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# A NEW CASTELNUOVO BOUND FOR TWO CODIMENSIONAL SUBVARIETIES OF $\mathbb{P}^r$

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ABSTRACT. Let $X$ be a smooth $n$-dimensional projective subvariety of $\mathbb{P}^r(\mathbb{C})$, $(r \geq 3)$. For any positive integer $k$, $X$ is said to be $k$-normal if the natural map $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathscr{O}_X(k))$ is surjective. Mumford and Bayer showed that $X$ is $k$-normal if $k \geq (n+1)(d-2)+1$ where $d = \deg(X)$. Better inequalities are known when $n$ is small (Gruson–Peskine, Lazarsfeld, Ran). In this paper we consider the case $n = r-2$, which is related to Hartshorne's conjecture on complete intersections, and we show that if $k \geq d + 1 + (1/2)r(r-1) - 2r$ then $X$ is $k$-normal and $I_X$, the ideal sheaf of $X$ in $\mathbb{P}^r$, is $(k+1)$-regular.

About these problems Lazarsfeld developed a technique based on generic projections of $X$ in $\mathbb{P}^{n+1}$; our proof is an application of some recent results of Ran's (on the secants of $X$): we show that in our case there exists a projection such generic as Lazarsfeld requires.

When $r \geq 6$ we also give a better inequality: $k \geq d - 1 + (1/2)r(r-1) - (r-1)[(r+4)/2]$ ( [ ] means: integer part); it is obtained by refining Lazarsfeld's technique with the help of some results of ours about $k$-normality.

## 1. INTRODUCTION

Let $X$ be a smooth, nondegenerate (i.e. not contained in a hyperplane), $n$-dimensional projective subvariety of $\mathbb{P}^r(\mathbb{C})$. For any positive integer $k$, $X$ is said to be $k$-normal if the natural map $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathscr{O}_X(k))$ is surjective, i.e. if the hypersurfaces of degree $k$ cut out a complete linear system on $X$. Let $d$ be the degree of $X$.

It is well known that for $k \gg 0$ every $X$ is $k$-normal, but people look for precise bounds; such bounds are often called Castelnuovo bounds after the classical work of Castelnuovo [C] (completed by Gruson–Lazarsfeld–Peskine [GLP]) concerning the case $n = 1$.

If $r \geq 2n+1$, the best possible linear inequality is: $X$ is $k$-normal if $k \geq d+n-r$ (see [L]). It was proved for $n = 1$ by Gruson–Lazarsfeld–Peskine [GLP], (for $X$ singular too); for $n = 2$ by Lazarsfeld [L]; for $n = 3$ by Ran [R2] when $r \geq 9$.

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For other values of $n$ we know only this result of Mumford: $X$ is $k$-normal if $k \geq (n+1)(d-2)+1$ (see [BM]).

For small codimensions other inequalities are known, but they have to do with $k$-normality for *small* $k$: if $n \geq (2/3)(r-1)$ $X$ is 1-normal if $r \geq 5$ (see [Z], this is the best possible value); if $n = r-2$ and $k \geq 2$, $X$ is $k$-normal if $r \geq 6$ and $r \geq \min\{k+4, 6k-2\}$ (see [AO1, AO2]); Peskine has an approach to: if $n = r-2$, $r \geq 5$, $X$ is $k$-normal if $k \leq r-4$ (see [S]). Finally we want to recall that $X$ is (a complete intersection and therefore) $k$-normal if $n = r-2$, $r \geq 6$, and $d \leq (r-1)(r+5)$ (see [HS]).

Obviously many of these results are surpassed if Hartshorne's conjecture about complete intersections is proved.

