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RATIONAL ORBITS ON THREE-SYMMETRIC PRODUCTS OF ABELIAN VARIETIES

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ABSTRACT. Let A be an n-dimensional Abelian variety, $n \geq 2$; let $\mathrm{CH}_0(A)$ be the group of zero-cycles of A, modulo rational equivalence; by regarding an effective, degree k, zero-cycle, as a point on $S^k(A)$ (the k-symmetric product of A), and by considering the associated rational equivalence class, we get a map $\gamma: S^k(A) \to \mathrm{CH}_0(A)$, whose fibres are called γ-orbits.

For any $n \geq 2$, in this paper we determine the maximal dimension of the γ-orbits when $k = 2$ or 3 (it is, respectively, 1 and 2), and the maximal dimension of families of γ-orbits; moreover, for generic A, we get some refinements and in particular we show that if $\dim(A) \geq 4$, $S^3(A)$ does not contain any γ-orbit; note that it implies that a generic Abelian four-fold does not contain any trigonal curve. We also show that our bounds are sharp by some examples.

The used technique is the following: we have considered some special families of Abelian varieties: $A_t = E_t \times B$ (E_t is an elliptic curve with varying moduli) and we have constructed suitable projections between $S^k(A_t)$ and $S^k(B)$ which preserve the dimensions of the families of γ-orbits; then we have done induction on n. For $n = 2$ the proof is based upon the papers of Mumford and Roitman on this topic.

1. INTRODUCTION

Let X be a d-dimensional smooth algebraic variety; a *cycle* Z of codimension r in X is defined to be an element of the free Abelian group $\mathbf{C}^r(X)$ generated by the irreducible subvarieties of codimension r on X. We are interested in zero-cycles, i.e. when $r = d$. Two zero-cycles Z_1 and Z_2 of X are *rationally equivalent* if there exists a cycle Z on $X \times \mathbf{A}^1$, which intersects each fibre $X \times \{t\}$ in some points such that Z_1 and Z_2 are obtained respectively by intersecting Z with the fibres $X \times \{0\}$ and $X \times \{1\}$. Note that this is in fact an equivalence relation and that the zero-cycles rationally equivalent to $\mathbf{0}$ (the zero of $\mathbf{C}^d(X)$) form a subgroup of $\mathbf{C}^d(X)$, (see [H, R_1]).

We denote by $\mathrm{CH}_0(X)$ the (Chow) group of zero-cycles on X, modulo rational equivalence. If $Z = \sum n_i P_i$ is a zero-cycle, where the P_i are points of X, we define the *degree* of Z to be $\sum n_i$. It is convenient to regard an *effective*

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zero-cycle $Z = \sum n_i P_i$ i.e. one where all the $n_i > 0$, as a point on the kth symmetric product $S^k(X)$ of X, where $k = \deg(Z)$. Then by taking the associated rational equivalence class, we obtain a map $\gamma: S^k(X) \to \mathrm{CH}_0(X)$; the fibres of this map will be called *γ-orbits*; the irreducible, connected, components of a γ-orbit will be called *γ-components*, (*γ-curves* if they have dimension 1, *γ-surfaces* if they have dimension 2, etc.).

Now let A be an Abelian variety, if we consider the Albanese morphism $\alpha_k: S^k(A) \to \mathrm{Alb}[S^k(A)] = A$ (i.e. $\alpha_k(x_1, x_2, \dots, x_k) = x_1 + x_2 + \cdots + x_k$), we have that the fibres of α_k are all isomorphic and that every γ-orbit of $S^k(A)$ is contained in exactly one fibre of α_k. Then, if we want to study the γ-orbits of $S^k(A)$, we have only to consider the γ-orbits contained in $K_k(A) = \ker(\alpha_k)$.

In [P] the author showed that for a generic Abelian variety A, with $\dim(A) \geq 3$, its Kummer variety, $K(A)$, does not contain any rational curve. By remarking that $K(A)$ is $K_2(A)$ in the previous notations, you can think that in $S^2(A)$ there are no one-dimensional γ-orbits, (where "dimension" means: maximal dimension of the γ-components of the γ-orbit, see §3). In fact, as Clemens pointed out, the technique used in [P] is related to the famous Mumford's paper [M] about the rational equivalence of zero cycles on a surface. So that, by those arguments, it is possible to show:

Theorem (1.1). *Let A be an Abelian variety,* $\dim(A) \geq 2$, *then*

(a) $S^2(A)$ *does not contain any two-dimensional γ-orbit*;

(b) *if A is generic and* $\dim(A) \geq 3$, $S^2(A)$ *does not contain any one-dimensional γ-orbit.*

The proof of (1.1) is essentially contained in [P]: you have only to change the words "rational curve" into "γ-curve", (see also (7.1)).

In this paper we study the γ-orbits of $S^3(A)$, $\dim(A) \geq 2$, and we obtain the following results:

Theorem (1.2). *Let A be an Abelian variety,* $\dim(A) \geq 2$, *then*

(a) *in $S^3(A)$ there are no d-dimensional γ-orbits with $d \geq 3$*;

(b) *in $K_3(A)$ there are no one-dimensional families of two-dimensional γ-orbits*;

(c) *if* $\dim(A) = 2$, *in $K_3(A)$ there are no three-dimensional families of one-dimensional γ-orbits.*

Remark (1.3). If $\dim(A) = 2$, in $S^3(A)$ there are some two-dimensional γ-orbits and some two-dimensional families of one-dimensional γ-orbits, see Examples (5.2) and (5.3); so that (1.2) is sharp.

Theorem (1.4). *Let A be a generic Abelian variety,* $\dim(A) \geq 3$, *then*

(a) *if* $\dim(A) = 3$, *in $S^3(A)$ there are no two-dimensional γ-orbits*;

(b) *if* $\dim(A) = 3$, *in $K_3(A)$ there are no two-dimensional families of one-dimensional γ-orbits*;

(c) *if* $\dim(A) \geq 4$, *in $S^3(A)$ there are no one-dimensional γ-orbits.*

The proof of (1.2), in §5, is based upon the results of Mumford and Roitman (see §3); but, to apply them, we have needed some linear algebra which we have condensed in §4.

To prove (1.4) we have considered some special families of Abelian varieties of this type: $A_t = E_t \times B$ (where E is usually an elliptic curve with varying moduli), and we have used the projections between $S^3(A_t)$ and $S^3(B)$ which preserve the dimension of the families of γ-orbits, then we have applied (1.2) to $S^3(B)$, (see §7).

Unfortunately we did not find an easy way to show that such projections do exist, not even when A is isogenous to a product of elliptic curves. So we were forced to prove the lemmas in §6; actually some proof could be shortened by using the De Franchis-Severi theorem (for curves and for surfaces, see [D-M]), but we have avoided this theorem, firstly since it is not strictly necessary, secondly since we hope to generalize our results to $S^k(A)$, $k \geq 4$.

Our theorems have the following corollary, which solves the problem put at the end of [P]:

Corollary (1.5). *Let A be a generic g-dimensional Abelian variety, $g \geq 4$. Then A is not a quotient of a Jacobian of a trigonal curve, in other words A does not contain trigonal curves.*

Proof. Let C be a trigonal curve such that there exists a surjective map

$$f\colon J(C) \to A.$$

By composing f with the Abel-Jacobi map, we get a nontrivial map $C \to A$, hence we have a finite map: $S^3(C) \to S^3(A)$; as C is trigonal we have another obvious map: $\mathbf{P}^1 \to S^3(C) \to S^3(A)$; this gives rise to a rational curve in $S^3(A)$, but it is not possible by (1.4)(c). □

Remark (1.6). Obviously the Jacobian of a trigonal curve contains a trigonal curve: the curve itself; (1.5) shows that, among Abelian varieties, the Jacobians of genus 4 curves are special also under this point of view.

