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A C^1 VIRTUAL ELEMENT METHOD FOR THE CAHN–HILLIARD EQUATION WITH POLYGONAL MESHES*

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Abstract. In this paper we develop an evolution of the C^1 virtual elements of minimal degree for the approximation of the Cahn–Hilliard equation. The proposed method has the advantage of being conforming in H^2 and making use of a very simple set of degrees of freedom, namely, 3 degrees of freedom per vertex of the mesh. Moreover, although the present method is new also on triangles, it can make use of general polygonal meshes. As a theoretical and practical support, we prove the convergence of the semidiscrete scheme and investigate the performance of the fully discrete scheme through a set of numerical tests.

Key words. virtual element method, Cahn–Hilliard equation

AMS subject classification. 65M99

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1. Introduction.

The study of the evolution of transition interfaces, which is of paramount importance in many physical/biological phenomena and industrial processes, can be grouped into two macro classes, each one corresponding to a different method of dealing with the moving free boundary: the sharp interface method and the phase field method. In the sharp interface approach, the free boundary is to be determined together with the solution of suitable partial differential equations where proper jump relations have to be imposed across the free boundary. In the phase field approach, the interface is specified as the level set of a smooth continuous function exhibiting large gradients across the interface.

Phase field models, which date back to the works of Korteweg [34], Cahn [13], Hilliard and Cahn [31, 32], Ginzburg and Landau [35], and van der Waals [45], have been classicaly employed to describe phase separation in binary alloys. However, recently Cahn–Hilliard-type equations have been extensively used in an impressive variety of applied problems, such as, among others, tumor growth [49, 40], origin of Saturn's rings [44], separation of diblock copolymers [15], population dynamics [17], image processing [9], and even clustering of mussels [36].

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Due to the wide spectrum of applications, the study of efficient numerical methods for the approximate solution of the Cahn–Hilliard equation has been the object of intensive research activity. Summarizing the achievements in this field is a tremendous task that goes beyond the scope of this paper. Here, we limit ourselves to some remarks on finite element (FE) based methods, as the main properties (and limitations) of these schemes are instrumental for motivating the introduction of our new approach.

Since the Cahn–Hilliard equation involves fourth-order spatial derivatives, conforming FE solutions are not popular because primal variational formulations of fourth-order operators are only well defined and integrable if the FE basis functions are piecewise smooth and globally C^1 continuous. There are a very limited number of FE methods (FEMs) possessing C^1-continuity applicable to complex geometries; see [25, 21] for the solution of the Cahn–Hilliard equation by C^1 FEMs. In order to avoid the well known difficulty met in the implementation of C^1 FEMs, another possibility is the use of nonconforming (see, e.g., [22]) or discontinuous (see, e.g., [48]) methods; the drawback is that in such cases the discrete solution will not satisfy a C^1 regularity. Alternatively, the most common strategy employed *in practice* to solve the Cahn–Hilliard equation with (continuous and discontinuous) FEMs is to use mixed methods (see, e.g., [23, 24] and [33] for the continuous and discontinuous setting, respectively). Clearly, the drawback of this approach is the increase of the number of degrees of freedom (dof), and thus of the computational cost. Very recently, the difficulty related to the practical use of C^1 basis functions has been addressed with success also in the framework of isogeometric analysis [29].

The aim of the present paper is to construct a new conforming C^1 discretization of the Cahn–Hilliard equation, easier to implement than standard C^1 FEMs and yielding a reduced number of dof compared to mixed FEMs. To this end, we introduce and analyze the C^1 virtual element method (VEM) for the approximate solution of the Cahn–Hilliard equation. This newly introduced method (see, e.g., [4] for an introduction to the method and [6] for the details of its practical implementation) is characterized by the capability of dealing with very general polygonal/polyedral meshes and the possibility of easily implementing highly regular discrete spaces. Indeed, by avoiding the explicit construction of the local basis functions, the VEM can easily handle general polygons/polyhedrons without complex integrations on the element. In addition, thanks to this added flexibility, it was discovered [12, 7] that virtual elements can also be used to build global discrete spaces of arbitrary regularity (C^1 and more) that are quite simple in terms of dof and coding. Other virtual element contributions are, for instance, [11, 3, 5, 8, 14, 27, 37, 38], while for a very short sample of other FEM-inspired methods dealing with general polygons we refer to [10, 16, 18, 19, 26, 28, 41, 42, 46, 47].

In the present contribution we develop a modification of the C^1 virtual elements (of minimal degree) of [7] for the approximation of the Cahn–Hilliard equation. Also taking inspiration from the enhancement techniques of [2], we define the virtual space in order to be able to compute three different projection operators, that are used for the construction of the discrete scheme. Afterwards, we prove the convergence of the semidiscrete scheme and investigate the performance of the fully discrete scheme numerically. We underline that, to our knowledge, this is the first application of the newborn virtual element technology to a nonlinear problem.

The paper is organized as follows. In section 2 we describe the proposed VEM. In section 3 we develop the theoretical error estimates. In section 4 we present the numerical tests, while in section 5 we draw some conclusions. Finally, Appendix A contains the proof of some technical results.

2. The continuous and discrete problems. In this section, after presenting the Cahn–Hilliard equation, we introduce the virtual element discretization. The proposed strategy takes the steps from the C^1 methods described in [12, 7] for the Kirchhoff and Poisson problems, respectively, combined with an enhancement strategy first introduced in [2]. The present virtual scheme makes use of three different projectors and of a particular construction to take care of the nonlinear part of the problem. Throughout the paper we use standard notation for Sobolev spaces [1].

2.1. The continuous problem. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. Let $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(x) = (1-x^2)^2/4$ and let $\phi(x) = \psi'(x)$; we consider the following Cahn–Hilliard problem: find $u(x,t) : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u - \Delta\big(\phi(u) - \gamma^2 \Delta u(t)\big) = 0 & \text{in } \Omega \times [0,T], \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}}\big(\phi(u) - \gamma^2 \Delta u(t)\big) = 0 & \text{on } \partial\Omega \times [0,T], \end{cases} \tag{2.1}$$

where $\partial_{\mathbf{n}}$ denotes the (outward) normal derivative and $\gamma \in \mathbb{R}^+$, $0 < \gamma \ll 1$, represents the interface parameter. We note that on the boundary of the domain we impose a no-flux type condition both to u and the so-called chemical potential $\phi(u) - \gamma^2 \Delta u$. We now introduce the variational form of (2.1) that will be used to derive the virtual element discretization. To this aim, we preliminarily define the following bilinear forms

$$\begin{aligned} a^{\Delta}(v,w) &= \int_\Omega (\nabla^2 v) : (\nabla^2 w)\,\mathrm{d}x && \forall v, w \in H^2(\Omega), \\ a^{\nabla}(v,w) &= \int_\Omega (\nabla v) \cdot (\nabla w)\,\mathrm{d}x && \forall v, w \in H^1(\Omega), \\ a^{0}(v,w) &= \int_\Omega v\, w\,\mathrm{d}x && \forall v, w \in L^2(\Omega), \end{aligned}$$

and the semilinear form

$$r(z; v, w) = \int_\Omega \phi'(z) \nabla v \cdot \nabla w \,\mathrm{d}x \quad \forall z, v, w \in H^2(\Omega),$$

where all the symbols above follow a standard notation. Finally, introducing the space

$$V = \big\{ v \in H^2(\Omega) \,:\, \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega \big\}, \tag{2.2}$$

the weak formulation of problem (2.1) reads as find $u(\cdot,t) \in V$ such that

$$\begin{cases} a^0(\partial_t u, v) + \gamma^2 a^{\Delta}(u,v) + r(u;u,v) = 0 & \forall v \in V, \\ u(\cdot,0) = u_0(\cdot). \end{cases} \tag{2.3}$$

In the theoretical analysis of section 3, we will work under the following regularity assumption on the solution of (2.3):

$$u \in C^1(0,T; H^4(\Omega) \cap V); \tag{2.4}$$

see, e.g., [39] for a possible proof under higher regularity hypotheses on the initial datum u_0.

2.2. A C^1 virtual element space. In the present section we describe the virtual element space $W_h \subset H^2(\Omega)$ that we will use in the next section to build a discretization of problem (2.3). The construction of W_h is performed in several steps. First, we introduce a local auxiliary space $\widetilde{V}_{h|E}$ that will be employed to build the local virtual element space $W_{h|E}$ (Definition 2.2). Next, we identify a set of dof $D1$ and $D2$ for $W_{h|E}$ (Lemma 2.3) and show an approximation property of the local space (Corollary 2.4). Finally, we introduce the global virtual element space W_h by gluing the local spaces (Definition 2.7). In parallel to the construction of W_h, we introduce three projectors, namely, $\Pi^{\Delta}_E, \Pi^{\nabla}_E, \Pi^0_E$, that will be employed to construct the discrete counterparts of the bilinear forms in (2.5). Moreover, we show that these projection operators can be computed making use only of the values of the dof $D1$ and $D2$ (Lemmas 2.1, 2.5, and 2.6).

