MODULAR p-ADIC L-FUNCTIONS ATTACHED TO REAL QUADRATIC FIELDS AND ARITHMETIC APPLICATIONS

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Abstract. Let $f \in S_{k_0+2}(\Gamma_0(Np))$ be a normalized N-new eigenform with $p \nmid N$ and such that $a_p^2 \neq p^{k_0+1}$ and $\mathrm{ord}_p(a_p) < k_0 + 1$. By Coleman's theory, there is a p-adic family $\mathbf{F}$ of eigenforms whose weight $k_0 + 2$ specialization is f. Let K be a real quadratic field and let ψ be an unramified character of $\mathrm{Gal}(\overline{K}/K)$. Under mild hypotheses on the discriminant of K and the factorization of N, we construct a p-adic L-function $\mathcal{L}_{\mathbf{F}/K,\psi}$ interpolating the central critical values of the Rankin L-functions associated to the base change to K of the specializations of $\mathbf{F}$ in classical weight, twisted by ψ. When the character ψ is quadratic, $\mathcal{L}_{\mathbf{F}/K,\psi}$ factors into a product of two Mazur-Kitagawa p-adic L-functions. If, in addition, $\mathbf{F}$ has p-new specialization in weight $k_0 + 2$, then under natural parity hypotheses we may relate derivatives of each of the Mazur-Kitagawa factors of $\mathcal{L}_{\mathbf{F}/K,\psi}$ at k_0 to Bloch-Kato logarithms of Heegner cycles. On the other hand the derivatives of our p-adic L-functions encodes the position of the so called Darmon cycles. As an application we prove rationality results about them, generalizing theorems of Bertolini-Darmon, Seveso, and Shahabi.

Contents

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1. Introduction

1.1. **Summary.** Let p be a prime, let N be a squarefree integer with $p \nmid N$, and let $E/\mathbb{Q}$ be an elliptic curve of conductor pN. Let K be a real quadratic field and let ϵ_K be the corresponding quadratic character. Under the *Stark-Heegner hypothesis*

$$\epsilon_K(p) = -1 \quad \text{and} \quad \epsilon_K(\ell) = +1 \quad \text{for all } \ell \mid N, \tag{1}$$

Darmon [12] presented a construction of a *Stark-Heegner* point $P_K \in E(K_p)$ which, he conjectured, actually belongs to $E(K)$ and governs the arithmetic of E over K in much the same way as a classical Heegner point governs the arithmetic of elliptic curves over imaginary quadratic fields. Even though largely conjectural, Darmon's construction provided the beginnings of a coherent approach for studying rational points and associated objects (Selmer classes, classical and p-adic L-functions, etc.) outside of usual framework of complex multiplication theory. In recent joint work [32] with Rotger, the second author gave a far-reaching generalization of Darmon's original construction and conjectures, replacing elliptic curves by higher weight modular forms, Mordell-Weil groups with Bloch-Kato Selmer groups, and the Stark-Heegner hypothesis (1) with a natural condition on signs in functional equations of L-functions. The goal of this paper is to provide strong theoretical evidence for the conjectures of [32], thus generalizing results of [5], by proving that the *conjectures* of loc. cit. are compatible with analogous *theorems* from CM-theory in situations where they overlap, as first discovered in [5]. We deduce such compatibilities using a class of weight variable p-adic L-functions that we construct in Part 2 of this paper. In Part 3, we recall the conjectures of [32], relate them to the p-adic L-functions of Part 1, and prove our main results.

1.2. **Setting.** Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, an odd prime number p, and embeddings

$$\sigma_\infty : \overline{\mathbb{Q}} \longrightarrow \mathbb{C}, \qquad \sigma_p : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p. \tag{2}$$

Let N be a positive integer with $p \nmid N$, let E be a p-adic field, and let Ω be an affinoid disk in the weight space $\mathcal{X}$ defined over the p-adic field E. A *Coleman Ω-family of cuspidal eigenforms of tame level* $\Gamma_0(N)$ is a formal q-expansion

$$\mathbf{F}(q) = \sum_{n \geq 1} \mathbf{a}_n q^n \in \mathcal{O}(\Omega)[[q]],$$

such that for all

$$k \in \Omega_{\mathrm{cl}} := \{n \in 2\mathbb{Z} : n \geq 0\} \cap \Omega,$$

there is a normalized eigenform $F_k \in S_{k+2}(\Gamma_0(Np), \overline{\mathbb{Q}})$ satisfying

$$F_k(q) = \sum_{n \geq 1} \mathbf{a}_n(k) q^n.$$

Since $\mathbf{a}_p(k)$ is rigid analytic, the function $\mathrm{ord}_p\, \mathbf{a}_p(k)$ may be assumed to be constant, up to shrinking in an open affinoid neighbourhood of any $k \in \Omega$. We assume this condition satisfied, and call this quantity the *slope of* $\mathbf{F}$.

Restricting to a case of particular interest, we assume that F_k is N-new for all $k \in \Omega_{\mathrm{cl}}$. It follows that, for each $k \in \Omega_{\mathrm{cl}}$, the form F_k is a *p-stabilized newform*, i.e., either F_k is p-new, or there is a newform $F_k^\sharp \in S_{k+2}(\Gamma_0(N))$ such that

$$F_k(q) = F_k^\sharp(q) - \frac{p^{k+1}}{\mathbf{a}_p(k)} F_k^\sharp(q^p). \tag{3}$$

If F_{k_0} is p-new, we set $F^\sharp_{k_0} = F_k$. There is at most one $k_0 \in \Omega_{\rm cl}$ at which F_{k_0} is p-new: F_k is p-new exactly when $\mathbf{a}_p(k_0) = \pm p^{k_0/2}$. That this equation can hold for at most one $k_0 \in \Omega_{\rm cl}$ follows from the constancy of the slope of $\mathbf{F}$. We set

$$\Omega_{\rm cl}^{p\text{-old}} = \{k \in \Omega_{\rm cl} : F_k \text{ is } p\text{-old}\}.$$

Our first goal in this paper is to study the p-adic variation of the central critical L-values associated to the newforms $F^\sharp_k$ as k varies over $\Omega_{\rm cl}$. By a theorem of Shimura [43, Theorem 1], there are nonzero *Shimura periods* $u^\pm(k) \in \mathbb{C}$ such that for every quadratic Dirichlet character χ,

$$L^*(F^\sharp_k, \chi, j) := \frac{(j-1)!\tau(\chi)c(\chi)^j}{(-2\pi i)^{j-1}u^\pm(k)} L(F^\sharp_k, \chi, j) \in \mathbb{Q}(F^\sharp_k), \qquad 1 \le j \le k+1, \tag{4}$$

where $\tau(\chi)$ is the Gauss sum associated to χ, $c(\chi)$ its conductor, $\mathbb{Q}(F^\sharp_k)$ is the field generated by the Fourier coefficients of $F^\sharp_k$, and the sign of the Shimura period is chosen so that

$$\pm(-1)^{j-1} = \chi(-1). \tag{5}$$

The quantity $L^*(F^\sharp_k, \chi, j)$ is called the *algebraic part of the special value.* Scaled appropriately by p-adic periods and Euler-like factors, these algebraic parts can be p-adically interpolated:

Theorem 1.1. *There exist nonzero p-adic periods $\lambda^\pm(k) \in E$ for $k \in \Omega_{cl}$ such that for every quadratic Dirichlet character χ there is a p-adic analytic function $L_{\mathbf{F},\chi}$ on $\Omega \times \mathbb{Z}_p$ satisfying*

$$\frac{L_{\mathbf{F},\chi}(k,j)}{\lambda^\pm(k)} = \begin{cases} \left(1 - \dfrac{\chi(p)p^{j-1}}{\mathbf{a}_p(k)}\right) L^*(F_k, \chi, j) & \textit{if } F_k \textit{ is p-new,} \\ \left(1 - \dfrac{\chi(p)p^{j-1}}{\mathbf{a}_p(k)}\right)\left(1 - \dfrac{\chi(p)p^{k-j+1}}{\mathbf{a}_p(k)}\right) L^*(F^\sharp_k, \chi, j) & \textit{if } F_k \textit{ is p-old.} \end{cases}$$

whenever $1 \le j \le k+1$ and the parity condition (5) *holds.*

Remark 1.2. For each $k \in \Omega_{\rm cl}$, there is a unique normalization of $L_{\mathbf{F},\chi}$ such that $\lambda^\pm(k) = 1$. When $\mathbf{F}$ has a p-new specialization at k_0 we normalize $L_{\mathbf{F},\chi}$ such that $\lambda^\pm(k_0) = 1$.

The function $L_{\mathbf{F},\chi}$ is called the 2-variable p-adic L-function associated to the Coleman family $\mathbf{F}$ and the character χ. It's construction is due to Mazur (unpublished notes) and Kitagawa [24] when the family $\mathbf{F}$ has slope zero, and to Stevens (see [3], [29] and [30]) in the general case. Considering the restriction

$$\mathcal{L}_{\mathbf{F},\chi}(k) := L_{\mathbf{F},\chi}(k, k/2+1), \qquad k \in \Omega, \tag{6}$$

of $L_{\mathbf{F},\chi}$ to the *critical line* $k = 2j-2$ we see that

$$\frac{\mathcal{L}_{\mathbf{F},\chi}(k)}{\lambda^\pm(k)} = \begin{cases} \left(1 - \dfrac{\chi(p)p^{k/2}}{\mathbf{a}_p(k)}\right) L^*(F_k, \chi, k/2+1) & \text{if } F_k \text{ is } p\text{-new,} \\ \left(1 - \dfrac{\chi(p)p^{k/2}}{\mathbf{a}_p(k)}\right)^2 L^*(F^\sharp_k, \chi, k/2+1) & \text{if } F_k \text{ is } p\text{-old.} \end{cases} \tag{7}$$

It is possible that $\mathcal{L}_{\mathbf{F},\chi}(k)$ is identically zero: Let $\omega_{N,k}$ be the eigenvalue of the Atkin-Lehner involution W_N acting on $F^\sharp_k$. The completed L-function

$$\Lambda(F^\sharp_k, \chi, s) := (2\pi)^{-s}\Gamma(s)L(F^\sharp_k, \chi, s)$$

satisfies the functional equation

$$\Lambda(F^\sharp_k, \chi, s) = (-1)^{k/2+1}\chi(-N)\omega_{N,k}(\mathfrak{f}(\chi)^2N)^{k/2+1-s}\Lambda(F^\sharp_k, \chi, k+2-s),$$

implying

$$\Lambda(F^\sharp_k, \chi, k/2+1) = (-1)^{k/2+1}\chi(-N)\omega_{N,k}\Lambda(F^\sharp_k, \chi, k/2+1).$$

It can be shown that for $k, k' \in \Omega_{\rm cl}$,

$$\omega_{N,k'} = (-1)^{(k+k')/2}\omega_{N,k}.$$

Therefore,

$$\omega_N := (-1)^{k/2+1}\omega_{N,k}$$

is independent of k and the sign in the functional equation of $\Lambda(F_k^\sharp, \chi, s)$ is $\chi(-N)\omega_N$. If $\chi(-N)\omega_N = -1$, then $L(F_k^\sharp, \chi, k/2+1) = 0$ for all k and $\mathcal{L}_{\mathbf{F},\chi}$ is identically zero by (7). It is therefore natural, studying $\mathcal{L}_{\mathbf{F},\chi}$, to work under the assumption

$$\chi(-N)\omega_N = +1. \tag{8}$$

Even though we exclude the possibility of (7) forcing $\mathcal{L}_{\mathbf{F},\chi}$ to be identically zero, it is still possible that the interpolation formula imposes an isolated zero on $\mathcal{L}_{\mathbf{F},\chi}$, namely, when the Euler-like factor

$$\left(1 - \frac{\chi(p)p^{k/2}}{\mathbf{a}_p(k)}\right)^2$$

vanishes. Suppose there is a weight $k_0 \in \Omega_{\mathrm{cl}}$ such that

$$\chi(p) = \frac{\mathbf{a}_p(k_0)}{p^{k_0/2}} = -\omega_{p,k_0}. \tag{9}$$

(There is at most one such k_0 by the constancy of the slope of $\mathbf{F}$.) By the above discussion together with our assumption that χ is quadratic, this can happen only when F_{k_0} is p-new. When this does happen, though, the quantity $\mathcal{L}''_{\mathbf{F},\chi}(k_0/2+1)$ is related to Heegner cycles. This result, which is proved in [4] and generalized to our setting in [38], is recalled in Theorem 1.4 below. We introduce the terminology necessary to state this relationship precisely.

We need to assume the existence of an auxiliary factorization $N = MQ$ with M and Q coprime and Q squarefree and with an *odd* number of primes factors. For such a decomposition to exist, we need the existence of a prime $q \parallel N$, in which case we may take $Q = q$. Let B_{Qp} be the unique indefinite quaternion algebra of discriminant Qp and write $S_{k_0+2}(\Gamma_0^{Qp}(M))$ for the space of weight k_0+2-modular forms on B_{Qp} of level $\Gamma_0^{Qp}(M)$. (See §3.1 for the notation.)

Let $\mathcal{M}_{k_0}$ (resp. $\mathcal{M}_{k_0}^{Qp}$) be the Chow motive associated to the space $S_{k_0+2}(\Gamma_0(Np))$ (resp. $S_{k_0+2}(\Gamma_0^{Qp}(M))$) and let $V(Np)$ (resp. $V^{Qp}(M)$) be its p-adic étale realization, viewed as a continuous, $\mathbb{Q}_p$-adic representation of $G_{\mathbb{Q}}$. By the Jacquet-Langlands correspondence, the Eichler Shimura relations and the Brauer-Nesbitt principle (see for example [22, Lemma 5.9]) there is an identification, both Hecke and Galois equivariant, of $V(Np)^{Qp\text{-new}} \simeq V^{Qp}(M)$. It induces an identification $V(Np)^{\mathrm{new}} \simeq V^{Qp}(M)^{\mathrm{new}}$ between the associated new parts. We fix such an identification once and for all and simply write V for either of these two identified representations.

The new subspace $S_{k_0+2}(\Gamma_0(Np))^{\mathrm{new}}$ has a natural $\mathbb{Q}$-structure, preserved by the Hecke operators, arising from q-expansion. Write $S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\mathrm{new}}$ for the $\mathbb{Q}_p$-space obtained by base change and let $\mathbb{T}_{\mathbb{Q}_p}^{\mathrm{new}}$ be the $\mathbb{Q}_p$-algebra of $S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\mathrm{new}}$ generated by all the Hecke operators. By Fontaine's theory, we may associate to (the restriction to a decomposition group at p of) V a filtered Frobenius module with a monodromy operator

$$D := D_{\mathrm{st}}(V)$$

which is a $\mathbb{T}_{\mathbb{Q}_p}^{\mathrm{new}}$-monodromy module defined over $\mathbb{Q}_p$ by results of [9]. Then D is a free module over $\mathbb{T}_{\mathbb{Q}_p}^{\mathrm{new}}$ of rank two. Let $\mathcal{L}$ be the Fontaine-Mazur $\mathcal{L}$-invariant of D (as defined in [26]) and let $(\cdot)^\vee$ mean $\mathbb{Q}_p$-dual in the following theorem.

Theorem 1.3. *There is an isomorphism of $\mathbb{T}_{\mathbb{Q}_p}^{new}$-monodromy modules defined over $\mathbb{Q}_p$*

$$D \xrightarrow{\sim} S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{new,\vee} \oplus S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{new,\vee}$$

under which the only non-trivial step in the filtration, namely F^jD for $j = 1, \ldots, k_0-1$, maps isomorphically onto

$$\{(-\mathcal{L}x, x) : x \in S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{new,\vee}\}.$$

Set $m = k_0/2+1$. If E is a finite extension of $\mathbb{Q}_p$, then the Bloch-Kato exponential gives an isomorphism

$$\exp : \frac{D \otimes E}{F^m(D \otimes E)} \xrightarrow{\sim} H^1_{\mathrm{st}}(E, V), \tag{10}$$

where

$$H^1_{\mathrm{st}}(E,V(m))=\ker\left(H^1(E,V(m))\longrightarrow H^1(E,V(m)\otimes\mathbf{B}_{\mathrm{st}})\right).$$

Theorem 1.3 gives an isomorphism

$$\frac{D\otimes E}{F^m(D\otimes E)}\overset{\sim}{\longrightarrow} S_{k_0+2}(\Gamma_0(Np),E)^{\mathrm{new},\vee},$$

where now $(\cdot)^\vee$ means E-dual. Composing this map with $\exp^{-1}$, we obtain a isomorphism

$$\log: H^1_{\mathrm{st}}(E,V(m))\longrightarrow S_{k_0+2}(\Gamma_0(Np),E)^{\mathrm{new},\vee}.$$

On the global side, we have the p-adic étale Abel-Jacobi map

$$\mathrm{cl}:=\mathrm{cl}^{k_0/2+1}_{0,L}:\mathrm{CH}^{k_0/2+1}_0(\mathcal{M}^{Qp}_{k_0}\otimes L)\longrightarrow H^1(L,V)$$

for any number field $L\subset\overline{\mathbb{Q}}$, where the Chow groups are taken with rational coefficients. Here, the i-Chow group of a motive $M:=(X,p,m)$, where X is a smooth scheme, p is an idempotent and m is an integer, is defined to be $CH^i(M):=Hom(\mathbf{1},M(i))$; if $m=0$, this is simply the p-component of the Chow group $CH^i(X)$. The subscript 0 denotes the subgroup of cycles that are homologically equivalent to zero. It is known that the image of cl is contained in the in the semistable Bloch-Kato Selmer group $\mathrm{Sel}_{\mathrm{st}}(L,V)$. (See (6.5) for its definition.) Let $\mathfrak{p}$ be the prime of $L\subset\overline{\mathbb{Q}}$ over p determined by the embedding $\sigma_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$ and let $L_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion. Then we may consider the composite $\log\mathrm{cl}$

$$\mathrm{CH}^{k_0/2+1}_0(\mathcal{M}^{Qp}_{k_0}\otimes L)\longrightarrow\mathrm{Sel}_{\mathrm{st}}(L,V(m))\longrightarrow H^1_{\mathrm{st}}(L_{\mathfrak{p}},V(m))\longrightarrow S_{k_0+2}(\Gamma_0(Np),L_{\mathfrak{p}})^{p\text{-new},\vee},$$

where $(\cdot)^\vee$ means $L_{\mathfrak{p}}$-dual.

Whenever $\mathcal{L}_{\mathbf{F},\chi}$ vanishes at k_0 to order at least two, $\mathcal{L}''_{\mathbf{F},\chi}(k_0)$ does not depend on the family $\mathbf{F}$ through F_{k_0} (see Remark 5.9). This double vanishing occurs when (8) and (9) hold, by results of [4] generalized in [38] to our setting. If $[F_{k_0}]$ is the companion (also called Galois conjugacy) class of F_{k_0}, the $[F_{k_0}]$-isotypic component $S_{k_0+2}(\Gamma_0(Np),\mathbb{C}_p)_{[F_{k_0}]}$, which is the sum of all $\sigma(F_{k_0})$-isotypic components for all $\sigma\in G_{\mathbb{Q}}$, descends to $\mathbb{Q}_p$ (and, indeed, to $\mathbb{Q}$). In other words, there is a space of modular forms $S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}_p)_{[F_{k_0}]}$ whose base change to $\mathbb{C}_p$ is $S_{k_0+2}(\Gamma_0(Np),\mathbb{C}_p)_{[F_{k_0}]}$ (see [32, §4.3]). We note that (8) and (9) are really conditions on $[F_{k_0}]$, because $\omega_N,\omega_{p,k_0}=\omega_{p,F_{k_0}}\in\mathbb{Q}$ and χ takes its values in $\{\pm1\}\subset\mathbb{Q}$. In particular, it makes sense to consider the subspace

$$S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}_p)^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}\subset S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}_p)^{\mathrm{new}}$$

spanned by new forms on which (8) and (9) hold. Similarly, we may consider the $\mathbb{Q}_p$-adic representation $V_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ (resp. $V_{[F_{k_0}]}$) of $G_{\mathbb{Q}}$ cut out by the conditions (8) and (9) (resp. attached to $F_{k_0}\in S_{k_0+2}(\Gamma_0(Np),\mathbb{C}_p)^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$). Let $\mathbb{Q}^\chi$ be the quadratic extension cut out by the character χ and let $\mathbb{Q}^\chi_{\mathfrak{p}}$ be its $\mathfrak{p}$-adic completion. Then $F_{k_0}\mapsto\mathcal{L}''_{\mathbf{F},\chi}(k_0)$ may be viewed as a $\mathbb{C}_p$-valued function on $S_{k_0+2}(\Gamma_0(Np),\mathbb{C}_p)^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ that we denote $\mathcal{L}''_\chi(k_0)$. Indeed, it restricts on $S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}^\chi_{\mathfrak{p}})^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ to a $\mathbb{Q}^\chi_{\mathfrak{p}}$-valued function

$$\mathcal{L}''_\chi(k_0): S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}^\chi_{\mathfrak{p}})^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}\to\mathbb{Q}^\chi_{\mathfrak{p}}$$

and similarly for its restriction to the $[F_{k_0}]$-component $S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}^\chi_{\mathfrak{p}})_{[F_{k_0}]}$. Let $(\cdot)^\chi$ denote the χ-component. Projecting onto the new subspaces where (8) and (9) hold (resp. the $[F_{k_0}]$-component), we may consider the composite $\log\mathrm{cl}(\cdot)|_{S^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}}$ (resp. $\log\mathrm{cl}(\cdot)|_{S_{[F_{k_0}]}}$):

$$\begin{aligned}&\mathrm{CH}^{k_0/2+1}_0(\mathcal{M}^{Qp}_{k_0}\otimes\mathbb{Q}^\chi)^\chi\longrightarrow\mathrm{Sel}_{\mathrm{st}}(\mathbb{Q}^\chi,V^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}(m))\longrightarrow H^1_{\mathrm{st}}(\mathbb{Q}^\chi_{\mathfrak{p}},V^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}(m))\\&\quad\longrightarrow S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}^\chi_{\mathfrak{p}})^{\mathrm{new},\vee}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p},\\&\mathrm{CH}^{k_0/2+1}_0(\mathcal{M}^{Qp}_{k_0}\otimes\mathbb{Q}^\chi)^\chi\longrightarrow\mathrm{Sel}_{\mathrm{st}}(\mathbb{Q}^\chi,V_{[F_{k_0}]}(m))\longrightarrow H^1_{\mathrm{st}}(\mathbb{Q}^\chi_{\mathfrak{p}},V_{[F_{k_0}]}(m))\longrightarrow S_{k_0+2}(\Gamma_0(Np),\mathbb{Q}^\chi_{\mathfrak{p}})^\vee_{[F_{k_0}]}.\end{aligned}$$

Let $\mathbb{T}^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ be the semisimple rational Hecke algebra of $S_{k_0+2}(\Gamma_0(Np))^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ and let $\mathbb{T}_{[F_{k_0}]}$ its projection onto the $[F_{k_0}]$-component for any $F_{k_0}\in S_{k_0+2}(\Gamma_0(Np),\mathbb{C}_p)^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$. Note that $\lambda_{F_{k_0}} : \mathbb{T}_{[F_{k_0}]} \simeq \mathbb{Q}(F_{k_0})$ for any $F_{k_0} \in [F_{k_0}]$, where $\lambda_{F_{k_0}}$ is the eigenpacket of F_{k_0}. In particular, $\sigma \circ \lambda_{F_{k_0}} = \lambda_{\sigma(F_{k_0})}$ for any $\sigma \in G_{\mathbb{Q}}$.

The following theorem is proved in [4], in the special case $k_0 = 0$, when one restricts to the subspace of $S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\text{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ spanned by modular forms with rational Fourier coefficients. It is proved in the general case in [38].

Theorem 1.4. *Suppose that there is a prime q with $q \parallel N$ and choose a factorization $N = MQ$ as above. Then the following facts hold.*

(1) *There is a cycle $y^\chi = y^\chi_{Qp} \in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes \mathbb{Q}^\chi)^\chi$ and $t = t_\chi^{Qp} \in \mathbb{T}^{new,\times}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ such that*

$$\mathcal{L}''_\chi(k_0) = t \cdot \log^2 \mathrm{cl}\,(y^\chi)\,|_{S^{new}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}}.$$

(2) *If $0 \neq \mathrm{cl}\,(y^\chi)_{[F_{k_0}]} \in \mathrm{Sel}_{st}(\mathbb{Q}^\chi, V_{[F_{k_0}]}(m))$, then*

$$\mathrm{Sel}_{st}(\mathbb{Q}^\chi, V_{[F_{k_0}]}(m))^\chi = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\,(y^\chi)_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p,$$

where $\mathrm{cl}\,(y^\chi)_{[F_{k_0}]}$ is the $[F_{k_0}]$-component of $\mathrm{cl}\,(y^\chi)$.

(3) *Suppose $k_0 = 0$. Then $0 \neq \mathrm{cl}\,(y^\chi)_{[F_{k_0}]}$ if and only if $L'(F_{k_0}, \chi, k_0/2+1) \neq 0$.*

(4) *If $t_{F_{k_0}} := \lambda_{F_{k_0}}(t) \in \mathbb{Q}(F_{k_0})$, then for any quadratic Dirichlet character ϵ such that*

$$\epsilon(N) = \chi(N),\ \epsilon(p) = -\chi(p) \text{ and } L(F_{k_0}, \epsilon, k_0/2+1) \neq 0,$$

the congruence

$$t_{F_{k_0}} \equiv L(F_{k_0}, \epsilon, k_0/2+1) \text{ in } \mathbb{Q}(F_{k_0})^\times / \mathbb{Q}(F_{k_0})^{\times 2}.$$

holds.

Proof. We simply remark that an inspection to the proof of [38, Theorem 6.1] shows that the theorem is true with our slightly more general condition (4). We also note that the case $k_0 = 0$ appears in detail in [4] only for modular forms with rational Fourier coefficients; however a generalization of this result to $S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\text{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ is possible, thanks to [4, Introduction, Remark 5]. See [20] for more details. □

1.3. **Main results I: interpolation and arithmetic applications.** Let $K \subset \overline{\mathbb{Q}}$ be a real quadratic field whose discriminant d_K is prime to Np. We make the following convenient but unnatural assumption, which can likely be removed with more effort:

Assumption 1.5. *If a prime ℓ divides N and is inert in K, then* $\mathrm{ord}_\ell(N) = 1$.

We introduce some notation concerning the field K. We write $N = N^+D$ as the product of the primes N^+ that are split in K and those primes D that are inert. Let $\mathfrak{p}$ be the prime ideal of K above p determined by σ_p. If p splits in K, let $\mathfrak{p}'$ be the other prime of K above p. The embedding σ_∞ also picks out a real embedding of K which we will also denote by σ_∞. Let H_K (resp. H_K^+) be the Hilbert class field (resp. narrow Hilbert class field) of K. The extension H_K^+/H_K has degree one or two. If the degree is two, let s denote the nontrivial element of $\mathrm{Gal}(H_K^+/H_K)$. If $H_K^+ = H_K$, let s denote the identity element of $\mathrm{Gal}(H_K^+/H_K)$.

Let

$$\psi : \mathrm{Gal}(H_K^+/K) \longrightarrow \overline{\mathbb{Q}}^\times$$

be a character. In this paper, we prove interpolation results in the spirit of Theorems 1.1 and 1.4 for central critical Rankin L-values $L(F_k^\sharp/K, \psi, k/2+1)$. Prototypes of these results have been proved by several authors [4, 5, 38, 41]. The goal of this paper is to unify the methods of these papers into a cohomological framework, simultaneously simplifying the treatment and generalizing the results. After stating our results, we will point out the precise overlap between these and the results of [4, 5, 38, 41].

Let ϵ_K be the Dirichlet character associated to K. If $k \neq k_0$ is in Ω_{cl}, then the completed twisted L-functions of $F_k^\sharp$ over K satisfies the functional equation

$$\Lambda(F_k^\sharp/K, \psi, k+2-s) = \epsilon_K(-N)\Lambda(F_k^\sharp/K, \psi, s).$$

Thus, the central critical values $L(F_k^\sharp/K,\psi,k)$ vanish for all $k\in\Omega_{\mathrm{cl}}$, $k\neq k_0$, when $\epsilon_K(-N)=-1$. Thus, to avoid interpolating the zero function for reasons of signs, we work under the assumption that $\epsilon_K(-N)=+1$. From here, our analysis falls into two cases as $\epsilon_K(p)=+1$ or $\epsilon_K(p)=-1$. Note that in the latter case, $\epsilon_K(-Np)=-1$ and $L(F_{k_0}/K,\psi,k/2+1)=0$. In either case, we have

$$L^*(F_k^\sharp/K,\psi,k/2+1):=\frac{(k/2)!\sqrt{d_K}}{(2\pi i)^k u^\pm(k)^2}L(F_k^\sharp/K,\psi,k/2+1)\in\mathbb{Q}(\psi,F_k^\sharp), \tag{11}$$

The sign of the Shimura period is $(-1)^{k/2}\psi(s)$. Our main interpolation result is the following:

Theorem 1.6. *There is a unique rigid analytic function $\mathcal{L}_{\mathbf{F}/K,\psi}\in\mathcal{O}(\Omega)$ such that for all $k\in\Omega_{cl}^{p\text{-}old}$,*

$$\frac{\mathcal{L}_{\mathbf{F}/K,\psi}(k)}{\lambda^\pm(k)^2}=\begin{cases}\left(1-\dfrac{p^k}{\mathbf{a}_p(k)^2}\right)^2 L^*(F_k^\sharp/K,\psi,k/2+1) & \text{if } \epsilon_K(p)=-1,\\[2ex] \left(1-\dfrac{\psi(\mathfrak{p})p^{k/2}}{\mathbf{a}_p(k)}\right)^2\left(1-\dfrac{\psi(\mathfrak{p}')p^{k/2}}{\mathbf{a}_p(k)}\right)^2 L^*(F_k^\sharp/K,\psi,k/2+1) & \text{if } \epsilon_K(p)=+1.\end{cases}$$

Moreover, if $\epsilon_K(p)=-1$ (resp. $\epsilon_K(p)=+1$) and F_{k_0} is p-new (resp. F_{k_0} is new at p and $\psi(\mathfrak{p})=\mathbf{a}_p(k_0)/p^{k/2}$, then $\mathcal{L}_{\mathbf{F}/K,\psi}$ vanishes at k_0 to order at least two.

Remark 1.7. Theorem 1.6 is expected to hold for all ring class characters with conductor prime to d_KNp. The proof, however, relies on a special value formula of [31] that is proved only in the special case of unramified characters.

