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On Sober Platonism

New Perspectives in Mathematical Platonism

Beyond Strong Ontological Assumptions

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*To you, the daring venturers and adventurers,
and whoever hath embarked with cunning sails upon frightful seas;
To you the enigma-intoxicated, the twilight-enjoyers,
whose souls are allured by flutes to every treacherous gulf.*

NIETZSCHE, FRIEDRICH, *Thus Spoke Zarathustra*

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Sintesi

In questo lavoro ho voluto analizzare una tendenza che ha preso forma negli ultimi anni tra le teorie platoniste nell'ambito della filosofia della matematica. I modelli da cui sono partita sono quattro autorevoli proposte, che mi sono parse paradigmatiche: il platonismo purosangue di Mark Balaguer, lo strutturalismo *ante rem* di Stewart Shapiro, la teoria degli oggetti di Edward Zalta e il trivialismo di Agustìn Rayo. Queste quattro teorie hanno in comune tra loro un atteggiamento di tipo platonista nei confronti degli oggetti matematici, ovvero, assumono che gli oggetti di cui parlano gli enunciati matematici esistano effettivamente. Al contrario però del platonismo matematico classico, le loro assunzioni ontologiche, pur manifestando l'adesione alla teoria platonista, sono talmente misurate da dare l'impressione che non si tratti neanche di teorie platoniste. Propongo dunque di chiamare 'platonismo sobrio' questi approcci, che supportano il platonismo, pur rinunciando a forti assunzioni ontologiche.

La caratteristica fondamentale di questa tendenza è che l'impegno nei confronti dell'esistenza degli oggetti matematici non costituisca più il punto di partenza di una teoria degli oggetti matematici, ma diventi condizione necessaria al verificarsi di un fatto: la mente umana attinge alla conoscenza matematica. Di conseguenza, gli oggetti matematici devono esistere e devono essere tali da rendere possibile un rapporto tra gli oggetti matematici e la mente umana. In realtà, ciò a cui queste teorie puntano è ottenere una descrizione della pratica matematica, che sia in grado di rispondere a domande

filosofiche, imponendo alla pratica matematica il minor numero possibile di assunzioni filosofiche.

Il primo capitolo di questo lavoro è dedicato all'analisi del platonismo matematico classico. Propongo di considerare tale corrente di pensiero come la somma di tre assunti principali: *Indipendenza* (gli oggetti matematici sono indipendenti dal pensiero e dalle pratiche umane), *Esistenza* (gli oggetti matematici esistono) e *Conoscenza* (gli oggetti matematici sono conoscibili). Quest'ultima tesi è ulteriormente divisa in tre sotto-tesi: *Teoria della Conoscenza, Riferimento e Verità*.

Nel secondo, terzo, quarto e quinto capitolo ho discusso le proposte dei quattro autori citati, accomunati, come si è visto, da un impegno ontologico solo implicito o sobrio nei confronti degli oggetti matematici. Le questioni filosofiche con oggetto l'esistenza degli oggetti matematici, la possibilità di accedere alla conoscenza matematica, il significato degli enunciati matematici e il riferimento dei termini in essi contenuti sono affrontate e considerate filosoficamente rilevanti da tali teorie. Ciononostante, il loro principale obiettivo è piuttosto l'elaborazione di una descrizione precisa della matematica nella sua autonomia.

Nell'ultimo capitolo ho definito il platonismo sobrio attraverso la sua adesione alle stesse tesi cui aderisce il platonismo classico, *Indipendenza, Esistenza e Conoscenza* (anche qui analizzata nelle tre sotto-tesi già menzionate). Attraverso una valutazione comparativa, risulta evidente che il platonismo sobrio assume in buona parte ciò che assume il platonismo classico. Il vero elemento di distinzione risiede nel rapporto tra filosofia e matematica, in quanto nel platonismo sobrio l'autonomia e la dignità della matematica sono chiaramente affermate. La filosofia arriva solo dopo, a giochi fatti e,

piuttosto che una prescrizione normativa di come si deve fare matematica, si limita a fornire una descrizione metodologica di come si fa matematica. Di là dai risultati che può avere fino a oggi conseguito, il platonismo sobrio promette di essere in grado di ridurre molto rapidamente la portata se non altro di alcuni di quei problemi che sono rilevanti per la filosofia della matematica, ma non lo sono per la pratica matematica.

In conclusione, il platonismo sobrio propone sia un innovativo approccio nella filosofia della matematica, sia un fecondo contributo alla riflessione della filosofia su discipline altre da sé: una proposta descrittiva, piuttosto che normativa, ma ricca di prospettive filosofiche.

Résumé

Ce travail vise à identifier et définir une nouvelle tendance du platonisme mathématique. Je suis parti de l'analyse de quatre propositions fiable et paradigmatique: le platonisme pur-sang de Mark Balaguer, le structuralisme *ante rem* de Stewart Shapiro, la théorie de l'objet abstrait de Edward Zalta et le trivialisme de Agustín Rayo. Ces quatre théories ont en commun une attitude platoniste, c'est à dire l'admission de l'existence d'objets mathématiques. Contrairement au platonisme mathématique classique, son engagement ontologique est si léger, ou bien sobre, qui donne l'impression de n'être même pas dans le platonisme. Pour cette raison, je propose d'appeler 'platonisme sobre' ces propositions qui supportant le platonisme, en renonçant à de fortes engagements ontologiques.

La principale caractéristique de cette tendance est que l'engagement de l'existence d'objets mathématiques ne est plus considéré comme le point de départ d'une théorie des objets mathématiques, mais devient une condition nécessaire d'un fait: l'esprit humain peu obtenir connaissance des mathématiques. Par conséquence, les objets mathématiques existent et doivent être tels qu'une relation entre les objets mathématiques et l'esprit humain soit possible et fiable. En fait, ces théories visent à obtenir une description de la pratique mathématique, qui est aussi capable de répondre à des questions philosophiques, sans imposer aucun argument philosophique aux pratiques mathématique.

Dans le premier chapitre, on analyse le platonisme mathématique classique. Je propose d'examiner cette ligne de pensée comme la somme de trois thèses principales: *Indépendance* (les objets mathématiques sont indépendant de la pensée et de les

pratiques), *Existence* (les objets mathématiques existent) et *Epistémologie* (les objets mathématiques sont connaissables). Cette dernière thèse est encore divisée en trois sous-thèses: *Théorie de la connaissance*, *Référence* et *Vérité*.

Le deuxième, troisième, quatrième et cinquième chapitre sont respectivement dédiés à l'examen des quatre théories citées, un par un engagement ontologique seulement implicite ou sobre vers des objets mathématiques. Ces théories sont explicitement platonistes, mais seulement sobrement engagées dans l'existence d'objets mathématiques. Elles traitent l'existence d'objets mathématiques, la possibilité d'accéder à la connaissance mathématique, le sens des énoncés mathématiques et la référence de leur termes en tant que questions philosophiquement pertinentes. Cependant, leur objectif principal est l'élaboration d'une description précise des mathématiques en tant que telles.

Dans le dernier chapitre, je définis le platonisme sobre à travers les façons dont il soutient la même thèse du platonisme classique, *Indépendance*, *Existence* et *Epistémologie* (encore analysée comme *Théorie de la connaissance*, *Référence* et *Vérité*). Grâce à une évaluation comparative, il est en train de devenir clair que le platonisme sobre assume en grande partie le même principe du platonisme mathématique classique. Le véritable élément de distinction réside dans la relation entre la philosophie et les mathématiques, parce que dans le platonisme sobre l'autonomie et la dignité des mathématiques sont clairement établies. Le platonisme sobre est donc défini comme une description méthodologique de la façon dont les mathématiques sont réalisées, plutôt que comme une prescription normative de la façon dont les mathématiques doivent être réalisées.

Au-delà des résultats qui peuvent être obtenus jusqu'à aujourd'hui, le platonisme sobre permet de réduire très rapidement l'importance au moins d'une partie de celles questions qui sont pertinentes à la philosophie des mathématiques, mais ne sont pas pertinentes pour la pratique mathématique.

En conclusion, le platonisme sobre réalise le but de fournir la philosophie et les mathématiques avec un propre domaine d'enquête.

Abstract

This work aims at analyzing a trend which in recent years has been developed in mathematical Platonism. I have identified four theories which seem to me paradigmatic of this new trend: Full-Blooded Platonism by Mark Balaguer, *ante rem* Structuralism by Stewart Shapiro, Abstract Objects Theory by Edward Zalta and Trivialism by Agustín Rayo. These four theories share a platonist attitude towards mathematical objects, assuming that mathematical objects, as the reference of the terms in mathematical statements, actually exist. But contrary to classical mathematical Platonism, their ontological assumptions are so moderate, or sober, as to give the impression that these theories aren't even genuinely platonist. I therefore propose to call 'Sober Platonism' those approaches that support Platonism, without endorsing strong ontological commitment.

The key feature of this trend is that the assumption of the existence of mathematical objects is no longer considered the starting-point of a theory of mathematical objects, but becomes a necessary condition to the occurrence of a fact: the human mind accesses to mathematical knowledge. Consequently, mathematical objects must exist and be such as to make possible a connection between mathematical objects and the human mind. Hence, the ultimate aim of Sober Platonism is to obtain a description of mathematics as practiced, which does not impose any philosophical constrain, but is able to answer philosophical questions.

The first chapter of this work is devoted to the analysis of classical mathematical Platonism. I propose to consider this line of thought as the sum of three major theses: *Independence* (mathematical objects are independent of human thought and practices),

Existence (mathematical objects exist) and *Epistemology* (mathematical objects are knowable). The latter thesis is further divided into three sub-theses: *Theory of Knowledge, Reference and Truth*.

In the second, third, fourth and fifth chapter I discussed the proposals of the four aforementioned authors, matched together by their implicit, or sober, ontological commitment towards mathematical objects. These four theories take into account the existence of mathematical objects, the possibility to access to mathematical knowledge, the meaning of mathematical statements and the reference of their terms as philosophically relevant questions. Their main objective, however, is rather the development of an accurate description of mathematics in its autonomy.

In the last chapter I have defined Sober Platonism through its adherence to the same theses to which classical Platonism adheres, *Independence, Existence and Epistemology* (again analyzed as *Theory of Knowledge, Reference and Truth*). After a comparative evaluation, it becomes clear that Sober Platonism assumes largely what is assumed by classical Platonism. The real element of distinction lies in the relationship between philosophy and mathematics, since in Sober Platonism the autonomy and dignity of mathematics are clearly established. The proper role of Philosophy is then to deliver a methodological description of how mathematics is performed, rather than a normative prescription of how mathematics should be performed.

Beyond the results that may be achieved until today, Sober Platonism promises to have what it takes to reduce the importance of at least some of those issues that seems to be relevant to the philosophy of mathematics, but are not relevant for mathematics as practiced.

In conclusion, Sober Platonism offers both an innovative approach in the philosophy of mathematics, and a fruitful contribution in providing both philosophy and mathematics with a genuine domain of inquiry.

*The mathematician cannot create things at will,
any more than the geographer can; he too can only discover
what is there and give it a name.*

FREGE, GOTTLÖB, (2002), *Die Grundlagen der Arithmetik*, section 98

Preliminary Remarks

The present work aims at understanding and analyzing a trend in contemporary mathematical Platonism I propose to call ‘Sober Platonism’. In chapter 1 I will offer an interpretation of classical mathematical Platonism as the conjunction of three theses. Thereafter, I will sketch a preliminary definition of Sober Platonism. The central chapters of this work, from the second to the fifth, chapters are devoted to the examination of Full-Blooded Platonism, *ante rem* Structuralism, Abstract Objects Theory and Trivialism. In the last chapter, I will disclose why these theories endorse Sober Platonism and I will propose an exhaustive definition of this trend.

As a first step, I want to justify three choices I made during the draft of this work.

The first choice is terminological. I decided to call the trend I analyzed Sober Platonism because it shows a general commitment with the existence of mathematical objects, and so it does Platonism, but their existence is open to different conceptualization and, first and foremost, is not assertive. The existence of mathematical objects is taken into account, but without strong assumptions. The reason is that Sober Platonism’s first goal is to provide a description of how mathematics is performed as philosophically satisfying as possible, but also respectful of the autonomy of mathematics as practiced. By now, it will be sufficient to say that the most distinctive features of Sober Platonism are at least four: the attention for mathematics as practiced, the adoption of plenitudinous ontologies, the embracement of non-uniqueness in reference, a serious attention to the role of mathematical language.

I also want to stress here that Sober Platonism is able to give up philosophical rigidity in order to better pattern after mathematics. As a collateral result, Sober Platonism carves

up a new role for philosophy that is far from deflationary, that will be further investigated in chapter 6.

The second choice I want to justify is the one that led me to consider the four theories to which I devoted chapters from second to fifth as Platonism. This choice is far from being foregone, for at least two reasons.

The first is that Balaguer and Rayo endorsed Platonism explicitly, but they likewise explicitly states that their endorsement is only contingent. The approach they defend is specifically said to work both with Platonism and with Anti-platonism, even if their propensity for Platonism is clear. Someone could write another thesis on why these theories can serve the two, mutually exclusive, theories, and the advantages involved by the adoption of Balaguer's Full Blooded Platonism or Full Blooded Anti-Platonism, and Rayo's Trivialist Anti-platonism or Trivialist Platonism.

The second reason is in a way suggested by the first reason: at the end of the day, there is plenty of good reasons for not being Platonist. No ultimate argument in favor of one position over the other has been found nowadays; none of the arguments has proven to be decisive and thus anti-Platonist theories have not been ruled out. Platonists haven't come up yet with a satisfying response to the question of mathematical knowledge, whereas Anti-Platonists need to explain the success mathematics enjoys in its various physical applications.

On the one hand, there are two main arguments in favor of Platonism. First, Platonist theories have no troubles in accounting for mathematical truth. Some mathematical statements are true and others are false and they bear their truth values in virtue of the Platonist mind-independent entities which figure as their truth-makers. The other

important argument in favor of Platonism is the well-known indispensability argument. On the other hand, Anti-platonism has many virtues too. The most significant consists in its ability to deal with the question of mathematical knowledge. If mathematics is a construction of the human mind, then it is straightforward that humans are able to acquire mathematical knowledge. The same goes for Anti-Platonist theories which take at least some mathematics to describe concrete entities and structures. Another advantage of Anti-Platonism is its ontological economy. There is no doubt that being able of providing a theory of mathematics without embracing a huge ontology of infinitely many mind-independent abstract objects is an advantage.

In conclusion, the adoption of platonism is contingent. But is not contingent for who is interested in maintaining that descriptive approach to mathematics that is so characteristic to Sober Platonists. In other words, what I mean here is that, as trivial as it could sound, Full Blooded Anti-Platonism or Trivialist Anti-Platonism couldn't be considered Sober Platonism, not only because of their ontological assumptions, but firstly because of the consequences their ontological assumptions have on the *status* of mathematics. And the reason is that those theories wouldn't be Sober before of not being Platonism.

The second choice I want to motivate is the exclusion of all the other forms of Platonism from my analysis.

My first criterion was to exclude all those platonism who are strongly committed with the existence of mathematical objects. As a second criterion, I decided to not consider any philosophical theory who bear a normative attitude towards mathematics.

Thanks to the combination of these two criteria, I excluded the theories by Plato,

Aristotle, Kant and all the others classical platonism in general, including Gödel's theories. Because of its strong ontological commitment, I excluded Frege and the Neologicists. I also excluded Non-Eliminative Structuralism dispensation of the *ante rem*'s version by Stewart Shapiro and for a very simple reason: in *Philosophy of Mathematics: Structure and Ontology* he proposes an approach to philosophy of mathematics that can be seen as the forerunner of Sober Platonism. This approach takes the name of Working Realism and will be further analyzed in chapter 4.

Finally, after a harsh decision, I also excluded Penelope Maddy's naturalism and Kit Fine's Procedural Postulationism. The reasons are more articulated than those for the exclusion of the other forms of Platonism and I will attempt to explain them here.

Penelope Maddy's work¹ consists in several works. Even if their theories were submitted to more than one extensive revisions, the main aim of Maddy's work is to deliver an epistemology who is able to solve the problem of access to knowledge of abstract objects. I refer here to the theory she called 'Compromise Platonism', who accounts for an intuitive and satisfactory epistemology for mathematical objects.

Maddy's solution to the problem of access led her to look for an explanation of how humans' mind grasps the first, elementary parts of mathematics starting from the perception of collections of concrete objects. That is why she made an important appeal to experimental findings in Neurology, Psychology and Cognitive Sciences, with the goal of shading a light on the processes that undergo the development of mathematical knowledge.

¹ See Maddy, P. (1980), "Perception and Mathematical Intuition", *Philosophical Review*, 89, 163–96; Maddy, P. (1989), "The Roots of Contemporary Platonism", *The Journal of Symbolic Logic*, 54, 1121–44 and Maddy P. (1990), *Realism in Mathematics*, Oxford University Press, Oxford, New York.

Maddy's idea is that, through the observation of collections, humans first experience and attain mathematical knowledge. Notice that in Maddy's account, set theory has a prominent role, both because it is appropriate and commonly used for founding mathematics, and because it is about collections of objects. And by the very observation of collections of objects that first raise the notion of set theory. So according to Maddy, is through set theory, that the human mind attains access to mathematics. But insofar as collections of concrete objects are perceivable, the basic notion of set theory, even the fundamental notion of sets, are abstract objects, and not the concrete objects which are the members of the set.

Mainly one concern leads me to the exclusion of Penelope Maddy's work by my analysis: that it is not clear, at least to me, what these mathematical objects are and, more importantly, if their existence is independent on the human mind. Indeed, imagine a world in which no human mind had ever had any perception of collection of concrete objects, or a world in which no human mind had ever existed. Do mathematical objects exist in such a world? Apparently, they would not exist. But if so, *Independence* is explicitly denied: mathematical objects are dependent on humans' mind. Moreover, her theory had been often criticized on the basis that it is not to be considered as genuinely Platonist, because of the concreteness of (at least some) abstract objects.

For these reasons, I decided to exclude Maddy's Compromise Platonism, not because it isn't a good candidate for the Sober sides, but because I'm not sure it could count as Platonism at all.

Up until the last minute I was torn about whether to include Procedural Postulationism² or not, mainly because it seems to possess several characteristics of Sober Platonism as I preliminarily defined it. Indeed, it agrees very naturally with the way mathematicians work, thanks to the use of postulation, a very powerful device in stating the axioms of mathematical theories. The main role for postulations is in epistemology, as the means by which mathematical knowledge is attained.

Unfortunately, Kit Fine developed an ontology for Procedural Postulationism that left behind several problems. Just to mention a few, if mathematical objects exist and postulation simply allows humans to get in contact with them, then it isn't possible to refer to a domain of everything. Therefore, the change in meaning, from a domain to another, of the quantifiers in the language will merely reflect the epistemological facts that certain kinds of entities are known or not-known. But unfortunately, if what exists depends on the postulational perspective endorsed, numbers can't be just out there in reality: from a postulational perspective, there are no entities (a domain is empty or has less elements before the execution of the procedure than after it) until a postulational procedure is executed. Such entities exist; the domain is extended to new objects just introduced by the postulates. And this is surrounded by an air of mystery.

In the few works Fine dedicated to the formulation of his theory, he didn't provide a clear and exhaustive justification. Several, fundamental aspects of Procedural Postulationism are left unspecified, in particular the process of executability or the

² See FINE, KIT (2006), *Our Knowledge of Mathematical Objects* in Gendler, T.Z., Hawthorne, J., *Oxford Studies in Epistemology*, Oxford, Clarendon Press, p.p. 89-109; FINE, KIT (2006), *Relatively Unrestricted Quantifiers* in Rayo, A., Uzquiano, G., *Absolute Generality*, Oxford, Clarendon Press, p.p. 89-109; FINE, KIT, (2008), *The Limits of Abstraction*, Oxford, Oxford University Press;

aforementioned mysterious come into being of numbers that occurs after postulations are executed.

Nevertheless, Fine pictures the mathematical realm in a very interesting and intuitive way. He embraces the existence of multiple ontologies and multiple ways to talk about the mathematical world, none of them occupying any privileged position over the others. In this way, Procedural Postulationism is able to give mathematical objects a narrow ontological *status*, so narrow that it could serve also for a version of Procedural Postulationism that embraces Anti-Platonism.

Notice that the possibility of developing both a realist and an antirealist version of Procedural Postulationism, as could be done with Full Blooded Platonism and Trivialism, is one of its advantages from the point of view of its inclusion in Sober Platonism. In any case, Fine explicitly picks the realist side, since he simply assumes that mathematical objects exist as abstract objects. But he doesn't provide any argument in favor of Platonism.

Procedural Postulationism, albeit its intuitive originality, comes with more than one ambiguous ontological assumption. These assumptions set it on the very boundary between platonism and anti-platonism: even if Procedural Postulationism would be able to deliver a satisfying account of *Epistemology*, Fine didn't provide a complete account of *Independence* and *Existence*.

Let me explain this point with further details. As defined in chapter 1, section 6, Sober Platonists have to accept both:

1. True mathematical statements exist and are knowable;
2. Such statements correctly describe some kind of mathematical reality;

3. Some kind of mathematical reality must exist.

Procedural Postulationism agrees with the first two assumptions, but seems to put the cart in front of the horse with the third assumption: unless Kit Fine would specify with more details the status of the objects brought into being by postulation's executions, it is not clear if the objects exist independently from the postulation's executions. Indeed, if mathematical objects are brought into being through the postulation's execution, then their existence depends on postulation's execution. And this is a clear negation of *Independence*.

Even if these and other worries, mostly due to the limited extension of details on Procedural Postulationism, persuaded me to exclude it from the range of Sober Platonism, I'm sincerely persuaded that it could actually be a very good candidate, once further explanation would be presented.

The third choice is complementary to the second. Indeed, I not only chose to exclude some Platonism, but I also selected some valiant candidates for Sober Platonism. After a separate analysis of each of these candidates, in chapter 6 I will motivate my choice with further details. By now, let me say that in general, all of the four theories I chose have a descriptive approach to mathematics and are not scared of underestimating the relevance of philosophical problems in order to fulfill mathematical needs.

There are also singular reasons why I chose each theory: Full Blooded Platonism doesn't hesitate in front of the adoption of the hugest (and heaviest) possible ontology, once it meshes adequately with mathematical practice. Trivialism proposes a different approach that, ultimately, allows to reformulate mathematical (and non-mathematical statements) in order to reanalyze their ontological commitment.

Structuralism admits an incredibly huge ontology, even with the *caveat* of isomorphism, so meeting the extreme freedom of mathematicians' work. According to Zalta's perspective, the work of Philosophy is exhausted by the formulation of a coherent and expressive description of such objects and the metaphysical realm they could live in. Asking whether these objects actually populate this realm or not, exceed the very possibility of philosophical inquiries.

In conclusion, Sober Platonism is firstly focused on providing an accurate account of mathematics by its own, through a methodological description of how mathematics is performed, rather than a normative prescription of how mathematics should be performed. In doing so, Sober Platonism establishes the guide-line of an attitude in philosophy that promises to be not only philosophically adequate, but also mathematically significant.

However, in this age of post-Moorean modesty, many of us are inclined to doubt that philosophy is in possession of arguments that might genuinely serve to undermine what we ordinarily believe. It may perhaps be conceded that the arguments of the skeptic appear to be utterly compelling; but the Mooreans among us will hold that the very plausibility of our ordinary beliefs is reason enough for supposing that there must be something wrong in the skeptic's arguments, even if we are unable to say what it is. In so far, then, as the pretensions of philosophy to provide a worldview rest upon its claim to be in possession of the epistemological high ground, those pretensions had better be given up.

FINE, KIT, (2002), *Questions of Realism*, page 5

Chapter 1: Platonism

1.1 From Plato to contemporary Platonism

The present work aims at understanding and analyzing what is contemporary Platonism in philosophy of mathematics. The term ‘Platonism’ is usually taken to indicate all those metaphysical theories affirming the existence of the objects they predicate about and is not meant to indicate Plato’s philosophy. For what regards philosophy of mathematics, classical Platonism affirms the existence of the objects mathematical statements predicate about. Moreover, it states the existence of mathematical objects and properties independently from the existence of humans’ language, thought and practices.

In the last decades, several philosophers had proposed new explicitly Platonist theories in philosophy of mathematics. Nevertheless, these theories endorsed positions in semantics, epistemology or ontology that seems quite far from the ones endorsed by classical Platonism in mathematics. The reason behind this shift rests both in the discovery of problems in classical Platonism and in the search for new argument to defend Platonism from the antirealists’ objections.

Before offering an analysis of the new forms of Platonism contemporary philosophy developed in philosophy of mathematics, I will try to deliver a detailed definition of classical Platonism.

Mainly, Plato dedicated two dialogues to the problem of accessing to knowledge of abstract objects: the *Phaedrus* and the *Theaetetus*. In the former, Plato states the existence of one single homogeneous world, in which the forms aren’t only knowable, but are the very condition for the possibility of knowledge. In the latter, Plato’s view is

overturned. He affirmed that there exist two different uneven worlds: the forms, that authentically exist as concrete objects; and the copy of these objects, that are abstract objects. The problem is that, according to the *Theaetetus*'s theory of knowledge, such a hierarchical distinction confines the proper objects of knowledge, the forms, in an inaccessible realm. But if there is no possibility to entering in contact with the proper object of knowledge, knowledge isn't possible.

The problem of knowledge in Plato's philosophy is also known as the 'Problem of the Third Man': the idea is, roughly, that in order for a subject of knowledge to enter in contact with the object of knowledge, there must be something in between that allows knowledge, that passes the information from the object to the subject. But once such a medium is found, it seems that it is also required something in between the subject and the medium, and the object and the medium, that passes the information, and so on. Therefore, the regression to infinity of such an approach entails makes knowledge impossible.

Contrary to Plato's idea, Platonist theories don't provide a hierarchy between a real world, inhabited by concrete objects, or forms, and a fading world, in which there exist abstract objects that are only shadows of the concrete objects. Platonist theories assumes the existence of two different kinds of objects: perceivable objects, who gain the status of concreteness, and not perceivable object, but for example referable-to objects, who have assigned the status of abstractness.

Consequently, while concrete objects, like humans, computers and buildings, occupy a place in space-time, abstract objects don't. Nevertheless, they are real, they actually

have some properties and relations and it is possible to successfully refer to such objects. That's why another term often used for Platonism is 'Realism'.

Platonism classically endorses also Realism in truth-value, as a consequence of Realism in ontology: mathematical objects exist independently on mathematicians' practice or linguistic or ontological dispositions; hence, mathematical statements have objective truth-values. Although this assumption isn't always explicit, it means that, since mathematical objects exist independently from mathematicians, mathematical statements' truth-values is independent on the minds. Therefore, contrary to Plato's ontology, Platonists theories don't aim at establishing different levels of existence, from the most to the less authentic.

Notice that the existence of a Platonist realm is by its very definition unverifiable: since it is abstract and independent, it is not possible to have any perception of it. And here stands the main strength of the opponent of Platonism, Anti-Platonism: how is it possible to attain knowledge of such a realm? Even assuming its existence, why and how do mathematical statements refer to mathematical objects?

In any case, even if Platonists have to find an answer to the previous questions, Anti-Platonism is puzzled too, mainly by the undeniable, as unreasonable as it could seem if mathematical objects don't exist, applicability of mathematics to science and everyday experience. In section 1.2 I will examine this topic with further details.

What I want to stress is that mathematical Platonists have the duty to explain how is it possible to access and rely on mathematical knowledge based on unaccessible objects, while anti-Platonists must explain how and why mathematics is reliable and applicable,

without accounting for the existence of some kind of reality whom mathematical statements describe.

Briefly, mathematical Anti-Platonism is the view according to which there are no abstract objects behind the name we use to indicate objects that aren't concrete³.

Realism is contrasted by 'Anti-Realism', term I will take to indicate, with approximation, almost the same that 'Anti-Platonism'. With this term, I refer to philosophical theses that try to answers questions about mathematics without postulating the existence and independence of mathematical objects, and, most of all, the objectivity of mathematical truth. Anti-Platonists' main aim is therefore to provide an account that explains how and why mathematics is significant, without appealing to the existence of a mathematical realm.

Several definitions of mathematical Platonism have been proposed by contemporary philosophers. Michael Dummett in his 1978⁴, page 202, stated that:

Platonism, as a philosophy of mathematics, is founded on a simile: the comparison between the apprehension of mathematical truth to the perception of physical objects, and thus of mathematical reality to the physical universe.

Again in 1991⁵, page 301, he declares:

Platonism is the doctrine that mathematical theories relate to systems of abstract objects, existing independently of us, and that the statements of

³ See section 1.3 for a complete analysis of the difference between abstract and concrete objects.

⁴ DUMMETT, MICHAEL, (1978) *Truth and Other Enigmas*, Harvard University Press, Cambridge.

⁵ DUMMETT, MICHAEL, (1991) *Frege: Philosophy of Mathematics*, Harvard University Press, Cambridge.

those theories are determinately true or false independently of our knowledge.

Stewart Shapiro (1997⁶ page 37), states in the framework of structuralist Platonism that [R]ealism in ontology or Platonism is the view that mathematical objects exist independently of mathematicians, and their minds, languages, and so on.

One of the most famous anti-platonists, Hartry Field, also proposes an interesting definition of Platonism (1989⁷, page 1):

A mathematical realist, or Platonist, (as I will use these terms) is a person who (a) believes in the existence of mathematical entities (numbers, functions, sets and so forth), and (b) believes them to be mind-independent and language-independent.

I propose to define mathematical Platonism as the philosophical thesis according to which mathematical objects exist as abstract object independently from humans' thought, language and practice. Consequently, mathematical knowledge is objective: mathematical statements have objective truth-values, independently from humans' thought, language and practice.

In the following of this chapter, I will analyze Platonism in more details, as the conjunction of three main thesis:

1. *Independence*: the independence of mathematical realm from anyone's thought and practice;

⁶ SHAPIRO, STEWART, (1997) *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, Oxford;

⁷ FIELD, HARTRY, (1989) *Realism, Mathematics and Modality*, Basil Blackwell, Oxford;

2. *Existence*: the existence of mathematical objects as abstract objects;
3. *Epistemology*: the successful reference and knowability of mathematical statements.

Epistemology is also linked to the processes through which knowledge is attained, the questions concerning the *status* of truth and to the problem of reference. For these reasons, I divided *Epistemology* in three sub-these: *Theory of Knowledge* (section 1.4.1), *Truth* (section 1.4.2) and *Reference* (section 1.4.3).

I will dedicate the final section to an introduction to the proper object of my work, the attitude of philosophy towards mathematics I proposed to call ‘Sober Platonism’.

1.2 Independence

Independence: the independence of mathematical realm from anyone’s thought and practice.

The first thesis, *Independence*, can also be conceived as stating that mathematics is not created and therefore is independent on anyone’s thought and practice; so, what mathematicians do when they formulate mathematical theories is a genuine enterprise of discovery. It is important to keep in mind that other kinds of abstract objects are instead dependent on humans’ thought and practice. This is the case, for example, for nations, laws, values, feelings, fictional characters. These are all abstract objects: philosophers who adopt ontological committalism, i.e. the view according to which an ontological commitment with references of terms follows from the very use of the terms, will have to commit with their existence.

Other philosophers, like physicalists, will say that they don’t exist, but are just useful

tools for expressing states of mind, or to organize and rule a community. Such abstract objects are of course dependent on humans' thought and practice. Even more: they are the very product of humans' thought and practice.

Mathematical abstract objects are different from the aforementioned abstract objects, precisely because of their independence. Independence of mathematical objects allows to account for an objective state of the matter for what concerns mathematical truth. If mathematics is independent, then mathematicians' enterprise of discovering and revising theories is authentic: they face an objective matter, a brute fact. Platonism aims at developing an account able to render the intuition mathematicians have that things stay in a determinate way in the mathematical realm and they are discovering this state of matter.

Independence comes with several concerns: if mathematical objects are independent on thought and practice, how is it possible to enter in contact with them? They are completely disconnected from the concrete world. But if so, how do we manage to attain mathematical knowledge? How do we know that it is reliable? Although the successful application of mathematical theories to our best science is a great advantage of Platonism, some may object that Platonism doesn't explain exhaustively enough how such a successful application is obtained: how happens that theories about an independent and disconnected world match so perfectly and usefully with theories about the concrete world?

At least two classes of problems follow from *Independence*: the first is the possibility of mathematical knowledge, and is best known as part of Benacerraf's Dilemma; the second and consequent regards the effective applicability and the constitutive role of

mathematical knowledge to science and scientific knowledge. As I already disclosed, the applicability of mathematics to science is also one of the main argument for Platonism.

The first class of problems regards epistemology, in general, and the problem of access in particular. Indeed, if mathematical objects exist independently of us, then there is no possibility of connection between the truth conditions assigned to mathematical statements and the way things actually stay in the mathematical realm. Then, how is it possible to justify and verify the correctness of truth conditions' assignment?

The second class follows from the consideration that, if mathematics is independent on anyone's thought and practice and not causally related, the effective applicability of mathematical theories to science seems unexplainable. On one hand, mathematics's connection with science is a motivation for Platonism: if our best interpretation of science makes successful use of mathematics, then mathematics itself is, at least indirectly, verified.

Moreover, it has been asserted that mathematics is not only useful in science, but also indispensable. This argument, known as the indispensability argument, was first explicitly formulated by Hilary Putnam in *Philosophy of Logic*⁸, but was asserted also by Quine in *On What There Is*⁹ and several further works on naturalism and naturalized epistemology.

On the other hand, applicability, and *a fortiori* indispensability, of mathematics in science seems unjustified. Platonism can legitimately benefit from application and

⁸ PUTNAM, HILARY (1971) *Philosophy of Logic*, Harper Torchbooks, New York.

⁹ QUINE, WILLARD VAN ORMAN, (1948), "On What There Is", *Review of Metaphysics*, 2, 21–38;

indispensability only if it is also able to explain why and how application and indispensability occur. In other words, Platonists have to explain how descriptions of a realm of abstract objects apply so successfully to descriptions of another realm of objects, the natural world, albeit there are neither contacts nor relations between the two. If Platonism reveals to have no such an explanation, it would be fairer to not taking advantage from applicability and indispensability, because they would seem to occur fortuitously, as an unexpected stroke of luck.

Independence is a desirable assumption for philosophy of mathematics, since it guarantees objectivity and reliability of mathematical knowledge. But, ironically, it denies the very possibility of mathematical knowledge. This puts mathematical Platonism in a quandary: if mathematical objects exist independently from human's thought and practice, its objectivity is guaranteed. At the same time, there is no way to access to such an objective matter. Theoretically, if it would be possible to obtain mathematical knowledge, then it will be reliable. But it isn't possible because of the very trait that makes it reliable. That is way deepest problems for Realism are in the epistemic front: how to know anything about an eternal, timeless, abstract, mathematical realm?

The main objection for contemporary Platonism is known as Benacerraf's dilemma and was raised by Paul Benacerraf in two famous papers: *What numbers could not be* (1965)¹⁰ and *Mathematical Truth* (1973)¹¹. The dilemma challenges Platonism both in

¹⁰ BENACERRAF, PAUL, (1965) *What Numbers Could not Be*, reprinted in Benacerraf, Paul, Putnam, Hilary, (1983) *Philosophy of Mathematics*, Cambridge University Press, Cambridge;

¹¹ BENACERRAF, PAUL, (1973) *Mathematical Truth*, *Journal of Philosophy* vol.70 pp. 661-679;

ontology and in epistemology, but it is deeply linked with *Independence* too. Indeed, Benacerraf's Dilemma as formulated in *Mathematical Truth* insists on this very point: in (1973, p. 666), he requires that:

[A]ny theory of mathematical truth be in conformity with a general theory of truth [. . .] which certifies that the property of sentences that the account calls 'truth' is indeed truth.

Benacerraf's first requirement is that semantics for statements of ordinary language must be the same as semantics for mathematical statements. But he also asks for the possibility of verifying that the statements considered true are indeed true, through a process of revision and justification. He adds this requirement because, in accordance with the causal theory of knowledge he endorses, it is possible to attain knowledge only through processes that could be verified and justified. And obviously, justification and verification require the possibility of entering in contact with the objects of knowledge. Ultimately, this means that knowledge is the understanding of the conditions under which statements are true or false. Again in *Mathematical Truth* (p. 667), Benacerraf suggests that

Since our knowledge is of truths, or can be so construed, an account of mathematical truth, to be acceptable, must be consistent with the possibility of having mathematical knowledge.

Here the possibility of having mathematical knowledge is explicitly related with the very possibility of understanding the truth conditions of mathematical statements. What Benacerraf is suggesting here is that truths have to be knowable. Hence, if there is no possibility of accessing to mathematical knowledge, because they are not casually

related with the subject of knowledge, it is not possible to objectively determine the truth conditions of mathematical statements.

Again, a criterion for knowledge of mathematical objects is needed, but at the same time impossible to obtain because of the very nature of mathematical objects.

1.3 Existence

Existence: the existence of mathematical objects as abstract objects.

Several reasons motivate the adoption of *Existence*. Some philosophers take the existence of mathematical objects as a philosophical *datum*: humans from every culture and country, and even other species, make extensive use of mathematical knowledge. The existence of mathematical objects is the condition for the successful reference of mathematical terms and for the meaning of mathematical statements. Indeed, if there are no mathematical objects, mathematical terms will not refer to anything at all, and there will be no reason for the reliability of mathematical knowledge. Moreover, there will be no way to justify and to apply mathematical knowledge.

According to Platonism, mathematical objects are discovered, rather than invented (as for Anti-Platonism), by mathematicians: there exist an objective realm of abstract objects, the mathematical objects, that have precise features and relations. As physicians derive the existence of atoms and bosons from their effects on other perceptible entities, mathematicians derive the existence of mathematical objects from the successful application of mathematical knowledge on useful descriptions of other perceptible objects. Note that this means that mathematical objects exist independently, and is therefore an influence of *Independence* over *Existence*: if mathematics is discovered, as

the natural world, mathematics is independent, as the natural world, from anyone's thought and practice. And trivially, if mathematical objects don't exist, they can't be neither independent nor dependent on anything. But the whole point on which I want to focus here is that if mathematical objects exist they can be either independent or dependent. If they are dependent, objectivity of mathematics is denied, while if they are independent, is admissible.

Since mathematical knowledge is traditionally believed to be *a priori*, and knowledge of concrete things *a posteriori*, if mathematical objects are believed to exist they are conceived as abstract objects. There is no general agreement on the criterion for discriminating between abstract and concrete objects and a couple of words are worth to be spent about this topic.

The distinction between abstract and concrete objects rest on several different criteria: objects are concrete if and only if they are perceived by the senses, or can take part in a causal chain, or occupy a portion of space-time. Abstract objects, as opposed to concrete objects, are not perceivable, cannot be part of a causal chain and their existence is not related to any particular point in space and time. Abstract and concrete objects are therefore mutually exclusive and mutually definable categories.

One important claim of Platonism is that, even if abstract objects are assigned with a weaker sense of existence in comparison with the one attributed to concrete objects, still mathematical objects possess some kind of objectivity. There actually is something to be right or wrong about in mathematics. For example, even if there isn't general agreement on what numbers are, there is general agreement on the truth conditions of sentences involving numbers, like ' $2+3=5$ ' or ' $2+3=4$ '. And this occurs even if

mathematical objects are abstracts, not perceivable in any way, completely disconnected from the knower's world.

The main objection against Platonist's ontology is due to another paper by Paul Benacerraf, *What Numbers Could Not Be*. Here he proposed an hypothetical situation in which two children, sons of two logicians, learned logic and set theory before being told about numbers. Then, they learned two different set-theoretical foundations of natural numbers: Von Neumann's and Zermelo's.