From now on, we will assume that Ω is a polygonal domain in $\mathbb{R}^2$. Let Ω_h represent a decomposition of Ω into general, possibly nonconvex, polygonal elements E with $\mathrm{diam}(E) = h_E$, where $\mathrm{diam}(E) = \max_{x,y\in E} \|x - y\|$. In the following, we will denote by e the straight edges of the mesh Ω_h and, for all $e \in \partial E$, $\mathbf{n}^e_E$ will denote the unit normal vector to e pointing outward to E. We will use the symbol $\mathbb{P}_k(\omega)$ to denote the space of polynomials of degree less than or equal to k living on the set $\omega \subseteq \mathbb{R}^2$. Finally, we will make use of the following local bilinear forms for all $E \in \Omega_h$,

$$\begin{aligned} a^{\Delta}_E(v,w) &= \int_E (\nabla^2 v) : (\nabla^2 w)\,\mathrm{d}x \quad && \forall v,w \in H^2(E),\\ a^{\nabla}_E(v,w) &= \int_E (\nabla v)\cdot(\nabla w)\,\mathrm{d}x && \forall v,w \in H^1(E),\\ a^{0}_E(v,w) &= \int_E v\,w\,\mathrm{d}x && \forall v,w \in L^2(E). \end{aligned} \tag{2.5}$$

Given an element $E \in \Omega_h$, the *augmented* local space $\widetilde{V}_{h|E}$ is defined by

$$\begin{aligned} \widetilde{V}_{h|E} = \Big\{ v \in H^2(E) \,:\, \Delta^2 v \in \mathbb{P}_2(E),\, v_{|\partial E} \in C^0(\partial E), v_{|e} \in \mathbb{P}_3(e)\ \forall e \in \partial E,\\ \nabla v_{|\partial E} \in [C^0(\partial E)]^2, \partial_{\mathbf{n}} v_{|e} \in \mathbb{P}_1(e)\ \forall e \in \partial E \Big\}, \end{aligned} \tag{2.6}$$

with $\partial_{\mathbf{n}}$ denoting the (outward) normal derivative. The space $\widetilde{V}_{h|E}$ is made of functions that are continuous and piecewise cubic on the boundary, with continuous gradient on the boundary, normal linear component on each edge, and such that its bi-Laplacian is a quadratic polynomial.

With the aim of identifying suitable dof for our virtual spaces, we now introduce two sets $D1$ and $D2$ of linear operators from $\widetilde{V}_{h|E}$ into $\mathbb{R}$. For all $v_h \in \widetilde{V}_{h|E}$ they are defined as follows:

(i) $D1$ contains linear operators evaluating v_h at the $n = n(E)$ vertices of E;

(ii) $D2$ contains linear operators evaluating ∇v_h at the $n = n(E)$ vertices of E.

Note that, as a consequence of definition (2.6), the output values of the two sets of operators $D1$ and $D2$ are sufficient to uniquely determine v_h and ∇v_h on the boundary of E.

Let us now consider the projection operator $\Pi^{\Delta}_E\colon \widetilde{V}_{h|E} \to \mathbb{P}_2(E)$, defined by

$$\begin{cases} a^{\Delta}_E(\Pi^{\Delta}_E v_h, q) = a^{\Delta}_E(v_h, q) \quad \forall q \in \mathbb{P}_2(E),\\ ((\Pi^{\Delta}_E v_h, q))_E = ((v_h, q))_E \quad \forall q \in \mathbb{P}_1(E), \end{cases} \tag{2.7}$$

for all $v_h \in \widetilde{V}_{h|E}$, where $(\!(\cdot,\cdot)\!)_E$ represents a euclidean scalar product acting on the function vertex values, i.e.,

$$(\!(v_h, w_h)\!)_E = \sum_{\substack{\nu \text{ vertexes} \\ \text{of } \partial E}} v_h(\nu)\, w_h(\nu) \quad \forall v_h, w_h \in C^0(E).$$

LEMMA 2.1. *The operator $\Pi_E^\Delta \colon \widetilde{V}_{h|E} \to \mathbb{P}_2(E)$ is well defined and uniquely determined on the basis of the information carried by the linear operators in $D1$ and $D2$.*

Proof. First, we note that the bilinear form $a_E^\Delta(\cdot,\cdot)$ has a nontrivial kernel, given by $\mathbb{P}_1(E)$, and the role of the second condition in (2.7) is to select an element of the kernel of the operator. It is easy to check that the operator Π_E^Δ is well defined, as for all $v_h \in \widetilde{V}_{h|E}$ it returns one (and only one) function $\Pi_E^\Delta v_h \in \mathbb{P}_2(E)$.

We now show that the operator Π_E^Δ is uniquely determined on the basis of the information carried by the linear operators in $D1$ and $D2$. It is sufficient to perform a double integration by parts on the right-hand side of (2.7), which gives

$$\begin{aligned} a_E^\Delta(v_h, q) &= \int_E \nabla^2 v_h : \nabla^2 q \mathrm{d}x = \int_{\partial E} \left(\nabla^2(q)\mathbf{n}_E^e\right) \cdot \nabla v_h \mathrm{d}s - \int_{\partial E} v_h(\mathrm{div}\nabla^2 q) \cdot \mathbf{n}_E^e \mathrm{d}s \\ &= \int_{\partial E} \left(\nabla^2(q)\mathbf{n}_E^e\right) \cdot \nabla v_h \mathrm{d}s, \end{aligned}$$

where we employed $q \in \mathbb{P}_2(E)$. We observe that the above term on the right-hand side only depends on the boundary values of v_h and ∇v_h. Moreover, we note that the same holds for the right-hand side of the second equation in (2.7), since it depends only on the vertex values of v_h. To conclude, as for any $v_h \in \widetilde{V}_{h|E}$, the output values of the linear operators in $D1$ and $D2$ are sufficient to define v_h and ∇v_h on the boundary, it turns out that the operator Π_E^Δ is uniquely determined on the basis of the evaluations performed by the linear operators in $D1$ and $D2$. $\square$

Now, starting from the augmented space $\widetilde{V}_{h|E}$ and employing the projector Π_E^Δ we define our virtual local spaces.

DEFINITION 2.2 (local virtual space). *The local virtual space is defined as follows:*

$$W_{h|E} = \left\{ v \in \widetilde{V}_{h|E} \;:\; a_E^0(\Pi_E^\Delta(v), q) = a_E^0(v, q) \quad \forall q \in \mathbb{P}_2(E) \right\}. \tag{2.8}$$

We observe that, since $W_{h|E} \subset \widetilde{V}_{h|E}$, the operator Π_E^Δ is well defined on $W_{h|E}$ and computable only on the basis of the output values of the operators in $D1$ and $D2$.

Moreover, we have the following result characterizing the dof of the local virtual space $W_{h|E}$.

LEMMA 2.3. *The set of operators $D1$ and $D2$ constitutes a set of dof for the space $W_{h|E}$.*

Proof. We start by noting that the space $\widetilde{V}_{h|E}$ is associated with a well-posed biharmonic problem on E with Dirichlet boundary data and standard volume loading, i.e.,

$$\begin{cases} -\Delta^2 v_h \text{ assigned in } E, \\ v_h \text{ and } \partial_{\mathbf{n}} v_h \text{ assigned on } \partial E. \end{cases}$$

Thus the dimension of $\widetilde{V}_{h|E}$ equals the dimension of the data space (loading and boundary data spaces). We now recall that, as already noted, the operators $D1$ and

$D2$ uniquely determine v_h and ∇v_h on the boundary of E and thus the cardinality $\#\{D1\} + \#\{D2\}$ exactly corresponds to the dimension of the boundary data in the above biharmonic problem. Therefore, since the loading data space has dimension equal to $\dim(\mathbb{P}_2(E))$, we have

$$\dim(\widetilde{V}_{h|E}) = \#\{D1\} + \#\{D2\} + \dim(\mathbb{P}_2(E)).$$

Now, we observe that the space $W_{h|E}$ is a subspace of $\widetilde{V}_{h|E}$ obtained by enforcing the constraints in (2.8), i.e., a set of m linear equations, with $m = \dim(\mathbb{P}_2(E))$. Since such equations could, in principle, not be linearly independent, all we can say on the dimension of $W_{h|E}$ is

$$\dim(W_{h|E}) \geq \dim(\widetilde{V}_{h|E}) - \dim(\mathbb{P}_2(E)) = \#\{D1\} + \#\{D2\}. \tag{2.9}$$

The proof is therefore complete if we show that any $v_h \in W_{h|E}$ that vanishes on $D1$ and $D2$ is indeed the zero element of $W_{h|E}$. Let $v_h \in W_{h|E}$ vanish on $D1$ and $D2$. First of all, this easily implies that v_h and ∇v_h are null on the boundary ∂E. Moreover, since the operator Π_E^{Δ} is linear and depends only on the output values of the operators in $D1$ and $D2$, it must hold $\Pi_E^{\Delta}(v_h) = 0$. Recalling definition (2.8), this in turn yields

$$\int_E v_h \, q \, \mathrm{d}x = 0 \quad \forall q \in \mathbb{P}_2(E). \tag{2.10}$$

Since $v_h \in W_{h|E} \subseteq \widetilde{V}_{h|E}$, we have $\Delta^2 v_h \in \mathbb{P}_2(E)$. Therefore, we can take $q = \Delta^2 v_h$ as a test function in (2.10). A double integration by parts, using also that v_h and ∇v_h are null on ∂E, then gives

$$0 = \int_E v_h \, \Delta^2 v_h \, \mathrm{d}x = \int_E \Delta v_h \, \Delta v_h \, \mathrm{d}x.$$

Thus $\Delta v_h = 0$ and the proof is complete by recalling again the boundary conditions on v_h. ◻

Since clearly $\mathbb{P}_2(E) \subseteq \widetilde{V}_{h|E}$ and the additional condition in (2.8) is satisfied by $\mathbb{P}_2(E)$ polynomials (Π_E^{Δ} being a projection on such polynomial space), the following result, which establishes the approximation properties of $W_{h|E}$, is easy to prove.