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2. Notations and conventions

$\oplus$	direct sum of vector spaces,
$\langle \mathbf{x}_1, \mathbf{x}_2, \dots \rangle$	C-vector space generated by $\mathbf{x}_1, \mathbf{x}_2, \dots$,
variety	by this term we mean a projective complex variety,
n-fold	n-dimensional variety (not necessarily smooth),
surface	two-fold,
curve	one-fold,
generic	by this word we mean: outside a countable union of proper analytic subvarieties,
K_V	canonical divisor of the variety V when it is smooth,
$V \times V$	Cartesian product of the variety V with itself,
V^k	k-Cartesian product of the variety V,
$S^k(V)$	k-symmetric product of the variety V,
$\mathscr{H}_n$	Siegel space of n-dimensional Abelian varieties.

3. Rational equivalence of zero-cycles

In this paragraph we recall the results of Roitman and Mumford we need in the sequel.

Proposition (3.1) (see [R_2]). *Let Z be a degree k effective zero-cycle on a smooth variety X, then the γ-orbit of X containing Z is a countable union of closed subsets of $S^k(X)$; such a set is usually called c-closed.*

We can define the *dimension* of a c-closed set as the maximal dimension of its irreducible components. In this way it is possible to define the dimension of the image: $\gamma(S^k(X)) \subseteq \mathrm{CH}_0(X)$, even though it is not an algebraic variety, as

$$d_k = \dim(S^k(X)) - \min\{\text{dimension of a fibre of } \gamma\}.$$

We say that $\mathrm{CH}_0(X)$ is *finite dimensional* if the set of integers d_k is bounded, otherwise we say that $\mathrm{CH}_0(X)$ is *infinite dimensional*.

In [M] Mumford proved that if X is a surface with geometric genus $p_g > 0$, then $\mathrm{CH}_0(X)$ is infinite dimensional. In [R_2] Roitman gave the following generalization:

Theorem (3.2). *Let X be a smooth variety; then there are integers $d(X)$ and $j(X) \geq 0$, and an integer k_0, such that for all $k \geq k_0$ we have $d_k = kd(X) + j(X)$. Moreover $d(X) \leq \dim(X)$, and $d(X) = 0$ if and only if $\mathrm{CH}_0(X)$ is finite dimensional.*

In [R_1 and R_2] Roitman proved the following:

Theorem (3.3). *Let X be a smooth variety, suppose that, for some positive integer q, there exists a nonzero global q-form ω on X. Then ω induces a q-form ω_k on $S^k(X)$ whose restriction to any γ-component of $S^k(X)$ is zero. Hence $d(X) \geq q$.*

We recall that the q-form ω_k quoted in (3.3) is defined as follows: we consider X^k and for any $i = 1, 2, \dots, k$ we consider the natural projection onto the ith factor $p_i: X^k \to X$, now the q-form $\sum p_i^*\omega$ is well defined at the generic point of $S^k(X)$ because it is invariant under the action of the symmetric group; so we set $\omega_k = \sum p_i^*\omega$. In the same papers Roitman also shows the following:

Theorem (3.4). *Let f_1, f_2 be two maps between a smooth variety V and $S^k(X)$ such that $\forall v \in V$ $f_1(v)$ is rationally equivalent to $f_2(v)$; let ω be a q-form defined on X; then $f_1^*(\omega_k) = f_2^*(\omega_k)$.*

The previous theorem allows us to prove this corollary.

Corollary (3.5). *Let V be a smooth n-dimensional variety; let $f: V \to S^k(X)$ be a map; suppose that there exists a map $p: V \to B$, where B is an $n - t$ dimensional variety, such that $\forall b \in B$, $f[p^{-1}(b)]$ is a t-dimensional γ-component of $S^k(X)$; let ω be a q-form defined on X. Then $f^*\omega_k = 0$ if $q > n - t$.*

Proof. We can always choose a suitable subvariety W of V such that $p_{|W}$ is finite over B; let $V^\#$ be $V \times_B W$ (fibre product). Let $p^\#: V^\# \to W$ and $\pi^\#: V^\# \to V$ the induced projections and $\sigma: W \to V^\#$ be the canonical section of $p^\#$; now we consider the maps h, $g\colon V^\# \to S^k(X)$ such that $h(v) = f[\pi^\#(v)]$ and $g(v) = h\{\sigma[p^\#(v)]\}$. Obviously $h(v)$ is rationally equivalent to $g(v)$ $\forall v \in V^\#$, and therefore, by (3.4), $h^*\omega_k = g^*\omega_k$. But $g^*\omega_k = (p^\#)^*\sigma^*h^*\omega_k$ and $\sigma^*h^*\omega_k = 0$ if $q > n-t$, as $\pi^\#$ is finite on V, $f^*\omega_k = 0$. □

4. Some linear algebra

Let V be $\mathbf{C}^2$, and let $\{dz, dw\}$ be a basis for V^*. Let L_2 be the kernel of the map $\sigma\colon V \oplus V \oplus V \to V$ given by summation. Consider the following two-form on L_2:

$$(\hat{\ }) \qquad \begin{aligned}&[dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + dz_3 \wedge dw_3]_{|L_2} \\ &= [2dz_1 \wedge dw_1 + 2dz_2 \wedge dw_2 + dz_1 \wedge dw_2 + dz_2 \wedge dw_1]_{|L_2} \\ &= [dz_1 \wedge d(2w_1 + w_2) + d(z_1 + 2z_2) \wedge dw_2]_{|L_2}.\end{aligned}$$

As $(\hat{\ })$ has maximal rank on L_2, we have that any locally isotropic subspace of $V \oplus V \oplus V$ for $(\hat{\ })$, has dimension 2 at most. In fact there are such two-dimensional maximal subspaces, for instance: $\{(\mathbf{v}, \rho\mathbf{v}, \rho^2\mathbf{v}),\ \mathbf{v} \in V,\ \rho \in \mathbf{C}$ with $1 + \rho + \rho^2 = 0\}$.

Now let W be $\mathbf{C}^n$, $n \geq 2$, and let L_n be the kernel of the map $\sigma\colon W \oplus W \oplus W \to W$ as before. Let U be a linear subspace of L_n such that for all projections $W \to V$, the induced map $L_n \to L_2$ sends U into a totally isotropic subspace of L_2 for $(\hat{\ })$. Then $\dim(U) \leq n$. In fact, for $n = 2$ this is true, for $n \geq 3$ we can proceed by induction on n: every projection $L_n \to L_{n-1}$ has kernel of dimension 2, so that $\dim(U) \leq n+1$; moreover if $\dim(U) = n+1$, the kernel of every projection $L_n \to L_{n-1}$ would lie in U, and this is not possible.

