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Quantum Noise from Reduced Dynamics

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We consider the description of quantum noise within the framework of the standard Copenhagen interpretation of quantum mechanics applied to a composite system environment setting. Averaging over the environmental degrees of freedom leads to a stochastic quantum dynamics, described by equations complying with the constraints arising from the statistical structure of quantum mechanics. Simple examples are considered in the framework of open system dynamics described within a master equation approach, pointing in particular to the appearance of the phenomenon of decoherence and to the relevance of quantum correlation functions of the environment in the determination of the action of quantum noise.

1. Introduction

The corner stone for the successful description of experiments with microscopic systems as statistical experiments was laid by Bohr through his probabilistic reading of the square modulus of the wavefunction, finally leading to the so called Copenhagen interpretation of quantum mechanics. This interpretation of quantum mechanics is often also termed orthodox, to stress the existence of alternative viewpoints, still compatible with present day most refined experiments on the foundations of quantum mechanics (see e.g. the special issue [1]). Further developments have deepened and strengthened the understanding of quantum mechanics as a theory describing experiments in a statistical framework. In this spirit it has become clear that quantum mechanics naturally leads to a new probabilistic description with respect to the classical one, sometimes termed quantum probability [2], so that from now on we will use the general term quantum theory, even though it actually started as an alternative to classical mechanics.

Quantum theory includes and extends the classical probabilistic description, so that bringing over ideas and concepts from classical probability theory through the quantum border is a fruitful path in order to further understand and explore the statistical structure of quantum theory, and leads to a reach variety of new

phenomena (for a presentation of quantum theory along these lines see e.g. [3,4]). Actually it is an amusing, and possibly telling, coincidence the fact that the book in which von Neumann laid the mathematical foundations of quantum theory [5] appeared almost at the same time as the contribution in which Kolmogorov laid the foundations of classical probability theory basing its axiomatic presentation on measure theory [6].

The Copenhagen interpretation, which tells us that the quantum description of physical systems brings with itself an intrinsic statistical aspect, can equally well describe composite systems, that is a situation in which one can distinguish between different parts of the overall system. Let us call system the subset of degrees of freedom we are interested in and can access experimentally, as well as environment the other degrees of freedom, still to be described with the aid of quantum theory. A relevant and interesting question is the quantum prediction for the dynamics of the relevant degrees of freedom we call system if one does not or can not observe the environmental degrees of freedom. In such a situation, on top of the in principle unavoidable statistical aspect due to the very nature of quantum theory, an additional source of randomness appears, which can be termed quantum noise [7,8]. This situation can be seen as the analogue of what happens in a classical setting when a given system undergoes a stochastic dynamics. However, in the classical case the dynamics of a small isolated system is in principle deterministic, and the statistical aspect in the description can always be seen as arising from the effect of a classical noise, possibly effectively describing the interaction with other classical degrees of freedom. In the quantum setting the action of quantum noise on the contrary builds on the original statistical description. Besides this, important constraints on the structure of the equations describing the quantum stochastic dynamics as well as on the properties of the quantum noise itself appear, essentially related to the non commutativity of observables, playing the role of random variables in quantum theory, and to the tensor product structure of the Hilbert space on which composite systems are described.

It is to be stressed that quantum noise, or actually more precisely noise in a quantum system, can describe a phenomenon which is typical of the quantum realm, namely decoherence. The latter can be understood as the dynamical loss of the capability to show up quantum interference effects in a given system basis, as a consequence of the interaction with other external quantum degrees of freedom. We recall that the term decoherence or dephasing is sometimes also used to describe more generically a loss of coherence or visibility which can be obtained within a classical description. As a result quantum noise can induce an effective classical dynamics for certain system observables, still not selecting a definite outcome so that it does not lead to a solution of the measurement problem.

An interesting issue within the description of randomness in the dynamics of a quantum system is also the distinction between noise which can be avoided by means of a more refined control or noise which is actually intrinsic to the quantum

description [9]. Most recently an approach has also been suggested [10] in order to discriminate between decoherence arising from an actual interaction with unobserved degrees of freedom and decoherence arising from modifications of quantum mechanics as suggested by collapse models or other alternative theories.