COROLLARY 2.4. *There holds*

$$\mathbb{P}_2(E) \subseteq W_{h|E}.$$

We now introduce two further projectors on the local space $W_{h|E}$, namely, Π_E^0 and Π_E^{∇}, that will be employed together with the above projector Π_E^{Δ} to build the discrete counterparts of the bilinear forms in (2.5).

First, let $\Pi_E^0 : W_{h|E} \to \mathbb{P}_2(E)$ be the standard L^2 projection operator on the space of quadratic polynomials in E. The following result addresses its computability in terms of the dof $D1$ and $D2$.

LEMMA 2.5. *The operator $\Pi_E^0 : W_{h|E} \to \mathbb{P}_2(E)$ is computable (only) on the basis of the values of the dof $D1$ and $D2$.*

Proof. We first remark that for all $v_h \in W_{h|E}$ the function $\Pi_E^0 v_h \in \mathbb{P}_2(E)$ is defined by

$$a_E^0(\Pi_E^0 v_h, q) = a_E^0(v_h, q) \quad \forall q \in \mathbb{P}_2(E), \tag{2.11}$$

where the bilinear form $a_E^0(\cdot,\cdot)$, introduced in (2.5), represents the $L^2(E)$ scalar product. Due to the particular property appearing in definition (2.8), the right-hand side in (2.11) is computable using $\Pi_E^{\Delta} v_h$, and thus $\Pi_E^0 v_h$ depends only on the values of the dof $D1$, $D2$ attained by v_h and ∇v_h. Actually, it is easy to check that on the space $W_{h|E}$ the projectors Π_E^{Δ} and Π_E^0 are the same operator (although for the sake of clarity we prefer to keep the notation different). ∎

Then, we introduce an additional projection operator that we will need in the following. We define $\Pi_E^{\nabla} : W_{h|E} \to \mathbb{P}_2(E)$ by

$$\begin{cases} a_E^{\nabla}(\Pi_E^{\nabla} v_h, q) = a_E^{\nabla}(v_h, q) \quad \forall q \in \mathbb{P}_2(E), \\ \displaystyle\int_E \Pi_E^{\nabla} v_h \,\mathrm{d}x = \int_E v_h \,\mathrm{d}x. \end{cases} \tag{2.12}$$

LEMMA 2.6. *The operator $\Pi_E^{\nabla} : W_{h|E} \to \mathbb{P}_2(E)$ is well defined and uniquely determined on the basis of the information carried by the linear operators in $D1$ and $D2$.*

Proof. We first remark that, since the bilinear form $a_E^{\nabla}(\cdot,\cdot)$ has a nontrivial kernel (given by the constant functions) a second condition is added in (2.12) in order to keep the operator Π_E^{∇} well defined. Moreover, it is easy to check that the right-hand side in (2.12) is computable on the basis of the values of the dof $D1$ and $D2$. For the first equation in (2.12), this can be shown with an integration by parts (similarly as already done for the Π_E^{Δ} projector), i.e.,

$$\int_E \nabla v_h \cdot \nabla q \mathrm{d}x = -(\Delta q)_{|E} \int_E v_h \mathrm{d}x + \int_{\partial E} v_h \, \partial_{\mathbf{n}} q \,\mathrm{d}s$$

and noting that the identity

$$\int_E v_h \mathrm{d}x = \int_E \Pi_E^0 v_h \mathrm{d}x$$

allows us to compute the integral of v_h on E using only the values of the dof $D1$ and $D2$. ∎

We are now in position to introduce the global discrete space that can be assembled in the classical FE fashion.

DEFINITION 2.7 (global virtual space). *The global virtual space is defined as follows:*

$$W_h = \{ v \in V \ : \ v|_E \in W_{h|E} \quad \forall E \in \Omega_h \}.$$

Note that, by gluing in the standard way the dof, the ensuing functions of W_h will have continuous values and continuous gradients across edges. Therefore, W_h is indeed contained in $H^2(\Omega)$ and yields a conforming solution. It turns out that the *global dof* of W_h are simply given by the following:

(i) evaluation of v_h at the vertices of the mesh Ω_h;

(ii) evaluation of ∇v_h at the vertices of the mesh Ω_h.

Thus the dimension of W_h is three times the number of vertices in the mesh. As a final note we observe that, in practice, it is recommended to scale the dof $D2$ by some local characteristic mesh size h_ν in order to obtain a better condition number of the final system. Indeed, without such scaling, the basis functions of W_h could have very different amplitudes for small values of the mesh size, thus easily leading to an ill-conditioned system. We remark that the proposed scaling ony affects the amplitude of the basis functions and not the space W_h or the involved bilinear forms. Therefore it has no direct effect on the accuracy of the method itself.

2.3. Virtual forms. The second key step in the contruction of the method is the definition of suitable discrete forms. Analogously to the FE case, these forms will be constructed element by element and will depend on the dof of the discrete space. Unlike in the FE case, these forms will not be obtained by some Gauss integration of the shape functions (that are unknown inside the elements) but rather using the projection operators that we defined in the previous section.

We start by introducing a discrete approximation of the three exact local forms in (2.5). By making use of the projection operators of the previous section, the development of the bilinear forms follows a standard approach in the virtual element literature. We therefore refer, for instance, to [4] for more details and motivations regarding this construction. Let $E \in \Omega_h$ be any element of the polygonal partition. We introduce the following (strictly) positive definite bilinear form on $W_{h|E} \times W_{h|E}$,

$$s_E(v_h, w_h) = \sum_{\substack{\nu \text{ vertexes} \\ \text{of } \partial E}} \Big(v_h(\nu)\, w_h(\nu) + (h_\nu)^2\, \nabla v_h(\nu) \cdot \nabla w_h(\nu) \Big) \quad \forall v_h, w_h \in W_{h|E},$$

where h_ν is some characteristic mesh size length associated with the node ν (for instance the maximum diameter among the elements having ν as a vertex).

Recalling (2.5), we then propose the following discrete (and symmetric) local forms

$$\begin{aligned} a^{\Delta}_{h,E}(v_h, w_h) &= a^{\Delta}_E(\Pi^{\Delta}_E v_h, \Pi^{\Delta}_E w_h) + h_E^{-2} s_E(v_h - \Pi^{\Delta}_E v_h, w_h - \Pi^{\Delta}_E w_h), \\ a^{\nabla}_{h,E}(v_h, w_h) &= a^{\nabla}_E(\Pi^{\nabla}_E v_h, \Pi^{\nabla}_E w_h) + s_E(v_h - \Pi^{\nabla}_E v_h, w_h - \Pi^{\nabla}_E w_h), \\ a^{0}_{h,E}(v_h, w_h) &= a^{0}_E(\Pi^{0}_E v_h, \Pi^{0}_E w_h) + h_E^{2}\, s_E(v_h - \Pi^{0}_E v_h, w_h - \Pi^{0}_E w_h) \end{aligned} \tag{2.13}$$

for all $v_h, w_h \in W_{h|E}$.

The consistency of the discrete bilinear forms is assured by the first term on the right-hand side of each relation, while the role of the second term $s_E(\cdot,\cdot)$ is only to guarantee the correct coercivity properties. Indeed, noting that the projection operators appearing above are always orthogonal with respect to the associated bilinear form, it is immediate to check the following consistency lemma.

LEMMA 2.8 (consistency). *For all the three bilinear forms in* (2.13) *it holds*

$$a^{\dagger}_{h,E}(p, v_h) = a^{\dagger}_E(p, v_h) \quad \forall p \in \mathbb{P}_2(E),\ \forall v_h \in W_{h|E},$$

where the symbol $\dagger$ *stands for the symbol* $\Delta, \nabla,$ *or* 0.

The lemma above states that the bilinear forms are exact whenever one of the two entries is a polynomial in $\mathbb{P}_2(E)$. In order to present a stability result for the proposed discrete bilinear forms, we need some mesh regularity assumptions on the mesh sequence $\{\Omega_h\}_h$.

Assumption 2.1. We assume that there exist positive constants c_s and c_s' such that every element $E \in \{\Omega_h\}_h$ is star shaped with respect to a ball with radius $\rho \geq c_s h_E$ and every edge $e \in \partial E$ has at least length $h_e \geq c_s' h_E$.

Under the above mesh regularity conditions, we can show the following lemma. Since the proof is standard and based on a scaling argument, it is omitted.

LEMMA 2.9 (stability). *Let Assumption* 2.1 *hold. There exist two positive constants* $c_\star, c^\star$ *independent of the element* $E \in \{\Omega_h\}_h$ *such that*

$$c_\star\, a^{\dagger}_E(v_h, v_h) \leq a^{\dagger}_{h,E}(v_h, v_h) \leq c^\star a^{\dagger}_E(v_h, v_h) \quad \forall v_h \in W_{h|E},$$

where the symbol $\dagger$ *stands for the symbol* $\Delta, \nabla,$ *or* 0.

Note that, as a consequence of the above lemma, it is immediate to check that the bilinear forms $a^{\dagger}_{h,E}(\cdot,\cdot)$ are continuous with respect to the relevant norm: H^2 for $(2.13)_1$, H^1 for $(2.13)_2$, and L^2 for $(2.13)_3$. The global discrete bilinear forms will be written (following the classical FE procedure)

$$a^{\dagger}_h(v_h, w_h) = \sum_{E\in\Omega_h} a^{\dagger}_{h,E}(v_h, w_h) \quad \forall v_h, w_h \in W_h,$$

with the usual multiple meaning of the symbol †.