Note that $\dim(U) = n$ is possible, for instance if $U = \{(\mathbf{w}, \rho\mathbf{w}, \rho^2\mathbf{w}),\ \mathbf{w} \in W,\ \rho \in \mathbf{C}$ with $1 + \rho + \rho^2 = 0\}$; we will see in (4.2) that it is the only possibility. Now we can prove the following:

Proposition (4.1). *In the same notation as before, let* $n = 3$, *let* $\{dz, dw, du\}$ *be a basis for* W^*; *consider the following three-form*:

$$(\hat{\ }\hat{\ }) \qquad dz_1 \wedge dw_1 \wedge du_1 + dz_2 \wedge dw_2 \wedge du_2 + dz_3 \wedge dw_3 \wedge du_3$$

and suppose that U *is totally isotropic for* $(\hat{\ }\hat{\ })$. *Then* $\dim(U) \leq 2$.

Proof. By contradiction we suppose that $\dim(U) = 3$, then by projecting W to V three times along the respective axes we see that:

$$\begin{aligned}U = \langle (a_1, 0, 0), (a_2, 0, 0), (a_3, 0, 0)\rangle + \langle (0, b_1, 0), (0, b_2, 0), (0, b_3, 0)\rangle \\ + \langle (0, 0, c_1), (0, 0, c_2), (0, 0, c_3)\rangle\end{aligned}$$

with: $\sum a_i = \sum b_i = \sum c_i = 0$. So the vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$, $\mathbf{c} = (c_i)$ in $\mathbf{C}^3$ lie in the plane P defined by the equation: $\sum x_i = 0$. Since for all projections $W \to V$, the induced map $L_n \to L_2$ sends U into a totally isotropic subspace of L_2 for $(\hat{\ })$, we have: $\sum a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$.

Since the symmetric bilinear form on $\mathbf{C}^3$ which has the identity associated matrix (with respect to the standard base) has rank 2 on P, we conclude from the above equations that either $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$ is $\mathbf{0}$, (this is impossible as we have supposed that $\dim(U) = 3$) or $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all multiples of the same vector

$\mathbf{w}$ with $\sum w_i = \sum (w_i)^2 = 0$. So that $\mathbf{w}$ can be taken to be some permutation of $(1, \rho, \rho^2)$. Hence we can write: $\mathbf{a} = A\mathbf{w}$, $\mathbf{b} = B\mathbf{w}$ and $\mathbf{c} = C\mathbf{w}$ for some nonzero complex numbers A, B, C. But if we apply $(\hat{}\hat{})$ to these three vectors we have that the result is zero if and only if $ABC = 0$, contradiction! □

By (4.1) it is very easy to prove the following:

Proposition (4.2). *In the previous notation: let* $n \geq 4$. *Then* $U \subseteq \{(\mathbf{w}, \rho\mathbf{w}, \rho^2\mathbf{w})$, $\mathbf{w} \in W$, $\rho \in \mathbf{C}$ *with* $1 + \rho + \rho^2 = 0\}$ *and if all projections of* W *into* $\mathbf{C}^3$ *send* U *into a totally isotropic subspace for* $(\hat{}\hat{})$, *we have that* $\dim(U) \leq 2$.

5. Proof of (1.2) and some examples

Let A be an n-dimensional Abelian variety. Firstly we want to recall some useful facts about $S^k(A)$.

There is an action of the additive group A on the variety $S^k(A)$: for every $a \in A$ we have $T_a \colon S^k(A) \to S^k(A)$ such that for every $(x_1, x_2, \dots, x_k) \in S^k(A)$ $(T_a(x_1, x_2, \dots, x_k) = (x_1 + a, x_2 + a, \dots, x_k + a)$. For every $a \in A$, T_a is an isomorphism of $S^k(A)$ which we will call translation, by abuse of language.

If we consider the nk-dimensional Abelian variety A^k, we have that there is a $(k!)$-covering $p \colon A^k \to S^k(A)$ which is obviously ramified on the points $(x_1, x_2, \dots, x_k)$ of $S^k(A)$ such that the x_i are not all distinct. Moreover there is another obvious $(k!)$-covering $\pi \colon A^{k-1} \to K_k(A)$ ($K_k(A)$ is the kernel of the Albanese map, see §1) such that $\pi(x_1, x_2, \dots, x_{k-1}) = (x_1, x_2, \dots, x_{k-1}, -x_1 - x_2 \cdots - x_{k-1})$. Remark that any d-dimensional γ-component in $K_k(A)$ gives rise to a d-fold in A^{k-1} via π.

Now we are able to prove (1.2); recall that, by the argument of §1, we have to study the γ-orbits contained in $K_3(A)$.

Proof of (1.2)(a). Let V be the dual of the Lie algebra of A, $\dim(V) = \dim(A) = n$, and we recall that, for any Abelian variety A, $\forall q \geq 1$, $H^{q,0}(A) = \Lambda^q(V)$.

For any $\omega \in \Lambda^q(V)$, $q \geq 2$, we consider the q-form $\phi(\omega)$ induced by ω on $S^3(A)$ in the following way: $\phi(\omega) = p_*(p_1^*\omega + p_2^*\omega + p_3^*\omega)$, where

$$p \colon A^3 \to S^3(A)$$

and p_1, p_2, p_3 are the projections of $A \times A \times A$ on A.

The tangent space U at every smooth point of any γ-orbit of $K_3(A)$ lies in L_n (see §4); $\phi(\omega)$ has to vanish on U, by Theorem (3.3), for *any* $\omega \in \Lambda^q(V)$, $q = 2, 3, \dots, n$; this means that the assumptions of (4.2) about the projections of U are satisfied. Hence $\dim(U) \leq 2$; therefore every γ-orbit has dimension 2 at most. □

Remark (5.1). The previous proof is based on the fact that all the forms belonging to $\phi(\Lambda^q(V))$, $q = 2, 3, \dots, n$, have to vanish on the tangent spaces at the smooth points of any γ-component of $K_3(A)$. So we can say that, if a d-fold, contained in $K_3(A)$, has the same properties, then $d \leq 2$.

Proof of (1.2)(b). If there would be such a family $\{S_t\}$, $t \in \mathbf{C}$, then in $K_3(A)$ we would get a three-fold T which would be filled by two-dimensional γ-components. By using the same notations as in the proof of (1.2)(a), we have that, by Corollary (3.5), the forms belonging to $\phi(\Lambda^q(V))$, $q = 2, 3, \dots, n$, have to vanish on the tangent spaces at the smooth points of T, but this implies that $\dim(T) \leq 2$ by Remark (5.1): contradiction! □

Proof of (1.2)(c). If there would be a family $\{C_\mathbf{r}\}$, $\mathbf{r} \in \mathbf{C}^3$, of one-dimensional γ-orbits in $K_3(A)$ then $K_3(A)$ would be filled by one-dimensional γ-components and this is not possible by (3.2) and (3.3). □

Now we prove, by some examples, that, when $\dim(A) = 2$, the one-dimensional γ-orbits can span a three-fold in $S^3(A)$, and that there are two-dimensional γ-orbits.

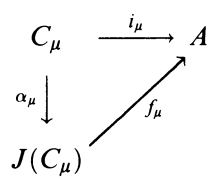
Example (5.2). Let A be an Abelian surface; let C be a nonhyperelliptic genus 3 (smooth, irreducible) curve on A. If we consider the divisor L supported by C, we get $L^2 = 4$ by the genus formula, and $h^0(L) = 2$ by the Riemann-Roch and Kodaira vanishing theorems.

So C moves in a pencil $\{C_\mu\}$ which has four base points: A, B, C, D. The adjunction formula yields: $K_L = L_{|L}$; so that $A+B+C+D$ is a canonical divisor on every curve C_μ of the pencil.