Let $[x]$ denote the integer part of a real number $x$. In this paper we show the following results:

**Theorem 1.1.** *Let $X$ be a nondegenerate, degree $d$, 2-codimensional, smooth, subvariety of $\mathbb{P}^r(\mathbb{C})$.*

*Then $X$ is $k$-normal if $k \geq d+1+(1/2)r(r-1)-2r$. If $r \geq 6$, $X$ is $k$-normal if $k \geq d-1+(1/2)r(r-1)-(r-1)[(r+4)/4]$.*

**Theorem 1.2.** *With the same assumptions of Theorem 1.1, let $I_X$ be the ideal sheaf of $X$.*

*Then $I_X$ is $(k+1)$-regular if $k \geq d+1+(1/2)r(r-1)-2r$; and if $r \geq 6$, $I_X$ is $(k+1)$-regular if $k \geq d-1+(1/2)r(r-1)-(r-1)[(r+4)/4]$.*

Note that 1.1 is better than Mumford's inequality in many cases. Our technique is very simple. We apply the ideas of Lazarsfeld contained in [L], which we follow step by step. The crucial point, as Lazarsfeld himself pointed out, is its Lemma 1.2. Here we use a result of Ran about the $r$-secants of $X$ (see [R3]).

When $r \geq 6$ our results from [AO1, AO2] allow us to improve the technique of Lazarsfeld by using a stronger result of regularity for the vector bundles introduced in [L].

## 2. Following Lazarsfeld

Let $P$ be a point in $\mathbb{P}^r$. Let $p: M \to \mathbb{P}^r$ be the blowing up of $\mathbb{P}^r$ at $P$. Denoting by $q: M \to \mathbb{P}^{r-1}$ the natural projection, for any positive integer $h$, one obtains a homomorphism $w_h: q_*(p^*\mathscr{O}_{\mathbb{P}^r}(h)) \to q_*(p^*\mathscr{O}_X(h))$ of sheaves on $\mathbb{P}^{r-1}$.

Let $f$ be the linear projection of $X$ centered at $P$, so that $f_*\mathscr{O}_X(h) = q_*(p^*\mathscr{O}_X(h))$. We choose homogeneous coordinates on $\mathbb{P}^r$ in such a way that $P$ is defined by $T_0 = T_1 = \cdots = T_{r-1} = 0$. Then $(T_r)^s$ determine sections in $H^0(\mathbb{P}^r, \mathscr{O}_X(s)) = H^0(\mathbb{P}^{r-1}, f_*\mathscr{O}_X(s))$.

Combining these with the canonical map $\mathscr{O}_{\mathbb{P}^{r-1}} \to f_*\mathscr{O}_X$, one deduces a homomorphism

$$w: \mathscr{O}_{\mathbb{P}^{r-1}}(-h) \oplus \mathscr{O}_{\mathbb{P}^{r-1}}(-h+1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^{r-1}} \to f_*\mathscr{O}_X; \tag{2.1}$$

$w$ may be identified with $w_h$.

Now for every $y \in \mathbb{P}^{r-1}$, let $L_y = p(q^{-1}(y))$ be the line $\langle P, y\rangle$, and let $X_y$ be the scheme-theoretic intersection $X \cap L_y$. $w_h \otimes \mathbb{C}(y)$ is identified with the restriction homomorphism $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(h)) \approx H^0(L_y, \mathscr{O}_{L_y}(h)) \to$

$H^0(L_y, \mathcal{O}_{Xy}(h))$. Suppose that

$$(*) \qquad H^1(L_y, I_{Xy/Ly}(h)) = 0,$$

then $w_h \otimes \mathbb{C}(y)$ is surjective and therefore $w_h$ is surjective too, (see [L, Lemma 1.2]).

Now let $E$ be the kernel of $w_h$, we have this exact sequence

$$(2.2) \qquad 0 \to E \to \mathcal{O}_{\mathbb{P}^{r-1}}(-h) \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(-h+1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{r-1}} \to f_*\mathcal{O}_X \to 0$$

of sheaves on $\mathbb{P}^{r-1}$. Since $f_*\mathcal{O}_X$ is a sheaf of $(r-2)$-dimensional Cohen-Macaulay modules over $\mathbb{P}^{r-1}$, $E$ is locally free, $\operatorname{rank}(E) = h+1$, $c_1(E) = -d - h(h+1)/2$. (In fact, the vector bundle map in the previous sequence (2.2) drops rank on a hypersurface of degree $d$.)