We now turn our attention to the semilinear form $r(\cdot;\cdot,\cdot)$, which we here write more explicitly:

$$r(z; v, w) = \sum_{E\in\Omega_h} r_E(z; v, w) \quad \forall z, v, w \in H^2(\Omega),$$
$$r_E(z; v, w) = \int_E (3z(x)^2 - 1)\nabla v(x) \cdot \nabla w(x)\,\mathrm{d}x \quad \forall E \in \Omega_h.$$

On each element E, we approximate the term $z(x)^2$ with its average, computed using the $L^2(E)$ bilinear form $a^0_{h,E}(\cdot,\cdot)$:

$$(z_h^2)|_E \simeq |E|^{-1} a^0_{h,E}(z_h, z_h),$$

where $|E|$ denotes the area of element E. This approach will turn out to have the correct approximation properties and, moreover, it preserves the positivity of z^2. We therefore propose the following approximation of the local nonlinear forms,

$$r_{h,E}(z_h; v_h, w_h) = \widehat{\phi'(z_h)}_{|E}\, a^{\nabla}_{h,E}(v_h, w_h) \quad \forall z_h, v_h, w_h \in W_{h|E},$$

where $\widehat{\phi'(z_h)}_{|E} = 3|E|^{-1}a^0_{h,E}(z_h, z_h) - 1$. The global form is then assembled as usual,

$$r_h(z_h; v_h, w_h) = \sum_{E\in\Omega_h} r_{h,E}(z_h; v_h, w_h) \quad \forall w_h, r_h, v_h \in W_h.$$

2.4. Discrete problem. We here outline the virtual element discretization of problem (2.3), that follows a Galerkin approach in space combined with a backward Euler in time. Let us introduce the space with boundary conditions,

$$W^0_h = W_h \cap V = \big\{v \in W_h \,:\, \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\big\}.$$

As usual, it is convenient to first introduce the semidiscrete problem

$$(2.14)\quad \begin{cases} \text{Find } u_h(\cdot,t) \text{ in } W^0_h \text{ such that} \\ a^0_h(\partial_t u_h, v_h) + \gamma^2 a^{\Delta}_h(u_h, v_h) + r_h(u_h;u_h, v_h) = 0 \quad \forall v_h \in W^0_h, \ \textit{a.e.} \text{ in } (0,T), \\ u_h(0,\cdot) = u_{0,h}(\cdot) \end{cases}$$

with $u_{0,h} \in W^0_h$ a suitable approximation of u_0 and where the discrete forms above have been introduced in the previous section.

In order to introduce the fully discrete problem, we subdivide the time interval $[0,T]$ into N uniform subintervals of length $k = T/N$ by selecting, as usual, the time nodes $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. We now search for $\{u^1_{hk}, u^2_{hk}, \ldots, u^N_{hk}\}$ with $u^i_{hk} \in W^0_h$ representing the solution at time t_i.

The fully discrete problem reads as follows: Given $u^0_{hk} = u_{0,h} \in W^0_h$, for $i = 1,\ldots,N$ look for $u^i_{hk} \in W^0_h$ such that

$$(2.15)\quad k^{-1}a^0_h(u^i_{hk} - u^{i-1}_{hk}, v_h) + \gamma^2 a^{\Delta}_h(u^i_{hk}, v_h) + r_h(u^i_{hk}, u^i_{hk}; v_h) = 0 \quad \forall v_h \in W^0_h.$$

3. Error analysis of the semidiscretization scheme. Throughout the subsequent discussion, we will employ the notation $x \lesssim y$ to denote the inequality $x \leq Cy$, C being a positive constant independent of the discretization parameters but that may depend on the regularity of the underlying continuous solution. Moreover, note that (unless needed to avoid confusion) in what follows the dependence of u and u_h on time t is left implicit and the bounds involving u or u_h hold for all $t \in (0, T]$.

In this section we present the convergence analysis of the semidiscrete virtual element formulation given in (2.14). Our theoretical analysis will deal only with the semidiscrete case since the main novelty of the present paper is the (virtual element) space discretization. The error analysis of the fully discrete scheme follows from the analysis of the semidiscrete case employing standard techniques as for the classical FE case (see, e.g., [43]).

The subsequent convergence analysis will be performed under the following regularity assumption on the semidiscrete solution u_h of (2.14) (see, e.g., [22] for a discussion on its validity).

Assumption 3.1. The solution u_h of (2.14) satisfies, for all $t \in (0, T]$,

$$\|u_h(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

with C a positive contant independent of h.

As a starting point, we recall the following approximation result; see [20] and [38, 4].

PROPOSITION 3.1. *Assume that Assumption* 2.1 *is satisfied. Then for every $v \in H^s(E)$ there exists $v_\pi \in \mathbb{P}_k(E)$, $k \geq 0$, and $v_I \in W_{h|E}$ such that*

$$\begin{aligned} &|v - v_\pi|_{H^\ell(E)} \lesssim h_K^{s-\ell} |v|_{H^s(E)}, \quad 1 \leq s \leq k+1, \ \ell = 0, 1, \ldots, s, \\ &|v - v_I|_{H^\ell(E)} \lesssim h_K^{s-\ell} |v|_{H^s(E)}, \quad s = 2, 3, \ \ell = 0, 1, \ldots, s, \end{aligned} \tag{3.1}$$

where the hidden constant depends only on k and on the constants in Assumption 2.1.

Let

$$\overline{\phi'(u)}_{|E} = 3|E|^{-1} a_E^0(u, u) - 1;$$

we define

$$\overline{r}_h(u; v_h, w_h) = \sum_{E \in \Omega_h} \overline{\phi'(u)}_{|E} a_{h,E}^\nabla(v_h, w_h).$$

We introduce the elliptic projection $P^h v \in W_h^0$ for $v \in H^4(\Omega)$ defined by

$$b_h(P^h v, \psi_h) = a^0(\gamma^2 \Delta^2 v - \nabla \cdot (\phi'(u) \nabla v) + \alpha v, \psi_h) \tag{3.2}$$

for all $\psi_h \in W_h^0$, where $b_h(\cdot, \cdot)$ is the bilinear form

$$b_h(v_h, w_h) = \gamma^2 a_h^\Delta(v_h, w_h) + \overline{r}_h(u; v_h, w_h) + \alpha\, a^0(v_h, w_h), \tag{3.3}$$

α being a sufficiently large positive parameter.

For the subsequent analysis, it is instrumental to introduce the following auxiliary problem: Find $\varphi \in V$ such that

$$b(\varphi, w) = a^0(u - P^h u, w) + a^\nabla(u - P^h u, w) \tag{3.4}$$

for all $w \in V$, where $b(\cdot,\cdot)$ is the bilinear form

$$b(v,w) = \gamma^2 a^{\Delta}(v,w) + r(u;v,w) + \alpha\, a^0(v,w). \tag{3.5}$$

We assume the validity of the following regularity result.

Assumption 3.2. Let φ be the solution of (3.4) with forcing term $g \in H^1(\Omega)$, i.e.,

$$b(\varphi,w) = a^0(g,w) + a^{\nabla}(g,w) \qquad \forall w \in V.$$

Then, there exists a positive constant C_Ω, only depending on Ω such that

$$\|\varphi\|_{H^3(\Omega)} \le C_\Omega \|g\|_{H^1(\Omega)}.$$

The proof of Assumption 3.2 can be found, for example, in [22, Theorem A.1] in the case of a rectangular domain Ω. Recalling that $P^h u$ is the elliptic projection defined in (3.2), in the following we will make use of Assumption 3.2 taking $g = u - P^h u$, i.e.,

$$\|\varphi\|_{H^3(\Omega)} \le C_\Omega \|u - P^h u\|_{H^1(\Omega)}. \tag{3.6}$$

3.1. Technical results. In this section we collect some technical results that will be useful to prove the main result (Theorem 3.5). The proof of the following lemma can be found in Appendix A.

LEMMA 3.2. *Let u be the solution to* (2.3) *and $P^h u$ be the elliptic projection defined in* (3.2). *Then, it holds*

$$\|u - P^h u\|_{H^2(\Omega)} \lesssim h, \tag{3.7}$$
$$\|u - P^h u\|_{H^1(\Omega)} \lesssim h^2, \tag{3.8}$$
$$\|u_t - (P^h u)_t\|_{H^2(\Omega)} \lesssim h, \tag{3.9}$$
$$\|u_t - (P^h u)_t\|_{H^1(\Omega)} \lesssim h^2. \tag{3.10}$$

LEMMA 3.3. *Let u be the solution to* (2.3) *and $P^h u$ be the elliptic projection defined in* (3.2). *Then, setting $\rho = u - P^h u$ and $\theta = P^h u - u_h$, it holds*

$$r_h(u_h;u_h,\theta) - \overline{r}_h(u;P^h u,\theta) \lesssim |\theta|_{H^1(\Omega)} \left(\|\theta\|_{L^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + |\theta|_{H^1(\Omega)} + h^2\right). \tag{3.11}$$

Proof. We preliminarily observe that using Lemma 3.2 and proceeding as in the proof of [22, (3.2c)] yield $P^h u \in W^{1,\infty}(\Omega)$, with norm bounded uniformly in time. Moreover, it holds

$$\begin{aligned} r_h(u_h;u_h,\theta) - \overline{r}_h(u;P^h u,\theta) &= r_h(u_h;P^h u,\theta) - \overline{r}_h(u;P^h u,\theta) + r_h(u_h;u_h - P^h u,\theta) \\ &= \sum_{E\in\Omega_h} (\widehat{\phi'(u_h)} - \overline{\phi'(u)})_{|E}\, a^{\nabla}_{h,E}(P^h u,\theta) \\ &\quad + r_h(u_h;u_h - P^h u,\theta) \\ &= A + B. \end{aligned}$$