The canonical model C'_μ of C_μ is a smooth plane quartic whose canonical series is cut by the lines, therefore the divisor of C'_μ corresponding to $A+B+C+D$ is cut on C'_μ by a line.

Now we consider a point P_μ on C_μ and the linear series g^1_3 corresponding to the linear series g^1_3 cut on C'_μ by the lines passing through the point corresponding to P_μ. So that for every $\lambda \in \mathbf{P}^1$ we have a divisor: $P_\mu + Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda}$ on C_μ. We choose an Abel map $\alpha_\mu \colon C_\mu \to J(C_\mu)$ such that $\alpha_\mu(P_\mu) = 0$, hence, by Abel theorem, $\alpha_\mu(Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda}) = \tau_{P,\mu}$ is constant with respect to λ. The 3-ples: $\alpha_\mu(Q_{\mu\lambda})$, $\alpha_\mu(R_{\mu\lambda})$, $\alpha_\mu(S_{\mu\lambda})$ in $J(C_\mu)$ gives rise to a rational curve in $S^3[J(C_\mu)]$ as λ moves in $\mathbf{P}^1$.

We consider the following commutative diagram



in which i_μ is the embedding of C_μ in A and f_μ is the homomorphism between Abelian varieties induced by α_μ. By using f_μ we get a rational curve in $S^3(A)$; by translating this curve by $f_\mu(\tau_{P,\mu})$ we get a rational curve $\gamma_{P,\mu}$ in $K_3(A)$.

Now we let P vary on C_μ: for every point P we get a curve $\gamma_{P,\mu}$ in $K_3(A)$; these curves are all distinct because the used linear series g^1_3 on C'_μ are distinct. Now let P vary on C_μ and let μ vary in $\mathbf{P}^1$: for every couple P, μ we get a curve $\gamma_{P,\mu}$ in $K_3(A)$; these curves are all distinct because they are made by points lying on different curves C_μ of A.

Obviously every curve $\gamma_{P,\mu}$ is contained in a γ-orbit of $K_3(A)$ and this example shows that in $K_3(A)$ there exist γ-orbits whose span is a three-fold.

Example (5.3). The previous example also shows that in $K_3(A)$ there exist some γ-orbits whose span is a surface. In fact for every curve C_μ of the previous example we can fix the point A, (one of the base points of the pencil $\{C_\mu\}$), and for every $\mu \in \mathbf{P}^1$ we get a rational curve $\gamma_{A,\mu} = \gamma_\mu$ in $K_3(A)$.

In this case, by recalling the construction of the linear series g_3^1, we have that for every $\mu \in \mathbf{P}^1$ there exists a $\lambda \in \mathbf{P}^1$ such that $Q_{\mu\lambda} = B$, $R_{\mu\lambda} = C$, $S_{\mu\lambda} = D$. Therefore: $\alpha_\mu(B+C+D) = \tau_{A,\mu}$ and $f_\mu[\alpha_\mu(B+C+D)] = f_\mu(\tau_{A,\mu}) = i_\mu(B+C+D)$ is independent from μ, hence the obtained curves in $S^3(A)$ belong to

$$\{(x, y, z) \in S^3(A) | x + y + z = i_\mu(B + C + D)\}$$

and all pass through the point: $(i_\mu(B), i_\mu(C), i_\mu(D))$ in $S^3(A)$.

So that the translated curves γ_μ in $K_3(A)$ all intersect between them. Therefore the curves γ_μ span a rational surface in $K_3(A)$ which is contained in a γ-orbit.

6. The lemmas

In this paragraph we prove some lemmas which will be useful in §7. We will need to study the projections of d-dimensional γ-components which are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, where V and W will be suitable Abelian varieties.

By the commutativity of the following diagram

$$\begin{array}{ccc} (V \times W) \times (V \times W) & \longrightarrow & W \times W \\ \downarrow \pi & & \downarrow \pi \\ K_3(V \times W) & \longrightarrow & K_3(W) \end{array} \tag{6.1}$$

we have to study the natural projections $(V \times W) \times (V \times W) \to W \times W$, this is the aim of the following two lemmas.

Let X be a smooth irreducible d-fold and let A be an n-dimensional Abelian variety; let $\sigma: X \to A \times A$ be a map, birational onto its image, such that $\sigma(X)$ generates $A \times A$. Assume that A is isogenous to $D \times D \times B$ where D and B are Abelian varieties of dimension q and $(n-2q)$ respectively. We fix two "dual" isogenies $D \times D \times B \to A \to D \times D \times B$ such that their composition is the multiplication by an integer; in this way we get a map $f \circ \sigma: X \to B \times B$ by composing the natural projection f with σ; let Y be $f[\sigma(X)]$; assume that

(*) the natural projection $f: A \times A \to B \times B$ is such that $Y = f[\sigma(X)]$ is a d-dimensional subvariety of $B \times B$.

Now let $\nu_i: D \to D \times D \to A$ be the composition of an embedding of D in $D \times D$ with the previously chosen isogeny; we can suppose that i varies in a countable set, in fact among all embeddings $D \to D \times D$ there are the following morphisms of algebraic groups: $\mathbf{d} \to (a\mathbf{d}, b\mathbf{d})$ (for any $\mathbf{d} \in D$ and for a fixed couple of coprime integers a, b). We set $B_i = [(D \times D)/\nu_i(D)] \times B$ and let X_i be the image of X under the composition of the natural projection $A \times A \to B_i \times B_i$ with σ.

In this situation we have the maps: $q_i^*: H^1(X_i, \mathbf{Q}) \to H^1(X, \mathbf{Q})$ and $\sigma^*: H^1(A \times A, \mathbf{Q}) \to H^1(X, \mathbf{Q})$; let Λ_i be the image of q_i^*, then

Lemma (6.2). *With the previous notations, there exists an index i at least (hence an embedding of D in $D \times D$) such that Λ_i contains the image of σ^*.*

Proof. Note that this proof actually shows more, i.e. Λ_i contains the image of $H^1(A \times A, \mathbf{Q})$ in $H^1(X, \mathbf{Q})$ save for a finite number of i.

For every i we have a diagram of equidimensional d-folds

$$\begin{array}{ccc} X & \longrightarrow & X_i \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

(the map $X_i \to Y$ is obtained by using the natural projection

$$B_i \to B_i/[(D \times D)/\nu_i(D)]$$

and by remarking that $B_i/[(D \times D)/\nu_i(D)]$ is isogenous to B). It follows that: $K[Y] \subset K[X_i] \subset K[X]$ so that there are only a finite number of birational models for the X_i. The maps in the following diagram are defined in the obvious way:

$$\begin{array}{ccccc} H^1(B \times B, \mathbf{Q}) & \longrightarrow & H^1(B_i \times B_i, \mathbf{Q}) & \longrightarrow & H^1(A \times A, \mathbf{Q}) \\ \downarrow & & \downarrow & & \downarrow \sigma^* \\ H^1(Y, \mathbf{Q}) & \longrightarrow & H^1(X_i, \mathbf{Q}) & \longrightarrow & H^1(X, \mathbf{Q}) \end{array}$$

and we remark that, as $\sigma(X)$ generates $A \times A$ and the natural projection $A \times A \to B_i \times B_i$ is surjective, the map $H^1(B_i \times B_i, \mathbf{Q}) \to H^1(X, \mathbf{Q})$ is injective for any i. Now if we choose two distinct, transverse, embeddings of D in $D \times D$ for which the corresponding fields $K[X_{i1}]$ and $K[X_{i2}]$, contained in $K[X]$, coincide, then we have that $H^1(X_{i1}, \mathbf{Q}) \to H^1(X, \mathbf{Q})$ and $H^1(X_{i2}, \mathbf{Q}) \to H^1(X, \mathbf{Q})$ are the same map; by the injectivity of $H^1(B_{ij} \times B_{ij}, \mathbf{Q}) \to H^1(X, \mathbf{Q})$, $j = 1, 2$, we have that $\Lambda_{i1} = \Lambda_{i2}$ must contain the span of the images of $H^1(B_{i1} \times B_{i1}, \mathbf{Q})$ and $H^1(B_{i2} \times B_{i2}, \mathbf{Q})$ in $H^1(X, \mathbf{Q})$, hence $\Lambda_{i1} = \Lambda_{i2}$ must contain the image of $H^1(A \times A, \mathbf{Q})$ in $H^1(X, \mathbf{Q})$. $\square$