In this contribution we will briefly describe the emergence of a dynamics driven by quantum noise in the framework of open quantum system theory, considering basic examples.

2. Reduced system dynamics

Let us consider the general framework of open quantum system dynamics [11], introducing a quantum system described on the Hilbert space \mathcal{H}_S , interacting with a quantum environment living in \mathcal{H}_E , as depicted in Fig. 1. If we denote with ρ_{SE} the total state and describe the interaction by means of the unitary operators $U(t)$ acting on $\mathcal{H}_S \otimes \mathcal{H}_E$, further assuming that the state at the initial time is factorized $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E$, we have that the reduced state of the system, describing the dynamics of the system's observables only, is given by

$$\rho_S(t) = \text{Tr}_E\{U(t)\rho_S(0) \otimes \rho_E U(t)^\dagger\}. \quad (1)$$

The assignment $\rho_S(0) \mapsto \rho_S(t)$ turns out to define a map which is in particular completely positive, that is remains positive when extended to act on a tensor product extension of the considered Hilbert space \mathcal{H}_S .

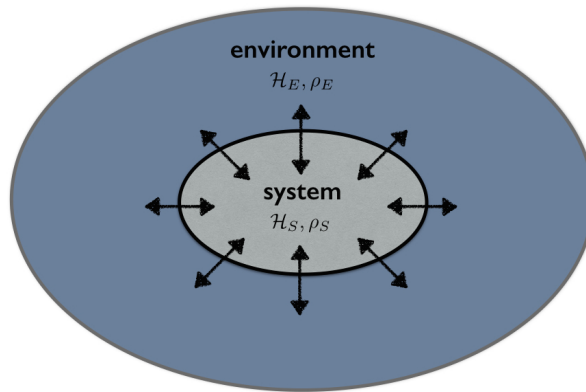


Fig. 1. A schematic illustration of an open system with Hilbert space \mathcal{H}_S and reduced state ρ_S , interacting with an environment described in the Hilbert space \mathcal{H}_E , with reduced state ρ_E .

In many situations of interest the reduced system state dynamics is well described by a time-local master equation of the form

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}(t)\rho_S(t), \quad (2)$$

where the superoperator $\mathcal{L}(t)$ is known as generators of the dynamics. For the case in which this superoperator is actually time independent, according to a famous result [12, 13] it is known to have the so called Gorini-Kossakowski-Sudarshan-Lindblad form

$$\frac{d}{dt}\rho_S(t) = -\frac{i}{\hbar}[H, \rho_S(t)] + \sum_{j,k} a_{jk} \left[L_j \rho_S(t) L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_j, \rho_S(t)\} \right], \quad (3)$$

where H is a self-adjoint operator on the space of the system and the matrix a_{jk} has to be positive. In particular together with the system operators $\{L_k\}$ this matrix defines the details of the system-environment interaction and depends on the quantum correlation functions of the environment. Properties of the quantum noise affecting the open system dynamics are therefore encoded in the operator structure of the r.h.s. of Eq. (3), as well as in the features of correlation functions of quantum operators, as we shall see in the examples. The situation described by Eq. (3) corresponds to a semigroup dynamics for the open quantum system, which can be considered Markovian, in the sense that the state of the system at a given time is enough to determine it at later times. Actually the proper definition of what should be considered as non-Markovian dynamics in a quantum framework, and therefore also of non-Markovian quantum noise, has newly become the object of an extensive research activity (see e.g. [14, 15]). It is important to stress that the strategy that we have here briefly outlined is certainly not the only approach to the description of quantum noise arising within the Copenhagen interpretation by the interaction of the system with other unobserved quantum degrees of freedom, for the presentation of other viewpoints and techniques see e.g. [7, 11, 16, 17]. A crucial point to stress is the measurement character of such time evolution, at variance with a standard unitary dynamics. Indeed as it has been shown within the framework of continuous measurement theory (see e.g. [18]), such a dynamics can be obtained as a result of measurements performed on the side of the system, and can be described introducing non commuting noises. A thorough quantum description of noise allows in particular the preservation of basic features of quantum mechanics, such as e.g. Heisenberg's commutation relations [19], which are generally not accounted for in phenomenological models which can be used to describe a stochastic dynamics.