Benacerraf noticed that the two young boys have been given correct accounts of the numbers. Nevertheless, they explicitly disagree on the concept of number, because they disagree on which particular set every number is. So, there are two possible scenarios in Benacerraf thought: either both the two guys are right and 3 is $(\{\{\{\emptyset\}\})$ and also $(\emptyset, (\emptyset), (\emptyset, (\emptyset)))$ and also other things; or only one of the guys is right, and so the other account somehow does not respect the conditions that describe a correct account of mathematics.

The first scenario could seem inadmissible, but it isn't. Indeed, is the one in which Structuralism is developed (see chapter 3). The second scenario is very articulated and lead Benacerraf, after a long discussion, to another question: if such a correct account exists, are there arguments that allow to demonstrate that it is definitely the correct one?

The point of Benacerraf's argument is then that, since there is no possibility to access to mathematical realm, there is no way to discriminate between different descriptions of the same mathematical object. Most contemporary Platonist believes there is no one correct answer, and give away uniqueness in reference for adopting Structuralism or

plenitudinous Platonism. These strategies are endorsed by all the platonists I propose to see as paradigmatic instances of Sober Platonism.

Indeed, the endorsement of *Existence* can be sustained in very atypical fashions. One of the most interesting example is Kit Fine's procedural postulationism. In *Our Knowledge of Mathematical Objects*, Kit Fine suggests to replace postulation of axioms by postulation of rules, instructions or procedures for the constitution of standard mathematical domains. These procedures are ontologically innocent, but suggest both a solution to the problem of nature of mathematical entities and a solution to the problem of mathematical knowledge. Fine formulated a new form of Platonism he called Procedural Postulationism, whose ontology is at least original.

According to Fine, the existence of mathematical objects is relative to the way mathematicians talk about them, that is, their postulational perspective. And is through the obtaining of a postulational perspective that mathematical objects constituting the consequent postulationally perspectival facts are brought into being.

Fine admits the existence of multiple ontologies and multiple ways to talk about the mathematical world, none of them occupying any privileged position over the others.

Moreover, the simple act of having a (consistent) postulational perspective allows for the existence of the objects that constitute the relative postulationally perspectival facts.

As a pleasant result, Fine managed to understand postulations first and foremost as the means by which mathematical knowledge is attained. But the existence of numbers is dependent on the postulational perspective adopted, and Fine explicitly states that, if numbers exist from a postulational perspective, they exist necessarily.

Unfortunately, his position in ontology raises several concerns. For, if mathematical objects are out there in reality and postulation simply allows human to get in contact with them, then it is possible to speak of a domain of everything, and the change in meaning of the quantifiers in the language will merely reflect the epistemological facts of certain kinds of entities being known or not-known. But unfortunately, if what exist depends on the postulational perspective endorsed, number can't be just out there in reality. Fine has indeed to endorse multiple ontologies, according to which, from a particular postulational perspective, there are no entities (in the sense that a domain is empty or has less elements before the execution of the procedure than after it) until a postulational procedure is executed. Thereafter, such entities exist; the domain is extended to new objects just introduced by the postulates. According to Fine, the objects of these domains are as mind-independent as the objects of an all-inclusive domain. The point is that a special mechanism is needed to get to them, through the extension of one domain in the other.

The multiple ontologies are therefore the only choice: there is no such a thing as *the* ontology, the privileged sum-total of what there is. When a domain is expanded, a shift in the understanding of what there is happens. The ontology embraced depends therefore on the postulational perspective, and no ontology has a privileged position. The different ontologies correspond thereof to the maximal consistent sets of facts that obtain at a given postulational perspective.

This form of Platonism brings with it several, urgent ontological concerns. Most of all in regards with the status of the new objects brought into being through the execution of procedural postulates. In the end, Procedural Postulationism appeals to a process of

creation of new objects that urgently needs to be justified. Moreover, Procedural Postulationism seems also to allow for there being different ways of expanding a subdomain of individuals. Indeed, the endorsement of the existence of mathematical objects, together with a notion of identity that goes beyond isomorphism, still presents the problem of deciding which one of the multiple, non-identical but structurally undistinguishable number definitions identifies the numbers that exist.

1.4 Epistemology

Epistemology: the successful reference and knowability of mathematical statements.

The main challenge for *Epistemology* is to achieve a connection between bearers of relevant mathematical beliefs and constituents of relevant mathematical facts. *Epistemology* takes with it some assumptions: first, mathematical knowledge is possible; second, if mathematical knowledge is possible, mathematical theories should be about some kind of truths. Thirdly and consequently, there must be some kind of connection between mathematical theories and the truths they describe. This connection is reference: mathematical theories describe truths because the terms in them have a meaning and a reference.

For these reasons, I decided to draw a further distinction in *Epistemology* and analyze it as the sum of three sub-theses:

3.1 *Theory of Knowledge*: mathematical knowledge is possible;

3.2 *Truth*: mathematical knowledge is knowledge of truth;

3.3 *Reference*: mathematical knowledge is about some kind of objects.

1.4.1 Theory of Knowledge

Theory of Knowledge: mathematical knowledge is possible.

Mathematical knowledge possesses some very specific features that distinguish it from general knowledge. To mention just a few, the objects of mathematical knowledge are always abstract, while general knowledge occurs both with concrete and abstract objects; mathematical knowledge is widely considered as necessary, while general knowledge can be both consistent and necessary. Nevertheless, it is to be kept in mind that the more an account of mathematical knowledge is conciliable with the account of general knowledge, the more it avoids objections and concerns (first of all, Benacerraf's).

Among the different theories of knowledge developed by philosophy throughout its entire history, one of the most commonly adopted by contemporary philosophers is the causal theory of knowledge, first formulated by Alvin Goldman in *A Causal Theory of Knowing*¹².

According to causal theory of knowledge, in order for a subject to have a justified belief regarding the truth of a statement, there must be some kind of phenomenon that had caused the effect of knowledge in the subject. Therefore, there must be some kind of relation between the truth of a statement and the belief of the truth of a statement by the subjects of knowledge. This relation generated many objections and doubts in the case of mathematical Platonism, mainly because the adoption of *Independence* brushed off

¹² GOLDMAN, ALVIN (1967) *A Causal Theory of Knowing*, The Journal Of Philosophy, vol. 64, no 12, pages 357-372.

the very possibility of having any relation with mathematical objects.

Nevertheless, at least two ways of obtaining access to mathematical objects have been proposed: the first is through a faculty generally named acquaintance¹³ or intuition; the second is by mean of descriptions.

These two kinds of knowledge are deeply diverse and are the object of a famous debate in epistemology. Briefly, the point is that, while acquaintance can provide foundational knowledge, description can't. Foundational knowledge is the kind of knowledge that is independent on previous knowledge. As a result, if a proposition can't be inferred from other, yet known, propositions, this proposition provides foundational knowledge. In this sense, knowledge by description seems to need some pieces of knowledge previous to the act of description: some property to attribute to the objects under description, and also some more fundamental concepts to be described or combined together.

Another point of distinction between acquaintance and description is that the former provides direct knowledge: the subject grasps the information directly, without the mediation of previous pieces of knowledge or processes of inference.

On the opposite, the latter provides non-direct or mediate knowledge, since the subject has to represent a piece of information making use of some abilities, like abstraction, language or other representational means.

Moreover, while acquaintance requires the existence of an object to get acquainted with, description can occur even with non-existing objects: its impossible to get acquainted

¹³ On the debate between knowledge by acquaintance and by description, see: Bonjour, Lawrence, 2005, "In Defense of the *a Priori*", in Matthias Steup and Ernest Sosa (eds.), *Contemporary Debates in Epistemology*, Malden, MA: Blackwell Publishing Ltd., 98–105; Russell, Bertrand, 1910–11, "Knowledge by Acquaintance and Knowledge by Description" *Proceedings of the Aristotelian Society*, 11: 108–128; Sellars, Wilfrid, 1975, "The Structure of Knowledge," in H. N. Castaneda (ed.), *Action, Knowledge, and Reality*, Indianapolis: Bobbs-Merrill.

with the square circle, since its existence is impossible. Nevertheless, it is possible to describe the object that possesses both the properties, like a squared circle (or a circled square).

Beware that such a description will not result as knowledge, since it is inconsistent and false. Indeed, knowledge is always knowledge of truths, as the aim of both acquaintance and description is to obtain true knowledge.

Another distinction is that, while knowledge by description needs some knowledge of truths as its ground, knowledge by acquaintance requires the existence of the objects that is going to be acquainted with, but it isn't necessarily knowledge of truths.

As Bertrand Russell famously stated at page 72 of *The Problems of Philosophy*¹⁴:

Knowledge of things, when it is of the kind we call knowledge by *acquaintance*, is essentially simpler than any knowledge of truths, and logically independent of knowledge of truths, though it would be rash to assume that human beings ever, in fact, have acquaintance with things without at the same time knowing some truth about them.

Knowledge by description, on the contrary, always involves, as we shall find in the course of the present chapter, some knowledge of truths as its source and ground.

On the one hand, acquaintance was and still is object to several controversies. The reason is that it simply seems unjustifiable that the objects of knowledge are just given to the subject of knowledge, whom is, for a stroke of luck, perfectly able to transform the presence of objects in knowledge of precisely those objects. An important trait of acquaintance is that, even if it is plausibly a subjective matter, of someone getting

¹⁴ RUSSELL, BERTRAND, (1912), *The Problems of Philosophy*, Oxford: Oxford University Press.

acquainted with something, it is also intersubjective, at least for mathematics.

If mathematical knowledge comes from acquaintance, than it isn't only a matter of a mathematician getting acquainted with a piece of mathematical knowledge. Rather, every mathematicians, and actually every subject, will be acquainted with the same object, with the same properties and relations. That is why acquaintance with mathematical objects must be intersubjective, rather than subjective. Nevertheless, from intersubjectivity to objectivity there is a not too short step.

The biggest problems for knowledge by acquaintance raise when it comes to *Independence*. Indeed, it explicitly denies the very possibility of entering in contact with mathematical objects. And without contact, acquaintance seems impossible. Hence, some Platonist, from Gödel to Zalta, appealed to intuition or acquaintance. Further details on the possibility of knowledge by acquaintance in platonism are to be found in section 6.4.1 of the present work.

On the other hand, knowing mathematical objects trough description is controversy too. Indeed, if mathematical objects exist as abstract objects, but we can't access to them, or be acquainted with them, because of their independence, then someone may ask what is the object of the description and on which criteria the description should be formulated. One possible answer is that knowledge by description proceed by attempts: every possible combination of properties is analyzed and, if it doesn't lead to contradiction, it could be considered as a description of mathematical objects. This strategy is very close to the one endorsed by Mark Balaguer I have analyzed in chapter 3.

In conclusion, the impression is that both acquaintance and description try to avoid the problem of *Independence*, appealing mainly to the necessary and *a priori* character of mathematical truth. Bounding mathematical knowledge to logical knowledge helps a lot

to fill this gap. But first and foremost, is the understanding of what is mathematical truth that allows for a fundamental step forward in investigating what is mathematical knowledge.

1.4.2 Truth

In 1973 Benacerraf published *Mathematical Truth*, as I have already mentioned. In this paper, he advocates that almost all accounts of the concept of mathematical truth serve the concern for having an homogeneous semantic theory, in which mathematical propositions have the same semantics than the rest of the language; and the concern that the account of mathematical truths meshes with a reasonable and shared epistemology. But almost all accounts serve one at the expense of the other.

Balaguer, in *A Platonist Epistemology*, page 303, argued that:

If all logically possible mathematical objects exist [...] then all we have to do in order to attain mathematical knowledge is [...] think about a mathematical object. Whatever we come up with, so long as it is consistent, we will have formed an accurate representation of some mathematical object.

Consequently, though mathematical objects are mind-independent, any view we have had of them would have been correct. But this conclusion implies that we are somehow granted to invent whatever mathematical object we can think of, without minding if it exists or has a role in mathematical practice.

Notice that ‘consistent’ does not mean the same as ‘true’, especially if we are committed with an exclusive concept of truth and to characteristic of logic such as non-

contradiction, bivalence and excluded middle. Furthermore, many views are consistent, and so the problem of the correctness of just one account is not only unsolved, but it also arises from a new point of view. That is why, classically, philosophers choose to first identify some standard by which the truth-values of mathematical statements can be assessed, and only then argue that some mathematical theorem meets this standard.

For example, Logicism founds mathematical truth on the logical truth, conceived as more fundamental. Another option is the Indispensability Argument provided by Quine and Putnam, that found mathematical truth on its indispensability in obtaining empirical truth. Another possibility is to appeal to the standards of mathematics itself, claiming that mathematics has in itself its justification, just as others theoretical branches of knowledge, like logic again, or experimental physics.

Hartry Field (1989) tempted a different approach. He tried to deliver a fictionalist and non-conservative answer to this very problem, claiming that, since it is not the function of mathematical theories to be true, the quandary doesn't arise. In Field (2005) he argued also that:

Mathematical theories, taken at face value, postulate mathematical objects that are mind-independent and bear no causal or spatiotemporal relation to us [...] that would explain why our beliefs about them tend to be correct; it seems hard to give any account of our beliefs about these mathematical objects that doesn't make the correctness of the beliefs a huge coincidence.

Therefore, the worry that, had our mathematical beliefs been different, they would have been false, appears as just persuading to the extent that it seems a stroke of luck that humans came to have the mathematical beliefs that they came to have.

Now, since different but still true mathematical theories are admitted, it seems hard, if possible, to see which theory is the correct one, the one with which is desirable to be ontologically committed with. Accordingly, the highly contingency of mathematical beliefs is to be acknowledged.

This acknowledgment is anything but harmless. Indeed, Hartry Field (2005) proposed that, had mathematical truths been different, mathematical beliefs would have been false:

The Benacerraf's problem [...] seems to arise from the thought that we would have had exactly the same mathematical [...] beliefs even if the mathematical [...] truths were different; it can only be a coincidence if our mathematical [...] beliefs are right, and this undermine those beliefs.

The demand for a strong necessity for mathematical truths collides with the existence of different but equally consistent mathematical beliefs. One possible way out is to provide a broader sense of possibility to mathematics. But the risk is to compare the exact science *par excellence* with disciplines such as, for example, ethics: indeed, it is absolutely no problematic that, had our moral beliefs been different, our moral beliefs wouldn't have been false.

Obviously, this is not feasible for mathematics: if mathematical truths weren't necessary, they wouldn't have been very useful. But it is even more senseless to renounce to some mathematical truth only with the aim of justifying the truth of some others mathematical truths.

Philosophers of mathematics are in a quandary: they can either accept the plurality of interpretations of the language arranged for mathematical theory, but deny that every

arithmetical expression is an interpretation to which corresponds a unique and only object. Or, they can claim that every arithmetical expression has a unique and only reference. Roughly, that every theory isolates a unique and only object.

In the first case, 3 , $((\emptyset))$ and $(\emptyset, (\emptyset), (\emptyset, (\emptyset)))$, are simply different arithmetical expressions that share the same reference. Indeed, following this perspective, there exists a single object, let's name it 'three'. Then, '3', ' $((\emptyset))$ ' and ' $(\emptyset, (\emptyset), (\emptyset, (\emptyset)))$ ' are different senses for the same reference, the third natural number. We need to indicate different senses because the object, although shares some characteristics, has several properties that are context-sensitive. Depending on the discourse, the object is named in a different way. The truth of mathematical beliefs is so dependent on the context in which they are formed.

This perspective may also be subjected to quinean indeterminacy of translation: how can philosophers of mathematics be sure that the reference of ' $((\emptyset))$ ' in Zermelo's language is exactly the same of the reference of ' $(\emptyset, (\emptyset), (\emptyset, (\emptyset)))$ ' in Von Neumann's? How do we secure reference to mathematical objects?

Unfortunately, this is not the entire story. There still remain some important epistemological quandaries: to mention one, accepting different but true mathematical beliefs involve the adoption of a coherence theory of truth. Indeed, if a philosopher of mathematics accepts the plurality of interpretations of mathematics, but denies that every interpretation corresponds to a specific and unique object (for example asserting that ' $((\emptyset))$ ' and ' $(\emptyset, (\emptyset), (\emptyset, (\emptyset)))$ ' have exactly the same reference), how can he justify the acceptance of a correspondence theory of truth? Mathematical belief aren't

true because they describe some specific object that is so and so and anything else. Rather, truth of mathematical beliefs lie in their coherence with a theory. And so their truth is again context-sensitive.

In conclusion, the philosopher has to adopt a coherence theory of truth to legitimate the truth of mathematical beliefs. But such a theory of truth entails, especially for mathematics, some gödelian questions that can weaken both the ontology and the epistemology developed by the philosopher of mathematics.

Alternatively, philosophers of mathematics can claim that every theory isolates a unique and only object. It is doubtless that such a conception determines a considerable ontological commitment: there is no object 'three', to which correspond a several number of representation, because every representation constructs a single object. If so, it is not acceptable that the mathematical expressions '3', '(((\emptyset)))' and '(\emptyset , (\emptyset), (\emptyset , (\emptyset)))' represent the same object.

One possible way out is to admit an infinite number of objects and categorize them in a kind of set-theoretic universals: the set isolated by conditions such as 'to occupy the third place in the series of natural numbers', or such as 'to entertain a one-to-one correspondence with the third natural number', contains, among its elements, '3', '(((\emptyset)))' and '(\emptyset , (\emptyset), (\emptyset , (\emptyset)))'. Such a set allows the philosopher of mathematics to claim that, although '3', '(((\emptyset)))' and '(\emptyset , (\emptyset), (\emptyset , (\emptyset)))' are different objects, he can consider them similarly, because they are part of the same set, thanks to some peculiar characteristics these objects share.

There are problems with this perspective too. If philosophers of mathematics claim that every theory isolates a unique and only object, there is a unique and only mathematical

truth in each and every theory. But many views are consistent and so consistency cannot work as the criterion for discriminate false and true mathematical beliefs (even if Kit Fine (2005) made a very interesting attempt in this direction with his procedural postulationist approach).

A possible solution is that, once mathematical Platonism is adopted, and therefore once a strong background ontology is supplied, philosophers can rely upon a correspondence theory of truth. Such a theory of truth can be very useful for a philosopher who argue that every theory isolates a unique object, and entails a strong sense of existence for mathematical objects.

Moreover, imagine that a philosopher accepts this view and commit himself with a mathematical universe that guarantee the truth and falsehood of every mathematical statement. But if it is so, mathematical objects must have some precise properties. How are we able to acquire knowledge of mathematical objects? What happens when a mathematician discovers a new object in the mathematical universe? And when he discovers a new property of an object yet known by the community of mathematicians? How can the philosopher account for this new peculiarity of the object as a mathematical truth?

1.4.3 Reference

Reference: mathematical statements are about some kind of objects.

Reference is the relation between a term, for example a name or a sign, and an object. A well-formed statement therefore describes something in virtue of the reference between the terms it uses and the objects it describes.

Ontological commitment is the thesis according to which we are ontologically committed to the reference for the terms that appears in statements we believe to be true (see also section 1.3). Since Platonists generally endorse ontological commitment, the development of a coherent theory of reference is one of its main goals. Moreover, the aforementioned abandonment of uniqueness in reference asks for a revision of standard semantics and truth-value Realism.

Standard semantics claims that the language used by mathematicians functions semantically like ordinary language. Therefore, it claims that the semantic functions of singular terms and quantifiers are to refer to objects and to range over objects. Standard semantics doesn't explicitly take any position in ontology: rather, it rests content with the descriptive claim that the language of mathematics appears to have the same semantic structure as ordinary language.

Sober Platonists extensively agree on standard semantics, but develop truth-value Realism as a further step in. Truth-value Realism holds that every well-formed statement has a unique and objective truth-value, independently from anyone's thought and practice. Once applied to mathematics, truth-value Realism provides a clue for the existence of mathematical objects: if the semantic function of singular terms is to refer to objects, and every well-formed mathematical statement has a unique, objective and independent truth-value, then a condition for the meaningfulness, and even more so for truth, of a mathematical statement is the existence of references for mathematical terms, i.e. the existence of mathematical objects. Indeed, if a statement contains a term that doesn't refer to anything, the statement results as being meaningless and as lacking in truth-value.

Nevertheless, truth-value Realism doesn't by itself necessary imply the existence of mathematical objects. As a result, both Platonism and Anti-Platonism can adopt it, even if it fits certainly better with Platonism. Anti-Platonists who adopt truth-value Realism will have to explain how a mathematical statement turns out to be true, even if its terms have no reference, i.e., if the objects it is about don't exist. Anti-Platonists can avoid this contradiction by rejecting standard semantics. Or, by stating that if there is no reference for ordinary statements, those statements will be false. But Anti-Platonists classically hold that, even if there is no reference for mathematical terms, it could be in anyway useful to consider some mathematical statements as true.

The adoption of non-uniqueness is partly a consequence of Benacerraf's objection in *What Numbers Could Not Be*. As I already stressed, in this paper Benacerraf argues that he isn't aware of the existence of a criterion able to determine which of the different sequences of natural numbers is the right reference of the sequence of natural numbers. The reason is that no particular sequence of sets stands out as the right one. There is no metaphysical advantage of one over the others: neither Zermelo's nor Von Neumann's provide a better reduction than the others.

As a result, Benacerraf claim that, from an arithmetical point of view, only the structural properties of a sequence matter to the question of whether it is the sequence of the natural numbers. In particular, any ω -sequence will be as good candidate as any other.

Non-uniqueness in reference isn't a pleasant result in Platonist philosophy of mathematics. Indeed, it seems to deny the very possibility of objectiveness in denying that mathematical theories describe unique collections of abstract objects. Moreover, it denies standard semantics.

The problem here stays in the very definition of Platonism: Platonists adopting uniqueness can claim over Platonists adopting non-uniqueness that is essential for Platonist theories that mathematical theories are taken as being about unique collections of mathematical objects. And clearly, this would happen only if there were unique references for these terms. Indeed, if a singular term doesn't have a unique referent, we are inclined to say that it doesn't refer at all, and also that it isn't a singular term, since it has no unique referent. But what non-uniqueness Platonists want to deny is exactly that our mathematical singular terms have unique referents, because all mathematically important facts regard the relations between mathematical objects.

In this sense, the adoption of non-uniqueness in reference, albeit became quite popular in the last decades (for example, it is explicitly and enthusiastically endorsed by Full-Blooded Platonism, see chapter 2, and Trivialism, see chapter 5), stands out as an important difference between classical and Sober Platonism.

Most of this happened because many consequences can be drawn from non-uniqueness in reference, both in epistemology and in theory of truth. Indeed, contemporary Platonism embracing non-uniqueness has to deliver an approach to mathematical knowledge that explains the relationships between the multiple references. In addition, the adoption of one theory over the others, for example of Zermelo's over Von Neumann's, isn't always explicit but can raise some problems. As I already suggested in section 1.3, to account for mathematical truth will not be so easy: the statement '3 contained 2' will be true in Zermelo's definition and false in Von Neumann's. In absence of further elucidations, like 'in Zermelo's definition, 3 contained 2', which truth-value is to be assigned to the statement?

1.5 Sober Platonism

The subject of this work is the attitude in contemporary philosophy which I propose to call ‘Sober Platonism’. A Platonism is Sober if and only if it assumes that:

1. Mathematics is a discipline with a proper domain of study;
2. The role of philosophy of mathematics is to interpret mathematics and philosophically justifying it, without imposing any direction or hindrance to mathematicians’ enterprise;
3. Mathematical objects exist, but there is no need to endorse any strong ontological commitment with them;
4. Mathematical objects are such that it is possible to obtain true knowledge of them.

I will analyze each assumption individually. Only thereafter, I will try to offer a general overview of what I called Sober Platonism.

Assumption 1 can appear quite naïve and presumptuous at the same time. Indeed, it may be asked the reason why there is need to explicitly state that mathematics has a proper domain, as any authentic discipline with its own dignity. The reason is that several philosophical positions, mainly on the Anti-Platonism side, have explicitly deny the very existence of the proper domain of mathematics. Addressing mathematical enquiry as a research about objects that don’t exist, or that are unknowable by the very essence they are supposed to possess, fulfill the work of mathematicians with an air of mystery. And indeed, the best efforts of Anti-Platonism are spent in the direction of justifying the

undeniable application and reliability of mathematics, despite it has been made of literally false statements made by terms who lack in reference because predicate properties to non-existing objects.

One of the main reason why Sober Platonism embraces Platonism is the advantages to which Platonism leads in exactly this direction. Indeed, Platonism in mathematics fits very comfortable with mathematical practice. Platonism allows to take mathematical statements at face-value and to supply mathematicians with a real domain: mathematicians do work and refer to numbers in exactly the same way architects do refer to bricks or windows. Sober Platonists dig further in this direction, aiming at describing the work of mathematicians without any attempt to impose philosophy-based restrictions on mathematicians' liberty.

Assumption 2, that philosophy has to limit itself to the interpretation of mathematics, is very linked to assumption 1, if it isn't its direct consequence. Once the authenticity and dignity of mathematics is stated, it would be completely unconceivable to philosophically dictate to mathematicians what mathematical objects are. As a result, philosophers of mathematics is redrafted to the formulation of a good philosophical account of what mathematicians do.

Indeed, once philosophy recognizes a proper domain to mathematics and endorses a descriptive approach towards the work of mathematicians, the goal of philosophers changed: a descriptive approach to mathematics can supply philosophers of mathematics with an authentic subject, a proper domain, if not even a brute fact.

But most importantly, the role of philosophers of mathematics changes and assumes a more modern and appropriate perspective. The Sober Platonist 'climbed down' into the

mathematical world and, as an external beholder, observes and analyses it. Only from this point of view way Sober Platonists are able to provide a philosophical guide to mathematical practice that mirrors adequately the use of mathematical language and the way humans obtain and apply mathematical knowledge.

Speaking about knowledge and application, there is another advantage of Platonism in general, and Sober Platonism in particular: it can come to terms both with mathematical practice and with common sense. Indeed, in the case of philosophy of mathematics, several arguments showed that endorsing Anti-Platonism can lead to doubting what is ordinarily believed to be true. This phenomenon is exemplified very emblematically in *Question of Realism*¹⁵, page 4, by Kit Fine:

The antirealist about numbers maintains:

There are no numbers.

But most of us, in our non-philosophical moments, are inclined to think:

There are prime numbers between 2 and 6.

And yet the second of these claims implies that there are numbers, which is incompatible with the first of the claims. The antirealist will be taken to dispute what we ordinarily accept, the realist to endorse it. Thus the antirealist about numbers will be taken to deny, or to doubt, that there are prime numbers between 2 and 6.

Sober Platonism makes extensive use of such kind of arguments, in order to demonstrate that being Anti-Platonist isn't only wrong or useless, but that it could even be unintelligible. One of the arguments that are more explicitly along these lines is

¹⁵ FINE, KIT, (2002), *Questions of Realism*, reprinted in Bottani, Andrea, Carrara, Massimiliano, Giaretta, Pierdaniele (2002) *Individuals, essence and identity. Themes of Analytic Metaphysics*, Alphen aan den Rijn, Netherlands, Kluwer Academic Publishers;

Rayo's Zero Argument (see section 5.3).

Here I'm not stating that Platonism is the most perfect philosophy of mathematics. Indeed, as I already pointed out, there is a plenty of objection against it. What I'm rather stating is that, for what concerns mathematics, Platonism and in particular Sober Platonism, endorses a more constructive approach. Far from the misrepresentations of mathematics that can be delivered from antirealist positions, Sober Platonism takes the responsibility of philosophical issues and approaches mathematics as it is genuinely represented by mathematicians. And if Platonism would have philosophical problems, it will be a problem for philosophy, not for mathematics, who will maintain its autonomy.

As it should be clear, assumptions 1 and 2 are general assumptions about the methodology and the role of philosophy in its approach to mathematics. Assumptions 3 and 4 regard the ways in which Sober Platonism endorses the generally Platonist positions, namely metaphysical Realism for assumption 3 and epistemological Realism for assumption 4.

Assumption 3, that mathematical objects exist even without the need to commit with a strong ontology, constitutes a new perspective on a very classical Platonist view. Indeed, as it is customary in discussing the foundations of mathematics, Platonism entails not just the acceptance of the existence of abstract entities or universals, but the stronger acceptance of metaphysical Realism with respect to them.

In philosophy of mathematics, endorsing metaphysical Realism means that the mathematical realm consists of a fixed totality of mind-independent objects. That is why its endorsement requires a stronger assumption in ontology: the mathematical realm is inhabited by objects, the ultimate bricks of ontology, that have some specified and fixed

properties. Now, metaphysical Realism goes hand by hand with epistemological Realism: indeed, if the world is constituted by a fixed totality of mind-independent objects, there actually is one and only one true and complete description of the way the world is.

As a result, a platonist interpretation of a theory of mathematical objects will take the truth or falsity of statements of the theory, in particular statements of existence, to be objectively determined independently on the possibilities of human knowing these truths or falsities.

Sober Platonists aren't friends of metaphysical Realism; rather, they assume multiple ontologies, an infinite number of ways for the world to be. Together with the embracement of non-uniqueness in reference, this comes to mean that the same term can correspond to more than one object, determined by different contexts. And consequently, Sober Platonists aren't friend of epistemological Realism neither.

Assumption 4, namely that mathematical objects are such that it is possible to obtain true knowledge of them, is kind of a weak version of epistemological Realism. Indeed, on the one hand, if there is no one, fixed totality of mind-independent objects that constitutes the mathematical realm, there couldn't be an objectively determined assignment of truth conditions to mathematical statements. But on the other hand, assuming a less fixed and statical notion of mathematical knowledge (and mathematical truth), allow Sober Platonist to approach mathematics in a less assertive and more descriptive fashion.

Ultimately, what Sober Platonism generally defends is a descriptive approach to mathematics: what mathematical objects are, is said by mathematicians; what is true in

mathematics, is said by mathematicians. If mathematicians assert some mathematical statements, those statement must have some mathematical reason for being asserted. Philosophy must rest content with the mere observation of mathematical practice and promotion of philosophical interpretations of it. In this perspective, it would be completely senseless for philosophers to dictate what are mathematical objects. Indeed, such a question is addressed to mathematicians, and not to philosophers.

The attitude proposed by Sober Platonism is far from a defeat of philosophy. Rather, drawing the limits of philosophical inquiry allow to focus on what philosophy can actually investigate.

In conclusion, Sober Platonists, as classical Platonists, are committed with the existence of mathematical objects, but from a different perspective. Indeed, if classical Platonists could say something along the lines of:

1. Mathematical objects exist as abstract, independent objects;
2. Such objects are correctly described by true mathematical statements;
3. True mathematical statements are knowable.

Sober Platonists will rather say that:

1. Mathematical statements are knowable;
2. Such statements truly describe some kind of mathematical reality;
3. Some kind of mathematical reality must exist.

At the end of the day, and unless Sober Platonists strongly attempted to provide a description as detailed as possible of mathematical reality, this is not their main aim. Rather, it's a consequence of their main aim, that is to philosophically justify a brute

fact: humans (and other animals) do attain mathematical knowledge. And this knowledge is knowledge of necessary truths.

The development of a satisfactory definition of Platonism turned out to be nothing but easy. Moreover, the adoption of *Independence* seems to be an entanglement for all the other aspect of Platonism. I'm persuaded that it constitutes the very point break between genuine and non-genuine Platonism.

In what follows, I will introduce four paradigmatic Sober Platonists approaches to philosophy of mathematics. I will then attempt an analysis of this four approaches and try to disclose the new role Sober Platonism assigns to philosophy. In a sense, it would reveal to be in assonance with the words famously used by Hegel in the preface to *Philosophy of Right*:

One more word about *teaching* what the world ought to be: Philosophy always arrives too late to do any such teaching. As the *thought* of the world, philosophy appears only in the period after actuality has been achieved and has completed its formative process. The lesson of the concept, which necessarily is also taught by history, is that only in the ripeness of actuality does the ideal appear over against the real, and that only then does this ideal comprehend this same real world in its substance and build it up for itself into the configuration of an intellectual realm. When philosophy paints its gray in gray, then a configuration of life has grown old, and cannot be rejuvenated by this gray in gray, but only understood; the Owl of Minerva takes flight only as the dusk begins to fall.

[I]f a mathematician comes up with a radically new (pure mathematical) theory, she can be criticized on the grounds that the theory is inconsistent or uninteresting or useless, but she cannot be criticized on the grounds that the objects of the theory do not exist

BALAGUER, MARK, (1995) *A Platonist Epistemology*, page 311

Chapter 2: Full-blooded Platonism

2.1 Introduction

The philosophical account that Mark Balaguer developed is built on some ontological basic notions he endorses in the very beginning of his major work, *Platonism and Anti-Platonism in Mathematics*¹⁶.

His philosophy of mathematics is substantively metaphysical, as he aims at reflecting on mathematical theories and mathematical practice in order to disclose whether they tell us anything about the mathematical world. But, although his project is mainly metaphysics, it takes the cue from some considerations about the epistemology of mathematics. Thus, it is from the developing of Balaguer's epistemology that its ontological theory is derived. Roughly, Balaguer affirms that, in order to solve the problem of access, the existence of the objects of knowledge must be assumed. As a result, such objects must be characterized in a fashion that allows human beings to attain knowledge of them. The underlying idea in Balaguer theory is that mathematical objects must exist since mathematical knowledge occurs. To be more explicit, since humans do attain mathematical knowledge, the objects of knowledge must exist and be knowable, even if it is not available any access to them.

Balaguer's ontological dissertation takes the cue from a definition of what is a mathematical object. He adopts the classical definition of abstract objects, as opposed to concrete objects: abstract objects are non-spatiotemporal, non-physical, non-mental and

¹⁶ BALAGUER, MARK (1998) *Platonism and Anti-Platonism in Mathematics*, Oxford University Press, New York.

a-causal. A mathematical object is an abstract object that would ordinarily be thought of as failing in the domain of mathematics.

Balaguer's perspective seems to embrace Platonism very spontaneously. His account could also look like a defense of Platonism against Anti-Platonism. But this is not the whole story: he claims that Platonism and Anti-Platonism, particularly in his versions of Platonism and in Hartry Field's Anti-Platonism, are both perfectly workable philosophies of mathematics. Moreover, he is convinced that humans could never discover a conclusive argument for or against mathematical Platonism. So, specifically, his position in the dispute between Platonists and anti-Platonists is neutral. In particular, he is convinced that the only valid options are both his Platonism and Fictionalism.

Balaguer (1998) proposes an original kind of Platonism that he calls *Full-Blooded Platonism* (henceforth, FBP). FBP differs from classical versions of Platonism in various aspects, but, above all, regarding how many mathematical objects there are. FBP is a plenitudinous Platonism, as it maintains that there exist all logically possible mathematical objects, even those that haven't yet been discovered. This claim makes FBP a non-standard version of Platonism, because classical versions of Platonism are non-plenitudinous, in the sense that they admit some kind of mathematical objects but not others.

Moreover, Balaguer argues that the structuralist philosophy of mathematics, in particular Shapiro's perspective, can be reduced to a (structuralist) version of FBP. He does so through the analysis of the two main positions in Platonism: Object-Platonism and Structuralism. The former is traditionally defended by Gödel, Frege and those Platonists who claim that there exists a mathematical realm populated by abstract

objects, which are described by mathematical theories. On the other hand, Structuralism claims that mathematical theories describes a structure, something like an objectless template, a system of positions that can be filled by any system of objects that exhibit a structure. Balaguer demonstrates that FBP is consistent with both, by showing that Object-Platonism and Structuralism are not distinct from a metaphysical point of view. The idea is that, since structuralists refer to positions in structures with singular terms, quantify over them in first-order languages, ascribe them properties, then positions in structure are to be taken as mathematical objects. Whatever kind of entities could inhabit the mathematical realm, mathematicians's use of mathematical language suggests that they treat them as objects. As a result, the difference between Object-Platonism and Structuralism collapses.

In Balaguer's account, any mathematical object, which possibly could exist, actually does exist, and this is what makes FBP a plenitudinous Platonism. Several features of FBP ask for a further examination: for example, the possibility to attain mathematical knowledge and what Balaguer really means with 'logically possible' Moreover, as FBP deals with plenitudinous ontology, it entails the attitude to embrace non-unique references for mathematical terms. Due to the originality of this approach, both a justification and an explanation are needed.

2.1.1 The Problem of Formalization

Balaguer's attempts to provide a formalized definition of FBP encounter some difficulties. He firstly provides a formalized definition of FBP at page 6 of his *Platonism and Anti-Platonism in Mathematics*:

$$(\forall x)[(x \text{ is a mathematical object} \ \& \ x \text{ is logically possible}) \rightarrow x \text{ exists}]$$

This definition makes use of a *de re* sort of possibility and seems to suggest that existence is a predicate and that there are possible objects that may or may not be actual objects. But Balaguer specifies he does not think there are any such thing as objects that don't exist, that is a pretty common position in ontology. He also claims that there aren't possible but not actual objects, at least in mathematics. Rather, all objects are ordinary, actually existing objects. The idea is that the ordinary actually existing mathematical objects exhaust all of the logical possibilities for such objects. In other words, there actually exist mathematical objects of all logically possible kinds.

Balaguer claims that FBP could be better captured in a second-order modal language (page 6, Balaguer (1998)):

Let 'x' be a first-order variable, 'Y' be a second-order variable, 'Mx' mean 'x is a mathematical object'; then:

$$(Y)[\Diamond(\exists x)(Mx \ \& \ Yx) \rightarrow (\exists x)(Mx \ \& \ Yx)]$$

Balaguer isn't convinced about this formula either, because it doesn't render properly the commitment engaged by FBP with the existence of all mathematical objects that possibly could exist. So, it does not entail that there exists any mathematical object, because it is silent on the question of whether it is possible that there exist mathematical objects at all. In the previous formalization, nothing is said to guarantee that the antecedent of the conditional will ever be true. But it is entirely trivial that the existence of mathematical objects is logically possible.

Thus, Balaguer (1998) proposes another formula at page 7:

$$(\exists x)(Mx) \ \& \ (Y)[\Diamond(\exists x)(Mx \ \& \ Yx) \rightarrow (\exists x)(Mx \ \& \ Yx)]$$

This formula has an existential quantifier that varies over the domain of mathematical objects. Finally, this formalization involves the existential commitment to mathematical objects Balaguer was looking for.

2.2 Epistemology

2.2.1 The Epistemological Argument

In *A Platonist Epistemology*¹⁷, Balaguer engages Benacerraf's (1973)¹⁸ challenge to mathematical Platonism. In Benacerraf's thought, the claim that mathematical theories are descriptions of a non-physical, a-spatial, a-temporal, mind-independent aspect of reality turns out to be incompatible with a naturalistic epistemology. His epistemological argument takes the form of a *prima facie* worry about human beings' ability to acquire knowledge of abstract objects being a *prima facie* argument against mathematical Platonism: Platonism cannot be true because it precludes the very possibility of mathematical knowledge.

According to Benacerraf (1973), the best epistemology is the causal theory of knowledge. Briefly, it dictates that in order for a person *S* to know that *p* it is necessary for *S* to be causally related to the fact that *p* in an appropriate way. See section 1.4 for a less sketchy description of theories of knowledge.

Balaguer (1998, page 22) reconstructs Benacerraf's epistemological argument based on causal theory of knowledge and looks for a way to reject it.