One observes an intesting phenomenon when ψ is a *genus character of* K, i.e., a quadratic character of $\mathrm{Gal}(H_K^+/K)$. The genus character ψ corresponds to a factorization $d_K=d_{K_1}d_{K_2}$ of d_K into two fundamental discriminants, corresponding to the fields K_1 and K_2, say, and a pair of Dirichlet characters

$$\chi_i:(\mathbb{Z}/d_{K_i}\mathbb{Z})^*\to\{\pm1\}$$

satisfying $\chi_1\chi_2=\epsilon_K$ and $\chi_i(\mathrm{Norm}(\mathfrak{q}))=\psi(\mathfrak{q})$ for all degree-one primes of K. In addition, one can also easily establish the following factorization formula for central critical values:

$$L^*(F_k^\sharp/K,\psi,k/2+1)=L^*(F_k^\sharp,\chi_1,k/2+1)L^*(F_k^\sharp,\chi_2,k/2+1),\qquad k\in\Omega_{\mathrm{cl}}. \tag{12}$$

Combining this factorization formula with (7) and Theorem 1.6, we obtain:

Corollary 1.8. *The following factorization holds on Ω:*

$$\mathcal{L}_{\mathbf{F}/K,\psi}=\mathcal{L}_{\mathbf{F},\chi_1}\cdot\mathcal{L}_{\mathbf{F},\chi_2}.$$

Combining Theorem 1.4 with Corollary 1.8 yields interesting arithmetic consequences. Suppose that F_{k_0} is p-new and that $\varepsilon_K(p)=-1$ (resp. $\varepsilon_K(p)=1$). Under our Assumption 1.5 and $\epsilon_K(-N)=1$, the complex L-functions $L(F_{k_0},\chi_i,s)$ have opposite signs (resp. the same sign) at $s=k_0/2+1$, $i=1,2$, equal to

$$(-1)^{k_0/2+1}\omega_{pN,k_0}\chi_i(-pN)=\omega_N\chi_i(-N)\,\omega_{p,k_0}\chi_i(p). \tag{13}$$

Note that, under our running Assumption 1.5 and $\varepsilon_K(-N)=\varepsilon_K(N)=1$,

$$\omega_N\chi_i(-N)=\omega_N\psi(\infty\mathfrak{n})=\omega_N\psi\left(\infty\mathfrak{n}^+\right),$$

where $\mathfrak{n}$ (resp. $\mathfrak{n}^+$) is a prime of K above N (resp. N^+), and ∞ is the class of complex conjugation in $G_{H_K/K}$. In particular, $\omega_N\chi_i(-N)$ does not depend on $i=1,2$, and (8) holds for χ_i. It follows from (13) that, in this case,

$$\text{sign } L(F_{k_0},\chi_i,s)=-1 \text{ is negative} \iff \text{(9) holds for } \chi_i.$$

Let H_K^ψ be the quadratic extension of K cut out by the character ψ, and let $H_{K,\mathfrak{p}}^\psi$ and $(\cdot)^\psi$ have the same meaning as above with $L=H_K^\psi$. Suppose that W is a $G_{H_K^\psi/\mathbb{Q}}$-module; since $\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\psi)=\chi_1\oplus\chi_2$, we have $W^\psi=W^{\chi_1}\oplus W^{\chi_2}$, where the left hand side is viewed as a $G_{H_K^\psi/K}$-module, while the right hand side is viewed as a $G_{H_K^\psi/\mathbb{Q}}$-module. This remark applies to $W = \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes H_K^\psi)$ and $W = \mathrm{Sel}_{\mathrm{st}}(H_K^\psi, V')$ for any $\mathbb{Q}_p$-adic representation of $G_\mathbb{Q}$, and gives

$$\begin{aligned}&\mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes H_K^\psi)^\psi = \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes \mathbb{Q}^{\chi_1})^{\chi_1} \oplus \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes \mathbb{Q}^{\chi_2})^{\chi_2},\\ &\mathrm{Sel}_{\mathrm{st}}(H_K^\psi, V') = \mathrm{Sel}_{\mathrm{st}}(\mathbb{Q}^{\chi_1}, V')^{\chi_1} \oplus \mathrm{Sel}_{\mathrm{st}}(\mathbb{Q}^{\chi_2}, V')^{\chi_2}.\end{aligned} \tag{14}$$

In case $\varepsilon_K(p) = -1$, we may order (χ_1, χ_2) in such a way that the sign of $L(F_{k_0}, \chi_1, s)$ is -1. The conjectures of Bloch and Beilinson predict that the corresponding Bloch-Kato Selmer groups have positive dimension. We expect them to be partially explained by Theorem 1.4, as it is shown in the subsequent Corollary 1.9. Indeed we see that conditions (8) and (9) on χ_1 are compatible with the above sign condition on $L(F_{k_0}, \chi_1, s)$. The factorization formula of Corollary 1.8, joint with (8) and (9) on χ_1, implies that $\mathcal{L}_{\mathbf{F}/K,\psi}$ vanishes at k_0 to order at least two. Similarly as above, $\mathcal{L}''_{\mathbf{F}/K,\psi}(k_0)$ does not depend on the lift $\mathbf{F}$ of F_{k_0} to an eigenfamily $\mathbf{F}$ (see Remark 5.9). In this case, $F_{k_0} \mapsto \mathcal{L}''_{\mathbf{F}/K,\psi}(k_0)$ gives rise to a function

$$\mathcal{L}''_\psi(k_0) : S_{k_0+2}(\Gamma_0(Np), H_{K,\mathfrak{p}}^\psi)^{\mathrm{new}}_{\chi(-N)=\omega_N, \chi(p)=-\omega_p} \to H_{K,\mathfrak{p}}^\psi$$

and we may consider the analogous maps $\log \mathrm{cl}(\cdot)|_{S^{\mathrm{new}}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}}$ and $\log \mathrm{cl}(\cdot)|_{S_{[F_{k_0}]}}$ for $L = H_K^\psi$ and coefficients in $H_{K,\mathfrak{p}}^\psi$. We write $\mathcal{L}''_{[F_{k_0}],\psi}(k_0)$ for the restriction of $\mathcal{L}''_\psi(k_0)$ to $S_{[F_{k_0}]} = S_{k_0+2}(\Gamma_0(Np), H_{K,\mathfrak{p}}^\psi)_{[F_{k_0}]}$.

Corollary 1.9. *Suppose that there is a prime q with $q \parallel N$ and choose a factorization $N = MQ$ as above. Let K be such that $\epsilon_K(p) = -1$ and $\epsilon_K(-N) = +1$. Let ψ be a genus character of K associated to the Dirichlet characters (χ_1, χ_2), ordered in such a way that the sign of $L(F_{k_0}, \chi_1, k_0/2+1)$ is negative. Suppose further that $\omega_N \chi_i(-N) = 1$ for one (or equivalently both) $i \in \{1, 2\}$. Then the following facts hold.*

(1) *There is a cycle*

$$y^\psi = y^\psi_{Qp} \in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes \mathbb{Q}^{\chi_1})^{\chi_1} \subset \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes H_K^\psi)$$

and a $t = t_\psi^{Qp} \in \mathbb{T}^{new,\times}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}$ such that

$$\mathcal{L}''_\psi(k_0) = t \cdot \log^2 \mathrm{cl}\left(y^\psi\right)|_{S^{new}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}}.$$

(2) *If $0 \neq \mathrm{cl}\left(y^\psi\right)_{[F_{k_0}]} \in \mathrm{Sel}_{st}(H_K^\psi, V_{[F_{k_0}]}(m))$,*

$$\mathrm{Sel}_{st}(\mathbb{Q}^{\chi_1}, V_{[F_{k_0}]}(m))^{\chi_1} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(y^\psi\right)_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p,$$

where $\mathrm{cl}\left(y^\psi\right)_{[F_{k_0}]}$ is the $[F_{k_0}]$-component of $\mathrm{cl}\left(y^\psi\right)$. If, further, we assume $L(F_{k_0}, \chi_2, k_0/2+1) \neq 0$ (equivalently, if $L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$),

$$\mathrm{Sel}_{st}(H_K^\psi, V_{[F_{k_0}]}(m))^\psi = \mathrm{Sel}_{st}(\mathbb{Q}^{\chi_1}, V_{[F_{k_0}]}(m))^{\chi_1} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(y^\psi\right)_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p.$$

(3) *Suppose $k_0 = 0$. Then $0 \neq \mathrm{cl}\left(y^\psi\right)_{[F_{k_0}]}$ if and only if $L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$.*

(4) *If $L(F_{k_0}, \chi_2, k_0/2+1) \neq 0$ (e.g., if $L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$), there is*

$$\widetilde{y}^\psi \in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0} \otimes \mathbb{Q}^{\chi_1})^{\chi_1} \subset \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H_K^\psi)$$

such that

$$\mathcal{L}''_{[F_{k_0}],\chi}(k_0) = 2\,\mathrm{cl}\left(\widetilde{y}^\psi\right)^2_{[F_{k_0}]}$$

and such that, if $0 \neq \mathrm{cl}\left(\widetilde{y}^\psi\right)_{[F_{k_0}]} \in \mathrm{Sel}_{st}(H_K^\psi, V_{[F_{k_0}]}(m))$,

$$\mathrm{Sel}_{st}(H_K^\psi, V_{[F_{k_0}]}(m))^\psi = \mathrm{Sel}_{st}(\mathbb{Q}^{\chi_1}, V_{[F_{k_0}]}(m))^{\chi_1} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(\widetilde{y}^\psi\right)_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p.$$

If, in addition, $k_0 = 0$, then $0 \neq \mathrm{cl}\left(\widetilde{y}^\psi\right)_{[F_{k_0}]}$ if and only if $L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$.

Proof. We may work with the $[F_{k_0}]$-component for all $F_{k_0} \in S_{k_0+2}(\Gamma_0(Np), \mathbb{C}_p)^{\text{new}}_{\chi_1(-N)=\omega_N, \chi_1(p)=-\omega_p}$. Thanks to (10), $H^1_{\text{st}}(H^\psi_{K,\mathfrak{p}}, V_{[F_{k_0}]}(m)) \subset H^1_{\text{st}}(E, V_{[F_{k_0}]}(m))$ for any finite extension $E/H^\psi_{K,\mathfrak{p}}$. In particular, the validity of parts (1) and (4) of the corollary are unaffected by viewing $\mathcal{L}''_\psi(k_0)$ and $\mathcal{L}''_{[F_{k_0}],\chi}(k_0)$ as E-valued functionals. Assuming that E contains the field generated by the Fourier coefficients of all $f \in [F_{k_0}]$ (via σ_p), $S_{k_0+2}(\Gamma_0(Np), E)_{[F_{k_0}]}$ decomposes as the direct sum of its λ-components on which $\mathbb{T}_{[F_{k_0}],p} := \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p$ acts through λ, for all $\lambda \in \text{Hom}_{\mathbb{Q}_p\text{-alg}}\left(\mathbb{T}_{[F_{k_0}],p}, E\right)$. There is no loss of generality in assuming that λ is obtained by means of the identification $\lambda_{F_{k_0}} : \mathbb{T}_{[F_{k_0}]} \simeq \mathbb{Q}(F_{k_0})$ followed by σ_p, and then passing to the completion. In this case, the λ-component of $\mathcal{L}''_{[F_{k_0}],\psi}(k_0)$ is $\mathcal{L}''_{\mathbf{F}/K,\psi}(k_0)$. It follows from Corollary 1.8, Theorem 1.1 and Theorem 1.4 applied to χ_1, that there exist $y^\psi = y^{\chi_1} \in \text{CH}_0^{k_0/2+1}(\mathcal{M}^{Qp}_{k_0} \otimes \mathbb{Q}^{\chi_1})^{\chi_1}$ and $t_{\chi_1} \in \mathbb{T}^{new,\times}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}$ such that, setting $t_{\chi_1,F_{k_0}} := \lambda\left(t_{\chi_1}\right)$,

$$\mathcal{L}''_{\mathbf{F}/K,\psi}(k_0) = \mathcal{L}''_{\mathbf{F},\chi_1}(k_0) \cdot \mathcal{L}_{\mathbf{F},\chi_2}(k_0) = t_{\chi_1,F_{k_0}} \sigma_p\left(2L^*(F_{k_0}, \chi_2, k_0/2+1)\right) \cdot \log \text{cl}\left(y^\psi\right)^2 (F_{k_0}).$$

Since the quantity $t_{F_{k_0}} := \sigma_p^{-1}\left(t_{\chi_1,F_{k_0}}\right) 2L^*(F_{k_0}, \chi_2, k_0/2+1) \in \mathbb{Q}(F_{k_0})$ satisfies $t_{\sigma(F_{k_0})} = \sigma\left(t_{F_{k_0}}\right)$ for all $\sigma \in G_{\mathbb{Q}}$, there is $t_{[F_{k_0}]} \in \mathbb{T}_{[F_{k_0}]}$ inducing $t_{\sigma(F_{k_0})}$ whenever $\lambda = \sigma_p \circ \lambda_{\sigma(F_{k_0})}$, where $\lambda_{\sigma(F_{k_0})} : \mathbb{T}_{[F_{k_0}]} \simeq \mathbb{Q}(\sigma(F_{k_0}))$ is the homomorphism determined by the eigenpacket $\lambda_{\sigma(F_{k_0})}$ of $\sigma(F_{k_0})$. Claim (1) follows. If $L(F_{k_0}/K, \chi_2, k_0/2+1) \neq 0$, then χ_2 satisfies the assumptions on ϵ appearing in Theorem 1.4 (4). It follows that $t_{F_{k_0}}/2 \in \mathbb{Q}(F_{k_0})^{\times 2}$. Since the Hecke action on the λ-component is through $\lambda = \sigma_p \circ \lambda_{F_{k_0}}$, our claimed equation in (4) follows setting $\widetilde{y}^\psi := \widetilde{\sqrt{t_{F_{k_0}}/2}}\, y^\psi$, where $\widetilde{\sqrt{t_{F_{k_0}}/2}}$ is any lift of $\sqrt{t_{F_{k_0}}/2} \in \mathbb{Q}(F_{k_0})^\times$ to a Hecke operator acting on the Chow groups. The second part of (4) follows from (2), (3), the definition of $\widetilde{y}^\psi$ and the fact that $L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$ if and only if $L'(F_{k_0}, \chi_1, k_0/2+1)$ when $L(F_{k_0}, \chi_2, k_0/2+1) \neq 0$.

The first assertion of (2) follows from Theorem 1.4 (2). The second assertion follows from the implication

$$L(F_{k_0}/K, \chi_2, k_0/2+1) \neq 0 \implies \text{Sel}_{\text{st}}(\mathbb{Q}^{\chi_2}, V_{[F_{k_0}]}(m))^{\chi_2} = 0,$$

proved in [23, Theorem 14.2 (2)], and (14). Part (3) is a restatement of Theorem 1.4 (3). □

We now assume $\varepsilon_K(p) = 1$. If $L(F_{k_0}, \chi_i, s)$ for $i = 1, 2$ have negative sign, $L'(F_{k_0}/K, \psi, k_0/2+1) = 0$ and we expect to have larger rank, again to be partially explained by Theorem 1.4, joint with the factorization (14), which is the cohomological version of the factorization of the complex L-functions. We already remarked that condition (8) on χ_i does not depend on $i = 1, 2$; as we assume $\varepsilon_K(p) = 1$, we see that the same is true for (9) relative to χ_i. In particular, (8) and (9) that are required for the application of Theorem 1.4 are simultaneously satisfied and $S_{k_0+2}(\Gamma_0(Np), H^\psi_{K,\mathfrak{p}})^{\text{new}}_{\chi_i(-N)=\omega_N,\chi_i(p)=-\omega_p}$ does not depend on $i = 1, 2$. It then follows from Corollary 1.8 that $\mathcal{L}_{\mathbf{F}/K,\psi}$ vanishes at k_0 to order at least two. Similarly as above, $\mathcal{L}^{(4)}_{\mathbf{F}/K,\psi}(k_0)$ does not depend on the lift $\mathbf{F}$ of F_{k_0} to an eigenfamily $\mathbf{F}$ (see Remark 5.9) and we may consider

$$\mathcal{L}^{(4)}_\psi(k_0) : S_{k_0+2}(\Gamma_0(Np), H^\psi_{K,\mathfrak{p}})^{\text{new},\vee}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p} \to H^\psi_{K,\mathfrak{p}}.$$

Corollary 1.10. *Suppose that there is a prime q with $q \parallel N$ and choose a factorization $N = MQ$ as above. Let K be such that $\epsilon_K(p) = +1$ and $\epsilon_K(-N) = +1$. Let ψ be a genus character of K associated to the Dirichlet characters (χ_1, χ_2) and suppose that the sign of $L(F_{k_0}, \chi_i, k_0/2+1)$ is negative for one (or equivalently both) $i \in \{1, 2\}$ and that $\omega_N \chi_i(-N) = 1$ for one (or equivalently both) $i \in \{1, 2\}$*

(1) *There exist cycles*

$$y_i^\psi = y^\psi_{Qp,i} \in \text{CH}_0^{k_0/2+1}(\mathcal{M}^{Qp}_{k_0} \otimes \mathbb{Q}^{\chi_i})^{\chi_i} \subset \text{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H^\psi_K)$$

and there exist $t_i = t^{Qp}_{i,\chi_i} \in \mathbb{T}^{new,\times}_{\chi_i(-N)=\omega_N,\chi_i(p)=-\omega_p}$, $i = 1, 2$, such that

$$\mathcal{L}^{(4)}_\psi(k_0) = 6t_1t_2 \cdot \log \text{cl}\left(y_1^\psi\right)^2_{|S^{new}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}} \log \text{cl}\left(y_2^\psi\right)^2_{|S^{new}_{\chi_1(-N)=\omega_N,\chi_1(p)=-\omega_p}}$$

(2) *If* $0 \neq \mathrm{cl}\left(y_i^{\psi}\right)_{[F_{k_0}]} \in \mathrm{Sel}_{st}(H_K^{\psi}, V_{[F_{k_0}]}(m))$,

$$\mathrm{Sel}_{st}(\mathbb{Q}^{\chi_i}, V_{[F_{k_0}]}(m))^{\chi_i} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(y_i^{\psi}\right)_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p,$$

where $\mathrm{cl}\left(y_i^{\psi}\right)_{[F_{k_0}]}$ *is the* $[F_{k_0}]$*-component of* $\mathrm{cl}\left(y_i^{\psi}\right)$*. If* $0 \neq \mathrm{cl}\left(y_i^{\psi}\right)_{[F_{k_0}]}$ *for* $i = 1, 2$

$$\mathrm{Sel}_{st}(H_K^{\psi}, V_{[F_{k_0}]}(m))^{\psi} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(y_1^{\psi}\right)_{[F_{k_0}]} \oplus \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot \mathrm{cl}\left(y_2^{\psi}\right)_{[F_{k_0}]} \simeq \left(\mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}\right)_p^2.$$

(3) *Suppose* $k_0 = 0$*. Then* $0 \neq \mathrm{cl}\left(y_i^{\psi}\right)_{[F_{k_0}]}$ *if and only if* $L'\left(F_{k_0}, \chi_i, k_0/2+1\right) \neq 0$*. Furthermore, they are both non-trivial if and only if* $L''\left(F_{k_0}/K, \psi, k_0/2+1\right) \neq 0$*.*

(4) *If* $t_{i,F_{k_0}} := \lambda_{F_{k_0}}(t_i) \in \mathbb{Q}(F_{k_0})$*, then for any quadratic Dirichlet character* ϵ *such that*

$$\epsilon(N) = \chi_i(N), \ \epsilon(p) = -\chi_i(p) \ \text{ and } L\left(F_{k_0}, \epsilon, k_0/2+1\right) \neq 0,$$

the congruence

$$t_{i,F_{k_0}} \equiv L\left(F_{k_0}, \epsilon, k_0/2+1\right) \ \text{ in } \mathbb{Q}\left(F_{k_0}\right)^{\times} / \mathbb{Q}\left(F_{k_0}\right)^{\times 2}.$$

holds.

Proof. As in the proof of Corollary 1.9, we fix F_{k_0} and a sufficiently large p-adic field E. It follows from Corollary 1.8 and Theorem 1.4 applied to χ_i, that there exist $y_i^{\psi} = y^{\chi_i} \in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes \mathbb{Q}^{\chi_i})^{\chi_i}$ and $t_i \in \mathbb{T}^{\mathrm{new},\times}_{\chi_i(-N)=\omega_N, \chi_i(p)=-\omega_p}$ such that, setting $t_{i,F_{k_0}} := \lambda_{F_{k_0}}(t_i)$,

$$\mathcal{L}^{(4)}_{\mathbf{F}/K,\psi}(k_0) = 6\mathcal{L}''_{\mathbf{F},\chi_1}(k_0) \cdot \mathcal{L}''_{\mathbf{F},\chi_2}(k_0) = 6 t_{1,F_{k_0}} t_{2,F_{k_0}} \cdot \log \mathrm{cl}\left(y_1^{\psi}\right)^2 (F_{k_0}) \log \mathrm{cl}\left(y_2^{\psi}\right)^2 (F_{k_0}).$$

The claim is now a restatement of Theorem 1.4. ∎

Remarks 1.11.

(1) When $k_0 = 0$, the image of the cycles appearing in Theorem 1.4, Corollary 1.9 and Corollary 1.10 factors through the appropriate Mordell-Weil group $A(L) \otimes \mathbb{Q}$ (see [20] for more details). After extending coefficients to $\mathbb{Q}_p$, their local restrictions factor through $A(L_{\mathfrak{p}}) \otimes \mathbb{Q}_p$ and the Bloch-Kato logarithm is compatible, up to the Kummer map, with the usual p-adic logarithm. Hence, in this case, our p-adic L-functions really control elements coming from

$$A(L) \otimes \mathbb{Q} \hookrightarrow A(L_{\mathfrak{p}}) \otimes \mathbb{Q}_p.$$

(2) When $\varepsilon_K(p) = -1$ and $D = 1$, Theorem 1.6, Corollary 1.8 and Corollary 1.9 were proved in [4], when $k_0 = 0$ for modular forms with rational Fourier coefficients, and in [38] for arbitrary even $k_0 \geq 0$. When $\varepsilon_K(p) = 1$ and $D = 1$, Theorem 1.6, Corollary 1.8 and Corollary 1.10 were proved in [41], when $k_0 = 0$ for modular forms with rational Fourier coefficients. The novelty of our methods, which allows us to work simultaneously in the case where D may be different from 1 and k_0 may be greater than 0, resides in two resources. The use of purely cohomological methods and, respectively, the use of Ash and Stevens results on slope decompositions.

1.4. **Main results II: the connection with Darmon cycles.** Suppose that p is inert in K, so that K_p is the unramified quadratic extension of $\mathbb{Q}_p$. Consider the Dp-new quotient $V(pN)^{Dp\text{-new}}$ of $V(pN)$. When $L = H_K^{\psi}$, σ_p induces $H_K^{\psi} \hookrightarrow K$ and the restriction map takes the form

$$\mathrm{res}_p : \mathrm{Sel}_{\mathrm{st}}(H_K^{\psi}, V(pN)^{Dp\text{-new}}(m)) \to H^1_{\mathrm{st}}(K_p, V(pN)^{Dp\text{-new}}(m)).$$

With ψ as above, the conjectures of Beilinson and Bloch, in conjunction with the conjectures of Bloch-Kato (see [8, Conjecture 5.15]), predict that

$$\dim_{\mathbb{T}_{[F_{k_0}]}} \mathrm{Sel}(H_K^{\psi}, V_{[F_{k_0}]}(m))^{\psi} = \mathrm{ord}_{s=k_0/2+1} L(F_{k_0}/K, \psi, s).$$

In particular, if $L(F_{k_0}/K, \psi, s)$ has sign -1, as ensured by Assumption 1.5 and the conditions $\epsilon_K(-N) = 1$ and $\epsilon_K(p) = -1$, then one expects $\mathrm{Sel}_{\mathrm{st}}(H_K^{\psi}, V(pN)^{Dp\text{-new}}(m))$ to be nonzero. In this situation, methods have been devised to construct local classes in $H^1_{\mathrm{st}}(K_p, V(pN)^{Dp\text{-new}}(m))$ that are conjectured to lie in the image of $\mathrm{Sel}_{\mathrm{st}}(H^{\psi}_K, V(pN)^{Dp\text{-new}}(m))$ under res_p (see [12], [18] and [32]). Since their construction is based on techniques of Darmon, we follow [32] in calling these elements of $H^1_{\mathrm{st}}(K_p, V(pN)^{Dp\text{-new}}(m))$ *Darmon classes*. In §6.4, we give evidence that the Darmon classes are indeed images of global Galois cohomology classes in $\mathrm{Sel}_{\mathrm{st}}(H^{\psi}_K, V(pN)^{Dp\text{-new}}(m))$, up to restricting to the new part, in the case where ψ is a genus character of K. Thanks to the Atkin-Lehner theory, there is no loss of generality in our restriction to the new part (see for example [40, §6.2], where the proof of the Teitelbaum conjecture is reduced its proof for the new part). As the proof will show, these global cohomology classes come from an appropriate rational Chow group. In some sense, this fact presents a slight strengthening of the conjectures formulated in [32], where only the more abstract $\mathrm{Sel}_{\mathrm{st}}$ group is involved. In particular, when $k_0 = 0$, the above Remark 1.11 applies and, as noticed after Theorem 6.11, we really get the rationality result at the level of Mordell-Weil groups.

This generalizes work of Bertolini and Darmon [5] under the hypotheses $k_0 = 0$, $D = 1$, and $\mathbb{Q}(f) = \mathbb{Q}$, and of Seveso [38] in the case $D = 1$. Results of a similar nature have been obtained simultaneously and independently by Longo and Vigni under the hypotheses $k_0 = 0$ and $\mathbb{Q}(f) = \mathbb{Q}$ (see [25]).

1.5. **Construction of p-adic L-functions.** Modular symbols are the main tool used in the construction of cyclotomic p-adic L-functions associated to modular forms [27]. In unpublished work, Stevens developed a theory of p-adic families of modular symbols that he applied to the construction of cyclotomic 2-variable p-adic L-functions. We describe some aspects of Stevens' theory since analogous ideas will be employed below.

Remark 1.12. Stevens' techniques can be readily adapted to define p-adic families of automorphic forms on definite quaternion algebras. These families, together with Chenevier's p-adic Jacquet-Langlands correspondence for definite quaternion algebras, can be used to construct 2-variable anticyclotomic p-adic L-functions. This is carried out in [4, 38].

Let Y be the $\mathbb{Q}_p$-manifold $\mathbb{Z}_p^\times \times \mathbb{Z}_p$(*see* [34, §9]) and let $\mathcal{D}(Y)$ be the space of E-locally analytic distributions on Y (as in [34, §11]). The diagonal action of $\mathbb{Z}_p^\times$ on Y endows $\mathcal{D}(Y)$ with the structure of a $\mathcal{D}(\mathbb{Z}_p^\times)$-Fréchet module. By the theorem of Amice and Vélu, the convolution algebra $\mathcal{D}(\mathbb{Z}_p^\times)$ is isomorphic to $\mathcal{O}(\mathcal{X})$, the coordinate ring of the weight space $\mathcal{X}$. If Ω is a subspace of $\mathcal{X}$,

$$\mathcal{D}(Y)_\Omega := \mathcal{O}(\Omega)\widehat{\otimes}_{\mathcal{O}(\mathcal{X})}\mathcal{D}(Y)$$

is a Fréchet $\mathcal{O}(\Omega)$-module. We define the space $\mathrm{Symb}_{\Gamma_0(Np)}\mathcal{D}(Y)_\Omega := Hom_{\Gamma_0(Np)}\left(\mathrm{Div}^0\,\mathbb{P}^1(\mathbb{Q}), \mathcal{D}(Y)_\Omega\right)$ to be the *space of Ω-families of modular symbols*. The family of norms defining the Fréchet $\mathcal{O}(\Omega)$-module structure on $\mathcal{D}(Y)_\Omega$ are $\Gamma_0(Np)$-invariant. Since $\mathrm{Div}^0\,\mathbb{P}^1(\mathbb{Q})$ is a finitely generated $\Gamma_0(Np)$-module, $\mathrm{Symb}_{\Gamma_0(Np)}\mathcal{D}(Y)_\Omega$ becomes an $\mathcal{O}(\Omega)$-Fréchet module as well. A key fact is that the operator U_p acts completely continuously on this space. Therefore, Coleman's theory of slope decompositions applies. It follows that there is an open affinoid Ω neighbourhood of k_0 in $\mathcal{X}$ and a Hecke-eigenvector $\Psi \in \mathrm{Symb}_{\Gamma_0(Np)}\mathcal{D}(Y)_\Omega$ whose weight $k_0 + 2$ specialization $\phi := \Psi_{k_0} \in \mathrm{Symb}_{\Gamma_0(Np)} V_{k_0}$ has the same system of Hecke eigenvalues as f. The eigenvector Ψ is the cohomological version of the Coleman family $\mathbf{F}$.

The p-adic families which we need to consider in this paper are Hecke eigenvectors Φ in $H^1(\Gamma_0, \mathcal{D}(Y)_\Omega)$ whose weight $k_0 + 2$ specializations $\varphi := \Phi_{k_0}$ have the same system of Hecke eigenvalues as f. Here, Γ_0 is a group of quaternionic units – see §2. The technical difficulty which arises in this situation is that $H^1(\Gamma_0, \mathcal{D}(Y)_\Omega)$ seems not to have a natural Fréchet module structure, making Coleman's theory of slope decompositions inapplicable. This difficulty has been resolved by Ash and Stevens in [1]. By applying Coleman's theory on the level of cochains with respect to a resolution consisting of finitely generated Γ_0-modules, they prove that the cohomology groups of Γ_0 possess canonical slope decompositions. This is the key step which allows for the construction of p-adic deformations of cohomology classes. Based on the very general machinery developed in [1], these issues are discussed in more detail in [19].

With an eigenvector $\Phi \in H^1(\Gamma_0, \mathcal{D}(Y)_\Omega)$ in hand, we construct p-adic L-functions $\mathcal{L}_{\Phi/K,\psi}$ using a combination of the methods of [4, 5, 38, 41] and group-cohomological techniques of the sort employed in [15, 18]. Note, however, that we have subscripted these p-adic L-functions with a Φ instead of with an $\mathbf{F}$. In order to justify changing this subscript, we must show that the specializations of Φ in classical weights correspond to those of $\mathbf{F}$ under the Jacquet-Langlands correspondence. This result, interesting in its own right, is proved in [19]. With this compatibility in hand, the interpolation property of $\mathcal{L}_{\mathbf{F}/K,\psi} = \frac{\mathcal{L}_{\Phi/K,\psi}}{d_K^{\kappa/2}\nu}$ (see the end of §5.1 for the definition of ν) is deduced in §5.1 and §5.2 from Popa's formula relating the central critical values $L(F_k^\sharp, \chi, k/2+1)$ to integrals of certain real-analytic cycles on Shimura curves.

Part 1. Real-analytic cycles on Shimura curves and special values of L-functions

2. Quaternions

2.1. Splittings and orders. We recall the following assumption on the real quadratic field K. Let

$$D = \prod(\ell : \ell \mid N,\ \epsilon_K(\ell) = -1).$$

By Assumption 1.5 and our running hypothesis that $\epsilon_K(-N) = +1$, D has an even number of prime factors. Therefore, there is a unique indefinite quaternion $\mathbb{Q}$-algebra B of discriminant D. Let $^* : B \to B$ be the involution and let $\mathrm{nrd} : B^\times \to \mathbb{Q}^\times$ denote the reduced norm: $\mathrm{nrd}(x) = xx^*$. When $B = \mathbf{M}_2(\mathbb{Q})$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \mathrm{nrd}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By construction, K is a splitting field of B, i.e., there is an isomorphism

$$\iota : B \otimes_{\mathbb{Q}} K \longrightarrow \mathbf{M}_2(K). \tag{15}$$

If $D = 1$, we choose ι to be the canonical isomorphism

$$\iota : \mathbf{M}_2(\mathbb{Q}) \otimes_{\mathbb{Q}} K \longrightarrow \mathbf{M}_2(K).$$

Let $\mathfrak{p}$ denote the prime ideal of $\mathcal{O}_K$ corresponding to the valuation on K induced by the embedding σ_p of (2). If $\epsilon_K(p) = 1$, we write $\mathfrak{p}'$ for the other prime ideal of $\mathcal{O}_K$ above p. Write $\iota_{\mathfrak{p}}$ for the composite

$$B \longrightarrow \mathbf{M}_2(K) \longrightarrow \mathbf{M}_2(K_{\mathfrak{p}}) = \begin{cases} \mathbf{M}_2(\mathbb{Q}_p) & \text{if } \epsilon_K(p) = +1, \\ \mathbf{M}_2(\mathbb{Q}_{p^2}) & \text{if } \epsilon_K(p) = -1. \end{cases}$$

(Here, $\mathbb{Q}_{p^2}$ denotes the quadratic, unramified extension of $\mathbb{Q}_p$.) In either case, the image of B is contained in $M_2(\mathbb{Q}_p)$, and $\iota_{\mathfrak{p}}$ induces an isomorphism

$$\iota_{\mathfrak{p}} : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow \mathbf{M}_2(\mathbb{Q}_p).$$

For an ideal I of a ring A, let

$$\mathcal{R}_0(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) : c \in I \right\}.$$

If $\mathfrak{m}$ is an ideal of $\mathcal{O}_K$, we set

$$R_0^D(\mathfrak{m}) = \{x \in B : \iota(x) \in \mathcal{R}_0(\mathfrak{m})\}.$$

The embedding $\iota_{\mathfrak{p}}$ induces an isomorphism

$$\iota_{\mathfrak{p}} : R_0^D(\mathfrak{m}) \otimes \mathbb{Z}_p \longrightarrow \mathcal{R}_0(p^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})}\mathbb{Z}_p).$$

Set

$$\Gamma_0^D(\mathfrak{m}) = \ker\left(\mathrm{nrd} : R_0^D(\mathfrak{m})^\times \longrightarrow \{\pm 1\}\right).$$

Let

$$N^+ = N/D.$$

Since $\epsilon_K(\ell) = +1$ for all $\ell \mid N^+$, we may choose an ideal $\mathfrak{n}^+$ of $\mathcal{O}_K$ of norm N^+. Of particular interest are the rings R and R_0. They are Eichler orders in B of levels N^+ and N^+p, respectively. We will use the following shorthand:

$$R = R_0^D(\mathfrak{n}^+), \qquad R_0 = R_0^D(\mathfrak{n}^+\mathfrak{p}), \qquad \Gamma = \Gamma_0^D(\mathfrak{n}^+), \qquad \Gamma_0 = \Gamma_0^D(\mathfrak{n}^+\mathfrak{p}). \tag{16}$$

The order R_0 contains a unique bilateral ideal of norm p, and this ideal is principal. Let $w_p \in R_0$ be a generator. It is characterized by the properties that

$$\mathrm{nrd}(w_p) = p, \qquad \iota_{\mathfrak{p}}(w_p) \equiv \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \pmod{p}. \tag{17}$$

2.2. **Embeddings and orientations.** Let $\mathcal{O} \subset K$ be an order in K, of conductor prime to $\mathfrak{m}D$ and the discriminant of K. We say that an embedding

$$j : K \to B$$

is an *optimal embedding* of $\mathcal{O}$ of level M if $j^{-1}\left(R_0^D(\mathfrak{m})\right) = \mathcal{O}$, where M is a positive generator of $\mathfrak{m} \cap \mathbb{Z}$, and let $\mathcal{E}(\mathcal{O}, R_0^D(M))$ be the set of such oriented embeddings. A necessary and sufficient condition for such embeddings to exist is that the primes dividing M are split in K (together with our running assumption on the primes dividing D). We assume that this is the case. Note that, replacing $\mathfrak{m}$ with an ideal $\widetilde{\mathfrak{m}} \mid M$ such that $\mathcal{O}/\mathfrak{m} \simeq \mathcal{O}/\widetilde{\mathfrak{m}}$, the order $R_0^D(M) := R_0^D(\mathfrak{m}) = R_0^D(\widetilde{\mathfrak{m}})$ is unchanged, thus justifying our notation for the set of optimal embeddings.

