	5665	4141	3886	—	—
10^{100}	21750	21735	20435	20327	19158	18957	16132	15700	8544	8124
10^{200}	67067	67051	64905	64795	62775	62572	57592	57153	43265	42382

TABLE 4. Least N for Buchmann's algorithm: according to Belabas–Friedman and the new algorithms with $\mathcal{R}^{(1)}_{\text{bas}}$ and $\mathcal{R}^{(2)}_{\text{bas}}$. Belabas–Friedman's data is reprinted from [5].

	$n=2$			$n=6$			$n=10$			$n=20$			$n=50$		
Δ	B.–F.	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{bas}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{bas}}$	$\mathcal{R}^{(2)}_{\text{bas}}$
10^5	1619	371	466	1632	211	256	—	—	—	—	—	—	—	—	—
10^{10}	3169	763	899	3181	529	601	3194	341	369	—	—	—	—	—	—
10^{20}	6838	1835	2054	6850	1478	1607	6861	1159	1216	—	—	—	—	—	—
10^{50}	21619	6961	7444	21629	6305	6631	21639	5678	5862	21665	4248	4141	—	—	—
10^{100}	56332	20776	21750	56341	19709	20435	56351	18668	19158	56374	16177	16132	56445	9704	8544
10^{200}	156151	64950	67067	156160	63189	64905	156169	61451	62775	156191	57198	57592	156256	45269	43265

TABLE 5. Least N for Buchmann's algorithm: according to Belabas–Friedman and the new algorithms with $\mathcal{R}^{(1)}_{\text{imp}}$ and $\mathcal{R}^{(2)}_{\text{imp}}$. Belabas–Friedman's data is reprinted from [5].

	$n=2$			$n=6$			$n=10$			$n=20$			$n=50$		
Δ	B.–F.	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{imp}}$	B.–F.	$\mathcal{R}^{(1)}_{\text{imp}}$	$\mathcal{R}^{(2)}_{\text{imp}}$
10^5	1619	361	451	1632	190	221	—	—	—	—	—	—	—	—	—
10^{10}	3169	752	884	3181	485	531	3194	310	317	—	—	—	—	—	—
10^{20}	6838	1824	2039	6850	1406	1504	6861	1085	1097	—	—	—	—	—	—
10^{50}	21619	6950	7429	21629	6231	6524	21639	5541	5665	21665	4088	3886	—	—	—
10^{100}	56332	20765	21735	56341	19634	20327	56351	18529	18957	56374	15879	15700	56445	9446	8124
10^{200}	156151	64939	67051	156160	63114	64795	156169	61310	62572	156191	56897	57153	156256	44710	42382

TABLE 6. Constants for $\sum_{\rho} |\rho(\rho+1)|^{-1}$ in Lemma 4.1.