Here how he reconstructs the argument:

¹⁷ BALAGUER, MARK (1995) "A Platonist Epistemology", *Synthese* vol 103 pp. 303-325;

¹⁸ BENACERRAF, PAUL, (1973) *Mathematical Truth*, *Journal of Philosophy* vol.70 pp. 661-679;

1. *Human beings exist entirely within space-time*
2. *If there exist any abstract mathematical objects, then they exist outside of space-time*

Therefore:

3. *If there exist any abstract mathematical objects, then human beings could not attain knowledge of them.*

Therefore:

4. *If mathematical Platonism is correct, then human beings could not attain mathematical knowledge*
5. *Human beings have mathematical knowledge.*

Therefore:

6. *Mathematical Platonism is not correct.*

The most popular response to Benacerraf's argument has been to reject causal theory of knowledge. But rejecting it, anti-Platonists can argue that 1 and 2 jointly imply that human beings could not be causally related to any mathematical objects, which are totally inaccessible to us: information cannot pass from mathematical objects to human beings. So, 1 and 2 alone give rise to a *prima facie* argument to believe 3. But Balaguer believes 3 is false, although 1 and 2 are indeed *prima facie* reasons to be suspicious about the reliability of our mathematical beliefs.

According to Balaguer, there are three strategies to undercut the reasons to believe that 3. The first strategy is to deny 1, asserting that human mind is capable of entering in contact with mathematical realm, and thereby acquiring information about that realm. Balaguer calls this path to abstract objects the 'non-spatiotemporal contact view'. This

strategy is undertaken by Kurt Gödel¹⁹, who thought that human beings acquire knowledge of abstract mathematical objects in much the same way in which they acquire knowledge of concrete physical objects. This happens thanks to a faculty analogue to sense perception, namely mathematical intuition. Gödel's thought was that, while data about physical objects arise from sense perception, the presence in us of mathematical data may be due to another kind of relationship between reality and humans. But there are several problematic aspects: it seems impossible for humans to receive information from mathematical objects, since the latter are abstract and causally inert; therefore they could not generate information-carrying signals. Indeed, such a signal has somehow to involve a sort of cross-realm contact. This is the reason behind Gödel's appeal to mathematical intuition. But even if minds are immaterial, it doesn't follow that they are into informational contact with mathematical objects.

Balaguer interprets Gödel's work in not only believing that human minds are immaterial, but that we are lead to this conclusion by reflecting on mathematics. That's why Balaguer concludes that immaterialism about the mind follows from Gödel's incompleteness theorem.

It is a digression of particular interest to see how Balaguer argues this. Gödel's incompleteness theorem states that, for any consistent axiomatic system, there are propositions that are undecidable in that system. He claims that there are no mathematical propositions that are absolutely undecidable, not just within some particular axiomatic system, but by any mathematical proof the human mind can

¹⁹ GÖDEL, KURT, (1951) *Some Basic Theorems on the Foundations of Mathematics and Their Implications*, in his *Collected Works Vol. III*, New York: Oxford University Press, 1995, pp. 304-323.

conceive. That is, the set of humanly provable mathematical propositions cannot be recursively axiomatized and the human mind cannot be reduced to a Turing machine.

The second strategy Balaguer experiments is to assert that 2 (the claim that, if there exist any abstract mathematical objects, then they exist outside of space-time) is false: human beings can acquire information about mathematical objects via physical perceptual means. This is the strategy undertaken by Penelope Maddy²⁰. According to her, human beings are capable of knowledge of mathematical objects by coming into contact with them *via* sense perception.

Maddy refers mainly to sets, some of which are, according to her account, spatio-temporally located and perceptible. She proposes to explain how humans proceed from perception of sets to knowledge of the axioms of set theory: our perceptions of sets lead to mathematical intuitions about sets. In this context, Benacerraf's worry may not arise, thanks to the bare claim that mathematical objects are perceptible. Here Balaguer answers a legitimate question: does Maddy genuinely endorse Platonism? Since Maddy's objects are concrete, her theory seems far from Platonism. Indeed, rejecting 2 traditionally entails abandoning Platonism. If Maddy endorses Anti-Platonism, she has to claim that sets are ordinary concrete objects. And the whole point of Benacerrafian argument is to see how knowledge of thing in space-time could lead to knowledge of thing outside of space-time. But Maddy maintains that sets are neither abstract objects nor ordinary concrete objects, embracing a sort of hybrid Platonism: there are sets that are concrete and sets that are abstract (e.g. the pure sets, built from the null set).

²⁰ MADDY, PENELOPE, (1980), *Perception and Mathematical Intuition*, *Philosophical Review* vol. 89 pp. 163-196.

The last strategy is to accept that 1 and 2 are true, but rejecting that they lead to 3: humans are not capable of any sorts of contact with mathematical objects, nonetheless can acquire knowledge of such objects. Therefore, an epistemology of abstract objects, an explanation of how human beings could acquire knowledge of abstract mathematical objects, is demanded. The challenge is to account for the fact that if mathematicians accept p , then p . Indeed, Platonists need to account for the reliability of mathematical beliefs, that is to justify that, if mathematicians accept a purely mathematical sentence p , then p truly describes part of the mathematical realm. In the following section, I will focus on Balaguer's solution.

2.2.2 The full-blooded solution

According to Balaguer, mathematicians can acquire knowledge without checking their discoveries against mathematical facts, since all the mathematical objects, which possibly could exist, actually do exist. If FBP is correct, all humans have to do in order to attain knowledge of mathematical objects is to conceptualize, think about or even dream such objects²¹.

Such a theory is explicitly inspired by Hartry Field's famous 'Argument of the Nepalese Village'. The argument starts from the assumption that, if all possible Nepalese villages exist, then it would be possible to attain knowledge of one of these villages without any access to it. Since a village is a Nepalese village if it is, for example, a group of up to ten houses somewhere in Nepal, and once every possible combination of those elements

²¹ This conception applies only to mathematical abstract objects, and not to abstract objects in general.

is known, Field concludes that this is sufficient for attaining knowledge of a Nepalese village with which there is no possibility of entering into contact. It would be sufficient to dream up a possible Nepalese village and that dream will perfectly represent one actually existing Nepalese village, since all possible Nepalese villages actually exist.

Balaguer and Field agree on the identification of the meaning of 'possible' with that of 'logically possible'. Logical possibility embraces everything that does not lead to contradiction. Moreover, given that, according to Balaguer, all objects are ordinary, actually existing objects, the ordinary actually existing mathematical objects exhaust all of the logical possibilities for such objects. Therefore, there actually exist mathematical objects of all logically possible kinds. This move allows to maintain 1 and 2, but deny 3. Moreover, the opening to all logically possible mathematical objects guarantees that FBP is incompatible with those sorts of Platonism that deny certain kinds of mathematical objects but assert that they are metaphysically possible.

Balaguer's next move is to provide a version of Platonism that is capable to resist Benacerrafian argument. He first argues that FBP-ists can account for the fact that human beings can know of certain purely mathematical theories that they are consistent: humans do have some skills at recognizing consistent from inconsistent theories. According to Balaguer, this is a matter of fact. And since it is so, FBP-ists can account for the fact that, if mathematicians accept a purely mathematical sentence p , then p is consistent.

It is important to keep in mind here that human abilities in recognizing consistent from inconsistent theories are not ultimately reliable. It is a matter of history that some theories, thought to be consistent, have revealed to be inconsistent after a new analysis.

Humans' ability to recognize consistent from inconsistent theories evolves. Balaguer does not tackle this problem here, but provides an account of the evolution of theories that could be applied to it. This subject is further specified in sections 2.4 and 2.5 of this chapter.

On the assumption that FBP is true, it is possible to deliver a coherent theory of knowledge and solve the problem of access. But the truth of FBP is still hypothesized. Then why and how FBP-ists are allowed to assume that FBP is true? That is, if T is a purely mathematical theory, which humans know to be consistent, then FBP-ists only have a FBP-ist account of their ability to know that if FBP is true, then T truly describes part of the mathematical realm. But FBP-ists don't have an account of humans' ability to know that T truly describes a part of mathematical realm, since they haven't yet said a word in support of humans' ability to know that FBP is true.

This objection demands for an internalist account of the reliability of our mathematical beliefs, but in order to meet Benacerraf epistemological challenge, both classical Platonists and FBP-ists need only to provide an externalist account of the reliability of our mathematical beliefs. A theory of the reliability of S 's belief is internal when it explains how S knows or reliably believes that her methods of belief acquisition are reliable. What is needed is an explanation of the reliability of S 's beliefs that can also explain how S can reliably believe that the explanation is true.

That is why Balaguer provides FBP with an external account, explaining the reliability of our mathematical beliefs by pointing out that humans use their knowledge of the consistency of purely mathematical theories in fixing our purely mathematical beliefs.

On the assumption that FBP is true, any method of fixing purely mathematical belief, which is so constrained by knowledge of consistency, is reliable.

Balaguer does not rest content of having reduced the Benacerrafian argument to externalist perspective thanks to the adoption of consistency as criterion for reliability in mathematics. He moves forward in defending FBP and argues that, even in internalist terms, FBP-ists can account for humans' ability to know that if there is an external physical world of the sort that gives rise to accurate sense perceptions, then a theory about the physical world R is true of that world.

Clearly, an appeal to sense perception does not yield necessarily to an internalist account of humans' ability to know that R is true of the physical world. Indeed, it does nothing to explain the reliability of humans' ability to know that there is an external physical world of the sort that gives rise to accurate sense perception. But an appeal to sense perception is sufficient for an externalist account of humans' ability to know that R is true of the physical world.

Those who believe that there is an external world of the sort that gives rise to accurate sense perception, can provide an externalist account of our empirical knowledge of physical objects by merely pointing out that we use sense perception as a mean of fixing our beliefs about the physical world. On the assumption that there is an external world of that sort, any method of fixing empirical belief that is so constrained by sense perception is reliable.

FBP, in respect to knowledge of mathematical objects, seems to be analogous to the belief in the existence of an external world of the specified sort. Balaguer here aims to identify sense perception with the ability to discriminate consistent from inconsistent

theories, and shows that both break down in attempt to account for knowledge that FBP or the belief in the existence of an external world of the sort that gives raise to accurate sense perception are true.

Anti-Platonists cannot allow the two situations to be analogous, because the whole point of Benacerraf's worry is to raise a special problem for abstract objects, i.e. that is not a problem for concrete, perceived objects. That is why Balaguer interprets the dilemma externalistically, because it raises only in externalist key: an internalist account is as problematic for physical objects as it is for mathematical, for concretes as for abstracts.

To briefly summarize, Balaguer proposal is that, in order to acquire knowledge of mathematical objects, all it is needed is to acquire knowledge that some purely mathematical theory is consistent.

There remain a few clarifications to be made, in order to offer a complete picture of Balaguer's theory. First, he challenges the following, questionable assumption that seems to be held in his account; namely, that humans are capable of thinking about mathematical objects or dreaming up stories about such objects, or formulating theories about them. The problem here is that it is not clear how humans could do this, unless before having acquired knowledge of mathematical objects, but also how do they have beliefs about and refer to mathematical objects.

Balaguer response make use of a distinction introduced by Hodes (1990)²², between thick and thin ontological commitments. According to Hodes, the concept of 'having beliefs about some objects' can be interpreted in two different ways: to have a belief that is *thickly* about an object, for which there must be an appropriate connection; to have a belief that is *thinly* about an object, for which there is no such need. This last

²² Hodes, Harold, (1990) *Ontological Commitments: Thick and thin* in Boolos, (1990) 347-407;

kind of belief is by itself sufficient to allow humans to formulate beliefs and theories that are (thinly) about mathematical objects.

The second clarification is that, if FBP is true and all consistent purely mathematical theories truly describes some collection of mathematical objects, or some part of the mathematical realm, it seems that such theories do not characterize unique parts of the mathematical realm. This point is very thorny, since once all consistent purely mathematical theories have multiple models, Platonists are committed to the thesis that such theories fail to pick out unique collections of mathematical objects. That is, reference is not uniquely fixed. But according to Balaguer, non-uniqueness is simply not a problem. Resting on the consideration that some beliefs being about some fact depends upon the interpretation, namely on the choice of the model for these beliefs, he denies that there are any unique collections of objects that correspond to what humans have in mind when they formulate mathematical beliefs and theories. The embracement of non-uniqueness is deeply analyzed in section 2.4 of the present chapter.

2.3 Consistency

Balaguer's notion of consistency plays a fundamental role in his form of Platonism: since existence depends upon consistency, all consistent purely mathematical theories truly describe some existent part of the mathematical realm. In what follows, I will focus on the criteria for determining if a sentence is purely mathematical and on defining Balaguer's notion of consistency.

The first question has a quite simple answer: a sentence or theory is purely mathematical if it contains exclusively mathematical terms, speaking of nothing but the

mathematical realm, predicating mathematical properties and relations to mathematical objects. Thus, ‘there are three apples on the table’ isn’t mathematical, while ‘the first three natural numbers are primes’ is mathematical.

The second question requires a longer discussion. To understand the idea of consistency Balaguer has in mind it will be useful to recall two quotations he himself reports in his (1998). The first is taken from Hilbert in a letter to Frege:

[I]f the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist.

This is for me the criterion of truth and existence²³.

The second is from Poincaré (1913):

[I]n mathematics the word exist...means free from contradiction²⁴.

Both Hilbert and Poincaré stated clearly that in mathematics existence and consistency are strongly bounded: if something is free from contradictions, then it exists.

Balaguer analyses also the work on consistency by Georg Kreisel²⁵ and Hartry Field²⁶, who both take consistency as a primitive term, governed by two rules: if a sentence is semantically consistent, i.e. it has a model, then it is consistent; and if a sentence is consistent, then it is syntactically consistent, i.e. cannot be refuted in a system of formal logic.

²³FREGE, GOTTLIB (1980) *Philosophical and Mathematical Correspondence*, University of Chicago Press, Chicago, pp. 39-40.

²⁴ POINCARÉ, HENRI, (1913), *The Foundation of Science*, The Science Press, Lancaster, p. 454.

²⁵ KREISEL, GEORG, (1967) *Informal Rigor and Completeness Proof* in LAKATOS, IMRE, *Problems in the Philosophy of Mathematics*, North Holland, Amsterdam;

²⁶ FIELD, HARTRY, (1989) *Realism, Mathematics and Modality*, Basil Blackwell, Oxford;

Anti-Platonists may object that, insofar as models and derivations are abstract objects, they are both notions only Platonists can appeal to. Thus, knowledge of consistency is understood as knowledge of abstract objects, namely models and derivations. FBP-ists seems to have accomplished nothing by reducing the question of how humans could know that our mathematical theories are true to the question of how humans could know that they are consistent. Moreover, if the existence of abstract objects depends upon the consistency of the objects (or theories) and the consistency of abstract objects (or theories) depends upon the existence of other abstract objects (such as models and derivations), then the argument is circular.

FBP-ists can claim that the notion of consistency at work in anti-Platonist theories, like the one proposed by Kreisel and Field, is that consistency is simply a primitive term. Therefore, it is not defined in any exhaustive way. Indeed, it is related to the two formal notions of semantic consistency and syntactic consistency; but the two formal notions alone don't provide a definition of the primitive notion. Rather, they both provide information about the extension of the primitive notion. From the definitions of the two formal notions and from the intuitive understanding of the primitive notion, Balaguer argues that, if a theory T is semantically consistent, then it is intuitively consistent, and if T is syntactically inconsistent, then it is intuitively inconsistent. Therefore, Balaguer aims at showing that, among first-order theories, the intuitive notion of consistency is coextensive with both formal notion of consistency. Combining his analysis with

Henkin's theorem, he obtains the attended result: among first-order theories, syntactic consistency implies semantic consistency²⁷.

In Balaguer account, there seems to be no reasons to believe the intuitive notion of consistency as being coextensive with the semantic notion. Even assuming that the two notions are coextensive, there are still good reasons for thinking that the semantic notion doesn't provide by itself a definition of the intuitive notion. Field (1989) argues in defense of this point very convincingly, by showing that the semantic notion is not able to capture the essence of the intuitive notion. In particular, he points out that there are certain theories for which is obvious that they are intuitively consistent, but not obvious that they are semantically consistent²⁸.

Advocates of the Kreisel-Field view might claim that the primitive notion of consistency is equivalent to a primitive notion of possibility. Consider that, for each different kind of possibility, we can define formal notions of syntactic and semantic

²⁷ Formally:

1) $\text{SemC}(T) \rightarrow \text{IntC}(T)$

2) $\text{Syn}\sim\text{C}(T) \rightarrow \text{Int}\sim\text{C}(T)$

3) $\text{SynC}(T) \rightarrow \text{SemC}(T)$

If 4) $\text{Sem}\sim\text{C}(T) \leftrightarrow \sim\text{SemC}(T)$, 5) $\text{Syn}\sim\text{C}(T) \leftrightarrow \sim\text{SynC}(T)$ and 6) $\text{Int}\sim\text{C}(T) \rightarrow \sim\text{IntC}(T)$, from this and 2) follows:

7) $\text{IntC}(T) \rightarrow \text{SynC}(T)$

Therefore:

8) $\text{IntC}(T) \leftrightarrow \text{SynC}(T)$

From this, adding:

9) $\text{SemC}(T) \leftrightarrow \text{SynC}(T)$

Therefore:

10) $\text{IntC}(T) \leftrightarrow \text{SemC}(T)$

8) and 10) express the coextensiveness of the intuitive consistency with both syntactic consistency and semantic consistency

²⁸ For instance, consider the set S of all the truths about sets that are storable in the language of set theory. S is obviously consistent in the intuitive sense, but it is not at all obvious that S is semantically consistent, e.g. that it has a model. Indeed, a model of S would have the set of all sets as its universe, but is well known that there is no such a thing. Accordingly to Henkin's theorem, if a sentence a follows semantically from a set of sentences S then there is a proof of a from S . Therefore, S does have a model, but an Henkin's model, produced by this proof. The result doesn't extend to cases where the language of S is higher-order. Moreover, the bare fact that the result has to be proven, shows that the semantic notion doesn't capture the essence of the intuitive notion.

consistency. Balaguer offers two examples at the end of page 70 of *Platonism and Anti-Platonism in Mathematics*:

1. A theory T is semantically conceptually consistent if and only if the union of $T+C$ of T and the set C of all conceptual truths has a model; and T is syntactically conceptually consistent if and only if there is no derivation of a contradiction from $T+C$ in any logically sound derivation system.
2. A theory T is semantically physically consistent if and only if the union $T+P$ of T and the set P of all physical laws has a model; and T is syntactically physically consistent if and only if there is no derivation of a contradiction from $T+P$ in any logically sound derivation system.

Assuming this, Balaguer is convinced that it is possible to say that there is an intuitive notion of possibility or consistency, corresponding to each such pair of formal notions. Thus, the Kreisel-Field intuitive notion is simply the broadest of these notions: is a notion of logical possibility, so broad that all of the other intuitive notions of possibility can be defined in terms of it.

Balaguer argues that it is acceptable for him to use Kreisel-Field primitive notion of consistency, even though it is genuinely anti-Platonist. Indeed, what is really important is that some anti-Platonist account of consistency is available and allows him to understand the notion of consistency in an anti-Platonist way. If there were no legitimate anti-Platonist account of consistency, this will be a problem both for Platonists and anti-Platonists, since no one would ever be able to account for the simple

fact that some of our theories are consistent and others are not. Such a view is totally unacceptable. But Balaguer claims that this notion is available and is a primitive notion, therefore it isn't defined in terms of abstract objects. According to him, this primitive notion is the notion of logical possibility: whatever is logically possible is consistent. This means that there is no need to enter in epistemic contact with objects predicated by a set of sentences in order to understand if the set of sentences is consistent. For example, no access to the first son of Julius Caesar is required in order to know that the sentence asserting him to be male and the sentence asserting it to be not female are consistent to each other. While no epistemic contact to him is required to know that the sentences asserting it to be blonde and not blonde aren't consistent to each other. In the same way, Balaguer claims that there is no need for epistemically access to the number 3 in order to know that the sentences asserting it to be prime and even are consistent to each other, while the sentences asserting it to be even and multiple of the number 4 are not.

Thus, if there is a legitimate anti-Platonist notion of consistency, FBP-ists too can account for the fact that human beings can know that certain purely mathematical theories are (anti-Platonistically) consistent.

Balaguer considers some possible objection in regards of consistency. First he considers that, while sentences and theories are abstract objects, FBP-ists restrict their attention to concrete tokens of sentences and theories. But, if T is a purely mathematical theory, although the claim that concrete tokens of T are consistent isn't about abstract objects, then T itself is about abstract objects. Indeed, it may be that humans need contact with

abstract objects in order to know that T is consistent, because it may be that humans need contact with T 's own ontology in order to know that T is consistent.

Balaguer replies that knowledge of consistency of a set of sentences does not require any sort of epistemic access to the objects the sentences are about. That is because knowledge of consistency is logical knowledge. If FBP is true, then mathematical knowledge can arise directly out of logical knowledge. Nevertheless, mathematical truth is not logical truth, because the existence claims of mathematics are not logically necessary.

A second worry is that FBP-ists do not have to account for how humans know which of our consistent purely mathematical theories truly describe mathematical objects and which do not. Since, according to FBP, all of our consistent purely mathematical theories truly describe mathematical objects, the question doesn't even raise. And again, if FBP is true, then knowledge of mathematical objects falls straight out of knowledge of the consistency of mathematical theories.

In conclusion, FBP-ists state that they do not need any access to a set of objects in order to know whether a set of sentences about these objects is consistent. That is, Balaguer reached his goal: epistemic contact is not required in order to obtain knowledge of consistency.

2.4 Mathematical Correctness and Mathematical Truth

Mark Balaguer, in *A theory of Mathematical Correctness and Mathematical Truth*²⁹ draws a distinction between mathematical correctness and mathematical truth. *Prima facie*, in mathematics correctness and truth may appear to describe the same notion: a correct theorem of mathematics is a true theorem in mathematics. If something is correct in mathematics, it is because it is proven, and if it is proven, then it is true.

This is why the notion of truth that underlies FBP takes a particular position in the debate between correspondence and coherence theories of truth. According to the former perspective, truth entails a correspondence with reality: a true sentence describes accurately an existing reality. That is, there exists a reality to which the singular terms that compose the sentences correctly refer to and accurately describe. On the other side, coherence entails truth. A coherence theory of truth does not entail the existence of a reality to which correct sentences rely. Accordingly, true sentences are coherent sentences, which do not lead to contradiction. Sentences that are true thanks to coherence theory of truth can refer vacuously: the singular terms that occur in a coherently true sentence do not need to refer to something existing. What is relevant is only the absence of contradiction.

To offer just two simple examples, a truth like ‘everything is identical with itself’ is coherently true, since it does not require to check that everything in the world is actually identical with itself, but only that the identity with itself in general does not lead to

²⁹ BALAGUER, MARK (2001), “A Theory of Mathematical Correctness and Mathematical Truth”, *Pacific Philosophical Quarterly* 82, pp. 87-114;

contradiction. Thus, to falsify this sentence all it is needed is to find just a thing that is not self-identical.

Instead, 'Costanza is Italian' is a true sentence due to the existence of an object to which the term 'Costanza' refers and to the correspondence of the term with an existing object, namely to the fact that there exist an object named 'Costanza' that truly owns the property of 'being Italian'. So, what is needed in order to falsify this sentence is either that there exist no such object or property (thus no reference for the singular terms that occur in it) or that there actually exist such object, but it does not own this property.

Finally, the interesting aspect is that in FBP the difference between coherence and correspondence theories of truth collapses. If a theory is coherent, i.e. it does not lead to contradiction, then the objects it talks about exist, so there is a correspondence, a reference for the terms that occur in it, although reference couldn't be uniquely determined, as explained in section 2.5.

As a result, the embracement of non-uniqueness in reference entails, for FBP, that consistency is sufficient for truth. If both a consistent mathematical sentence and its consistent negation are true, a genuine contradiction doesn't raise, because since the consistent mathematical sentences describe parts of the mathematical realm, FBP-ists can claim that they simply describe different parts.

2.4.1 How to deal with undecidable sentences in Mathematics

In his account of mathematical truth, Balaguer argues also in favor of an account of mathematical correctness that agrees with his conception of correspondence for coherent truth. In order to obtain an account of mathematical correctness, he faces

directly the problem raised by the existence of certain mathematical sentences that are undecidable in all of the currently accepted axiomatic mathematical theories. These sentences give rise to open mathematical questions that cannot be answered by our axiomatic theories, for example questions related to the *continuum* hypothesis.

The debate can be stated in the following way: either these questions have objectively correct answer, or these questions have no objectively correct answer. The former position is endorsed by the objectivists, who do claim the existence of a correct and ultimate answer, while the latter is endorsed by the anti-objectivists, who don't feel the need to claim that there actually is a correct and ultimate answer.

This problem is also related to the question of whether mathematical Platonism or Anti-Platonism is true. Indeed, if there are mathematical objects, these objects are so and so, and, in the mathematical realm, there is a correct answer to the question of whether the *continuum* hypothesis is true or false. It is just a matter of finding out the correct answer. On the contrary, if there are no such things as mathematical objects, then there is nothing for the *continuum* hypothesis or other open questions to be wrong or right about. So there is no correct answer, but different, hopefully coherent, theories that propose different possible answers.

Mark Balaguer suggests to address the question in a different way: whether a given mathematical question has an objective answer does not depend on the existence of any mathematical objects. Indeed, his proposal fits both with Platonism and Anti-Platonism.

In *A Theory of Mathematical Correctness and Mathematical Truth*, Balaguer suggests to make use of an Intention-Based Partial Objectivism, characterized by three principles (page 90, 91 and 92):

(COR) A mathematical sentence is objectively correct just in case it is 'built into', or follows from, the notions, conceptions, intuitions, and so on that we have in connection with the given branch of mathematics.

Roughly, a mathematical sentence is correct if it is built into our full conception of the mathematical objects, that is, the sum of all our mathematical thoughts and what follows from these thoughts. Notice that, according to (COR), it is perfectly acceptable for a mathematical sentence to be undecidable in all of our axiomatic theories, but still remaining objectively correct, since it has been built into the notions, conceptions, intuitions and so on that we have in the given branch of mathematics.

(INCOR) A mathematical sentence is objectively incorrect just in case it is inconsistent with the notions, conceptions, intuitions, and so on that we have in connection with the given branch of mathematics.

Some open questions in mathematics are such that none of their possible answers are built into the notions, conceptions, intuitions, and so on that we have in the given branch of mathematics. This is the case rendered by (INCOR).

The last principle is:

(NEUT) Many mathematical sentences are objectively correct, and many are objectively incorrect, but it may be that there are some mathematical sentences that are neither objectively correct nor objectively incorrect, because (a) they do not follow from the notions, conceptions, intuitions, and so on that we have in the given branch of mathematics, and (b) they are not inconsistent with these notions, conceptions, intuitions and so on.

According to it, some sentences can be objectively true in some branches of mathematics, while being objectively false in some others.

The theory of undecidable sentences provided by Mark Balaguer fits well with mathematical practice, since it explains our change of mind in mathematics and leaves space for our conceptions to evolve: it is possible to be neutral (NEUT) regarding a sentence, and then develop (COR) or (INCOR) for it, conveying it in a correct or incorrect sentence, a truth or falsehood of mathematics.

Balaguer argues that FBP is in the same line with his theory of undecidable sentences. Indeed, according to FBP, every consistent purely mathematical theory truly describes some collection of mathematical objects. In doing mathematics, humans talk about some particular collection of mathematical objects and state that it is objectively correct only if it is true in the intended interpretation or standard model, determined by the intentions they have in the pertinent branch of mathematics. Thus, to say that a model is standard is to say that it is the one intended, with an important *caveat*: the intended model is standard, but not every standard model is intended. Balaguer concludes that, since every consistent purely mathematical theory truly describes some collection of mathematical objects, every such theory is true in a language that interprets it in a way such that the given theory is about the objects that it is about. This could seem a little obscure, but Balaguer means here a very simple fact: correctness or incorrectness of mathematical theories depend at least partially upon facts about humans, about what they intend to say.

Accordingly, FBP can adopt a notion of truth *simpliciter*, a kind of Realism in semantics, fixing the intended interpretation or model of a theory, which can come up

correctly or not. That is, mathematical statements truly refer partly because of how we interpret them and partly because there exist objects that are described by the statements, as long as the theories containing the referring statements are consistent.

2.5 Non-uniqueness

FBP-ist epistemology seems to deliver a response to Benacerraf (1973) epistemological argument: roughly, knowledge of mathematical abstract objects is successfully attained even without any contact, since knowledge of mathematical objects is logical knowledge. But Benacerraf had presented also another difficulty for Platonism, namely the argument of multiple reductions, or non-uniqueness in reference. This argument appeals to the Platonist suggestion that our mathematical theories describe collections of abstract objects. Accordingly, mathematical theories would have to define unique mathematical objects and pick out them among the objects in the mathematical universe. That is to say that singular terms in mathematics have to refer uniquely. And Benacerraf's (1965) is an argument based on the failure of the Platonist ability to uniquely refer.

Balaguer (1998) analyses Benacerraf's argument as follows (page 76, 77):

1. *If there are sequences of abstract objects that satisfy the axioms of Peano Arithmetic, then there are infinitely many such sequences;*
2. *There is nothing 'metaphysically special' about any of these sequences that makes it stand out from the others as the sequence of natural numbers;*

Therefore:

3. *There is no unique sequence of abstract objects that is the natural numbers;*

But:

4. *Platonism entails that there is a unique sequence of abstract objects that is the natural numbers;*

Therefore:

5. *Platonism is false;*

With the intent to defend FBP, Balaguer individuates two strategies, based on the rejection of the vulnerable parts of the argument, that according to Balaguer are 2 and 4.

To reject 2 is to affirm that, even if there were infinitely many sequences of abstract objects that satisfy the axioms of Peano Arithmetic, one of these sequences is in a way ‘metaphysically special’ and deserves to stand out from the others as the legitimate sequence of natural numbers.

Benacerraf (1965) argues that no sequence of sets stands out as the sequence of natural numbers: neither Zermelo’s nor Von Neumann’s provide a better reduction than the others. Benacerraf extends the point by claiming that, from an arithmetical point of view, only the structural properties of a sequence matter to the question of whether it is the sequence of the natural numbers. In particular, any ω -sequence will be as good candidate as any other.

Accordingly, Balaguer also points out that Peano’s axioms don’t capture everything is known about the natural numbers. Balaguer proposes to take into consideration what he calls the full conception of natural numbers, that is the total sum of humans’ intuitions,

notions and conceptions regarding the natural numbers. The enterprise of defining what is contained in the full conception of natural numbers and what is not seems controversial. Nevertheless, Balaguer believes that there is nothing problematic about Platonists appealing to full conception of natural numbers, motivating this with the claim that the idea goes hand in hand with the platonic conception of mathematics.

The point of Benacerraf's argument is that if all the ω -sequences were laid out before us, we could have no good reason for singling one of them out as the sequence of natural numbers. Any ω -sequence that don't satisfy the full conception of natural numbers can be ruled out, but Platonists can't claim that all ω -sequences but one can be ruled out. Moreover, since Platonists endorse that abstract objects exist independently of us, they must admit that there are very likely numerous kinds of abstract objects that we have never thought about. As a consequence, there are very likely numerous ω -sequences that satisfy the full conception of natural numbers and differ from one another only in ways that no human being has ever imagined since now. Hence, Balaguer has to conclude that 2 is true.

He then tries the other strategy, rejecting 4, the claim that Platonism entails that there is a unique sequence of abstract objects that is the natural numbers. The claim that mathematical theories truly describe collections of abstract mathematical objects entails that mathematical theories truly describe unique collections of abstract mathematical objects. But according to Balaguer, there is absolutely no reason to believe this strong claim. Platonists can simply accept that mathematical theories truly describe collections of abstract mathematical objects, while rejecting that mathematical theories truly

describe *unique* collections of abstract mathematical objects. That is, while mathematical theories describe collections of abstract objects, none of them (eventually only by blind lucky) describes a unique collection of such objects.

Here Balaguer's strategy: Platonists can avoid the non-uniqueness problem simply adopting non-uniqueness Platonism. This strategy fits very well with the plenitudinous character of full-blooded Platonism, which can face successfully both the problem of multiple reductions and the epistemological argument. Furthermore, if the mathematical realm is as populated as full-blooded Platonism would assumed it, then it seems extremely unlikely that any of our mathematical theories, or any possible mathematical theory, are uniquely satisfied.

Thereafter, Balaguer argues that there are no good reasons for favoring uniqueness over non-uniqueness Platonism. He first claims that uniqueness Platonists can claim over non-uniqueness's that mathematical theories are to be taken as being about unique collections of mathematical objects. This would be acceptable only if there were unique references for these terms. Indeed, if a singular term doesn't have a unique referent, if it refers ambiguously, we are inclined to say that it doesn't refer at all, and above all that it isn't a singular term. But what non-uniqueness Platonists want to deny is exactly that our mathematical singular terms have unique referents, because all mathematically important facts are structural facts, the one that are about the relations between mathematical objects. Since it is possible to capture the structural facts that mathematicians look for, without picking out unique collections of objects, it doesn't matter if our mathematical theories fail to pick out unique collections of objects.

Even if the non-uniqueness account of mathematics that Balaguer developed can be satisfyingly supported, there remains some concerns uniqueness Platonist can raise regarding the use of singular terms without unique referents in mathematics. Uniqueness Platonists can claim that, in abandoning unique references, non-uniqueness-Platonists abandon the adoption of a standard semantics that matches the one for ordinary discourse. But the ability to adopt such a semantic has always been one of the main motivations for Platonism. In response, Balaguer denies that the non-uniqueness-Platonism's appeal to non-unique reference leads to the abandonment of standard semantics. What non-uniqueness Platonists merely claim is that, in using singular terms and providing them with a standard semantics, mathematicians make an assumption that is false, but nevertheless convenient. Mathematicians simply aren't interested in the differences between the various ω -sequences that satisfy their full conception of natural numbers, since all of these sequences are indistinguishable with respect to the sort of facts and properties they are trying to characterize while doing arithmetic.

Moreover, uniqueness Platonists can object that mathematicians seem to have unique objects in mind when they use singular terms. Indeed, considerations of this sort can be used to argue against non-uniqueness Platonists, for it seems that mathematicians have unique collections of objects in mind when they construct their theories. But FBP does not account for the existence of a unique sequence of objects that mathematicians have in mind when they are doing arithmetic. Given the Platonist thesis that the mathematical realm exists independently of mathematicians and their theorizing, FBP-ists reach the result that there may be more than one ω -sequence that satisfies mathematicians' full conception of natural numbers and differ from one another only in ways that no human

being has ever imagined. If there are numerous ω -sequences that satisfy our conception, then arithmetical beliefs are thinly³⁰ about all of those ω -sequences. There is no privileged ω -sequence that is such that mathematicians have in mind it and only it.

Moreover, if full conception of natural numbers doesn't pick out a unique object, it is simply not a problem, because mathematicians can still accomplish what they want to accomplish in arithmetic, namely characterize the structural facts that they aim to characterize.

Balaguer considers also the objection according to which, given the right background, any mathematical object could play the role of any position in any mathematical structure. Therefore, non-uniqueness Platonists have to allow that every mathematical singular term refers to every mathematical object. This is simply false, since the mere consideration that all mathematically important facts are structural facts does not entail that these are the only facts relevant to the determination of mathematical reference. The author of this objection would seem to think of mathematical objects as bare particulars: taken in themselves, without any interpretation given, mathematical objects are all indistinguishable from one another. But who adopts such a conception has to admit that '3' really could refer to any mathematical object whatsoever, because all of these objects would be undistinguishable from one another. This claim does not fit with any sort of reasonable theory of mathematics.

To briefly summarize, Balaguer provides an account for non-uniqueness in mathematics in response to the difficulties raised from the attempt of defining exhaustively a part of the mathematical realm. But, the intentions mathematicians have in mathematical

³⁰ Recall the difference between having beliefs that are thinly and thickly about objects at the end of section 2.2 of the present chapter.

contexts are anything like the intentions that humans have in empirical contexts. There are no unique collections of objects that correspond to what mathematicians have in mind when they formulate mathematical beliefs and theories. Hence, Platonists are challenged to account for the fact that if mathematicians accept p , then p . But Platonists do not have to account for there being a perfect correlation between mathematical beliefs and mathematical facts, simply because there isn't such a correlation to account for.

2.6 A Defense of Full-Blooded Platonism

Several objections have been raised against FBP. Balaguer analyses some of them in different passages of his work, particularly in *Platonism and Anti-Platonism in Mathematics*, beginning from page 58. In this last section, I will briefly consider the most significant questions and Balaguer's answers.

The first objection is against the many cases in which consistent purely mathematical theories contradict one another. For example, assuming that $ZF+C$ and $ZF+\sim C$ are both consistent, FBP entails that they both truly describes part of the mathematical realm. Thus, FBP seems to lead to the contradictory result that both $ZF+C$ and $ZF+\sim C$ are true. Balaguer responds that both $ZF+C$ and $ZF+\sim C$ actually describes parts of the mathematical realm, but the whole poin is that they describe different parts, different kinds of universes of sets. That is why FBP assigns different sorts of entities to the expressions of C in the two different cases.

Balaguer responds then to an objection that could follow from his answer, namely the objection according to which FBP seems to sacrifice the objectivity of mathematics,

entailing that undecidable sentences like the *continuum* hypothesis do not have determinate truth-values. But in FBP-ists terms, to say that there is no objectively correct answer to the continuum hypothesis is just to say that the notion of set on the table isn't strong enough to settle the question. Indeed, FBP can account for the existence of open questions, while classical Platonists cannot, and this is a strong argument in favor of FBP, because it makes FBP more lined up with mathematical practice. The reason is that FBP can allow mathematicians to say whatever they want to say regarding each different open question. If multiple answers to an open question are consistent with the intentions, concepts and intuitions that mathematicians have, then different answers to the question will be true in different standard models, and so the question will not have a unique, objectively correct answer.

A third objection states that FBP's entailment that among purely mathematical theories, consistency is sufficient for truth, seems to represent a shift in the meaning of the word 'true'. Balaguer promptly argues, in agreement with mathematical practice, that what mathematicians standardly mean when they say that a sentence is true is that it is true in the standard model for the given branch of mathematics. That is, models are just parts of the mathematical realm and to say that a sentence S is true in a model M is just to say that S is true of some particular part of the mathematical realm. But FBP-ists specify that there is nothing metaphysically special about standard models, since they are just the intended interpretation.

As a result, FBP-ists can maintain that a mathematical sentence is true *simpliciter*, or correct, if and only if it is true in all of the standard models for the given branch of mathematics; and it is incorrect if and only if it is false in all of these models; and if it is

true in some of these models and false in others, then it is neither correct nor incorrect: is neutral.

Another objection is how human beings can acquire knowledge of what the various standard models are. According to FBP, standard models haven't any metaphysical priority, but only sociological or psychological ones. Mathematicians formulate axioms that are intuitively pleasing and try to settle open questions by constructing proofs that rely upon currently accepted propositions, that is propositions that they already have reasons to believe to hold in the standard model.

Balaguer tries to respond to whom who charge FBP of forbidding to speak of all sets, for it seems that every set theory is about a restricted universe of sets. But again, FBP is neutral with respect to the question of whether there is a unique amalgamated universe that contains all and only things that legitimately count as sets. Actually, FBP seems to prohibit from making claims about the entire mathematical realm, justifying itself by saying that there is nothing mathematically interesting about the entire mathematical realm, because mathematical realm is too vast and diverse. Still, mathematicians, but also common people, might want to say something about the entire mathematical realm. In this case, FBP-ists have to remember that they never commit themselves in stating that all consistent purely mathematical sentences and theories truly describe the entire mathematical realm. So, it seems impossible to speak simultaneously of the entire mathematical realm in any mathematically interesting way. Obviously, there are philosophically and metaphysically interesting way of speaking of the entire mathematical realm. FBP is one of these.