We say that $j \in \mathcal{E}(\mathcal{O}, R_0^D(M))$ is $\mathfrak{m}$-*oriented* if the diagram

(18)
$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{j} & R_0^D(\mathfrak{m}) & \xrightarrow{\iota} & \mathcal{R}_0(\mathfrak{m}) \\ & \searrow & & & \downarrow \\ & & & & \mathcal{O}/\mathfrak{m} \end{array}$$

commutes, where the vertical arrow is the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto a$ and the diagonal arrow is the natural projection. For a prime $l \mid D$, let B_l be the l-adic completion of B, and fix once and for all an identification $B_l/\mathfrak{m}_{B_l} = \mathbb{F}_{l^2}$, where $\mathfrak{m}_{B_l}$ is the unique maximal bilateral ideal of the unique maximal order $\mathcal{O}_{B_l}$ of B_l. Note that $\mathrm{Hom}_{\mathbb{Z}_l\text{-alg}}(\mathcal{O}, \mathbb{F}_{l^2}) = \left\{\delta_l^{\pm}\right\}$ is the set with two elements. Any $j \in \mathcal{E}(\mathcal{O}, R_0^D(M))$ induces $\mathfrak{d}_l(j) : \mathcal{O} \to \mathbb{F}_{l^2}$ by means of the identification $B_l/\mathfrak{m}_{B_l} = \mathbb{F}_{l^2}$. If $\mathfrak{d} = (\delta_l^{\varepsilon_l})_{l \mid D}$ with $\varepsilon_l \in \{\pm 1\}$ is a choice of homomorphisms one for every $l \mid D$, we say that $j \in \mathcal{E}(\mathcal{O}, R_0^D(M))$ is $\mathfrak{d}$-*oriented* if $\mathfrak{d}_l(j) = \mathfrak{d}_l$ for every l, where $\mathfrak{d}_l := \delta_l^{\varepsilon_l}$ is the l-component of $\mathfrak{d}$. Let $\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$ be the subset of such optimal $\mathfrak{m}\mathfrak{d}$-oriented embeddings. Clearly, $R_0^D(M)^\times$ acts on $\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$ by conjugation on the target.

Let $\mathrm{Cl}^+(\mathcal{O})$ be the narrow ideal class group of $\mathcal{O}$. There is a faithful, transitive action of $\mathrm{Cl}^+(\mathcal{O})$ on $\Gamma_0^D(M)\backslash\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$. Let $\mathfrak{b}$ be an ideal of $\mathcal{O}$ relatively prime to $D\mathfrak{m}$ and the conductor of $\mathcal{O}$. Then $R_0^D(\mathfrak{m})j(\mathfrak{b})$ is an invertible left ideal of an indefinite quaternion order, and thus is principal, say $R_0^D(\mathfrak{m})j(\mathfrak{b}) = R_0^D(\mathfrak{m})b$. By the norm theorem, $R_0^D(\mathfrak{m})$ contains elements of norm -1, so we may assume that $\mathrm{nrd}(b) > 0$. Since $j(\mathcal{O})j(\mathfrak{b})R_0^D(\mathfrak{m}) \subset j(\mathfrak{b})R_0^D(\mathfrak{m})$, the image $j(\mathcal{O})$ is contained in the left order of $j(\mathfrak{b})R_0^D(\mathfrak{m})$. But this left order is just $bR_0^D(\mathfrak{m})b^{-1}$, so we conclude that $j(\mathcal{O}) \subset b^{-1}R_0^D(\mathfrak{m})b$. Therefore, we may define

$$([\mathfrak{b}] \cdot j) : \mathcal{O} \longrightarrow R_0(\mathfrak{m}), \qquad ([\mathfrak{b}] \cdot j)(x) = bj(x)b^{-1}.$$

One may show that $[\mathfrak{b}] \cdot j \in \mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$ and that the class of $[\mathfrak{b}] \cdot j$ in $\Gamma_0^D(M)\backslash\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$ depends only on $[\mathfrak{b}]$.

For a prime $l \mid MD$, let $w_l \in R_0^D(M)$ be an element of norm l; it normalizes $\Gamma_0^D(M) := \Gamma_0^D(\mathfrak{m}) = \Gamma_0^D(\mathfrak{m}')$. In particular, it induces by conjugation a well defined action on $\Gamma_0^D(M)\backslash\mathcal{E}(\mathcal{O}, R_0^D(M))$. If $t \mid MD$ is a squarefree integer, we may uniquely write $\mathfrak{m} = \mathfrak{t}\mathfrak{t}_c$, where $\mathfrak{t}$ and $\mathfrak{t}_c$ are coprime, $\mathfrak{t} \cap \mathbb{Z} = t\mathbb{Z}$, and $D = D_t D_{t,c}$, where D_t and $D_{t,c}$ are coprime and $D_t = \gcd(D, t)$. Define $w_t := \prod_{l \mid t} w_l$ and let W_t be the action induced by w_t on $\Gamma_0^D(M)\backslash\mathcal{E}(\mathcal{O}, R_0^D(M))$. Set $W_t(\mathfrak{m}) := \bar{\mathfrak{t}}\mathfrak{t}_c$, where $\bar{\mathfrak{t}}$ is the conjugate of $\mathfrak{t}$ by the non-trivial automorphism τ of $K/\mathbb{Q}$, and $W_t(\mathfrak{d}) = \left((\bar{\mathfrak{d}}_l)_{l \mid D_t}, (\mathfrak{d}_l)_{l \mid D_{t,c}}\right)$, where $\bar{\mathfrak{d}}_l := \mathfrak{d}_l \circ \tau$. Let $w \in R_0^D(M)$ be an element of reduced norm -1. Again, w normalizes $\Gamma_0^D(M)$ and we let T_∞ be the corresponding operator on $\Gamma_0^D(M)$-equivalence classes. The element τ induces an action by the rule $\tau(j) = \bar{j} := j \circ \tau$. We let $\mathcal{W}$ be the group generated by the involutions W_t and T_∞.

If $j \in \mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M))$, there exists a unique extension $j_\mathfrak{p}$ of j such that the following diagram commutes:

(19)
$$\begin{array}{ccccc} \mathcal{O}_K \otimes \mathbb{Z}_p & & & & \\ \uparrow & & & \overset{j_\mathfrak{p}}{\searrow} & \\ \mathcal{O}_K & \xrightarrow[j]{} & R_0^D(\mathfrak{m}) & \xrightarrow[\iota_\mathfrak{p}]{} & \mathcal{R}_0(p^{\mathrm{ord}_\mathfrak{p}(\mathfrak{m})}\mathbb{Z}_p) \end{array}$$

When $R_0(\mathfrak{m}) = R_0(\mathfrak{n}^+)$ (resp. $R_0(\mathfrak{n}^+\mathfrak{p})$) we write $\mathcal{E}^*(\mathcal{O}, R)$ (resp. $\mathcal{E}^*(\mathcal{O}, R_0)$) for the set of such embeddings, where $*$ may be empty or $\mathfrak{m}\mathfrak{d}$.

In §3 we will need to consider a slightly different type of embeddings. Suppose that p is inert and set

$$\widetilde{\mathcal{O}} := \mathcal{O}\,[1/p]\,,\; \widetilde{R} := R\,[1/p] = R_0\,[1/p] \text{ and } \widetilde{\Gamma} := \widetilde{R}_1^{\times},$$

where $\widetilde{R}_1^{\times}$ is the subgroup of norm one elements in $\widetilde{R}^{\times}$. We define the set of optimal embeddings of $\widetilde{\mathcal{O}}$ of level M, that we write $\mathcal{E}(\widetilde{\mathcal{O}}, \widetilde{R})$, by the requirement $j^{-1}(\widetilde{R}) = \widetilde{\mathcal{O}}$, while the subset of optimal $\mathfrak{n}^+\mathfrak{d}$-oriented embeddings is defined in exactly the same way.

We recall how to define an orientation at p. Let $\mathcal{T}$ be the Bruhat-Tits tree attached to $\mathbf{GL}_2(\mathbb{Q}_p)$, whose set of vertices we denote by $\mathcal{V}$. Set $L_* := \mathbb{Z}_p^2$ and $v_* := [L_*] \in \mathcal{V}$ and write $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) for the set of those $v \in \mathcal{V}$ that lie at an even (resp. odd) distance from v_*. We let $\mathbf{GL}_2(\mathbb{Q}_p)$ act from the left on $\mathcal{V}$ by viewing the elements of $\mathbb{Q}_p^2$ as column vectors. If $j \in \mathcal{E}(\widetilde{\mathcal{O}}, \widetilde{R})$, the group $K^{\times}$ acts on $\mathcal{V}$ by means of $j_p := j_{\mathfrak{p}}$ with a unique fixed point $v_j \in \mathcal{V}$. Let $\mathcal{E}_{\pm}^{\mathfrak{n}^+\mathfrak{d}}(\widetilde{\mathcal{O}}, \widetilde{R})$ be the set of optimal $\mathfrak{n}^+\mathfrak{d}$-oriented embeddings such that $v_j \in \mathcal{V}^{\pm}$. Again there is a faithful, transitive action of $\mathrm{Cl}^+(\mathcal{O})$ on $\widetilde{\Gamma}\backslash\mathcal{E}_{\pm}^{\mathfrak{n}^+\mathfrak{d}}(\widetilde{\mathcal{O}}, \widetilde{R})$, as well as involutions W_t for every squarefree integer $t \mid pN^+D$, an involution T_∞ and an action of τ, all defined in the same way as above. We define signs $W_t(\pm) = \mp$ if $p \mid t$ and $W_t(\pm) = \pm$ otherwise.

We record in the following lemma some basic facts about embeddings and orientations, whose proof we leave to the reader.

Lemma 2.1.

(1) *The* $\mathrm{Cl}^+(\mathcal{O})$*-actions commute with the* $\mathcal{W}$*-actions. We have* $\overline{W_t \cdot [j]} = W_t \cdot \overline{[j]}$, $\overline{T_\infty \cdot [j]} = T_\infty \cdot \overline{[j]}$ *and* $\overline{[\mathfrak{b}] \cdot j} = \overline{[\mathfrak{b}]} \cdot \overline{j}$. *The involution* W_t *induces bijections*

$$W_t \;:\; \Gamma_0^D(M)\backslash\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M)) \to \Gamma_0^D(M)\backslash\mathcal{E}^{W_t(\mathfrak{m}\mathfrak{d})}(\mathcal{O}, R_0^D(M)),$$
$$W_t \;:\; \widetilde{\Gamma}\backslash\mathcal{E}_{\pm}^{\mathfrak{n}^+\mathfrak{d}}(\widetilde{\mathcal{O}}, \widetilde{R}) \to \widetilde{\Gamma}\backslash\mathcal{E}_{W_t(\pm)}^{W_t(\mathfrak{n}^+\mathfrak{d})}(\widetilde{\mathcal{O}}, \widetilde{R}),$$

T_∞ *preserves orientation and* τ *induces bijections*

$$\tau \;:\; \Gamma_0^D(M)\backslash\mathcal{E}^{\mathfrak{m}\mathfrak{d}}(\mathcal{O}, R_0^D(M)) \to \Gamma_0^D(M)\backslash\mathcal{E}^{W_{MD}(\mathfrak{m}\mathfrak{d})}(\mathcal{O}, R_0^D(M)),$$
$$\tau \;:\; \widetilde{\Gamma}\backslash\mathcal{E}_{\pm}^{\mathfrak{n}^+\mathfrak{d}}(\widetilde{\mathcal{O}}, \widetilde{R}) \to \widetilde{\Gamma}\backslash\mathcal{E}_{W_{MD}(\pm)}^{W_{MD}(\mathfrak{n}^+\mathfrak{d})}(\widetilde{\mathcal{O}}, \widetilde{R}).$$

(2) *For every* $[j] \in \Gamma_0^D(M)\backslash\mathcal{E}(\mathcal{O}, R_0^D(M))$ *there is a unique* $\sigma_{[j]} \in \mathrm{Cl}^+(\mathcal{O})$ *such that* $W_{MD}\overline{[j]} = \sigma_{[j]}\,[j]$ *and* $\sigma_{[\mathfrak{b}]\cdot[j]} = \sigma_{[j]}[\mathfrak{b}]^{-2}$. *In particular the image* σ *of* $\sigma_{[j]}$ *in* $\mathrm{Cl}^+(\mathcal{O})\,/\,\mathrm{Cl}^+(\mathcal{O})^2$ *is a well defined element. If* $\mathcal{O} = \mathcal{O}_K$, *then* $\sigma = \infty\mathfrak{m}\mathfrak{d}$ *in* $\mathrm{Cl}^+(\mathcal{O})\,/\,\mathrm{Cl}^+(\mathcal{O})^2$, *where* ∞ *is the class of complex conjugation in* $\mathrm{Cl}^+(\mathcal{O})$ *and* $\mathfrak{d}$ *is the unique squarefree ideal of* $\mathcal{O}$ *dividing* D. *Hence, for a genus character* ψ *attached to the Dirichlet characters* (χ_1, χ_2), *the value*

$$\psi\left(\sigma_{[j]}\right) = \psi\left(\infty\mathfrak{m}\mathfrak{d}\right) = \chi_i\left(-MD\right)$$

does not depend on $[j]$.

(3) *Suppose that* p *splits (resp. is inert) in* K. *Then the natural inclusion induces a bijection, commuting with all the actions described above,*

$$\Gamma_0\backslash\mathcal{E}^{\mathfrak{n}^+\mathfrak{d}\mathfrak{p}}(\mathcal{O}, R_0) \xrightarrow{\sim} \Gamma\backslash\mathcal{E}^{\mathfrak{n}^+\mathfrak{d}}(\mathcal{O}, R) \text{ (resp. } \Gamma\backslash\mathcal{E}^{\mathfrak{n}^+\mathfrak{d}}(\mathcal{O}, R) \xrightarrow{\sim} \mathcal{E}_{+}^{\mathfrak{n}^+\mathfrak{d}}(\widetilde{\mathcal{O}}, \widetilde{R})\text{)}.$$

Suppose p splits in K and that $j \in \mathcal{E}(\mathcal{O}_K, R_0)$. It follows from (18) that

$$j(\mathfrak{p}) \subset \left\{\alpha \in R : \iota_{\mathfrak{p}}(\alpha) \equiv \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \pmod{p}\right\}. \tag{20}$$

The set on the right is a left R-ideal of norm p, generated by P, say. Since R contains elements of norm -1, we may assume that $\operatorname{nrd} P = p$. Finally, it's worth noting that the ideal $Rj(\mathfrak{p})$ does not depend on j so long as the image of j is contained in R_0.

Again assume that p splits in K and that $j \in \mathcal{E}(\mathcal{O}_K, R_0)$. Let P_j be a generator of $R_0 j(\mathfrak{p}')$ as a left R_0-ideal. (Unlike $Rj(\mathfrak{p})$, the ideal $R_0 j(\mathfrak{p}')$ does depend on j.) By reasoning similar to that of the last paragraph,

$$R_0 j(\mathfrak{p}') = R_0 P_j \subset \left\{ \alpha \in R_0 : \iota_{\mathfrak{p}}(\alpha) \equiv \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \right\}.$$

Although the set on the right is a left R_0-ideal, it is not invertible as its localization at p is not principal. Thus, $R_0 P_j$ is an invertible subideal of this set. Since the orientation of j implies $R_0 P_j \neq R_0 w_p$, it follows that there is an integer t_j, unique modulo p, such that

$$\mathcal{R}_0(p\mathbb{Z}_p) j_{\mathfrak{p}}(\mathfrak{p}') = \mathcal{R}_0(p\mathbb{Z}_p) \begin{pmatrix} 1 & t_j \\ 0 & p \end{pmatrix}. \tag{21}$$

2.3. Rational representations of $B^\times$.

2.3.1. *The split case.* Consider the matrix algebra $\mathcal{B} := \mathbf{M}_2(E)$ over a field E and let

$$\mathcal{B}^0 = \{\alpha \in \mathcal{B} : \mathrm{trd}(\alpha) = 0\}. \tag{22}$$

Define a left action of $\mathcal{B}^\times$ on $\mathcal{B}^0$ by

$$\alpha \cdot b = \alpha b \alpha^* \tag{23}$$

This action induces a map

$$\widetilde{\ } : \mathcal{B}^\times \longrightarrow \mathbf{GL}(\mathcal{B}^0) \approx \mathbf{GL}_3(E),$$

the so-called symmetric-square lift. Explicitly,

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad+bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}.$$

The matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of $\mathcal{B}^0$. Thus,

$$\mathrm{Sym}^r \mathcal{B}^0 = E[X, Y, Z]_r,$$

where $E[X, Y, Z]_r$ is the space of degree r homogeneous polynomials in X, Y, and Z. $\mathbf{GL}_3(E)$ acts on $E[X, Y, Z]$ by linear change of variable:

$$(A \circ P)(X, Y, Z) = P((X, Y, Z)A).$$

A routine computation shows the actions of $\mathcal{B}^\times$ and $\mathbf{GL}_3(E)$ on $\mathrm{Sym}^r \mathcal{B}^0$ and $E[X, Y, Z]_r$ are compatible with respect to the symmetric-square lift:

$$\alpha \cdot P = \widetilde{\alpha} \circ P, \qquad P \in \mathrm{Sym}^r \mathcal{B}^0 = E[X, Y, Z]_r.$$

For a representation V and an integer m, let $V(m)$ be the representation whose underlying space is V, but with the twisted action defined by the rule $\alpha \cdot v := \mathrm{Nr}(\alpha)^m \alpha v$, where Nr is the reduced norm. We also write E for the trivial representation with undelying vector space the base field E. With these notations we may consider the trace form on $\mathcal{B}$, regarded as a $\mathcal{B}^\times$-representation by (23), as a $\mathcal{B}^\times$-equivariant morphism of representations

$$\langle -, - \rangle : \mathcal{B} \otimes_E \mathcal{B} \to E(2),\ \langle \alpha, \beta \rangle = \mathrm{trd}(\alpha\beta^*) = \mathrm{Tr}(\alpha\beta^*),$$

where Tr denotes the usual trace of a matrix. The trace form induces an orthogonal decomposition

$$\mathcal{B} = E(1) \perp \mathcal{B}^0$$

where $E(1)$ is identified with the space generated by the identity matrix and the restriction of this pairing to $\mathcal{B}^0$ is nondegenerate. The dual basis of $\{X, Y, Z\}$ with respect to this pairing is $\{L_X, L_Y, L_Z\} = \{Y, X, -\frac{1}{2}Z\}$. The trace form induces pairings $\langle \cdot, \cdot \rangle_r$ on $\mathrm{Sym}^r \mathcal{B}^0$ defined by

$$\langle \alpha_1 \cdots, \alpha_r, \beta_1, \cdots \beta_r \rangle_r = \sum_{\sigma \in S_r} \langle \alpha_1, \beta_{\sigma(1)} \rangle \cdots \langle \alpha_n, \beta_{\sigma(n)} \rangle. \tag{24}$$

Using the evaluations

$$\langle X,X\rangle=\langle Y,Y\rangle=\langle X,Z\rangle=\langle Y,Z\rangle=0,\ \langle X,Y\rangle=1,\text{ and }\langle Z,Z\rangle=-2 \tag{25}$$

one deduces that

$$\langle X^iY^jZ^k, X^{i'}Y^{j'}Z^{k'}\rangle_r=\begin{cases}(-2)^k i!j!k!, & \text{if } i=j',\ j=i', \text{ and } k=k',\\ 0, & \text{otherwise.}\end{cases}\qquad (i+j+k=i'+j'+k'=r)$$

In particular, we see that the pairing $\langle\cdot,\cdot\rangle_r$ on $\operatorname{Sym}^r\mathcal{B}^0$ is nondegenerate.

Define the *hyperbolic Laplacian* (or Casimir operator)

$$\Delta : E[X,Y,Z]_r \longrightarrow E[X,Y,Z]_{r-2}(2) \quad\text{by}\quad \Delta P=\frac{\partial}{\partial X}\frac{\partial}{\partial Y}-\frac{\partial^2}{\partial Z^2}.$$

That Δ is $\mathcal{B}^\times$-equivariant follows from standard properties of Casimir operators, or from a direct calculation. The hyperbolic Laplacian admits a coordinate free description, that we will exploit in an essential way in the nonsplit case – see §2.3.2.

Lemma 2.2. *Viewing Δ as a map from $\operatorname{Sym}^r\mathcal{B}^0$ into $\operatorname{Sym}^{r-2}\mathcal{B}^0(2)$, we have*

$$\Delta(P)=\sum_{1\le i<j\le r}\langle\alpha_i,\alpha_j\rangle\alpha_1\cdots\widehat{\alpha}_i\cdots\widehat{\alpha}_j\cdots\alpha_r, \tag{26}$$

where $P=\alpha_1\cdots\alpha_r\in\operatorname{Sym}^r\mathcal{B}^0$.

Proof. Since $\{X,Y,Z\}$ is a basis of $\mathcal{B}^0$, we may assume P has the form

$$P=\underbrace{X\cdots X}_{a}\cdot\underbrace{Y\cdots Y}_{b}\cdot\underbrace{Z\cdots Z}_{c},\qquad a+b+c=r.$$

By (25), the right hand side of (26) equals

$$ab\langle X,Y\rangle\underbrace{X\cdots X}_{a-1}\cdot\underbrace{Y\cdots Y}_{b-1}\cdot\underbrace{Z\cdots Z}_{c}+\binom{c}{2}\langle Z,Z\rangle\underbrace{X\cdots X}_{a}\cdot\underbrace{Y\cdots Y}_{b}\cdot\underbrace{Z\cdots Z}_{c-2}$$
$$=abX^{a-1}Y^{b-1}Z^c-c(c-1)X^aY^bZ^{c-2}=\left(\frac{\partial}{\partial X}\frac{\partial}{\partial Y}-\frac{\partial^2}{\partial Z^2}\right)P.$$ ☐

Define

$$\Delta^* : E[X,Y,Z]_{r-2}(2)\longrightarrow E[X,Y,Z]_r \quad\text{by}\quad \Delta^*Q=(Z^2-4XY)Q.$$

Lemma 2.3. *Δ^* is adjoint to Δ, i.e.,*

$$\langle\Delta P,Q\rangle_{r-2}=\langle P,\Delta^*Q\rangle_r.$$

Proof. Just compute both sides with $P=X^iY^jZ^k$, and $Q=X^{i'}Y^{j'}Z^{k'}$. ☐

Set

$$\mathcal{H}_r=\ker\big(\Delta : E[X,Y,Z]_r\longrightarrow E[X,Y,Z]_{r-2}(2)\big).$$

We call $\mathcal{H}_r$ the space of *hyperbolic harmonic polynomials.*

Proposition 2.4. *We have the following $\mathcal{B}^\times$-invariant orthogonal decompositions:*

$$\operatorname{Sym}^r\mathcal{B}^0\cong\begin{cases}\mathcal{H}_r\perp\mathcal{H}_{r-2}(2)\perp\cdots\perp\mathcal{H}_2(r-2)\perp E(r), & \text{if } r \text{ is even,}\\ \mathcal{H}_r\perp\mathcal{H}_{r-2}(2)\perp\cdots\perp\mathcal{H}_3(r-3)\perp\mathcal{H}_1(r-1), & \text{if } r \text{ is odd.}\end{cases}$$

This decomposition is natural with respect to change of base field.

Proof. By Lemma 2.3,

$$\operatorname{Sym}^r\mathcal{B}^0=\mathcal{H}_r\perp\Delta^*((\operatorname{Sym}^{r-2}\mathcal{B}^0)(2)).$$

By dimension counting, $\Delta^*((\operatorname{Sym}^{r-2}\mathcal{B}^0)(2))\cong(\operatorname{Sym}^{r-2}\mathcal{B}^0)(2)$. The desired decomposition now follows by induction. The naturality is obvious. ☐

We wish to connect the spaces $\mathcal{H}_r$ with more standard models of $\mathcal{B}^\times$-representations. For a row vector $v = (x, y) \in E^2$, we set

$$v^* = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

If $A \in \mathcal{B}$, then

$$(vA)^* = A^* v^*. \tag{27}$$

Let $P_r = P_r(E) = E[x, y]_r$, equipped with the left action of $\mathbf{GL}_2(E)$ defined by

$$(AP)(x, y) = P((x, y)A) = P(ax + cy, bx + dy), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is well-known that $P_r \cong \operatorname{Sym}^r E^2$ is an irreducible representation of $\mathcal{B}^\times$. Define

$$P_- : \mathcal{B}^0 \longrightarrow P_2 \quad \text{by} \quad P_A(x, y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x & y \end{pmatrix}^*.$$

Thus, if $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ then

$$P_A(x, y) = bx^2 - 2axy - cy^2.$$

It is clear from this formula that $A \mapsto P_A$ is a bijection. If follows from (27) that $A \mapsto P_A$ is in fact an isomorphism of left $\mathcal{B}^* = \mathbf{GL}_2(E)$-modules.

Define

$$\pi : \operatorname{Sym}^r \mathcal{B}^0 \cong \operatorname{Sym}^r P_2 \to P_{2r} \quad \text{by} \quad \pi(P_1 \otimes \cdots \otimes P_r) = P_1 \cdots P_r.$$

This map is $\mathcal{B}^\times$-equivariant. Since π is evidently nonzero and has irreducible codomain, it's surjective.

Proposition 2.5. *The map π restricts to an isomorphism of $\mathcal{H}_r$ onto $\mathcal{P}_{2r}$.*

Proof. It is easy to see that $\mathcal{H}_r$ and P_{2r} both have dimension $2r + 1$. Therefore, we need only establish the surjectivity of $\pi|_{\mathcal{H}_r}$. By the irreducibility of P_{2r}, it suffices to show that $\pi|_{\mathcal{H}_r}$ is nonzero. One verifies directly that $X^r \in \ker \Delta = \mathcal{H}_r$. On the other hand, $P_X(x, y) = -x^2$, so $\pi(X^r) = (-1)^r x^{2r} \neq 0$. □

2.3.2. *The nonsplit case.* We now revert to the notation of §2.1. Define B^0, the left action of B^* on B^0, and the trace form on $\operatorname{Sym}^r B^0$ as in (22), (23), and (24), respectively. Define

$$\Delta : \operatorname{Sym}^r B^0 \longrightarrow (\operatorname{Sym}^{r-2} B^0)(2)$$

as in (26). It is $B^\times$-equivariant and respects the trace forms on the domain and codomain. Let $\mathcal{H}_r^B$ be the kernel of Δ. It is easy to check that Δ is natural with respect to change of base field. The following results follow easily from this fact, together with the results of §2.3.1 — just choose a splitting field of B.

Proposition 2.6. *We have the following $\mathcal{B}^\times$-invariant orthogonal decompositions:*

$$\operatorname{Sym}^r B^0 \cong \begin{cases} \mathcal{H}_r^B \perp \mathcal{H}_{r-2}^B(2) \perp \cdots \perp \mathcal{H}_2^B(r-2) \perp E(r), & \text{if } r \text{ is even,} \\ \mathcal{H}_r \perp \mathcal{H}_{r-2}^B(2) \perp \cdots \perp \mathcal{H}_3^B(r-3) \perp \mathcal{H}_1^B(r-1), & \text{if } r \text{ is odd.} \end{cases}$$

$\mathcal{H}_r^B$ is an irreducible representation of $B^\times$.

Let $j : K \to B$ be a $\mathbb{Q}$-algebra embedding. We obtain an induced map

$$j_* : \operatorname{Sym}^r K^0 \longrightarrow \operatorname{Sym}^r B^0,$$

where K^0 is the set of trace-zero elements of K. Let $\delta \in \mathcal{O}_K$ be such that $\delta^2 = d_K$ and $\sigma_\infty(\delta) > 0$. Let

$$\operatorname{pr} : \operatorname{Sym}^r B^0 \longrightarrow \mathcal{H}_r^B$$

be the orthogonal projection arising from the decomposition of Proposition 2.6. We make the following key definition:

$$Q_j^r := \operatorname{pr} j_* \delta^r \in \mathcal{H}_r^B. \tag{28}$$

Since j splits B, it induces an identification

$$\mathcal{H}_r^B \otimes_{\mathbb{Q}} K \cong P_{2r}(K).$$

In what follows, we can unambiguously identify Q_j^r with its image in $P_{2r}(K)$. Our notation is justified by the facts that, in $P_{2r}(K)$ (and over any other splitting field), $Q_j^r = (Q_j)^r$. It is easily checked that the polynomial

Q_j^r may be characterized, up to sign, by the property of being a generator for the one dimensional subspace of $P_{2r}(K)$ on which B acts via nrd^r such that $\langle Q_j^r, Q_j^r\rangle = (-2)^r r! d_K$; the choice $\sigma_\infty(\delta) > 0$ fixes the sign. This characterization descend to $\mathbb{Q}$.

3. Shimura curves

3.1. **Modular forms and Hecke operators.** Let ι_∞ to be the composite

$$\iota_\infty : B \xrightarrow{\iota} \mathbf{M}_2(K) \xrightarrow{\sigma_\infty} \mathbf{M}_2(\mathbb{R}).$$

Via ι_∞, we may view the groups $\Gamma_0^D(\mathfrak{m})$ as a subgroups of $\mathbf{SL}_2(\mathbb{R})$. As such, they act on the complex upper half-plane $\mathfrak{h}$. The quotients

$$X_0^D(\mathfrak{m})(\mathbb{C}) := \Gamma_0^D(\mathfrak{m})\backslash\mathfrak{h}$$

are Riemann surfaces, compact if and only if $D > 1$. As the notation suggests, there are algebraic *Shimura curves* $X_0^D(\mathfrak{m})$ defined over $\mathbb{Q}$ whose loci of complex points are identified with $\Gamma_0^D(\mathfrak{m})\backslash\mathfrak{h}$. Let $k \geq 0$ be an integer and let $S_{k+2}^D(\mathfrak{m})$ be the space of cusp forms for $\Gamma_0^D(\mathfrak{m})$ of weight $k+2$. When $D = 1$, we will drop the superscript D and write $S_{k+2}(M)$, where

$$M = \mathrm{nrd}(\mathfrak{m}).$$

The space $S_{k+2}^D(\mathfrak{m})$ is endowed with an action of a commutative algebra of *Hecke operators* which we now describe. Define the semigroup

$$S_0^D(\mathfrak{m}) = \left\{\alpha \in R_0^D(\mathfrak{m}) : \mathrm{nrd}\,\alpha \neq 0 \text{ and, for all } v \nmid D\infty,\ M \mid c_v(\alpha) \text{ and } (a_v(\alpha), M) = 1\right\}.$$

Note that $\Gamma_0^D(\mathfrak{m})$ is the subgroup of invertible elements of $S_0^D(M)$ with positive reduced norm. Let $\ell > 0$ be a prime with $\ell \nmid D$. By the Strong Approximation Theorem, we may find an element λ of $S_0^D(M)$ such that $\mathrm{nrd}\,\lambda = \ell$. The quotient $\Gamma_0^D(\mathfrak{m})\backslash\Gamma_0^D(\mathfrak{m})\lambda\Gamma_0^D(\mathfrak{m})$ is finite and we may choose representatives $\lambda_i \in S_0^D(M)$ such that

$$\Gamma_0^D(\mathfrak{m})\lambda\Gamma_0^D(\mathfrak{m}) = \coprod_i \Gamma_0^D(\mathfrak{m})\lambda_i.$$

For an element $g \in S_{k+2}^D(\mathfrak{m})$, define

$$g|T_\ell = \sum_i f|_{k+2}\lambda_i,$$

where

$$(g|_{k+2}\alpha)(\tau) = (\mathrm{nrd}\,\alpha)^{k+1}(c\tau+d)^{-(k+2)}g(\iota_\infty(\alpha)\tau), \qquad \alpha \in B_+^\times, \qquad \iota_\infty(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A standard argument shows that $g|T_\ell$ is independent of the choices made above and is an element of $S_{k+2}^D(\mathfrak{m})$. Thus, T_ℓ is a well-defined endomorphism of $S_{k+2}^D(\mathfrak{m})$. We define

$$\mathbb{T}(S_{k+2}^D(\mathfrak{m})) = \mathbb{C}\big[\{T_\ell : \ell > 0 \text{ prime and } \ell \nmid D\}\big] \subset \mathrm{End}_{\mathbb{C}}\, S_{k+2}^D(\mathfrak{m})$$

The spaces $S_{k+2}^D(\mathfrak{m})$ admit rational structures. Since $X_0^D(\mathfrak{m})$ admits a canonical model over $\mathbb{Q}$ and we assume k to be even, we may identify $S_{k+2}^D(\mathfrak{m})$ with

$$H^0(X_0^D(\mathfrak{m})_{\mathbb{Q}}, (\Omega^1_{X_0^D(\mathfrak{m})/\mathbb{Q}})^{\otimes(k+2)/2}) \otimes_{\mathbb{Q}} \mathbb{C} = H^0(X_0^D(\mathfrak{m})(\mathbb{C}), (\Omega^1_{X_0^D(\mathfrak{m})(\mathbb{C}))})^{\otimes(k+2)/2}).$$

by associating to $g \in S_{k+2}^D(\mathfrak{m})$ the k-fold differential form $g(\tau)(2\pi i\, d\tau)^{\otimes(k+2)/2}$. If F is any number field, we define $S_{k+2}^D(\mathfrak{m})_F$ to be the image of $H^0(X_0^D(\mathfrak{m})_F, (\Omega^1_{X_0^D(\mathfrak{m})/F})^{\otimes(k+2)/2})$ in $S_{k+2}^D(\mathfrak{m})$. If $D = 1$, then the q-expansion principle states that $S_{k+2}(M)_F$ is simply the subspace of $S_{k+2}(M)$ consisting of forms whose Fourier coefficients belong to F. We have:

$$S_{k+2}^D(\mathfrak{m})_F \otimes_F \mathbb{C} = S_{k+2}^D(\mathfrak{m}) \tag{29}$$

The *Jacquet-Langlands correspondence* identifies the systems of Hecke-eigenvalues occuring in $S_{k+2}^D(\mathfrak{m})$ with those occuring in spaces of classical cusp forms:

Theorem 3.1 (Jacquet-Langlands correspondence)**.** *There is an isomorphism*

$$\cdot^B : S_{k+2}(DM)^{D\text{-}new} \longrightarrow S_{k+2}^D(\mathfrak{m})$$

that intertwines the action of $\mathbb{T}(S_{k+2}(DM)^{D\text{-}new})$ *with that of* $\mathbb{T}(S_{k+2}^D(\mathfrak{m}))$.