Let us first estimate the term A which can be written as follows:

$$\begin{aligned} A &= \sum_{E\in\Omega_h} (\widehat{\phi'(u_h)} - \widehat{\phi'(P^h u)} + \widehat{\phi'(P^h u)} - \overline{\phi'(P^h u)} + \overline{\phi'(P^h u)} - \overline{\phi'(u)})_{|E}\, a^{\nabla}_{h,E}(P^h u,\theta) \\ &= \sum_{E\in\Omega_h} (I + II + III)_{|E}\, a^{\nabla}_{h,E}(P^h u,\theta). \end{aligned}$$

Using Lemma 2.8 we obtain

$$\begin{aligned}
A &\lesssim \sum_{E\in\Omega_h} (|I| + |II| + |III|)_{|E} \|P^h u\|_{H^1(E)} |\theta|_{H^1(E)} \\
&\lesssim \|P^h u\|_{W^{1,\infty}(\Omega)} \sum_{E\in\Omega_h} |E|^{1/2} (|I| + |II| + |III|)_{|E} |\theta|_{H^1(E)} \\
&\lesssim \|P^h u\|_{W^{1,\infty}(\Omega)} \left(\sum_{E\in\Omega_h} |E| (I^2 + II^2 + III^2)_{|E} \right)^{1/2} |\theta|_{H^1(\Omega)} \\
&\lesssim \|P^h u\|_{W^{1,\infty}(\Omega)} (A_I + A_{II} + A_{III}) |\theta|_{H^1(\Omega)}, \qquad (3.12)
\end{aligned}$$

where $A_{(\cdot)} = (\sum_{E\in\Omega_h} |E| (\cdot)^2_{|E})^{1/2}$. Using the definition of $\widehat{(\cdot)}$ and Lemma 2.8 we obtain

$$\begin{aligned}
I &= \frac{3}{|E|} (a^0_{h,E}(u_h, u_h) - a^0_{h,E}(P^h u, P^h u)) \\
&= \frac{3}{|E|} (a^0_{h,E}(u_h - P^h u, u_h + P^h u)) \\
&\lesssim \frac{3}{|E|} \|u_h - P^h u\|_{L^2(E)} (\|u_h\|_{L^2(E)} + \|P^h u\|_{L^2(E)}) \qquad (3.13)
\end{aligned}$$

which implies

$$\begin{aligned}
A_I &\lesssim \sum_{E\in\Omega_h} \left(\frac{1}{|E|} \|\theta\|^2_{L^2(E)} \left(\|u_h\|_{L^2(E)} + \|P^h u\|_{L^2(E)} \right)^2 \right)^{1/2} \\
&\lesssim (\|u_h\|_{L^\infty(\Omega)} + \|P^h u\|_{L^\infty(\Omega)}) \left(\sum_{E\in\Omega_h} \|\theta\|^2_{L^2(E)} \right)^{1/2} \\
&\lesssim \|\theta\|_{L^2(\Omega)}, \qquad (3.14)
\end{aligned}$$

where in the last step we employed Assumption 3.1 on the regularity of u_h.

Similarly, using the definition of $\overline{(\cdot)}$ we have

$$\begin{aligned}
III &= \frac{3}{|E|} \left(\int_E (P^h u)^2 \mathrm{d}x - \int_E u^2 \mathrm{d}x \right) \\
&\leq \frac{3}{|E|} \|P^h u - u\|_{L^2(E)} \left(\|u\|_{L^2(E)} + \|P^h u\|_{L^2(E)} \right)
\end{aligned}$$

which yields

$$A_{III} \lesssim (\|u_h\|_{L^\infty(\Omega)} + \|P^h u\|_{L^\infty(\Omega)}) \left(\sum_{E\in\Omega_h} \|P^h u - u\|^2_{L^2(E)} \right)^{1/2} \lesssim \|P^h u - u\|_{L^2(\Omega)}.$$

Finally, employing $q \in \mathbb{P}_2(E)$ together with Lemma 2.8 and the interpolation estimates of Proposition 3.1, it is easy to prove that the following holds:

$$
\begin{aligned}
II &= \frac{3}{|E|}\left(a^0_{h,E}(P^hu, P^hu) - a^0(P^hu, P^hu)\right) \\
&= \frac{3}{|E|}\left(a^0_{h,E}(P^hu - q, P^hu) - a^0(P^hu - q, P^hu)\right) \\
&\lesssim \frac{1}{|E|}\|P^hu - q\|_{L^2(E)}\|P^hu\|_{L^2(E)} \\
&\lesssim \frac{1}{|E|}\left(\|P^hu - u\|_{L^2(E)} + \|u - q\|_{L^2(E)}\right)\|P^hu\|_{L^2(E)} \\
&\lesssim \frac{1}{|E|}\left(\|P^hu - u\|_{L^2(E)} + h^2\right)\|P^hu\|_{L^2(E)}
\end{aligned}
$$

which implies

$$
\begin{aligned}
A_{II} &\lesssim \|P^hu\|_{L^\infty(\Omega)}\left(\left(\sum_{E\in\Omega_h}\|P^hu - u\|^2_{L^2(E)}\right)^{1/2} + h^2\right) \\
&\lesssim \|P^hu - u\|_{L^2(\Omega)} + h^2. \qquad (3.15)
\end{aligned}
$$

Employing the above estimates for A_I, A_{II}, and A_{III} into (3.12) and recalling that $\|P^hu\|_{W^{1,\infty}(\Omega)}$ is uniformly bounded in time, we get

$$
A \lesssim \left(\|\theta\|_{L^2(\Omega)} + \|P^hu - u\|_{L^2(\Omega)} + h^2\right)|\theta|_{H^1(\Omega)}. \qquad (3.16)
$$

To conclude, it is sufficient to estimate B. Using the definition of $r_h(\cdot;\cdot,\cdot)$ together with Lemma 2.9 and Assumption 3.1 we have

$$
B \lesssim \|u_h\|_{L^\infty(\Omega)}|\theta|^2_{H^1(\Omega)} \lesssim |\theta|^2_{H^1(\Omega)}. \qquad (3.17)
$$

∎

LEMMA 3.4. *Let $v_h \in W^0_h$ and $\varepsilon > 0$. Then there exists a constant C_ε depending on ϵ such that it holds*

$$
|v_h|^2_{H^1(\Omega)} \le \varepsilon|v_h|^2_{H^2(\Omega)} + C_\varepsilon\|v_h\|^2_{L^2(\Omega)}. \qquad (3.18)
$$

Proof. It is straightforward to observe that it holds

$$
\begin{aligned}
|v_h|^2_{H^1(\Omega)} &= \sum_E \int_E \nabla v_h \cdot \nabla v_h \mathrm{d}x = \sum_E \left\{\int_{\partial E} v_h \frac{\partial v_h}{\partial n}\, ds - \int_E \Delta v_h v_h\, dx\right\} \\
&= \sum_E \left\{-\int_E \Delta v_h v_h\, dx\right\} \le \varepsilon\|\Delta v_h\|^2_{L^2(\Omega)} + C_\varepsilon\|v_h\|^2_{L^2(\Omega)}, \qquad (3.19)
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality and the fact that $W^0_h \subset H^2(\Omega)$. ∎

3.2. Error estimates. We are now ready to prove the following convergence result.

THEOREM 3.5. *Let u be the solution to (2.3) and u_h the solution to (2.14). Then for all $t \in [0, T]$ it holds*

$$
\|u - u_h\|_{L^2(\Omega)} \lesssim h^2. \qquad (3.20)
$$

Proof. As usual, the argument is based on the following error decomposition,

$$u - u_h = (u - P^h u) + (P^h u - u_h) =: \rho + \theta. \tag{3.21}$$

In view of Lemma 3.2, we only need to estimate $\|\theta\|_{L^2(\Omega)}$. Proceeding as in [22], we first observe that it holds

$$\begin{aligned}
a_h^0(\theta_t, \chi_h) + \gamma^2 a_h^\Delta(\theta, \chi_h) &= a_h^0((P^h u - u_h)_t, \chi_h) + \gamma^2 a_h^\Delta(P^h u - u_h, \chi_h) \\
&= a_h^0((P^h u)_t, \chi_h) + \gamma^2 a_h^\Delta(P^h u, \chi_h) \\
&\quad - [a_h^0((u_h)_t, \chi_h) + \gamma^2 a_h^\Delta(u_h, \chi_h)] \\
&= a_h^0((P^h u)_t, \chi_h) + \gamma^2 a_h^\Delta(P^h u, \chi_h) + r_h(u_h, u_h; \chi_h).
\end{aligned}$$