Lemma (6.3). *With the same assumptions as in* (6.2), *we get the same thesis if we consider $F^1H^1(\_, \mathbf{C})$, (in the sense of mixed Hodge structures, see* [G]), *instead of $H^1(\_, \mathbf{Q})$.*

Remark (6.4). Note that, if $\dim(X) = 1$, $(*)$ is always satisfied, (save, obviously, when $A = E \times E \times B$, E elliptic curve, and $X = E$).

Now let Δ be an analytic scheme $(0 \in \Delta)$, and $h: \mathbf{A} \to \Delta$ a proper fibration such that $h^{-1}(t)$, $t \in \Delta$, is an Abelian variety isogenous to $\mathbf{D}_t \times \mathbf{B}$, $\mathbf{B}$ fixed, ($h^{-1}(0)$ isogenous to $\mathbf{D}_0 \times \mathbf{B}$).

The infinitesimal variation of the Hodge structures induces the following map $\phi: H^{1,0}(\mathbf{D}_0) \to \mathrm{Hom}(T_\Delta(0), H^{0,1}(\mathbf{D}_0))$, such that for any $\mu \in H^{1,0}(\mathbf{D}_0)$ and for any $\mathbf{t} \in T_\Delta(0)$, $\phi(\mu)(\mathbf{t})$ is the derivative of μ along $\mathbf{t}$. We have the following:

Lemma (6.5). *With the previous assumptions, consider the commutative diagram*

$$\begin{array}{ccc} q_t\colon X_t & \longrightarrow & Z \\ {\scriptstyle \sigma_t}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ f_t\colon \mathbf{D}_t \times \mathbf{B} & \longrightarrow & \mathbf{B} \end{array}$$

where X_t are varieties parametrized by t, σ_t are maps birational onto their images, $\sigma_t(X_t)$ generates $\mathbf{D}_t \times \mathbf{B}$ for any t, f_t is the natural projection, q_t is induced by f_t, ι is an inclusion and Z is fixed. Assume that ϕ is injective; then

$$\sigma_0^*[H^{1,0}(\mathbf{D}_0)] \cap q_0^* F^1 H^1(Z) = 0 \in F^1 H^1(X_0).$$

Proof. If μ belongs to that intersection, $\phi(\mu) = 0$ as $F^1H^1(Z)$ is independent from t; as ϕ is injective we have $\mu = 0$. □

Now let Δ be an open set of $\mathscr{H}_n$, $(0 \in \Delta)$, we will call a "(Δ, m, G)-situation" (for $\mathscr{H}_n$) the following data:

(i) a bundle of Abelian varieties over Δ: $\mathbf{A} \times_\Delta \mathbf{A} \times_\Delta \cdots \times G$ (m times) where $\mathbf{A}$ is the tautological Abelian bundle over Δ and G is a constant Abelian variety; (by abuse of notation we write $G = G \times \Delta$ and $\mathbf{A}^m \times G = \mathbf{A} \times_\Delta \mathbf{A} \times_\Delta \cdots \times G$ (m times));

(ii) a family of d-dimensional varieties $k\colon \mathbf{X} \to \Delta$ over Δ;

(iii) a morphism of Δ families $\sigma\colon \mathbf{X} \to \mathbf{A}^m \times G$, i.e. a commutative diagram as follows:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\ \sigma\ } & \mathbf{A}^m \times G \\ & {\scriptstyle k}\searrow \quad \swarrow{\scriptstyle h} & \\ & \Delta & \end{array}$$

(we set $X_t = k^{-1}(t)$ and $\mathbf{h}^{-1}(t) = (A_t)^m \times G$ for any $t \in \Delta$);

(iv) the assumption that the image $\sigma_t(X_t)$ generates $(A_t)^m \times G$ as a group, for any $t \in \Delta$.

We remark that, if conditions (i), (ii), (iii) are satisfied, the bundle of Abelian varieties generated by the images $\sigma_t(X_t)$ must be isomorphic to $\mathbf{A}^{m'} \times G'$ where $m' \leq m$ and G' is an Abelian subvariety of G; so that, by changing the bundle, we always get a (Δ, m', G')-situation. With the above warning we can say that to have a (Δ, m, G)-situation is equivalent to have a d-dimensional variety in $A^m \times G$ where A is generic in Δ; (i.e. for any $t \in \Delta$ we have a d-fold X_t in $(A_t)^m \times G)$. Actually we usually will consider only the case: $m = 2$, $G = 0$, (hence $\mathbf{h} = h \times_\Delta h$); for the sake of simplicity, from now on, this case will be simply called "Δ-situation."

Lemma (6.6). *We suppose to be in a Δ-situation; we choose A isogenous to $D \times D \times B$, (as in Lemma (6.2)), and for any linear embedding $\nu_i\colon D \to D \times D$ we fix an isogeny between A and $\nu_i(D) \times [(D \times D)/\nu_i(D)] \times B$.*

Let $\Delta_i = \{t \in \Delta |$ the fibre of $h \times_\Delta h$ is $A_t \times A_t$ where A_t is isogenous to $\nu_i(D) \times D_t \times B$, $D_t \in \mathscr{H}_q\}$; let A_0 be isogenous to A by the isogeny induced by the previously fixed one. This defines an embedding $\nu_i^*\colon \mathscr{H}_q \to \mathscr{H}_n$, such that $\Delta_i = \Delta \cap [\nu_i^*(\mathscr{H}_q)]$; we set $B_i = \nu_i(D) \times B$.

For any $t \in \Delta_i$, let $f_{i,t}\colon A_t \times A_t \to B_i \times B_i$ be the natural projection; if we assume $(*)$ for the natural projection $f_{i,0}\colon A \times A \to B \times B$ and $\sigma_0(X_0)$, we have *that, save a finite number of* i *at most,* $f_{i,t}[\sigma_t(X_t)]$ *is not a fixed subvariety of* $B_i \times B_i$.