3.2. **Eichler-Shimura cohomology groups.** For a ring R, let $P_k(R)$ denote the subgroup of $R[X,Y]$ consisting of homogeneous polynomials of degree k. The group $\mathbf{GL}_2(R)$ acts on $P_k(R)$ from the left by the rule

$$(\gamma \cdot P)(X,Y) = P(aX+cY, bX+dY).$$

Let $V_k(R)$ be the R-linear dual of $P_k(R)$. Fix a base-point $\tau \in \mathfrak{h}$. For a modular form $g \in S_{k+2}^D(\mathfrak{m})$, define $c_{g,\tau} : \Gamma_0^D(\mathfrak{m}) \to V_k(\mathbb{C})$ by

$$c_{g,\tau}(\gamma)(P) = \int_\tau^{\gamma_\infty \tau} g(z)P(1,z)dz, \qquad P \in P_k(\mathbb{C}).$$

A standard computation shows that $c_{g,\tau}$ is a 1-cocycle:

$$c_{g,\tau}(\gamma_1\gamma_2) = c_{g,\tau}(\gamma_1)|\gamma_2 + c_{g,\tau}(\gamma_2).$$

Its cohomology class

$$c_f \in H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))$$

does not depend on τ. If $D=1$, the class c_f actually belongs to the parabolic subspace $H^1_{\mathrm{par}}(\Gamma_0(M), V_k(\mathbb{C}))$. (If V is a right $\Gamma_0(M)$-module, we define $\Gamma_0(M)_c$ to be the stabilizer in $\Gamma_0(M)$ of a representative c of a $\Gamma_0(M)$-equivalence class of cusps and then

$$H^1_{\mathrm{par}}(\Gamma_0(M), V) = \ker\left(H^1(\Gamma_0(M), V) \xrightarrow{\mathrm{res}} \bigoplus_{\text{cusps } c} H^1(\Gamma_0(M)_c, V)\right).)$$

The cohomology groups $H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))$, $D \neq 1$, and $H^1_{\mathrm{par}}(\Gamma_0(M), V_k(\mathbb{C}))$, are called *Eichler-Shimura cohomology groups.*

The Eichler-Shimura cohomology groups also admit actions of Hecke operators. Anticipating our needs later in this paper, we describe this in more generality than is necessary for the current discussion. Let V be a right $S_0^D(M)$-module. Let ℓ and λ be as in §3.1 and let $\gamma \in \Gamma_0(M)$. As $\lambda_i\gamma \in \Gamma_0^D(\mathfrak{m})\lambda\Gamma_0^D(\mathfrak{m})$, there is an index i^γ and an element $\gamma_i \in \Gamma_0^D(\mathfrak{m})$ such that

$$\lambda_i\gamma = \gamma_i\lambda_{i^\gamma}.$$

For 1-cocycle ξ on $\Gamma_0^D(\mathfrak{m})$ with values in V, we define $\eta : \Gamma_0^D(\mathfrak{m}) \to V$ by

$$\eta(\gamma) = \sum_i \xi(\gamma_i)|\lambda_i.$$

A standard computation shows that η is a 1-cocycle whose cohomology class depends only on that of ξ. Thus, we obtain an endomorphism

$$T_\ell : H^1(\Gamma_0^D(\mathfrak{m}), V) \longrightarrow H^1(\Gamma_0^D(\mathfrak{m}), V).$$

These cohomology groups also admit an action of a "Hecke operator at infinity". Let w be an element of $R_0^D(M)$ of norm -1. As w normalizes $\Gamma_0^D(\mathfrak{m})$ and $t^2 \in \Gamma_0^D(\mathfrak{m})$, it induces an involution

$$T_\infty : H^1(\Gamma_0^D(\mathfrak{m}), V) \longrightarrow H^1(\Gamma_0^D(\mathfrak{m}), V).$$

When $D=1$, T_∞ preserves the parabolic subspace. If $H \subset H^1(\Gamma_0^D(\mathfrak{m}), V)$ is any subspace which is stable under T_∞, we define $H^\pm$ for the eigenspace of T_∞ acting on H for the eigenvalue ± 1. The operators T_∞ and T_ℓ commute. Therefore, T_ℓ preserves the eigenspaces $H^1(\Gamma_0^D(\mathfrak{m}), V)^\pm$.

Let

$$\mathrm{ES}^\pm : S_{k+2}^D(\Gamma_0(M)) \longrightarrow H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))^\pm$$

be the composition of the map $g \mapsto c_g$ with the projection

$$H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C})) \longrightarrow H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))^\pm, \qquad c \mapsto \frac{1}{2}(c \pm c|T_\infty).$$

We may descend to an arbitrary $\mathbb{Q}$-algebra R and consider the cohomology groups $H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(R))$, and $V_k^B(R) := \mathcal{H}_r^{B,\vee}(R)$. Over any splitting $\mathbb{Q}$-algebra R, an identification $B \otimes_{\mathbb{Q}} R \simeq \mathbf{M}_2(R)$ induces $V_k^B(R) \simeq V_k(R)$, and the cohomology groups can be identified as in the case $R = \mathbb{C}$. Since $S_0^D(M)$ acts on $V_k^B(R)$, on $H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(R))$ and on its eigenspaces $H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(R))^{\pm}$, they all admit actions of the Hecke operators T_ℓ for $\ell \nmid MD$, U_ℓ, W_ℓ for $\ell | M$ and W_ℓ for $\ell | D$. Let

$$\mathbb{T}(H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(R))) \subset \operatorname{End}_R H^1(\Gamma_0^D(\mathfrak{m}), \mathcal{H}_r^{B,\vee}(R))$$

be the R-subalgebra generated by these Hecke operators. Define Hecke algebras $\mathbb{T}(H^1(\Gamma_0^D(\mathfrak{m}), \mathcal{H}_r^{B,\vee}(R))^{\pm})$ similarly.

Theorem 3.2 (Eichler-Shimura isomorphism)**.** *The map* $\mathrm{ES}^{\pm}$ *intertwines the actions of* $\mathbb{T}(S_{k+2}^D(\mathfrak{m}))$ *with that of* $\mathbb{T}(H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))^{\pm})$*. When* $D \neq 1$*, this map is an isomorphism. When* $D = 1$*, it maps* $S_{k+2}(M)$ *isomorphically onto* $H^1_{par}(\Gamma_0(M), V_k(\mathbb{C}))^{\pm}$.

We may compare the rational structure arising from algebraic de Rham cohomology with the one coming from Betti cohomology by choosing a Hecke equivariant $\mathbb{Q}$-isomorphism $S_{k+2}^D(\mathfrak{m})_{\mathbb{Q}} \simeq H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(\mathbb{Q}))^{\pm}$, uniquely determined up to a Hecke equivariant $\mathbb{Q}$-automorphism. Together with the Jacquet-Langlands correspondence, this gives rise to periods that we are going to describe. For simplicity, we restrict our consideration to a normalized new eigenform $g \in S_{k_0+2}(DM)^{\mathrm{new}}$. By multiplicity-one, Theorem 3.1 and (29),

$$\dim_{\mathbb{Q}(g)} (S_{k+2}^D(\mathfrak{m})_{\mathbb{Q}(g)})_g = 1.$$

Let g^B be a basis of this space. (If $D = 1$, we take $g^B = g$.) Since it arises from the rational structure on $X_0^D(\mathfrak{m})$, we say that g^B is *arithmetically normalized*. By Theorem 3.2,

$$\dim_{\mathbb{Q}(g)} H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(\mathbb{Q}(g)))_g^{\pm} = \dim_{\mathbb{C}} H^1(\Gamma_0^D(\mathfrak{m}), V_k(\mathbb{C}))_g^{\pm} = 1.$$

Letting $\phi^{\pm}(g^B)$ be a basis of $H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(\mathbb{Q}(g)))^{\pm,g}$, there is a nonzero scalar $u^{\pm}(g) \in \mathbb{C}$ such that

$$u^{\pm}(g^B)\phi^{\pm}(g^B) = \mathrm{ES}^{\pm}(g^B). \tag{30}$$

We will use the shorthand

$$\phi_k^{\pm} = \phi^{\pm}(F_k^B), \qquad \phi_k^{\pm,\sharp} = \phi^{\pm}((F_k^{\sharp})^B) \qquad (k \in \Omega_{\mathrm{cl}}).$$

By (3) and the Jacquet-Langlands correspondence, $\phi_k^{\pm}$ is p-old for $k \neq k_0$ and

$$\phi_k^{\pm} = \mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^{\pm,\sharp} - \mathbf{a}_p(k)^{-1} (\mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^{\pm,\sharp})|W_p. \tag{31}$$

3.3. **Homology classes and values of L-functions.** Fix a $\delta \in \mathcal{O}_K$ with $\delta^2 = d_K$ and $\sigma_\infty(\delta) > 0$. For $j \in \mathcal{E}(\mathcal{O}_K, R)$ and $k \in \Omega_{\mathrm{cl}}$, let $Q_j^{k/2} \in \mathcal{H}_{k/2}(\mathbb{Q})$ be the element that, up to $\mathcal{H}_{k/2}^B \otimes_{\mathbb{Q}} K \cong P_k(K)$, corresponds to the $\frac{k}{2}$-th power of

$$Q_j(x, y) = cx^2 + (d - a)xy - by^2 \in P_2(K), \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \iota(j(\delta)). \tag{32}$$

One may verify that if $\alpha \in B^{\times}$, then

$$\alpha \cdot Q_j = (\mathrm{nrd}\, \alpha) Q_{\alpha j \alpha^{-1}}. \tag{33}$$

It follows that if u is a generator of $\mathcal{O}_1^{\times}$ such that $\sigma_\infty(u) > 1$ (a fundamental unit of K, when $\mathcal{O} = \mathcal{O}_K$), then

$$j(u) \cdot Q_j = Q_j.$$

Set

$$\Gamma_j := \langle j(u) \rangle \subset \Gamma.$$

We may define the weight-k cycle

$$C_{j,k} := j(u) \otimes Q_j^{k/2} \in Z_1(\Gamma_j, \mathcal{H}_r(\mathbb{Q})). \tag{34}$$

Lemma 3.3.

(1) *The homology class*
$$\mathrm{cores}_{\Gamma_j}^{\Gamma}[C_{j,k}] \in H_1(\Gamma, \mathcal{H}_{k/2}(\mathbb{Q}))$$
depends only on the Γ*-conjugacy class of* j*.*

(2) *If* $\alpha \in B^\times$*, then*
$$\alpha \cdot C_{j,k} = (\mathrm{nrd}\,\alpha)^{k/2} C_{\alpha j \alpha^{-1},k}.$$
(This is an identity of elements of $Z_1(\alpha^{-1}\Gamma\alpha, \mathcal{H}_{k/2}(\mathbb{Q}))$*.)*

Proof. The first statement is clear. The second follows from the definition of the action and equation (33). □

Motivated by statement (1) of the above lemma, we introduce the shorthand

(35) $$C_{[j],k} := \mathrm{cores}_{\Gamma_j}^{\Gamma}[C_{j,k}]$$

Take $\mathcal{O} = \mathcal{O}_K$ for the remainder of this section. For a character $\psi : \mathrm{Cl}_K^+ \longrightarrow \overline{\mathbb{Q}}^\times$, define the ψ-twisted cycle
$$C_{[j],k}^{\psi} = \sum_{\sigma \in \mathrm{Cl}_K^+} \psi(\sigma) C_{\sigma\cdot[j],k} \in H_1(\Gamma, \mathcal{H}_{k/2}(\mathbb{Q}(\psi))).$$
The behaviour of the homology classes under $C_{[j],k}^{\psi}$ under T_∞ will play an important role. Let $s \in \mathrm{Cl}_K^+$ correspond to the generator of $\mathrm{Gal}(H_K^+/H_K)$.

Lemma 3.4. *The homology class* $C_{[j],k}^{\psi}$ *belongs to the eigenspace of* $H_1(\Gamma, V_2(K(\gamma)))$ *on which* T_∞ *acts with eigenvalue* $(-1)^{k/2}\psi(s)$*.*

Consider the natural pairing
$$\langle\cdot,\cdot\rangle : H_1(\Gamma_0^D(\mathfrak{m}), \mathcal{H}_r(\mathbb{Q}(\psi))) \times H^1(\Gamma_0^D(\mathfrak{m}), V_k^B(\mathbb{Q}(g))) \longrightarrow \mathbb{Q}(\psi, g).$$

Theorem 3.5 ([31, Theorem 5.41])**.** *If* $k \neq k_0$ *is in* Ω_{cl}*, then there is a non-zero element* $\eta(k) = \eta_{\sigma(F_k^\sharp)} \in \mathbb{Q}(F_k^\sharp)$ *depending only on* $F_k^\sharp$ *and* B *such that*
$$\langle \phi_k^{\pm,\sharp}, C_{[j],k}^{\psi}\rangle\langle \phi_k^{\pm,\sharp}, C_{[j],k}^{\psi^{-1}}\rangle = d_K^{k/2}\eta(k) L^*(F_k^\sharp/K, \psi, k/2+1).$$

The following remark is a direct application of the existence of the rational structures $\mathcal{H}_r$, and the fact that $Q_j^r \in \mathcal{H}_r(\mathbb{Q})$.

Remark 3.6. By the rationality of $\mathcal{H}_{k/2}(\mathbb{Q})$ and the fact that $Q_j^{k/2} \in \mathcal{H}_{k/2}(\mathbb{Q})$, we have
$$\sigma(\eta_{F_k^\sharp}) = \eta_{\sigma(F_k^\sharp)}$$
for all $\sigma \in G_{\mathbb{Q}}$.

Part 2. p-adic L-functions

4. Families of cohomology classes

Just like modular forms can vary in p-adic families, so too can their corresponding group cohomology classes. We develop aspects of this theory following [2]. As in the introduction, we let Y be the $\mathbb{Q}_p$-manifold $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ (see [34, §9]), let $\mathcal{A}(Y)$ be the space of E-locally analytic functions on Y (see [34, §9]) and let $\mathcal{D}(Y)$ be its strong dual, the space of E-locally analytic distributions on Y (see [34, §11]). The space Y admits a right action of the semigroup
$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbb{Z}_p) : ad - bc \neq 0,\ p \mid c,\ p \nmid a \right\}$$
given by
$$(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+cy, bx+dy)$$
and a left action of $\mathbb{Z}_p^\times$ given by
$$t(x,y) = (tx, ty).$$

Note that this action of $\mathbb{Z}_p^\times$ on Y agrees with that of the diagonally embedded $\mathbb{Z}_p^\times \subset \Sigma_0(p)$:

$$t(x,y) = (x,y)\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Thus, $\mathcal{A}(Y)$ is a $(\Sigma_0(p), \mathbb{Z}_p^\times)$-bimodule and $\mathcal{D}(Y)$ is a $(\mathbb{Z}_p^\times, \Sigma_0(p))$-bimodule.

Let $\mathcal{X}$ be the p-adic weight space. It is a rigid analytic $\mathbb{Q}_p$-variety such that, for all p-adic fields F,

$$\mathcal{X}(F) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, F^\times).$$

If $\kappa \in \mathcal{X}(F)$ and $t \in \mathbb{Z}_p^\times$, we will often write t^κ in place of $\kappa(t)$. Let $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ be the structure sheaf of $\mathcal{X}$. There is a natural embedding of $\mathbb{Z}$ into $\mathcal{X}(E)$ given by

$$k \mapsto (x \mapsto x^k).$$

The weight space $\mathcal{X}$ is the union of $p-1$ affinoid subvarieties $\mathcal{X}_i$, $i \in \mathbb{Z}/(p-1)\mathbb{Z}$, that can be indexed such that an integer k is in $\mathcal{X}_i(\mathbb{Q}_p)$ if and only if $k \equiv i \pmod{p-1}$. Since Ω is a disk, we must have

$$\Omega \subset \mathcal{X}_{k_0}. \tag{36}$$

Consider the Fourier transform

$$\widehat{\cdot} : \mathcal{D}(\mathbb{Z}_p^\times) \longrightarrow \mathcal{O}(\mathcal{X}), \qquad \widehat{\mu}(\kappa) = \int_{\mathbb{Z}_p^\times} \kappa(t) d\mu(t). \tag{37}$$

Theorem 4.1 (Amice, Velu)**.** *The Fourier transform $\mu \mapsto \widehat{\mu}$ is an isomorphism.*

Let Ω be an affinoid subvariety of $\mathcal{X}$. We may view $\mathcal{O}(\Omega)$ as a $\mathcal{D}(\mathbb{Z}_p^\times)$-module via the Fourier transform. We set

$$\mathcal{D}(Y)_\Omega = \mathcal{O}(\Omega) \widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)} \mathcal{D}(W).$$

Note that $\mathcal{D}(Y)_{\mathcal{X}} = \mathcal{D}(Y)$. The same notation $\mathcal{D}(\cdot)_\Omega$ makes sense for every $\mathbb{Q}_p$-manifold in $\mathbb{Q}_p^2$ which is stable under the $\mathbb{Z}_p^\times$-action described above. The space $\mathcal{D}(Y)_\Omega$ inherits the right $\Sigma_0(p)$-action from $\mathcal{D}(Y)$ making it an $(\mathcal{O}(\Omega), \Sigma_0(p))$-bimodule. Since $\Sigma_0(p)$ contains $S_0^D(\mathfrak{n}^+\mathfrak{p})$, the cohomology group

$$\mathbb{W}_\Omega := H^1(\Gamma_0, \mathcal{D}(Y)_\Omega)$$

is a left $\mathcal{O}(\Omega)$ module, and admits a right action of the Hecke operators T_ℓ for $\ell \nmid N$, U_ℓ, W_ℓ for $\ell | pN^+$ and W_ℓ for $\ell | D$. We write $\mathbb{T}(\mathbb{W}_\Omega)$ for the $\mathcal{O}(\Omega)$-subalgebra of $\mathrm{End}_{\mathcal{O}(\Omega)} \mathbb{W}_\Omega$ generated by these operators.

The utility of the space $\mathcal{D}(Y)_\Omega$ lies in the fact that it admits $\Sigma_0(p)$-equivariant homomormorphisms to the classical weight module $V_k(E)$ for every $k \in \Omega_{\mathrm{cl}}$. For integers $k \geq 0$, define

$$\rho_k : \mathcal{D}(Y) \longrightarrow V_k(E), \qquad \rho_k(\mu)(P) = \mu(P) = \int_Y P(x,y)\mu(x,y) \qquad (P \in P_k(E)).$$

Letting $\mathcal{O}(\Omega)$ act on $V_k(E)$ via the evaluation-at-k homomorphism $\mathcal{O}(\Omega) \to E$, we obtain an analogous map

$$\rho_k : \mathcal{D}(Y)_\Omega \longrightarrow V_k(E), \qquad k \in \Omega_{\mathrm{cl}}.$$

(See also [19] for a more conceptual discussion.) Being $\Sigma_0(p)$-equivariant (as is easily checked), these homomorphism induce Hecke-equivariant *specialization maps*

$$\rho_k : \mathbb{W}_\Omega \longrightarrow H^1(\Gamma_0, V_k(E)), \qquad k \in \Omega_{\mathrm{cl}}.$$

Just like the Eichler-Shimura cohomology groups, $\mathbb{W}_\Omega$ admits an involution T_∞ and a corresponding $\mathbb{T}(\mathbb{W}_\Omega)$-module decomposition

$$\mathbb{W}_\Omega = \mathbb{W}_\Omega^+ \oplus \mathbb{W}_\Omega^-.$$

The specialization maps ρ_k respect the $\pm$-eigenspace decompositions.

Theorem 4.2 (Ash-Stevens)**.** *There exists a $\mathbb{T}(\mathbb{W}_\Omega)$-eigenvector $\Phi^\pm \in H^1(\Gamma_0, \mathcal{D}(Y))^\pm$ such that*

$$\rho_{k_0}^Y(\Phi^\pm) = \phi_{k_0}^\pm.$$

Further, for every $k \in \Omega_{cl} - \{k_0\}$ such that $k > k_0$, there are scalars $\mu^\pm(k) \in E^\times$ satisfying

$$\Phi_k^\pm := \rho_k^Y(\Phi^\pm) = \mu^\pm(k)\phi_k^\pm.$$

Proof. The proof of this theorem, which is well known when $D = 1$, is given in [19] as an application of the techniques and results of Ash and Stevens. $\square$

It is indeed proved that $\Phi^{\pm}$ is a U_p-eigenvector of bounded slope: $\Phi^{\pm} \mid U_p = \mathbf{a}_p \Phi^{\pm}$ where $\mathbf{a}_p \in \mathcal{O}(\Omega)$ is such that $\operatorname{ord}_p \mathbf{a}_p < k_0 + 1$ (and may be taken to have constant slope, up to shrinking Ω). In particular, $\mathbf{a}_p$ has no zeros and is therefore invertible in $\mathcal{O}(\Omega)$. Let

$$X = \mathbb{Z}_p^2 - p\mathbb{Z}_p^2 \qquad \text{and} \qquad Z = p\mathbb{Z}_p \times \mathbb{Z}_p^{\times}$$

We note that

$$X = Y \sqcup Z, \qquad Z = Y \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}, \qquad \text{and} \qquad X = \bigsqcup_i \Gamma_0 \gamma_i,$$

where $\{\gamma_i\}$ is a system of representatives for $\Gamma_0 \backslash \Gamma$.

Lemma 4.3.

(1) *The mapping*

$$S : \mathcal{A}(X) \longrightarrow E[\Gamma] \otimes_{E[\Gamma_0]} \mathcal{A}(Z), \qquad S(f) = \sum_i \gamma_i \otimes (\gamma_i^{-1} \cdot f)|_Z$$

is an isomorphism of left $E[\Gamma]$-modules.

(2) *The mapping*

$$S : \operatorname{Hom}_{E[\Gamma_0]}(E[\Gamma], \mathcal{D}(Z)) \longrightarrow \mathcal{D}(X)$$

is an isomorphism of right $E[\Gamma]$-modules.

(3) *S^{-1} induces an isomorphism*

$$\rho_Z^X : H^1(\Gamma, \mathcal{D}(X)) \to H^1(\Gamma_0, \mathcal{D}(Z)).$$

Proposition 4.4 (Key calculation)**.** *The following diagram commutes:*

$$\begin{array}{ccccc} & & H^r(\Gamma, \mathcal{D}(X)_\Omega) & & \\ & \swarrow^{\rho_Z^X} & & \searrow^{\rho_Y^X} & \\ H^r(\Gamma_0, \mathcal{D}(Z)_\Omega) & \xrightarrow[W_p^{-1}]{} & H^r(\Gamma_0, \mathcal{D}(Y)_\Omega) & \xrightarrow[U_p]{} & H^r(\Gamma_0, \mathcal{D}(Y)_\Omega) \end{array}$$

Remark 4.5. This is the only cohomological calculation we must perform on the level of cocycles.

Proof. This diagram is so natural that it actually commutes on the level of cochains for any resolution computing the cohomology: Let $Q_\bullet \to \mathbb{Z} \to 0$ be a resolution of $\mathbb{Z}$ by projective $\mathbf{GL}_2(\mathbb{Q}_p)$-modules and let $\phi \in \operatorname{Hom}_\Gamma(Q_r, \mathcal{D}(X))$. Then:

$$\begin{aligned} (\rho_Z^X(\phi)|W_p^{-1}|U_p)(q)(h) &= \sum_{a=0}^{p-1} \rho_Z^X(\phi)(q\pi_a^*(w_p^*)^{-1})(w_p^{-1}\pi_a \cdot h) \\ &= \sum_{a=0}^{p-1} \phi(q\pi_a^{-1}w_p)(\iota_Z(w_p^{-1}\pi_a \cdot h)) \\ &= \sum_{a=0}^{p-1} \phi(q)(\pi_a^{-1}w_p \cdot \iota_Z(w_p^{-1}\pi_a \cdot h)) \\ &= \sum_{a=0}^{p-1} \phi(q)(\iota_{Y_a}(h)) \\ &= \phi(q)(\iota_Y(h)) \\ &= \rho_Y^X(\phi)(q)(h). \end{aligned}$$

$\square$

Since ρ_Z^X is the isomorphism of Shapiro's lemma, we can make the following key definition:

(38) $$\Phi^{\pm,\sharp} = \mathbf{a}_p^{-1}(\rho_Z^X)^{-1}(\Phi^{\pm}|W_p) \in H^1(\Gamma, \mathcal{D}(X)_\Omega)^{\pm}.$$

Corollary 4.6. *Let π be any element of $S_0^D(\mathfrak{n}^+\mathfrak{p})$ of reduced norm p. Then*

$$\Phi^{\pm} = \mathbf{a}_p \rho^X_{Y\pi}(\Phi^{\pm,\sharp})|\pi^{-1}.$$

Proof. Let $\gamma = w_p^{-1}\pi$, and observe that $\gamma \in \Gamma$. The diagram

$$\begin{array}{ccc} H^r(\Gamma, \mathcal{D}(X)_\Omega) & \xrightarrow{\gamma} & H^r(\Gamma, \mathcal{D}(X)_\Omega) \\ \downarrow \rho^X_Z & & \downarrow \rho^X_{Y\pi} \\ H^r(\Gamma_0, \mathcal{D}(Z)_\Omega) & \xrightarrow[\gamma]{} & H^r(\pi^{-1}\Gamma_0\pi, \mathcal{D}(Y\pi)_\Omega) \end{array}$$

is obviously commutative, and the top arrow is actually the identity map since $\gamma \in \Gamma$. Therefore

$$(\rho^X_{Y\pi})(\Phi^{\pm,\sharp}) = \rho^X_Z(\Phi^{\pm,\sharp})|\gamma = \mathbf{a}_p^{-1}\Phi|W_p|\gamma = \mathbf{a}_p^{-1}\Phi^{\pm}|\pi. \qquad \square$$

Set

$$\Phi^{\pm,\sharp}_k = \rho^X_k(\Phi^{\pm,\sharp}).$$

Proposition 4.7. *If $k \in \Omega_{cl} - \{k_0\}$, then*

$$\left(1 - \frac{p^k}{\mathbf{a}_p(k)^2}\right)\Phi^{\pm}_k = \mathrm{res}^{\Gamma}_{\Gamma_0}\Phi^{\pm,\sharp}_k - \mathbf{a}_p(k)^{-1}(\mathrm{res}^{\Gamma}_{\Gamma_0}\Phi^{\pm,\sharp}_k)|W_p. \tag{39}$$

In particular, $\Phi^{\pm}_k$ is p-old for $k \in \Omega_{cl} - \{k_0\}$.

Proof. To avoid clutter, we drop the $\pm$ from the notation. We have:

$$\begin{aligned} \mathrm{res}^{\Gamma}_{\Gamma_0}\Phi^{\sharp}_k &= \mathrm{res}^{\Gamma}_{\Gamma_0}\rho^X_k(\Phi^{\sharp}) && \text{by definition of } \Phi^{\sharp}_k \\ &= \rho^Z_k(\rho^X_Z(\Phi^{\sharp})) + \rho^Y_k(\rho^X_Y(\Phi^{\sharp})) && \text{as } X = Y \sqcup Z \\ &= \rho^Z_k(\mathbf{a}_p^{-1}\Phi|W_p) + \rho^Y_k(\Phi) && \text{by Proposition 4.4} \\ &= \rho^Y_k(\mathbf{a}_p^{-1}\Phi)|W_p + \Phi_k && \text{as } W_p \circ \rho^Y_k = \rho^Z_k \circ W_p \\ &= \mathbf{a}_p^{-1}(k)\,\Phi_k|W_p + \Phi_k && \text{by definition of } \Phi_k. \end{aligned} \tag{40}$$

Applying W_p to (40), we obtain

$$\begin{aligned} (\mathrm{res}^{\Gamma}_{\Gamma_0}\Phi^{\sharp}_k)|W_p &= \mathbf{a}_p^{-1}(k)\,\Phi_k|W_p^2 + \Phi_k|W_p \\ &= \mathbf{a}_p^{-1}(k)\,p^k\Phi_k + \Phi_k|W_p, \end{aligned} \tag{41}$$

as $W_p^2 = p$ acts on $H^1(\Gamma_0, V_k(E))$ by multiplication by p^k. Combining (40) and (41) yields (39). $\square$

Corollary 4.8 (cf. [5, Theorem 3.5])**.** *Suppose $k \in \Omega_{cl} - \{k_0\}$. Then*

$$\Phi^{\pm,\sharp}_k = \left(1 - \frac{p^k}{\mathbf{a}_p(k)^2}\right)\mu^{\pm}(k)\phi^{\pm,\sharp}_k.$$

Proof. By Theorem 4.2 and Proposition 4.7, we have

$$\phi_k = \mathrm{res}^{\Gamma}_{\Gamma_0}(C^{-1}\Phi^{\pm,\sharp}_k) - \mathbf{a}_p(k)^{-1}(\mathrm{res}^{\Gamma}_{\Gamma_0}(C^{-1}\Phi^{\pm,\sharp}_k))|W_p,$$

where

$$C = \mu^{\pm}(k)\left(1 - \frac{p^k}{\mathbf{a}_p(k)^2}\right).$$

But by (31), this equation is also satisfied by with $C^{-1}\Phi^{\pm,\sharp}_k$ replaced by $\phi^{\pm,\sharp}_k$. But it is well known that the map

$$(\mathrm{res}, W_p \circ \mathrm{res}) : H^1(\Gamma, V_k(E)) \oplus H^1(\Gamma, V_k(E)) \longrightarrow H^1(\Gamma_0, V_k(E))$$

is injective. Therefore, $\phi^{\pm,\sharp}_k = C^{-1}\Phi^{\sharp}_k$. $\square$

Define $\mathcal{D}(X)^0_\Omega$ by the exact sequence

$$0 \longrightarrow \mathcal{D}(X)^0_\Omega \longrightarrow \mathcal{D}(X)_\Omega \xrightarrow{\rho^X_{k_0}} V_{k_0}(E) \longrightarrow 0. \tag{42}$$

The p-newness of f manifests itself as follows:

Lemma 4.9. *There is a unique Hecke eigenvector in $H^1(\Gamma, \mathcal{D}(X)^0_\Omega)$ that maps to $\Phi^\sharp$.*

Proof. See [15, Lemma 9]. □

Thus, we may unambigously treat $\Phi^\sharp$ as an element of $H^1(\Gamma, \mathcal{D}(X)^0_\Omega)$ whenever convenient.