Using (3.3) and (3.2) it holds

$$\begin{aligned}
\gamma^2 a_h^\Delta(P^h u, \chi_h) &= b_h(P^h u, \chi_h) - \overline{r}_h(u; P^h u, \chi_h) - \alpha a^0(P^h u, \chi_h) \\
&= a^0(\gamma^2 \Delta^2 u - \nabla \cdot (\phi'(u) \nabla u) + \alpha u, \chi_h) \\
&\quad - \overline{r}_h(u; P^h u, \chi_h) - \alpha a^0(P^h u, \chi_h) \\
&= a^0(\gamma^2 \Delta^2 u - \nabla \cdot (\phi'(u) \nabla u), \chi_h) - \overline{r}_h(u; P^h u, \chi_h) + \alpha a^0(\rho, \chi_h).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
a_h^0(\theta_t, \chi_h) + \gamma^2 a_h^\Delta(\theta, \chi_h) &= a_h^0((P^h u)_t, \chi_h) + a^0(\gamma^2 \Delta^2 u - \nabla \cdot (\phi'(u) \nabla u), \chi_h) \\
&\quad + r_h(u_h; u_h, \chi_h) - \overline{r}_h(u; P^h u, \chi_h) + \alpha a^0(\rho, \chi_h) \\
&= - a_h^0(\rho_t, \chi_h) + a^0(u_t + \gamma^2 \Delta^2 u - \nabla \cdot (\phi'(u) \nabla u), \chi_h) \\
&\quad + r_h(u_h; u_h, \chi_h) - \overline{r}_h(u; P^h u, \chi_h) + \alpha a^0(\rho, \chi_h) \\
&= \alpha a^0(\rho, \chi_h) - a_h^0(\rho_t, \chi_h) + r_h(u_h; u_h, \chi_h) - \overline{r}_h(u; P^h u, \chi_h).
\end{aligned}$$

Taking $\chi_h = \theta$ in the above equality we get

$$a_h^0(\theta_t, \theta) + \gamma^2 a_h^\Delta(\theta, \theta) = \alpha a^0(\rho, \theta) - a_h^0(\rho_t, \theta) + r_h(u_h; u_h, \theta) - \overline{r}_h(u; P^h u, \theta) \tag{3.22}$$

which, combined with the stability properties of $a_h^\Delta(\cdot, \cdot)$ and $a_h^0(\cdot, \cdot)$ (see Lemma 2.9), implies the following crucial inequality,

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(\Omega)} + \gamma^2 |\theta|^2_{H^2(\Omega)} \\
&\qquad \lesssim (\alpha \|\rho\|_{L^2(\Omega)} + \|\rho_t\|_{L^2(\Omega)}) \|\theta\|_{L^2(\Omega)} + r_h(u_h, u_h; \theta) - \overline{r}_h(u; P^h u, \theta).
\end{aligned}$$

Employing Lemmas 3.2, 3.3, and 3.4 we obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(\Omega)} + \gamma^2 |\theta|^2_{H^2(\Omega)} \lesssim h^4 + \|\theta\|^2_{L^2(\Omega)}, \tag{3.23}$$

which yields the required estimate for $\|\theta\|_{L^2(\Omega)}$ after an application of Gronwall's lemma. ∎

4. Numerical results. The time discretization is performed by the backward Euler method. The resulting nonlinear system (2.15) at each time step is solved by the Newton method, using the l^2 norm of the relative residual as a stopping criterion. The tolerance for convergence is $1e-6$. For the simulations, we have used a MATLAB code and a Fortran90 parallel code based on the PETSc library. The parallel tests were run on the FERMI linux cluster of the CINECA consortium (www.cineca.it).

Table 1
Test 1: *H^2, H^1, and L^2 errors and convergence rates α computed on four quadrilateral meshes discretizing the unit square.*

h	$\lvert u_h - u\rvert_{H^2(\Omega)}$	α	$\lvert u_h - u\rvert_{H^1(\Omega)}$	α	$\lVert u_h - u\rVert_{L^2(\Omega)}$	α
1/16	1.35e-1	–	8.57e-2	–	8.65e-2	–
1/32	5.86e-2	1.20	2.20e-2	1.96	2.20e-2	1.97
1/64	2.79e-2	1.07	5.53e-3	1.99	5.52e-3	1.99
1/128	1.38e-2	1.02	1.37e-3	2.01	1.37e-3	2.01

Table 2
Test 2: *Degrees of freedom, H^1, and L^2 errors computed with a discontinuous Galerkin discretization and with our VEM discretization on the unit square.*

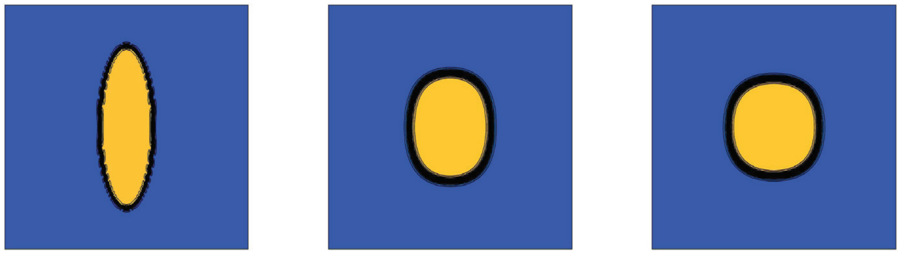
	DG			VEM		
h	dof	$\lvert u_h - u\rvert_{H^1(\Omega)}$	$\lVert u_h - u\rVert_{L^2(\Omega)}$	dof	$\lvert u_h - u\rvert_{H^1(\Omega)}$	$\lVert u_h - u\rVert_{L^2(\Omega)}$
1/16	1536	7.21e-3	7.23e-5	867	2.56e-2	9.06e-4
1/32	6144	3.63e-3	1.82e-5	3267	6.70e-3	2.31e-4
1/64	24576	1.82e-3	4.57e-6	12675	1.74e-3	5.90e-5
1/128	98304	9.12e-4	1.14e-6	49923	4.46e-4	1.49e-5

4.1. Test 1: Convergence to exact solution. In this test, we study the convergence of our VEM discretization applied to the Cahn–Hilliard equation with a forcing term f obtained by imposing as exact solution $u(x, y, t) = t\cos(2\pi x)\cos(2\pi y)$. The parameter γ is set to $1/10$ and the time step size Δt is $1e-7$. The H^2, H^1, and L^2 errors are computed at $t = 0.1$ on four quadrilateral meshes discretizing the unit square.

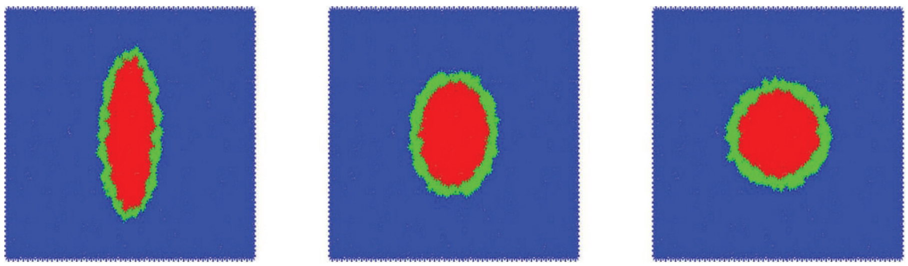
The results reported in Table 1 show that in the L^2 norm the VEM method converges with order 2, as predicted by Theorem 3.5. In the H^2 and H^1 norms, the method converges with orders 1 and 2, respectively, as can be expected according to the FEM theory and the approximation properties of the adopted virtual space.

4.2. Test 2: Comparison with a discontinuous Galerkin (DG) discretization. We compare here our VEM discretization with a P^1 mixed DG discretization. The problem considered is, as in the previous example, the Cahn–Hilliard equation with a forcing term f obtained by imposing as exact solution $u(x, y, t) = t\cos(2\pi x)\cos(2\pi y)$. The parameter γ is set to $1/10$ and the time step size Δt is $1e-7$. The simulation is run for 10 time steps. The H^1 and L^2 errors are computed at the final time step on four triangular (for the DG method) and quadrilateral (for the VEM) meshes discretizing the unit square.

The results reported in Table 2 show that, since the DG method is first order convergent while the VEM is second order convergent in the H^1 norm, for meshes still quite coarse ($h = 1/64$, $h = 1/128$) the VEM is more accurate than the DG method even with half of dof. Regarding the L^2 norm instead, both methods are second order convergent, with the DG method slightly more accurate than the VEM. However, we observe that in the L^2 norm our VEM is suboptimal with respect to its interpolation properties. Indeed, even if second order polynomials belong to the VEM space, according to the Aubin–Nitsche theory, we cannot recover the third order convergence in the L^2 norm because our primal formulation is posed in H^2. On the other hand, the mixed DG first order method is optimal with respect to its interpolation properties, and it can recover the second order convergence in the L^2 norm.



(a) Quadrilateral mesh of $16384 = 128 \times 128$ elements (49923 dof).



(b) Triangular mesh of 8576 elements (13167 dof).

FIG. 1. *Test* 3: *evolution of an ellipse at three temporal frames* ($t = 0, 0.5, 1$).