2.1. *Decoherence models*

For the sake of example we will now briefly consider two quantum dynamics which can be addressed within the previously introduced framework, and show how quantum noise can induce decoherence on the system degrees of freedom, determined by the environmental correlation functions.

Let us first consider a massive quantum particle interacting through collisions with a background ideal quantum gas. In such a setting for a sufficiently dilute gas memory effects can be safely neglected, so that indeed the dynamics can be taken to be Markovian. It can therefore be assumed that the dynamics can be described

by an equation of the form Eq. (3), upon suitable microscopic or phenomenological determination of the different coefficients and operators. In this case the interaction can be naturally taken of the form [20]

$$V = \int d^3\mathbf{x} \int d^3\mathbf{y} N_S(\mathbf{x}) v(\mathbf{x} - \mathbf{y}) N_E(\mathbf{y}), \quad (4)$$

where $N_S(\mathbf{x})$ and $N_E(\mathbf{y})$ denote the number operator density for system and environment respectively. In this situation it can be shown [21] that tracing over the gas degrees of freedom the master equation takes on the form

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & -\frac{i}{\hbar} [H_0, \rho_S(t)] \\ & + \int d^3\mathbf{q} \mu(\mathbf{q}) \left[e^{\frac{i}{\hbar} \mathbf{q} \cdot \hat{\mathbf{x}}} \sqrt{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}}))} \rho_S(t) \sqrt{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}}))} e^{-\frac{i}{\hbar} \mathbf{q} \cdot \hat{\mathbf{x}}} \right. \\ & \left. - \frac{1}{2} \{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}})), \rho_S(t)\} \right], \end{aligned} \quad (5)$$

where H_0 is the free kinetic Hamiltonian, $\mu(\mathbf{q}) = (2\pi)^4 \hbar^2 n |\tilde{v}(\mathbf{q})|^2$, with n gas particle density and $\tilde{v}(\mathbf{q})$ Fourier transform of the interaction potential. In the expression $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ denote position and momentum operators of the test particle, so that the unitary operators $e^{\frac{i}{\hbar} \mathbf{q} \cdot \hat{\mathbf{x}}}$ describe momentum translations, while

$$\begin{aligned} S(\mathbf{q}, E) = & \frac{1}{2\pi\hbar} \int dt \int d^3\mathbf{x} e^{\frac{i}{\hbar}(Et - \mathbf{q} \cdot \mathbf{x})} \frac{1}{N} \int d^3\mathbf{y} \langle N_E(\mathbf{y}) N_E(\mathbf{x} + \mathbf{y}, t) \rangle \\ = & \frac{1}{2\pi\hbar} \frac{1}{N} \int dt e^{\frac{i}{\hbar} Et} \langle \varrho_{\mathbf{q}}^\dagger \varrho_{\mathbf{q}}(t) \rangle \end{aligned} \quad (6)$$

upon defining

$$\varrho_{\mathbf{q}} = \int d^3\mathbf{x} e^{-\frac{i}{\hbar} \mathbf{q} \cdot \mathbf{x}} N_E(\mathbf{x}). \quad (7)$$

The master equation is fixed by the function $S(\mathbf{q}, E)$ defined in Eq. (6), also known as dynamic structure factor [22], which is actually the Fourier transform of the density-density correlation function of the environment, which appears operator-valued being evaluated in $E(\mathbf{q}, \hat{\mathbf{p}})$, with $E(\mathbf{q}, \mathbf{p}) = (\mathbf{p} + \mathbf{q})^2 / (2M) - \mathbf{p}^2 / (2M)$ energy transfer in a single collision, M mass of the gas particle. Note that one has the identity