Lastly, Balaguer considers a classical objection also made against fictionalism: what is the difference between true and false mathematical statements, since FBP does not

characterize in different ways $2+2=4$ and $2+2=5$? They seem to be both true of part of the mathematical realm, but in fact a mathematician who wants to demonstrate that $2+2=5$ will not provide a good proof, since he will have to use at least one of the terms in this sentence in a non-standard way, assigning it a meaning that will result to be different from the meaning standardly assigned to it. Accordingly, epistemic principles that are applicable in empirical contexts aren't applicable in mathematical contexts. That is, in mathematics it is false that, in order for a person S to know that p , there has to be a counterfactual relationship between S 's belief that p and the fact that p , so that if things would have been different, then S would have believed differently.

It is precisely this misunderstanding that Balaguer accuses of originating Benacerrafian worries regarding the possibility for human to acquire knowledge of objects without any access to them. FBP provides a path to knowledge of abstract objects, at least mathematical ones, that does not require access to them or a counterfactual relationship between subject and object of knowledge.

In conclusion, FBP can vaunt many virtues, first of all his ability to reconcile the objectivity of mathematics with the extreme freedom that mathematicians have. It does so thanks to the admission of infinitely many different universes of sets, while classical Platonists have to allow that one universe of sets could correspond to any of the FBP-ist universes. Consequently, FBP can also endorse the existence of open questions with or without correct answer. But FBP-ist can also account for the legitimacy of pragmatic modes of justification, since they preserve consistency. FBP privileges inclusiveness and broadness in comparing theories, reconciling the objectivity of mathematics with the legitimacy in mathematics of pragmatic modes of justification.

From an ontological point of view, FBP is committed with an infinite number of objects, leaving aside any reductionist and simplifying tendency. Therefore, it pays the cost of a gigantic ontology, in exchange for being able to describe mathematics in an exhaustive manner.

*Every theory, philosophical or otherwise, must take some notions for granted...
[T]he notions of existence, object, and identity occur
in just about every philosophical work, usually without further ado...
[S]hould we conclude that every one already has clear
and distinct ideas of them? Is any attempt
to articulate such notions a waste of time and effort?*

SHAPIRO, STEWART (1997), *Philosophy of Mathematics: Structure and Ontology*,
page 71.

Chapter 3: Structuralism

3.1 Structuralism

Structuralism proposes to conceive mathematics as the science of structures: mathematical theories describe abstract structures and mathematical theorems are about places in structures, conceived as abstract objects. As a result, mathematical statements describe authentic objects. Nevertheless, Structuralism is opposed to the so called ‘Object Platonism’, who considers mathematical objects in Structuralism as inauthentic objects, even though they can serve as reference for mathematical terms. According to Structuralism, mathematical objects are places in structures and therefore have no inner essence or properties they authentically possess. Therefore, Structuralism assigns to mathematical objects only those properties they possess because of the relations they have each with others.

This conception is, *prima facie*, very related to 19th-20th-century mathematical practice. In order to understand the historical reasons behind the formulation of such a view, it is important to keep in mind that, during the nineteenth century, mathematics underwent a significant transformation. The same goes for philosophy of mathematics, which mirrors this transformation in particular on new ideas of necessity and a priority. Several scholars argue in favor of an account for the necessary and *a priori* nature of mathematics, without invoking kantian intuition, but aiming at understanding necessity and a priority in formal terms³¹: by now, it is enough to say that necessary truth is truth by definition, while *a priori* knowledge is knowledge of how to use the language.

³¹ Alberto Coffa, in his *From Geometry to Tolerance: Sources of Conventionalism in Nineteenth Century Geometry*, identifies the semantic tradition, represented by Bolzano, Frege, Hilbert and Wittgenstein.

In Kant's *Critique of Pure Reason*³², both geometry and arithmetic are intended as synthetic *a priori*. But mathematics at the end of the 19th century is no more the same subject Kant was thinking about. For example, geometry is not yet the study of a form of intuition of space, but has been transformed into the study of freestanding structures, with ideal elements that behave in different ways and inhabit different kind of spaces from the past ideas. For example, the discovery of the non-euclidean geometry is a typical example of a new way of thinking mathematics, but also allows to picture and to understand how geometry moved from its empirical and physical feature.

As Stewart Shapiro points out in chapter 5 of *Philosophy of Mathematics: Structure and Ontology* (1997), mathematicians can no more rely either on perceptual intuition or on visualization when dealing with new elements like imaginary points, projection located at infinity, and so on. This is strictly bounded to one particular feature that plays a prominent role in Shapiro's reconstruction of history of contemporary mathematics. From the reflection on contemporary mathematical objects, introduced by postulation, implicit definition or construction, he derived that, in contemporary mathematics, there is no more room for intuition.

As an illustrative case, Shapiro (1997) reports a debate that took place in the beginning of 20th century: two mathematicians defended the role of perception and intuition and begun a debate against two logicians, who were strongly resisting such an account of mathematics, and geometry in particular. The protagonists of this debate are Henri Poincaré as opposed to Bertrand Russell, and David Hilbert against Gottlob Frege.

³² Kant, Immanuel, (1978) *Critique of Pure Reason*, Cambridge University Press, Cambridge;

Poincaré³³ noticed that it is impossible to figure out whether physical space is Euclidean by an experiment, because measurement can only be done on physical objects by the mean of physical objects. The choice of a particular geometry has to be done only in the key of convenience: once geometry is the science of forms, matter does not matter.

On the other side of Poincaré conventionalism, the young Bertrand Russell held that mathematical definitions have fixed meaning: they identify objects with properties in space that stand in certain relations to other items. In his *Introduction to Mathematical Philosophy*³⁴, pages 59 and 60, Russell highlighted that mathematicians are obliged to adopt some kind of Structuralism, even if the philosopher would not:

[T]he mathematician need not concern himself with the particular being or intrinsic nature of his point, lines and planes...[A] “point” ...has to be something that as nearly as possible satisfies our axioms, but it does not have to be “very small” or “without parts”... If we can, out of empirical material, construct a logical structure, no matter how complicated, which will satisfy our geometrical axioms, that structure may legitimately be called a “point.” We must not say that there is nothing else that could legitimately be called a “point”; we must only say: “This object we have constructed is sufficient for the geometer; it may be one of many objects, any of which would be sufficient, but that is no concern of ours, since this object is enough to vindicate the empirical truth of geometry, in so far as geometry is

³³ POINCARÉ, HENRI, (1899) “*Des fondements de la géométrie*”, *Revue de Métaphysique et de Morale*, 7, 251-279, POINCARÉ, HENRI, (1899) “*Sur le principes de la géométrie*”, *Revue de Métaphysique et de Morale*, 7: 251-279, Paris.

³⁴ RUSSELL, BERTRAND, (1919) *Introduction to Mathematical Philosophy*, Dover, New York.

not a matter of definition.” We may say of two similar relations, that they have the same “structure”.

Russell’s position is, concisely, that what matters in mathematics is not the intrinsic nature of mathematical objects, but the logical nature of their interrelations. This perspective has to be seen as an important antecedent to Structuralism.

It is interesting to note that Russell emphasized the aspects according to which mathematicians are forced to adopt some kind of Structuralism, even if the philosopher would not. For mathematical purpose, the only important thing about a relation is if it holds and how, whereas philosophy is more interested in the essence, or nature, of the relation.

Shapiro (1997) spends some words also on the work of Richard Dedekind both on natural and real numbers, suggesting to consider it as an important antecedent of Structuralism. Shapiro is referring to Dedekind (1872)³⁵, where rational numbers are introduced through the characterization of three fundamental properties they possess:

1. Order: if $a > b$ and $b > c$, then $a > c$;
2. Density: if $a \neq b$, then between a and b there exist infinite rationals;
3. Section: if a is a given rational, all the others rationals are subdivided in two classes A_1 and A_2 , each of which containing infinite elements such that every number smaller than a belongs to the first class, and every number greater than a belongs to the second, whereas a belongs to the first or to the second class.

³⁵ DEDEKIND, RICHARD (1872), *Continuity and Irrational Numbers*, in Beman, W. W., (1963) *Essays on the theory of Numbers* New York, Dover.

Dedekind noticed that, once a segment is fixed and used as a measurement unit, it is possible to associate a point in a line to every rational numbers and even the points in the line will possess analogous properties as order, density and section³⁶. Dedekind identifies the essence of continuity in the inverse of the property of section: if there is a partition of the line in two classes in which every element of a class stands to the left of every element of the other class, then it will exist one and only one point that produces this partition.

Dedekind defines real numbers as couples of non-empty and disjoint subsets A_1 and A_2 , the union of whom is the set of rational numbers, such that, for every element a of A_1 and b of A_2 , $a < b$. Sections that are produced by no-rational numbers produce an irrational number. In this way, Dedekind can state that to each section corresponds one number, rational or irrational. Keep in mind that Dedekind didn't identify real numbers with the cuts, but are a representation of them.

According to Shapiro (1997), the cuts exemplify the real-number structure. And in Dedekind's perspective, numbers are something that mind creates, objects produced by a free creation made through an abstraction of sorts, based on relations that identify between objects.

Several scholars endorsed Structuralism during the last decades. In particular, it is worthy to mention Marcus Giaquinto (2002), Geoffrey Hellman (1996, 2001), Colin McLarty (1993), Carl Parsons (1990) and Michael Resnik (1997). The version proposed by Stewart Shapiro acquires interest for the purpose of this work, because he explicitly

³⁶ The inverse property of correspondence is not satisfied, because in the line exists an infinity of points to which it is not possible to associate a rational number.

defines his philosophy of mathematics as Platonist and defended one of the main thesis of Platonism, i.e. the thesis I called *Existence*, with new and powerful arguments, but without assuming the existence of mathematical objects.

Shapiro has a particular approach to mathematical ontology. He express very explicitly his support to the line of thought he calls ‘philosophy last if at all’³⁷ (henceforth PL), as opposed to the view he calls ‘philosophy first’³⁸ (henceforth PF).

Following PF, philosophy determines and precedes mathematical practice, in the sense that defining or constructing what mathematics is all about fixes the way mathematics is to be done.

According to PL, philosophy has to participate in providing orientation and direction, to deliver an accurate account of mathematics and its place in our intellectual lives, to interpret mathematics and, if needed, to philosophically justify it. But mathematics can get by on its own in a world without philosophy. In fewer words, PF is normative, while PL is descriptive.

Shapiro supports to PL and Platonism as a consequence: since the adoption of *Existence* provides a good guide to mathematical practice and mirrors adequately the use of mathematical language, and Platonism dictates the adoption of *Existence*, Platonism is the right ontology for mathematics. If it has philosophical problems, is a problem for philosophy, not for mathematics.

Shapiro engages himself in defending both the autonomy of mathematics and the role of philosophy of mathematics, trying to draw the boundary line and assign to each

³⁷ Shapiro indicates as supporters of PL Carnap (1950), Quine (1981), Lewis (1993)

³⁸ See for example Hempel and Oppenheim (1948)

discipline the appropriate domain. In what follows, I will analyze his strategy and evaluate his results.

3.2 Existence

Stewart Shapiro defines his form of Structuralism as *ante rem*. This aspect of his thought about mathematics is of particular relevance for the purpose of this work, because is fundamental in defining Shapiro's Structuralism as a form of Platonism.

Shapiro proposes to approach Platonism as Working Realism, an approach he defines as aiming at defending the autonomy of mathematics from philosophical concerns. Although ontology, epistemology, semantics, application's problems are philosophically relevant, Working Realism is firstly focused on providing an accurate account of mathematics by its own. For, Working Realism is a methodological description of how mathematics is performed, rather than a normative prescription of how mathematics should be performed.

Shapiro delivers a theory of the work of mathematicians along the lines of Working Realism. Such a theory is implicit in the operations of the mathematicians, which use at least classical mathematics without dealing with epistemological or methodological problems, but defending mathematics and its principles in the name of practical reasons: mathematics works, its principles are useful and deliver coherent results. So, such a mathematical spontaneous Realism predicates that mathematical objects exist and mathematical statements refer not vacuously.

This *a posteriori* argument is very common, according to Shapiro, in mathematical practice. Working Realism is very lined up with mathematical practice and

methodologies like, to mention just a few, impredicative definitions, the axiom of choice and laws of classical logic, such as excluded middle.

Shapiro recognized three level of Working Realism: third person, first person and normative. First and third person Working Realism are descriptive: the mathematicians conform their job to the methodology, using it uncritically, in the former, and critically in the latter, but they are not committing themselves with the use of those principles in further researches. Therefore, it is plausible that some methodologies are used in mathematics because they are helpful for some purpose without, as Shapiro (1997) observed, further investigations on their validity.

Contrarily, normative Working Realism acknowledges and accepts explicitly the methodological principles, grounding Mathematics on them and working on conforming mathematical practice to them. As a consequence, normative Working Realism is to be considered as internal to the framework of mathematics, while philosophical questions are external.

Thereafter, Shapiro indicates three senses of the dichotomy internal/external. The first is internal/external to the framework, following Rudolph Carnap (1950). In the internal sense, the existence of numbers is an analytic truth because it is a consequence of the framework. External questions play no role in Carnap view: they are simply non-sense, since he posed question on the pertinence of the use of certain framework. The second sense, due to Arthur Fine (1986), is internal to science. Fine's theory, Natural Ontological Attitude or NOA, invokes an interpretation of Mathematics and science to clarify what Mathematics and science predicate. The third sense derives from Putnam's (1980) internal Realism as opposed to metaphysical Realism. In this case, internal and

external notions depend on the conceptual scheme. Shapiro doesn't pick up explicitly one of these senses, but he seems to implicitly adhere to Fine's theory.

Shapiro aims at accentuating the link between Working Realism, Structuralism and the idea of mathematical objects relatively to a system of axioms. For example, natural numbers may be objects within Arithmetic, but names for natural numbers may not designate objects in other theory or framework of Mathematics. As a consequence, only if the context is held fixed, there can be room for a determinate statement of identity between objects; otherwise, identity is based on convenience or resemblance. Identity between natural numbers is determinate, but there is no general criterion for identity. Indeed, identity is systematically ambiguous, and the reason for this ambiguity resides in the variation of the notion of object from theory to theory. In this sense, cross-identifications aren't ultimately definable. They are matter of decision or convenience, rather than matter of discovery.

Within arithmetic, numbers may be conceived as objects and a truth-value may be assigned to any well-formed identity in the language of arithmetic. Such identity conditions lead to the conclusion that the idea of a single, fixed universe composed by objects *a priori*, that exist independently from our ability to talk or to have knowledge of them, is firmly rejected.

Working Realism takes mathematical language at face value. Theorems of mathematics are about objects that have the same *status* of everyday objects. Several well known theories by Willard Van Orman Quine, like relativity of ontology, inscrutability of reference and ontology through variables have such an essential assumption, that it is possible to give different models in order to analyze the same content. Thus, the best

one can achieve is that all models of a theory are isomorphic, in which case the ontology is determined up to isomorphism.

There is an isomorphism when there is a one-to-one correspondence from the objects and relations of one system to the objects and relations of the other that preserves the relations. Isomorphism is different from identity: if two objects are the same, in the sense that everything true about one object is true about the other, the objects are identical: indeed, two objects that share exactly the same properties, are the same object. As a result, two objects that are identical are one and the same object, while two objects that shares an isomorphism are not: they remain two different objects.

Moreover, isomorphism regards structure, rather than individual objects: if two structures are isomorphic, then there is a one-to-one correspondence from the objects and relations of one to the objects and relations of the other that preserves the relations. For example, the natural numbers can be mapped into the even natural numbers by the one-to-one correspondence, assigning to each n of the set of natural numbers, the number $2n$ of the set of even natural numbers. Since the binary operation ‘+’ is defined on the natural numbers and is preserved by the mapping, the sets of natural numbers and even natural numbers are isomorphic under the binary operation ‘+’.

The structuralists’ adherence to Working Realism has notably consequences on how mathematical objects are conceived according to Shapiro’s *ante rem* Structuralism.

As I already disclosed, mathematical objects are places in structure. They do not have properties beyond the ones assigned them by the brute fact that they occupy a particular position into a structure. In this sense, the context must be determined by the structure. Indeed, only once the context is fixed, identity between numbers can be determinate,

while identity between numbers and other sort of objects, including positions in other structures or way of representing numbers, is not determinate.

Two different places in different structures can correspond up to the existence of an isomorphism between the two structures. The essence of any single number resides in its relations with the other numbers that belong to the same structure it belongs to. This means that it is senseless to ask what the number 5 is, without contextualizing it into a structure.

Shapiro also indicates two views in which it is possible to identify places in structures with objects, basing his argument on two ways places can belong to structures. According to the view Shapiro called 'places-are-offices', a background ontology supplies objects that fill the places of the structures. The places of the very structure under discussion can be the objects of the background ontology.

According to the view called 'places-are-objects' there are singular terms that occur in statements about the respective structure, independently of any exemplification. Singular terms have to be taken at face value: they refer directly to places, taken as objects. For example, in a finite structure instantiated by concrete objects, places of a structure can be taken as objects considering the portion of space and time they name as an objects of the web of space and time. It is also possible, and actually very common in mathematics, that places in a structure are occupied by other structures.

Further clarifications of what an objects actually is for Structuralism can come from an example: $\{\{\{\emptyset\}\}\}$ and $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ can occupy the place of the number 3 in the natural number structure, but neither are the number three. Indeed, the number three is an empty place in the natural number structure, isolated by the relations that the

natural numbers' structure assigns to the number three.

Therefore, it is simply senseless in Structuralism to talk of an object on its own, without the contextualization given by its structure. It may seem easier to explain first what is a structure and secondly what is an object. Shapiro (1997) delivers an intuitive account of what a structure is, then defines objects as places in the structure and only lastly clarifies the notion of structure. I decided to respect his order.

But before introducing structures, it is useful to analyze two examples that help elucidating what objects are in Structuralism.

Imagine that a nice old woman drives an even older car. After thousands of kilometers, the engine breaks up and she has to reline it with new pieces. Then, the window breaks up and she has to buy a new one. To make a long story short, in a couple of months she had to replace every single piece of the car with a new one. But, still, it is a car and it works, no matter if she had to change the pieces. The only thing that is important is that all the objects that constitute a car are at their places. She can change the objects, not the places. The fulfillment of the relations that hold between the objects that compose the car allows the respect of the working principles and guarantees the conformity of the car. Briefly, it makes a car what it is. One may ask if the new car is or not the same old car. I believe that, although the question can be of interest for general ontology, from a structuralist point of view the only important thing is that the car(s) instantiate(s) the same structure.

Now imagine a football team. It is composed by eleven people: a doorkeeper, some players on the front line whose task is to score goals, some playmakers, some midfielders and some defenders. Is it relevant who is the doorkeeper or the midfielders?

Yes, if you want to win the Champions League. But it is not relevant if you are just aiming at forming a football team, that is a system of roles that follows certain rules. Offices are relevant and people endorsing the offices can change without changing the system.

A structure is composed by places that share relations and that are not existing if the structure doesn't exist. That is why Shapiro takes the notion of structure as primitive, at least at the beginning: a structure is the abstract form of a system. More openly, a structure is the net composed by the interrelationships among places, once any features of them that do not affect how they relate to each other in the system has been ruled out. Several definitions of structure are available, also in the non-mathematical fields. Thus Structuralism's first effort is to offer a criterion that discriminates mathematical structures from other structures. According to Shapiro, mathematical structures possess a few of properties that makes them mathematical:

1. Mathematical structures are known deductively, leaving aside whatever system a structure could be the structure of;
2. Mathematical structures have only formal or structural relations and properties;
3. Mathematical structures are freestanding, that is they are independent on the existence of objects that exemplify them.

Moreover, according to Shapiro (1997, page 100) there is no difference between simulating and exemplifying a mathematical structure. Unfortunately, as far as I see, Shapiro hasn't yet provided a clear (at least to me) explanation of what he means by 'simulating a structure'. I believe he means something like 'creating a structure without

referring to any particular object or office, just listing down the axioms'. Therefore, this could mean that there is no difference between implicitly defining some objects and referring to objects already defined through a model of a system of axioms.

The 'freestandingness' of mathematical structures is one of the most crucial aspects of Shapiro's Structuralism. For, if it is not relevant for his theory if a structure is exemplified or not, both as a system and in the physical world, then structures exist independently on the existence of some objects that exemplify them. It is freestandingness the point-break in which Shapiro's Structuralism acquires the *ante rem* trait that qualifies it as Platonism.

Structuralism, at this point, has to face another question: since the same structure can be exemplified by more than one system, a structure is one-over-many, like universals. And, as happens with all the partisans of universals, also the partisans of Structuralism have to challenge the problem of explaining the relations between universals and particulars, in terms of structures and systems that exemplify the structures. Shapiro (1997) held that there are two main positions, one due to Plato, the other due to Aristotle. According to the former, universals are *a priori* and independent on any instantiations. The latter predicates the opposite: the existence of universals is ontologically dependent on the existence of instantiations of them. As a result, Plato's universals are *ante rem*, while Aristotle's are *in re*. Easy to guess, Shapiro supports Plato's view: structures are independent and their existence is apart from the existence of objects that instantiate them.

Some attempts were done also in the direction of obtaining *in re* structures. It is the case of Eliminative Structuralism. The strategy adopted is to consider statements about a kind of mathematical objects as general statements about structures of a certain type and to look for a way of eliminating reference to mathematical objects like numbers. This is the kind of Structuralism Hellman called ‘Structuralism without structures’. Such a name is due to the understanding of statements on structures as convenient shorthand to speak about systems without committing neither with objects, nor with structures.

Eliminative Structuralism can be formulated also in a modal way: mathematics is taken to be about all possible systems of a certain type. In such a formulation, if mathematical objects exist, their existence is metaphysically necessary. Existence and possible existence of the items in the background ontology are equivalent, but this means that mathematical existence is rendered in terms of logical possibility, and logical possibility is usually given in terms of existence in the set theoretic hierarchy, that is Mathematics. Defining Mathematics in terms of Set Theory is broadly considered circular.

Ante rem Structuralism has to do with much of the troubles entailed by Platonism, but can provide mathematics with a reliable subject, that exist independently of its instantiation, and with a reliable theory of meaning, as mathematical statements are taken to be literally true.

Before introducing the axioms of Structuralism, let me briefly summarize what I said till here: Structuralism is the philosophy of mathematics according to which there exist systems that exemplify structures. Places in the structure are mistakenly believed to be objects from Object Platonism. But places can be also offices and in any case, they don’t have an inner, independent nature. A structure is therefore any collection of

places, functions and relations on those places. Two systems have the same structure if they share an isomorphism that preserves the structure³⁹. In conclusion, mathematics is the science of freestanding structures.

3.4 Axioms

Shapiro (1997) can, after the previous elucidations, provide the axioms for his theory of structures, at page 91. The first axiom is:

Infinity: there is at least one structure that has an infinite number of places;

This axiom guarantees that in Shapiro's ontology there are structures, places, functions and relations, a structure having places and a finite number of functions and relations on those places.

The axioms for substructures (page 94 Shapiro 1997) are:

Subtraction: if S is a structure and R a relation on S , then there is a structure S' isomorphic to the system that consists of the places, functions and relations of S except R . If S is a structure and f is a function on S , then there is a structure S'' isomorphic to the system consisting of the places, functions and relations of S except f ;

Subclass: if S is a structure and c is a subclass of the places of S , then there is a structure isomorphic to the system that consist of c but with no relations

³⁹ Here Shapiro's Structuralism is particularly controversial, because it allows to derive some counterintuitive conclusions. For example, let consider the case of Frege's Arithmetic (henceforth FA), and Peano's Axioms at second-order (henceforth PA2). This two have no isomorphic models, since PA2 is categorical, while FA isn't. According to Shapiro, they don't exemplify the same structure (natural numbers), *a fortiori* because FA's non categoricity means that it cannot exemplify a structure.

and functions;

Addition: if S is a structure and R is any relation on the places of S , then there is a structure S' isomorphic to the system that consists of the places, functions and relations of S together with R . If S is a structure and f is any function from the places of S to the places of S , then there is a structure S'' isomorphic to the system that consists of the places, functions and relations of S together with f ;

These axioms assure that it is possible to eliminate or modify places, relations and functions and add relations and functions.

Shapiro's next move is to guarantee the existence of large structures. This is the role of the following axiom:

Powerstructure: let S be a structure and s the collection of its places. Then there is a structure T and a binary relation R such that for each subset $S' \subseteq S$, there is a place x of T such that $\forall z(z \in S' \equiv Rxz)$;

This axiom means that each subset of the places of S is related to a place of T , and so there are at least as many places in T as there are subset of the places of S . Thus, the collection of places of T is at least as large as the powerset of the places of S .

To ensure even larger structures, Shapiro formulates the Replacement Axiom, asserting that it exists a structure as large as the result of replacing each place x of S with the collection of places of a structure S_x :

Replacement: let S be a structure and f a function such that, for each place x of S , fx is a place of a structure, which we may call S_x . Then there is a

structure T that is at least the size of the union of the places in the structures S_x . That is, there is a function g such that for every place z in each S_x , there is a place y in T such that $gy=z$;

To ensure that any coherent theory characterizes a structure, Shapiro has first to face the problem of defining what ‘coherent’ means in his theory. He adds the following axiom:

Coherence: if ϕ is a coherent formula in a second-order language, then there is a structure that satisfies ϕ ;

But he had also to specify that, since he is using second-order logic, because of the failure of the completeness theorem, there are not satisfiable⁴⁰ consistent second-order theories.⁴¹ But coherence is very different from consistency, first of all because consistency can only be expressed in modal terms or by considering derivations as abstract objects, and then denying the existence of some of them. And also because there is no guarantee for completeness at second-order. Nevertheless, Coherence is enough for ensuring categoricity.

To avoid this puzzle, Shapiro proposes to add the Reflection axiom, that guarantees that, if ϕ is any first or second-order sentence in the language of structure theory, then:

Reflection: if ϕ , then there is a structure S that satisfies the other axioms of structures theory and ϕ .

⁴⁰A theory is satisfiable if there is a model for it.

⁴¹ For example, the conjunction P of the axioms of Peano arithmetic together with the statement that P is not consistent.

This axiom entails that, if ϕ is the conjunction of the other axioms of structure-theory, there exists a structure the size of the second inaccessible cardinal⁴² and it goes on from there.

Shapiro's axioms can support Structuralism's conception of mathematics and can offer a domain big enough. He succeeds in delimiting his ontological engagement only to structures, thanks to the Infinity Axiom. This axiom guarantees a minimal background ontology, which can be offered from set theory as well as from category theory, as Shapiro (1997) suggests at page 96. Moreover, again at page 96, he claims that:

[W]e need not assume any more about the background ontology of mathematics than is required by Structuralism itself.

Such a reductionist attitude simplifies the ontology, but requires an effort to specify how mathematics is built from structures. This is the role of the axioms, to allow and rule operations on structures.

Shapiro's *ante rem* Structuralism is thus a powerful form of Platonism: by asserting the existence of structures even beyond the existence of any exemplifications of them, it provides a genuine domain for mathematics. Structures are abstract objects that exist 'freestandingly' and independently from the human possibility to have any access to them. The idea of structures as composed by places can make structures look as composed objects. Composed objects cannot serve as the fundamental of reality; at least their components could. But structures aren't really conceived as composed. Indeed, a

⁴² There is no Russell's paradox because, as a system is a collection of places in structure, some systems are too big to exemplify structures.

structure is a net of relations that constitute a single object, each of its parts being essential for the identity and existence of the structure.

The axioms of structure theory, together with Peano's axioms for arithmetic, are shown by Shapiro to be coherent: according to him, the subject matter of arithmetic is a single abstract structure, common to any infinite collection of objects that has a successor relation with a unique initial object and satisfies the second-order induction principle. As a consequence, the essence of a natural number is just in its relations with the other natural numbers.

3.5 Epistemology

Most platonists, including Stewart Shapiro, agree that mathematical objects are abstract objects. Abstract objects, contrary to concrete objects, are not located in space and time and can't be part of a causal chain. As I explained in chapter 1, these features have the important role of guaranteeing objectivity and independence to mathematics. Unfortunately, they are also the cause of several worrying quandaries in epistemology, first of all the problem of access. Indeed, it seems hard to account for the possibility of attaining knowledge of a-spatial, a-temporal, a-causal objects, belonging to a realm which is independent on anyone's thought and practice.

The challenge for Structuralism is then to develop an account able to explain and justify the phenomenon of mathematical knowledge. Shapiro proposes to endorse the casual theory of knowledge, even if alternatives are available, as I already pointed out in section 1.4.

According to the causal theory of knowledge, any faculty the knower can invoke in pursuit of knowledge must involve only natural processes subject to ordinary scientific scrutiny. This means that the knower needs some kind of relation with his/her objects of knowledge. This demanded relation is pretty difficult to realize with abstract objects. Briefly, a path to knowledge of abstracts objects is needed, but it seems hard to obtain because of their very nature⁴³.

Several scholars tried to face this issue, formulating a wide range of possible ways to access to knowledge of abstract objects. Shapiro (1997) reported just a few of them: Kurt Gödel's⁴⁴ special faculty named 'mathematical intuition', analogous to sense perception, delivers a way to grasp mathematical objects; Penelope Maddy's⁴⁵ theory on the existence of some concrete and perceivable mathematical objects that, denying the general abstractness of mathematical objects, transforms a part of mathematical knowledge in knowledge of concrete objects, in order to avoid the difficulty related to the way abstract objects are thought to be. Shapiro presents also Hilary Putnam's⁴⁶ and Michael Resnik's⁴⁷ perspective on mathematical objects as theoretical entities, postulated, but not perceived, and therefore known by postulation.

⁴³ For a less sketchy exposure of epistemological issues in Platonism, see Chapter 1.

⁴⁴ GÖDEL, KURT, (1964) *What is Cantor's Continuum Problem?* 1964, reprinted in Benacerraf and Putnam (1983).

⁴⁵ MADDY, PENELOPE, (1990) *Realism in Mathematics* Oxford University Press, Oxford.

⁴⁶ PUTNAM, HILARY, (1971) *Philosophy of Logic*, Harper Torchbooks, New York.

⁴⁷ RESNIK, MICHAEL, (1990) *Beliefs about Mathematical Objects in Physicalism in Mathematics*, Kluwer Academic Publisher, Dordrecht.

The differences between these solutions to the problem of access to knowledge of abstract objects delivered by the above scholars and Stewart Shapiro's proposal are essential.

The abstract objects with which Shapiro is ontologically committed with are mathematical structures. According to Structuralism, mathematical structures exist even if there is no exemplification of them. But some structures are in fact exemplified and so they seem to possess concreteness, that can easily be driven to avoid the problem of access to knowledge. But this is misleading. First because, for the rest of the structures, the ones that are not concretely instantiated, the problem remains. Accordingly, in an exemplified structure, mathematical objects as places in structures would exist and be concrete, while in the non-exemplified structures the places would be abstract objects. And secondly, the concrete exemplification of a structure is not to be considered as a concrete structure: a structure is abstract by definition, because is a form, the form of a system. The objects that constitute the system exemplified by the structure may (or may not) be concrete: a structure can be exemplified and its places can be represented by concrete objects, but this won't ever make the structure concrete.

As long as epistemology, and ontology, are normative and have to combine with mathematics as practiced, Shapiro states fiercely (1997, page 112):

I take the existence of mathematical knowledge to be something close to a philosophical *datum*, just about incorrigible. If an epistemology entails that mathematical knowledge is impossible, I would be inclined to reject the epistemology. Of course, the confidence in mathematical knowledge does

not guarantee that a successful epistemology will be consistent with Structuralism, or with the particular epistemic tactics invoked below.

The tactics Shapiro is referring to in the above quote are three paths to attaining knowledge of structures he proposed. But first he provides an account of the existence of structures, based on the ability to coherently discuss a structure. This ability will then be proposed as evidence for structure's existence.

In the following, I will analyze these three paths to knowledge of structures, as expounded in Shapiro (1997).

The first tactic is pattern recognition. Shapiro defines it as a process in which, due to something like ordinary sensory perception, humans possess the faculty of recognising patterns and learn information about them. According to Shapiro, pattern recognition leads to the understanding that some patterns are freestanding and *ante rem*.

For example, Shapiro considers small cardinal numbers and shows the pattern that belongs to them: for each natural number n there is a structure that is exemplified by all systems consisting of n objects. If the 3 *pattern* is the structure common to all collections of three objects: the corner of all triangles, or the three pyramids in Gyza's Necropolis.

Since everything can be counted, systems of all sorts exemplify the cardinal patterns, thank to the freestanding nature, the topic neutrality and the universal applicability of small cardinal numbers structures.

Pattern recognition is the procedure that allow to disclose the structure behind a system by apprehending the system. This activity requires, first, to apprehend a system that exemplifies a structure, and second, to comprehend that the system exemplifies the

structures. But these steps don't seem to be implied by pattern recognition alone. Rather, in order to obtain knowledge of the structures behind the number systems, the subject must encounter and analyze collections of objects. Shapiro points out that, even if pattern recognition by itself does not deliver anything resembling *a priori* knowledge, it is arguable that we can obtain *a priori* knowledge of finite structures: at page 116 Shapiro (1997) claims

[J]ust as we can know *a priori* that all green objects are colored, we can know *a priori* that any system exemplifying the 4 pattern is larger than any system exemplifying the 3 pattern. Still, I will look elsewhere for the sources of the idea that mathematical knowledge is *a priori*.

Unluckily, the kind of knowledge he refers to in the previous quote is too weak for his purpose, that is why he searches elsewhere the reasons for the aprioricity of mathematics.

Thus, Shapiro proposes to provide humans with a faculty that resembles pattern recognition but goes beyond simple abstraction. For example, the finite cardinal structures exhibit a pattern that can be projected. Shapiro ascribes to humans the faculty of displaying finite structures' patterns. By several exposures to these finite patterns, the subject grasps the possibility of projection, understands that he or she has seen just a few of the instances of this pattern and realizes that there are much more instances than what it is possible to count.

Sequences that go on indefinitely are, according to Shapiro, the path to the understanding of infinite structures. Shapiro considers the natural numbers structure as a paradigmatic pattern: for each sequence of natural numbers, there is a unique next-

longer sequence, and so there is no longest sequence. Reasoning on ever-increasing sequences of strokes and formulating the notion of a sequence of strokes that does not end, Shapiro claims that humans have access to infinite structures.

The second way to structures is linguistic abstraction. Shapiro (1997 page 121) relates to Robert Kraut's book *Indiscernibility and Ontology*⁴⁸ and states that:

[W]hat is 'discernible' depends on the conceptual resources available. The result is a theory of relativity of objects, quite consonant with the present relativity of objects and the relativity of system and structure.

Shapiro is convinced that Kraut's work provides another path to knowledge of structures and offers some examples. Consider a mathematician who uses an impoverished version of English that does not discern between equinumerous collections, an interpretation of Leibniz's theory of identity of indiscernible, in which anything true of one collection is true of any other collection of the same cardinality. The collections are so the very same objects, since it is not possible to discern among them. What Shapiro aims to highlight here is that the relation of equinumerosity between collections of objects is an equivalence relation that divides the domain into mutually exclusive collections, or equivalence classes. Shapiro intends equivalence classes that exemplify a structure as abstract object, and treats its places as objects dependent on the structure.

Then Shapiro proposes to formulate a sublanguage for which the equivalence is congruence, and suggests that, in the sublanguage, this equivalence relation is treated as the identity relation. At first blush, a structure characterized by such a sublanguage is

⁴⁸ KRAUT, ROBERT, (1980) *Indiscernability and Ontology*, *Synthese* 44: 113-135.

not freestanding: only objects in the original ontology can be counted and only properties of those objects have numbers. But once a structure is so characterized, it is freestanding, as it exists independently of the existence of any objects beyond those of the original ontology that exemplify the structures.

Shapiro's linguistic abstraction is explicitly similar to abstraction in Frege's and the Neologicists' works. Indeed, Logicians and Neologicists predicate that it is possible to introduce abstract objects by abstraction over an equivalence relation on a base class of entities. This conception is the one from which Frege derives his principles of abstraction, from the one for directions to Hume's Principle. More importantly, Shapiro's projection is comparative and aims at finding out the properties that are possessed by both the systems, while Frege's is based on the classification of objects in concepts through the idea of equinumerosity.

The third way to structures is implicit definition, defined as the simultaneous characterization of a number of items through their relations to each other. This is a way to structures that reveal a strong *ante rem* appeal. Since objects themselves do not matter, only structures do, derivation of structures from definition is independent on their matter, because definition indicates a general form, thus characterizes the structure. If the axioms are part of a successful implicit definition, then they characterize a structure and are true of it. Implicit definition and deduction also supports the claim of most platonists that mathematical knowledge is *a priori*. The repeated references to language offer an explanation of why and how a definition can deliver knowledge, also assuring the existence of the objects of knowledge. Since the ability to coherently discuss a structure is evidence for the existence of the structure, it is the language that

marks down and determines a structure. That is why implicit definition and Structuralism goes hand by hand.

According to Shapiro, an implicit definition must encounter two requirements to fit with Structuralism. The first, that Shapiro calls 'existence', states that at least one structure must satisfy the axioms, in order to infer the existence of the objects isolated by an implicit definition from the ability to coherently discuss it. Recalling the axioms of theory of structure, if ϕ is a coherent sentence in second-order language, then there is a structure that satisfies ϕ . A structure exists if there is a coherent characterization of it: if it is possible to coherently define a structure, then it exists. Briefly, coherence implies existence. But the problem is now shifted on the definition of consistency, and Shapiro provides two options. The first is deductive coherence: if one cannot derive contradictory consequences from a set of axioms, then those axioms describe at least one structure. The second option is satisfiability: to say that a sentence ϕ is satisfiable is to say that there exists a model of ϕ , interpreting 'exists' as 'is a member of the set-theoretic hierarchy'. The problem with satisfiability is that the set-theoretic hierarchy is also a structure, and so it is circular: through which mean is then possible to deliberate on the existence and coherence of set-theoretic hierarchy? One option is to take Set Theory's coherence as presupposed, but Shapiro prefers to settle thing once and for all and taking coherence to be primitive. The point is therefore that, in Shapiro's perspective, coherence can and do serve as the criterion for structure existence.

The second requirement, as Shapiro calls it, 'uniqueness', is much less problematic: according to it, at most one structure, up to isomorphism, is described. Since all models of PA2 are categorical, because sharing isomorphism among systems is sufficient for

having the same structure, a categorical theory determines a single structure if anything at all. That is, uniqueness is maintained up to isomorphism. As goes for Shapiro (1991), second-order model theory provides a picture of the semantics of mathematical languages that is sufficient for the purpose of Structuralism. Shapiro finds what is needed for fulfilling the uniqueness requirement in mathematics itself.

The essential idea that stands at the base of the epistemology of Shapiro's *ante rem* Structuralism is that the existence of a way to knowledge of structures is guaranteed by the fact that humans are indeed able to know, characterize, describe and apply structures, establishing if places in a structure participate or not in certain properties or relations. As a result, is Mathematics itself, through its soundness, usefulness and applicability that guarantee the possibility to have mathematical knowledge. Philosophy simply can't deny it. At least, it can find a way to justify why and explain how this knowledge occurs.