Proof. We proceed by contradiction: if (6.6) is false, then for any i, $f_{i,t}[\sigma_t(X_t)]$ is a fixed d-fold X_i in B_i for any t, and X_i generates B_i. Then we have the following commutative diagram:

$$\begin{array}{ccc} q_{i,0}\colon X_0 & \longrightarrow & X_i \\ \downarrow \sigma_0 & & \downarrow \iota_i \\ f_{i,0}\colon (D \times D \times B)^2 & \longrightarrow & B_i \times B_i . \end{array}$$

Note that we can apply Lemma (6.5) because we are in a Δ-situation, so we have that $\sigma_0^*[H^{1,0}(D^4)] \cap (q_{i,0})^* F^1 H^1(X_i) = 0 \in F^1 H^1(X_0)$ but, by Lemma (6.3), $(q_{i,0})^* F^1 H^1(X_i)$ contains $\sigma_0^*[H^{1,0}(D^4)]$ except for a finite number of i, contradiction! □

Lemma (6.7). *We are supposed to be in a* Δ*-situation; but now we choose* A *isogenous to* $D^m \times B$, *and we consider the countable set of the linear embeddings* $\nu_i\colon D^p \to D^m$ ($p \leq m$, *positive integers,* $D \in \mathscr{H}_q$, $B \in \mathscr{H}_{n-mq}$). *For any embedding* ν_i *we fix an isogeny between* A *and* $\nu_i(D^p) \times [D^m/\nu_i(D^p)] \times B$; *let* $\Delta_i = \{t \in \Delta |$ *the fibre of* $h \times_\Delta h$ *is* $A_t \times A_t$ *where* A_t *is isogenous to* $F_t \times [D^m/\nu_i(D^p)] \times B$, $F_t \in \mathscr{H}_{pq}\}$, A_0 *is isogenous to* A *as in the previous cases. This defines an embedding* $\nu_i^*\colon \mathscr{H}_{pq} \to \mathscr{H}_n$ *such that:* $\Delta_i = \Delta \cap [\nu_i^*(\mathscr{H}_{pq})]$; *we set:* $B_i = [D^m/\nu_i(D^p)] \times B$.

For any $t \in \Delta_i$, let $f_{i,t}\colon A_t \times A_t \to B_i \times B_i$ be the natural projection; if we assume $(*)$ for the natural projection $f_{i,0}\colon A \times A \to B \times B$ and $\sigma_0(X_0)$, we have that, save a finite number of i at most, $f_{i,t}[\sigma_t(X_t)]$ is not a fixed subvariety of B_i.

Proof. See the proof of (6.6). □

To apply the above lemmas we need condition $(*)$; this is a crucial point: it allows us to avoid the use of the De Franchis-Severi theorem. When X is of general type and $d = 1$ or 2, this theorem would assure the existence of a *finite* number of subfields $K[X_i]$ of $K[X]$ (see the proof of Lemma (6.2)), without the assumption that f is generically finite, i.e., roughly speaking, without fixing a shield $Y = f[\sigma(X)]$.

We use the following remark: consider diagram (6.1): our natural projections between $(V \times W) \times (V \times W)$ and $W \times W$ are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, so that to verify $(*)$ it suffices to verify the corresponding statement for projections between $K_3(V \times W)$ and $K_3(W)$, and vice versa. This explains the statements of the following other lemmas.

Lemma (6.8). *Let* S *be a* γ*-surface in* $K_3(E \times E)$ *where* E *is a generic elliptic curve* (*in the sense of moduli*); *let* S' *be the pullback of* S *in* $E^2 \times E^2$; *let* E_{pq} *be a fixed embedding of* $E \times E$ *in* $E^2 \times E^2$ *such that* $E_{pq} = \{px, qx, py, qy\}$ *where* $(x, y) \in E \times E$ *and* p, q *are coprime integers. Then there exist infinitely many couples* (p, q) *such that* E_{pq} *intersects* S' *properly. In these cases the natural projection* $E^2 \times E^2 \to (E^2 \times E^2)/E_{pq}$ *is generically finite on* S' (*and the induced map* $K_3(E \times E) \to K_3[(E \times E)/\{px, qy\}]$ *is generically finite on* S).

Proof. We will prove that there exists a couple (p, q) at least, such that E_{pq} intersects S' properly, but, in fact, our proof will also show that the intersection is proper save for a finite number of couples.

We proceed by contradiction; we recall that if two surfaces in E^4 does not intersect properly then, for every generic point of the first surface, there passes a translate of the second one which intersects the former one along a curve. In fact the intersection cycle of two surfaces in E^4 depends only on their homology class, and the homology class is invariant under translations.

We fix a generic point P of S', if every E_{pq} does not intersect S' properly then, $\forall p, q$, there exists a translate of E_{pq} passing through P and cutting S' along a curve; hence, by looking at the tangent spaces, we have that in the Lie algebra of E^4 there are: a vector space generated by $(p, q, 0, 0)$ and $(0, 0, p, q)$, $\forall p, q$, and the vector space $\langle(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\rangle$ (corresponding to the tangent space to S' at P), such that the matrix:

$$\begin{matrix} p & 0 & a_1 & b_1 \\ q & 0 & a_2 & b_2 \\ 0 & p & a_3 & b_3 \\ 0 & q & a_4 & b_4 \end{matrix}$$

is always singular. Now we show that, for generic E, this situation is not possible.

As (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are independent, it is possible to choose a base for the Lie algebra such that: $a_1 = b_2 = 1$, $b_1 = a_2 = 0$; otherwise is not possible that the previous matrix is singular $\forall p, q$. Now it is easy to see that it is possible only if $b_3 = a_4 = 0$ and $b_4 = a_3 = \rho$, with $\rho \in \mathbf{C}$. As S' is the pullback in $E \times E \times E \times E$ of a γ-component S in $K_3(E \times E)$ which is not contained in the branching locus of π, the skew symmetric two-form $(\widehat{\ })$ considered in §4 has to vanish on the tangent space at the generic point P of S' by (3.3), hence: $1 + \rho + \rho^2 = 0$ and ρ is a constant, independent from P.

This means that the only surfaces in E^4 which does not intersect properly E_{pq} $\forall p, q$, are, up to translations, those Abelian surfaces S' which are the embeddings of $E \times E$ in E^4 such that $S' = \{x, y, \rho x, \rho y\}$ where $(x, y) \in E \times E$ and $\rho \in \mathbf{C}$ with $1 + \rho + \rho^2 = 0$; but this implies that E has an endomorphism: $x \to \rho x$ $\forall x \in E$, with $1 + \rho + \rho^2 = 0$, and this is not possible for generic E. □

Lemma (6.9). *Let S be a γ-surface in $K_3(E \times E \times E)$ where E is a generic elliptic curve; let S' be the pullback of S in $E^3 \times E^3$; let $E(p, q, r, p', q', r')$ be a fixed embedding of $E^2 \times E^2$ in $E^3 \times E^3$ such that $E(p, q, r, p', q', r') = \{px + p'y, qx + q'y, rx + r'y, pz + p'w, qz + q'w, rz + r'w\}$ where $(x, y, z, w) \in E^2 \times E^2$, and (p, q, r), (p', q', r') are triple of coprime integers, and such that the following matrix has rank 2:*

$$\begin{matrix} p & q & r \\ p' & q' & r'. \end{matrix}$$

Then there exist infinitely many choices (p, q, r, p', q', r') such that $E(p, q, r, p', q', r')$ intersects S' properly. In these cases the natural projection

$$E^3 \times E^3 \to (E^3 \times E^3)/E(p, q, r, p', q', r')$$

is generically finite on S',

$$(K_3(E \times E \times E) \to K_3[(E \times E \times E)/\{px + p'y, qx + q'y, rx + r'y\}]$$

is generically finite on S).