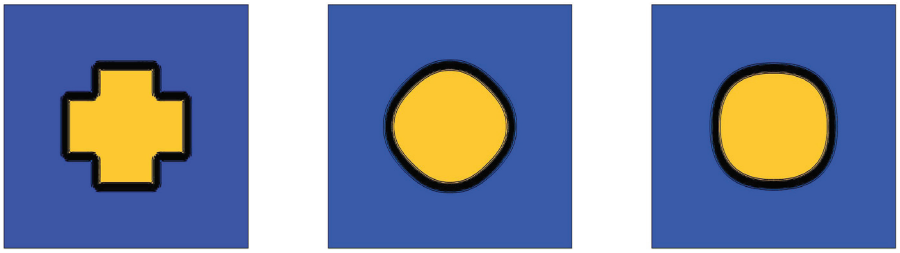
4.3. Test 3: Evolution of an ellipse. In this test, we consider the Cahn–Hilliard equation on the unit square with $\gamma = 1/100$. The time step size Δt is $5e-5$. The initial datum u_0 is a piecewise constant function whose jump set is an ellipse:

$$u_0(x, y) = \begin{cases} 0.95 & \text{if } 9(x-0.5)^2 + (y-0.5)^2 < 1/9, \\ -0.95 & \text{otherwise .} \end{cases}$$

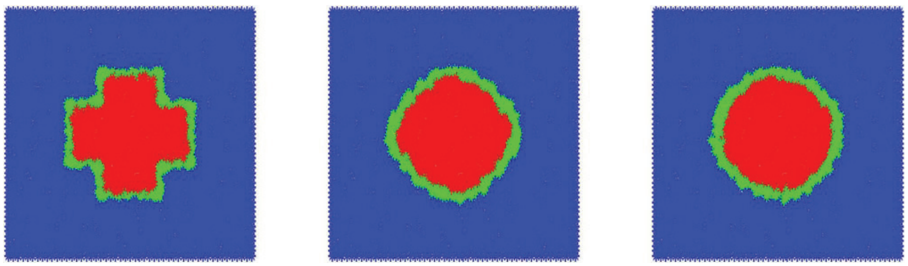
Both a structured quadrilateral mesh and an unstructured triangular mesh (generated with the mesh generator of the MATLAB PDEToolbox) are considered, with 49923 and 13167 dof, respectively. As expected the initial datum u_0 with the ellipse-shaped jump set evolves to a steady state exhibiting a circular interface; see Figures 1(a) and 1(b). Thereafter, no motion will occur as the interface has constant curvature.

4.4. Test 4: Evolution of a cross. We use here the same domain and the parameters as in Test 3. The initial datum u_0 is a piecewise constant function whose jump set has the shape of a cross; see Figures 2(a), 2(b), and 2(c) (t = 0). The same quadrilateral and triangular meshes of Test 2 are considered, with 49923 and 13167 dof, respectively, and a Voronoi polygonal mesh (including quadrilaterals, pentagons, and hexagons; see Figure 3 as an example) with 59490 dof. As in the ellipse example, the initial datum u_0 with a cross-shaped jump set evolves to a steady state exhibiting a circular interface; see Figures 2(a), 2(b), and 2(c).

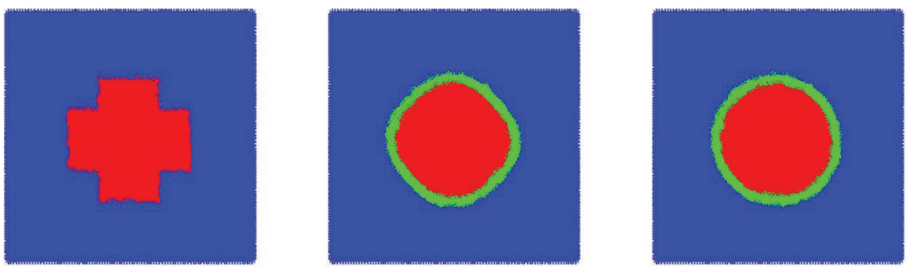
4.5. Test 5: Spinoidal decomposition. Spinodal decomposition is a physical phenomenon consisting of the separation of a mixture of two or more components to bulk regions of each. It occurs when a high-temperature mixture of different components is rapidly cooled. To model this separation the initial datum u_0 is chosen to be a uniformly distributed random perturbation between -1 and 1; see Figures 4(a), 4(b), 4(c) ($t = 0$). The same parameters as in Test 3 are used. We remark that the three initial random configurations are different. We consider a quadrilateral mesh



(a) Quadrilateral mesh of $16384 = 128 \times 128$ elements (49923 dof).



(b) Triangular mesh of 8576 elements discretizing the unit square (13167 dof).



(c) Voronoi polygonal mesh (quadrilaterals, pentagons, hexagons) of 10000 elements (59490 dof).

FIG. 2. *Test* 4: *evolution of a cross at three temporal frames* ($t = 0, 0.05, 1$).

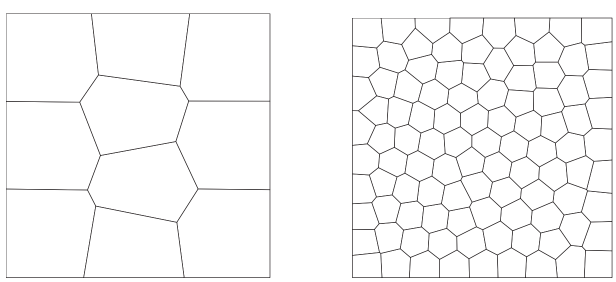
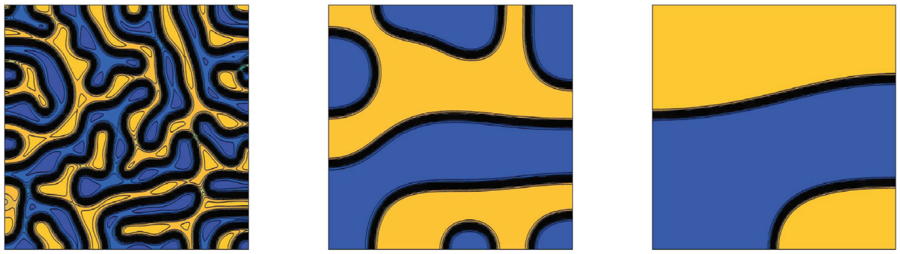
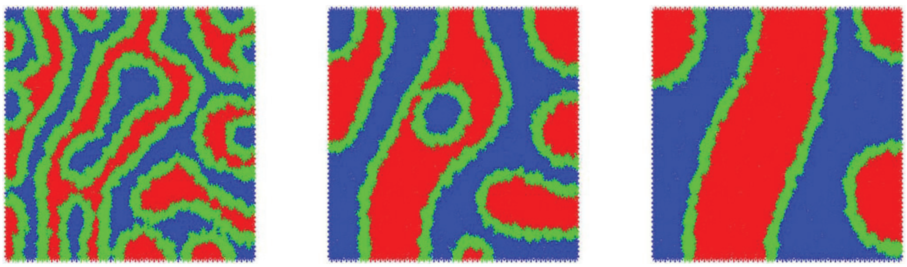


FIG. 3. *Examples of Voronoi polygonal meshes (quadrilaterals, pentagons, hexagons) with* 10 *(left) and* 100 *(right) elements.*

with 49923 dof (Figure 4(a)), a triangular mesh with 13167 dof (Figure 4(b)), and a polygonal mesh with 59590 dof (Figure 4(c)). The separation of the two components into bulk regions can be appreciated quite early; see Figures 4(a), 4(b), 4(c) ($t = 0.01$). This initial separation happens over a very small time scale compared to the motion thereafter. Then, the bulk regions begin to move more slowly, and separation will



(a) Quadrilateral mesh of $16384 = 128 \times 128$ elements (49923 dof).



(b) Triangular mesh of 8576 elements (13167 dof).

(c) Voronoi polygonal mesh (quadrilaterals, pentagons, hexagons) of 10000 elements (59490 dof).

FIG. 4. *Test* 5*: spinoidal decomposition at three temporal frames* ($t = 0.01, 0.05, 5$ *for the quadrilateral and Voronoi polygonal meshes,* $t = 0.075, 0.25, 1.25$ *for the triangular mesh).*

continue until the interfaces develop a constant curvature. In the quadrilateral (Figure 4(a)) and triangular (Figure 4(b)) mesh cases, the final equilibrium configuration is the square divided into two rectangles, while in the polygonal (Figure 4(c)) mesh case the final equilibrium configuration is clearly a circle. The fact that different final configurations are obtained starting from different initial random configurations is consistent with the results in [30].

5. Conclusions. We developed a C^1 virtual element method of minimal degree for the approximation of the Cahn–Hilliard equation. The proposed method can make use of general polygonal meshes. Moreover, it has the advantages of being conforming in H^2 and making use of a very simple set of dof, namely, 3 dof per vertex of the mesh. We proved the convergence of the semidiscrete scheme and showed the good performance of the (fully discrete) associated scheme through a set of numerical tests on problems with known solution and benchmarks from the literature. Potential research directions for the near future could include the analysis of the Cahn–Hilliard equations in three dimensions and the solution of topology optimization problems on polygonal meshes via the use of Cahn–Hilliard equations.

Appendix A. This section is devoted to the proof of Lemma 3.2.

Proof of Lemma 3.2. It is worth observing that the solution u to (2.3) satisfies

$$b(u,\psi_h) = a^0(\gamma^2\Delta^2 u - \nabla\cdot(\phi'(u)\nabla u) + \alpha u, \psi_h) \tag{A.1}$$

for all $\psi_h \in W_h^0$.

We first prove (3.7). Let $u_I \in W_h^0$ be a generic element to be made precise later. We preliminarily remark that, using $P^h u - u_I \in W_h^0$ together with Lemma 2.9 and choosing α sufficiently large, we obtain

$$b_h(P^h u - u_I, P^h u - u_I) \gtrsim \|P^h u - u_I\|^2_{H^2(\Omega)}. \tag{A.2}$$

Moreover, employing (3.2) and (A.1) yields

$$b_h(P^h u, \psi_h) = a^0(F,\psi_h) = b(u,\psi_h) \qquad \forall \psi_h \in W_h^0 \tag{A.3}$$

with $F = \gamma^2\Delta^2 u - \nabla\cdot(\phi'(u)\nabla u) + \alpha u$.