Now we have the following fact, whose proof is in [L, Lemma 1.5]:

**Lemma 2.3.** *For any integer $k$ such that $k \geq h$, $X$ is $k$-normal if*

$$H^1(\mathbb{P}^{r-1}, E(k)) = 0.$$

The previous construction is due to Gruson and Peskine; the following idea is due to Lazarsfeld. Recall that a coherent sheaf $F$ on some projective space $\mathbb{P}$ is said to be $m$-regular if $H^i(\mathbb{P}, F(m-i)) = 0$ for $i > 0$. Suppose that, for a positive integer $x$:

(**) there is an exact sequence $0 \to E \to B \to A \to 0$ of vector bundles on $\mathbb{P}^{r-1}$ where $A^*$ is $(-x+1)$-regular and $B^*$ is $(-x)$-regular.

Then by Proposition 2.4 of [L], $E$ is $\{-c_1(E) - x[\operatorname{rank}(E)] + x\}$-regular.

Actually in [L] the proof is given when $x = 2$, but the general case follows immediately from Lazarsfeld's proof.

## 3. Proofs of Theorems 1.1 and 1.2

Obviously we have to prove the theorems only when $X$ is not a complete intersection.

First we choose an integer $h$ such that condition $(*)$ is satisfied. By Corollary 2 of [R3] we know that through a generic point $P$ of $\mathbb{P}^r$ there are no lines that are $r$-secants (or more than $r$-secants) for $X$. So if we project $X$ from $P$ on a generic hyperplane, we have that $(*)$ is satisfied for $h \geq r-1$. From now on we fix a generic point $P$, a projection $f$, as in §2, and the integer $h = r-1$.

Exactly as in [L, Lemma 2.1], we can consider the graded module $F = \bigoplus H^0(\mathbb{P}^{r-1}, f_*\mathcal{O}_X(s)) = \bigoplus H^0(\mathbb{P}^r, \mathcal{O}_X(s))$ over the homogeneous coordinate ring $\mathbb{C}[T_0, T_1, \ldots, T_{r-1}]$ of $\mathbb{P}^{r-1}$. The exact sequence (2.1) gives rise to generators of $F$: one in degree 0, one in degree 1, and so on. These can be expanded to a full set of generators of $F$ by adding (say) $p$ more generators in degrees $a_1, a_2, \ldots, a_p$. By setting $A = \bigoplus \mathcal{O}_{\mathbb{P}^{r-1}}(-a_i)$, this system of generators determines upon sheafifying an exact sequence:

$$(3.1) \qquad 0 \to B \to A \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(-r+1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{r-1}} \to f_*\mathcal{O}_X \to 0,$$

which defines a vector bundle $B$ on $\mathbb{P}^{r-1}$. Comparing (2.2) with (3.1), one sees that $E$ is isomorphic to the kernel of the surjective map $B \to A$. So we get an exact sequence $0 \to E \to B \to A \to 0$ of vector bundles on $\mathbb{P}^{r-1}$.

In [L, Proposition 2.4] it is proved that condition $(**)$ is satisfied for $A$ and $B$ with $x = 2$. So we have that $E$ is $\{d + (r-1)r/2 - 2r + 2\}$ regular. In particular $H^1(\mathbb{P}^{r-1}, E(k)) = 0$ if $k \geq \{d + (r-1)r/2 - 2r + 1\}$, so that by Lemma 2.3, the first part of Theorem 1.1 is proved.

To prove the first part of Theorem 1.2, we remark that we get the $(p+1)$-regularity of $I_X$ if we have the $p$-normality of $X$, and, by using 2.2, the $(p+1)$-regularity of $E$, (see [L, p. 427]).

Now to prove the second part of 1.1 and 1.2 we have only to show that, when $r \geq 6$, condition $(**)$ is satisfied for $A$ and $B$ with $x[(r+h)/h]$. To prove that $B^*$ is $(-x)$-regular, we have to prove that $H^i(\mathbb{P}^{r-1}, B(x-i-1)) = 0$ for $i = 0, 1, \ldots, r-2$. For $i = 0$ we get the vanishing because there are no syzygies of degree $1, 2, \ldots, r$ among the generators of $F$ because there are no hypersurfaces of degree $1, 2, \ldots, r$ that contain $X$ (otherwise $X$ is a complete intersection, see [R1]). For $i = 1$ we get the vanishing by the construction of $B$. For $i \geq 2$, by using (3.1), by putting $q = i - 1$, we have only to show that $H^q(X, \mathscr{O}_X(x-2-q)) = 0$ for $q = 1, 2, \ldots, r-3$; now if $x - 2 < q$ we use Kodaira vanishing, if $x - 2 = q$ we use Barth theorem, if $x - 2 > q \geq 1$ we use [AO2].