$$\langle \varrho_{\mathbf{q}}^\dagger \varrho_{\mathbf{q}}(t) \rangle = \frac{1}{2} \langle \{ \varrho_{\mathbf{q}}^\dagger, \varrho_{\mathbf{q}}(t) \} \rangle + \frac{1}{2} \langle [\varrho_{\mathbf{q}}^\dagger, \varrho_{\mathbf{q}}(t)] \rangle, \quad (8)$$

where the last contribution is non vanishing just due to the operator nature of the environmental quantities in the quantum description. This quantity depending on the density fluctuations can be directly related to the dynamic response function of the environment $\chi''(\mathbf{q}, E)$ according to the fluctuation-dissipation formula

$$S(\mathbf{q}, E) = \frac{1}{\pi} \frac{1}{1 - e^{\beta E}} \chi''(\mathbf{q}, E). \quad (9)$$

The considered master equation describes both dissipation and decoherence effects in the stochastic dynamics of the particle undergoing quantum Brownian motion. To put into evidence decoherence effects in the position representation it is convenient to consider a simplified expression in which we treat momentum as a classical variable, so that Eq. (5) takes the much simpler expression

$$\frac{d}{dt}\rho_S(t) = \int d^3\mathbf{q}\tilde{\mu}(\mathbf{q}) \left[e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}\rho_S(t)e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \rho_S(t) \right], \quad (10)$$

with $\tilde{\mu}(\mathbf{q})$ a suitable positive density and its solution in the position matrix elements can be written as

$$\langle \mathbf{x} | \rho_S(t) | \mathbf{y} \rangle = e^{-\Lambda[1-\Phi(\mathbf{x}-\mathbf{y})]t} \langle \mathbf{x} | \rho_S(0) | \mathbf{y} \rangle, \quad (11)$$

with $\Phi(\mathbf{x})$ the characteristic function of the probability distribution of momentum transfers between test and gas particles, and Λ a collision rate [23]. As a result off-diagonal matrix elements in the position representation are suppressed with elapsing time. This means in particular that if the system is initially in a coherent superposition of spatially separated states the quantum noise can drive the system to a classical mixture, which is a typical decoherence effect. This kind of models can explain decoherence effects in interference experiments with massive particles [24, 25]. Note that a similar result for the dynamics of the statistical operator $\rho_S(t)$ arises in dynamical reduction models [26], however only the average effect can be compared, in such models one has a localization effect acting on the single realizations, leading to a possible solution of the measurement problem [27].

As a further example showing the relevance of quantum correlation functions in the description of a noisy quantum dynamics we consider an exactly solvable model of decoherence [11]. In this case one considers a two-level system interacting with a bosonic reservoir according to the coupling

$$V = \sigma_z \sum_k (g_k b_k^\dagger + g_k^* b_k), \quad (12)$$

where besides the standard Pauli operator we have introduced complex coupling coefficients g_k , as well as the creation and annihilation operators b_k and b_k^\dagger obeying the standard canonical commutation relations. If we assume the bosonic reservoir to be in a thermal state the reduced dynamics can be exactly worked out and leads to a master equation which is in a form similar to Eq. (3), albeit with a time dependent coefficient

$$\frac{d}{dt}\rho_S(t) = -\frac{i}{\hbar}[H_0, \rho_S(t)] + \gamma(t)[\sigma_z \rho_S(t) \sigma_z - \rho_S(t)], \quad (13)$$

where $H_0 = \hbar\omega_0\sigma_z$ is the free system Hamiltonian. The time dependent coefficient $\gamma(t)$ is determined again from a correlation function depending on the environment

operators appearing in the interaction term Eq. (12) and given by

$$\begin{aligned}\alpha(t) &= \sum_k |g_k|^2 (\langle b_k(t) b_k^\dagger \rangle + \langle b_k^\dagger(t) b_k \rangle) \\ &= \int_0^\infty d\omega J(\omega) \left\{ \coth\left(\frac{\beta}{2}\hbar\omega\right) \cos(\omega t) - i \sin(\omega t) \right\},\end{aligned}\quad (14)$$