j	$a_j \cdot 10^7$	j	$a_j \cdot 10^7$
1	250548071	41	3648003867198618158032688666281279907332926401
2	-40769390315	42	$-5353733754976758827081327735207850440805276490$
3	5795175723671	43	7411481592406340412123547436619012373828347148
4	-642251894123528	44	$-9680502044407712819603502062322026576945648999$
5	54218815728127329	45	11931531864054985817793303920405879163945577143
6	-3508878919641771688	46	$-13877909647596266697860364708436232764310634475$
7	177001043449933176447	47	15232402120827550086363255671423471322396554451
8	-7094015596077453633868	48	$-15775334247682258723942059247603410917983570659$
9	230165538494597837675083	49	15412228661977544619641915478149603219261905955
10	$-6150059294311314135993327$	50	$-14200388071264097711591911264486344481572166054$
11	137429722146979678372903545	51	12334252439899208072837355489763427886826472535
12	$-2603418437013270575777900517$	52	$-10094621415908831481370162399779502133799211521$
13	42312055609004270243526202076	53	7779879804978723458319595088819777701055258725
14	$-596228700498573506460507915379$	54	$-5642269216814651472704347110867628137752200021$
15	7352229299660977983271586796428	55	3847449822637486914738835007382155275278296082
16	$-79991031610893192700264201347849$	56	$-2464415859390293757851604168538551569779024142$
17	773449451301413812623754497322110	57	1481149204040503957548445332963392835676301519
18	$-6689469356634480595952166290773419$	58	$-834221598218683553855012914220968482480130787$
19	52049469989158830787111938359894141	59	439683909946941169248931270316639282116138102
20	$-366215235328303748457085063911062452$	60	$-216506399095273447319941898397799604643721683$
21	2340727373433875029268033585654013101	61	99419464389596242263022332671287321846845659
22	$-13647569726889979888635481117558851444$	62	$-42484842271343000074946235138759717939948915$
23	72856233722227845138595374556506130213	63	16854882803935701426471290624390658493962917
24	$-357308755193444577424236430238048816629$	64	$-6191128565930702299565499949693343368179853$
25	1614746152052189321203222537039119640756	65	2099040048795249597742846189024188906401966
26	$-6742815290893858601169185495146599758906$	66	$-654534994786055122946588844807706357840291$
27	26081651135346560764877059272421175463793	67	186947926780001735965997901059271943644593
28	$-93662343928951334238283477190373026235970$	68	$-48675054083574200340006259345314988135384$
29	312909679670641654646206585298379548969664	69	11488195908804597573088392774662830983598
30	$-974320711195668140488233654168711408974431$	70	$-2441545378407438626531756675121759469076$
31	2832324810202406292456790051806622242980143	71	463522993226953954733282029125741713819
32	$-7698430960693278611182246394416801692267482$	72	$-77845294874645427333933115933295893005$
33	19591848109684436395280873748247452691475281	73	11425492896861966116655614216587059464
34	$-46741161307608105954759866712283645186774375$	74	$-1443037809175186486864654360574474088$
35	104654143256889695138455737518470291722254806	75	153673972446397363248771006965862929
36	$-220128737522779177621064610160993365216868168$	76	$-13418974865215897151246028213990153$
37	435354172671749489946963445292948033362127211	77	922600572073108333758203469960875
38	$-810197596515155479177768566395714272810920262$	78	$-46833786494978206937017577663173$
39	1419759138597775056528221649897613967888797952	79	1560648037479364896275100707017
40	$-2344042914614942938251851695053817440107394201$	80	$-25610063982827093894391815027$

TABLE 7. Constants for $\sum_\rho |\rho(\rho+1)(\rho+2)|^{-1}$ in Lemma 4.1.

j	$a_j \cdot 10^7$	j	$a_j \cdot 10^7$
1	116043280	21	44212581087391037851257051242
2	−15019134746	22	−82776719893697522350544625956
3	1306482026256	23	136740298375301487499890195367
4	−76315741770330	24	−199275732886794715825307355765
5	3116274365157230	25	255978158207512987528401996503
6	−92621169453588672	26	−289344568336337774362395707820
7	2074954505670798718	27	287063511581435319315875881542
8	−36069656819440696263	28	−249076192247094252611008694964
9	498302313581120124204	29	188100860940650555126546470265
10	−5579712481960141840354	30	−122861964251233620242612405716
11	51471550420429886034202	31	68841615651370858094138826161
12	−396486283111534949768375	32	−32737240356857723641726749028
13	2579203445202845079404723	33	13026982181479475895165888915
14	−14302917461736234191777842	34	−4255456128181013051676843112
15	68148701393954628073176631	35	1111002720718102002015316745
16	−280819396505042268256263139	36	−222834382207437523098078851
17	1006203468485334827305158167	37	32228595062589755026085278
18	−3148890161469033145131905085	38	−2991080884530620994922737
19	8637410243724442351566255216	39	133739429590971377317925
20	−20823652528449395097665316823	—	—

TABLE 8. Constants for $-B_{\mathbb{K}} = \sum_\rho \rho^{-1}$ in Lemma 3.1.

j	$a_j \cdot 10^7$	j	$a_j \cdot 10^7$
1	149178011	6	−189514259129
2	−1773766184	7	205612934195
3	11465438478	8	−140312989024
4	−45115091060	9	54661946795
5	114102793523	10	−9271031235

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