5. p-adic L-functions: interpolation properties

Since $\mathbb{Q}_p$ splits B, we may identify $\mathcal{H}^B_{k/2}(\mathbb{Q}_p) \simeq P_k(\mathbb{Q}_p)$. Let $j \in \mathcal{E}(\mathcal{O}, R)$ and let δ, $Q_j \in P_2(\mathbb{Z}_p)$, and u be as in §3.3. Setting

$$X_j = \{(x, y) \in \mathbb{Z}_p^2 : Q_j(x, y) \in \mathbb{Z}_p^\times\},$$

we may consider the closed $\mathcal{O}(\Omega)$-submodule $\mathcal{A}_\Omega(X_j) \subset \mathcal{O}(\Omega)\widehat{\otimes}\mathcal{A}(X_j)$ of those functions $\Theta : \Omega(E) \times X_j \to E$ that are rigid analytic in the Ω-variable, locally analytic in the X_j-variable and satisfy $\Theta(\kappa, t(x, y)) = t^\kappa \Theta(\kappa, (x, y))$. Since $\kappa \in \mathcal{X}_{k_0}$, up to shrinking Ω around k_0, there is a unique $s \in \mathbb{Z}_p$ such that

$$t^\kappa = [t]^{k_0} \langle t \rangle^s,$$

where $[\cdot]$ and $\langle\cdot\rangle$ are the projections of $\mathbb{Z}_p^\times$ onto μ_{p-1} and $1 + p\mathbb{Z}_p$, respectively. Since k_0 is even, $k_0/2$ is an integer. Thus, we may define $\kappa/2 \in \Omega$ by $t^{\kappa/2} = [t]^{k_0/2} \langle t \rangle^{s/2}$. Note that

$$\Theta^j(\kappa, (x, y)) := Q_j(x, y)^{\kappa/2} \in \mathcal{A}_\Omega(X_j).$$

As explained in [19], the elements of $\mathcal{D}(X_j)_\Omega$ naturally integrate elements of $\mathcal{A}_\Omega(X_j)$. Indeed there is an $\mathcal{O}(\Omega)$-bilinear integration pairing, compatible with specialization to $k \in \Omega$, which gives rise to a commutative diagram

$$\begin{array}{cccccc} \mathcal{D}(X_j)_\Omega & \otimes_{\mathcal{O}(\Omega)} & \mathcal{A}_\Omega(X_j) & \to & \mathcal{O}(\Omega) \\ \eta_k \downarrow & & \downarrow e_k & & \downarrow e_k \\ \mathcal{D}(X_j)_k & \otimes & \mathcal{A}_k(X_j) & \to & E \end{array} \tag{43}$$

where $\eta_k := k\widehat{\otimes}1$ (viewing k as a homomorphism $k : \mathcal{O}(\Omega) \to E$) and e_k is the evaluation at k (so that $\rho^{X_j}_k = \nu_k \circ \eta_k$, for ν_k the restriction to homogeneous polynomials of degree k). Here $\mathcal{A}_k(X_j)$ (resp. $\mathcal{D}(X_j)_k$) is defined in the same way as the space $\mathcal{A}_\Omega(X_j)$, just setting $\kappa = k$ in the definition (resp. viewing E as a $\mathcal{D}(\mathbb{Z}_p^\times)$-module via the Fourier transform followed by $k : \mathcal{O}(\Omega) \to E$ and setting $\mathcal{D}(X_j)_k := E\widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)}\mathcal{D}(X_j)$). Since the pairing is Γ_j-invariant, it induces an $\mathcal{O}(\Omega)$-bilinear pairing

$$H^1(\Gamma_j, \mathcal{D}(X_j)_\Omega) \otimes_{\mathcal{O}(\Omega)} H_1(\Gamma_j, \mathcal{A}_\Omega(X_j)) \to \mathcal{O}(\Omega).$$

As we have $j(u) \cdot Q_j = Q_j$ and $\Gamma_j \simeq \mathbb{Z}$,

$$C_j := \gamma_j \otimes_{\Gamma_j} \Theta^j \in Z_1(\Gamma_j, \mathcal{A}_\Omega(X_j)),$$

and we may consider its class $[C_j] \in H_1(\Gamma_j, \mathcal{A}_\Omega(X_j))$. We define

$$\mathcal{L}_{\Phi^\pm/K,[j]} := \left\langle \rho^X_{X_j}(\Phi^{\pm,\sharp}), [C_j] \right\rangle \in \mathcal{O}(\Omega). \tag{44}$$

Our notation is justified by the following lemma.

Lemma 5.1. *$\mathcal{L}_{\Phi^\pm/K,[j]}$ only depends on the class $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O}, R)$ of j.*

Proof. Note that, for all $k \in \Omega_{\mathrm{cl}}$, $e_k(C_j) = C_{k,j}$ and that, for every $\gamma \in \Gamma$, $X_{\gamma j} = X_j\gamma^{-1}$ and $[C_{k,\gamma j}] = \gamma \cdot [C_{k,j}]$, so that (43) gives

$$\begin{aligned}\mathcal{L}_{\Phi^{\pm}/K,\gamma j}(k) &= \left\langle \rho_k^{X_{\gamma j}} \rho_{X_{\gamma j}}^{X} \Phi^{\pm,\sharp}, [C_{k,\gamma j}] \right\rangle = \left\langle \rho_k^{X_j\gamma^{-1}} \rho_{X_j\gamma^{-1}}^{X} \Phi^{\pm,\sharp}, \gamma\cdot[C_{k,j}] \right\rangle \\ &= \left\langle \left(\rho_k^{X_j\gamma^{-1}} \rho_{X_j\gamma^{-1}}^{X} \Phi^{\pm,\sharp}\right) |\gamma, [C_{k,j}] \right\rangle = \left\langle \rho_k^{X_j\gamma^{-1}\gamma} \left(\rho_{X_j\gamma^{-1}}^{X} \Phi^{\pm,\sharp}\right) |\gamma, [C_{k,j}] \right\rangle \\ &= \left\langle \rho_k^{X_j} \left(\rho_{X_j\gamma^{-1}\gamma}^{X\gamma} \left(\Phi^{\pm,\sharp}|\gamma\right)\right), [C_{k,j}] \right\rangle = \left\langle \rho_k^{X_j} \rho_{X_j}^{X} \Phi^{\pm,\sharp}, [C_{k,j}] \right\rangle = \mathcal{L}_{\Phi^{\pm}/K,j}(k).\end{aligned}$$

The claimed independence follows from the density of classical weights. $\square$

Let ψ be a character of $\mathrm{Cl}^+(\mathcal{O})$ and define

$$\mathcal{L}_{\Phi/K,\psi,[j]} = \sum_{\sigma\in \mathrm{Cl}_K^+} \psi(\sigma)\mathcal{L}_{\Phi/K,\sigma\cdot[j]} \tag{45}$$

The dependence on j of $\mathcal{L}_{\Phi/K,[j]}$ is relatively minor:

$$\mathcal{L}_{\Phi/K,\psi,\delta\cdot[j]} = \psi(\delta^{-1})\mathcal{L}_{\Phi/K,\psi,[j]}.$$

Therefore,

$$\mathcal{L}_{\Phi/K,\psi} := \mathcal{L}_{\Phi/K,\psi,[j]}\mathcal{L}_{\Phi/K,\psi^{-1},[j]} \tag{46}$$

is independent of j.

Remark 5.2. Suppose that $\mathcal{L}$ vanishes to order at least n at k_0, for $\mathcal{L} = \mathcal{L}_{\Phi^{\pm}/K,[j]}$, $\mathcal{L}_{\Phi/K,\psi,[j]}$ or $\mathcal{L}_{\Phi/K,\psi}$. Then $\phi_{k_0}^{\pm} \mapsto \mathcal{L}^{(n)}(k_0)$ does not depend on the lift Φ of $\phi_{k_0}^{\pm}$ to an eigenfamily of bounded slope $\leq h$, for any $h < k_0 + 1$.

Proof. If $\Psi \in H^1(\Gamma_j, \mathcal{D}(X_j)_\Omega)$, set $\mathcal{L}_{\Psi/K,[j]} := \langle \Psi, [C_j]\rangle$ and then define $\mathcal{L}_{\Psi/K,\psi,[j]}$ and $\mathcal{L}_{\Psi/K,\psi}$ by formulas (45) and (46). By $\mathcal{O}(\Omega)$-bilinearity of the integration pairing,

$$\mathcal{L}_{\alpha\Psi/K,[j]} = \alpha\mathcal{L}_{\Psi/K,[j]},\ \mathcal{L}_{\alpha\Psi/K,\psi,[j]} = \alpha\mathcal{L}_{\Psi/K,\psi,[j]} \text{ and } \mathcal{L}_{\alpha\Psi/K,\psi} = \alpha^2\mathcal{L}_{\Psi/K,\psi}.$$

Let $I_{k_0} \subset \mathcal{O}(\Omega)$ be the ideal of functions that vanish at k_0. Then, by the control theorem proved in [19], $\Phi^{\pm}$ is well defined up to an element of $I_{k_0}H^1(\Gamma_0, \mathcal{D}(Y)_\Omega)^{\leq h}$. The associated $\rho_{X_j}^{X}(\Phi^{\pm,\sharp})$ is well defined up to an element of $I_{k_0}H^1(\Gamma_j, \mathcal{D}(X_j)_\Omega)$. Thus, it suffices to show that $d_{k_0}^{(n)}\mathcal{L}_{\alpha\Psi/K} = 0$ for $\alpha \in I_{k_0}$ and $\Psi \in H^1(\Gamma, \mathcal{D}(X_j)_\Omega)$ if $\mathcal{L}_{\Psi/K}$ vanishes to order at least $n-1$ at k_0, where

$$d_{k_0} := \frac{d^n}{d\kappa^n}\Big|_{\kappa=k_0}$$

and $\mathcal{L}_{\alpha\Psi/K} = \mathcal{L}_{\alpha\Psi/K,[j]}$, $\mathcal{L}_{\alpha\Psi/K,\psi,[j]}$ or $\mathcal{L}_{\alpha\Psi/K,\psi}$. For $\beta = \alpha$ or α^2, we have

$$\mathcal{L}_{\alpha\Psi/K}^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \beta^{(n-i)}\mathcal{L}_{\Psi/K}^{(i)}.$$

Evaluating at k_0 and using the fact that $\mathcal{L}_{\Psi/K}$ vanishes to order at least $n-1$ we see that $d_{k_0}^{(n)}\mathcal{L}_{\alpha\Psi/K} = \beta(k_0)\mathcal{L}_{\Psi/K}^{(n)}(k_0)$ and, since $\beta(k_0) = 0$, we obtain $d_{k_0}^{(n)}\mathcal{L}_{\alpha\Psi/K} = 0$. $\square$

In the following two subsections, we assume $\mathcal{O} = \mathcal{O}_K$.

5.1. **p-adic L-functions when $\epsilon_K(p) = +1$.** In this subsection, we assume p splits in K. Thanks to Lemma 2.1 (3), we may assume without loss of generality that $\mathcal{L}_{\Phi^\pm/K,[j]}$ is obtained from $j \in \mathcal{E}(\mathcal{O}, R_0)$.

Lemma 5.3. *We have* $X_j = Y - Y\begin{pmatrix} 1 & t_j \\ 0 & p \end{pmatrix} = Y - YP_j$ *for* $j \in \mathcal{E}(\mathcal{O}, R_0)$.

Proof. We first observe that X_j can be defined as above for any $\mathbb{Z}_p$-algebra embedding j of $\mathbb{Z}_p^2$ into $\mathcal{R}_0(p\mathbb{Z}_p)$. Suppose

$$j_0 : \mathbb{Z}_p^2 \longrightarrow \mathcal{R}_0(p\mathbb{Z}_p), \qquad j_0(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

is the diagonal embedding. Then $Q_{j_0}(x,y) = \alpha xy$ for some $\alpha \in \mathbb{Z}_p^\times$ and

$$X_{j_0} = \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = Y - Y\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

Recall the extension $j_\mathfrak{p}$ of j to a map

$$j_\mathfrak{p} : \mathbb{Z}_p^2 = \mathcal{O}_{K,\mathfrak{p}} \oplus \mathcal{O}_{K,\mathfrak{p}'} = \mathcal{O}_K \otimes \mathbb{Z}_p \longrightarrow \mathcal{R}_0(p\mathbb{Z}_p).$$

Write $\mathbb{Z}_p^2$ as $\mathbb{Z}_p u \oplus \mathbb{Z}_p v$ where u and v are eigenvectors of $j_\mathfrak{p}(\mathbb{Z}_p^2)$, uniquely determined up to multiplication by an element of $\mathbb{Z}_p^{\times 2}$, ordered so that

$$uj_\mathfrak{p}(x,y) = ux, \qquad vj_\mathfrak{p}(x,y) = vy.$$

By (18) and (21),

$$j_\mathfrak{p}(a,p) \equiv \begin{pmatrix} a & at_j \\ 0 & 0 \end{pmatrix} \pmod{p}.$$

Noting that, up to multiplying (u,v) by an element of $\mathbb{Z}_p^{\times 2}$,

$$\begin{pmatrix} 1 & t_j \end{pmatrix} j_\mathfrak{p}(a,p) = a\begin{pmatrix} 1 & t \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \end{pmatrix} j_\mathfrak{p}(a,p) \equiv 0 \begin{pmatrix} 0 & 1 \end{pmatrix} \pmod{p},$$

we must have

$$u \equiv \begin{pmatrix} 1 & t_j \end{pmatrix}, \qquad v \equiv \begin{pmatrix} 0 & 1 \end{pmatrix} \pmod{p}.$$

Therefore,

$$j_\mathfrak{p} = A^{-1} j_0 A, \qquad \text{where} \qquad A = \begin{pmatrix} u \\ v \end{pmatrix}.$$

It follows that

$$X_j = X_{j_\mathfrak{p}} = X_{A^{-1}j_0A} = X_{j_0}A = YA - Y\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

Now $YA = Y$ as $A \in \mathcal{R}_0(p\mathbb{Z})^\times$, and the congruence

$$A \equiv \begin{pmatrix} 1 & t_j \\ 0 & 1 \end{pmatrix} \pmod{p}$$

implies that

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} A \begin{pmatrix} 1 & t_j \\ 0 & p \end{pmatrix}^{-1} \in \mathcal{R}_0(p\mathbb{Z}_p)^\times.$$

The result follows. □

Proposition 5.4. *Suppose* $k \in \Omega_{cl}$. *Then*

$$\mathcal{L}_{\Phi^\pm/K,[j]}(k) = \begin{cases} \langle \phi_{k_0}^\pm, C_{[j],k_0} \rangle - p^{k_0/2}\mathbf{a}_p^{-1}(k_0)\langle \phi_{k_0}^\pm, C_{\mathfrak{p}'\cdot[j],k_0} \rangle, & \text{if } k = k_0, \\ \mu^\pm(k)\Bigg(\left(1 + \dfrac{p^k}{\mathbf{a}_p(k)^2}\right)\langle \phi_k^{\pm,\sharp}, C_{[j],k} \rangle - \\ \qquad \dfrac{p^{k/2}}{\mathbf{a}_p(k)}\left(\langle \phi_k^{\pm,\sharp}, C_{\mathfrak{p}\cdot[j],k} \rangle + \langle \phi_k^{\pm,\sharp}, C_{\mathfrak{p}'\cdot[j],k} \rangle\right)\Bigg), & \text{if } k \neq k_0. \end{cases}$$

Proof. To reduce notational clutter, we omit the $\pm$ from the notation. Unwinding (44) by means of (43), we have

$$\mathcal{L}_{\Phi/K,[j]}(k) = \langle \rho_k^{X_j} \rho_{X_j}^X \Phi^\sharp, [C_{j,k}] \rangle.$$

By Lemma 5.3, $X_j = Y - YP_j$. Therefore,

$$\rho_k^{X_j} \rho_{X_j}^X \Phi^\sharp = \mathrm{res}^{\Gamma_0}_{\Gamma_j} \rho_k^Y \rho_Y^X \Phi^\sharp - \mathrm{res}^{P_j^{-1}\Gamma_0 P_j}_{\Gamma_j} \rho_k^{YP_j} \rho_{YP_j}^X \Phi^\sharp$$

and

$$\mathcal{L}_{\Phi/K,j}(k) = \langle \rho_k^Y \rho_Y^X \Phi^\sharp, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle - \langle \rho_k^{YP_j} \rho_{YP_j}^X \Phi^\sharp, \mathrm{cores}^{P_j^{-1}\Gamma_0 P_j}_{\Gamma_j}[C_{j,k}] \rangle \tag{47}$$

We evaluate these terms separately. First, by Proposition 4.4

$$\begin{aligned} \langle \rho_k^Y \rho_Y^X \Phi^\sharp, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle &= \langle \rho_k^Y(\Phi), \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}]] \rangle \\ &= \mu(k) \langle \phi_k, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle. \end{aligned} \tag{48}$$

Moving on to the second term in (47), by Corollary 4.6 and Lemma 3.3 (2)

$$\begin{aligned} \langle \rho_k^{YP_j} \rho_{YP_j}^X \Phi^\sharp, \mathrm{cores}^{P_j^{-1}\Gamma_0 P_j}_{\Gamma_j}[C_{j,k}] \rangle &= \langle (\rho_k^{YP_j} \rho_{YP_j}^X \Phi^\sharp)|P_j^{-1}, P_j \cdot \mathrm{cores}^{P_j^{-1}\Gamma_0 P_j}_{\Gamma_j}[C_{j,k}] \rangle \\ &= p^{k/2} \langle \rho_k^Y(\rho_{YP_j}^X(\Phi^\sharp)|P_j^{-1}), \mathrm{cores}^{\Gamma_0}_{P_j \Gamma_j P_j^{-1}}[C_{P_j j P_j^{-1},k}] \rangle \\ &= p^{k/2} \langle \rho_k^Y(\mathbf{a}_p^{-1}\Phi), \mathrm{cores}^{\Gamma_0}_{P_j \Gamma_j P_j^{-1}}[C_{P_j j P_j^{-1},k}] \rangle \\ &= \mu(k) p^{k/2} \mathbf{a}_p^{-1}(k) \langle \phi_k, \mathrm{cores}^{\Gamma_0}_{P_j \Gamma_j P_j^{-1}}[C_{P_j j P_j^{-1},k}] \rangle. \end{aligned} \tag{49}$$

If $k = k_0$, then $\mu(k_0) = 1$, $\mathrm{cores}^{\Gamma_0}_{\Gamma_j} = C_{[j],k_0}$, and $\mathrm{cores}^{\Gamma_0}_{P_j \Gamma_j P_j^{-1}}[C_{P_j j P_j^{-1},k_0}] = C_{\mathfrak{p}'\cdot[j],k_0}$. Thus,

$$\mathcal{L}_{\Phi/K,j}(k_0) = \mu(k_0) \langle \phi_{k_0}, C_{[j],k} \rangle - \mu(k_0) p^{k_0/2} \mathbf{a}_p^{-1}(k_0) \langle \phi_{k_0}, C_{\mathfrak{p}'\cdot[j],k_0} \rangle$$

Now suppose $k \neq k_0$. Continuing from (48), using (31),

$$\begin{aligned} \langle \rho_k^Y \rho_Y^X \Phi^\sharp, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle &= \mu(k)(\langle \mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^{\pm,\sharp}, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle - \mathbf{a}_p(k)^{-1} \langle (\mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^{\pm,\sharp})|W_p, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle) \\ &= \mu(k)(\langle \phi_k^{\pm,\sharp}, C_{[j],k} \rangle - \mathbf{a}_p(k)^{-1} \langle (\mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^{\pm,\sharp})|W_p, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle). \end{aligned} \tag{50}$$

Noting that $Rw_p = RP$, we have

$$(\mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^\sharp)|W_p = \mathrm{res}^{w_p^{-1}\Gamma w_p}_{\Gamma_0}(\phi_k^\sharp|W_p) = \mathrm{res}^{P^{-1}\Gamma P}_{\Gamma_0}(\phi_k^\sharp|P).$$

Therefore,

$$\begin{aligned} \langle (\mathrm{res}^{\Gamma}_{\Gamma_0} \phi_k^\sharp)|W_p, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle &= \langle \phi_k^\sharp|P, \mathrm{cores}^{P^{-1}\Gamma P}_{\Gamma_0} \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle \\ &= \langle \phi_k^\sharp, P \cdot \mathrm{cores}^{P^{-1}\Gamma P}_{\Gamma_j}[C_{j,k}] \rangle \\ &= \langle \phi_k^\sharp, \mathrm{cores}^{\Gamma}_{P\Gamma_j P^{-1}}(P \cdot [C_{j,k}]) \rangle \\ &= p^{k/2} \langle \phi_k^\sharp, \mathrm{cores}^{\Gamma}_{\Gamma_{P^{-1}jP}} \cdot [C_{PjP^{-1},k}] \rangle \\ &= p^{k/2} \langle \phi_k^\sharp, C_{[\mathfrak{p}]\cdot[j],k} \rangle \quad . \end{aligned} \tag{51}$$

Combining (50) and (51), we get

$$\langle \rho_k^Y \rho_Y^X \Phi^\sharp, \mathrm{cores}^{\Gamma_0}_{\Gamma_j}[C_{j,k}] \rangle = \mu(k)(\langle \phi_k^\sharp, C_{[j],k} \rangle - \frac{p^{k/2}}{\mathbf{a}_p(k)} \langle \phi_k^\sharp, C_{[\mathfrak{p}]\cdot[j],k} \rangle) \tag{52}$$

Moving on to the second term in (47) and continuing from (49),

$$\langle \rho_k^{YP_j} \rho_{YP_j}^X \Phi^\sharp, \mathrm{cores}_{\Gamma_j}^{P_j^{-1}\Gamma_0 P_j}[C_{j,k}] \rangle = \mu(k) p^{k/2} \mathbf{a}_p^{-1}(k) (\langle \mathrm{res}_{\Gamma_0}^{\Gamma} \phi_k^\sharp, \mathrm{cores}_{P_j \Gamma_j P_j^{-1}}^{\Gamma_0} [C_{P_j j P_j^{-1}, k}] \rangle -$$

$$\mathbf{a}_p(k)^{-1} (\mathrm{res}_{\Gamma_0}^{\Gamma} \phi_k^\sharp) | W_p, \mathrm{cores}_{P_j \Gamma_j P_j^{-1}}^{\Gamma_0} [C_{P_j j P_j^{-1}, k}] \rangle) \qquad \text{(by (31))}$$

$$= \mu(k) p^{k/2} \mathbf{a}_p(k)^{-1} \left(\langle \phi_k^\sharp, C_{\mathfrak{p}' \cdot [j], k} \rangle - \frac{p^{k/2}}{\mathbf{a}_p(k)} \langle \phi_k^\sharp, C_{[\mathfrak{p}] \cdot [\mathfrak{p}'] \cdot [j], k} \rangle \right) \quad \text{(by (51))}$$

$$= \mu(k) \left(\frac{p^{k/2}}{\mathbf{a}_p(k)} \langle \phi_k^\sharp, C_{\mathfrak{p}' \cdot [j], k} \rangle - \frac{p^k}{\mathbf{a}_p(k)^2} \langle \phi_k^\sharp, C_{[j], k} \rangle \right) \tag{53}$$

Plugging (52) and (53) into (47), we obtain the desired formula

$$\mathcal{L}_{\Phi/K,[j]}(k) = \mu(k) \left(\left(1 + \frac{p^k}{\mathbf{a}_p(k)^2} \right) \langle \phi_k^\sharp, C_{[j],k} \rangle - \frac{p^{k/2}}{\mathbf{a}_p(k)} \left(\langle \phi_k^\sharp, C_{[\mathfrak{p}] \cdot [j], k} \rangle + \langle \phi_k^\sharp, C_{[\mathfrak{p}'] \cdot [j], k} \rangle \right) \right). \qquad \square$$

Recall the function $\mathcal{L}_{\Phi/K,\psi}$ defined in (46).

Proposition 5.5. *Let $k \in \Omega_{cl}$. Then*

$$\mathcal{L}_{\Phi/K,\psi}(k) = \begin{cases} d_K^{k/2} \eta(k) \left(1 - \frac{\psi(\mathfrak{p}) p^{k/2}}{\mathbf{a}_p(k)} \right) \left(1 - \frac{\psi(\mathfrak{p}') p^{k/2}}{\mathbf{a}_p(k)} \right) L^*(F_k/K, \psi, k/2+1) & \text{if } F_k \text{ is } p\text{-new}, \\ d_K^{k/2} \eta(k) \mu(k)^2 \left(1 - \frac{\psi(\mathfrak{p}) p^{k/2}}{\mathbf{a}_p(k)} \right)^2 \left(1 - \frac{\psi(\mathfrak{p}') p^{k/2}}{\mathbf{a}_p(k)} \right)^2 L^*(F_k^{\#}/K, \psi, k/2+1) & \text{if } F_k \text{ is } p\text{-old}. \end{cases}$$

Proof. We prove the stated result for $k \in \Omega_{\mathrm{cl}}^{p\text{-old}}$. The argument in the other case is similar. By Proposition 5.4,

$$\frac{1}{\mu(k)} \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^\ell \mathcal{L}_{\Phi/K, \mathfrak{p}^\ell \cdot [j]}(k) = \sum_{\ell=0}^{t-1} \Big\{ \psi(\mathfrak{p})^\ell (\left(1 + \frac{p^k}{\mathbf{a}_p(k)^2} \right) \langle \phi_k^\sharp, C_{\mathfrak{p}^\ell \cdot [j], k} \rangle -$$

$$\frac{p^{k/2}}{\mathbf{a}_p(k)} \left(\langle \phi_k^\sharp, C_{\mathfrak{p}^{\ell+1} \cdot [j], k} \rangle + \langle \phi_k^\sharp, C_{\mathfrak{p}^{\ell-1} \cdot [j], k} \rangle \right)) \Big\}$$

$$= \left(1 + \frac{p^k}{\mathbf{a}_p(k)^2} \right) \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^\ell \langle \phi_k^\sharp, C_{\mathfrak{p}^\ell \cdot [j], k} \rangle -$$

$$\frac{p^{k/2}}{\mathbf{a}_p(k)} \Big\{ \psi(\mathfrak{p}') \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^{\ell+1} \langle \phi_k^\sharp, C_{\mathfrak{p}^{\ell+1} \cdot [j], k} \rangle +$$

$$\psi(\mathfrak{p}) \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^{\ell-1} \langle \phi_k^\sharp, C_{\mathfrak{p}^{\ell-1} \cdot [j], k} \rangle \Big\}$$

$$= \left\{ 1 + \frac{p^k}{\mathbf{a}_p(k)^2} - \frac{p^{k/2}}{\mathbf{a}_p(k)} \left(\psi(\mathfrak{p}) + \psi(\mathfrak{p}') \right) \right\} \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^\ell \langle \phi_k^\sharp, C_{\mathfrak{p}^\ell \cdot [j], k} \rangle$$

$$= \left(1 - \frac{\psi(\mathfrak{p}) p^{k/2}}{\mathbf{a}_p(k)} \right) \left(1 - \frac{\psi(\mathfrak{p}') p^{k/2}}{\mathbf{a}_p(k)} \right) \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^\ell \langle \phi_k^\sharp, C_{\mathfrak{p}^\ell \cdot [j], k} \rangle. \tag{54}$$

Choose a system $\{\delta\} \subset \mathrm{Cl}_K^+$ of representatives for $\mathrm{Cl}_K^+/\langle \mathfrak{p} \rangle$. Then

$$\begin{aligned}
\frac{\mathcal{L}_{\Phi/K,\psi,[j]}(k)}{\mu(k)} &= \mu(k)^{-1} \cdot \sum_{\sigma \in \mathrm{Cl}_K^+} \psi(\sigma) \mathcal{L}_{\phi,\sigma\cdot[j]}(k) \\
&= \sum_{\delta} \psi(\delta) \left(\mu(k)^{-1} \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^{\ell} \mathcal{L}_{\phi,\mathfrak{p}^{\ell}\cdot\delta\cdot[j]}(k) \right) \\
&= \left(1 - \frac{\psi(\mathfrak{p})p^{k/2}}{\mathbf{a}_p(k)}\right)\left(1 - \frac{\psi(\mathfrak{p}')p^{k/2}}{\mathbf{a}_p(k)}\right) \sum_{\delta} \psi(\delta) \sum_{\ell=0}^{t-1} \psi(\mathfrak{p})^{\ell} \langle \phi_k^{\sharp}, C_{\mathfrak{p}^{\ell}\cdot\delta\cdot[j],k} \rangle \quad \text{(by (54))} \\
&= \left(1 - \frac{\psi(\mathfrak{p})p^{k/2}}{\mathbf{a}_p(k)}\right)\left(1 - \frac{\psi(\mathfrak{p}')p^{k/2}}{\mathbf{a}_p(k)}\right) \sum_{\sigma \in \mathrm{Cl}_K^+} \psi(\sigma) \langle \phi_k^{\sharp}, C_{\sigma\cdot[j],k} \rangle.
\end{aligned}$$

The desired result now follows from the definition of $\mathcal{L}_{\Phi/K,\psi}$ together with Theorem 3.5. ☐

Define

$$\nu(k) = \frac{\eta(k)\mu(k)^2}{\lambda(k)^2}, \qquad k \in \Omega_{\mathrm{cl}}. \tag{55}$$

Note that ν is independent of K and ψ.

Lemma 5.6. *The function ν extends to an element of $\mathcal{O}(\Omega)^{\times}$.*

Proof. Suppose ψ is a genus character corresponding to the quadratic Dirichlet characters χ_1 and χ_2 as described in the paragraph following Remark 1.7. Then we may thus rephrase Proposition 5.5 as:

$$\mathcal{L}_{\Phi/K,\psi}(k) = \begin{cases} d_K^{k/2}\nu(k)\left(1 - \frac{\chi_1(p)p^{k/2}}{\mathbf{a}_p(k)}\right)\left(1 - \frac{\chi_2(p)p^{k/2}}{\mathbf{a}_p(k)}\right) L^*(F_k/K,\psi,k/2+1) & \text{if } F_k \text{ is } p\text{-new}, \\ d_K^{k/2}\nu(k)\lambda(k)^2\left(1 - \frac{\chi_1(p)p^{k/2}}{\mathbf{a}_p(k)}\right)^2\left(1 - \frac{\chi_2(p)p^{k/2}}{\mathbf{a}_p(k)}\right)^2 L^*(F_k^{\sharp}/K,\psi,k/2+1) & \text{if } F_k \text{ is } p\text{-old}. \end{cases}$$

It now follows from (7) and (12) that

$$\mathcal{L}_{\Phi/K,\psi}(k) = d_K^{k/2}\nu(k)\mathcal{L}_{\mathbf{F},\chi_1}(k)\mathcal{L}_{\mathbf{F},\chi_2}(k).$$

for all $k \in \Omega_{\mathrm{cl}}$, for all real quadratic fields K satisfying Assumption 1.5 and $\epsilon_K(-N) = 1$, and all pairs (χ_1, χ_2) of quadratic Dirichlet characters corresponding to genus characters of K. Nonvanishing results of [28] may be applied to show that for each $k \in \Omega_{\mathrm{cl}}$, there is a pair (χ_1, χ_2) as above such that $\mathcal{L}_{\Phi/K,\psi}(k)$, $\mathcal{L}_{\mathbf{F},\chi_1}(k)$, and $\mathcal{L}_{\mathbf{F},\chi_2}(k)$ are all nonzero. For details, see [4, Proposition 5.2]. The result follows. ☐

Finally we define

$$\mathcal{L}_{\mathbf{F}/K,\psi}(k) = \frac{\mathcal{L}_{\Phi/K,\psi}(k)}{d_K^{k/2}\nu(k)}, \qquad k \in \Omega.$$

The $\epsilon_K(p) = +1$ case of Theorem 1.6 now follows from Proposition 5.5.

5.2. **p-adic L-functions when $\epsilon_K(p) = -1$.** In this subsection, we assume p is inert in K and take $j \in \mathcal{E}(\mathcal{O}, R)$.

Lemma 5.7 ([4, Lemma 3.7])**.** *We have $Q_j(x,y) \in \mathbb{Z}_p^{\times}$ if and only if $(x,y) \in X$, i.e. $X_j = X$.*

Proposition 5.8. *Suppose $k \in \Omega_{cl}^{p\text{-}old}$. Then*

$$\mathcal{L}_{\Phi^{\pm}/K,j}(k) = \mu^{\pm}(k)\left(1 - \frac{p^k}{\mathbf{a}_p(k)^2}\right)\langle \phi_k^{\pm,\sharp}, C_{[j],k} \rangle.$$

If F_{k_0} is p-new, then $\mathcal{L}_{\Phi/K,j}(k_0) = 0$.

Proof. Unwinding (44) by means of (43), we see that

$$\mathcal{L}_{\Phi^{\pm}/K,j}(k) = \langle \Phi_k^{\pm,\sharp}, C_{[j],k} \rangle, \qquad k \in \Omega_{\mathrm{cl}}^{p\text{-old}}.$$

The desired result now follows from Corollary 4.8. If F_{k_0} is p-new, then $\Phi_{k_0}^{\sharp} = 0$ by Lemma 4.9. Therefore,

$$\mathcal{L}_{\Phi/K,\psi,[j]}(k_0) = \langle \Phi_{k_0}^{\sharp}, C_{[j],k_0} \rangle = 0. \qquad \square$$

By arguments analogous to those of Proposition 5.5, we see that

$$\mathcal{L}_{\Phi/K,\psi}(k) = d_K^{k/2} \eta(k) \mu^{\pm}(k)^2 \left(1 - \frac{p^k}{\mathbf{a}_p(k)^2}\right)^2 L^*(F_k^{\sharp}/K, \psi, k/2+1). \tag{56}$$

Set

$$\mathcal{L}_{\mathbf{F}/K,\psi}(k) = \frac{\mathcal{L}_{\Phi/K,\psi}(k)}{d_K^{k/2}\nu(k)}. \tag{57}$$

The $\epsilon_K(p) = -1$ case of Theorem 1.6 now follows from (56).