3.6 Reference

In Stewart Shapiro's Platonism, appeals to language turn up pretty often, particularly in regard with the epistemic techniques that provide access to structures. The reason is that the ability to know a structure has been identified with the ability to refer to relationships among its places. And to understand the language of the theory and the definitions provided by it requires a correct and proper use of its language. Two quotes from Shapiro (1997) will help to clarify the point. Shapiro states at page 137 that:

[A] structure is not determined by the places in it, considered in isolation from each other, but rather by the relations among the places. In essence,

these relations are embodied in the language. In fact, the correct use of the language *determines* what the relations are.

Here it could seem obscure what Shapiro refers to with the word ‘correct’, also because, as he specifies at the end of page 139:

Understanding how to use ordinary language involves understanding, at some level, of reference.

The problem is getting more complicated, but Shapiro hurries to specify that it is sufficient to have a model-theoretic definition and a correct account of reference in order to obtain a satisfactory account of the truth conditions of sentences in natural languages, and getting closer to the truth conditions of sentences in formal languages.

Shapiro is not claiming that the language ‘creates’ the ontology. In his account, mathematics is objective and exist *a priori* and independently from our ability to recognize it or talk about it, but it is language that ultimately opens the access to knowledge of mathematical objects.

Briefly, it is because human knowledge has a linguistic feature that mathematics is known through language. But this doesn’t ascribe to mathematics itself any contingent and not independent property, in Shapiro account.

This consideration implement both Realism in ontology and Realism in truth-value. The central notion of model theoretic semantic is that of satisfaction or truth in a model. Shapiro defines reference in a structuralist model-theoretic semantic as a function between the singular terms of the language and the background ontology. Several

classical problems⁴⁹ in semantics focus on how terms refer to objects. These problems get urgent in Structuralism, because, as I already made explicit in this section, knowing a structure is the same as being able to use its language, and understanding how to use a language implies to be able to guarantee reference, and meaning, to singular terms in the language.

The work of Shapiro is a strong defense of mathematics as practiced, and an act of liberation from any external ontological, semantical, epistemological concerns. One *leit motiv* of his theory is that mathematics exists, because mathematicians use it, discover it, apply it to reality. A good philosophy of mathematics is then the one who takes the step from this very point and from it provides mathematics with philosophical interpretation, theoretical frameworks, theory of knowledge and, most of all, language. Here stands his adhesion with the principles of PL.

Shapiro's Structuralism is mathematical Platonism, because he asserts that mathematics has an objective, freestanding, immutable, abstract domain. But he is not interested in undertaking the enterprise of justify it in the light of platonist ontology, epistemology or semantic. Mathematics explains itself and gets itself a naïve ontology. Sure enough, Shapiro's epistemology and ontology are platonist: objects exist, mathematical statements are true and significant.

Therefore, what is relevant is that a justification of these platonist beliefs does not come from any philosophical argument. Shapiro provides several convincing philosophical justifications and arguments in favor of Structuralism. But the strongest seems to come

⁴⁹ For example, inscrutability of reference, relativity of ontology, indeterminacy of translation. Ontological relativity and inscrutability reference arise because there is more than one way to regiment the same part of a given language. Shapiro's notion of isomorphism between structures could be seen as avoiding this problem.

from an explicit appeal to authority: since mathematicians take mathematics as having a proper domain and mathematical statements as being truth and significant, so it is.

3.7 Structuralism and Caesar's Problem

The Julius Caesar's problem was first formulated by Gottlob Frege. In paragraph 55 of his *Grundlagen der Arithmetik*⁵⁰, he claims:

[W]e can never, to take a crude example, decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that conqueror of Gaul is a number or is not.

A few paragraphs later (*Grundlagen* 66), while he analyses the 'direction-of-a-line abstraction principle'⁵¹, Frege returns to the problem:

[I]t will not, for instance, decide for us whether England is the same as the direction of the Earth's axis... [N]aturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction.

The problem resides in the difficulty that arises once Frege tried to define numbers as objects and to find a criterion that clarifies the kind of objects the numbers are. The leader of the Roman army Julius Caesar is intuitively not a number, but Frege needs a clear principle that discriminates between the number of columns of the Pantheon, which is a number, and Socrates, which is not. Frege's solution is based on the

⁵⁰ FREGE, G. (1884), *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl*, Breslau 1974. English translation in *The Foundations of Arithmetic*, AUSTIN, J. L., Oxford: Basil Blackwell.

⁵¹ The principle is: 'the direction of line x = the direction of line y if and only if x is parallel to y .'

definitions of numbers as extensions of second-order concepts. If numbers are so defined, and Julius Caesar or Socrates are not the extensions of second-order concepts, [Julius Caesar] and [Socrates] are not numbers.

In Shapiro's perspective, mathematical objects are places in structure and possess properties ascribed by the structure. Hence, it is simply not a legitimate question in Structuralism to ask whether Julius Caesar is the number 2, because the first is an object, but not a place in a structure, while the second is an object and a place in a structure. Shapiro identifies here a categorical mistake, due to the misunderstanding of the very notion of a mathematical object. Since Julius Caesar is not a place in the structure of natural numbers, he isn't the number 2 or whatever mathematical object. And obviously this is true holding that it is possible to ascribe numerical properties to objects: there can be three apples in the basket, holding that the essence of '3' is not equal to 'apples in the basket'.

The difference between Frege's and Shapiro's approach to Julius Caesar's problem is ultimately found in the way Logicism (and Structuralism) defines the notion of mathematical objects and identity. Shapiro's solution of Julius Caesar's problem is non-conservative, because he didn't admit the subsistence of the problem. According to Shapiro, identity between natural numbers is determinate, while identity between numbers and other sorts of objects is not.

In Frege's analysis of Julius Caesar's problem there is no need for determinate answers, because identity is systematically ambiguous, since the notion of object varies from theory to theory. The definition of mathematical objects delivered by Frege extends only

to arithmetic, while Structuralist definition is broader and applies to almost the whole contemporary mathematics.

In conclusion, Structuralism is able to deliver a definition of what mathematical objects are that excludes the rise of problems like the Julius Caesar's: as objects in structures, mathematical objects are exhaustively defined once so is the structure.

3.8 Structuralism and Formalism

Structuralism is widely considered as agreeing with a trend in philosophy of mathematics known as 'formalism' and linked with the work of David Hilbert⁵². Formalism is traditionally characterized as a form of Anti-Realism, and in particular as the forefather of the contemporary metaphysical views collected under the name of Nominalism. But Shapiro's *ante rem* Structuralism is settled in the framework of Platonism. And is precisely for this reason that the ontological consequences that formalist and structuralist derive from a fundamental and common idea give place to this relevant disagreement.

Nevertheless, I believe the aspects in which Structuralism and Formalism agree are way more than the ones in which they disagree.

Both Formalism and Structuralism states that in mathematics the relations between the objects are more important than the properties possessed by the objects. And both Formalism and Structuralism affirm that because objects don't really possess properties beyond the relational ones.

⁵² HILBERT, DAVID, (1923), "Die logischen Grundlagen der Mathematik", *Mathematische Annalen* 88: 151-165.

What is different is the way in which Formalism and Structuralism vehicle the reliability of mathematics: according to the latter, *ante rem* structures are the ‘authorities’ that guarantee mathematical knowledge, application, meaning to mathematical statements. According to formalism, mathematics comes out as the result of a combination of formulas that is proved to be consistent, but that comes from a stroke of luck.

Indeed, Formalism and *ante rem* Structuralism are cogently different also in ontology: according to Hilbert, there aren’t mathematical objects, to which refer mathematical statements. Singular terms are empty names useful to express and apply mechanical rules. According to Shapiro, structures exist and are freestanding.

The fundamental idea behind Formalism is that mathematics predicates rules of transformation of certain formulas in other formulas. Although this idea is pretty much assonant with *ante rem* Structuralism, Formalism developed it in a very different direction. The relevance of consistency lead Hilbert and the formalists to the idea that mathematics is not a body of propositions about a realm of abstract objects that exist, but rather, is just made by rules and formal procedures. In this perspective, there is no ontological commitment to any objects.

Shapiro (2000, pages 41-48) proposes to divide formalism into two trends: term-formalism and game-formalism. The former assigns reference to symbols, who don’t represent numbers or other hypothetically existing entities in mathematics, to singular terms contained in mathematical propositions.

On the latter perspective, game-formalism, mathematics is a method of transforming formulas made of symbols without meaning, in accordance with fixed rules.

In term-formalism, mathematics is about abstract objects, mathematical symbols, and syntactic rules. In game-formalism, mathematics is all about rules. Mathematical reasoning is considered to be so close to mechanical manipulation of abstract signs, its results being independent on any references. What matters is that rules have been respected.

Hilbert, in his *Grundlagen der Geometrie* (1903), claims that geometry is not about objects in possess of certain properties, which satisfy certain axioms. Rather, it is about a structure which places can be occupied only by objects that respect the conditions imposed by the structure. Hilbert is convinced that singular terms in mathematical statements need no reference, because it is not relevant what the objects are, once they satisfy the axioms.

In conclusion, I argue that formalism has lot in common with Structuralism in particular because of the general concept they share about mathematics. Indeed, mathematics is seen by both as a whole of rules, which is independent on the objects that could eventually exemplify or happen to respect the rules.

«I see metaphysics as an a priori science that is prior to mathematics: whereas mathematical theories are about particular abstract objects (e.g., the natural numbers, the ZF sets, etc.) and particular relations and operations (e.g., successor, membership, group addition, etc.), metaphysics is about abstract objects in general and relations in general. So metaphysics should be free of mathematical primitives, though primitive mathematical terms and predicates might be imported into metaphysics when those primitives are accompanied by principles that identify the denotations of the terms and predicates as entities already found in the background metaphysics [...]

The metaphysical and epistemological problems about mathematics cannot be solved by an appeal to set theory or model theory, for that is just more mathematics and therefore part of the data to be explained. Such problems must be solved by an appeal to a more general theory of abstract objects and relations [...]

The metaphysics and epistemology of mathematics should be consistent with whatever conclusion mathematicians (including set theorists, category theorists, etc.) draw with respect to the existence of such a theory. Some philosophers might wonder how this is possible, but the theory described below shows that it is.»

ZALTA, EDWARD, (2007) *Reflections on Mathematics*, page 3

Chapter 4: Objects Theory

4.1 Naïve Objects Theory

Edward Zalta first presented Objects Theory in his book *Abstract Objects: an Introduction to Axiomatic Metaphysics*⁵³. After that, it has been the subject of several other books and papers⁵⁴ published in the last thirty years. Zalta formulated Objects Theory with the aim of delivering a rigorous solution to an ancient problem. The ancient problem regards what are objects in general and if there exist abstract objects, properties and relations; the rigorous solution is achieved through the construction of a system of axioms. In other words, Zalta's theory aims at the development of an axiomatic metaphysics capable of justifying and explaining reference to several kinds of objects, including abstract objects and, among these, mathematical theories.

Zalta himself retraces the origin of the trend of thought he followed in order to theorize its axiomatic metaphysics in the very beginning of his book. Explicit mention is made along the entire book both to Plato's Forms, Leibniz's Monads, Meinong's theory and Possible Worlds theory.

Zalta based his account on the distinction between *a priori* datas, which are metaphysical hypothesis intuitively believed to be true, and *a posteriori* datas, consisting in sentences concerning fictional characters and triads of sentences assigning identity to concepts. *A priori* datas regards abstract objects, while *a posteriori* datas are

⁵³ ZALTA, EDWARD, (1983), *Abstract Objects: an Introduction to Axiomatic Metaphysics*, D. Reidel, Dordrecht;

⁵⁴See ZALTA, EDWARD (1999), "Natural Numbers and Natural Cardinals as Abstract Objects: A Partial Reconstruction of Frege's *Grundgesetze* in Objects Theory", *Journal of Philosophical Logic*, 28, 619–60, and also ZALTA, EDWARD (2000), "Neo-logicism? An Ontological Reduction of Mathematics to Metaphysics", *Erkenntnis*, 53, 219–65.

about concrete objects.

The development of a formal language allows Zalta to consider the references of terms denoting objects as abstract objects, and the references of terms denoting properties as properties, mainly in the line of Frege's theory of reference.

The starting point of Objects Theory is what Zalta called 'Naïve Objects Theory' (henceforth NOT), a name borrowed from the debated work by Meinong⁵⁵ and Mally⁵⁶ on ontology.

NOT is based on the following comprehension principle:

NOT Comprehension Principle: for every describable set of properties, there is an object which exemplifies just the members of the set.

The very possibility of describing a set of properties implies the existence of an object that satisfies the conditions described. It also discriminates, among all the properties, the ones that belong to the set and the ones that don't. NOT is presented in the following second-order predicate calculus.

Three metaphysical primitive notions are endorsed: objects: x, y, z

n -place relation: F^n, G^n, H^n

$x_1 \dots x_n$ exemplify F^n : $F^n x_1 \dots x_n$.

Properties are defined as 1-place relations. The atomic statements of the formal language are provided by exemplification ' $F^n x_1 \dots x_n$ '.

⁵⁵ MEINONG, ALEXIUS, (1904) 'Uber Gegenstandstheorie', translated by Lev, Terrell and Chisholm, *Realism and the Background of Phenomenology*, Glencoe: The Free Press, (1960), pp. 76-117.

⁵⁶ MALLY, ERNST, (1912), *Gegenstandstheoretische Grundlagen der Logik und Logistik*, Leipzig: Barth.

The language uses three primitive logical constants:

it is not the case that ϕ : $\sim \phi$

if ϕ then ψ : $\phi \rightarrow \psi$

every x (every F^n) is such that ϕ : $(\forall x)\phi$, $(\forall F^n)\phi$.

$(\forall x)\phi$ and $(\forall F^n)\phi$ are frequently abbreviated respectively as $(x)\phi$ and $(F^n)\phi$.

NOT can also be used without any reference to the notion of sets or membership, as Zalta highlights in the following example from pages 7 and 8 of *Abstract Objects: an Introduction to Axiomatic Metaphysics*:

Consider the following open formula ‘Socrates exemplifies F^1 ’. If we let ‘ s ’ denote Socrates, then we can represent this condition on properties in our language as ‘ F^1s ’. Now we can form the following description of a set: the set of all properties F^1 such that Socrates exemplifies F^1 , i.e., $\{ F^1 \mid F^1s \}$. This describes the set of properties that satisfy (in Tarski's sense) the open condition ‘ F^1s ’. The set contains properties like being a philosopher, being Greek, being snub-nosed, etc. Here's another example, where ‘ p ’ denotes Plato. Take the open condition ‘both Socrates exemplifies F^1 and Plato exemplifies F^1 ’ (‘ $F^1s \ \& \ F^1p$ ’) and form the set abstract: the set of all properties F^1 such that both Socrates exemplifies F^1 and Plato exemplifies F^1 , i.e., $\{ F^1 \mid F^1s \ \& \ F^1p \}$. The set described here contains such properties as being a philosopher and being Greek as well, but it would not contain the property of being snub-nosed, since Plato did not exemplify that property.

Since it is possible to describe a set of properties from every expressible conditions on properties, thanks to NOT, for each such set of properties there is an object

exemplifying all and only the properties in the set. An axiom schema can capture this:

$$(\text{NOT}') : (\exists x)(F^l)(F^l x \equiv \phi), \text{ where } \phi \text{ has no free } x\text{'s}$$

The following instances of the axiom schema NOT' guarantee that some objects correspond to the sets of properties as in the example above from Zalta (1983):

$$\text{a) } (\exists x)(F^l)(F^l x \equiv F^l s)$$

$$\text{(b) } (\exists x)(F^l)(F^l x \equiv F^l s \ \& \ F^l p)$$

Leibniz's Law of identity of indiscernible completes the theory:

$$(\text{LL}) \ x = y \equiv (F^l)(F^l x \equiv F^l y)$$

It states the notorious thesis according to which two objects determined by exactly the same properties are the same object: every particular set of properties would determine one and unique object.

Unfortunately, NOT presents several serious problems, in particular regarding n -place relations. The most compelling is its inconsistency with the abstraction schema:

$$(\exists F^n)(x) \dots (x)(F^n x \dots x \equiv \phi), \text{ where } \phi \text{ has no free } F^n\text{'s}$$

Two typical instances of this schema are:

$$1. \ (\exists F)(x)(Fx \equiv \sim Gx)$$

$$2. \ (\exists F)(x)(Fx \equiv Gx \ \& \ Hx).$$

1. states that any given property G will have a negation, while 2. that any two given properties G and H will have a conjunction. Zalta then analyses the following instance of the abstraction schema:

$$(\exists F)(x)(Fx \equiv Rx \ \& \ \sim Rx).$$

This axiom individuates the property objects exemplify if and only if they exemplify redness and don't exemplify redness. It is then possible to form a set which members

are the two properties R and $\sim R$. So there is an object which exemplifies just the members of the set described by the two properties R and $\sim R$. The property K is called the description of this set, that is, the property of describing the set described by the two properties R and $\sim R$. The assumption that something exemplifies such an arbitrary property K produces an immediate contradiction. But (NOT') ensures just that:

$$(\exists x)(F)(Fx \equiv F = K).$$

Nevertheless, NOT is intuitive and seems very promising. That is why Zalta starts from it, explicitly aiming at challenging with the many problems NOT encompasses and building up from its ruins a new theory of objects.

Following a suggestion from Ernst Mally⁵⁷, Zalta introduces two different relations between objects and properties. In Mally's terms, properties determine objects, while objects satisfy properties. But, properties can determine an object without necessarily being satisfied by it. The point is that the properties that determine an object, determine its identity. On the other side, an object satisfies certain properties independently and subsequently from its identity condition. This is the case of inconsistent sets of properties, such as the well-known russellian example of the square circle:

$$(\exists x)(F^I)(F^I x \equiv F^I = R^I \vee F^I = S^I)$$

Such an object would be determined by the property of roundness or squareness, or of roundness and squareness, but will fail to satisfy both of them at the same time: it could satisfy roundness and not squareness, or squareness and not roundness, but will fail to satisfy both of them, while still being determined by both.

⁵⁷ MALLY, ERNST, (1912), *Gegenstandstheoretische Grundlagen der Logik und Logistik*, Leipzig: Barth.

Zalta gives up Mally's terminology, using instead of 'determine' and 'satisfy', 'encode' and 'exemplify' respectively. Therefore, in Zalta's terms, an object exemplifies a property if it satisfies the property; and encodes a property if it is determined by that property. This distinction is explicitly represented in the language of the theory by a distinction Zalta settles in atomic formulas of the language. An improvement of NOT comes together with the distinction between encoding and exemplifying. Therefore, the primitive notions are:

Metaphysical:

object: x, y, z, \dots

n -place relation: F^n, G^n, H^n, \dots

$x_1 \dots x_n$ exemplify F^n : $F^n x_1 \dots x_n$

x encodes F^1 : $x F^1$.

Logical:

it is not the case that ϕ : $\sim \phi$

if ϕ then ψ : $\phi \rightarrow \psi$

every x (every F^n) is such that ϕ : $(\forall x)\phi, (\forall F^n)\phi$.

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existence: $E!$.

With these primitive notions at hand, Zalta can proceed in the formulation of Objects Theory. First, he defines a property as a one-place relation. For example, the property of being abstract ($A!x$) is henceforth the property x encodes if it fails to exemplify existence.

$(x)(\sim E!x \rightarrow xA!)$

An abstract object is then the object encoded by one among any possible condition on properties. Two objects x and y are *identical_E* ($x =_E y$) if and only if they both exemplify existence and are encoded by the same properties.

More explicitly:

(I) Conditions of existence for abstract objects: for every expressible condition on properties, there is an abstract object that encodes just the properties meeting the condition:

$(\exists x)(A!x \ \& \ (F^I)(xF^I \equiv \phi))$, where ϕ has no free x 's.

(II) Identity conditions for all objects: two objects are identical if and only if they are identical_E or they are both abstract and encode the same properties.

$x = y \equiv x =_E y \vee (A!x \ \& \ A!y \ \& \ F^I(xF^I \equiv yF^I))$.

(III) Identity conditions for properties: two properties are identical if and only if the same objects encode them.

$F^I = G^I \equiv (x)(xF^I \equiv xG^I)$.

Following the example at page 7 and 8 of *Abstract Objects: an Introduction to Axiomatic Metaphysics*, these three principles allow Zalta to define an abstract object, which encodes just the properties Socrates exemplifies:

$((\exists x)(A!x \ \& \ (F)(xF \equiv Fs)))$

saving the intuition that this object is not identical with the concrete objects Socrates who lived and died in Athens.

The same goes for the square circle: there is an abstract object which encodes just the

properties of being round and being square, without paining contradiction. This objects is individuated by the following formula:

$$((\exists x)(A!x \ \& \ (F)(xF \equiv F = R \vee F = S)))$$

but it doesn't exemplify neither the property of being round nor the property of being square. Although the theory presupposes that this object fails to exemplify existence, this is compatible with the contingent fact that no existing object exemplifies all the properties this abstract object encodes.

The contradiction is overtaken even for the abstraction principle for complex relations. The property K is generated in the same manner as above but the theory will now guarantee that there is an object which encodes K but fails to exemplify K :

$$K((\exists x)(A!x \ \& \ (F)(xF \equiv F = K))).$$

4.2 Elementary Objects Theory

4.2.1 Object Calculus

Zalta formulates Elementary Objects Theory using standard second-order language, to which he adds only one syntactic modification to express the difference between encoding and exemplifying.

The language contains two kinds of primitive terms, one for objects and one for relations. Primitive object-terms are $a, b, c...$ for names and $x, y, z...$ for variables. Primitive relation-terms are $P^n, Q^n, R^n...$ for names and $F^n, G^n, H^n...$ for variables. Few others primitive assumptions are needed. First, ' $E!$ ' is the name for a particular one-place relation, whose meaning is 'to concretely exist' or 'to be a concrete object'. Second, $=_E$ is the name for a particular two-place relation, the relation of concrete

identity. And third, a formula ϕ is propositional if and only if ϕ has neither encoding sub-formulas nor sub-formulas with quantifiers binding relation variables.

Zalta uses quotation marks inside parentheses ('...'), to define readings or abbreviations of formulas. That is the reason why definitions of the object language appear with the label ' D_n '.

Examples of formulas are:

' P^3axb ' (a , x , and b exemplify relation P^3);

' aG ' (a encodes property G);

' $\sim(\exists x)(xQ \ \& \ Qx)$ ' (there is no object both encoding and exemplifying Q);

' $(x)(E!x \rightarrow \sim(\exists F)xF)$ ' (every object which exemplifies existence fails to encode any properties).

Zalta also defines $\phi(\alpha_1 \dots \alpha_n)$ to designate a formula which major may not have $\alpha_1, \dots, \alpha_n$ occurring free. The expression $\phi_{\alpha_1}^{\tau_1} \dots \alpha_n^{\tau_n}$ designates the formula which results in substituting, for each i , $1 \leq i \leq n$, τ_i for each free occurrence of α_i in ϕ . Zalta defines inductively propositional formulas, object-term and n -place relation-term at page 17 of *Abstract Objects: an Introduction to Axiomatic Metaphysics* as follows:

1. All primitive object-terms are object-terms and all primitive n -place relation-terms are n -place relation-terms;
2. Atomic exemplification: If ρ^n is any n -place relation-term, and $o_1 \dots o_n$ are any object-terms, $\rho^n o_1 \dots o_n$ is a (propositional) formula (read: ' $o_1 \dots o_n$ exemplify relation ρ^n ');
3. Atomic encoding: If ρ^l is any one-place relation-term and o is any object-term, $o\rho^l$ is

a formula (read: 'o encodes property ρ' ');

4. Molecular: If ϕ and ψ are any (propositional) formulas, then $(\sim\phi)$

and $(\phi\rightarrow\psi)$ are (propositional) formulas;

5. Quantified: If ϕ is any (propositional) formula, and α is any (object) variable, then

$(\forall\alpha)\phi$ is a (propositional) formula;

6. Complex n -place relation-terms: If ϕ is any propositional formula

with n -free object variables $v_1\dots v_n$ then $[\lambda v_1\dots v_n\phi]$ is an n -place relation-term.

λ -expressions are used to express in the object language that a and b exemplify being

two objects x and y such that x exemplifies P and y exemplifies Q :

$[\lambda xy Px \& Qy]ab$

or that an object x encodes failing-to-exemplify- R :

$x[\lambda y \sim Ry]$

The role of λ -expressions is fundamental in Objects Theory. Roughly, they transform formulas in predicates of the object language, thanks to comprehension principle and exemplification, and allow speaking about formulas without moving to the metalanguage. The role of λ -expressions will be further investigated in the following of this chapter.

The first definition of Objects Theory is then:

D_1 x is abstract $(A!x) =_{df} [\lambda y \sim E!y]x$.

To be read as 'x exemplify to be an object that is not concrete'. The property of being abstract is therefore a property only objects that don't exemplify any property can encode. Abstract objects cannot exemplify properties, but they can encode properties.

This is the relevant difference between abstract and concrete objects: the former only encode properties, while the latter only exemplify properties. A more precise and perspicuous definition of what is for an object to be concrete or abstract is given by the semantics of Objects Theory.

Zalta delineates the semantics for Objects Theory defining an interpretation as the 6-tuple $\langle \mathcal{D}, \mathcal{R}, ext_{\mathcal{R}}, \mathcal{L}, ext_{\mathcal{A}}, \mathcal{F} \rangle$ meeting the following conditions:

- \mathcal{D} is the non-empty class indicating the domain of objects. The metalinguistic variables that range over members of the domain is o ;
- \mathcal{R} is the non-empty class indicating the domain of relations. It is the union of a sequence of non-empty classes $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$. The metalinguistic variables that range over the elements of \mathcal{R} is r^n . Moreover, \mathcal{R} must be closed under all the logical function prescribed by \mathcal{L} ;
- $ext_{\mathcal{R}}$ is a function called ‘the exemplification extension of r^n ’ that maps each $r \in \mathcal{R}_n$ into the power set of \mathcal{D}^n ;
- \mathcal{L} is a class of logical functions operating on the members of \mathcal{R}_n and \mathcal{D} to produce the complex relations that denote the λ -expressions. There are six logical functions: $PLUG_i$ (i -plug), $UNIV_i$ (i -universalization), $CONV_{i,j}$ (i,j -conversion), $REF_{i,j}$ (i,j - reflection), $COND$ (conditionalization) and NEG (negation). These logical functions guarantee a rich variety of complex relations in the domain of relations;
- $ext_{\mathcal{A}}$ is a function which imposes a structure on the domains \mathcal{D} and \mathcal{R} . Since

every property \mathcal{R}_i has an encoding extension ($'ext_{\mathcal{A}}'$), the encoding extension of a property is a set of members of \mathcal{D} , which encode the property; $ext_{\mathcal{A}}$ is then a function that maps \mathcal{R}_i into the power set of \mathcal{D} ;

- \mathcal{F} is a function that maps the simple names of the language into elements of the appropriate domain. For each object name k , $\mathcal{F}(k) \in \mathcal{D}$. For each relation name k^n , $\mathcal{F}(k^n) \in \mathcal{R}_n$.

Zalta can now offer a more perspicuous notion of abstract and concrete objects: being ' $E!$ ' a simple property name, $\mathcal{F}(E!) \in \mathcal{R}_1$ and so $ext_{\mathcal{R}} \mathcal{F}(E!) \subseteq \mathcal{D}$. The resulting subset of \mathcal{D} is the set of existing objects (\mathcal{E}). The complement of \mathcal{E} on \mathcal{D} is then the set of abstract objects (\mathcal{A}). So the Objects Theory is built on a domain of objects that includes existing and non-existing objects.

Since, for any object x :

$$E!x \vee \neg E!x$$

objects are classified into two complementary classes: concrete existing objects and abstract not-existing objects. Note that the abstract not-existing objects are still part of \mathcal{D} even if they don't exist. The reason, as you can recall from the end of the previous section, is that abstract objects can encode the property of existing, while they will never exemplify it. Only concrete objects exemplify properties, including, *a fortiori*, the property of existing. But the simple fact that abstract objects can't exemplify any property, neither the property of existing, does not mean they are not part of the domain of Objects Theory. They have precise being condition and identity condition that allow to speak about and refer to abstract objects unambiguously.

Zalta's next step is to introduce Objects Theory's logic. He first presents some schemata for logical axioms. Such schemata are very classical in second-order language and are:

Propositional schemata:

1. $\phi \rightarrow (\psi \rightarrow \phi)$;
2. $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$;
3. $(\sim\phi \rightarrow \sim\psi) \rightarrow ((\sim\phi \rightarrow \psi) \rightarrow \phi)$;

Quantificational schemata:

4. $(\alpha)\phi \rightarrow \phi_\alpha^\tau$ where τ is substitutable for α ;
5. $(\alpha)(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\alpha)\psi)$, with α not free in ϕ .

Lambda schemata:

λ -equivalence: where ϕ is any propositional formula, the following is an axiom:

$$(x_1) \dots (x_n) ([\lambda v_1 \dots v_n \phi] x_1 \dots x_n \equiv \phi_{v_1}^{x_1} \dots_{v_n}^{x_n})$$

The second λ -schema, λ -identity, requires two more definitions:

$$D_2 F^l = G^l =_{df} (x)(xF^l \equiv xG^l)$$

states that properties are identical if the same objects encode them. Zalta needs to generalize the definition of property identity in order to analyze the relation identity.

When $n > 1$:

$$D_3 F^n = G^n =_{df} (x_1) \dots (x_{n-1}) ([\lambda y F^n y x_1 \dots x_{n-1}] = [\lambda y G^n y x_1 \dots x_{n-1}] \& [\lambda y F^n x_1 y x_2 \dots x_{n-1}] =$$

$$[\lambda y G^n x_1 y x_2 \dots x_{n-1}] \& \dots \& [\lambda y F^n x_1 y x_2 \dots x_{n-1} y] = [\lambda y G^n x_1 y x_2 \dots x_{n-1} y]).$$

The second λ -schemata is:

λ -identity: where ρ^n is any relation-term and $v_1...v_n$ are any object variables, the

following n -place relation term, subject to \exists -introduction, is an axiom:

$$[\lambda v_1...v_n \rho^n v_1...v_n] = \rho^n$$

The rules of inference required are Modus Ponens (from ϕ and $(\phi \rightarrow \psi)$, infer ψ) and Universal Introduction (from ϕ , infer $(\alpha)\phi$).

From Modus Ponens and Quantificational Schemata 4 Zalta derives Universal Elimination. He also makes use of Existential Introduction and Existential Elimination, in addition to rules for Introduction and Elimination for \sim , $\&$, \vee , and \equiv .

Taking a closer look to λ -expressions, two rules of inference are derived from λ -equivalence: if ϕ is any propositional formula with object-terms $o_1...o_n$ and $v_1...v_n$ are object variables substitutable for $o_1...o_n$ respectively, then the following are rules of inference:

(1) λ -Introduction: from ϕ , we may infer $[\lambda v_1...v_n \phi_{o_1}^{v_1}:::o_n^{v_n}] o_1...o_n$;

(2) λ -Elimination: from $[\lambda v_1...v_n \phi_{o_1}^{v_1}:::o_n^{v_n}] o_1...o_n$ we may infer ϕ .

Applying existential introduction to λ -equivalence, Zalta formulates a useful logical theorem on relations:

'RELATIONS': where ϕ is a propositional formula which has no free F^n 's, but has $x_1...x_n$ free, the following comprehension schema is a theorem:

$$(\exists F^n)(x_1)...(x_n)(F^n x_1 \dots x_n \equiv \phi).$$

Looking at the instances of this theorem, Zalta can determine what complex properties and relations there are. At page 32 of *Abstract Objects: an Introduction to Axiomatic Metaphysics*, he states:

RELATIONS, **D₂**, and jointly constitute a full-fledged theory of relations.

We no longer need to suppose that relations are ‘creatures of darkness’.

They have precise ‘being’ conditions and precise identity conditions.

Two definitions, **D₄** and **D₅**, are added to **D₁**, **D₂** and **D₃**: **D₄** is a generalization of **D₁**, defining the general conditions for identity between objects:

D₄ $x = y =_{df} x =_E y \vee (A!x \wedge A!y \wedge (F)(xF \equiv yF))$

D₅ defines the notions of ‘blueprint’ and ‘correlate’, which meaning will be specified by the axioms:

D₅ x is the blueprint of y and y is the correlate of x (‘*Blue(x,y)*’ and ‘*Cor(y,x)*’) =_{df} $(F)(xF \equiv Fy)$.

4.2.2 Axioms

Objects Theory has an infinite number of axioms: two axioms and two schemata. Zalta suggests to consider the axioms as *a priori* truths.

The first axiom prescribes that two objects bear the identity_E relation to one another if and only if they both exist and exemplify the same properties:

AXIOM 1. (‘E-IDENTITY’): $x =_E y \equiv E!x \ \& \ E!y \ \& \ (F)(Fx \equiv Fy)$.

E-IDENTITY completes the work of **D₄** as it allows to demonstrate the rule for the Identity Introduction between all kinds of variables:

‘IDENTITY INTRODUCTION’: $\alpha = \alpha$, where α is any variable.

The second axiom guarantees that no existing objects encode properties:

AXIOM 2. (‘NO-CODER’): $E!x \rightarrow \sim (\exists F)x F$.

The third axiom, together with the rule of Elimination of Negation, demonstrates the rule for Identity Elimination:

AXIOM(S) 3. (‘IDENTITY’): $\alpha = \alpha \rightarrow (\phi(\alpha, \alpha) \equiv \phi(\alpha, \beta))$, where $\phi(\alpha, \beta)$ is the result of replacing some, but not necessarily all, free occurrences of α by β in $\phi(\alpha, \alpha)$, provided β is substitutable for α in the occurrences of α it replaces.

The fourth axiom is the most important for Objects Theory, since it guarantees that, for every expressible set of properties, there is an abstract object which encodes just the members of the set⁵⁸:

AXIOM(S) 4. (‘A-OBJECTS’): for any formula ϕ where x is not free, the following is an axiom:

$(\exists x)(A!x \ \& \ (F)(x F \equiv \phi))$.

Zalta offers an example of the functioning of AXIOM(S) 4 at page 34 of *Abstract Objects: an Introduction to Axiomatic Metaphysics*:

If we let ‘ $F = R \vee F = S$ ’ be our formula ϕ , and suppose that ‘ R ’ denotes roundness and ‘ S ’ denotes squareness, then our axiom guarantees that there is a ‘round square’ as follows:

$(\exists x)(A!x \ \& \ (F)(x F \equiv F=R \vee F=S))$.

⁵⁸ As Zalta explicitly emphasizes several times in *Abstract Objects: an Introduction to Axiomatic Metaphysics*, the theory in general, and the schema in particular, do not need any commitment to sets. The appeals to sets are motivated by reasons of ease of formulation and in the metalanguage. In the object language, read AXIOM 4 with no appeal to set as: for every condition on properties, there is an abstract object which encodes just the properties which meet the conditions stated by the axiom.

Suppose a_o is such an object. It is easy to see that a_o must be unique. For suppose some other distinct abstract object, say a_1 , encoded exactly roundness and squareness. By D_4 , it would follow that either a_1 encoded a property a_o did not, or viceversa, contrary to hypothesis.

A-OBJECTS leads to interesting consequences. Using the standard abbreviation ‘ $(\exists x)\psi$ ’ for $(\exists x)(\psi \ \& \ (y)(\psi^y \rightarrow y = x))$ and given **D₄**, Zalta can prove from A-OBJECTS the following theorem:

UNIQUENESS: For any formula ϕ where x is not free, the following is a theorem:

$$(\exists!x)(A!x \ \& \ (F)(xF \equiv \phi)).$$

Given any object a , A-OBJECTS yields to an object which encodes all the properties a fails to exemplify. Given any two objects a and b , A-OBJECTS yields to an object which encodes just the properties a and b have in common and the properties exemplified by either a or b . It also yields to the relational properties that hold between a and b with the instance:

$$(\exists x)(A!x \ \& \ (F)(xF \equiv (\exists G)((Gab \ \& \ F = [\lambda xGxb]) \vee (Gba \ \& \ F = [\lambda xGbx])))).$$

Objects Theory can so provide a unique way to plug to abstract objects.

From $\phi = [F \neq F]$, we point at the object that fails to encode any properties, that is the empty object. From $\phi = [F = F]$ to the universal object, which encodes every property. To see how, Zalta appeals to the notion of blueprint and correlator defined by **D₅**: suppose ‘ a_o ’ is a name for an object o . Then A-OBJECTS guarantees the existence of an abstract object, which encodes exactly the properties o exemplifies:

$$(\exists x)(A!x \ \& \ (F)(xF \equiv Fa_o))$$

Zalta proposes to call this object o 's *blueprint*, and the object o the *correlate* of the blueprint. A-OBJECTS then guarantees that every object, existing or abstract, has a unique blueprint:

$$(\forall y)(\exists! x)(A!x \ \& \ (F)(xF \equiv Fy)).$$

In order to make clear the expressive power of definite descriptions in Objects Theory, Zalta suggests to introduce the description operator ι : being ϕ any formula with one free x -variable, $(\iota x)\phi$ (read ‘the object x such that ϕ ’), is a complex object-term of our language.

Descriptions like $(\iota x)\phi$ denotes therefore the unique object which satisfies ϕ , if there is one, and as not denoting anything if there is none. The following axiom schema guarantees that atomic formulas or defined identity formulas ψ in which there occurs a description $(\iota x)\phi$ are true if and only if there is a unique object satisfying ϕ and there is something which satisfies both ϕ and ψ :

DESCRIPTIONS: where ψ is any atomic formula or defined object identity formula with one free object variable v , the following is a proper axiom:

$$\psi_{v^{(\iota x)\phi}} \equiv (\exists! x)\phi \ \& \ (\exists x)(\phi \ \& \ \psi_v)$$

E-IDENTITY, NO-CODER, IDENTITY and A-OBJECT complete the Elementary Objects Theory. Within this theory, abstract objects are given a precise kind of existence, that allows plugging to any of them and naming them in a unique way. The question is now how to establish which, among the abstract objects, are mathematical.

4.3 Modal Theory of Abstract Objects

4.3.1 Modal Object Calculus

Modal Object Calculus uses almost the same language of Elementary Object Calculus, with some extensions. First, it has the modal operator for necessity \Box and \Diamond , with $\Diamond\phi$ abbreviating $\sim\Box\sim\phi$. Second, names and variables for propositions are introduced, allowing the superscripts on the primitive relation-terms to reach zero: P^o, Q^o, R^o, \dots for names and F^o, G^o, H^o, \dots for variables for propositions.

Zalta defines inductively propositional formulas, object-terms and n -place relation-terms for Modal Object Calculus at pages 59 and 60 of *Abstract Objects: an Introduction to Axiomatic Metaphysics*:

1. All primitive object-terms are object-terms and all primitive n -place relation-terms are n -place relation-terms;
2. If ρ^o is any zero-place relation-term, ρ^o is a (propositional) formula;
3. Atomic exemplification: If ρ^n is any n -place relation-term, and $o_1 \dots o_n$ are any object-terms, $\rho^n o_1 \dots o_n$ is a (propositional) formula;
4. Atomic encoding: If ρ^1 is any one-place relation-term and o is any object-term, $o\rho^1$ is a formula;
5. Molecular, Quantified and Modal: If ϕ and ψ are any (propositional) formulas and α is any (object) variable, then $(\sim\phi)$, $(\phi \rightarrow \psi)$, $(\forall\alpha)\phi$, and $(\Box\phi)$ are (propositional) formulas;

6. Objects Description: If ϕ is any formula with one free object-variable x , then $\iota(x)\phi$ is an object-term;
7. Complex n -place relation-terms: If ϕ is any propositional formula and $v_1...v_n$ are any object variable which may or may not be free in ϕ , then $[\lambda v_1...v_n \phi]$ is an n -place relation-term ($n \geq 1$) and ϕ itself is zero-place relation-term.