where the two contributions at the r.h.s. come from the evaluation of the correlation function relying on a decomposition as the one considered in Eq. (8), where now the commutator part typically related to dissipation amounts to a \mathbb{C} -number term. We have further introduced the so called spectral density $J(\omega)$, formally defined as $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$, with ω_k the frequency of the bosonic modes appearing in the free Hamiltonian of the environment $\sum_k \hbar\omega_k b_k^\dagger b_k$, which actually allows to go over to a continuum limit embodying in itself dependence of the coupling strength on the environment frequencies as well as on the distribution of the environmental modes. For the decoherence dynamics described by the present model only the anticommutator part of the correlation function related to decoherence is relevant and one has in particular

$$\begin{aligned}\gamma(t) &= \Re \int_0^t d\tau \alpha(t - \tau) \\ &= \int_0^\infty d\omega J(\omega) \coth\left(\frac{\beta}{2}\hbar\omega\right) \frac{\sin(\omega t)}{\omega}.\end{aligned}\quad (15)$$

In view of the interaction term Eq. (12) one immediately sees that the diagonal matrix elements of the statistical operator in the basis of eigenvectors of the system Hamiltonian are constant, while coherences are generally suppressed according to

$$\langle 1 | \rho_S(t) | 0 \rangle = e^{-\Gamma(t)} e^{i\omega_0 t} \langle 1 | \rho_S(0) | 0 \rangle, \quad (16)$$

where bra and ket denote the eigenvectors of the σ_z operator, and the decoherence function $\Gamma(t)$ is still determined by the spectral density and the correlation function of the environment through the expression

$$\begin{aligned}\Gamma(t) &= \int_0^t d\tau \gamma(\tau) \\ &= \int_0^\infty d\omega J(\omega) \coth\left(\frac{\beta}{2}\hbar\omega\right) \frac{1 - \cos(\omega t)}{\omega^2}.\end{aligned}\quad (17)$$

As a result one has a general description of the decoherence dynamics of a two-level system coupled to bosonic degrees of freedom, allowing for a phenomenological modelling of the effective reservoir through the suitable definition of a spectral density. Also in this model we have seen how the quantum stochastic dynamics is driven by correlation functions of the environment operators, which embody the noisy action of the quantum environment.

3. Conclusions and outlook

If one considers a non isolated quantum system, its dynamics shows up an additional layer of stochasticity, on top of the probabilistic quantum description, which arises due to the interaction with the unobserved quantum environmental degrees of freedom. In this perspective quantum noise can be described applying the standard Copenhagen formulation of quantum mechanics to the overall degrees of freedom. In this framework one is able to describe in a consistent way both dissipative and decoherence effects. The latter lead from a quantum probabilistic setting to a classical one, in which the interference capability of selected quantum degrees of freedom is suppressed. As a result one recovers a classical behaviour for certain degrees of freedom, however still not solving the measurement problem, which has to face the fact that macroscopic objects do appear in definite states, rather than in superpositions or classical mixture states. Different techniques and approaches can be devised in order to describe the quantum noisy dynamics of such open quantum systems and in this contribution we have considered two paradigmatic examples within the framework of a master equation approach. It appears how the action of quantum noise in this description typically depends on the features of two-point correlation functions of the quantum operators of the environment involved in the interaction term.

An important open issue in this and other descriptions of quantum noise within the standard Copenhagen interpretation is the formulation and characterization of memory effects. Recent work on the subject [14, 15] has put quantum non-Markovianity in connection with properties of the statistical operator of the open system undergoing a stochastic dynamics, or of the mapping describing the reduced dynamics. This is at variance with the classical case, in which there is a clearcut definition of Markovian noise in the framework of classical stochastic processes, which cannot be directly used in the quantum framework [28]. Future characterization of quantum noise in view of its memory properties might well be connected with expression and features of multitime correlation functions of environmental quantum operators. A better understanding of the description of quantum noise will also be useful in view of a comparison between orthodox quantum mechanics and modifications of it, such as dynamical reduction models, aiming to a solution of the quantum measurement problem and leading to distinct experimental predictions, which are in principle detectable. Indeed determination of experimental bounds on the value of the parameters appearing in such models, as well as their extension to a non-Markovian regime, are the object of an intense research activity [29].

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