It will be convenient to impose the following normalization at k_0, in order to appropriately state the results in terms of classical Coleman families, as it is done in the introduction. We note that the $\mathbb{Q}$-structure $S_{k_0+2}(pN^+D, \mathbb{Q})$ obtained by means of the q-expansion coincides with the de Rham $\mathbb{Q}$-structure $S_{k_0+2}(pN^+D)_{\mathbb{Q}}$, by the q-expansion principle.

Remark 5.9. By multiplicity one, we may fix a Hecke equivariant identification (unique up to Hecke equivariant $\mathbb{Q}$-automorphisms):

$$S_{k_0+2}(pN^+D, \mathbb{Q})^{\mathrm{new}} = S_{k_0+2}(pN^+D)_{\mathbb{Q}}^{\mathrm{new}} \xrightarrow{\sim} H^1(\Gamma_0^D(\mathfrak{p}\mathfrak{n}^+), V_k^B(\mathbb{Q}))^{\pm,\mathrm{new}}.$$

Working p-adically we may promote this correspondence to families (passing through new modular forms at k_0): this is a consequence of the Jacquet-Langlands correspondence investigated in [19] and the multiplicity one result [19, Corollary 11.4]): it means that, working over any finite Galois extension $E/\mathbb{Q}_p$, the correspondence $\mathbf{F} \mapsto \Phi$ is such a way that $\sigma(\mathbf{F}) \mapsto \sigma(\Phi)$, where $\sigma \in G_{E/\mathbb{Q}_p}$. Furthermore, the pairing (44) is $G_{E/\mathbb{Q}_p}$-equivariant, since it makes sense working with Ω defined over $E = \mathbb{Q}_p$. On the other hand, $\mathcal{L}_{\Phi/K,\psi,[j]}$ is obtained by pairing Φ with the class $[C_{j,\psi}] := \sum_{\sigma} \psi(\sigma)[C_{\sigma\cdot j}]$, which can only be defined assuming that $E \supset H_{K,\mathfrak{p}}$, from which it follows that we have $\sigma\left(\mathcal{L}_{\Phi/K,\psi}\right) = \mathcal{L}_{\sigma\Phi/K,\psi}$ when $\sigma \in G_{E/H_{K,\mathfrak{p}}}$. Finally, according to the subsequent Lemma 6.10, the value $\eta_{F_{k_0},D} := \nu(k_0)$ only depends on (F_{k_0}, D) and satisfies $\sigma\left(\eta_{F_{k_0},D}\right) = \eta_{\sigma\left(F_{k_0}\right),D}$. Under the assumption of Corollary 1.9 we have $\mathcal{L}_{\mathbf{F}/K,\psi}(k_0) = \mathcal{L}'_{\mathbf{F}/K,\psi}(k_0) = 0$ for every $F_{k_0} \in S_{k_0+2}(\Gamma_0(Np), \mathbb{C}_p)^{\mathrm{new}}_{\chi(-N)=\omega_N,\chi(p)=-\omega_p}$ and (57) gives $\mathcal{L}''_{\mathbf{F}/K,\psi}(k_0) = \frac{\mathcal{L}''_{\Phi/K,\psi}(k_0)}{d_K^{k_0/2}\eta_{F_{k_0}}}$. We deduce from Remark 5.2 that $F_{k_0} \mapsto \mathcal{L}''_{\mathbf{F}/K,\psi}(k_0)$ is a well defined function. It also respects $H_{K,\mathfrak{p}}$-structures by the above discussion. A similar remark also applies to the Mazur-Kitagawa p-adic L-functions considered in Theorem 1.4 and to the function $F_{k_0} \mapsto \mathcal{L}^{(4)}_{\mathbf{F}/K,\psi}(k_0)$ considered in Corollary 1.10.

Part 3. Derivatives of p-adic L-functions

In this part of the paper, we focus on a p-new point k_0, under the assumption that $\varepsilon_K(p) = -1$. The aim is to establish a relation between the derivatives at k_0 of our p-adic L-functions, that are known to vanish, and the so called Darmon cycles.

We fix, once and for all, a p-adic field $E/\mathbb{Q}_p$, and let the spaces of functions be E-valued, even if the notation will not reflect this fact. The same abuse is in force for the spaces of distributions to be considered. We let $E^{\mathrm{ur}} := E \cap \mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified subextension of E. The notation $(\cdot)^{\vee}$ will always mean E-dual.

This part of the paper is organized as follows. In §6, we state Theorem 6.7 and derive its consequences: the main result provides evidences to the rationality conjectures about Darmon cycles. Then, in §7, we prove this theorem, except up to showing that two previously defined Abel-Jacobi maps agree. This is the content of the last section, thus closing the circle.

6. p-adic L-functions: relations with Darmon classes

In this section, we first introduce a canonical arithmetic p-adic Abel-Jacobi map

$$\log_{v_*} AJ : H_1\left(\Gamma, \Delta_{v_*}\left(P_{k_0}\right)\right) \longrightarrow \mathbb{H}_{k_0}^{\vee},$$

where $\mathbb{H}_{k_0} := H^1_{par}\left(\Gamma_0, V_{k_0}\right)^{p\text{-new},\pm}$. Our interest in this map relies on the fact that it is directly linked with the Abel-Jacobi images of the so called Darmon cycles, whose definition is recalled in §§6.2. While the precise relationship with the derivatives of our p-adic L-functions is described by Theorem 6.7, whose proof is deferred to §7, here we derive its consequences in §§6.4.

6.1. **The arithmetic p-adic Abel-Jacobi map.** Recall that we let $\mathbf{GL}_2(\mathbb{Q}_p)$ acts from the left on the Bruhat-Tits tree $\mathcal{T}$, that we set $L_* := \mathbb{Z}_p^2$, $v_* := [L_*] \in \mathcal{V}$ and write $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) to denote the set of those $v \in \mathcal{V}$ that are at even (resp. odd) distance from v_*. Let $\mathbf{GL}_2^+(\mathbb{Q}_p) \subset \mathbf{GL}_2(\mathbb{Q}_p)$ be the subgroup of those elements g such that $\mathrm{ord}_p \det g$ is even. Consider the *unramified p-adic upper half-plane*

$$\mathcal{H}_p^{\mathrm{ur}} = \mathbb{P}^1(\mathbb{Q}_p^{\mathrm{ur}}) - \mathbb{P}_1(\mathbb{Q}_p).$$

Actually, $\mathcal{H}_p$ is a $\mathbb{Q}_p$-rigid analytic space such that $\mathcal{H}_p(K) = \mathbb{P}^1(K) - \mathbb{P}^1(\mathbb{Q}_p)$ for complete field extensions $K/\mathbb{Q}_p$. The group $\mathbf{GL}_2(\mathbb{Q}_p)$ acts from the left on $\mathcal{H}_p^{\mathrm{ur}}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

The reduction map

$$r : \mathcal{H}_p^{\mathrm{ur}} \to \mathcal{V}$$

is then $\mathbf{GL}_2(\mathbb{Q}_p)$-equivariant. Set $\mathcal{H}_{p,\pm}^{\mathrm{ur}} := r^{-1}\left(\mathcal{V}^{\pm}\right)$ and $\mathcal{H}_{p,v}^{\mathrm{ur}} := r^{-1}\left(v\right)$ for $v \in \mathcal{V}$. Note that, since $r\sigma = r$ for every $\sigma \in G_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p}$, the Galois group $G_{\mathbb{Q}_p^{\mathrm{ur}}/E^{\mathrm{ur}}}$ acts on the spaces $\mathrm{Div}\,\mathcal{H}_{p,*}^{\mathrm{ur}}$ and $\mathrm{Div}^0\,\mathcal{H}_{p,*}^{\mathrm{ur}}$ of divisors and degree zero divisors supported on $\mathcal{H}_{p,*}^{\mathrm{ur}}$, where $*$ may be empty, $\pm$ or $v \in \mathcal{V}$. For $*$ empty, $\pm$ or $v \in \mathcal{V}$, set $\Delta_* = \left(\mathrm{Div}\,\mathcal{H}_{p,*}^{\mathrm{ur}}\right)^{G_{\mathbb{Q}_p^{\mathrm{ur}}/E^{\mathrm{ur}}}}$, $\Delta_*^0 = \left(\mathrm{Div}^0\,\mathcal{H}_{p,*}^{\mathrm{ur}}\right)^{G_{\mathbb{Q}_p^{\mathrm{ur}}/E^{\mathrm{ur}}}}$, $\Delta_*(P_{k_0}) = \Delta_* \otimes P_{k_0}\left(E\right)$ and $\Delta_*^0(P_{k_0}) = \Delta_*^0 \otimes P_{k_0}\left(E\right)$. As remarked, the group $\mathbf{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{H}_p$ from the left by fractional linear transformations. We may view $\Delta_*(P_{k_0})$ and $\Delta_*^0(P_{k_0})$ as $\mathbf{GL}_2(\mathbb{Q}_p)$-modules (resp. $\mathbf{GL}_2^+(\mathbb{Q}_p)$-modules or $\mathbf{GL}_2(L)$-modules) when $*$ is empty (resp. $* = \pm$ or $* = v = [L]$) by the usual tensor product action. For future reference, we record the following exact sequence of $\mathbf{GL}_2(\mathbb{Q}_p)$-modules (resp. $\mathbf{GL}_2^+(\mathbb{Q}_p)$-modules or $\mathbf{GL}_2(L)$-modules) when $*$ is empty (resp. $* = \pm$ or $* = v = [L]$):

$$0 \to \Delta_*^0(P_{k_0}) \overset{i}{\to} \Delta_*(P_{k_0}) \overset{\deg}{\to} P_{k_0} \to 0. \tag{58}$$

Set $W := \mathbb{Q}_p^2 - \{0\}$ and let $\mathcal{A}\left(W\right)_{k_0,k_0}$ be the space of locally analytic functions on W that are homogeneus of degree k_0 under the action of $\mathbb{Q}_p^{\times}$. Write $\mathcal{D}(W)_{k_0,k_0}$ to denote the strong E-dual space of $\mathcal{A}\left(W\right)_{k_0,k_0}$. Let $\mathcal{D}(W)^0_{k_0,k_0} \subset \mathcal{D}(W)_{k_0,k_0}$ be the $\mathbf{GL}_2(\mathbb{Q}_p)$-subspace of those distributions that are zero on $P_{k_0}\left(E\right) \subset \mathcal{A}\left(W\right)_{k_0,k_0}$ and let $\mathcal{D}(W)^{0,b}_{k_0,k_0} \subset \mathcal{D}(W)^0_{k_0,k_0}$ be the subspace of bounded distributions (see [32, §3] and [40, Remark 5]).

Let $\ell = \log\left(\langle\cdot\rangle\right)$ or ord and, for every $\tau_1, \tau_2 \in \mathcal{H}_p^{\mathrm{ur}}$ and $P \in P_{k_0}\left(E\right)$, define

$$\theta_\ell^{\tau_2-\tau_1,P} : W \to \mathbb{C}_p, \qquad \theta_\ell^{\tau_2-\tau_1,P}\left(x,y\right) := \ell\left(\frac{y+\tau_2 x}{y+\tau_1 x}\right) P\left(x,y\right).$$

We extend the definition of $\theta_\ell^{\tau_2-\tau_1,P}$ by linearity, so that for $d \in \Delta_*^0$ and $P \in P_{k_0}\left(E\right)$, we may view $\theta_\ell^{d,P}$ as a function $\theta_\ell^{d,P} : W \to E$. Note that $\theta_\ell^{\tau_2-\tau_1,P}\left(t\left(x,y\right)\right) = t^{k_0}\theta_\ell^{\tau_2-\tau_1,P}\left(x,y\right)$ for every $t \in \mathbb{Q}_p^{\times}$; in particular $\theta_\ell^{d,P} \in \mathcal{A}\left(W\right)_{k_0,k_0}$ and the quantity

$$I_\ell^0\left(\mu, d \otimes P\right) := \mu\left(\theta_\ell^{d,P}\right) \in E,\ \mu \in \mathcal{D}(W)_{k_0,k_0},$$

makes sense.

Lemma 6.1. *The pairing*

$$I_\ell^0 : \mathcal{D}(W)^0_{k_0,k_0} \otimes \Delta^0_*(P_{k_0}) \to E$$

is $\boldsymbol{GL}_2(\mathbb{Q}_p)$-invariant (resp. $\boldsymbol{GL}_2^+(\mathbb{Q}_p)$-invariant or $\boldsymbol{GL}(L)$-invariant) for $*$ empty (resp. $\pm$ or $v = [L]$).

Proof. Exploiting the relation $g \mid y + \tau x = (c_g\tau + d_g)(y + (g\tau)x)$ gives the equivariance of the pairing I_ℓ^0. $\square$

Let $\pi : W \to \mathbb{P}^1(\mathbb{Q}_p)$ be the canonical projection $\pi(x,y) := y/x$. Write $\mathcal{E}$ to denote the set of oriented edges of the Bruhat-Tits tree $\mathcal{T}$. If $e \in \mathcal{E}$ let $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$ be the open compact subset corresponding to the ends originating from e and define $W_e := \pi^{-1}(U_e)$. Set $V_{k_0} := P_{k_0}^\vee$ and let $\mathcal{C}_{har}(\mathcal{E}, V_{k_0})$ be the space of harmonic cocycles, i.e. maps $c_* : \mathcal{E} \to V_{k_0}$ such that $c_{\overline{e}} = -c_e$ and $\sum_{s(e)=v} c_e = 0$ for every $v \in \mathcal{V}$.

Consider the $\mathbf{GL}_2(\mathbb{Q}_p)$-equivariant morphism

$$R : \mathcal{D}(W)^0_{k_0,k_0} \to \mathcal{C}_{har}(\mathcal{E}, V_{k_0}) \qquad (59)$$
$$R(\mu)_e(P) := \mu\left(P\chi_{W_e}\right).$$

If $e \in \mathcal{E}$ write $\rho_e : \mathcal{C}_{har}(\mathcal{E}, V_{k_0}) \to V_{k_0}$ to denote the evaluation morphism and set

$$R_e : \mathcal{D}(W)^0_{k_0,k_0} \overset{R}{\to} \mathcal{C}_{har}(\mathcal{E}, V_{k_0}) \overset{\rho_e}{\to} V_{k_0}.$$

Note that, if we define $\overline{e}_\infty \in \mathcal{E}$ by the rule $U_{\overline{e}_\infty} := \mathbb{Z}_p$, $W_{\overline{e}_\infty} = p^\infty Y$ and the stabilizer of $\overline{e}_\infty$ in $\widetilde{\Gamma}$ is Γ_0.

Lemma 6.2. *$R_{\overline{e}_\infty}$ induces in cohomology an isomorphism*

$$R_{\overline{e}_\infty} : H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,b}_{k_0,k_0}\right) \overset{\simeq}{\to} H^1(\Gamma_0, V_{k_0})^{p\text{-}new}.$$

Proof. Let $\mathcal{A}_{k_0}(\mathbb{Q}_p)$ be the space of locally analytic functions on $\mathbb{Q}_p$ that extend to meromorphic functions on $\mathbb{P}^1(\mathbb{Q}_p)$ with poles at ∞ of order at most k_0, and let $\mathcal{D}_{k_0}(\mathbb{Q}_p)$ be its dual, equipped with the strong topology. The association $F \mapsto f_F(z) := F(1,z)$ realizes an E-linear topological identification $\mathcal{A}(W)_{k_0,k_0} \overset{\simeq}{\to} \mathcal{A}_{k_0}(\mathbb{Q}_p)$, with inverse the association $f \mapsto F_f(x,y) := x^{k_0} f(y/x)$ (when $x = 0$, $F_f(x,y) := y^k \lim_{z\to 0} z^k f(1/z)$). With this identification the claim follows from Lemma 2.8 and Theorem 3.5 of [32]. $\square$

If M is a Hecke module which has a decomposition into Eisenstein and cuspidal parts, we write M_c to denote its cuspidal part. Since $H^1(\Gamma_0, V_{k_0})^{p-new}$ posseses such a decomposition with $H^1(\Gamma_0, V_{k_0})^{p-new}_c$ $= H^1_{par}(\Gamma_0, V_{k_0})^{p-new}$, $R_{\overline{e}_\infty}$ gives rise to

$$R_{\overline{e}_\infty} : H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,b}_{k_0,k_0}\right)_c \overset{\simeq}{\to} H^1(\Gamma_0, V_{k_0})^{p-new}_c =: \mathbb{H}_{k_0}.$$

We let $\mathbb{T}_{k_0}$ be the Hecke $\mathbb{Q}_p$-algebra generated by the Hecke operators T_ℓ for $\ell \nmid Np$, U_ℓ for $\ell | pN^+$ and W_ℓ for $\ell | D$, acting on $\mathbb{H}_{k_0}$.

Consider the following piece of the long exact sequence in $\widetilde{\Gamma}$-cohomology obtained from (58)

$$\cdots \longrightarrow H_2(\widetilde{\Gamma}, P_{k_0}) \overset{\partial}{\longrightarrow} H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0})) \overset{i}{\longrightarrow} H_1(\widetilde{\Gamma}, \Delta(P_{k_0})) \longrightarrow H_1(\widetilde{\Gamma}, P_{k_0}) \longrightarrow \cdots.$$

By [32, Lemma 3.10], the map i induces an isomorphism

$$\bar{i} : H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0}))/\operatorname{im}\partial \overset{\sim}{\longrightarrow} H_1(\widetilde{\Gamma}, \Delta(P_{k_0})).$$

Since I_ℓ^0 restricted to $\mathcal{D}(W)^0_{k_0,k_0}$ is $\widetilde{\Gamma}$-equivariant, when restricted to $\mathcal{D}(W)^{0,b}_{k_0,k_0}$, it induces by cap product $I_\ell^0 : H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0})) \to H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,b}_{k_0,k_0}\right)^\vee$. We define

$$AJ_\ell^0 : H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0})) \overset{I_\ell^0}{\to} H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,b}_{k_0,k_0}\right)^\vee \to H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,b}_{k_0,k_0}\right)_c \overset{R^{-1}_{\overline{e}_\infty}}{\to} \mathbb{H}^\vee_{k_0},$$

where the middle arrow is the projection onto the cuspidal part.

Theorem 6.3 ([32, Corollary 3.13])**.** *There is a unique $\mathcal{L} \in \mathbb{T}_{k_0}$ such that*

$$(I^0_{\log} - \mathcal{L} I^0_{\mathrm{ord}})\operatorname{im}\partial = 0.$$

Set $\log AJ^0 := AJ^0_{\log} - \mathcal{L}AJ^0_{\mathrm{ord}}$. Thus, we may consider the following commutative diagram, where $\log \overline{AJ}^0$ is induced by $\log AJ^0$:

$$\begin{array}{ccccccc} H_1\left(\Gamma, \Delta^0_{v_*}(P_{k_0})\right) & \stackrel{\iota_{v_*}}{\to} & H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0})) & & & & \\ \downarrow i & & \downarrow i & \stackrel{\pi}{\searrow} & & & \\ H_1\left(\Gamma, \Delta_{v_*}(P_{k_0})\right) & \stackrel{\iota_{v_*}}{\to} & H_1(\widetilde{\Gamma}, \Delta(P_{k_0})) & \stackrel{\bar{i}^{-1}}{\to} & H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0}))/\operatorname{im}\partial & \stackrel{\log \overline{AJ}^0}{\to} & \mathbb{H}^\vee_{k_0}. \end{array} \tag{60}$$

We define the arithmetic *Abel-Jacobi maps*

$$\log AJ : H_1(\widetilde{\Gamma}, \Delta(P_{k_0})) \longrightarrow \mathbb{H}^\vee_{k_0} \text{ and } \log_{v_*} AJ : H_1\left(\Gamma, \Delta_{v_*}(P_{k_0})\right) \longrightarrow \mathbb{H}^\vee_{k_0}$$

to be the composites in the above commutative diagram.

6.2. **Darmon classes and conjectures.** Let $V^D_{k_0}(N^+p)$ be the $\mathbb{Q}_p$-adic representation attached to weight k_0+2 cusp forms $S^D_{k+2}(\mathfrak{n}^+\mathfrak{p})$ and let $V^D_{k_0}(N^+p)^{p\text{-new}}$ be its p-new part; this construction is explained in [35] and [9]. By Fontaine's theory, we may associate to (the restriction to a decomposition group at p of) $V^D_{k_0}(N^+p)^{p\text{-new}}$ a filtered Frobenius module with a monodromy operator

$$D^D_{k_0}\left(N^+p\right)^{p\text{-new}} := \mathbb{D}_{\mathrm{st}}\left(V^D_{k_0}\left(N^+p\right)\right)^{p\text{-new}}.$$

Since p divides the level exactly, it is a $\mathbb{T}^{p\text{-new}}_{\mathbb{Q}_p}$-monodromy module over $\mathbb{Q}_p$ by results of [9].

On the other hand, as explained in [32, §4.2], the space $\mathbb{D}_{k_0} := \mathbb{H}^\vee_{k_0} \oplus \mathbb{H}^\vee_{k_0}$ has a natural structure of $\mathbb{T}^{p\text{-new}}_{\mathbb{Q}_p}$-monodromy module defined over $\mathbb{Q}_p$, whose filtration is given by

$$F^0\mathbb{D}_{k_0} = \mathbb{D}_{k_0}, \qquad F^j = \{(-\mathcal{L}x, x) : x \in \mathbb{H}^\vee_{k_0}\} \qquad (1 \le j \le k-1).$$

The following result was proved by the second author:

Theorem 6.4 ([40])**.** *There is an isomorphism of $\mathbb{T}^{p\text{-}new}_{\mathbb{Q}_p}$-monodromy modules defined over $\mathbb{Q}_p$*

$$\mathbb{D}_{k_0} \stackrel{\simeq}{\to} D^D_{k_0}\left(N^+p\right)^{p\text{-}new}$$

such that the diagram

$$\begin{array}{ccc} \mathbb{H}_{k_0}(E)^\vee \oplus \mathbb{H}_{k_0}(E)^\vee & \xrightarrow{\sim} & D^D_{k_0}(N^+p)^{p\text{-}new} \otimes E \\ \Big\downarrow {\scriptstyle (x,y)\mapsto x+\mathcal{L}y} & & \Big\downarrow \\ \mathbb{H}_{k_0}(E)^\vee & \xrightarrow{\sim} & \dfrac{D^D_{k_0}\left(N^+p\right)^{p\text{-}new}\otimes E}{F^{k_0/2+1}(D^D_{k_0}(N^+p)^{p\text{-}new}\otimes E)} \end{array}$$

commutes for every local field extension $E/\mathbb{Q}_p$.

As in the introduction, we will use the shorthand $m = k_0/2+1$. The Bloch-Kato exponential yields an isomorphism

$$\exp : \frac{D^D_{k_0}(\Gamma_0)^{p\text{-new}} \otimes E}{F^m(D^D_{k_0}(\Gamma_0)^{p\text{-new}} \otimes E)} \xrightarrow{\sim} H^1_{\mathrm{st}}(E, V^D_{k_0}\left(N^+p\right)^{p\text{-new}}(m)).$$

Let

$$\mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k_0})) \longrightarrow H^1_{\mathrm{st}}(E, V^D_{k_0}\left(N^+p\right)^{p\text{-new}}(m))$$

be the composite

$$H_1(\widetilde{\Gamma}, \Delta(P_{k_0})) \xrightarrow{\log AJ} \mathbb{H}_{k_0}(E)^\vee \xrightarrow{\sim} \frac{D^D_{k_0}(\Gamma_0)^{p\text{-new}} \otimes E}{F^m(D^D_{k_0}(\Gamma_0)^{p\text{-new}} \otimes E)} \xrightarrow{\exp} H^1_{\mathrm{st}}(E, V^D_{k_0}\left(N^+p\right)^{p\text{-new}}(m)).$$

The map AJ is a group cohomological analogue of the étale Abel-Jacobi map (see the discussion in the introduction and in §5 of [32]), justifying our notation for the map we denote $\log AJ$.

Let δ, $Q_{\widetilde{j}}$, and u be as in §3.3, where $\widetilde{j} \in \mathcal{E}\left(\widetilde{\mathcal{O}}, \widetilde{R}\right)$. Recall

$$\sigma_\infty : K \hookrightarrow \mathbb{R} \text{ and } \sigma_\infty : K \hookrightarrow \mathbb{Q}_{p^2}$$

we already fixed in the introdution. The torus $K^{\times}$ acts on $\mathcal{H}_p^{\mathrm{ur}}$ with two fixed points $\tau_{\widetilde{j}}, \overline{\tau}_{\widetilde{j}} \in \mathcal{H}_p \cap \sigma_p(K)$ via $\widetilde{j}$ and the identification $B \otimes \mathbb{Q}_p = \mathbf{M}_2(\mathbb{Q}_p)$. Of course we have $v_{\widetilde{j}} = r\left(\tau_{\widetilde{j}}\right) = r\left(\overline{\tau}_{\widetilde{j}}\right)$, and we may order $\left(\tau_{\widetilde{j}}, \overline{\tau}_{\widetilde{j}}\right)$ in such a way that $K^{\times}$ acts on the tangent space at $\tau_{\widetilde{j}}$ via $z \mapsto z/\overline{z}$. We fix $u \in \mathcal{O}_1^{\times}$ in such a way that $\sigma_\infty(u) > 0$. Then $D_{\widetilde{j},k_0} := j(u) \otimes (\tau_j \otimes \sigma_p\left(\delta^{-k_0/2}\right) Q_{\widetilde{j}}^{k_0/2})$ belongs to $Z_1(\widetilde{\Gamma}, \Delta(P_{k_0}))$, and its image in $H_1(\widetilde{\Gamma}, \Delta(P_{k_0}))$ is an element $D_{[\widetilde{j}],k_0}$ that only depends on the class of $\widetilde{j}$ in $\widetilde{\Gamma}\backslash\mathcal{E}\left(\widetilde{\mathcal{O}}, \widetilde{R}\right)$. We remark that, if $\left[\widetilde{j}\right] \in \widetilde{\Gamma}\backslash\mathcal{E}_+\left(\widetilde{\mathcal{O}}, \widetilde{R}\right)$, thanks to Lemma 2.1 (3), there is $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O}, R)$ mapping to $\left[\widetilde{j}\right]$. The element $D_{j,k_0} := j(u) \otimes (\tau_j \otimes \sigma_p\left(\delta^{-k_0/2}\right) Q_j^{k_0/2})$ gives an element $D_{[j],k_0} \in H_1(\Gamma, \Delta(P_{k_0}))$ such that

$$\log_{v_*} AJ\left(D_{[j],k_0}\right) = \log AJ\left(D_{[\widetilde{j}],k_0}\right). \tag{61}$$

The Darmon conjecture, phrased precisely below, asserts that the local cohomology classes $AJ(D_{[\widetilde{j}],k_0})$ are restrictions of global classes. We now define the appropriate global cohomology groups.

Definition 6.5. Let H be a number field. The *semistable Selmer group of V over H* is

$$\mathrm{Sel}_{\mathrm{st}}(H, V) := \ker\left(H^1(H, V) \overset{\prod_v \mathrm{res}_v}{\longrightarrow} \prod_v \frac{H^1(H_v, V)}{H^1_{\mathrm{st}}(H_v, V)}\right).$$

As before, let $\left[\widetilde{j}\right] \in \widetilde{\Gamma}\backslash\mathcal{E}_+\left(\widetilde{\mathcal{O}}, \widetilde{R}\right)$ and let $H := H_{\mathcal{O}}^+$ be the narrow ring class field of $\mathcal{O}$. Let $\mathfrak{P}$ be the prime of H above p induced by σ_p. Since $H/\mathbb{Q}$ is a Dihedral extension, it induces an identification $K_p \cong H_{\mathfrak{P}}$ and a restriction map

$$\mathrm{res}_p : H^1(H, V_{k_0}^D\left(N^+p\right)^{p\text{-new}}(m)) \longrightarrow H^1(K_p, V_{k_0}^D\left(N^+p\right)^{p\text{-new}}(m)).$$

Also note that, since $H_{\mathfrak{P}}$ is Galois over K, the group $\mathrm{Gal}(H/K) \cong \mathrm{Cl}^+(\mathcal{O})$ acts on $H^1(H, V_{k_0}^D\left(N^+p\right)^{p\text{-new}}(m))$.

Conjecture 6.6. *There is a class $s_{[j]} \in \mathrm{Sel}_{st}(H, V_{k_0}^D(N^+p)^{p\text{-}new}(m))$ such that*

$$\mathrm{AJ}(D_{[\widetilde{j}],k_0}) = \mathrm{res}_p\, s_{[j]}.$$

Moreover, the Shimura reciprocity law

$$s_{\sigma\cdot[j]} = s_{[j]}^{\sigma}$$

holds for all $\sigma \in \mathrm{Gal}(H/K)$.

Since their construction is based on techniques of Darmon, we call the local classes

$$AJ(D_{[\widetilde{j}],k_0}) \in H^1_{\mathrm{st}}(K_p, V_{k_0}^D\left(N^+p\right)^{p\text{-new}})$$

Darmon classes. It follows from Lemma 2.1 (1) and (61) that Conjecture 6.6 is equivalent to the same statement restricted to $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O}, R)$. Therefore, we will restrict ourself to the consideration of

$$AJ_{v_*} : H_1(\Gamma, \Delta(P_{k_0})) \overset{\log_{v_*} AJ}{\longrightarrow} \mathbb{H}_{k_0}(E)^{\vee} \overset{\sim}{\longrightarrow} \frac{D_{k_0}^D(\Gamma_0)^{p\text{-new}} \otimes E}{F^m(D_{k_0}^D(\Gamma_0)^{p\text{-new}} \otimes E)} \overset{\exp}{\longrightarrow} H^1_{\mathrm{st}}(E, V_{k_0}^D\left(N^+p\right)^{p\text{-new}}(m)).$$

If ψ is a character of $G_{H/K}$ and $[j] \in \Gamma\backslash\mathcal{E}^{\mathfrak{n}^+\mathfrak{d}}(\mathcal{O}, R)$, we define, in $H_1(\Gamma, \Delta(P_{k_0}))^{\psi^{-1}}$ and $H_1(\widetilde{\Gamma}, \Delta(P_{k_0}))^{\psi^{-1}}$[1],

$$D_{[j],k_0}^{\psi} := \sum_{\sigma \in G_{H_K^+/K}} \psi(\sigma)\, D_{[j],k_0},\ D_{[\widetilde{j}],k_0}^{\psi} := \sum_{\sigma \in G_{H_K^+/K}} \psi(\sigma)\, D_{[\widetilde{j}],k_0}.$$

Conjecture 6.6 predicts that

$$AJ_{v_*}\left(D_{[j],k_0}^{\psi}\right) = AJ\left(D_{[\widetilde{j}],k_0}\right) \in \mathrm{res}_p\left(\mathrm{Sel}(H_K^{+,\psi}, V_{k_0}^D\left(N^+p\right)^{p\text{-new}}(m))^{\psi^{-1}}\right).$$

[1]Here $(\cdot)^{\psi^{-1}}$ means the ψ^{-1}-component of $(\cdot) \otimes \mathbb{Z}[\psi]$, where $\mathbb{Z}[\psi]$ is obtained adding the values of ψ.

6.3. **Darmon classes and derivatives of p-adic L-functions.** Since p is inert, we know that $\mathcal{L}_{\Phi/K,[j]}$ and $\mathcal{L}_{\Phi/K,\psi,[j]}$ vanish at k_0, while $\mathcal{L}_{\Phi/K,\psi}$ vanishes at k_0 to order at least two. Thanks to Remark 5.2, $\varphi_{k_0} \mapsto \mathcal{L}'_{\Phi/K,[j]}(k_0)$ and $\varphi_{k_0} \mapsto \mathcal{L}'_{\Phi/K,\psi,[j]}(k_0)$ give well defined elements $\mathcal{L}'_{JL,[j]}(k_0) \in \mathbb{H}^{\text{new},\vee}_{k_0}(K_p)$ and $\mathcal{L}'_{JL,\psi,[j]}(k_0) \in \mathbb{H}^{\text{new},\vee}_{k_0}(K_p)^{\psi^{-1}}$, while $\varphi_{k_0} \mapsto \mathcal{L}''_{\Phi/K,\psi}(k_0)$ gives a well defined quadratic form $\mathcal{L}''_{JL,\psi}(k_0) : \mathbb{H}^{\text{new}}_{k_0}(K_p)^{\psi^{-1}} \to K_p$. The proof of the following result is postponed to §7.