To show that $A^*$ is $(-x+1)$-regular, by definition of $A$, we have only to show $H^0(\mathbb{P}^{r-1}, A(x-2)) = 0$.

By [AO2, R1] we can say that, for $t = 1, 2, \ldots, [(r-4)/4]$,

$$H^0(X, \mathscr{O}_X(t)) \cong H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(t));$$

for the same values of $t$ we have that

$$H^0(\mathbb{P}^{r-1}, B(t)) = H^1(\mathbb{P}^{r-1}, B(t)) = 0,$$

so that by using (3.1), we have:

$$\begin{aligned} H^0(X, \mathscr{O}_X(t)) &\cong H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(t)) \cong H^0(\mathbb{P}^{r-1}, f_*\mathscr{O}_X(t)) \\ &\cong H^0(\mathbb{P}^{r-1}, A(t)) \oplus H^0(\mathbb{P}^{r-1}, \mathscr{O}_{\mathbb{P}^{r-1}}(t-h)) \\ &\quad \oplus H^0(\mathbb{P}^{r-1}, \mathscr{O}_{\mathbb{P}^{r-1}}(t-h+1)) \oplus \cdots \oplus H^0(\mathbb{P}^{r-1}, \mathscr{O}_{\mathbb{P}^{r-1}}(t-1)) \\ &\quad \oplus H^0(\mathbb{P}^{r-1}, \mathscr{O}_{\mathbb{P}^{r-1}}(t)). \end{aligned}$$

As $h = r - 1 > t$, we get $H^0(\mathbb{P}^{r-1}, A(t)) = 0$ for $t = 1, 2, \ldots, [(r-4)/4]$ and therefore, $H^0(\mathbb{P}^{r-1}, A(x-2)) = 0$ for $x = [(r+4)/4]$.

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## References

[AO1] A. Alzati and G. Ottaviani, *A linear bound on the t-normality of codimension two subvarieties of* $\mathbb{P}^n$, J. Reine Angew. Math. **409** (1990), 35–40.

[AO2] ——, *A vanishing theorem for the ideal sheaf of codimension two subvarieties of* $\mathbb{P}^n$, Proc. Conf. "Projective Varieties Trieste," (1989) (to appear).

[BM] D. Bayer and D. Mumford, *What can be computed in algebraic geometry*?, Proc. Computer Algebra as a Tool for Research in Math. and Phys., Courant Institute, New York, April 1984.

[C] G. Castelnuovo, *Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica*, Rend. Circ. Mat. Palermo (2) 7 (1893), 89–110.

[GLP] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo and the equations defining space curves*, Invent. Mat. **72** (1983), 491–506.

[HS] A. Holme and M. Schneider, *A computer aided approach to codimension two subvarieties of* $\mathbb{P}^n$ , $n \geq 6$ , J. Reine Angew. Math. **357** (1987), 205–220.

[L] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. **55** (1987), 423–429.

[R1] Z. Ran, *Systèmes linéaires complets de sections hypersurfaces sur les variétés projectives de codimension* 2 , C. R. Acad. Sci. Paris Sér. I, Math. **298** (1984), 111–112.

[R2] ——, *Local differential geometry and generic projections of threefolds*, J. Differential Geom. **32** (1990), 131–137.

[R3] ——, *The* (*dimension* +2)-*secant lemma*, preprint, 1989.

[S] M. Schneider, *Vector bundles and low-codimensional submanifolds of projective space*: *a problem list*, preprint, 1990.

[Z] F. L. Zak, *Projections of algebraic varieties*, Mat. Sb. **116** (1981), 593–602 (Russian); English transl.: Math. USSR Sb. **44** (1983), 535–544.

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