In comparison with the ones of Elementary Object Calculus, some clauses remained the same, while other changed in order to express the modal operators. Nevertheless, the most important differences are in the clause for complex n -place relation-terms. In Modal Object Calculus modal formulas appear after λ 's if the formula is propositional, λ 's bind variables are not free in the ensuing formula and propositional formulas can be treated as relation-terms. But there is also one definition more, the second one, describing what propositional formulas are.

Zalta delineates the semantics for Modal Objects Theory defining an interpretation as the 8-tuple $\langle \mathcal{W}, w_o, \mathcal{D}, \mathcal{R}, ext_w, \mathcal{L}, ext_{\mathcal{L}}, \mathcal{F} \rangle$ meeting the following conditions:

- \mathcal{W} is the non-empty class indicating the class of possible worlds. The metalinguistic variables that range over members of the domain is o ;
- w_o is an element of \mathcal{W} representing the actual world;
- \mathcal{D} is the non-empty class indicating the domain of objects;
- \mathcal{R} is the non-empty class indicating the domain of relations. It is the union of a sequence of non-empty classes $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$. Each \mathcal{R}_n is the class of n -place

relations, with \mathcal{R}_1 being the class of properties and \mathcal{R}_0 the class of propositions.

Moreover, \mathcal{R} must be closed under all the logical function prescribed by \mathcal{L} ;

- ext_w is a function that maps each $\mathcal{R}_n \times \mathcal{W}$ into the power set of \mathcal{D}^n , where $n \geq 1$ and $\mathcal{R}_0 \cdot \mathcal{W}$ into $\{T, F\}$. $ext_w(r^n)$ is called ‘the exemplification extension of r^n at w ’;
- \mathcal{L} is a class of logical functions operating on the members of \mathcal{R}_n and \mathcal{D} to produce the complex relations that denote the λ -expressions. \mathcal{L} restricts the ext_w of the complex relations resulting from all the logical functions at every possible world.
- $ext_{\mathcal{A}}$ is a function which maps \mathcal{R}_i into the power set of \mathcal{D} . It is called the encoding extension \mathcal{A} of r_i ;
- \mathcal{F} is a function that maps the simple names of the language into elements of the appropriate domain. For each object name k , $\mathcal{F}(k) \in \mathcal{D}$. For each relation name k^n , $\mathcal{F}(k^n) \in \mathcal{R}_n$.

Zalta offers now a new definition of abstract and concrete objects. In Modal Object Calculus, Zalta defines:

D₁ x is abstract ($A!x$) =_{df} $[\lambda y \Box \sim E!y]x$

D₂ x is a possibly existing object =_{df} $\Diamond E!x$

Abstract objects in Modal Objects Theory are those objects that don’t exist and couldn’t have existed.

Being $ext_w(\mathcal{F}(E!))$ the set of existing objects at w (‘ \mathcal{E}_w ’), $ext_{w_0}(\mathcal{F}(E!))$ is the set of existing objects at the actual world. Then $\{o \mid (\exists w)(o \in ext_w(\mathcal{F}(E!)))\}$ is the set of possibly existing objects (\mathcal{PE}). The complement of \mathcal{PE} on \mathcal{D} is finally the set of

abstract objects (\mathcal{A}). Abstract objects are therefore the complement set of the set of objects that could possibly exist or have existed in some possible world. If an object couldn't exist or have existed in any possible world, then it is abstract. Note that the sets of existing and possibly existing objects are obtained from the domain only through the exemplification expansion ext_w , leaving obviously apart the encoding expansion $ext_{\mathcal{A}}$. As in Elementary Objects Theory, abstract objects are those objects that encode but fail to exemplify.

The most interesting part of Modal Objects Theory regards propositions, which are no more confined to the metalanguage, but find a place into the object language as 0-place properties. Propositions aren't true or false: they are just propositions about the world, with no free-variable formulas to fill and check their satisfaction. By taking propositions as 0-place properties, Zalta can consider propositions about the world, that cannot be satisfied, because there is no place for something to satisfy in a propositional formula: propositions are rather expressed by closed formulas. If ' p ' ranges over propositions, then an object x may encode the proposition p by encoding the propositional property [being such that p]. So for example, the propositional property [being the actual world], makes true all the properties encoded by the actual world.

Modal Objects Theory is slightly different from Elementary Objects Theory also for its logic. The schemata for logical axioms are classical:

Propositional schemata:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$;
2. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$;
3. $(\sim\varphi \rightarrow \sim\psi) \rightarrow ((\sim\varphi \rightarrow \psi) \rightarrow \varphi)$;

Quantificational schemata:

4.1 $(\alpha)\phi \rightarrow \phi_{\alpha\tau}$ where τ contains no descriptions and is substitutable for α ;

4.2 $(\alpha)\phi \rightarrow (\psi_{\beta\tau} \rightarrow \phi_{\alpha\tau})$ where ψ is any atomic formula and τ both contains no descriptions and is substitutable for α, β ;

That is, Logical axioms 4.1 and 4.2 serve for non-denoting descriptions and terms which may contain such descriptions. Note also that here \Box is interpreted semantically as universal generalization over the domain of worlds. That is why commuting a box with a universal quantifier is just as valid as commuting two universal quantifiers.

Similarly, diamond commutes with existential quantifiers.

5. $(\alpha)(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\alpha)\psi)$, with α not free in ϕ .

Modal schemata:

6. $\Box\phi \rightarrow \phi$

7. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

8. $\Diamond\phi \rightarrow \Box\Diamond\phi$

9. $\Box(\alpha)\phi \equiv (\alpha)\Box\phi$

10. $(x)(F)(\Diamond xF \rightarrow \Box xF)$

Logical axiom 10 is typical to Modal Objects Theory, since it regards the modal logic of encoding. It guarantees that if an object encodes a property at some possible world, it will encode that property at all possible worlds.

Lambda schemata:

λ -equivalence: where ϕ is any propositional formula containing no descriptions, the following is an axiom:

$$(x_1) \dots (x_{n-1}) ([\lambda v_1 \dots v_n \phi] x_1 \dots x_n \equiv \phi_{v_1}^{x_1} \dots v_n^{x_n})$$

Also in Modal Objects Theory, **D2** and **D3** are required for the second λ -schema, regarding λ -identity:

λ -identity: where ρ^n relation-term and $v_1 \dots v_n$, $v'_1 \dots v'_n$ are distinct object variables not free in ρ^0 ,

$$[\lambda v_1 \dots v_n \rho^n v_1 \dots v_n] = \rho^n \ \& \ [\lambda v_1 \dots v_n \rho^0] = [\lambda v'_1 \dots v'_n \rho^0].$$

In this case, although the first conjunct has the same meaning as in the Elementary Theory, the second conjunct has a new one. Indeed, $[\lambda x_1 F^0]$ and $[\lambda x_2 F^0]$ will denote semantically identical properties, so the encoding extensions of such properties must be the same. The second conjunct of λ -identity generalizes to the case where relations are denoted by λ -expressions with more than one vacuously bound variable.

Description schemata:

λ -descriptions₁: where ψ is any atomic formula or conjunction of atomic formulas:

$$\psi_v^{(ix)} \phi \rightarrow (\exists y)(\phi_{x^y} \ \& \ \psi_{x^y}).$$

λ -descriptions₂: where ψ atomic formula:

$$\psi_v^{(ix)} \phi \rightarrow (y)(\phi_{x^y} \rightarrow \psi_{x^y})$$

λ -descriptions₃: where ψ is any atomic with v_1 free and χ is any formula with v_2 free,

$$\psi_v^{(ix)} \phi \rightarrow \sim((\exists y)(\phi_{x^y} \ \& \ \chi_{x^y}) \ \& \ (\exists y)(\phi_{x^y} \ \& \ \sim \chi_{x^y})).$$

Note that instances of these last three schemata regarding descriptions are not logically

true, because definite descriptions are defined as rigid designators. They can only be examples of logical truths, which are not necessary true.

The rules of inference required are, as for Elementary Objects Theory, Modus Ponens (from ϕ and $(\phi \rightarrow \psi)$, ψ) and Universal Introduction (from ϕ , $(\alpha)\phi$). The other rules are formulated as for Elementary Objects Theory, with some exception: for the derived rules of λ -introduction and λ -elimination, definite descriptions can't occur in ϕ ; the theorem schema for relations is derivable without imposing the restriction that $x_1 \dots x_n$ be free in ϕ , but it has to be restricted to ϕ containing no descriptions:

RELATIONS: where ϕ is any propositional formula which has no free F^n 's and no descriptions, the following is a logical theorem:

$$(\exists F^n) \Box (x_1) \dots (x_n) (F^n x_1 \dots x_n \equiv \phi).$$

This comprehension schema is particularly noteworthy because from it is derivable:

PROPOSITIONS: where ϕ is any propositional formula which has no free F^0 's and no descriptions, the following is a logical theorem:

$$(\exists F^0) \Box (F^0 \equiv \phi).$$

that allows to define the identity condition for propositions:

$$\mathbf{D3} \quad F^0 = G^0 =_{df} [\lambda y F^0] = [\lambda y G^0]$$

Two propositions are therefore identical if and only if the property of being such that F^0 is encoded by all and only the objects encoded by the property of being such that G^0 .

4.3.2 Axioms

The Modal Objects Theory has an infinite number of axioms: two proper axioms and three schemata.

The first axiom prescribes that two objects bear the identity_E relation to one another if and only if they both exist and exemplify the same properties:

AXIOM 1. ('E-IDENTITY'): $x =_E y \equiv \Diamond E!x \ \& \ \Diamond E!y \ \& \ \Diamond (F)(Fx \equiv Fy)$

AXIOM 2. ('NO-CODER'): $\Diamond E!x \rightarrow \Box \sim (\exists F)x F$

Axioms 1 and 2 regard possible objects. Zalta suggests to consider each possible world as semantically equivalent to a model of Elementary Objects Theory. Hence, for every world there are objects, which exist in that world, and objects, which fail to exist in that world. But between those objects there also are the ones that necessarily exist (or the ones that necessarily fail to exist) and the ones that could have existed (or fail to have existed) in some other possible world.

Before stating the third axiom, a definition is needed:

D₆ $x = y =_{df} x =_E y \vee (A!x \ \& \ A!y \ \& \ \Box (F)(xF \equiv yF))$

D₆ is a general identity criterion for abstract and concrete objects, asserting that two objects are identical either if they are concretely identical or if they are abstract and necessarily encode the same properties.

AXIOM(S) 3. ('IDENTITY'): $\alpha = \beta \rightarrow (\phi(\alpha, \alpha) \equiv \phi(\alpha, \beta))$, where $\phi(\alpha, \beta)$ is the result of replacing some, but not necessarily all, free occurrences of α by β in $\phi(\alpha, \alpha)$, provided β is substitutable for α in the occurrences of α it replaces.

Axiom 3 has much more expressive power in Modal than in Elementary Objects Theory, since it deals with a language enriched with many new terms. Moreover, it has some powerful instances like the following:

$$F^0 = G^0 \rightarrow (F^0 \Box \equiv \Box G^0)$$

stating that, if two propositions are identical at a world, then they are necessarily identical at that world.

AXIOM(S) 4. ('A-OBJECTS'): for any formula ϕ where x is not free, the following is an axiom:

$$(\exists x)(A!x \ \& \ (F)(xF \equiv \phi)).$$

Axiom 4 has greater significance in Modal than in Elementary Objects Theory, because the modal closures of Axiom 4 are axioms and the following counts as an axiom schema:

$$\Box(\exists x)(A!x \ \& \ (F)(xF \equiv \phi)), \text{ where } \phi \text{ has no free } x\text{'s.}$$

stating that, given a world ω and a condition on properties ϕ , there is an abstract object at ω which encodes just the properties satisfying ϕ at ω . So for example, a formula like Fs ('Socrates exemplifies F ') is satisfied by different properties at different worlds. This means that at each world there is a particular A-object which encodes just the properties Socrates exemplifies at that world. Otherwise, a formula like ' $(F)(RF \vee SF)$ ' (F encodes roundness or encodes squareness) is satisfied by the same two properties, roundness and squareness, at each world, and the round square of one world will be identical with the round square of any other world. The conclusion is then that the set of abstract objects remains the same from world to world, while the set of concrete objects changes.

AXIOM(S) 5. ('DESCRIPTIONS'): where ψ is any atomic or defined object or

identity formula with one free object variable v , the following is an axiom:

$$\psi_v^{(x)}\phi \equiv (\exists!y)\phi_x^y \ \& (\exists y)(\phi_x^y \ \& \ \psi_x^y)$$

E-IDENTITY, NO-CODER, IDENTITY, A-OBJECT and DESCRIPTIONS complete the Modal Objects Theory. Zalta himself showed how it can be applied to several philosophical theories, like Plato's Forms, Leibniz's Monads and the contemporary fictionalist approach to metaphysics. In the following section I will evaluate its more immediate applications in philosophy of mathematics and try to figure out what mathematical objects turn out to be (section 4.4) and how to obtain mathematical knowledge (section 4.5) according to Objects Theory.

4.4 Ontology

The axiomatic metaphysics Zalta presents in *Abstract Objects* aims at providing an account of objects in general, and mathematical objects in particular. But, while epistemological or semantical issues are almost introduced, Zalta is mainly focused on a solution for classical problems regarding existence. Here is what he writes in *Abstract Objects* page 50:

The assertion that some particular thing fails to exemplify existence (or being) strangely carries with it a commitment to the existence (or being) of the very thing which serves as the subject of the assertion. This is partly a result of trying to keep the theory of language as simple as possible - we try to account for the truth of a simple sentence by supposing that the objects denoted by the object-terms are in an extension of the relation denoted by

the relation-term. But when we have a true non-existence claim, talk about
‘the object denoted by the object name’ seems illegitimate.

Ultimately, this means that Zalta looks for a technique to translate existence (and non-existence) claims from the natural language to the language of Objects Theory in an effective and intuitive way. The two modes of predication, corresponding to two different ways of translating the copula ‘is’ as encoding or exemplifying, response precisely to such a need. Indeed, Zalta shows in several parts of his work that, when translating certain statements of natural language into the language of the theory, a few of them containing the copula ‘is’ expresses different truth-values when translated using an encoding formula or an exemplification formula.

Both encoding and exemplifying formulas are related to the primitive notion of existence, but with different meanings. Recall that exemplifying regards objects which concretely exist and are bounded to at least one particular world, possibly existent objects that happen to exist in some possible world. Encoding formulas, whereas, regard objects whose non-existence is not related to any possible world, but happens in every possible worlds. Objects that encode properties fail to exemplify existence by definition, but this doesn’t mean that they don’t exist at all. Actually, Zalta’s intuition is that if an object encodes some properties, it must participate in some kind of existence.

Zalta takes existence as a predicate, rather than a property. The distinction ultimately resides in the possibility to state that a predicate P is such that, for any object a , a encodes P but doesn’t exemplify it.

Zalta enriches with some details his characterization of abstract objects in a subsequent work with Bernard Linsky, 'Naturalized Platonism vs Platonized Naturalism'⁵⁹. Here Linsky and Zalta suggest a theory for abstract objects that is different from the one for concrete objects in three ways: first, abstract objects shouldn't be subject to a distinction between appearance and reality, while concrete should. Indeed, the properties concrete objects have can't be directly inferred from the way they appear, but they have to be discovered through empirical inquiry. Second, concrete objects are sparse and only empirical observation can guarantee their existence. Abstract objects' existence is instead guaranteed by the possibility to characterize them through the comprehension principle. Thirdly, concrete objects are ontologically complete, in the sense that they have all the properties they have, and the negation of all the properties they don't have, even the ones anyone will ever know. Abstract objects instead are a plenum: if they are to be described by their properties, and the comprehension principle guarantees that there is an abstract object for every group of properties, then there are as many abstract objects as there could possibly be. The following quote from 'Naturalized Platonism vs Platonized Naturalism', page 14, can help clarifying the point:

The comprehension principle asserts the existence of a wide variety of abstract objects, some of which are complete with respect to the properties they encode, while others are incomplete in this respect. For example, one instance of comprehension asserts that there exists an abstract object encoding just the properties Clinton exemplifies. This object is complete

⁵⁹ LINSKY, BERNARD, ZALTA, EDWARD, (1995), "Naturalized Platonism vs Platonized Naturalism", *Journal of Philosophy* vol. 92, pp. 525-555;

because Clinton either exemplifies F or exemplifies the negation of F , for every property F . Another instance of comprehension asserts that there is an abstract object that encodes just the two properties: *being blue* and *being round*. This object is incomplete because for every *other* property F , it encodes neither F nor the negation of F . But though abstract objects may be partial with respect to their encoded properties, they are all complete with respect to the properties they *exemplify*.

In this sense, there is a plenitude of abstract objects. Zalta is far from considering plenitudinousness of objects as not ontological parsimonious. Indeed, in order to satisfy the constraints of ontological parsimony, as few objects as possible must be added to the domain in a non-arbitrary way. This is the reason why, in his account of abstract objects, the only way to add as few objects as possible in a non-arbitrary way is to add all possible abstract objects. Notice here that the adoption of an extensional definition of objects leads to the unpleasant result that there are more and more objects, because every time an object is defined, there are also new properties that depends on that object and that can, on their turn, define new objects and so on.

The crucial point resides in the distinction between the two ways of speaking about existence: using the predicate $E!$, Zalta speaks about concrete objects and the exemplifying technique. Using the existential quantifier, Zalta refers to abstract object and to the encoding technique and the comprehension principle for abstract objects, as guaranteed by the predicate $A!$.

Among the abstract objects to which Objects Theory guarantees existence, there also are mathematical objects. According to Zalta, mathematical objects just are a particular

class of abstract objects. The existence of these objects, like that of any abstract object, is a purely logical, metaphysical and linguistic fact, guaranteed by the metaphysics and the logic of Objects Theory. So, for example, there are natural numbers at every possible world, though they will be different abstract objects at each world.

For instance, the natural number 'one' encodes all and only those properties exemplified by exactly one object. In other possible worlds, different properties are exemplified by exactly one object. So the natural number 'one' at those other worlds will be a different abstract object. Here is Zalta's strategy: to define world-relative natural numbers using the possible worlds that are also analyzed in Objects Theory: the natural number 1 at w^1 is the abstract object that encodes all and only those properties F that are exemplified by a unique object at w .

The problem is now to define the class of mathematical objects. Zalta proposes to consider mathematical theories as kind of abstract objects that satisfy two conditions: first, a mathematical theory T is an abstract object for which there exists a concrete object that authored it; second, a theory is an object denoted by a term in the language that encodes all and only the propositional properties that are true in that theory.

Some concerns can raise in regards of the fact that, though the extension of the set of abstract objects that are mathematical objects depends on which mathematical theories are actually authored by mathematicians, there is a class of possible mathematical objects that can be abstracted from possible mathematical theories. A possible mathematical theory T is an abstract object that encodes only propositional properties and such that there might be a mathematician who authored T . In this sense, the domain of possible mathematical objects is considered as the real domain of mathematics. This

aspect is interesting because goes hand by hand with the intuition that mathematics arises not just from what mathematicians actually do, but rather from what they might do.

Moreover, this means that no matter how a mathematician might formulate a mathematical theory, Objects Theory identifies the mathematical individuals and mathematical relations described by such a theory as abstract individuals and abstract relations.

Therefore, a mathematical theory T is the abstract object that encodes the propositional properties constructed out of certain mathematical propositions in the following way: let $p(x)$ be an open formula in x and p be any proposition; Objects Theory presents both the monadic property ‘being an x such that $p(x)$ ’, where $p(x)$ has a free occurrence of x , and the 0-place property $[\lambda y p(x)]$, to be read as ‘being an x such that p ’. This last is the property every object possesses just in case the world in which it exists is such that p . A mathematical theory is then the abstract object that encodes propositional properties according to the mathematical propositions of the theory.

Theories are descriptive propositions about some kind of reality; for mathematical propositional properties, this reality is the realm of mathematical entities. These entities are mathematical objects, that Zalta defines as the abstract objects that encode all and only the monadic properties P that are such that T encodes the property constructed out of the proposition $p[P(k_T)]$ for every mathematical theory T .

$p[P(k_T)]$ indicates the proposition that encodes the monadic properties encoded by any object of theory T . The theorems of each mathematical theory T are introduced in Objects Theory by indexing the individual terms and predicates used in T to T and

preceding the theory operator ' $T [\lambda y \varphi^*]$ ', to be read as 'in theory T , ...', to each theorem. Then, ' $T [\lambda y \varphi^*]$ ' is an encoding claim for which φ is the usual translation of '...' into the encoding-free formulas of classical logic and φ^* is just φ with all the terms and predicates of T indexed to T . For any primitive or defined individual term κ used in theory T , the object κ_T is identified with the abstract object that encodes all and only the properties F and satisfying the formula 'In theory T , $F\kappa_T$ '.

This thesis appears to be a version of mathematical Platonism, for it attempts to justify the belief that mathematics actually describes a realm of abstract objects. Zalta offers a way of formally constructing such a realm of platonic objects resulting from the reification, through λ -propositions, of some propositional properties about a metaphysical realm inhabited by abstract objects.

According to Objects Theory, natural language attributes properties even to objects that fail to exist. Objects are abstract or concrete depending on the modes of predication for properties, encoding and exemplifying. They attribute different kinds of being to objects and locate them in different places: concrete objects exist in the actual or in other possible worlds, while abstract objects exist in every possible world. Even if the existence of objects is dependent on the existence of theories, in their turn dependent on the existence of their author, this does not mean that an abstract object exist in some but not in other possible world, independently from the existence of its author.

Indeed, since mathematicians made those choices that leded them to formulate those theories, their theories are not only actual mathematical theories but, *a fortiori*, possible mathematical theories. It is not a contingent fact that it is possible that they made those

choices, it is only contingent as to which choices they made. The class of abstract objects remains the same in every possible world.

Zalta's idea is that, in a world in which no one have ever formulated a theory T , the objects of T exist as abstract objects, but they are not also mathematical objects. In this way, the class of abstract objects remains the same from world to world, while the class of mathematical objects may be contingent as to which abstract objects they are, but at a deeper level, it is not contingent that they are abstract objects.

Some objections can be moved against the ontology proposed by Objects Theory.

First, the encoding relation and the consequent kind of existence can seem suspicious. Indeed, abstract objects exist in a way that is both weaker and stronger than concrete objects: it is weaker because it tolerates the existence of impossible objects, such as the square circle, kind of objects philosopher usually try to exclude from ontology. But is also stronger, since it predicates that abstract objects exist in every possible world, in an eternal, incorruptible place. Things are different for mathematical objects: if someone authors a theory T stating that some abstract objects encode some mathematical properties, then those abstract objects are also mathematical. If no one formulates the theory T , the abstract objects encoding the properties exist, but are not mathematical.

Recall that mathematical objects encode all and only the properties the relevant theories attribute them. But the properties the theories attribute them depend upon the authors of the theories. This seems to mean that mathematical objects aren't mind-independent entities, because they exist thanks to the very act of mathematicians' minds postulating mathematical properties and objects that encode those properties.

Actually, Zalta denies such a conclusion and guarantees the mind-independence and objectivity of mathematical objects through *a priori* comprehension principle.

Mathematicians author mathematical theories through an *a priori* device that guarantees objectivity and apriority to the abstract objects with which mathematicians get acquaintance.

Moreover, the impossibility to modify mathematical theories could go against mathematical history. Imagine that a group of mathematicians formulates a theory T^1 . Some mathematical abstract objects will encode the properties the theory assigned them. Now, if T^1 turns out to be inconsistent, mathematicians will correct it and formulate a new theory, say T^2 , with new mathematical abstract objects encoding the new properties T^2 assigned to them. In this case, Objects Theory will not ask for the revision of the objects of T^1 in the object of T^2 , but it will simply point to new objects, making thus possible to speak again of T^1 , even if it turned out to be inconsistent. Recall that in Zalta's theory, consistency is a necessary condition for theories. This is because, if a theory is inconsistent, every formula is a theorem of it. Therefore, any object a of the theory will encode any possible propositional property. But there is at most one such object a . If every inconsistent theory has the same theorems, any inconsistent theory will have the object a , because it encodes every possible propositional property. Therefore, if a theory turns out to be inconsistent, mathematicians were wrong at the very beginning, when they define it as a theory.

Linsky and Zalta insisted on this line of thought in the following quotation from 'Naturalized Platonism vs Platonized Naturalism', page 21 and 22:

Consider [...] the situation in which two mathematicians 'disagree' about whether the Axiom of Foundation is 'true'. It seems clear [...] that mathematicians are simply talking about different sets. The appearance of disagreement is explained by the common vocabulary. What each has in

mind is perfectly real, but each party to the disagreement mistakes their limited portion of reality for the whole of reality. [...] So a mistake about the objects of a theory is *not* a successful discovery of a truth about some different objects.

According to Zalta's perspective, the work of Philosophy is exhausted by the formulation of a coherent and expressive description of such objects and the metaphysical realm they could live in. Asking whether these objects actually populate this realm or not, exceed the very possibility of philosophical inquiries. What is still within such possibilities is the delivery of a consistent and trustworthy epistemology for mathematical objects.

In the next section, I will evaluate how Objects Theory attempts to provide access and reliability to mathematical knowledge.

4.5 Epistemology

Zalta faces the problem of mathematical knowledge only at the very end of his major work, *Abstract Objects: An Introduction to Axiomatic Metaphysics*. Even there, he only suggests further line of research in epistemology for Objects Theory, without explicitly providing an epistemological account for abstract objects. Epistemology for Objects Theory is more deeply analyzed in 'Naturalized Platonism vs Platonized Naturalism'. But, again, even there Zalta seems to suggest that he would rest content with a simple postulation of an appropriate relation of acquaintance with abstract objects to guarantee reliable knowledge of them. Indeed, at page 156 of *Abstract Objects: An Introduction to Axiomatic Metaphysics*, he declares:

The idea is that we analyze worshipping Zeus, searching for the fountain of youth, thinking about Hamlet, etc., in terms of acquaintance with these objects plus different intellectual (possibly propositional) attitudes we adopt toward them.

In ‘Naturalized Platonism vs Platonized Naturalism’, page 547, Zalta precises:

Knowledge of particular abstract objects does not require any causal connection to them, but we know them on a one-to-one basis because *de re* knowledge of *abstracta* is by description. All one has to do to become so acquainted *de re* with an abstract object is to understand its descriptive, defining condition, for the properties that an abstract object encodes are precisely those expressed by their defining condition. So our cognitive faculty for acquiring knowledge of *abstracta* is simply the one we use to understand the comprehension principle. [...] The comprehension and identity axioms of Principled Platonism are the link between our cognitive faculty of understanding and abstract objects.

The comprehension principle is synthetic, since it’s not part of the meaning of ‘abstract’, ‘encodes’, and ‘property’ that for every condition on properties there is an abstract object that encodes just the properties satisfying the condition. And it is also *a priori*, since it can’t be confirmed or refuted by some empirical evidence.

If knowledge of abstract objects comes from the description of them provided by their authors, then understanding the definition of an abstract object is enough for reaching acquaintance *de re* with it. Moreover, Linsky and Zalta also suggest that such knowledge is to be considered as *de re* knowledge, with interesting consequences for

theory of truth. Indeed, mathematical knowledge through Objects Theory will be *de re* knowledge obtained via synthetic *a priori* principles.

Zalta defines then the notion of truth in a theory in the following way: a proposition p is true in a theory T ($T \models p$) if and only if T encodes the property ‘being such that p ’.

Formally:

$$T \models p =_{df} T[\lambda y p]$$

‘In theory T , a is F ’ is read as ‘the proposition that a exemplifies F is true in theory T ’, or that ‘ $T \models Fa$ ’. Now, since the comprehension principle is synthetic *a priori*, so are also the true propositional properties of mathematical theorems, just because they are treated as encoding claims. Indeed, mathematical theorems turn out to be true because there actually are mathematical objects with the properties theorems ascribe them. Then, Linsky and Zalta have an argument for the aprioricity of mathematical knowledge, and knowledge of *abstracta* in general, while knowledge of concrete objects will be *a posteriori*.

The kind of acquaintance relationship Zalta proposes for abstract objects has some interesting qualities: first, it has to be a-causal, since abstract objects have no relationship with the possible worlds. But, there could be some kind of causal relationship with the authors of mathematical theories, providing the definition that allows for the acquaintance relation.

Second, acquaintance relationship with abstract objects is different from acquaintance relationship with properties. Indeed, Zalta idea is that no matter what world is the actual one, it is always possible to recognize whether or not an object in that world possesses a property or not.

Even if no one has ever thought about the object that encodes both the property of being square and the property of being round, everyone is acquainted with the two properties alone. Getting acquainted with an object, through the description of its properties, is then a further step in abstraction.

The third interesting quality of acquaintance relationships is that, since the only proper knowledge of abstract objects is *a priori* knowledge, getting acquainted with the properties objects are defined to encode is sufficient for gathering necessary true knowledge about them.

In conclusion, it becomes clear why Zalta reserves so little interest for epistemological questions: according to Objects Theory, there is nothing extraordinary in gathering knowledge of abstract objects. No special faculty is needed to attain knowledge of mathematical objects and relations or to recognize the truth of mathematical statement. The only faculties required are the ones for understanding language and drawing inferences.

[F]or all I have said, it may well be possible to characterize an 'objective' notion of correctness, one which is relative neither to a conception of logical space nor to the aims of a particular community.

*I don't know how to do so myself,
but would be delight if it could be done*

RAYO, AGUSTÌN, (2013) *The Construction of Logical Space*, page 63

Chapter 5: Trivialism

5.1 Metaphysicalism and Compositionism

Agustin Rayo developed a new theory in philosophy of mathematics. He takes the step for formulating its new approach, called 'Trivialism, by debating two philosophical positions, metaphysicalism and compositionism. I will follow its path and introduce first metaphysicalism and compositionism, and only thereafter Trivialism and it relates with metaphysicalism and compositionism.

Metaphysicalism is the view according to which there actually is a privileged way in which carving up the world. This idea of carving up the world is quite classical in Platonism, from Plato's *Phaedrus* (265 d-e) to Gottlob Frege's *Grundgesetze der Arithmetik*. Rayo doesn't point to some particular philosopher or tendency: rather, he explicitly admits that there could be no metaphysicalist ever. But, if there is someone who believe that the truth of an atomic statement depends upon the existence of a certain kind of correspondence between the logical form of the statement and the metaphysical structure of reality, he will be a metaphysicalist. Rayo suggests to understand different ways of carving up the world as different interpretation given to the way things stay in reality.

According to metaphysicalism, even if it is admitted that a given fact can be carved up into constituents in more than one way, there is one and only one of them that corresponds to how facts really stand in the actual world.

This definition reveals the double-headed nature of metaphysicalism: it is both a linguistic and a metaphysical thesis, in which the assumption of the existence of one and

privileged metaphysical structure imposes a constraint on the way reference can successfully work. But, Rayo points out, metaphysicalism seems to suggest that the endorsement of one rather than another formulation of the same content is based on considerations that have no consequences on Metaphysics. Rather, this occurs for rhetorical reasons, such as style or effectiveness of communication or also conventions. But obviously such considerations have no consequences on views about metaphysical structure, once statements aren't considered to be true only if their logical form is in correspondence with the metaphysical structure of the world.

There also is a brasher argument: metaphysicalism endorses that there actually is one and only one objectively correct way in which the world can (objectively) be. Therefore, it must exist something over and above the syntactic properties of the different ways to interpreting the world. But if so, according to metaphysicalism, two statements with different logical form can't pick out the same part of logical space, contrary at least to classical philosophy of language and intuition: same truth conditions pick out same ways the world to be, regardless to rules of the languages in which the two logical forms are expressed, or even to the very languages. This argument will have a fundamental role in Rayo's philosophy, especially in his idea of 'just is'-statements.

Rayo suggests to reject metaphysicalism, since it rules out the possibility of existence of a plurality of way for the world to be carved in because of merely syntactic considerations. Instead, Rayo proposes to embrace non-objectiveness, the thesis that the way the world is represented depends only upon the way the world is described. And it is particularly so in the case of philosophy of mathematics, because of the very nature of mathematical entities. Ultimately, what Rayo is stating here is that there can't be an

objective language-independent fact of the matter about whether there are numbers or not.

The second position is compositionism. Compositionism provides a way to analyse object-talk, defined as a system of singular terms and the corresponding variables and quantifiers. Rayo explains compositionism as the conjunction of two constraints at page 14 and 15 of *The Construction of Logical Space*⁶⁰. The first constraint is:

Singulartermhood: the following three conditions are jointly sufficient for an expression t to count as a genuine singular term:

1. Syntax: t behaves syntactically like a singular term: it generates grammatical strings when placed in the right sorts of syntactic contexts.
2. Truth conditions: truth conditions have been assigned to every statement involving t that one wishes to make available for use.
3. Logical Form: this assignment of truth conditions is such as to respect any inferential connections that are guaranteed by the logical forms of the relevant statements. In particular:

If φ and ψ have been assigned truth conditions, and if ψ is a logical consequence of φ (that is, if logical form guarantees that ψ is true if φ is), then satisfaction of the truth conditions assigned to φ is at least as a strong requirement on the world as the satisfaction of the truth conditions assigned to ψ .

Second constraint is:

⁶⁰ RAYO, AGUSTÍN, (2013), *The Construction of Logical Space*, Oxford University Press, Oxford;

Reference: Assume t satisfies Conditions 1–3 above. Then the following additional condition is sufficient for t to have a referent:

4. True Existential: the world is such as to satisfy the truth conditions that have been associated with the statement ' $\exists x(x = t)$ ' (or an inferential analogue thereof).

Roughly, for a term to be non-empty it is sufficient that the truth conditions assigned to it are satisfied. Accordingly, a singular term t , specified thanks to the three constraints, is non empty if the truth conditions assigned to $\exists x(x=t)$ are satisfied.

Rayo's point here is that the ability of languages to use singular terms makes possible to describe the world in terms of objects. That is, an atomic statement can be true even if there is no correspondence between its logical form and the metaphysical structure. As a consequence, languages with singular terms explicit the existence of objects, but this doesn't mean that there wouldn't be object if languages weren't able to express singular terms. The existence of objects regards exclusively non-linguistic parts of reality. Therefore, a language includes singular terms and quantifiers ranging over singular-term-positions just because they are useful in specifying truth conditions.

Rayo explains his account by mean of the reformulation of a famous example from Frege *Grundgesetze der Arithmetik* at page 15 and 16 of Rayo (2013):

[I]magine the introduction of a new family of singular terms 'the direction' of a ', where a names a line. The only atomic statements involving direction*-terms one treats as well-formed are those of the form 'the direction' of $a =$ the direction* of b ', but well-formed formulas are closed under negation, conjunction and existential quantification. A statement φ is

said to have the same truth conditions as its nominalization $[\varphi]N$, where nominalizations are defined as follows:

- $[\text{'the direction}^* \text{ of } a = \text{the direction}^* \text{ of } b']N = \text{'}a \text{ is parallel to } b\text{'}$.
- $[\text{'}x_i = \text{the direction}^* \text{ of } a']N = \text{'}z_i \text{ is parallel to } a\text{'}$.
- $[\text{'}x_i = x_j']N = \text{'}z_i \text{ is parallel to } z_j\text{'}$.
- $[\text{'}\exists x_i(\varphi)\text{'}]N = \text{'}\exists z_i([\varphi]N)\text{'}$.
- $[\text{'}\varphi \wedge \psi']N = \text{the conjunction of } [\varphi]N \text{ and } [\psi]N$.
- $[\text{'}\neg\varphi']N = \text{the negation of } [\varphi]N$.

It is easy to verify that every condition on the compositionalist's list is satisfied. Notice, in particular, that since $[\text{'}\exists x(x = \text{the direction}^* \text{ of } a)\text{'}]N$ is $\text{'}\exists z(z \text{ is parallel to } a)\text{'}$, and since every line is parallel to itself, all that is required for the truth conditions of $\text{'}\exists x(x = \text{the direction}^* \text{ of } a)\text{'}$ to be satisfied is that a exist.

The compositionalist is therefore in a position to claim that the sole existence of a is enough to guarantee both that the singular term $\text{'the direction}^* \text{ of } a\text{'}$ refers and that it refers to the direction^{*} of a . The reason resides in the linguistic stipulation endorsed by the compositionalist. Indeed, what the compositionalist claims is that the same fact can be fully described both by saying $\text{'}a \text{ is parallel to } b\text{'}$ or $\text{'the direction}^* \text{ of } a = \text{the direction}^* \text{ of } b\text{'}$.

Following Rayo's thought, the existence of lines is reason enough to guarantee the existence of directions^{*}, because for the direction^{*} of a to be self-identical is equivalent

to for the direction* of a to exist. And for a to be self-parallel is also equivalent to for a to exist. So, for the direction* of a to be self-identical *just is* for a to be self-parallel.

Nevertheless, Rayo specifies a few lines later that, in order to get this, all it takes for the direction of a (as opposed to the direction* of a) to exist is for a to exist. For in doing so, the assumption that the statement ‘the direction of a exists’ has the same truth conditions as the statement ‘ a exists’ is all is needed.

From the conjunction of his analysis of metaphysicalism and compositionism, Rayo develops a trivialist theory of reference that basically comes from the rejection of metaphysicalism and the embracement of compositionism.

Metaphysicalism is to be rejected for at least two reasons: first, it makes a difference where there is none, by believing that differences in form lead to differences in content. Secondly, and consequently, there can’t be a privileged metaphysical structure or an objective way the world is, since the notion of singularterm-hood provided by languages and their uses is unconstrained and doesn’t allow speaking of objects in a way that guarantees 100% successful reference.

In Rayo (2013) the approach he had already proposed in several papers is systematically exposed. This approach is Trivialist Platonism, an original position in Metaphysics and in Philosophy of Mathematics. Rayo’s theory is very expressive and, although its main focus is on philosophy of Mathematics, his approach can be applied to many other fields.

Because of its broadness and explanatory power, Trivialism can be seen from several different angles. In an interesting way, Trivialism echoes logical empiricists’ anti-metaphysical theses. Participating to their spirit of simplification and reductionism,

Rayo proposes to reject the Quinean definition of analyticity for mathematical statements: mathematical truths do not demand anything from the world to satisfy their truth conditions; therefore, they are analytically true (see section 5.3 for more details on Rayo's critique to analyticity).

Accordingly, there is no way the world is that can affect the truth conditions of analytical statements, among which there are mathematical statements. But if so, there is no reason for questioning about the truth conditions of analytical and mathematical statements: they have trivial truth conditions. In the case of Mathematics, mathematical truths (or falsities) turn out to be true (or false) no matter how the world is.

Rayo's Trivialism has much broader applications than those in the Philosophy of Mathematics. Indeed, Rayo's approach suggests a pragmatic or common sense attitude toward ontological commitment, an intuitive epistemology and a compositional homophonic semantics that allows 'outscoping' the commitment beyond the domain of the actual world.

According to Rayo, for mathematical statements to have trivial truth conditions mathematical objects must necessarily exist. Worlds with no numbers are considered as inconsistent. At the same time, nothing is required of the world to satisfy mathematical truth.

Rayo's idea is, in short, that what is needed is just to understand whether the truth conditions of a mathematical statement are trivial and, in the case of a positive answer, the statement will result true. Evidently, determining whether the statement is trivial is not an easy task, above all considering the natural evolution of Mathematics. But in a Trivialist context, if a mathematical statement has trivial truth conditions (e.g. is a

necessary truth), than it is difficult to understand how it can turn out to be wrong, or false. In order to doing so, Rayo needs to provide an account of logical space and of the ‘just is’-statements that are supposed to construct it.