Proof. We can proceed as in the proof of Lemma (6.8). □

Lemma (6.10). *Let* E *be a generic elliptic curve and let* T *be a three-fold in* $K_3(E^3)$ *which is filled by a two-dimensional family of* γ*-curves; let* T' *be the pullback of* T *in* $E^3 \times E^3$; *let* E_{pqr} *be a fixed embedding of* $E \times E$ *in* $E^3 \times E^3$ *such that* $E_{pqr} = \{px, qx, rx, py, qy, ry\}$ *where* $(x, y) \in E \times E$ *and* p, q, r *are coprime integers. Then there exist infinitely many triples* (p, q, r) *such that* E_{pqr} *does not intersect* T' *or intersects* T' *in a finite number of points. In these cases the natural projection* $E^3 \times E^3 \to (E^3 \times E^3)/E_{pqr}$ *is generically finite on* T' (*and the induced map* $K_3(E^3) \to K_3[E^3/\{px, qx, rx\}]$ *is generically finite on* T).

Proof. By arguing as in Lemma (6.8) we get that the only three-folds in $E^3 \times E^3$ which does not intersect properly E_{pqr} $\forall p, q, r$ are, up to translations, those Abelian three-folds T' which are the embeddings of $E \times E \times E$ in $E^3 \times E^3$ such that $T' = \{x, y, z, sx, sy, sz\}$ where $(x, y, z) \in E \times E \times E$ and $s \in \mathbf{C}$ with $s(s+1) = 0$.

This would imply that, in $K_3(E^3)$, T would be given by the unordered triples: $\{P, sP, -(s+1)P\}$, where $s = 0$ or $s = -1$ and $P \in E^3$; in any case we could define an embedding $\lambda\colon T \to K_2(E^3)$ such that

$$\lambda(\{P, sP, -(s+1)P\}) = \{P, -P\};$$

$\lambda(T)$ would be a three-fold filled out by γ-curves; but this is not possible by (1.1)(b): recall that E is generic and the locus of nonsimple Abelian three-folds is dense in $\mathscr{H}_3$.

7. Proof of (1.4)

For the sake of simplicity, in every Δ-situation considered in §7 we will identify X_t with $\sigma_t(X_t)$.

Proof of (1.4)(a). We proceed by contradiction: we assume that for any three-dimensional Abelian variety A, $S^3(A)$, and therefore $K_3(A)$, contains a γ-surface; by their pullback via π we have a surface in any A^2, so we are in a Δ-situation. Then we can construct a fibration $h \times_\Delta h\colon \mathbf{A} \times_\Delta \mathbf{A} \to \Delta \subset \mathscr{H}_3$ as in §6. We want to apply Lemma (6.6) with $D = B = E$, E generic elliptic curve. To have (∗) we use Lemma (6.9): we can fix an Abelian variety A isogenous to $E \times E \times E$, such that, when we project the γ-surface X contained in $K_3(E \times E \times E)$ into $K_3(E)$ (the last factor), by the natural projection, we obtain another γ-surface Y. This means that the natural projection $f\colon A \times A \to B \times B$ satisfies (∗).

Now let E_{pq} be the image in $E \times E$ of the embedding ν_{pq} of E such that $\nu_{pq}(x) = (px, qx)$ $\forall x \in E$, (p, q) is a couple of coprime integers. We fix an isogeny between A and $E_{pq} \times B_{pq}$ where B_{pq} is $[(E \times E)/E_{pq}] \times E$. Let $\Delta_{pq} = \Delta \cap [\nu_{pq}^*(\mathscr{H}_1)]$ the open subset of Δ such that the fibre over $t \in \Delta_{pq}$ is

$A_t \times A_t$ where A_t is isogenous to $E_t \times B_{pq}$ (A_0 isogenous to A by the previously fixed isogeny) and E_t is an elliptic curve whose moduli depend on t.

Let φ_t be the natural projection between $K_3(A_t)$ and $K_3(B_{pq})$, by our assumption there is a γ-surface X_t in every A_t and $X_0 = X$. For small t, we can assume that $Y_t = \varphi_t(X_t)$ is a γ-surface of $K_3(B_{pq})$; in fact $Y_0 = \varphi_0(X_0) = \varphi_0(X)$ is a surface in $K_3(B_{pq})$ because X projects into a surface in $K_3(E)$.

By Lemma (6.6), we can choose (p, q) such that $\{Y_t\}$ is a one-dimensional family of γ-surfaces of $K_3(B_{pq})$ (i.e. the union of the Y_t span a three-fold in $K_3(B_{pq})$); but $\dim(B_{pq}) = 2$ and this is a contradiction with (1.2)(b). □

Remark (7.1). Here we want to give a short outline of the proof of (1.1)(b) when $\dim(A) = 3$. Firstly we need (1.1)(a) for dimension 2: this is just an application of (3.2) and (3.3): if (1.1)(a) were false, for the generic point of $S^2(A)$ would pass a positive dimensional γ-orbit, but then d_2 would be strictly less than 4.

Now we proceed by contradiction: we assume that for the generic Abelian three-fold A, $S^2(A)$, and therefore $K_2(A)$ (which is the Kummer variety $K(A)$ of A), contains a γ-curve. By their pullback via π we get a curve in any A; by using these we can build a family of curves that gives rise to a Δ-situation. By arguing as in the proof of (1.4)(a) we can choose a suitable projection from $K_2(A_t) = K(A_t)$ onto $K(E \times E)$, where A_t is isogenous to $E_t \times E \times E$, E generic elliptic curve, in such a way that the images of our curves cover $K(E \times E)$. Since the image of a γ-orbit is a γ-orbit, we get a contradiction with (1.1)(a).

Proof of (1.4)(b). We proceed by contradiction: we assume that for any three-dimensional Abelian variety $S^3(A)$, and therefore $K_3(A)$ contains a three-fold filled by γ-curves: by their pull-back via π we have a three-fold in any A^2. So we are in a Δ-situation and we can construct a fibration $h \times_\Delta h : \mathbf{A} \times_\Delta \mathbf{A} \to \Delta \subset \mathscr{H}_3$ as in §6. Pay attention: now we proceed in a very similar way to the proof of (1.4)(a), but we cannot use Lemma (6.6) in that manner.

We fix an Abelian variety A isogenous to $E \times E \times E$, E generic elliptic curve. Let E_{pqr} be the image in $E \times E \times E$ of the embedding ν_{pqr} of E such that $\nu_{pqr}(x) = (px, qx, rx)$ $\forall x \in E$, (p, q, r) is a triple of coprime integers; let F_{pqr} be $(E \times E \times E)/E_{pqr}$, we fix an isogeny between A and $E_{pqr} \times F_{pqr}$.

By Lemma (6.10) we can assume that, when we project the three-fold T, filled by γ-curves, contained in $K_3(E \times E \times E)$, into $K_3(F_{pqr})$, by the natural projection, we obtain another three-fold $T^{\#}$ with the same property.

Let $\Delta_{pqr} = \Delta \cap [\nu^*_{pqr}(\mathscr{H}_1)]$ the open subset of Δ such that the fibre over $t \in \Delta_{pqr}$ is $A_t \times A_t$ where A_t is isogenous to $E_t \times F_{pqr}$ (A_0 isogenous to A) and E_t is an elliptic curve whose moduli depend on t.

Let φ_t be the natural projection between $K_3(A_t)$ and $K_3(F_{pqr})$, by our assumption there is a three-fold T_t, filled by γ-curves, in every A_t and $T_0 = T$. Moreover $\varphi_0(T_0) = \varphi_0(T) = T\#$ is a three-fold in $K_3(F_{pqr})$ by the previous remarks. Therefore, by choosing a smaller disk, we can assume that $T^{\#}_t = \varphi_t(T_t)$ is three-fold in $K_3(F_{pqr})$.