Theorem 6.7. *The following equality holds, for an element $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O},R)$*

$$\mathcal{L}'_{JL,[j]}(k_0) = \frac{\delta^{k_0/2}}{2}\left(\log_{v_*} AJ\left(D_{[j],k_0}\right) + (-1)^{k_0/2+1}\log_{v_*} AJ\left(D_{\overline{[j]},k_0}\right)\right). \tag{62}$$

Corollary 6.8. *The following equalities hold, for a character ψ of $G_{H^+_{\mathcal{O}}/K}$ and $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O},R)$,*

$$\begin{aligned}
\mathcal{L}'_{JL,\psi,[j]}(k_0) &= \frac{\delta^{k_0/2}}{2}\left(\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right) + (-1)^{k_0/2+1}\log_{v_*} AJ\left(D^{\psi^{-1}}_{\overline{[j]},k_0}\right)\right)\\
\mathcal{L}''_{JL,,\psi}(k_0) &= \frac{d_K^{k_0/2}}{2}\left(\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right) + (-1)^{k_0/2+1}\log_{v_*} AJ\left(D^{\psi^{-1}}_{\overline{[j]},k_0}\right)\right)\\
&\quad\cdot\left(\log_{v_*} AJ\left(D^{\psi^{-1}}_{[j],k_0}\right) + (-1)^{k_0/2+1}\log_{v_*} AJ\left(D^{\psi}_{\overline{[j]},k_0}\right)\right).
\end{aligned}$$

Proof. The first equation readly follows from Theorem 6.7 and

$$\begin{aligned}
D^{\psi^{-1}}_{\overline{[j]},k_0} &= \sum_{\sigma\in G_{H/K}} \psi^{-1}(\sigma)\, D_{\sigma\overline{[j]},k_0} = \sum_{\sigma\in G_{H_{\mathcal{O}}/K}} \psi^{-1}(\overline{\sigma})\, y_{\overline{\sigma[j]}}\\
&= \sum_{\sigma\in G_{H_{\mathcal{O}}/K}} \psi(\sigma)\, y_{\overline{\sigma[j]}}.
\end{aligned}$$

Since $\mathcal{L}_{\Phi/K,\psi} = \mathcal{L}_{\Phi/K,\psi,[j]}\mathcal{L}_{\Phi/K,\psi^{-1},[j]}$ and $\mathcal{L}_{\Phi/K,\psi,[j]}(k_0) = \mathcal{L}_{\Phi/K,\psi^{-1},[j]}(k_0) = 0$, $\mathcal{L}'_{\Phi/K,\psi}(k_0) = 0$. Using this information we see that $\mathcal{L}''_{\Phi/K,\psi}(k_0) = 2\mathcal{L}'_{\Phi/K,\psi,[j]}(k_0)\,\mathcal{L}'_{\Phi/K,\psi^{-1},[j]}(k_0)$. Hence the second relation follows from the first. □

We now specialize to a genus character ψ attached to the pair of Dirichlet characters (χ_1,χ_2). Recall that $\chi_i(-N) = \psi(\infty\mathfrak{n})$ does not depend on $i = 1, 2$.

Corollary 6.9. *The following equality holds, for a genus character ψ of $G_{H^+_K/K}$ attached to the pair of Dirichlet characters (χ_1,χ_2) and $[j] \in \Gamma\backslash\mathcal{E}(\mathcal{O},R)$,*

$$\mathcal{L}''_{JL,\psi}(k_0) = \begin{cases} 2d_K^{k_0/2}\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)^2 & \text{if } \chi_i(-N) = \omega_N\\ 0 & \text{if } \chi_i(-N) = -\omega_N.\end{cases}$$

Proof. Lemma 2.1 (2) implies that $D^{\psi^{-1}}_{\overline{[j]},k_0} = \chi_i(-N)\,W_N D^{\psi^{-1}}_{[j],k_0}$. Since $\psi = \psi^{-1}$ and

$$\left(\log_{v_*} AJ\right)(W_N C) = \omega_{N,k_0}\log_{v_*} AJ(C)$$

(for every C), Corollary 6.8 yields

$$\mathcal{L}''_{\Phi/K,\psi}(k_0) = \frac{d_K^{k_0/2}}{2}\left(1 + (-1)^{k_0/2+1}\chi_i(-N)\,\omega_{N,k_0}\right)^2\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)\left(\varphi_{k_0}\right)^2.$$

The claim follows from this formula. □

6.4. **Darmon classes and their rationality.** We end this section with an application of our p-adic L-functions to Conjecture 6.6.

6.4.1. *Admissible modular parametrizations: changing the quaternion algebras.* In what follows, we restrict ourselves to the new part $V_{k_0}^{D}(N^+p)^{\mathrm{new}}$ of $V_{k_0}^{D}(N^+p)^{p\text{-new}}$. As we remarked in the introduction this is not a serious restriction for the purposes of providing evidences to Conjecture 6.6. Suppose that there is a prime $q \parallel N$ and let $N = MQ$ be a factorization such that Q is squarefree and divisible by an *odd* number of primes, as in the introduction. There is an Hecke equivariant identification of $G_{\mathbb{Q}}$-modules

$$V := V_{k_0}(Np)^{\mathrm{new}} \simeq V_{k_0}^{Qp}(M)^{\mathrm{new}} \simeq V_{k_0}^{D}\left(N^+p\right)^{\mathrm{new}}. \tag{63}$$

According to Theorems 1.3 and 6.4, (63) induces an Hecke equivariant identification of filtered Frobenius modules defined over $\mathbb{Q}_p$

$$\mathbb{D}_{k_0}^{\mathrm{new}} \simeq S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\mathrm{new},\vee,2}.$$

On the tangent spaces, this identification induces the following commutative diagram

$$\begin{array}{ccc} S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\mathrm{new},\vee,2}/F^{k_0/2+1} & \simeq & \mathbb{D}_{k_0}^{\mathrm{new}}/F^{k_0/2+1} \\ \downarrow & & \downarrow \\ S_{k_0+2}(\Gamma_0(Np), \mathbb{Q}_p)^{\mathrm{new},\vee} & \simeq & \mathbb{H}_{k_0}^{\vee,\mathrm{new}}, \end{array} \tag{64}$$

where the vertical arrows are $(x, y) \mapsto x + \mathcal{L}y$ and the lower row is defined in such a way that it makes the diagram commutative. By elementary linear algebra (see for example the proof of [38, Lemma 6.5]) we may assume that, after a base change from $\mathbb{Q}_p$ to a local field E such that $\sigma_p(\mathbb{Q}(F_{k_0})) \subset E$, the $\sigma_p \circ \lambda_{F_{k_0}}$-component of the lower row identification appearing in (64) is dual to $\varphi_{k_0} \mapsto F_{k_0}$ given by Remark 5.9. This fact will be implicit in the proof of the subsequent Theorem 6.11. Then, for every number field L and a prime $\mathfrak{p}$ of L over p, we may consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes L) & & & & \\ \downarrow \mathrm{cl}_{0,L}^{k_0/2+1} & & & & \\ \mathrm{Sel}_{\mathrm{st}}(L, V(m)) & \longrightarrow & H^1_{\mathrm{st}}(L_{\mathfrak{p}}, V(m)) & \longrightarrow & S_{k_0+2}(\Gamma_0(Np), L_{\mathfrak{p}})^{p\text{-new},\vee} \\ \| & & \| & & \| \\ \mathrm{Sel}_{\mathrm{st}}(L, V_{k_0}^{D}(N^+p)^{p\text{-new}}(m)) & \to & H^1_{\mathrm{st}}(L_{\mathfrak{p}}, V_{k_0}^{D}(N^+p)^{p\text{-new}}(m)) & \to & \mathbb{H}_{k_0}^{\vee,\mathrm{new}}(L_{\mathfrak{p}}). \end{array} \tag{65}$$

We write $\log \mathrm{cl}^{JL}$ (resp. cl^{JL}) for the composition going from $\mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}^{Qp} \otimes L)$ to $\mathbb{H}_{k_0}^{\vee,\mathrm{new}}(L_{\mathfrak{p}})$ (resp. $\mathrm{Sel}_{\mathrm{st}}(L, V_{k_0}^{D}(N^+p)^{p\text{-new}}(m))$).

Recall the function $\nu = \nu_{\mathbf{F},D}$ that enters in the factorization formula $\mathcal{L}_{\Phi/K,\psi}(\kappa) = d_K^{\kappa/2}\nu(\kappa)\mathcal{L}_{\mathbf{F},\chi_1}(\kappa)\mathcal{L}_{\mathbf{F},\chi_2}(\kappa)$ of a genus character ψ, that only depends on $\mathbf{F}$ and D. Its value $\eta_{F_{k_0},D} := \nu(k_0)$ at k_0 only depends on F_{k_0} and D.

Lemma 6.10. *We have $\eta_{F_{k_0},D} \in \mathbb{Q}(F_{k_0})^{\times 2}$ and $\eta_{\sigma(F_{k_0}),D} = \sigma\left(\eta_{F_{k_0},D}\right)$ for all $\sigma \in G_{\mathbb{Q}}$.*

Proof. We first remark that $\nu_{\mathbf{F},1} = 1$. Indeed, in this case $\eta(k) = 1$ for every $k \in \Omega_{\mathrm{cl}}^{p\text{-old}}$, as it follows from the explicit formula [31, Theorem 6.3.1]. In particular, we may assume that there exists a prime $q \parallel D$, and hence consider a factorization $D = D_1D_2$ with D_i divisible by an odd number of prime factors. If (χ_1', χ_2') is a pair of quadratic Dirichlet characters, let $\psi_{\chi_1',\chi_2'}$ be the associated genus character and let $d_{\chi_i'}$ be the discriminant of χ_i'.

To evaluate $\eta_{F_{k_0},D}$, choose a pair (χ_1, χ_2) such that $\chi_1(p) = \chi_2(p) = \omega_{p,k_0}$, $\chi_1(-N) = \chi_2(-N) = \omega_N$ and such that the complex L-functions of χ_i do not vanish at $k_0/2+1$, whose existence follows by non-vanishing results of [28]. Then $\mathcal{L}_{\psi_{\chi_1,\chi_2}} := \mathcal{L}_{\Phi/K,\psi}$, $\mathcal{L}_{\mathbf{F},\chi_1}$ and $\mathcal{L}_{\mathbf{F},\chi_2}$ do not vanish at k_0 and we have

$$\nu_{\mathbf{F},D}(\kappa) = \frac{\mathcal{L}_{\psi_{\chi_1,\chi_2}}(\kappa)}{d_{\chi_1}^{\kappa/2}d_{\chi_2}^{\kappa/2}\mathcal{L}_{\mathbf{F},\chi_1}(\kappa)\mathcal{L}_{\mathbf{F},\chi_2}(\kappa)}. \tag{66}$$

Let χ_i' be a quadratic Dirichlet character such that $\chi_i'(p) = \chi_i(p)$, $\chi_i'(-1) = -\chi_i(-1)$, $\chi_i'(l) = \chi_i(l)$ for every $l \mid N^+$, $\chi_1'(l) = -\chi_1(l)$ and $\chi_2'(l) = \chi_2(l)$ for $l \mid D_1$ and $\chi_1'(l) = \chi_1(l)$ and $\chi_2'(l) = -\chi_2(l)$ for $l \mid D_2$. Note that, since the number of primes dividing D_i is odd,

$$\chi_1'(-N) = \chi_1'(-1)\chi_1'\left(N^+\right)\chi_1'(D_1)\chi_1'(D_2) = \chi_1(-N) = \omega_N$$

and similarly $\chi_2'(-N) = \chi_2(-N) = \omega_N$. Hence, thanks to [28], we may arrange (χ_1', χ_2') in such a way that the complex L-functions of χ_i' do not vanish at $k_0/2+1$. Note that $\psi_{\chi_i,\chi_i'}$ is the genus character of an imaginary quadratic field $K_{\chi_i,\chi_i'}$ of discriminant $d_{\chi_i} d_{\chi_i'}$ such that the primes dividing N^+D_2 (resp. N^+D_1) are split in $K_{\chi_1,\chi_1'}$ (resp. $K_{\chi_2,\chi_2'}$), while the primes dividing D_1 (resp. D_2) are inert in $K_{\chi_1,\chi_1'}$ (resp. $K_{\chi_2,\chi_2'}$). It follows that $N = N^+D_2 \cdot D_1$ (resp. $N = N^+D_1 \cdot D_2$) is an admissible factorization and [4, Proposition 5.1] (or [38, Theorem 5.21] for arbitrary Coleman' families) yields the existence of $\mathcal{L}_{\psi_{\chi_i,\chi_i'}}, \nu_{\mathbf{F},D_i} \in \mathcal{O}(\Omega)$ such that

$$\mathcal{L}_{\psi_{\chi_i,\chi_i'}}(\kappa) = d_{\chi_i}^{\kappa/2} d_{\chi_i'}^{\kappa/2} \nu_{\mathbf{F},D_i}(\kappa)\, \mathcal{L}_{\mathbf{F},\chi_i}(\kappa)\, \mathcal{L}_{\mathbf{F},\chi_i'}(\kappa). \tag{67}$$

The condition placed on $\chi_i'(p) = \omega_{p,k_0}$ prevents the p-adic L-function $\mathcal{L}_{\mathbf{F},\chi_i'}$ from having an exceptional zero. It follows that we may recover $\mathcal{L}_{\mathbf{F},\chi_i}$ from (67). Similarly, $\psi_{\chi_1',\chi_2'}$ is the genus character of a real quadratic field $K_{\chi_1',\chi_2'}$ of discriminant $d_{\chi_1'} d_{\chi_2'}$ such that the primes dividing N split in $K_{\chi_1',\chi_2'}$ and, since we already remarked that $\nu_{\mathbf{F},1} = 1$,

$$\mathcal{L}_{\psi_{\chi_1',\chi_2'}}(\kappa) = d_{\chi_1'}^{\kappa/2} d_{\chi_2'}^{\kappa/2} \mathcal{L}_{\mathbf{F},\chi_1'}(\kappa)\, \mathcal{L}_{\mathbf{F},\chi_2'}(\kappa). \tag{68}$$

Inserting the expression of $\mathcal{L}_{\mathbf{F},\chi_i}$ recovered from (67) in (66) and then using (68) yields

$$\nu_{\mathbf{F},D}(\kappa) = \frac{\mathcal{L}_{\psi_{\chi_1,\chi_2}}(\kappa)\, \mathcal{L}_{\psi_{\chi_1',\chi_2'}}(\kappa)}{\mathcal{L}_{\psi_{\chi_1,\chi_1'}}(\kappa)\, \mathcal{L}_{\psi_{\chi_2,\chi_2'}}(\kappa)}.$$

Now the claim follows from [4, Theorem 3.12] (or [38, Corollary 5.19] for arbitrary Coleman' families), implying that $a_{i,F_{k_0}} := \mathcal{L}_{\psi_{\chi_i,\chi_i'}}(k_0) \in \mathbb{Q}(F_{k_0})^{\times 2}$ satisfies $a_{i,\sigma(F_{k_0})} = \sigma(a_{i,F_{k_0}})$ for all $\sigma \in G_{\mathbb{Q}}$, and our Theorem 1.6, joint with the analogue of Theorem 3.5 at $k = k_0$ and the defining relation $\mathcal{L}_{\Phi/K,\psi}(\kappa) = d_K^{\kappa/2} \nu(\kappa)\, \mathcal{L}_{\mathbf{F}/K,\psi}(\kappa)$, implying that $b_{F_{k_0}} := \mathcal{L}_{\psi_{\chi_1,\chi_2}}(k_0) \in \mathbb{Q}(F_{k_0})^{\times 2}$ satisfies $b_{\sigma(F_{k_0})} = \sigma(b_{F_{k_0}})$ for all $\sigma \in G_{\mathbb{Q}}$ and similarly for the quantity $b'_{F_{k_0}} := \mathcal{L}_{\psi_{\chi_1',\chi_2'}}(k_0)$. □

6.4.2. *The rationality of Darmon classes.* For a newform F_{k_0}, we write $V^D_{[F_{k_0}]} = V^D_{k_0}(N^+p)^{p\text{-new}}_{[F_{k_0}]}$ for the $[F_{k_0}]$-isotypic component of $V^D_{k_0}(N^+p)^{p\text{-new}}$. We also write

$$AJ_{v_*,[F_{k_0}]} : H_1(\Gamma, \Delta(P_{k_0})) \longrightarrow H^1_{\mathrm{st}}(K_p, V^D_{[F_{k_0}]}(m))$$

for the $[F_{k_0}]$-isotypic component of AJ_{v_*}, and a similar notation is in force for cl^{JL}.

Theorem 6.11. *Suppose that there is a prime q with $q \parallel N$ and choose a factorization $N = MQ$ as in §6.4.1. Let K be such that $\epsilon_K(p) = -1$ and $\epsilon_K(-N) = +1$. Let ψ be a genus character of K associated to the pair of Dirichlet characters (χ_1, χ_2), ordered in such a way that the sign of $L(F_{k_0}, \chi_1, k_0/2+1)$ is negative, and suppose that $\omega_N \chi_i(-N) = 1$ for one (or equivalently both) $i \in \{1,2\}$. Then the following facts hold:*

(1) *There is a cycle*

$$d^{\psi}_{[F_{k_0}]} = d^{\psi}_{Qp,[F_{k_0}]} \in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}^{Qp}_{k_0} \otimes \mathbb{Q}^{\chi_1})^{\chi_1} \subset \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}^{Qp}_{k_0} \otimes H^{\psi}_K)$$

such that, setting $s^{\psi}_{[F_{k_0}]} := \mathrm{cl}^{JL}_{[F_{k_0}]}\left(d^{\psi}_{[F_{k_0}]}\right)$,

$$\mathrm{res}_p\left(s^{\psi}_{[F_{k_0}]}\right) = AJ_{v_*,[F_{k_0}]}\left(D^{\psi}_{[j],k_0}\right).$$

(2) *If* $0 \neq s^{\psi}_{[F_{k_0}]} \in \mathrm{Sel}_{st}(H^{\psi}_K, V^D_{[F_{k_0}]}(m))$,

$$\mathrm{Sel}_{st}(H^{\psi}_K, V^D_{[F_{k_0}]}(m))^{\psi} = \mathrm{Sel}_{st}(\mathbb{Q}^{\chi_1}, V^D_{[F_{k_0}]}(m))^{\chi_1} = \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p \cdot s^{\psi}_{[F_{k_0}]} \simeq \mathbb{T}_{[F_{k_0}]} \otimes \mathbb{Q}_p.$$

(3) $0 \neq s^{\psi}_{[F_{k_0}]} \Leftrightarrow L'(F_{k_0}/K, \psi, k_0/2+1) \neq 0$ *when* $k_0 = 0$.

Proof. We are in the setting of Corollary 1.9; in particular, as recalled in the introduction, $\mathcal{L}_{\mathbf{F},\chi_1}$ vanishes to order at least two at k_0. The factorization formula $\mathcal{L}_{\Phi/K,\psi}(\kappa) = d_K^{\kappa/2}\nu_{\mathbf{F}}(\kappa)\mathcal{L}_{\mathbf{F},\chi_1}(\kappa)\mathcal{L}_{\mathbf{F},\chi_2}(\kappa)$, joint with Corollary 6.9, gives

$$2d_K^{k_0/2}\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)(\varphi_{k_0})^2 = \mathcal{L}''_{\Phi/K,\psi}(k_0) = d_K^{k_0/2}\nu(k_0)\mathcal{L}_{\mathbf{F},\chi_2}(k_0)\cdot\mathcal{L}''_{\mathbf{F},\chi_1}(k_0). \tag{69}$$

Since $\mathcal{L}_{\mathbf{F},\chi_2}$ does not have any exceptional zero at k_0, Theorem 1.1 implies that $\mathcal{L}_{\mathbf{F},\chi_2}(k_0) = 2L^*(F_{k_0},\chi_2,k_0/2+1)$ is zero if and only if $L(F_{k_0},\chi_2,k_0/2+1)=0$. Since $L(F_{k_0},\chi_2,k_0/2+1)=0$ if and only if $L(f,\chi_2,k_0/2+1)=0$ for all $f\in[F_{k_0}]$, in this case (69) applied to all f-components for all $f\in[F_{k_0}]$ shows that $\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)=0$ and the claim follows setting $d^{\psi}_{[F_{k_0}]}=0$. Otherwise, Corollary 1.9 (4) applies: joint with the defining relation $\mathcal{L}_{\Phi/K,\psi}(\kappa) = d_K^{\kappa/2}\nu_{\mathbf{F}}(\kappa)\mathcal{L}_{\mathbf{F}/K,\psi}(\kappa)$ implying $\mathcal{L}''_{\Phi/K,\psi}(\kappa) = d_K^{k_0/2}\eta_{F_{k_0}}\mathcal{L}_{\mathbf{F}/K,\psi}(k_0)$, it gives the existence of $\widetilde{y}^{\psi}\in \mathrm{CH}_0^{k_0/2+1}(\mathcal{M}_{k_0}\otimes\mathbb{Q}^{\chi_1})^{\chi_1}$ such that

$$2d_K^{k_0/2}\eta_{F_{k_0}}\,\mathrm{cl}\left(\widetilde{y}^{\psi}\right)^2(F_{k_0}) = d_K^{k_0/2}\eta_{F_{k_0}}\mathcal{L}''_{\mathbf{F},\chi}(k_0) = 2d_K^{k_0/2}\log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)(\varphi_{k_0})^2.$$

Thanks to Lemma 6.10 there is an Hecke operator $\eta\in\mathbb{T}^{\times 2}_{[F_{k_0}]}$ inducing $\eta_{F_{k_0}}$ on the F_{k_0}-component. Since the lower row identification appearing in (64) is dual to $\varphi_{k_0}\mapsto F_{k_0}$, the commutative diagram (65) implies that this relation taken over all f-components is equivalent to

$$\eta\,\mathrm{cl}^{JL}_{[F_{k_0}]}\left(\widetilde{y}^{\psi}\right)^2 = \log_{v_*} AJ\left(D^{\psi}_{[j],k_0}\right)^2.$$

Thanks to Lemma 6.10 it makes sense to set $d^{\psi}_{[F_{k_0}]} := \widetilde{\sqrt{\eta}}\widetilde{y}^{\psi}$, where $\widetilde{\sqrt{\eta}}$ is any lift of $\sqrt{\eta}\in\mathbb{T}^{\times}_{[F_{k_0}]}$ to the Hecke algebra acting on Chow groups. Claim (1) follows and the other assertions follow from the corresponding assertions in Corollary 1.9 (4) and our definition of $d^{\psi}_{[F_{k_0}]}$. ☐

Remark 6.12. When $k_0=0$, $\mathrm{cl}^{JL}_{[F_{k_0}]}$ factors through the Mordell-Weil group $A_{[F_{k_0}]}\left(H_K^{\psi}\right)\otimes\mathbb{Q}$. It is also true that $AJ_{v_*,[F_{k_0}]}$ factors through $A_{[F_{k_0}]}(K_p)\otimes\mathbb{Q}$ (see [18]). Since $(\cdot)\otimes\mathbb{Q}\hookrightarrow(\cdot)\otimes\mathbb{Q}_p$ is injective and the Bloch-Kato logarithm is compatible, up to the Kummer map, with the usual p-adic logarithm, our result actually gives evidences to the conjectures as formulated in [18]. See the final remark of [20] for the generalization of Greenberg's theory as formulated in [18] to include non rational eigenvalues, i.e. the case where $A_{[F_{k_0}]}$ may be of dimension greater than one.

7. Proof of Theorem 6.7

In this section, we first introduce a faux Abel-Jacobi map

$$\log_{v_*}\mathbf{AJ} : H_1(\Gamma,\Delta_{v_*}(P_{k_0}))\longrightarrow\mathbb{H}^{\vee}_{k_0}.$$

While $\log_{v_*} AJ$ was related with the Abel-Jacobi image of Darmon cycles, $\log_{v_*}\mathbf{AJ}$ is very close to the derivatives of our p-adic L-functions. Indeed, the proof of Theorem 6.7, can be divided in the following two steps: prove the claimed formula with $\log_{v_*} AJ$ replaced by $\log_{v_*}\mathbf{AJ}$ and then proof the equality $\log_{v_*} AJ = \log_{v_*}\mathbf{AJ}$. While the first step is carried on in this section, the latter is proved in §8.

7.1. **The faux Abel-Jacobi map.** Let $\mathcal{A}(W)_{k_0}$ be the space of locally analytic functions on W that are homogeneus of degree k_0 under the action of p. Write $\mathcal{D}(W)_{k_0}$ to denote the strong E-dual space of $\mathcal{A}(W)_{k_0}$ and set $\mathcal{D}(W)_{\Omega,k_0} := \mathcal{O}(\Omega)\,\widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^{\times})}\mathcal{D}(W)_{k_0}$. The restriction map $\mathcal{A}(W)_{k_0}\to\mathcal{A}(X)$ is easily checked to be a $\mathbf{GL}_2(\mathbb{Z}_p)$-equivariant identification, thus inducing a $\mathbf{GL}_2(\mathbb{Z}_p)$-equivariant $\mathcal{O}(\Omega)$-linear identification $\mathcal{D}(X)_\Omega\to\mathcal{D}(W)_{\Omega,k_0}$, whose inverse we denote by ρ_X^W[2] (see [40, before Lemma 7]). When $\Omega=\{k_0\}$, $\mathcal{D}(W)_{k_0,k_0}$ is identified with the strong E-dual space of $\mathcal{A}(W)_{k_0,k_0}$ introduced before Lemma 6.1, thus

[2]Note that, in spite of the notation, ρ_X^W has a different nature than ρ_Y^X. The latter is given by the rule $\rho_Y^X(\mu)(F) := \mu(F\chi_Y)$, where $F\chi_Y$ is F extended by zero. The first is given by $\rho_X^W(\mu)(F) := \mu\left(\widetilde{F}\right)$, where $\widetilde{F}$ is obtained from F extending the function by p^{k_0}-homogeneity.

justifying the overlap in the notation (see [40, Lemma 4]). We note that, as we assume $k_0 \in \Omega$, there is a $\mathbf{GL}_2(\mathbb{Q}_p)$-equivariant specialization map

$$\eta_{k_0} := k_0 \widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)} 1 : \mathcal{D}(W)_{\Omega,k_0} \to \mathcal{D}(W)_{k_0,k_0}.$$

We denote by $\mathcal{D}(W)^0_{\Omega,k_0} \subset \mathcal{D}(W)_{\Omega,k_0}$ (resp. $\mathcal{D}(W)^{0,b}_{\Omega,k_0} \subset \mathcal{D}(W)^0_{\Omega,k_0}$) the $\mathbf{GL}_2(\mathbb{Q}_p)$-subspace of those elements whose specialization at k_0 (under η_{k_0}) belongs to $\mathcal{D}(W)^0_{k_0,k_0}$ (resp. $\mathcal{D}(W)^{0,b}_{k_0,k_0}$), and we let $\mathcal{D}(X)^0_\Omega \simeq \mathcal{D}(W)^0_{\Omega,k_0}$ (resp. $\mathcal{D}(X)^{0,b}_\Omega \simeq \mathcal{D}(W)^{0,b}_{\Omega,k_0}$) be the corresponding space under the identification $\rho^W_X : \mathcal{D}(W)_{\Omega,k_0} \simeq \mathcal{D}(X)_\Omega$. Since the polynomials of homogeneus degree k_0, when viewed as functions on W, correspond to the same space of polynomials viewed as functions on X, $\mathcal{D}(X)^0_\Omega$ agrees with our previusly defined $\mathcal{D}(X)^0_\Omega$.

Let $\langle\cdot\rangle : \mathbb{Q}_p^{\mathrm{ur}\times} \to 1 + p\mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}}$ be the projection onto the principal units. Up to shrinking Ω around k_0 in an open affinoid defined over E, we may write $\kappa \in \Omega$ as $\kappa(t) = [t]^{k_0} \langle t \rangle^s$. Then we define $\langle t \rangle^{\kappa - k_0} := \exp((s-k_0)\log(\langle t \rangle))$, where $t \in \mathbb{Q}_p^{\mathrm{ur}\times}$. For every $\tau \in \mathcal{H}_p^{\mathrm{ur}}$ and $P \in P_{k_0}(E)$, define

$$\Theta^{\tau,P} : \Omega \times W \to \mathbb{C}_p$$
$$\Theta^{\tau,P}(\kappa,(x,y)) := \langle y + \tau x \rangle^{\kappa - k_0} P(x,y)\, 3.$$

We extend the definition of $\Theta^{\tau,P}$ by linearity so that, for $d \in \Delta_*$ and $P \in P_{k_0}(E)$, we may view $\Theta^{d,P}$ as a function $\Theta^{d,P} : W \to \mathcal{O}(\Omega)$, the target being $\mathcal{O}(\Omega)$ because $d \in \Delta_*$ is fixed by $G_{\mathbb{Q}_p^{\mathrm{ur}}/E^{\mathrm{ur}}}$ and $P \in P_{k_0}(E)$. Let $\mathcal{A}_\Omega(X)$ be the space defined in a similar way as $\mathcal{A}_\Omega(X_j)$ was defined at the beginning of §5. As above, by [19], the elements of $\mathcal{D}(X)_\Omega$ naturally integrate elements of $\mathcal{A}_\Omega(X)$. Note that $\Theta^{\tau,P}(\kappa, t(x,y)) = \kappa(t)\Theta^{\tau,P}(\kappa,(x,y))$ for every $t \in \mathbb{Z}_p^\times$ and $\Theta^{\tau,P}(\kappa, p(x,y)) = p^{k_0}\Theta^{\tau,P}(\kappa,(x,y))$; in particular $\Theta^{d,P}$ satisfies the same properties, $\Theta^{d,P}_{|X} \in \mathcal{A}_\Omega(X)$ and the quantity

$$\mu\left(\Theta^{d,P}\right) := \rho^W_X(\mu)\left(\Theta^{d,P}_{|X}\right) \in \mathcal{O}(\Omega),\ \mu \in \mathcal{D}(W)_{\Omega,k_0} \tag{70}$$

makes sense. We set $I(\mu, d \otimes P) := d_{k_0}\left[\mu\left(\Theta^{d,P}\right)\right] \in E$, where $d_{k_0} := \frac{d}{d\kappa}[\cdot]_{\kappa = k_0}$. Let E_τ be the field generated by $\tau \in \mathcal{H}_p^{\mathrm{ur}}$ over E and let Ω_{E_τ} be the base change of Ω over E to E_τ; we abusively write $\mu\left(\Theta^{\tau,P}\right) \in \mathcal{O}(\Omega_{E_\tau})$ and $I(\mu, \tau \otimes P) := d_{k_0}\left[\mu\left(\Theta^{\tau,P}\right)\right] \in E_\tau$ for any $\tau \in \mathcal{H}_p^{\mathrm{ur}}$, by viewing μ as an element of $\mathcal{D}(W)_{\Omega,k_0} = 1\widehat{\otimes}\mathcal{D}(W)_{\Omega,k_0} \subset \mathcal{D}(W)_{\Omega_{E_\tau},k_0}$.

Lemma 7.1. *The pairing*

$$I : \mathcal{D}(W)^0_{\Omega,k_0} \otimes \Delta_*(P_{k_0}) \to E$$

is $\boldsymbol{GL}_2(\mathbb{Q}_p)$-invariant (resp. $\boldsymbol{GL}_2^+(\mathbb{Q}_p)$-invariant or $\boldsymbol{GL}(L)$-invariant) for $*$ empty (resp. $\pm$ or $v = [L]$). Furthermore $I(\alpha\mu, C) = \alpha(k_0) I(\mu, C)$, for $\alpha \in \mathcal{O}(\Omega)$, $\mu \in \mathcal{D}(W)^0_{\Omega,k_0}$ and $C \in \Delta_*(P_{k_0})$.

Proof. Using the relation $g \mid y + \tau x = (c_g\tau + d_g)(y + (g\tau)x)$, the invariance of the pairing can be proved. The second assertion follows form [7, Lemma 4.11]. □

Consider the composition

$$\rho^{W/Y}_{k_0} : H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right) \overset{\rho^W_X}{\to} H^1\left(\Gamma, \mathcal{D}(X)^0_\Omega\right) \overset{\rho^X_Y}{\to} H^1\left(\Gamma_0, \mathcal{D}(Y)_\Omega\right) \overset{\rho^Y_{k_0}}{\to} H^1\left(\Gamma_0, V_{k_0}\right).$$

If $\varphi_{k_0} \in H^1_{par}(\Gamma_0, V_k)^{p\text{-new}}$, by Proposition 4.4, $\widetilde{\varphi_{k_0}} := \left(\rho^W_X\right)^{-1}\left(\Phi^\#\right) \in H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)$ satisfies

$$\rho^{W/Y}_{k_0}\left(\widetilde{\varphi_{k_0}}\right) = \rho^Y_{k_0}\left(\rho^X_Y\left(\Phi^\#\right)\right) = \rho^Y_{k_0}(\Phi) = \varphi_{k_0}.$$

Since I is Γ-invariant when restricted to $\mathcal{D}(W)^0_{\Omega,k_0}$, it induces by cap product

$$I : H_1(\Gamma, \Delta_{v_*}(P_{k_0})) \to H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)^\vee.$$

We define

$$\log_{v_*} \mathbf{AJ} : H_1(\Gamma, \Delta_{v_*}(P_{k_0})) \to \mathbb{H}^\vee_{k_0}$$

by the rule $\log_{v_*} \mathbf{AJ}(C)\left(\varphi_{k_0}\right) := I\left(\widetilde{\varphi_{k_0}}, C\right)$, where $\widetilde{\varphi_{k_0}} \in H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)$ is obtained as above from a lift $\Phi \in H^1\left(\Gamma_0, \mathcal{D}(Y)_\Omega\right)^{\leq h}$ of $\varphi_{k_0} \in \mathbb{H}_{k_0}$ such that $\Phi \mid U_p = \mathbf{a}_p \Phi$ (take $h < k_0 + 1$).