Thus, using (A.3) and letting u_π be a discontinuous piecewise quadratic polynomial, we get

$$\begin{aligned}
&b_h(P^h u - u_I, P^h u - u_I)\\
&\quad = b_h(P^h u, P^h u - u_I) - b_h(u_I, P^h u - u_I)\\
&\quad = b(u, P^h u - u_I) - b_h(u_\pi, P^h u - u_I) + b_h(u_\pi - u_I, P^h u - u_I)\\
&\quad = b(u, P^h u - u_I) - \overline{b}(u_\pi, P^h u - u_I) + b_h(u_\pi - u_I, P^h u - u_I),
\end{aligned}$$

where in the last equality we apply the consistency result contained in Lemma 2.8 to the bilinear form

$$\overline{b}(v,w) = \sum_{E\in\Omega_h} \gamma^2 a_E^{\Delta}(v,w) + \overline{\phi'(u)}_{|E}\, a_E^{\nabla}(v,w) + \alpha\, a_E^0(v,w).$$

From the above identity, using (A.2) we get

$$\begin{aligned}
\|P^h u - u_I\|^2_{H^2(\Omega)} \lesssim\ & b(u, P^h u - u_I) - \overline{b}(u, P^h u - u_I) + \overline{b}(u - u_\pi, P^h u - u_I)\\
&+ b_h(u_\pi - u_I, P^h u - u_I).
\end{aligned} \tag{A.4}$$

Let us now estimate each term on the right-hand side of (A.4). From the definitions of the bilinear forms $b(\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot)$, and employing the interpolation estimates given in Proposition (3.1), we obtain

$$\begin{aligned}
b(u, P^h u - u_I) - \overline{b}(u, P^h u - u_I) &= \sum_{E\in\Omega_h} \int_E (\phi'(u) - \overline{\phi'(u)}_{|E})\nabla u \cdot \nabla(P^h u - u_I)\mathrm{d}x\\
&\lesssim h\|P^h u - u_I\|_{H^2(\Omega)}.
\end{aligned} \tag{A.5}$$

Moreover, choosing u_I and u_π such that (see Proposition 3.1)

$$\|u - u_I\|_{H^2(E)} + \|u - u_\pi\|_{H^2(E)} \lesssim h \tag{A.6}$$

and employing the continuity properties of $\overline{b}(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ we get

$$\overline{b}(u - u_\pi, P^h u - u_I) + b_h(u_\pi - u_I, P^h u - u_I) \lesssim h\|P^h u - u_I\|_{H^2(\Omega)}. \tag{A.7}$$

Substituting (A.5) and (A.7) into (A.4) and using the triangle inequality together with (A.6) we get (3.7).

We now prove (3.8). Taking $w = u - P^h u$ in (3.4) yields

$$\|u - P^h u\|^2_{H^1(\Omega)} = b(\varphi, u - P^h u) = b(\varphi - \varphi_I, u - P^h u) + b(\varphi_I, u - P^h u). \tag{A.8}$$

We now estimate each term on the right-hand side of the above equation. Choosing, according to Proposition (3.1), φ_I such that $\|\varphi - \varphi_I\|_{H^2(\Omega)} \lesssim h$, using (3.6), and employing the continuity property of the bilinear form $b(\cdot,\cdot)$ together with (3.7) we get

$$b(\varphi - \varphi_I, u - P^h u) \lesssim \|\varphi - \varphi_I\|_{H^2(\Omega)} \|u - P^h u\|_{H^2(\Omega)} \lesssim h^2 \|u - P^h u\|_{H^1(\Omega)}. \tag{A.9}$$

Using (A.1) with $\varphi_I \in W_h^0$ we get $b(\varphi_I, u) = b_h(\varphi_I, P^h u)$ which implies

$$\begin{aligned} b(\varphi_I, u - P^h u) &= b_h(\varphi_I, P^h u) - b(\varphi_I, P^h u) \\ &= \gamma^2 (a_h^{\Delta}(\varphi_I, P^h u) - a^{\Delta}(\varphi_I, P^h u)) + \overline{r}_h(u; \varphi_I, P^h u) - r(u; \varphi_I, P^h u) \\ &= \gamma^2 A_1 + A_2. \end{aligned}$$

Let u_π and φ_π be piecewise discontinuous quadratic polynomials such that $\|u - u_\pi\|_{H^2(E)} \lesssim h$ and $\|\varphi - \varphi_\pi\|_{H^2(E)} \lesssim h$. Applying twice the consistency result contained in Lemma 2.8 together with (3.7) we obtain

$$\begin{aligned} A_1 &= \sum_{E \in \Omega_h} a^{\Delta}_{h,E}(\varphi_I - \varphi_\pi, P^h u - u_\pi) - \sum_{E \in \Omega_h} a^{\Delta}_E(\varphi_I - \varphi_\pi, P^h u - u_\pi) \\ &\lesssim (\|\varphi - \varphi_I\|_{H^2(E)} + \|\varphi - \varphi_\pi\|_{H^2(E)})(\|P^h u - u\|_{H^2(E)} + \|u - u_\pi\|_{H^2(E)}) \\ &\lesssim h^2 |\varphi|_{H^3(\Omega)} |u|_{H^3(\Omega)} \lesssim h^2 \|u - P^h u\|_{H^1(\Omega)}. \end{aligned} \tag{A.10}$$

Let us now estimate the term A_2. Using the definitions of $r(\cdot;\cdot,\cdot)$ and $\overline{r}_h(\cdot;\cdot,\cdot)$ we get

$$\begin{aligned} A_2 &= \sum_{E \in \Omega_h} \overline{\phi'(u)}_{|E} \left(a^{\nabla}_{h,E}(\varphi_I, P^h u) - a^{\nabla}_E(\varphi_I, P^h u) \right) \\ &\quad + \int_E \left(\overline{\phi'(u)}_{|E} - \phi'(u) \right) \nabla \varphi_I \cdot \nabla P^h u \mathrm{d}x \\ &=: A_{2,1} + A_{2,2}. \end{aligned}$$

Proceeding as in the bound of A_1 and employing assumption (2.4) on the regularity of u we obtain

$$A_{2,1} \lesssim h^2 \|u - P^h u\|_{H^1(\Omega)}. \tag{A.11}$$

Finally, we estimate the term $A_{2,2}$. By employing the orthogonality property of projectors and denoting by $\overline{(\cdot)}$ the projection of $(\cdot)$ on constants we get

$$\begin{aligned} A_{2,2} &= \sum_{E \in \Omega_h} \int_E \left(\overline{\phi'(u)}_{|E} - \phi'(u) \right) \left(\nabla \varphi_I \cdot \nabla P^h u - \overline{\nabla \varphi} \cdot \overline{\nabla u} \right) dx \\ &= \sum_{E \in \Omega_h} \int_E \left(\overline{\phi'(u)}_{|E} - \phi'(u) \right) (\nabla \varphi_I - \overline{\nabla \varphi}) \cdot \nabla P^h u \, \mathrm{d}x \\ &\quad + \int_E \left(\overline{\phi'(u)}_{|E} - \phi'(u) \right) \overline{\nabla \varphi} \cdot (\nabla P^h u - \overline{\nabla u}) \, dx. \end{aligned} \tag{A.12}$$

Using the interpolation estimates given in Proposition (3.1), and employing (3.6) and (3.7) together with the following inequalities

$$\begin{aligned}
\|\nabla\varphi_I - \overline{\nabla\varphi}\|_{L^2(E)} &\le \|\nabla\varphi_I - \nabla\varphi\|_{L^2(E)} + \|\nabla\varphi - \overline{\nabla\varphi}\|_{L^2(E)} \lesssim h\|\varphi\|_{H^2(E)},\\
\|\nabla P^h u\|_{L^2(E)} &\le \|\nabla P^h u - \nabla u\|_{L^2(E)} + \|\nabla u\|_{L^2(E)} \lesssim (1+h)\|u\|_{H^2(E)},\\
\|\overline{\nabla\varphi}\|_{L^2(E)} &\le \|\overline{\nabla\varphi_I} - \nabla\varphi\|_{L^2(E)} + \|\nabla\varphi\|_{L^2(E)} \lesssim (1+h)\|\varphi\|_{H^2(E)},\\
\|\nabla P^h u - \overline{\nabla u}\|_{L^2(E)} &= \|\nabla(P^h u - u) + (\nabla u - \overline{\nabla u})\|_{L^2(E)}\\
&\lesssim \|P^h u - u\|_{H^2(E)} + h\|u\|_{H^2(E)},\\
\|\overline{\phi'(u)}_{|E} - \phi'(u)\|_{L^\infty(E)} &\lesssim h|\phi'(u)|_{W^{1,\infty}(E)},
\end{aligned}$$

we obtain

$$A_{2,2} \lesssim h^2\|P^h u - u\|_{H^1(\Omega)}. \tag{A.13}$$

Combining (A.10), (A.11), (A.13), (A.9) with (A.8) we obtain (3.8).

We are left to show (3.9)–(3.10). To this aim it is sufficient to observe that it holds

$$b_h((P^h u)_t, \psi_h) = b(u_t, \psi_h) + a^0(\phi''(u)u_t\nabla u, \nabla\psi_h) - \sum_{E\in\Omega_h} \partial_t\overline{(\phi'(u))}_{|E} a^{\nabla}_{h,E}(P^h u, \psi_h)$$

for all $\psi_h \in W_h^0$. Then proceeding as before and using

$$\|\partial_t\overline{(\phi'(u))}_{|E} - \phi''(u)u_t\|_{L^\infty(E)} = \|\overline{6uu_t} - 6uu_t\|_{L^\infty(E)} \lesssim h$$

we obtain the thesis. ∎

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