5.2 Constructing Logical Space

Rayo’s theory starts from the consideration that there are ways for the world to be and ways for the world to be represented. Such a distinction is fundamental in its consequences and is the proper subject of Rayo’s investigation. According to him, a statement is a way in which the world can be represented. Setting forth a statement makes a distinction amongst ways for the world to be, and singles out one side of this distinction as the true side. Therefore, a statement is true if it singles out the region that corresponds to the way the world actually is. This conception of truth carries with it the notion of logical space, being the set of all distinctions that describe the way the world actually is.

These distinctions are dichotomies, since they cut the ways the world could be in two different parts. For example, one such a distinction could be the distinction between Venus to be Hesperus and Venus not to be Hesperus. Making this distinction entails to divide every possible way the world could be in two parts: on one side all the ways in which Venus is Hesperus, on the other all the ways in which Venus isn’t Hesperus. And once the distinction is made, it is possible to ask which part in which the world could be there actually is.

According to Rayo, both commitment to and truth of everyday, scientific and philosophical discourse depend on the part that happens to be actualized. Indeed, the

statement that asserts Venus to be Hesperus turns out to be true or false depending on the way the world actually is. A set of sides of distinctions characterizes a world. So, to be true is to be committed with the side of the distinction that represents the world the way it actually is.

Logical space is built up by means of the acceptance of such distinctions. But also by understanding if and when two or more statements coincide. For example, ‘Venus is Hesperus’ and ‘Venus is Phosphorus’ are two different ways in which the world is actually represented. But the distinction between Venus to be Hesperus and Venus not to be Hesperus, and Venus to be Phosphorus and Venus not to be Phosphorus, cuts only apparently the ways the world could be in different parts. Indeed, ‘Venus to be Hesperus’ and ‘Venus to be Phosphorus’ are two different ways of representing the same side of the distinctions. For Venus to be Hesperus just is for Venus to be Phosphorus.

This technique of distinguishing between ways for the world to be and ways for the world to be represented appeals to the notion of ‘just is’-statement, who associates different ways for the world to be represented that have the same truth conditions.

An example from the first page of *Neofregeanism Reconsidered*⁶¹ can help understanding Rayo’s point:

Consider the creation of the world. In the first six days God created light, oceans, animals, planets and everything. On the seventh day, instead of resting, she created mathematical objects. So it is possible to imagine a world with no mathematical objects: a world in which God rested at day seven and everything remain the same from day six.

⁶¹ RAYO, AGUSTÍN, (2011), “Neofregeanism Reconsidered”, in EBERT, PHILIP, ROSSBERG, MARCUS, (2014) *Abstractionism in Mathematics*, Oxford University Press, Oxford.

The point is that according to such a creation myth, God had had to do something more to create mathematical objects, something that wasn't already there when she finished to create light, oceans, animals, planets and everything. This position is wrong according to Rayo. He proposes an alternative creation myth, in which for the number of the planets to be eight just is for there to be eight planets. There is nothing God have to add to the planets to let them be eight planets. When God created eight planets she already made it the case that the number of the planets was eight.

'Just is'-statements are the cornerstones of logical space. But the process through which we accept 'just is'-statements depends upon the best available hypotheses concerning the way the world actually is. Indeed, to obtain knowledge of the world, according to Rayo, is to define the set of 'just is'-statements accepted. This process is both empirical, based upon observations of the way the world actually is, and logical, based on the set of statements that constitutes the truths of pure logic.

5.3 'Just is'-statements

Two statements and one occurrence of the 'just-is' operator compose a 'just is'-statement. This operation asserts that the statement on the right and the statement on the left of the operator are truth conditionally equivalent: that is, they have the same truth conditions. 'Just is'-statements are therefore symmetric just like identity statements, since they are true if their left-hand and right-hand parts actually refer to the same thing. But, they are not to be seen as anything like real definitions or having any metaphysical significance. Moreover, Rayo spends a big effort in stating that they are not analytic or knowable a priori.

In order to explicit the non-analyticity of ‘just is’-statements, Rayo (2013), page 35 and 36, proposes a new perspective on the debate between Carnap⁶² and Quine⁶³ in regards of meaning postulates.

Carnap argued that true statements can be divided into two groups, the ones that are true in virtue of the meaning of their constituent vocabulary, and the ones that are true in virtue of the way the world turns out to be. The former are analytical *a priori*, while the latter are synthetic *a posteriori*. But, famously, Quine objected that the notion of meaning postulates is not robust enough to guarantee the dichotomicity of Carnap distinction.

According to Rayo, if Quine was correct in stating that we have no robust enough notion of meaning postulates, Carnap was correct too in individuating such a distinction. But above all, Rayo is convinced that the correct criterion isn’t determined by the distinction between analytic and synthetic: as Quine famously showed in *Two Dogmas of Empiricism*, things are way more complicated that in Carnap’s proposal. The problem doesn’t rest in the notion of meaning postulates, but in the criteria by means of which the distinction was drawn. According to Rayo, the right way is rather the one that appeals to ‘just is’-statements in place of analyticity.

Here how Rayo shows why: since a set of statements is analytically consistent if it is logically consistent with the set of analytic truths, logical space is then the maximal analytically consistent sets. Instead, a statement is synthetic if some maximal

⁶² CARNAP, RUDOLF (1950-1956), ‘Empiricism, Semantics and Ontology’, *Revue Internationale de Philosophie*, 4: 20–40. Reprinted in *Meaning and Necessity*, Chicago: University of Chicago Press, 2nd edition, 1956, pp. 205–221;

⁶³ QUINE, WILLARD VAN ORMAN, (1951) “Two Dogmas of Empiricism”, *Philosophical Review*, 60, 20-43;

analytically consistent sets, but not others, logically entail it. This is the purpose of scientific inquiries: to shed a light on the truth conditions of synthetic statements.

Moreover, Rayo suggests to consider a set of statements as metaphysically consistent if it is logically consistent with the set of true 'just is'-statements. Then, logical space turns out to consist of the sum of the maximal metaphysically consistent sets.

A meaningful statement is counted as non-trivial if some maximal metaphysically consistent sets, but not others, logically entail it. So, the purpose of scientific inquiries is further detailed: to shed a light on the truth conditions of non-trivial statements.

That is why the truth of a 'just is'-statement cannot be known *a priori*. Rather, scientific inquiry suggests which 'just is'-statements to accept, according to costs and benefits considerations. Indeed, is precisely the acceptance of a 'just-is'-statement that reduces the size of logical space. At the same time, it increases the number of ways the world is represented with which the ways the world actually is match.

'Just is' statements can do the work of both analytics and synthetics statements, since they can be derived both from logical and empirical observations, as Rayo (2013) explicitly states at page 43:

Even if none of the decisions one makes in adopting a family of 'just is'-statements is wholly independent on empirical considerations, some decisions are more closely tied to empirical considerations than others. And when it comes to 'just is'-statements corresponding to logical truths, one would expect the focus to be less on particular empirical matters and more on the question of how to best organize one's methods of inquiry. So there is room for a picture whereby an epistemically responsible subject can accept

‘just is’-statements on the basis of considerations that aren’t grounded very directly in any sort of empirical investigation.

To briefly summarize, it turns out that, in Rayo’s account, is the process of rejection and acceptance of ‘just is’-statements that shapes the conception of logical space in which theories are developed and, more importantly, in which they reveal to be true or false.

Therefore, the notion of truth presupposes the notion of logical space. But the ‘just is’-statements accepted build the logical space, delivering a representation of the world. This representation has the role of drawing the borderline of consistent scenarios, leaving outside inconsistency.

Examples of ‘just is’-statements are:

SIBLINGS: for two people to be siblings just is for them to share a parent.

WATER: For something to be water just is for it to be H₂O.

DINOSAURS: For there to be no dinosaurs just is for the number of dinosaurs to be zero.

Rayo shows that, for example, the acceptance of ‘just is’-statements like DINOSAURS allows considering question like ‘I can see that there are no dinosaurs. But is it also true that the number of the dinosaurs is Zero?’ as resting on false presuppositions. Indeed, DINOSAURS states that there is no difference between the non-existence of dinosaurs and dinosaurs being counted by the number Zero.

But the most important ‘just is’-statement for the purpose of Trivialism for philosophy of mathematics is:

NUMBERS: There being exactly n Fs just is for the numbers of Fs to be n .

NUMBERS is schematic. One of its instances is DINOSAURS, since it states that there is no difference between there being exactly zero *Fs* (say dinosaurs) and the number of *Fs* (say dinosaurs) being zero.

Evidently, NUMBERS works only if the meaning of ‘there are exactly 0 *Fs*’ counts as the same of the meaning of ‘there is no *Fs*’. But if it is so, ‘there are *n Fs*’ has to be read as a statement quantified through numeric quantifier, whom are defined by recurrence involving numbers. The validity of NUMBERS is fundamental also because the Zero Argument follows directly from it:

THE ZERO ARGUMENT: assume, for reductio, that there are no numbers.

By NUMBERS, for the number of numbers to be zero just is for there to be no numbers. So, from the assumption follows that the number of numbers is zero. So zero exists. So at least a number exists. Contradiction.

This argument is very important, since it leads to the first trivialist mathematical Platonism, on the number zero, and proves that zero trivially exists. Obviously, NUMBERS can be used to show that each number must trivially exist since the world has to be consistent. A world with no numbers is considered absurd by Rayo’s theory.

To be more precise, thanks to the equivalence, expressed by NUMBERS, between using noun numerals and adjective numerals, Rayo demonstrates that numbers exist, because their existence is trivial. Once numbers belong to a realm that isn’t casually related with the concrete world, nothing is required from the world for numbers to exist. And it is in this very point that the adoption of *Independence* is made explicit in Rayo’s work.

As a consequence, from NUMBERS follows that mathematical Platonism is true because the existence of numbers is trivially true. That is to say that numbers trivially

exist if it is trivial to assume a 'just is'-statement from which follows that numbers trivially exist.

As it is quite evident, the notion of 'just is'-statements is crucial to demonstrate such a thesis. Indeed, in adopting a family of 'just is'-statements one is, according to Rayo, making decisions that are closely dependent on empirical observations and on the logical structure and organization of the world as perceived or relatively to a particular set of tasks or analysis.

Therefore, the use of Logic and Mathematics in the very act of discriminating the accepted 'just is'-statements from the non-accepted ones is fundamental. For example, endorsing classical logic, $p \leftrightarrow \sim\sim p$ is a logical truth. Therefore, it determines the acceptance of the 'just is'-statement:

BIMPLICATION: For p to be the case just is for $\sim\sim p$ to be the case.

For example:

RAIN: For raining to be the case just is for the negation of the negation of raining to be the case.

But the point is that the acceptance of $p \leftrightarrow \sim\sim p$ will not find the agreement of friends of intuitionistic logic, therefore they will not accept BIMPLICATION and all its instances.

As it should be clear by now, Rayo's idea is that the acceptance of 'just is'-statements is never independent on empirical matters, but some 'just is'-statements are more tied to logical and mathematical aspects than others. It is particularly so with regard to the 'just is'-statements that concern Logic and Mathematics. These 'just is'-statements are the ones that allow an epistemically responsible subject to believe that numbers exist and mathematical statements are true, even without empirical supports.

The crucial role of ‘just is’-statements for Trivialism resides in their use in outscoping techniques. ‘Just is’-statements offer a new mean for interpreting truth conditions, such as that p is true at world w if and only if w is such that the truth conditions for p obtain. Rayo suggests to interpret this criterion as supplying the conditions a world have to satisfy for p to be true as far as the non-mathematical facts are concerned, thanks to the use of a particular compositional semantics (section 5.4).

According to Rayo, mathematical vocabulary works as a metatheoretical test that ensures the satisfaction of the previous criterion. In trivialist semantics, the test is to be performed outside the scope of the actual world, henceforth represented by $[\dots]_w$ (read ‘it is true at w that ...’). This is the outscoping technique, that shows that there is no need to the resources used to perform the test to be present in w , in particular in the case of pure mathematical ‘just is’-statements.

In doing so, Rayo offers a new reading of the ‘just is’-statements:

For ϕ to be true at w just is $[\phi]_w$

Thanks to the outscoping technique, a trivial semantic clause is assigned to every truth of pure Mathematics, so that nothing remains in the scope of $[\dots]_w$. If there is no non-mathematical vocabulary to remain within the scope of $[\dots]_w$, application of the semantic clauses yields the result that the statement is true at a world w just in case w satisfies a metalinguistic formula in which *all* the vocabulary has been outscoped. For example, in

$$2+2=4$$

$[\dots]_w$ does not occur, and therefore it has no free variables. But a formula with no free variables is satisfied by all objects if it is true, and by no objects if it is false.

Since the metalinguistic formula ‘ $2+2=4$ ’ is, in fact, true, it will be satisfied by all objects, including w independently of what is w . But if it is satisfied by any arbitrarily w , nothing is required of w in order for the statement $2+2=4$ to be true at w .

As a result, the semantics clause for ‘ $2+2=4$ ’ is:

‘ $2+2=4$ is true at w if and only if $2+2=4$ ’

5.3.1 Connections of ‘just is’-statements

In order to complete the definition of ‘just is’-statements and provide a reliable account of the truth condition of them, Rayo (2013) analyses the connections ‘just is’-statements have with the notions of possibility, inconsistency, why-closure and sameness of truth conditions.

1. Possibility:

Rayo appeals to distinctions in the line of the ones made by Kripke in *Naming and Necessity*⁶⁴ between types of possibilities. Possibility can be *de mundo* or *de repraesentatione*. The former is a property of the ways for the world to be, while the latter is a property of the ways for the world to be represented. In this sense, logical possibility is *de repraesentatione*, while metaphysical possibility is *de mundo*. Rayo’s point is then that, since metaphysical possibility is the most inclusive form of possibility *de mundo* there is, going beyond it means falling into absurdity.

This notion of metaphysical possibility is strictly connected with Rayo’s account of ‘just is’-statements because a statement describes a metaphysically possible scenario if and

⁶⁴ KRIPKE, SAUL, (1980) *Naming and Necessity* Harvard University Press, Cambridge MA;

only if it is logically consistent with the set of true 'just is'-statements. Therefore, the truth conditions of a 'just is'-statement like 'for it to be the case that φ *just is* for it to be the case that ψ ' are determined by the acceptance or rejection of its corresponding modal statement ' $\Box(\varphi \leftrightarrow \psi)$ '.

2. Inconsistency:

Rayo suggests to define the kind of consistency that regards the sets of statements that are consistent with the true 'just is'-statements as metaphysical consistency. Here again Rayo is appealing to *de mundo* and *de repraesentatione* distinctions. Consistency *de mundo* requires that a statement isn't representing the world as satisfying an absurdity. Consistency *de repraesentatione* requires only that a statement isn't representing the world as satisfying an *a priori* absurdity.

This way of putting things regarding absurdity is strictly connected with Rayo's account of 'just is'-statements, because a statement is taken to represent the world as satisfying an absurdity if and only if it is logically inconsistent with the set of true 'just is'-statements. But this, together with the previous considerations about possibility, means that to go beyond metaphysical possibility isn't only to fall into absurdity but also to fall into inconsistency *de mundo*. The decision of accepting or rejecting a 'just is'-statement is, then, a decision about where to place the limits of absurdity in the way the world could be, and not only in the way the world could be represented.

3. Why-closure

Imagine that someone can actually see what it takes to satisfy the truth conditions of a statement φ , but aims to attain a richer understanding of why the world is such as to satisfy its truth conditions. Rayo suggests to treat such a statement as why-closed, so that there is no available way of making sense of the question ‘Why is it the case that φ ?’.

There is a connection between why-closure and ‘just is’-statements, because a statement is why-closed if and only if it is a logical consequence of the set of true ‘just is’-statements. Indeed, once the ‘just is’-statement ‘for φ to be the case *just is* for ψ to be the case’ is accepted, the corresponding biconditional ‘ $\varphi \leftrightarrow \psi$ ’ turns out to be why-closed. Rayo means that, if one accepts a ‘just is’-statement, he is thereby relieved from the need to explain certain facts. At page 54 of Rayo (2013), he explains this by mean of an example:

Suppose it is agreed on all sides that Hesperus (and Phosphorus) exist.

Someone says: ‘I can see as clearly as can be that Hesperus is Phosphorus; what I want to understand is *why*.’ It is not just that one wouldn’t know how to comply with such a request—one finds oneself unable to make *sense* of it. The natural reaction is to either find a charitable reinterpretation of the question (‘Why does one planet play both the morning-star and the evening-star roles?’) or reject it altogether (‘What do you mean *why*? Hesperus *just is* Phosphorus’.)

4. Sameness of truth conditions

Rayo defined a statement's truth conditions as a requirement on the world. If such a requirement is satisfied by the world being as the statement represents it to be, the statement is true. If it is not satisfied, the statement is false. But if the statements φ and ψ have the same truth conditions, the 'just is'-statement 'for it to be the case that φ *just is* for it to be the case that ψ ' is accepted. If the assumption that a statement's truth conditions fail to be satisfied leads to absurdity, the statement is said to have trivial truth conditions. And when it is so, there is no requirement on the world that can satisfy the statement's truth conditions.

The notions of logical space, 'just is'-statements and truth conditions are very correlated and fundamental for the theoretical architecture of Trivialism. These three notions constitute the treble core of Rayo's theory: truth presupposing logical space, logical space being built by 'just is'-statements, 'just is'-statements being true if composed by two sides with the same truth conditions.

Moreover, according to Rayo, to state something is to make a distinction between regions of logical space, to single out one region and to affirm the statement to be true if the region of logical space it singles out actually represents the world the way it is. Consequently, the distinction between truth and falsity turns out to be just the distinction between regions of logical space that include or not the world the way it is.

Now, the truth conditions of a contingent statement are satisfied by the way the world actually is. But for 'just is'-statements, truth conditions are entirely determined by the conception of logical space endorsed.

This is the case for mathematical statement. Rayo proposes to assign to true pure mathematical statements a trivial semantic clause, and to false pure mathematical statements an impossible semantic clause. He does so in order to achieve the result that true pure mathematical statements are true no matter how the world actually is, while false pure mathematical statements wouldn't be satisfied in whatever possible world. To be more explicit, truth of a pure mathematical statement in a world doesn't depend on any other non-mathematical statements' truth conditions in that world.

Some concerns can rise in regard with the definition of the notion of objective truth, since truth or falsity of 'just is'-statements will be determined by both empirical and logical considerations. According to Rayo, the idea of objectively truth or correctness is hard to define. Indeed, if the truth or falsity of 'just is'-statements is defined by the construction of logical space, and if there are many possible constructions, which one delivers the objectively correct one?

Rejecting metaphysicalism, Rayo's response is that there is no objectively correct construction of logical space. Rather, the question of which set of 'just is'-statement is correct makes sense only in the context of a particular purpose. In a few words, accepting a set of 'just is'-statements is a practical matter that depends only on the possibility to achieve a fruitful and applicable theory.

The point is that, according to Rayo, there is no ways to succeed in satisfactorily define the notion of objectively correct or true. In addition, according to Rayo, the very fact that nothing is required of the world in order for the truth conditions of mathematical statements to be met implies that knowledge of true pure mathematical statements is always relative to a particular conception of logical space. In this sense, the distinction

between true and false is just the distinction between regions of logical space that include or not the way in which the actual world is.

The ontological assumptions of Trivialism entails that, in extending the language, trivialists aren't extending or changing the world. The effect of extending the language is, rather, that trivialists acquire additional resources to describe the world.

In the case of pure Mathematics, trivialists acquire additional resources for express statements with trivially satisfiable truth conditions and terms that refer to objects to which trivialists were previously unable to refer.

5.4 Trivialist semantics

In chapter 3 of Rayo (2013), a trivialist compositional semantics for applied Mathematics is constructed.

The language is two-sorted first-order language with identity, L . It contains:

- the identity-symbol '=';
- arithmetical variables (' n_1 ', ' n_2 ', . . .);
- individual-constants ('0');
- function-letters ('S', '+' and '×');
- non-arithmetical variables (' x_1 ', ' x_2 ', . . .);
- constants ('Caesar', 'Earth');
- predicate-letters ('Human(. . .)', 'Planet(...)');
- the function letter ' $\#_v(. . .)$ '.

' $\#_v(. . .)$ ' is to be read 'the number of $v(. . .)$ '. It takes in its single argument-place a first-order predicate to form a first-order arithmetical term.

Let σ be a variable assignment and w be a world. $\delta_{\sigma,w}(t)$ is the denotation function assigning a referent to a term t relative to σ and w . Then $Sat(\phi, \sigma, w)$ is the satisfaction predicate that expresses the satisfaction of ϕ relative to σ in w . While $[\phi_w]$ express that ϕ is true at world w .

Denotation of arithmetical terms is defined thanks to:

1. $\delta_{\sigma,w}([n_i]) = \sigma([n_i])$
2. $\delta_{\sigma,w}('0') = \text{the number zero}$
3. $\delta_{\sigma,w}([S(t)]) = \delta_{\sigma,w}(t) + 1$
4. $\delta_{\sigma,w}([(t_1 + t_2)]) = \delta_{\sigma,w}(t_1) + \delta_{\sigma,w}(t_2)$
5. $\delta_{\sigma,w}([(t_1 \times t_2)]) = \delta_{\sigma,w}(t_1) \times \delta_{\sigma,w}(t_2)$
6. $\delta_{\sigma,w}([\#_{x_i}(\phi(x_i))]) = \text{the number of } z\text{s such that } Sat([\phi(x_i)], \sigma^{z/[x_i]}, w)$
7. $\delta_{\sigma,w}([\#_{n_i}(\phi(n_i))]) = \text{the number of } m\text{s such that } Sat([\phi(n_i)], \sigma^{m/[n_i]}, w)$

Denotation of non-arithmetical terms:

1. $\delta_{\sigma,w}([x_i]) = \sigma([x_i])$
2. $\delta_{\sigma,w}('Caesar') = \text{Gaius Julius Caesar}$
3. $\delta_{\sigma,w}('Earth') = \text{the planet Earth}$

Satisfaction in the system is guaranteed by the formulas below:

1. $Sat([\exists n_i \phi], \sigma, w) \leftrightarrow \text{there is a number } m \text{ such that } Sat(\phi, \sigma^{m/[n_i]}, w)$
2. $Sat([\exists x_i \phi], \sigma, w) \leftrightarrow \text{there is a } z \text{ such that } ([\exists y(y = z)]_w \wedge Sat(\phi, \sigma^{z/[x_i]}, w))$

3. $Sat(\lceil t_1 = t_2 \rceil, \sigma, w) \leftrightarrow \delta_{\sigma, w}(t_1) = \delta_{\sigma, w}(t_2)$ (for t_1, t_2 arithmetical terms)

4. $Sat(\lceil Planet(t) \rceil, \sigma, w) \leftrightarrow [\delta_{\sigma, w}(t) \text{ is a planet}]_w$ (for t a non-arithmetical term)

5. $Sat(\lceil \phi \wedge \psi \rceil, \sigma, w) \leftrightarrow Sat(\phi, \sigma, w) \wedge Sat(\psi, \sigma, w)$

6. $Sat(\lceil \neg \phi \rceil, \sigma, w) \leftrightarrow \neg Sat(\phi, \sigma, w)$

Trivialist semantics is built for the outscoping technique, an interpretation for mathematical discourse that does not lead to commitment to numbers. According to the system Rayo proposes, the truth condition of any given statement is a function of the truth conditions of non-mathematical statements of the language. This semantics has the advantage of mirroring perfectly trivialist thesis.

The first thing to note is that, in Rayo's semantics, arithmetic is assumed in the meta-theory. That is why he makes full and extensive use of mathematical vocabulary in the metalanguage and also why trivialist semantics is to be adopted only by someone who already accepts mathematical vocabulary.

The second aspect is that, according to Rayo's Trivialism, a statement's truth conditions depend on the set of consistent scenarios accepted. The actual world takes part of a consistent region of logical space, and the determination of truth in the actual world entails a specification of truth conditions for every statement in the language.

Thirdly, it is because mathematical vocabulary never occurs within the scope of $[\dots]_w$ that Rayo's semantics leads to trivial consequences: even though mathematical vocabulary is used to specify the satisfaction clauses, the terms in the range of w can be characterized entirely in non-mathematical terms.

The reason is that nothing is required of the world in order to satisfy the truth conditions of mathematical statements. All that is needed is to establish a connection between mathematical and non-mathematical descriptions of the world. In doing so, the consistent scenarios at which the number of the planets is zero turn out to be precisely the consistent scenarios at which there are no planets.

The outscoping technique allows mathematical vocabulary to always occur outside the scope of $[\dots]_w$. Outscoping limits the epistemic resources needed to know if a given arithmetical statement would be true at a world w , to knowledge of which non-mathematical predicates apply to which objects.

Consider the object-language statement $\#_x(\text{Planet}(x)) = 0$. In trivialist semantics, the statement will be true at a world w just in case w satisfies the following metalinguistic formula:

the number of z s such that $[z \text{ is a planet}]_w = 0$

Since arithmetic is assumed in the meta-theory, all that is required of w in order for the metalinguistic formula to be satisfied is that it contains no planets.

In homophonic semantics, $\#_x(\text{Planet}(x)) = 0$ will be true at w just in case w satisfies the following metalinguistic formula:

$[\text{the number of } z\text{s such that } z \text{ is a planet} = 0]_w$

Since arithmetical vocabulary occurs within the scope of $[\dots]_w$, what is required of w in order for the metalinguistic formula to be satisfied is that it contains the number zero, and that, at w , zero numbers the planets.

But, if it is true that for the number of the planets to be zero just is for there to be no planets, then the two requirements on w will coincide. So it will be true that all that is

required of w to verify ' $\#x(\text{Planet}(x)) = 0$ ' is that it contain no planets.

Recall that in trivialist semantics arithmetical reasoning is used in the metatheory, hence, in proving the result, truth of arithmetic is assumed.

For the case of statements of pure arithmetic, Rayo (2013, page 87) proposes as an example the object-language statement ' $1 + 1 = 2$ '. He then notices that there is no non-mathematical vocabulary to remain within the scope of ' $[\dots]_w$ '. But if so, the resulting formula will have no free variables and will be satisfied by all objects if it is true, and no objects if it is false. Since the metalinguistic formula ' $1 + 1 = 2$ ' is true, it will be satisfied by every possible world. And this is exactly the result Rayo aims at reaching: an arbitrary truth of pure arithmetic turns out to be true at w independently of what w is like, and an arbitrary falsehood of pure arithmetic turn out to be false at w independently of what w is like.

This result seems very satisfactorily, since it maintains the a-causality, absolute generality and independence of mathematical objects and knowledge.

Concerns may be raised regarding the incompatibility of standard semantics and trivialist semantics. But there is no contradiction in stating that the truth conditions of a mathematical statement are accurately defined both by standard semantics and trivialist semantics, because the truth conditions assigned are the same.

In standard semantics, for the truth conditions of $1+1 = 2$ to be satisfied there must be numbers. In trivialist semantics, the truth conditions are satisfied no matter how the world happens to be. But if, as for Trivialism, the existence of numbers is a trivial affair, the commitment to numbers entailed by ' $1+1 = 2$ ' is trivial too and will be satisfied no matter how the world turns out to be.

5.5 Is it really trivial?

Rayo theorizes Trivialism as a strong and expressive philosophy of Mathematics, equipping it with powerful argument and astonishing simplicity.

The central thesis is that, if mathematical statements are true, then they are necessarily true. But the power and the originality of Rayo's argumentation rest in the way he manage to understand the notion of mathematical truth, as necessity free from commitments with the way the world could be.

For example, if a mathematical statement has trivial truth conditions, and therefore is a necessary truth, of course it cannot turn out to be false. But trivialist Platonism starts from the assumption that mathematical statements are true and reaches the conclusion that they are trivially true. Mathematical statements have trivial truth conditions because they carry commitment about mathematical objects, but commitment to mathematical objects isn't commitment at all, because they maintain their truth condition no matter how the world turns out to be.

One of the main challenges for Trivialism is certainly to explain what the point of mathematical knowledge is, as it deals with trivialities. Rayo (2013, chapter 7) proposes a detailed account of what is to have knowledge in general, and mathematical knowledge in particular.

In a trivialist perspective, nothing is required of the world in order for the truth conditions of a truth of pure mathematics to be satisfied. Since truths of pure mathematics are necessarily and trivially true, in every possible world, there is no way for a world to confirm or disconfirm any pure mathematical statement. Pure

mathematical statements are therefore something outside and above any possible world. But how such trivialities are known? Rayo suggests to bound the notion of ‘just is’-statements with the process of cognitive accomplishment. Indeed, accepting:

DINOSAURS: for the number of dinosaurs to be zero just is for there to be no dinosaurs

is a non-trivial cognitive accomplishment.

A cognitive accomplishment is the acquisition of information transfer abilities from one way for the world to be represented to another way for the world to be represented. And this is exactly what cognitive accomplishment in logic and mathematics consists in: the information-transfer abilities between different modes of presentation of a given region in logical space.

Indeed, learning mathematics isn’t trivial at all, but truths of pure mathematics are true throughout logical space. So cognitive accomplishments in mathematics cannot only consist of ruling out ways for the world to be, but is also the acquisition of information-transfer abilities.

As a consequence, Rayo suggests that knowledge is fragmented, in the sense that a subject can have access to a piece of information for some purposes but not others. In Rayo (2013, page 105), a formal model for cognitive accomplishments in logic and mathematics is presented through a model for fragmented cognitive state represented by the ordered-triple $\langle T, b, \alpha \rangle$ with T consisting in the *tasks* that the subject might be involved in. Each task in T will correspond to a different fragment in the subject’s cognitive state: b corresponds to the belief-function mapping each element in T to a ‘belief-state’; and α is the relation that expresses accessibility amongst members of T ,

while $\alpha(t_1, t_2)$ captures the fact that the belief-state corresponding to task t_1 is accessible for the purposes of carrying out task t_2 .

Rayo states that the cognitive accomplishments of a subject $\langle T, b, \alpha \rangle$ can be modeled as updates of the belief-function b , or of the accessibility relation α . Changes in b represent changes in the information that is available to the subject of knowledge for the purposes of a given task; changes in α represent changes in the subject's information-transfer abilities.

For example, Rayo shows that deductions are modeled as the acquisition of information-transfer abilities since they increase the range of the accessibility relation, α . And here he introduces an interesting distinction between pure and applied mathematics: in pure mathematics, deductions are acquisitions of information-transfer abilities that increase the accessibility between tasks aimed at answering language-related question concerning the truth of a particular statement; while in applied mathematics, deductions increase the accessibility between tasks not solely language-related.

Rayo (2013, page 109) explains this by mean of an example:

[S]uppose φ is a truth of pure *mathematics*. According to the mathematical trivialist, the truths of pure mathematics have truth conditions which are no less trivial than the truth conditions of logical truths. So [...] the difficulty in answering the question is *not* that it is hard to figure out how things stand regarding the question's subject matter. It has all to do with linguistic processing: it is a matter of determining whether the particular arrangement of vocabulary in φ results in trivial truth conditions.

The case for deductions is particularly interesting since it is constrained by the notion of consistency. Indeed, the problem of evaluating the consistency of an axiom system is largely a problem of deduction. Rayo is persuaded that the ability to evaluate the consistency of a mathematical system will turn on the sorts of deductions available within the system. So, the ability to evaluate the consistency of a system coincides with the ability to consolidate fragmented information regarding the system. In Rayo's (2013, page 114) words, if the truth of a statement φ is entailed by the truth of the axioms, then:

any cognitive system that represents the world as being such that the relevant vocabulary is used in a way that renders the axioms true and the logic classical will *thereby* represent the world as being such that the relevant vocabulary is used in a way that renders φ true.

5.6 Trivialism, Platonism and Nominalism

Agustín Rayo (2013) sets up the debate in philosophy of Mathematics along two fundamental axes: the existence of mathematical objects on one side, and the ontological commitment to mathematical statements on the other. Following the former axe, Platonism is the view that mathematical objects exist, while Nominalism is the view that there aren't mathematical objects.

Following the latter axe, Committalism is the view that mathematical statements carry commitments to mathematical objects. According to Non-Committalism, on the contrary, the assertion of mathematical statements doesn't carry any commitment to mathematical objects.

Thanks to a crosscheck analysis of these two aspects, Rayo presents four possible positions: Platonism & Committalism, Nominalism & Non-Committalism, Platonism & Non-Committalism and Nominalism & Committalism. The last two positions aren't very common, that is why Rayo focuses on the first two, Platonism & Committalism and Nominalism & Non-Committalism. These two are very popular also because they have a big advantage over the others: if Nominalism & Non-Committalism delivers a good epistemology, denying the existence of abstract and inaccessible objects, Platonism & Committalism is more perspicuous in accounting for the reliability mathematical discourse.

Rayo introduces Trivialism in this very framework, attempting to re-establish the debate in the framework of a Trivialism/non Trivialism dichotomy. In accounting for mathematical knowledge for example, Trivialism has an advantage both if associated with Platonism & Committalism and with Nominalism & Non-Committalism, because mathematical knowledge is obtained thanks to the same devices used to attain logical knowledge.

Rayo is also convinced that Nominalist & Non-Committalist can't be trivialist, because they don't endorse a doctrine that assigns to the logical structure of a mathematical statement a portion of the structure of reality. Contrariwise, Platonist & Committalist can endorse Trivialism, since they offer a matching ontology. But they can be trivialist only endorsing a particular type of Platonism, the one that Rayo calls 'Subtle Platonism'.

The difference between subtle and classical Platonists is that, while the former believe that a world with no mathematical objects is possible, and therefore consistent, subtle

Platonists believe that such a world would be neither possible nor consistent, because of the Zero Argument. As a consequence, while classical Platonists believe that mathematical objects exist contingently, subtle Platonists believe that they exist necessarily. But if numbers exist necessarily, then the existence of numbers is no longer a factual question: in trivialist words, nothing is required of the world in order for the truth conditions of a truth of pure mathematics to be satisfied.

According to Rayo, subtle Platonists have reason to embrace compositionism, especially because, once they would, there is no pressure for thinking that a mixed identity-statement such as ‘the number of the planets = Julius Caesar’ should have well-defined truth conditions. So Frege’s §66 of the *Grundlagen der Geometrie* is denied: even if the truth value of mixed entities isn’t established, mathematical concept can still being defined and characterized.

Trivialism adheres to subtle Platonism: its main aim is to account for the existence of objects and for the triviality of truth conditions for mathematical statements. Even if, from a mathematical point of view, no particular reason favors subtle Platonism over its rivals, Rayo prefers it for many reasons, the most interesting being that he believes subtle Platonism isn’t subject to Benacerraf’s Dilemma.

In spite of his main thesis and of his defense of subtle Platonism from benacerrafians jaws of death, there are several parts in Rayo’s argumentation that lead Trivialism far away from what Platonism is traditionally thought to be. Sometimes Trivialism winks directly even to nominalism. For example, there is more than one point in common between ‘just is’-statements and some classical nominalist tool, like paraphrases. Much of nominalist work was done in order to deliver a reliable way of rephrasing

mathematical statements so that it doesn't require a way the world to be for the statements to result true. And funny enough, is precisely a philosopher admittedly Platonist to find one of the most effective solution to a nominalist problem.

Rayo proposes to imagine a Platonist interested in understanding which is the nominalist content Nominalism wishes to associate with mathematical statements. The Platonist accepts mathematical vocabulary and can adopt trivialist semantics. This semantics is able to outscope the content of mathematical statements, reading just their trivialist content, from both the actual and the possible worlds.

As a result, trivialist semantics transforms our ability to engage in mathematical practice into an ability to identify uncontroversially determinate truth conditions for mathematical statements.

But the Platonist may also be interested in seeing if the operation of carving off trivial contents from mathematical statements delivers interesting result. The Platonist understands the outscoped semantic clause corresponding to each mathematical statement as a result of carving the mathematical part from the relevant claim.

Rayo also points out that two main results are obtained from the application of trivialist semantics with its outscoping techniques to Nominalism. First, Nominalism can use paraphrases obtained through 'just is'-statements for extract nominalist contents. And consequently, the notion of nominalist content achieves a rigorous definition that can be used to address serious philosophical concerns. For example, which mathematical claims are relevant for one's knowledge of the world. Indeed, a trivial mathematical claim like DINOSAURS imposes non-trivial demands on the actual world. But can also be used to deliver a rigorous theory of mathematical Nominalism.

To settle the argument in a trivialist way, I wonder if Rayo would accept the following ‘just is’-statement:

TRIVIALISM AND NOMINALISM: For the world to satisfy the nominalist content of a given mathematical statement just is for the world to satisfy the truth conditions that would be assigned to a statement by a trivialist semantic theory.

5.7 Trivialism and Neofregeanism

Trivialism winks to Logicism and Neofregeanism, so explicitly that it can be seen as a vindication of Logicism. Sure enough, Trivialism delivers a convincingly way to reach the result that pure mathematical truths have the same conditions of satisfiability of pure logical truths. They are trivially satisfiable. In the same way, pure mathematical falsehood have the same truth conditions of pure logical falsehoods, that is to say that there is no way to satisfy them.

The most interesting consequence of this new shed of trivialist’s light on Logicism is that, as Rayo points out at page 26 of *Nominalism, Trivialism, Logicism*⁶⁵:

Admittedly, one also gets the result that a truth of pure arithmetic can carry commitment to numbers. But because the existence of numbers is a trivial affair, there is room for thinking of numbers as ‘logical objects’, as in Frege’s *Grundgesetze*.

Rayo dedicated another recent work to Trivialism and Neofregeanism, *Neofregeanism Reconsidered*, published in 2011.

⁶⁵ Rayo, Agustín, (2014), “Nominalism, Trivialism, Logicism” in *Philosophia Mathematica*;

In this work, Rayo rephrases Neofregeanism's main thesis, Hume's Principle, and gives it a trivialist interpretation.

Once Hume's Principle delivers an implicit, satisfactory definition of $\#_x(F(x))$, its truth is knowable *a priori* and referents of number terms exist and are such as Platonism describes them.

As Rayo distinguishes between subtle and classical Platonism, he proposes to distinguish also between subtle and classical Neofregeanism. Subtle Neofregeanist wouldn't rest content with the acceptance of Hume's Principle, but they would also accept it as a 'just is'-statement:

$$\#_x(F(x)) = \#_x(G(x)) \equiv_{F,G} F(x) \approx_x G(x)$$

HUME'S PRINCIPLE: For the number of the *F*s to equal the number of the *G*s just is for the *F*s to be in one-to-one correspondence with the *G*s.

As for the subtle Platonist who accepts NUMBERS, the first consequence of accepting HUME'S PRINCIPLE is that a world in which there are no numbers is inconsistent. This is proven thanks to a light modification of the Zero Argument: trivially, objects, say planets, are in one-to-one correspondence with themselves. But for the planets to be in one-to-one correspondence with themselves just is for the number of the planets to be self-identical. So numbers exist after all.

On the other side, classical neofregeanists accept Hume's Principle but don't accept HUME'S PRINCIPLE, resisting the idea of the impossibility of a world with no numbers, if Hume's Principle isn't expressed in a free logic. Otherwise, the existential import of Hume's Principle will sacrifice its analyticity.

Rayo suggests that Neofregeanists, first of all Frege, are in the very end subtle Neofregeanists. Arguing in favor of this thesis, he proposes to read abstraction principles in terms of ‘just is’-statements. So, the famous principle of direction in Frege’s *Grundlagen* §64 becomes:

PARALLEL: For the direction of line a to equal the direction of line b just is for a and b to be parallel.

Recall that ‘just is’-statements are the result of the act of carving, of delimiting the logical space in ways that represent the way the actual world is. And Frege have often refer to the very operation of content-recarving discussing abstraction principle, as pointed out also in Rayo (2011, page 13).