Lemma 7.2. $\log_{v_*} \mathbf{AJ}$ *is well defined, i.e. it does not depend on the choice of the lift* $\widetilde{\varphi_{k_0}} \in H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)$.

Proof. Let $I_{k_0} \subset \mathcal{O}(\Omega)$ be the ideal of functions that vanish at k_0. Then, by the control theorem proved in [19], Φ is well defined up to an element of $I_{k_0} H^1\left(\Gamma_0, \mathcal{D}(Y)_\Omega\right)^{\leq h}$. The associated $\widetilde{\varphi_{k_0}}$ is well defined up to an element of $I_{k_0} H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)$. Thus, it suffices to show that $I\left(\alpha\widetilde{\varphi}, C\right) = 0$ for $\alpha\widetilde{\varphi} \in I_{k_0} H^1\left(\Gamma, \mathcal{D}(W)^0_{\Omega,k_0}\right)$. But this follows immediately from Lemma 7.1: if $\alpha \in I_{k_0}$

$$I\left(\alpha\widetilde{\varphi}, C\right) = \alpha\left(k_0\right) I\left(\widetilde{\varphi}, C\right) = 0.$$

□

Remark 7.3. Strictly speaking we have defined $\log_{v_*} \mathbf{AJ}$ as a map with values in the new quotient $\mathbb{H}^{\mathrm{new},\vee}_{k_0}$ of $\mathbb{H}^\vee_{k_0}$, since we need to apply Theorem 4.2. However, a lift of $\log_{v_*} \mathbf{AJ}$ to a map with values in $\mathbb{H}^\vee_{k_0}$ is certainly possible. Indeed, we may take a basis of $\mathbb{H}_{k_0}$ by eigenvectors for the Hecke algebra generated by the Hecke operators prime to N. Then we may uniquely write such an eigenvector φ_{k_0} as a linear combination of eigenvectors arising from new eigenvectors of some level Mp with $M \mid N$, and apply Theorem 4.2 to these new eigenvectors. Working similarly as in the proof of [40, §6.2] shows that the required lift exists.

7.2. **Faux Abel-Jacobi map and derivatives of p-adic L-functions.** If $\tau_1, \tau_2 \in \mathcal{H}^{\mathrm{ur}}_p$ and $P \in P_{k_0}(E)$, define

$$\Theta^{\tau_1,\tau_2,P} : \Omega \times W \to \mathbb{C}_p$$
$$\Theta^{\tau,P}\left(\kappa, (x,y)\right) := \langle y + \tau_1 x\rangle^{\frac{\kappa-k_0}{2}} \langle y + \tau_2 x\rangle^{\frac{\kappa-k_0}{2}} P(x,y).$$

Again $\Theta^{\tau_1,\tau_2,P}$ satisfies the same homogeneity properties as $\Theta^{\tau,P}$. In particular, for an element $\mu \in \mathcal{D}(X)_{\Omega,k_0}$, $\mu\left(\Theta^{\tau_1,\tau_2,P}_{|X}\right) \in \mathcal{O}\left(\Omega_{E_{\tau_1,\tau_2}}\right)$ makes sense, if E_{τ_1,τ_2} is the field generated by τ_1 and τ_2 over E and $\Omega_{E_{\tau_1,\tau_2}}$ the base change of Ω to E_{τ_1,τ_2}. On the other hand, we already considered the quantity $\mu\left(\Theta^{\tau,P}_{|X}\right) \in \mathcal{O}\left(\Omega_{E_\tau}\right)$ appearing in the definition of the integration pairing I. We set $I_X\left(\mu, \tau \otimes P\right) := d_{k_0}\left[\mu\left(\Theta^{d,P}_{|X}\right)\right] \in E_\tau$ and extend the definition by linearity to $\Delta_*\left(P_{k_0}\right)$, thus obtaining a $\mathbf{GL}_2\left(\mathbb{Z}_p\right)$-invariant E-valued pairing

$$I_X : \mathcal{D}(X)^0_{\Omega,k_0} \otimes \Delta_*(P_k) \to E.$$

It is clear from (70) that

$$\log_{v_*} \mathbf{AJ}(C)\left(\varphi_{k_0}\right) = I_X\left(\Phi^\#, C\right), \tag{71}$$

where $\Phi^\# \in H^1\left(\Gamma, \mathcal{D}(X)^0_\Omega\right)$ is obtained from a lift $\Phi \in H^1\left(\Gamma_0, \mathcal{D}(Y)_\Omega\right)^{\leq h}$ of $\varphi_{k_0} \in \mathbb{H}_{k_0}$ such that $\Phi \mid U_p = \mathbf{a}_p \Phi$ (take $h < k_0 + 1$).

Lemma 7.4. *If* $\tau_1, \tau_2 \in \mathcal{H}^{ur}_p$, $P \in P_{k_0}(E)$ *and* $\mu \in \mathcal{D}(X)_{\Omega,k_0}$

$$d_{k_0}(\mu\left(\Theta^{\tau_1,\tau_2,P}_{|X}\right)) = \frac{1}{2}(I_X\left(\mu, \tau_1 \otimes P\right) + I_X\left(\mu, \tau_2 \otimes P\right)).$$

Proof. See [39, Proposition 3.1] □

Proof of Theorem 6.7. Write $Q_j(X,Y) = A\left(X + \tau_j Y\right)\left(X + \overline{\tau}_j Y\right)$. Since $Q_j(x,y) \in \mathbb{Z}_p^\times$ for $(x,y) \in X$,

$$Q_j(x,y)^{\kappa/2} = Q_j(x,y)^{\frac{\kappa-k_0}{2}} Q_j(x,y)^{k_0/2} = \langle Q_j(x,y)\rangle^{\frac{\kappa-k_0}{2}} Q_j(x,y)^{k_0/2} = \langle A\rangle^{\frac{\kappa-k_0}{2}} \Theta^{\tau_j,\overline{\tau}_j,Q_j^{k_0/2}}.$$

Let us write $c_{\Phi^\#}$ for a cocycle representing $\Phi^\#$ and set $\gamma_j := j(u)$. By definition

$$\mathcal{L}_{\Phi/K,[j]}(\kappa) = c_{\Phi^\#}\left(\gamma_j\right)\left(Q_j(x,y)^{\kappa/2}\right) = \langle A\rangle^{\frac{\kappa-k_0}{2}} c_{\Phi^\#}\left(\Theta^{\tau_j,\overline{\tau}_j,Q_j^{k_0/2}}\right)(\kappa).$$

Note also that $\left(\Theta^{\tau_j,\overline{\tau}_j,Q_j^{k_0/2}}\right)(k_0) = \mathcal{L}_{\Phi/K,[j]}(k_0) = 0$, so that Lemma 7.4 applied to $c_{\Phi^\#}\left(\gamma_j\right) \in \mathcal{D}(X)_\Omega$ yields

$$\mathcal{L}'_{\Phi/K,[j]}(k_0) = \frac{1}{2}\left(I_X\left(c_{\Phi^\#}\left(\gamma_j\right), \tau_j \otimes Q_j^{k_0/2}\right) + I_X\left(c_{\Phi^\#}\left(\gamma_j\right), \overline{\tau}_j \otimes Q_j^{k_0/2}\right)\right). \tag{72}$$

Since $c_{\Phi^\#}\left(\gamma_j\right) = -c_{\Phi^\#}\left(\gamma_j^{-1}\right)\gamma_j^{-1}$ (by the cocyle relation), $\gamma_j^{-1}\overline{\tau}_j = \overline{\tau}_j$, $\gamma_j^{-1}Q_j^{k_0/2} = Q_j^{k_0/2}$ and the pairing I_X is $\mathbf{GL}_2(\mathbb{Z}_p)$-invariant when we restrict to $\mathcal{D}(X)_\Omega^0$,

$$I_X\left(c_{\Phi^\#}\left(\gamma_j\right), \overline{\tau}_j \otimes Q_j^{k_0/2}\right) = -I_X\left(c_{\Phi^\#}\left(\gamma_j^{-1}\right), \overline{\tau}_j \otimes Q_j^{k_0/2}\right).$$

Note that $\left(\gamma_{\overline{j}}, \tau_{\overline{j}}, Q_{\overline{j}}\right) = \left(\gamma_j^{-1}, \overline{\tau}_j, -Q_j\right)$, so that

$$I_X\left(c_{\Phi^\#}\left(\gamma_j\right), \overline{\tau}_j \otimes Q_j^{k_0/2}\right) = (-1)^{k_0/2+1} I_X\left(c_{\Phi^\#}\left(\gamma_{\overline{j}}\right), \tau_{\overline{j}} \otimes Q_{\overline{j}}^{k_0/2}\right). \tag{73}$$

By definition and (71),

$$I_X\left(c_{\Phi^\#}\left(\gamma_j\right), \tau_j \otimes Q_j^{k_0/2}\right) = \delta^{k_0/2} I_X\left(\Phi^\#, D_{[j],k_0}\right) = \delta^{k_0/2}\log_{v_*}\mathbf{AJ}\left(D_{[j],k_0}\right)\left(\varphi_{k_0}\right),$$

$$I_X\left(c_{\Phi^\#}\left(\gamma_{\overline{j}}\right), \tau_{\overline{j}} \otimes Q_{\overline{j}}^{k_0/2}\right) = \delta^{k_0/2}\log_{v_*}\mathbf{AJ}\left(D_{\overline{[j]},k_0}\right)\left(\varphi_{k_0}\right). \tag{74}$$

Combining (72), (73) and (74) yields the claimed equality (62), except that $\log_{v_*} AJ$ is replaced by $\log_{v_*}\mathbf{AJ}$. The full proof of (62) is then achieved by the following result, shown in the next section.

Proposition 7.5. $\log_{v_*} AJ = \log_{v_*}\mathbf{AJ}$.

8. Equality of the arithmetic and the faux Abel-Jacobi maps

We prove in this section Proposition 7.5 asserting the coincidence of the previously defined Abel-Jacobi maps. Let $\mathcal{T}^p$ be the abstract Hecke algebra generated by the operators T_ℓ for $\ell \nmid Np$, U_ℓ for $\ell | N^+$ and W_ℓ for $\ell | D$, acting on the homology and cohomology groups by double cosets operators. Then $\log_{v_*} AJ, \log_{v_*}\mathbf{AJ} \in Hom_{\mathcal{T}^p}\left(H_1\left(\Gamma, \Delta_{v_*}\left(P_{k_0}\right)\right), \mathbb{H}_{k_0}^\vee\right)$. Consider the following piece of the long exact sequence in Γ-cohomology obtained from (58)

$$\cdots \longrightarrow H_2(\Gamma, P_{k_0}) \xrightarrow{\partial} H_1(\Gamma, \Delta_{v_*}^0(P_{k_0})) \xrightarrow{i} H_1(\Gamma, \Delta_{v_*}(P_{k_0})) \xrightarrow{\deg} H_1(\Gamma, P_{k_0}) \longrightarrow \cdots. \tag{75}$$

The strategy of the proof is simple: first prove that $\log_{v_*} AJ \circ i = \log_{v_*}\mathbf{AJ} \circ i$ and then show that this equality suffices to deduce the full equality $\log_{v_*} AJ = \log_{v_*}\mathbf{AJ}$. This latter implication is granted by the following proposition.

Proposition 8.1. *The composition with i induces an inclusion*

$$\mathrm{Hom}_{\mathcal{T}^p}\left(H_1\left(\Gamma, \Delta_{v_*}\left(P_{k_0}\right)\right), \mathbb{H}_{k_0}^\vee\right) \hookrightarrow \mathrm{Hom}_{\mathcal{T}^p}\left(H_1\left(\Gamma, \Delta_{v_*}^0\left(P_{k_0}\right)\right), \mathbb{H}_{k_0}^\vee\right).$$

Proof. We extract from the long exact sequence (75) the following short exact sequence

$$0 \longrightarrow \operatorname{coker}\partial \longrightarrow H_1(\Gamma, \Delta_{v_*}(P_{k_0})) \longrightarrow \operatorname{im}\deg \longrightarrow 0$$

and apply the functor $\mathrm{Hom}_{\mathcal{T}^p}(-, \mathbb{H}_{k_0}^\vee)$ to it, yielding

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}^p}(\operatorname{im}\deg, \mathbb{H}_{k_0}^\vee) \longrightarrow \mathrm{Hom}_{\mathcal{T}^p}(H_1(\Gamma, \Delta(P_{k_0})), \mathbb{H}_{k_0}^\vee) \longrightarrow$$
$$\mathrm{Hom}_{\mathcal{T}^p}(\operatorname{coker}\partial, \mathbb{H}_{k_0}^\vee) \longrightarrow \mathrm{Ext}^1_{\mathcal{T}^p}(\operatorname{im}\deg, \mathbb{H}_{k_0}^\vee) \longrightarrow \cdots$$

Since only cuspidal, p-new systems of $\mathcal{T}^p$-eigenvalues occur in $\mathbb{H}_{k_0}^\vee$ while the systems occuring in $\operatorname{im}\deg \subset H_1(\Gamma, P_{k_0})$ are either Eisenstein or p-old, we have

$$\mathrm{Hom}_{\mathcal{T}^p}(\operatorname{im}\deg, \mathbb{H}_{k_0}^\vee) = \mathrm{Ext}^1_{\mathcal{T}^p}(\operatorname{im}\deg, \mathbb{H}_{k_0}^\vee) = 0.$$

The result follows. □

It remains to be checked that $\log_{v_*} AJ \circ i = \log_{v_*} \mathbf{AJ} \circ i$ and this will be achieved by showing that both of them equals a thirt map, namely

$$AJ^0_{\log,v_*} : H_1\left(\Gamma, \Delta^0_{v_*}\left(P_{k_0}\right)\right) \overset{\iota_{v_*}}{\to} H_1(\widetilde{\Gamma}, \Delta^0(P_{k_0})) \overset{AJ^0_{\log}}{\to} \mathbb{H}^\vee_{k_0}.$$

This is the content of the subsequent lemmas 8.2 and 8.4.

Lemma 8.2. *We have* $\log AJ \circ i = \log AJ^0$ *and* $\log_{v_*} AJ \circ i = \log AJ^0 \circ \iota_{v_*} =: \log_{v_*} AJ^0$. *Furthermore,* $\log_{v_*} AJ^0 = AJ^0_{\log} \circ \iota_{v_*} =: AJ^0_{\log,v_*}$.

Proof. The equalities $\log AJ \circ i = \log AJ^0$ and $\log_{v_*} AJ \circ i = \log_{v_*} AJ^0$ follow from the defining diagram (60). The equality $\log_{v_*} AJ^0 = AJ^0_{\log,v_*}$ follows from [6, Lemma 2.5]. Indeed, up to the identification $R : H^1\left(\widetilde{\Gamma}, \mathcal{D}\left(W\right)^{0,b}_{k_0,k_0}\right) \overset{\simeq}{\to} H^1\left(\widetilde{\Gamma}, \mathcal{C}_{har}\left(\mathcal{E}, V_{k_0}\right)\right)$ of [32, Theorem 3.5], it implies that I^0_{ord} is induced by

$$I^0_{\mathrm{ord}}\left(c, \tau_2 - \tau_1 \otimes P\right) := \sum_{e: r(\tau_1) \to r(\tau_2)} c_e\left(P\right).$$

It follows that $I^0_{\mathrm{ord}} = 0$ when we restrict to $\Delta^0_{v_*}\left(P_{k_0}\right)$, so that $I^0_{\mathrm{ord}} \circ \iota_{v_*} = 0$ and

$$AJ^0_{\log} \circ \iota_{v_*} = AJ^0_{\log} \circ \iota_{v_*} - \mathcal{L} AJ^0_{\mathrm{ord}} \circ \iota_{v_*} = AJ^0_{\log} \circ \iota_{v_*}.$$

□

Before proving Lemma 8.4 and, hence, completing the proof of Proposition 7.5, we need the following result. The proof of this commutativity, which is implicit in [7, proof of Lemma 4.10], is left to the reader.

Lemma 8.3. *The following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{D}\left(W\right)^0_{\Omega,k_0} & \otimes & \Delta_*(P_{k_0}) & \overset{I}{\to} & E \\ \eta_{k_0}\downarrow & & \uparrow i & & \| \\ \mathcal{D}\left(W\right)^0_{k_0,k_0} & \otimes & \Delta^0_*(P_{k_0}) & \overset{I^0_{\log}}{\to} & E. \end{array}$$

Lemma 8.4. $\log_{v_*} \mathbf{AJ} \circ i = AJ^0_{\log,v_*}$.

Proof. Set $\psi_{k_0} := R^{-1}_{\overline{e}_\infty}\left(\varphi_{k_0}\right)$ and consider the morphism $\iota^{\widetilde{\Gamma}}_{\Gamma} : H^1\left(\widetilde{\Gamma}, \mathcal{D}\left(W\right)^{0,b}_{k_0,k_0}\right) \to H^1\left(\Gamma, \mathcal{D}\left(W\right)_{k_0,k_0}\right)$ induced by the restriction and the inclusion $\iota : \mathcal{D}\left(W\right)^{0,b}_{k_0,k_0} \subset \mathcal{D}\left(W\right)_{k_0,k_0}$ in either orders. We claim that we may choose $\widetilde{\varphi_{k_0}}$ is such a way that

$$\eta_{k_0}\left(\widetilde{\varphi_{k_0}}\right) = \iota^{\widetilde{\Gamma}}_{\Gamma}\left(\psi_{k_0}\right). \tag{76}$$

This fact may be deduced from [40, Theorem 13] as follows (choose $k_0/2 \le h < k_0 + 1$ in order to apply the result). For that we recall the definition of some relevant spaces of distributions that appear in [40]. Set $L_* = L^+_* := \mathbb{Z}^2_p$, $L^-_* := p^{-1} L_* w_p = \mathbb{Z}_p \times p^{-1}\mathbb{Z}_p$ and $v^\pm_* := [L^\pm_*] \in \mathcal{V}^\pm$; it will be convenient to set $\Gamma^+ := \Gamma$ and $\Gamma^- := w_p^{-1} \Gamma w_p$. We note that $p^{-1} Z w_p = Y$ and that, setting $X^- := L^-_* - pL^-_* = p^{-1} X w_p$, there is a Shapiro's isomorphism $\rho^{X^-}_Y : H^i\left(\Gamma^-, \mathcal{D}(X^-)_{\Omega,k_0}\right) \simeq H^i\left(\Gamma_0, \mathcal{D}(Y)_{\Omega,k_0}\right)$ which is the analogous of the Shapiro's isomorphism $\rho^{X^+}_Z : H^i\left(\Gamma^+, \mathcal{D}(X^+)_{\Omega,k_0}\right) \simeq H^i\left(\Gamma_0, \mathcal{D}(Z)_{\Omega,k_0}\right)$ we already considered for $X^+ := X$. We denote by $\mathcal{C}\left(\mathcal{V}^\pm, \mathcal{D}_{\Omega,k_0}\right)$ the right $\mathbf{GL}^+_2(\mathbb{Q}_p)$-module of maps $\mu_* : \mathcal{V}^\pm \to \mathcal{D}(W)_{\Omega,k_0}$. By Shapiro's Lemma, the evalutation morphism $\rho_{v^\pm_*} : \mathcal{C}\left(\mathcal{V}^\pm, \mathcal{D}_{\Omega,k_0}\right) \to \mathcal{D}(W)_{\Omega,k_0}$, followed by the $\mathbf{GL}(L^\pm_*)$-equivariant identification $\rho^W_{X^\pm} : \mathcal{D}(W)_{\Omega,k_0} \simeq \mathcal{D}(X^\pm)_\Omega$ (see [40, before Lemma 7]), induces

$$\rho_{X^\pm, v^\pm_*} : H^i\left(\widetilde{\Gamma}, \mathcal{C}\left(\mathcal{V}^\pm, \mathcal{D}_{\Omega,k_0}\right)\right) \overset{\rho_{v_*}}{\simeq} H^i\left(\Gamma^\pm, \mathcal{D}\left(W\right)_{\Omega,k_0}\right) \overset{\rho^W_{X^\pm}}{\simeq} H^i\left(\Gamma^\pm, \mathcal{D}\left(X^\pm\right)_\Omega\right).$$

A further composition with Shapiro's isomophisms $\rho^{X^+}_Z$ and $\rho^{X^-}_Y$ yields

$$\begin{aligned} \rho^+ \quad &: \quad = \rho^{X^+}_Z \circ \rho_{X^+, v^+_*} : H^i\left(\widetilde{\Gamma}, \mathcal{C}\left(\mathcal{V}^+, \mathcal{D}_{\Omega,k_0}\right)\right) \overset{\simeq}{\to} H^i\left(\Gamma_0, \mathcal{D}\left(Z\right)_\Omega\right), \\ \rho^- \quad &: \quad = \rho^{X^-}_Y \circ \rho_{X^-, v^-_*} : H^i\left(\widetilde{\Gamma}, \mathcal{C}\left(\mathcal{V}^-, \mathcal{D}_{\Omega,k_0}\right)\right) \overset{\simeq}{\to} H^i\left(\Gamma_0, \mathcal{D}\left(Y\right)_\Omega\right). \end{aligned}$$

Let $\iota_{\pm}:\mathcal{D}(W)_{k_0,k_0}^{0,b}\subset\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{k_0,k_0}\right)$ be the $\mathbf{GL}_2^+\left(\mathbb{Q}_p\right)$-equivariant inclusion obtained by viewing the elements of $\mathcal{D}(W)_{k_0,k_0}^{0,b}$ as constant functions on $\mathcal{V}^{\pm}$. The specialization map η_{k_0} induces

$$\eta_{k_0,*}:\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{\Omega,k_0}\right)\to\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{k_0,k_0}\right)$$

by the rule $\eta_{k_0,*}\left(\mu_*\right)_v:=\eta_{k_0}\left(\mu_v\right)$. Define

$$\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{\Omega,k_0}\right)^{p-new}:=\left\{\mu_*\in\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{\Omega,k_0}\right):\eta_{k_0,*}\left(\mu_*\right)\in\operatorname{im}\left(\iota_{\pm}\right)\right\}.$$

It is easily checked that $\iota_{\pm}:\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{\Omega,k_0}\right)^{p-new}\subset\mathcal{C}\left(\mathcal{V}^{\pm},\mathcal{D}_{\Omega,k_0}\right)$ is a $\mathbf{GL}_2^+\left(\mathbb{Q}_p\right)$-submodule (see [40, end of §2]). We write ρ_{X,v_*} for the $\mathbf{GL}_2(\mathbb{Z}_p)$-equivariant morphism

$$\rho_{X,v_*}:\mathcal{C}\left(\mathcal{V}^{+},\mathcal{D}_{\Omega,k_0}\right)^{p-new}\overset{\rho_{v_*}}{\to}\mathcal{D}(W)_{\Omega,k_0}^{0,b}\overset{\rho_X^W}{\simeq}\mathcal{D}\left(X\right)_{\Omega}^{0,b}.$$

Recall the eigenvector $\Phi:=\Phi^{\pm}\in H^i\left(\Gamma_0,\mathcal{D}\left(Y\right)_{\Omega}\right)$ lifting $\varphi_{k_0}:=\varphi_{k_0}^{\pm}$ and set

$$\Phi_{*,-}:=\left(\rho^{-}\right)^{-1}\left(\Phi\right)\in H^1\left(\widetilde{\Gamma},\mathcal{C}\left(\mathcal{V}^{-},\mathcal{D}_{\Omega,k_0}\right)\right),$$

which is the family denoted $\mathbf{c}_{*,-}^{U,k_0}$ in [40, before definition 15]. In [40, §3.2] it is defined an operator $\mathbf{V}^-$ sitting into the following commutative diagram

$$\begin{array}{ccc} H^i\left(\widetilde{\Gamma},\mathcal{C}\left(\mathcal{V}^{-},\mathcal{D}_{\Omega,k_0}\right)\right) & \overset{p^{k_0}\cdot\mathbf{V}^-}{\to} & H^i\left(\widetilde{\Gamma},\mathcal{C}\left(\mathcal{V}^{+},\mathcal{D}_{\Omega,k_0}\right)\right)\\ \rho^-\parallel & & \parallel\rho^+\\ H^i\left(\Gamma_0,\mathcal{D}\left(Y\right)_{\Omega}\right) & \overset{U_pW_p}{\to} & H^i\left(\Gamma_0,\mathcal{D}\left(Z\right)_{\Omega}\right).\end{array}$$

Let $\mathbf{a}_p\in\mathcal{O}\left(\Omega\right)$ be the eigenvalue of the U_p operator acting on Φ. Since $\mathbf{a}_p\left(k_0\right)=-\omega_{p,k_0}p^{k_0/2}$, with $\omega_{p,k_0}\in\{\pm1\}$ the eigenvalue of the Atkin-Lehner involution acting on F_{k_0}, up to shrinking Ω in a neighbourhood of k_0, we may assume that $\mathbf{a}_p\in\mathcal{O}\left(\Omega\right)^{\times}$ is a norm multiplicative element. Setting $\Phi_{*,+}:=\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,-}\mid\mathbf{V}^-$ (denoted $\mathbf{c}_{*,+}^{U,k_0}$ in [40, before definition 15]), we see that

$$\begin{aligned}\rho_Z^X\left(\mathbf{a}_p\left(k_0\right)\rho_{X^+,v_*^+}\left(\Phi_{*,+}\right)\right) &= \mathbf{a}_p^{-1}\rho^+\left(p^{k_0}\Phi_{*,-}\mid\mathbf{V}^-\right)=\mathbf{a}_p^{-1}\rho^-\left(\Phi_{*,-}\right)\mid U_pW_p\\ &= \mathbf{a}_p^{-1}\Phi\mid U_pW_p=\Phi\mid W_p.\end{aligned}$$

In other words,

$$\begin{aligned}&\Phi^{\#}=\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\rho_{X^+,v_*^+}\left(\Phi_{*,+}\right)\in H^1\left(\Gamma,\mathcal{D}\left(X\right)_{\Omega}\right),\\ &\text{i.e. }\widetilde{\varphi_{k_0}}=\left(\rho_X^W\right)^{-1}\left(\rho_{X^+,v_*^+}\left(\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,+}\right)\right)=\rho_{v_*}\left(\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,+}\right).\end{aligned}$$

But since we have $\rho_{k_0}^{W/Y}\left(\widetilde{\varphi_{k_0}}\right)=\varphi_{k_0}$, $\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,+}$ specializes to φ_{k_0} under the map considered in [40, Theorem 13 (b^+)]. This result implies that there is a canonical $\Phi_{*,+}^{p-new}\in H^1\left(\widetilde{\Gamma},\mathcal{C}\left(\mathcal{V}^{+},\mathcal{D}_{\Omega,k_0}\right)^{p-new}\right)$ such that $\iota_+\left(\Phi_{*,+}^{p-new}\right)=\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,+}$. It follows that

$$\widetilde{\varphi_{k_0}}=\rho_{v_*}\left(\mathbf{a}_p\left(k_0\right)\mathbf{a}_p^{-1}\Phi_{*,+}\right)=\iota\left(\rho_{v_*}\left(\Phi_{*,+}^{p-new}\right)\right)\in\iota\left(H^1\left(\Gamma,\mathcal{D}(W)_{\Omega,k_0}^{0,b}\right)\right).$$

Then (76) follows from [40, Theorem 13 (c^+)]: it gives the equality $\eta_{k_0,*}\left(\Phi_{*,+}^{p-new}\right)=\psi_{k_0}\in H^1\left(\widetilde{\Gamma},\mathcal{D}(W)_{\Omega,k_0}^{0,b}\right)$ and then

$$\begin{aligned}\eta_{k_0}\left(\widetilde{\varphi_{k_0}}\right) &= \eta_{k_0}\left(\rho_{v_*}\left(\iota_+\left(\Phi_{*,+}^{p-new}\right)\right)\right)=\rho_{v_*}\left(\eta_{k_0,*}\left(\iota_+\left(\Phi_{*,+}^{p-new}\right)\right)\right)\\ &= \rho_{v_*}\left(\iota_+\left(\eta_{k_0,*}\left(\Phi_{*,+}^{p-new}\right)\right)\right)=\rho_{v_*}\left(\iota_+\left(\psi_{k_0}\right)\right).\end{aligned}$$

Now the claimed equality follows from the fact that $\rho_{v_*}\circ\iota_+=\iota_{\Gamma}^{\widetilde{\Gamma}}$, because $\rho_{v_*}\circ\iota_+$ is the inclusion $\mathcal{D}(W)_{\Omega,k_0}^{0,b}\subset\mathcal{D}(W)_{\Omega,k_0}$ as Γ-modules.

We are now ready to prove the Lemma. Consider the following commutative diagram, induced by Lemma 6.1:

$$\begin{array}{ccccc} H_1(\widetilde{\Gamma},\Delta^0(P_{k_0})) & \overset{I^0_{\log}}{\to} & H^1\left(\widetilde{\Gamma},\mathcal{D}\left(W\right)^0_{k_0,k_0}\right)^\vee & \overset{\iota^\vee}{\to} & H^1\left(\widetilde{\Gamma},\mathcal{D}\left(W\right)^{0,b}_{k_0,k_0}\right)^\vee \\ \uparrow \iota_{v_*} & & \uparrow \left(\mathrm{res}^{\widetilde{\Gamma}}_{\Gamma}\right)^\vee & & \uparrow \left(\mathrm{res}^{\widetilde{\Gamma}}_{\Gamma}\right)^\vee \\ H_1\left(\Gamma,\Delta^0_{v_*}\left(P_{k_0}\right)\right) & \overset{I^0_{\log}}{\to} & H^1\left(\Gamma,\mathcal{D}\left(W\right)^0_{k_0,k_0}\right)^\vee & \overset{\iota^\vee}{\to} & H^1\left(\Gamma,\mathcal{D}\left(W\right)^{0,b}_{k_0,k_0}\right)^\vee \end{array}$$

Note that the morphism $I^0_{\log}$ (resp. $I^0_{\log,v_*} := I^0_{\log}\circ\iota_{v_*}$) appearing in the definition of $AJ^0_{\log}$ (resp. $AJ^0_{\log,v_*}$) is here the composition $\iota^\vee\circ I^0_{\log}$ (resp. $\iota^\vee\circ I^0_{\log}\circ\iota_{v_*}$) appearing in the above diagram; we are sorry for the redundancy in the notation, but since we are really interested in $I^0_{\log,v_*}=\iota^\vee\circ I^0_{\log}\circ\iota_{v_*}$, there will be no fear of confusion. The diagram implies $I^0_{\log,v_*}=\left(\iota^{\widetilde{\Gamma}}_{\Gamma}\right)^\vee\circ I^0_{\log}$, i.e.

$$I^0_{\log,v_*}\left(\cdot,C\right)=I^0_{\log}\left(\iota^{\widetilde{\Gamma}}_{\Gamma}\left(\cdot\right),C\right),\text{ for }C\in H_1\left(\Gamma,\Delta^0_{v_*}\left(P_{k_0}\right)\right). \tag{77}$$

Note that $I^0_{\log}$ appears in the first row of the following commutative diagram, whose existence follows from Lemma 8.3:

$$\begin{array}{ccc} H_1\left(\Gamma,\Delta^0_{v_*}\left(P_{k_0}\right)\right) & \overset{I^0_{\log}}{\to} & H^1\left(\Gamma,\mathcal{D}\left(W\right)^0_{k_0,k_0}\right)^\vee \\ i\downarrow & & \downarrow\eta^\vee_{k_0} \\ H_1(\Gamma,\Delta_{v_*}(P_{k_0})) & \overset{I}{\to} & H^1\left(\Gamma,\mathcal{D}\left(W\right)^0_{\Omega,k_0}\right)^\vee \end{array}$$

It follows that $I\circ i=\eta^\vee_{k_0}\circ I^0_{\log}$, i.e.

$$I\left(\cdot,i\left(C\right)\right)=I^0_{\log}\left(\eta_{k_0}\left(\cdot\right),C\right) \tag{78}$$

Now the claim follows from (78), (76) and (77):

$$\begin{aligned}\left(\log_{v_*}\mathbf{AJ}\circ i\right)\left(C\right)\left(\varphi_{k_0}\right) \quad : &= I\left(\widetilde{\varphi_{k_0}},i\left(C\right)\right)=I^0_{\log}\left(\eta_{k_0}\left(\widetilde{\varphi_{k_0}}\right),C\right) \\ &= I^0_{\log}\left(\iota^{\widetilde{\Gamma}}_{\Gamma}\left(\psi_{k_0}\right),C\right)=I^0_{\log,v_*}\left(\psi_{k_0},C\right)=:\left(AJ^0_{\log,v_*}\right)\left(C\right)\left(\varphi_{k_0}\right).\end{aligned}$$

□

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