We can use Lemma (6.6) (and Remark (6.4)), to assure that there exist triples (p, q, r) (for instance with $r = 0$) such that *every* one-dimensional family $\{C_t\}$ of γ-curves of $K_3(A_t)$ projects into another similar family of $K_3(F_{pqr})$. We choose one of these triples.

Now we consider two cases: if $T_t^\#$ is a variable three-fold in $K_3(F_{pqr})$, by the previous condition, we would get a three-dimensional family of γ-curves in $K_3(F_{pqr})$, but $\dim(F_{pqr}) = 2$ and this is forbidden by (1.2)(c).

If $T_t^\# = T_0^\#$ is a fixed three-fold in $K_3(F_{pqr})$ then, by the previous condition, infinitely many γ-components pass through any point of $T_0^\#$, hence we would have a one-dimensional family of γ-surfaces in $K_3(F_{pqr})$ at least, and this is not possible by (1.2)(b). □

Proof of (1.4)(c). Firstly we assume that $\dim(A) = 4$ and we proceed by contradiction: we assume that for any four-dimensional Abelian variety A, $S^3(A)$, and therefore $K_3(A)$ contains a γ-curve; by their pullback via π we have a curve in any A^2. So we have a Δ-situation and then we can construct a fibration $h \times_\Delta h \colon \mathbf{A} \times_\Delta \mathbf{A} \to \Delta \subset \mathscr{H}_4$ as in §6.

We want to use Lemma (6.7) with $D = B = E$, E generic elliptic curve, $p = 2$, $m = 3$; note that $(*)$ is satisfied by Remark (6.4).

We fix an Abelian variety A isogenous to $E \times E \times E \times E$. For any embedding $\nu_i \colon E^2 \to E^3$ let $\Delta_i = \Delta \cap [\nu_i^*(\mathscr{H}_2)]$ the open subset of Δ such that the fibre over $\mathbf{r} \in \Delta_i$ is $A_\mathbf{r} \times A_\mathbf{r}$ where $A_\mathbf{r}$ is isogenous to $D_\mathbf{r} \times [E^3/\nu_i(E^2)] \times E$ ($A_\mathbf{0}$ isogenous to A as usual), $D_\mathbf{r}$ is an Abelian surface and $D_\mathbf{0}$ is isogenous to $E \times E$; in this case we set: $B_i = [E^3/\nu_i(E^2)] \times E$. By our assumptions there exists a one-dimensional γ-component $C_\mathbf{r}$ in any $K_3(A_\mathbf{r})$.

Now we use Lemma (6.7) and we have that, for all i except a finite number, when we project $K_3(A_\mathbf{r})$ into $K_3(B_i)$, by the natural projection, we get that *every* one-dimensional family of γ-curves projects into a one-dimensional family of γ-curves.

So we get a three-dimensional family of γ-curves in $K_3(B_i)$; they cannot cover all $K_3(B_i)$ by Theorem (3.2) (recall that B_i is isogenous to E^2); they cannot cover a three-fold, otherwise this three-fold would be filled by γ-surfaces and this is not possible by (1.2)(b); so the only possibility is the following: they all project in a fixed surface S in $K_3(B_i)$, which is a γ-component.

Note that, by Lemma (6.8), we can suppose that S project into a fixed surface $S^\frown$ when we project $K_3(B_i) = K_3([E^3/\nu_i(E^2)] \times E)$ into $K_3(E)$ by the natural projection on the last factor, hence $S^\frown$ is $K_3(E)$.

Now we choose $D_\mathbf{r} = E_\sigma \times E_s$ (σ, s belonging to the moduli space of elliptic curves) and generic embeddings ν_i in *infinitely many different ways*; for *any* choice, by using all the previous arguments, we get

—a γ-curve $C_{\sigma,s}$ in any $K_3(E_\sigma \times E_s \times E \times E)$,

—a surface S_s in $K_3(E_s \times E \times E)$, ($S_s$ covered by γ-curves),

—a fixed surface S in $K_3(E \times E)$ into which *all* S_s project,

—a fixed surface $S^\frown$ in $K_3(E)$ into which S projects,

(we always use natural projections).

We want to prove that this is a contradiction to Lemma (6.6).

These facts create a situation which is very similar to a Δ-situation, (see §6): actually, in this case, we have only a one-dimensional family of Abelian varieties: $A_s \times A_s = (E_s \times E \times E) \times (E_s \times E \times E)$ and a surface in every $A_s \times A_s$ which is the pullback, via π, of the surface S_s contained in $K_3(E_s \times E \times E)$. So we have a fibration defined only over an open set $\Delta_i' \subset \mathscr{H}_1$ and the surfaces $\{S_s\}$ are the fibres over Δ_i'. However, by looking at the proof of (6.6), it is obvious that it is true even in this case.

But, in our case, we also have that, if we choose A isogenous to $E \times E \times E$, there are *infinitely many embeddings* $\mu_{pq}: E \to E^2$ $(\forall x \in E \quad \mu_{pq}(x) = (px, qx)$, p, q coprime integers, *the* μ_{pq} *are induced by the* ν_i) such that, when we choose a family $\{A_s \times A_s\}$, s belonging to a suitable open set $\Delta_{pq} \subset \mathscr{H}_1$, (depending on Δ_i'), such that $\forall s \in \Delta_{pq}$, A_s is isogenous to $E_s \times \mu_{pq}(E) \times E$ (as usual A_0 is isogenous to $[E^2/\mu_{pq}(E)] \times \mu_{pq}(E) \times E$, isogenous to A), then

—$K_3(A_s)$ contains a surface S_s for any s,

—S_0 projects into a surface S' in $K_3(E)$ by the natural projection on the last factor, (hence condition $(*)$ is satisfied),

—all surfaces S_s project into a fixed surface S in $K_3[\mu_{pq}(E) \times E]$.

This is a contradiction to Lemma (6.6)!

Now we assume that $\dim(A) = n \geq 5$ and we proceed by induction on n. Suppose that for any n-dimensional Abelian variety A, $S^3(A)$, and therefore $K_3(A)$ contains a γ-curve; then it is true for those Abelian n-folds which are isogenous to $E \times B$ where E is a generic elliptic curve and B is a generic Abelian $(n-1)$-fold. It is easy to see that, by choosing a suitable isogeny, we also get a γ-curve in $K_3(B)$, and this is a contradiction to our induction hypothesis. □

REFERENCES

[D-M] M. Martin-Deschamps and R. Lewin-Ménégaux, *Surfaces de type général dominées par une variété fixe*, Bull. Soc. Mat. France **110** (1982), 127–146.

[G] P. Griffiths, ed., *Topics in transcendental algebraic geometry*, Ann. of Math. Studies, no. 106, Princeton Univ. Press, 1984.

[H] R. Hartshorne, *Equivalence relations on algebraic cycles* and *subvarieties of small codimension*, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R.I., 1975, pp. 129–164.

[M] D. Mumford, *Rational equivalence of zero-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1969), 195–204.

[P] G. P. Pirola, *Curves on generic Kummer varieties*, Duke Math. J. **59** (1989), 73–80.

[R_1] A. A. Roitman, *On* Γ*-equivalence of zero-dimensional cycles*, Math. USSR-Sb. **15** (1971), 555–567.

[R_2] ——, *Rational equivalence of zero-cycles*, Math. USSR-Sb. **18** (1972), 571–588.

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