In last analysis, Trivialism reveals to have in store several approaches that can be helpful and ease Neofregeanist’s pursuit of a consistent way of deducing mathematical truths from logical truths.

“Philosophy, like all other studies, aims primarily at knowledge. The knowledge it aims at is the kind of knowledge which gives unity and system to the body of the sciences, and the kind which results from a critical examination of the grounds of our convictions, prejudices, and beliefs. But it cannot be maintained that philosophy has had any very great measure of success in its attempts to provide definite answers to its questions. If you ask a mathematician, a mineralogist, a historian, or any other man of learning, what definite body of truths has been ascertained by his science, his answer will last as long as you are willing to listen.

But if you put the same question to a philosopher, he will, if he is candid, have to confess that his study has not achieved positive results such as have been achieved by other sciences. It is true that this is partly accounted for by the fact that, as soon as definite knowledge concerning any subject becomes possible, this subject ceases to be called philosophy, and becomes a separate science.”

RUSSELL, BERTRAND, (1912), *The Problems of Philosophy*, page 185

Chapter 6: Concluding Remarks

6.1 Sober Platonism

In section 1.5 I proposed a preliminary definition of Sober Platonism as the trend in mathematical platonism characterized by a descriptive attitude towards the content of philosophical reasoning. In the case of mathematical platonism, towards mathematical objects.

Moreover, I suggested that the ontological commitment endorsed by Sober Platonism is softer than the one endorsed by classical Platonism, thanks to the appeal to several arguments demonstrating that a world without mathematical objects wouldn't be consistent. But Sober Platonists don't rest content of having demonstrated that a world without mathematical objects would be inconsistent. They also hypothesized that, even if such a world would be consistent, rejecting mathematics for philosophical reasons wouldn't be acceptable in any case. That is one of reason why I characterized Sober Platonism as the attitude in philosophy of mathematics that aims at understanding and interpreting mathematics as practiced, without imposing any philosophical argument. It will be absurd to make such an imposition, as the following words by Lewis (1993, page 15) caustically affirms:

I laugh to think how *presumptuous* it would be to reject mathematics for philosophical reasons.

Philosophy is a way of thinking that can be applied to itself, but is mostly applied to something else, as in the case of philosophy of science, inflected in philosophy of biology, of medicine and so on, but also of philosophy of history, philosophy of

psychology and much more.

I am persuaded that the attitude of philosophy towards the other discipline has a prominent influence on the quality of the results philosophy can obtain. If philosophical reflection takes on a too normative behavior, it runs the risk of nullifying the significance of any hypothetical result such reflection can reach. The reason is that this hypothetical result could be legitimately charged because its subject had become very far from the original. To be more clear, a too normative philosophical approach runs the risk of drifting apart from the real subject of philosophical reflection. This point seems to echo the worlds used by Russell in *Logical Atomism*⁶⁶, page 339:

We shall be wise to build our philosophy upon science, because the risk of error in philosophy is pretty sure to be greater than in science. If we could hope for certainty in philosophy the matter would be otherwise, but so far as I can see such a hope would be chimerical.

There is no need to specify that philosophical reflection about something different from philosophy, mathematics for example, is not mathematics and it will never be. But, if philosophers of mathematics are too impositive and normative towards mathematics, their reflections will distance too much for having any possibility of grasping the philosophically interesting aspects of mathematics.

The four theories I have analyzed and characterized as Sober Platonism all are very well attentive of not crossing the risk-threshold. That is way they behave so soberly and embrace the general descriptive attitude towards mathematics.

I have recognized some common characteristic all the four theories manifested that, in

⁶⁶ RUSSELL, BERTRAND, (1924) *Logical Atomism*, in *Logic and Knowledge*, ed. R.C. Marsh. London: Allen & Unwin, 1956.;

my analysis, play an important role in their sobriety. All these characteristics are evoked and extensively analyzed in the rest of this chapter, but let me briefly say a word on each of them here.

First, they all have a special consideration for mathematical language. In particular, Sober Platonism understands mathematical language as the most important way in which mathematics is known, rather than as a mean through which mathematics is expressed. If mathematical knowledge is knowledge by description (see section 6.4), then the tool needed for description, i.e. language, is the tool needed for knowledge. For instance, in Shapiro's theory, mathematics is known through language because human's knowledge has a linguistic feature. Also Zalta's distinction between encoding and exemplifying is just a difference in meaning, but with great consequences, since it accounts for the development of a precise definition of two different ways in which objects can exist.

Secondly, none of them felt the need to extensively justify their ontological assumptions. The reason is that, according to Sober Platonism, the existence of mathematical objects is no more under discussion, and for two very simple reasons. The first is that Sober Platonists take the existence of mathematical objects as a consequence of a brute fact: mathematical knowledge occurs. That is why Sober Platonists are way more interested in finding a way to fashion mathematical objects so to explain how mathematical knowledge obtains, than in justifying their ontological assumptions. And this means also that Sober Platonists are inclined to undertake some path that could be seen as philosophically unorthodox, like the explicit violation of the principle of ontological parsimony entailed by the adoption of plenitudinous ontologies, for which I

refer to section 6.3.

The second reason is that Sober platonists are persuaded that the existence of mathematical objects would not bring any additional information on them. Indeed, both ordinary speakers and mathematicians don't care much about the ontology of mathematical objects, and perceive the questions on their existence as irrelevant for the content of their mathematical statements, at least during non-philosophical conversations.

If the existence of mathematical objects would not bring any additional information on them, spending too many efforts in committing with their existence will not take philosophy of mathematics anywhere. On the contrary, taking the existence of mathematical objects as a *datum* derivable from the observation of linguistic behaviors of ordinary speakers and mathematicians, and developing a philosophical justification of it, could lead to fruitful fields for philosophy of mathematics, as Sober platonists theories revealed.

At third place, Sober Platonists share a positive attitude in regards of the embracement of non-uniqueness in reference. Beware that what is characterizing of Sober Platonism is not the embracement itself, but the approach to the embracement. Indeed, as a consequence of Benacerraf's Dilemma, non-uniqueness in reference for mathematical statements is widely accepted among philosophers of mathematics. But Sober Platonist embraced it not as an unlucky concession to the Dilemma, but as a philosophically welcomed and fruitful approach (section 6.4.3).

Fourthly, the source of mathematical knowledge is individuated by all Sober Platonists in logical knowledge. Even if this idea isn't that new (just think about Logicism), Sober

Platonism understands the way in which mathematical knowledge arises from logical knowledge in new and persuading way. Mostly, logic and language are the source of mathematical knowledge. But Sober Platonists proposed several theories of knowledge that take into account the epistemological and neurological processes through which the human mind achieves mathematical knowledge. The resulting picture of mathematical knowledge is very connected and influenced by the way humans' attain it. As a consequence, mathematical knowledge depends at least partially by humans' mathematical reasoning. This could weaken the independency of mathematical objects, but Sober Platonists, through the development of new accounts of mathematical independency and mathematical knowledge, can save the objectivity of mathematics and keep endorsing *Independence* (section 6.4.2).

In the rest of this chapter, I will analyze Sober Platonism, following the analysis of classical Platonism I did in chapter 1. Theoretically, it is possible to compare classical Platonism and Sober Platonism by comparing the way in which the two theories endorse the three general theses of Platonism, i.e. *Independence*, *Existence* and *Epistemology*.

6.2 Independence

Independence: the independence of mathematical realm from anyone's thought and practice.

Sober Platonism endorses a slightly modified version of *Independence*. The differences introduced are meant to overstep the problems involved by the classical version of the thesis. In particular, the challenge is to preserve the authenticity of mathematicians'

enterprise but maintaining the knowability of mathematical objects. Hence, Sober Platonism aims at accounting for the objectivity of mathematical statements and explaining how mathematical knowledge is attained.

In order to reach its goals, Sober Platonists developed mainly two strategies. The first strategy is to assume *Independence*, but providing a sort of cross-realm portal through which obtaining access to mathematical knowledge. The cross-realm portal is supplied by logical knowledge, as in the case of Full-Blooded Platonism (henceforth FBP). According to Balaguer's account, mathematical objects exist independently, and in theory there is no way to access to mathematical knowledge. Notice that the assumption of the independent existence of mathematical objects serves the only concern of guaranteeing reference and theoretically objective truth conditions to mathematical statements. Therefore, if there exist all logically possible mathematical objects, no matter how and where, its sufficient to think about such objects in order to attain knowledge of them. Moreover, the mathematical terms will refer to logically possible mathematical objects, whose existence is guaranteed by the ontological assumption.

If it sounds difficult to get this through your mind, imagine a world whose inhabitants can only access to empirical knowledge, or also a world in which nobody knows of any mathematical theory if it is logically consistent. In such worlds, there is no way for anybody to ever get a piece of mathematical knowledge, but mathematical objects will keep existing. Therefore, *Independence* is maintained.

The second strategy developed by Sober Platonists is to enrich the formulation of *Independence*. This is the strategy followed by *ante rem* Structuralism, Objects Theory and Trivialism.

In *ante rem* Structuralism, the independence of mathematics resides in its freestandingness. As I already explained in section 3.3, mathematical structures are freestanding because their existence is independent on the existence of the objects that occupy the places in structures. In other words, it is not relevant if a structure is exemplified or not, because structures exist independently from the existence of any objects exemplifying them. As a result, the existence of structures is not only independent on anyone's thought, practice or language, but even from the existence of everything else. If the world wouldn't have existed, structures would have existed in any case.

The formulation of *Independence* delivered by Zalta is different: according to him, the existence of objects is dependent on the existence of theories, in their turn dependent on the existence of their author. *Prima facie*, this seems to suggest that *Independence* is denied. But the impression is false. Indeed, Zalta endorses *Independence* in regard of abstract objects. In his account, abstract objects, before than mathematical objects, exist independently from anyone's thought and practice. His assumption is both stronger and weaker than the assumption of classical platonism: it is stronger because it assumes that abstract objects exist necessarily in every possible world; and it is weaker because it assumes that mathematical objects are not the same in every possible world, even if they would exist as abstract objects. As a result, in Zalta's account, abstract objects are prior to mathematical objects.

The choices mathematicians make in the actual world drive them to the formulation of (actual) mathematical theories, among all the possible mathematical theories, namely those theories that can be formulated starting from each abstract object. Therefore, in

Zalta's perspective, mathematics is contingent: what constitutes mathematics depends on each and every world; but it is not contingent what kind of abstract objects mathematical objects are. Indeed, not every kind of object can be a mathematical object, only abstracts can. And abstract objects remain the same from world to world. Thereon, the collection of every possible mathematical objects remains the same and remain independent on worlds, mathematicians or applications.

Rayo's formulation of *Independence* is logical first than metaphysical. Indeed, the independence of mathematical realm from anyone's thought and practice is expressed as the impossibility for anything in the world to influence the truth-values of mathematical statements. The reason is that nothing is required from the world for the truth-value of mathematical statements to be satisfied. Hence, the existence of mathematical objects and their independence is nothing but a consequence of the independence of mathematical statements' truth-value from any non-trivial affair.

Rayo is committed also with metaphysical independence, since he claims that worlds with no number would be inconsistent by mean of the Zero Argument. Notice that the kind of inconsistency Rayo refers to here is inconsistency *de mundo*⁶⁷.

As it may be evident, also the second strategy makes extensive appeal to logic

The strategies developed by Sober Platonism are not confuting *Independence*, but they are rather enriching it with notions that makes it more lined up with mathematics. Mathematics is a brute fact, not only independent, but almost stolid: there is no way the world can be that could ever have any influence on mathematics, because of the very nature of mathematics.

⁶⁷ See section 5.3.1 for the difference between inconsistency *de mundo* and inconsistency *de repraesentatione* in Trivialism.

In doing so, Sober Platonism is able to guarantee to mathematics its distinctive *a priori* character and a proper domain mathematicians can discover by mean of the very intellectual mean used for every kind of scientific enterprise, namely logical reasoning.

In conclusion, Sober Platonism appeals to logic both for justifying the independence of mathematics as a form of logical reasoning, and for providing a way to access mathematical knowledge.

6.3 Existence

Existence: the existence of mathematical objects as abstract objects.

Existence is the platonist thesis Sober Platonists are less likely to support as it is supported by classical Platonism. The reason is mainly that in Sober accounts the existence of mathematical objects is not any more under discussion, since, according to them, there is no fact of the matter as to whether numbers exist. The existential claims are shifted to logic and coherence becomes the criterion for existence. What Sober platonists are rather investigating is what mathematical objects could possibly be.

For this reason, Sober Platonist is actually endorsing *Existence*, but only after having interpreted and modified it, both in the linguistic and in the logical side, so to weaken the intensity of the resulting ontological commitment. Recall that such appeal to language interpretation is one of the distinctive features of Sober Platonism. Moreover, that the philosophical interpretation of mathematics is one of the main purpose of Sober Platonism, as it makes extensive appeal to mathematical language and the way in which it is used by mathematicians.

Language and ontological commitment are deeply interlocked in philosophy, especially in the platonist side. From a strictly philosophical point of view, ontological commitment serves the concern of guaranteeing the meaningfulness of mathematical statements. Theoretically, every term that composes a true statement has to refer not vacuously for the entire statement to be true and meaningful. If mathematical statements are true and meaningful, there must be some kind of relations between the mathematical terms and the objects they refer to. But if there must be a relation between terms and the objects the terms refer to, such objects must exist in some sense.

The ontological commitment carried by language is interpreted by Sober Platonists so to result less strict and straightforward. The reasons behind this need of softening ontological commitment is primarily motivated by sober platonist's efforts for resembling as close as possible mathematics as practiced. Accordingly, Sober Platonism's aims isn't only to interpret mathematics and its language, but also to provide a philosophically rigorous language that is also capable of delivering the most accurate description of mathematics as practiced.

For this reason, Sober Platonism makes use of a theory of ontological commitment that accounts for different kinds of relations between the way the world is represented and the way the world is. Here I'm referring to Balaguer's adoption of Hodes' theory on thin and thick ontological commitments. The thick one is a proper commitment, because it requires an actual connection between terms and the objects they refer to, so to have knowledge of the true conditions of the statements containing the term. On the opposite, thin commitment isn't requiring any kind of relation between terms and objects: statements thinly committed with objects must be consistent in order to be true, or

contradictory so to be false. But in no way the thin commitment requires the existence of the objects to commit with. For further details, see Balaguer's use of thick and thin commitment in section 2.2.2.

Balaguer's adoption of thick and thin commitment is very similar to Zalta's distinction between exemplification and encodement. Being more a linguistic than an ontological matter, what the distinction ultimately does is supplying a language for predicating two types of existence to objects, and so defining two different ways in which objects may exist. But it doesn't mean to involve any ontological commitment to any kind of existence. Zalta is not stating that there exist abstract objects. He is rather stating what abstract objects are and how it is possible to speak of them in rigorous terms.

Indeed, the distinction between encoding and exemplifying has been introduced with the sole goal of providing the linguistic skills needed to translate existence claims from natural language in rigorous way. In addition, these two ways of translating the predicate of existence are useful in the description of abstracts objects, that are precisely defined as the complement-set of the set of possibly existing objects on the domain of objects (see section 4.4).

Sober Platonism so provides a less ontologically committing interpretation of the platonist commitment to mathematical objects. Even more explicitly, Agustín Rayo reaches the conclusion that no ontological commitment at all is needed. All one have to do is just to enlarge mathematical language, so that it would be possible to fully express mathematics. Notice that, in Rayo's account, extending the language supplies additional linguistic resources that allow trivialists to describe the mathematical world with much more details and rigor. Nevertheless, it wouldn't extend or change what philosophers

have to be ontologically committed with, for a reason that will be specified in the following lines.

Sober Platonism also assumes the coherence of mathematical theories as the criterion for the existence of the objects they describe. The result is that its ontology will count a huge amount of objects, so it has to adopt plenitudinous ontologies. But such ontologies are often charged because of their lack of ontological parsimony. Taking a closer look to the objection, it is easy to see that Sober Ontology will rest content with a small number of type of objects, once it has at disposal a big number of tokens. Sober Platonists's idea is that, since the domain of possible mathematical objects is considered as the real domain of mathematics, every possible type of mathematical objects is to be included in mathematical ontology. Accordingly, ontological parsimony is maintained, since Sober Platonists argue that ontological parsimony demands to limit the kind of entities one posits, not their number. That is why, in Rayo's theory, extending the language isn't extending the world: because the world has already reached its maximum size. Extending the language would at least be a matter of tokens, not of types.

Balaguer and Zalta are both far from considering plenitudinousness of objects as not ontologically parsimonious. Indeed, in order to satisfy the constraints of ontological parsimony, as few objects as possible must be added to the domain in a non-arbitrary way. But if it is so, in their account of abstract objects the only way to add as few objects as possible in a non-arbitrary way is to add all possible abstract objects.

As a consequence of the openness to plenitudinous ontologies, Sober Platonists also accepted to give up the existence of one and privileged metaphysical structure and the uniqueness in reference for mathematical terms (see section 6.4.2). Once the

fundamental role of language in mathematics is taken into serious consideration, it becomes evident that there can't be an objective language-independent fact of the matter about whether there are numbers. Moreover, from a strictly mathematical point of view, the question about the existence of numbers would not bring any additional information on them. It is philosophy who need to account for their existence. According to mathematics, the existence of mathematical objects is just an undeniable fact.

It is also worth analyzing how Zalta and Linsky described abstract objects in 'Naturalized Platonism vs Platonized Naturalism'⁶⁸: as a first condition, they posit that abstract objects can't be distinguished by the distinction between apparent and real. Abstract objects participate with a very thorny type of existence: at least intuitively, if something belongs to the category of appearance, it is not really existing, it is not part of reality. And indeed abstract objects exist as not being part of what (concretely) exist. The second condition is that the characterization of abstract objects takes place thanks to the comprehension principle; therefore, nothing empirical can affect the existence of abstract object. The third condition is that abstract objects are fully defined: for every property, they either possess it or its negation. This means that abstract objects are a plenum: if they are to be described by their properties, and the comprehension principle guarantees that there is an abstract object for every group of properties, then there are as many abstract objects as there could possibly be. Notice that in this regards, the ontology provided by Objects Theory is very consonant with FBP's commitment with the existence of all logically possible mathematical objects.

⁶⁸ LINSKY, BERNARD, ZALTA, EDWARD, (1995), "Naturalized Platonism vs Platonized Naturalism", *Journal of Philosophy* vol. 92, pp. 525-555;

A few words are worth to be spent also in the topic of objecthood. In mathematical Platonism there are two trends for what concern the kind of entities numbers are: either they are objects, or they are something else. Sober Platonism doesn't involve the endorsement of a specific position on the dispute: some sober platonists adopt object-platonism, some don't. But Sober Platonism delivers interesting analysis of the dispute. For instance, in Rayo's account, what seems to be a metaphysical question reveals to have much to do with language. Indeed, Rayo points his finger to the ability of languages to use singular terms, charging it of being the solely reason behind the fact that mathematicians are used to describe mathematical theories in terms of objects.

Shapiro's *ante rem* Structuralism can be considered object-Platonism only if mathematical structures are given the status of objects. Indeed, since mathematical objects occupy places in structures in the places-are-offices way, it could seem that Structuralism endorses a very bizarre conception of objects, if it doesn't deny it at all, since it is ontologically committed only with the existence of structures. Therefore, it only aims at guaranteeing the existence of the structure of natural numbers, not the natural numbers as individual objects. It is exactly this the very sense of *ante rem* Structuralism: natural numbers aren't individual objects you can commit to.

Rather, they are a structure that can be exemplified but several systems, none of them being in any way the natural numbers. At least, such systems will represent the structure of natural numbers. But structuralism is definitely not committed with the existence of mathematical objects, in a classical theory of objects. Nevertheless, some attempts to reduce Structuralism to object-platonism have been made, for example by Mark Balaguer (see section 2.1).

In conclusion, sober ontology is very oriented towards mathematics, whether it deals with structures or with objects or, in theory, with other kinds of entities. That is why it works very satisfyingly in mathematics, but I would not say that it could work as well as it does for mathematics in other branches of ontology. As far as I can see, Sober ontology is tailor-made on mathematical platonism, and it fits very comfortably on it. Nevertheless, sober approaches could work extremely well also with other kind of abstract objects, if they would be readjusted to fit with new subjects.

6.4 Epistemology

Epistemology: the successful reference and knowability of mathematical statements.

According to Sober Platonism, *Epistemology* comes as a brute fact, rather than as an assumption. The success, and even more so the possibility, of mathematical knowledge is right there in front of everyone. Therefore, the role of philosophers is here more than descriptive, its apologetic. If the aim of Sober ontology is to fashion abstract objects so to make possible the explanation of how cognitive subjects have epistemic access to them, the role of Sober epistemology is to explain how cognitive subjects succeeded to have epistemic access to them.

Ultimately, the challenge is to achieve a connection between bearers of relevant mathematical beliefs and constituents of relevant mathematical facts. This challenge is accomplished by Sober Platonism thanks to the formulation of a theory of knowledge, disclosing how humans attain reliable mathematical knowledge, a theory of truth, that guarantees the truth of mathematical beliefs, and a theory of reference, so to make clear

how mathematical statements are bounded to mathematical facts. Hence, the outline I followed for *Epistemology* in classical platonism in chapter 1 is maintained here, and the main thesis is divided in three sub-theses: *Theory of Knowledge*, *Truth* and *Reference*.

6.4.1 Theory of Knowledge

Theory of Knowledge: mathematical knowledge is possible.

Sober Platonism is open to a great variety of theories of knowledge, on condition that the source of at least part of mathematical knowledge is logical knowledge.

As I have already stressed several times, logic has an essential role in Sober Platonism, both as the source of mathematical knowledge and as the mean through which humans overstep the distance between the empirical and the mathematical realities, so to access to mathematical knowledge. Having said that, Sober Platonists developed a rich range of theories of knowledge, but mainly in the trend of knowledge by description. Nevertheless, Zalta's appeals to knowledge by acquaintance might be misleading and prompt that mathematical knowledge can also come from acquaintance in Sober Platonism. In the following, I will explain why Zalta's theory of knowledge, and Sober Platonists' in general, are to be seen as both suggesting a theory of knowledge by description.

First of all, let me state that Zalta is not particularly concerned about developing an articulated theory of knowledge. He mainly claims that humans attain knowledge of mathematical objects simply by becoming acquainted with them through the comprehension principle. Such comprehension principle is synthetic and a priori and,

thanks to it, for every condition on properties there is an abstract object that encodes just the properties satisfying the condition.

Even if it seems straightforward that Zalta's theory of knowledge appeals to acquaintance, it must be kept in mind that the acquaintance comes from the comprehension principle. And the comprehension principle works with nothing but definitions. Intuitively, definitions and descriptions are different just because it is possible to define objects that don't exist, while description seems to require the existence of the objects of description. But once the definition of an object is the sufficient condition for its existence, at least as abstract object, this distinction loses significance, if it doesn't collapse. In particular, becoming acquainted with an object consists in understanding its descriptive condition. Therefore, Zalta's theory of knowledge reveals at most a mixed identity between acquaintance and description. If I were asked which of the two processes is subordinate to the other, I would answer that, as far as I see, description is definitely antecedent to acquaintance. Indeed, the first piece of mathematical knowledge is the comprehension principle, that is a definition of objects. Only thereafter and through the descriptions provided by the axioms, human mind starts grasping complete, necessary and *a priori* knowledge of mathematical objects.

Contrary to Zalta, Mark Balaguer purports explicitly a theory of knowledge by description. But in order to see how it works, the existence of the objects of knowledge, i.e. mathematical objects, must be firstly assumed. From this assumption, Balaguer derived that, since all the mathematical objects, which possibly could exist, actually do

exist, is sufficient to coherently define a mathematical object to attain reliable knowledge of it.

The identification of the meaning of 'possible' with 'logically possible' allows to assume the existence of all logically possible objects and supplies a reference for each and every consistent definition of mathematical objects. As a result, mathematical knowledge turns out to be exactly this: the ability to discriminate between mathematical consistency and inconsistency. And since humans have this ability, they can access to knowledge of mathematical objects simply by providing them with a consistent definition. Therefore, mathematical knowledge seems to consist more of competence than informations. Mathematical theories are the result of the application of this competence, namely the ability to discriminate between consistency and inconsistency. Its proper field is therefore all logically possible objects, obtained as the outcomes of performing the ability to discriminate consistency and inconsistency.

As satisfying at this could seem, it is still under the assumption that the objects of mathematical knowledge exist. But how this assumption can be justified? I already explained in section 2.2.2 of this work that Balaguer succeeds in identify his assumption with the assumption that exist a physical world that give raise to accurate sense perception. Thereon, he identifies sense perception with the ability to discriminate consistent from inconsistent theories. Ultimately, he shows that both aren't able to justify the assumption that the objects of mathematical knowledge exist and the objects of empirical knowledge exist.

What I want to state here is that, even if Balaguer's assumption is reduced to an externalist perspective, and thereon also Anti-platonists are puzzled, two clues aren't a

test. Proving that there is no evidence of the reliability of sense perception in human's forming of beliefs about the physical world doesn't prove anything about the effectiveness of assuming the existence of all logically possible mathematical objects on the reliability of beliefs about these objects. Rather, it places two problems instead of solving one.

In my perspective, this is the biggest limit of Full-blooded Platonism, especially because it shows that the only result obtained by a huge and compromising ontological assumption is showing that humans' beliefs about the external world are as reliable as the ones about mathematical objects. But the point I want to stress here is that the external world is made by concrete objects we collectively perceive, while the mathematical realm is made by abstract objects, we can at least collectively hypothesize and try to describe.

In accordance with FBP theory of knowledge, Shapiro's Structuralism assumes that mathematical knowledge comes from implicit definition, understood as the ability to simultaneously and coherently characterize a number of items through their relations to each other. Even in this case, the connection with mathematical language and mathematicians' linguistic habits is very explicit.

Recall also that the possibility to coherently define a structure isn't only a necessary and sufficient condition for attaining knowledge from it, but it is also the condition under which it exists. Since the ability to coherently discuss a structure is evidence for the existence of the structure, it is the language that characterizes and determines a structure. And it is again through the use of language that structures are known.

The satisfaction of the two requirements Shapiro imposes on implicit definition is the condition of possibility of mathematical knowledge. The first requirement guarantees that at least one structure satisfies the axioms. In doing so, Shapiro pledges that mathematical knowledge is always knowledge of something, again by the inference from the ability to coherently discuss a structure to the existence of the structure. The second requirement guarantees that implicit definitions can describe at most one structure each, up to isomorphism. In this way, knowledge of structures is determinate, and so it is also reference to it.

Mathematical knowledge is by definition also in Rayo's account. Indeed, knowledge occurs through the definition of the set of 'just is'-statements accepted in a determinate world. The acceptance of 'just is'-statements is never completely independent from empirical matters, but some 'just is'-statements are more tied to logical aspects than others. Those 'just is'-statements are the ones that concern Logic and Mathematics. As a result, mathematical knowledge doesn't turn out to be completely *a priori* in Trivialism, since the process of determining the set of 'just is'-statements accepted is logical and, at least partially, empirical. For further details on the acceptance of set of 'just is'-statements, see section 6.4.1.

Rayo bounds the notion of 'just is'-statements with the process of non-trivial cognitive accomplishment, i.e. the acquisition of information transfer abilities from one way for the world to be represented to another way for the world to be represented. Therefore, the acquisition of mathematical knowledge occurs as the acquisition of information transfer abilities between different modes of presentation of a given region in logical space. And the regions of logical space are determined by the definition of the set of

‘just is’-statements that holds in accordance with mathematical theories. Hence, mathematical knowledge resides in the process decision mathematicians take while accepting the ‘just is’-statements that concern mathematics.

In conclusion, the analysis of the different theories of knowledge proposed by Sober Platonists makes evident the importance of the role of mathematical language in attaining mathematical knowledge. Mathematical knowledge consists in the ability to discuss mathematical theories and their coherence, both internally and in relation with others mathematical theories. This ability also guarantees the existence of the objects of knowledge in accordance with the endorsement of *Existence*.

6.4.2 Truth

Truth: mathematical knowledge is knowledge of truth.

Sober Platonism endorses *Truth* without any extensive transformation. That mathematical knowledge results in knowledge of truth is widely accepted by any Sober Platonist. Nevertheless, they provide *Truth* with a notion of truth that is highly original and of great interest from a philosophical point of view.

First of all, truth is always understood as context-sensitive, mainly because of the adoption of non-uniqueness in reference and plenitudinous ontologies.

Secondly, from the adoption of coherence as the criterion for existence, the difference between coherentism and correspondentism in truth-theory crumbles: if a theory is coherent, i.e. it does not lead to contradiction, then the objects it talks about exist; hence, the terms contained in its theorem refer to existing objects. But if it is so, there also is a correspondence, even if not uniquely determined, between mathematical

theorems and mathematical objects. Notice that correspondence holds as nothing but a lucky outcome of the choices Sober Platonist did in settling truth-theory and the criterion for existence. What I mean is that correspondence isn't replacing in anyway coherence as the criterion for truth of mathematical statements. Rather, it is reinforcing the power or coherence as the criterion for truth (and existence). Indeed, correspondence doesn't only explain truth, but it also justifies and guarantees objective truth.

As a paradigmatic case, consider *ante rem* Structuralism. According to it, the criterion for existence is coherence. Moreover, mathematical knowledge is the ability of coherently discuss a mathematical structure. Therefore, a mathematical truth is a coherent use of mathematical language whose content is a mathematical structure.

In Shapiro's account, structures are determined by human ability to talk or to have knowledge of them. Therefore, the kind of knowledge mathematics can aim too is context-sensitive truth, but the context is determined by the language chosen by mathematicians when they theorize about mathematical structures.

Even if the relevance of mathematical language isn't so explicit, also for Trivialism the truth of a statement has much to do with mathematical language or, more precisely, with mathematical interpretation. In general, a statement is true if it singles out the region that corresponds to the way the world actually turns out to be. True mathematical statements single out coherent statements that correspond to coherent way for the world to be.

The outscoping technique assigns to every truth of pure Mathematics a trivial semantic clause. As I have already stressed in chapter 5.3, the outscoping technique allows to

exclude from the scope of a world w all the non-mathematical vocabulary contained in the 'just is'-statements under analysis. As a result, the statement is true at a world w just in case w satisfies a metalinguistic formula in which the vocabulary has all been outscoped. Such formula will have no free variables and would then be satisfied by any object if it is true, and by no object if it is false. And since trivialist Platonism interpret mathematical statements as being true, the formula will be true and satisfied by all objects, including w independently of which it is. Therefore, nothing in the way the world could turn out to be will in any way affects the truth conditions of a pure mathematical statement. In this sense, mathematical statements carry ontological commitment to objects. But they are interpreted in a such way that nothing that could or could not happen in a world will ever imply the satisfaction or dissatisfaction of the truth conditions of mathematical statements.

Balaguer accounts for a theory of truth that is very lined-up with the general Sober platonist attitude I described. He argues that every consistent purely mathematical theory truly describes some collection of mathematical objects. Therefore, in mathematics correspondence and coherence collapses and end to identify the same set of true statements. Even Balaguer reserves an important role to mathematical language in accounting for mathematical truth. He claims that every consistent purely mathematical theory is true in a language that interprets it so that it is about the objects it is intended to be about. In doing so, the truth conditions of mathematical statements depend not only on their internal coherence, but also on the choices mathematicians made while they express what they intend to express. Accordingly, mathematical

theories are true partly because they are coherent, and partly because mathematicians interpreted them so that they are coherent.

Zalta delivers an account of mathematical truth that comes straightforwardly from the way he has interpreted the comprehension principle. According to Objects Theory, mathematical theorems are true because they are encoding claims that ascribe to mathematical objects the properties described by the theorems. But since mathematical knowledge is attained through the comprehension principle, and the comprehension principle is synthetic *a priori*, also mathematical knowledge will consist in synthetic *a priori* truths. See section 4.5 for further details.

6.4.3 Reference

Reference: mathematical statements are about some kind of objects.

Even if Sober Platonism can be intended as a very homogeneous trend, in some aspects it varies considerably. In this section, I will analyze Shapiro's and Balaguer's endorsement of *Reference*. My aim is here to show how the two endorsement realized two very different theories of reference. Recall from section 1.4.3 that reference is the relation between a term and an object. A well-formed statement describes something in virtue of the reference between the terms used and the objects described. Moreover, according to standard semantics, the language of mathematics must work in the same way as every day language. Therefore, as for every day language, singular terms and quantifiers must semantically function as reference to objects and range over objects.

One of the biggest problems for platonism in reference is due to Benacerraf's famous paper *What Numbers Could Not Be*. Benacerraf point is, very roughly, that reference

appears to be not uniquely determined in mathematics, since it is hard to identify a criterion that discriminates between the different ω -sequences. The relevance of Benacerraf's problem is so wide that threatens to invalidate the entire platonist enterprise, denying the objectivity and meaningfulness of mathematics itself, by denying the possibility of referring unambiguously to mathematics. For a less sketchy exposure, see again section 1.4.3.

Sober Platonists endorse that the ambiguity of mathematical reference isn't only to be accepted, but it is also to be endorsed. The embracement of non-uniqueness in reference comes as a consequence of the acceptance of plenitudinous ontologies. For, if the mathematical realm counts as many objects as Sober ontologies prescribe, it seems extremely unlikely that any mathematical theory is uniquely satisfied.

One exception is Shapiro's *ante rem* Structuralism, who accepts uniqueness in reference but with some *caveat* that seems to settle it more on the side of the non-uniqueness Platonism than of that of uniqueness Platonism. Recall that in Structuralism knowing a structure is the same as being able to use its language. But the first requirement of a reliable use of language is an effective theory of reference. Shapiro claims that more than one system can exemplify a structure. Therefore, it seems that reference is not uniquely determined. But, Shapiro rushes to specify that reference to structures is determined up to isomorphism. Therefore, reference isn't uniquely fixed up to the singular systems, but is only fixed up to the forms in which systems exhibit structures. At first sight, Shapiro's theory of reference seems to guarantee reference only to the forms in which objects can be arranged, instead of referring to the very objects. But remind that, according to Shapiro, the forms in which objects can be arranged are the

very objects of mathematics, i.e. mathematical structure. These mathematical objects are endorsed by different systems, but still exhibit the same structure. Accordingly, if two systems exhibit two isomorphic structures, they are actually exhibiting the same structure, the very same object, because, again, they share isomorphism.

As a result, since only structural facts are mathematically significant, it doesn't matter if it is not possible to pick out unique collections of objects, because these collections are indeed a unique objects from the point of view of their structural properties and relations. Hence, reference is uniquely fixed up to the level of structures, the real mathematical objects, but is not uniquely fixed on the level of the systems exhibiting the structures. Notice that is the kind of ontology provided by Shapiro that allows him to avoid the embracement of non-uniqueness in reference, as it avoid also the embracement of plenitudinous ontologies.

FBP's adoption of plenitudinous ontology goes hand by hand with its adoption of non-uniqueness in reference. According to Balaguer, non-uniqueness in reference is a desirable trait in philosophy of mathematics, since it allows for a better description of mathematics as practiced. Here Balaguer's point is very similar to Shapiro's: mathematicians aren't interested in the differences between ω -sequences, since they all are indistinguishable with respect to the sort of facts and properties they are trying to characterize while doing arithmetic. Therefore, the terms used to describe an ω -sequence fail to refer to a unique object when they refer to objects that are similar from the point of view of mathematics as practiced. Here again, there is a philosophical problem: 'Why can mathematical statements still have a meaning while using terms whose reference isn't uniquely fixed?', with a mathematical answer: 'Because mathematicians while doing mathematics aren't interested in finding a unique referent

for their terms, but to refer to as many objects with the demanded characteristic there are'. As a consequence, FBP can be legitimately charged of the abandonment of standard semantics. But Balaguer specifies that, indeed, in using singular terms and providing them with a standard semantics, mathematicians make an assumption that is false. Nevertheless, it is convenient because thanks to it, mathematical language appears to behave like everyday language.

On the same line are also Zalta's and Rayo's theories of reference. Zalta developed an axiomatic language that justifies and explains reference to abstract objects in general, by considering the references of terms denoting objects as abstract objects, and the references of terms denoting properties as properties. In this way, the connection between terms and objects is straightforward, and so it is the connection between reference, language and knowledge.

Rayo's approach is very utilitarian: a language includes singular terms and quantifiers ranging over singular-term-positions only because they are useful in specifying truth conditions. There is no metaphysical view that can be derived or founded on reflections about the use of language. Moreover, reference between terms and objects is determined by 'just is'-statements. In a sense, the very meaning of 'just is'-statements isn't only identifying sameness of truth conditions, but also sameness of reference. For example, 'Venus is Phosphorus' and 'Venus is Hesperus' form the 'just-is'statement:

MORNING STAR: For Venus to be Phosphorus just is for Venus to be Hesperus

This 'just is'-statement tells something about the reference of the objects in it. And it says that there is no unique reference for the object called 'the morning star'. Indeed, it is also the evening star and actually it isn't a star at all, but a planet, the second planet in

our Solar System. Through the use of ‘just is’-statement, reference between objects and terms is so further specified.

In the same way, ‘just is’-statements in mathematics tell something about the way reference to mathematical terms is fixed. And since the acceptance of ‘just is’-statements is something that come at least partially from empirical matter, what mathematical terms must refer to is something that depends at least partially on the work of mathematicians. Hence, it is not uniquely fixed because it depends on the conception of logical space endorsed.

6.5 Conclusion

In conclusion, there is a consideration I didn’t already made. Quite emblematically, all the four Sober Platonists acknowledges that the philosophy of mathematics they propose differentiates themselves from classical Platonism. Shapiro and Rayo made this differentiation explicit because they admittedly baptize their new theories with the names of ‘Working Realism’, for Shapiro’s, and ‘Subtle Platonism’, for Rayo’s.

Shapiro’s definition of Working Realism is introduced by the formulation of two approaches philosophy can have towards its non philosophical content: ‘philosophy last if at all’ and ‘philosophy first’. In ‘philosophy first’, the role of philosophy is to define and draw the limits of mathematical possibility from a metaphysical and an epistemological point of view. In this account, philosophy precedes mathematics and fixes the way mathematics is to be done. Therefore, ‘philosophy first’ is a normative approach to mathematics.

On the contrary, ‘philosophy last if at all’ states the contingency of philosophical

reasoning for mathematics. Philosophy can come only thereafter, and, with a descriptive approach, try to formulate a philosophical account of how mathematics is known and interpreted.

Shapiro locates Working Realism in the line of ‘philosophy last if at all’. Working Realism is interested in the work of mathematicians, in their methodology and professional habits, but avoid the imposition of philosophical concerns on the analysis of mathematics. See section 3.2 for a less sketchy discussion of Working Realism.

Also Rayo’s approach is explicitly distancing itself from classical Platonism, by defining its philosophy as ‘Subtle Platonism’ (section 5.6). According to classical Platonists, the existence of mathematical objects is contingent, therefore a world with no mathematical objects is possible. But according to Subtle Platonists, who endorse the Zero Argument, mathematical objects exist necessarily, and a world without mathematical objects wouldn’t be even intelligible.

In conclusion, and far from having provided any complete description of how Sober Platonism is and where its development could bring in different areas of philosophy, I am persuaded that this new approach will not be given up any time soon. Its value resides in its open-mindedness and in its willingness of facing mathematics’ autonomy from a philosophical point of view. As Balaguer (1998) fiercely states at page 63:

[T]he point of philosophy of mathematics is to interpret mathematical practice, not to place metaphysically based restrictions on it.

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