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Semilinear elliptic equations on complete manifolds with boundary with some applications to Geometry and General Relativity

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Introduction

The aim of this work is to study the properties of positive smooth solutions of nonlinear problems of the type

$$\begin{cases} \Delta u - f(x,u) = 0 & \text{on } M, \\ H(x,u,\partial_\nu u) = 0 & \text{on } \partial M; \end{cases} \tag{0.1}$$

where $(M, \partial M, \langle\ ,\ \rangle)$ is a noncompact Riemannian manifold with (possibly empty or noncompact) smooth boundary; $f(\cdot,\cdot)$ and $H(\cdot,\cdot,\cdot)$ are continous functions respectively responsible of the nonlinearity of the equation and of the mixed (possibly nonlinear) boundary condition (∂_ν denotes the exterior normal derivative with respect to ∂M). This kind of nonlinear problems arise quite naturally in many branches of mathematics, for instance in studying conformal deformations of Riemannian manifolds with boundary [**25, 34, 35**], finding optimal constants for Sobolev trace embeddings [**25, 37, 64**], and reaction-diffusion equations [**16, 33**].

The models for $f(\cdot,\cdot)$ and $H(\cdot,\cdot,\cdot)$ that we have in mind are

$$f(x,u) = \sum_{i=1}^{n_f} a_i(x) u^{p_i}$$

and

$$H(x,u,\partial_\nu u) = \gamma(x)\partial_\nu u + \sum_{i=1}^{n_H} g_i(x) u^{q_i},$$

where $a_i \in C^0(M)$, $g_i, \gamma \in C^0(\partial M)$, and $p_i, q_i \in \mathbb{R}$. For instance, Yamabe-type [**65, 71, 18**] and Lichnerowicz [**42, 52, 53, 57**] equations belong to this class of nonlinearities $f(\cdot,\cdot)$ as special cases. Moreover, the boundary conditions considered include homogeneous Dirichlet and nonlinear Robin. In particular, since ∂M is not assumed to be connected, it is possible to have different kind of boundary conditions on different components of the boundary.

We specialize our analysis of positive solutions of (0.1) to a choice of $f(\cdot,\cdot)$ which is meaningful for applications in both Geometry and General Relativity, namely the

Lichnerowicz-type equation

$$\Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0 \tag{0.2}$$

with $a(x)$, $b(x)$, $c(x) \in C^0(M)$, $\tau < 1 < \sigma$, with the sign restrictions

$$b(x) > 0 \qquad \text{out of a compact set}$$

and

$$c(x) \geq 0\,,$$

we note that with the choice $c(x) \equiv 0$, equation (0.2) becomes a general Yamabe-type equation (see [**71**], for instance).

Our preferred choice for the boundary conditions will be the following semilinear Neumann condition

$$H(x, u, \partial_\nu u) = \partial_\nu u + \sum_{i=1}^{n_H} g_i(x)u^{q_i} \tag{0.3}$$

where $q_i \in \mathbb{R}$ are such that $q_i < q_{i+1}$.

Recently some attention has been put on this kind of problems, see for instance references [**14, 44, 46, 73**], in particular by researchers in mathematical General Relativity. The special feature of our analysis is that, no curvature restrictions are assumed on the manifold $(M, \partial M, \langle\ ,\ \rangle)$, indeed, almost all the results rely on spectral assumptions on particular Schrödinger operators and suitable volume growth at infinity of geodesic balls. We stress on this point because it is not just a *cosmetic* generalization, but it is crucial in order to deal with a general nonempty boundary ∂M, since, in this case (as follows from [**9, 10, 11**]) there are no effective curvature comparison theorems at hand.

More precisely, for problem (0.1), under fairly general conditions on the manifold $(M, \partial M, \langle\ ,\ \rangle)$ and on the coefficients $a(x)$, $b(x)$, $c(x)$, $g_i(x)$, σ, τ, and q_i we prove:

(1) Existence of solutions;
(2) Comparison and uniqueness results;
(3) *A priori* estimates.

(1) Is obtained under spectral assumptions on the naturally associated Schrödinger operators $L = \Delta \pm a(x)$ or a suitable control of $a(x)$, $b(x)$, $c(x)$ with respect to the volume growth of geodesic balls at infinity. We note that in the case of compact manifolds existence for solutions of such equations can be carried out with variational techniques that in our case do not apply. Thus we will prove existence of solutions using a version of the

classical sub/super solution method taylored to fit our needs.

(2) Are obtained with different techniques. From the one hand, the uniqueness is a consequence of a quite general comparison result that descends from a suitable form of the weak maximum principle (in the spirit of [**66**], [**71**], and [**5**]) developed in this work for noncompact manifolds with boundary. On the other hand, a careful analysis of the Schrödinger operator $L = \Delta + a(x)$, as in the recent papers [**18**] and [**19**], leads analogously to a pair of comparison/uniqueness results.

(3) Are a consequence of a new L^∞ estimate for positive solutions of (0.2) in the spirit of Theorem B of [**65**]. The main novelty here is that our technique permits to deal with the case of a possibly negative coefficient $b(x)$. Another key ingredient in our estimates is a particular simmetry property of equation (0.2) that leads to useful bilateral *a priori* estimates.

As we already said, an important motivation to study problems like (0.1) comes from Theoretical Physics. Indeed, in the analysis of Einstein field equations in General Relativity the initial data set for the non-linear wave system plays an essential role. These initial data have to satisfy the Einstein constraint conditions that can be expressed in a geometric form as follows. Let $(M, \widehat{g})$ be a Riemannian manifold and $\widehat{K}$ a symmetric 2-covariant tensor on M. Then $(M, \widehat{g})$ is said to satisfy the Einstein constraint equations with non-gravitational energy density $\widehat{\rho}$ and non-gravitational momentum density $\widehat{J}$ if

$$\begin{cases} S_{\widehat{g}} - \left|\widehat{K}\right|^2_{\widehat{g}} + \left(\mathrm{tr}_{\widehat{g}}\, \widehat{K}\right)^2 = 2\widehat{\rho} \\ \mathrm{div}_{\widehat{g}}\, \widehat{K} - \nabla\, \mathrm{tr}_{\widehat{g}}\, \widehat{K} = \widehat{J}\,. \end{cases} \tag{0.4}$$

Here $S_{\widehat{g}}$ stands for the scalar curvature of the metric $\widehat{g}$.

A common way to look for solutions of (0.4) is to exploit the conformal method introduced by Lichnerowicz [**51**] and York [**78, 79**]. This means that we generate an initial data set $\left(M, \widehat{g}, \widehat{K}, \widehat{\rho}, \widehat{J}\right)$ satisfying (0.4) by first choosing the following conformal data:

- a Riemannian manifold $(M, \langle\ ,\ \rangle)$;
- a symmetric 2-covariant tensor σ required to be traceless and transverse with respect to $\langle\, ,\, \rangle$, that is, for which $\mathrm{tr}_{\langle\, ,\, \rangle}\, \sigma = 0$ and $\mathrm{div}_{\langle\, ,\, \rangle}\, \sigma = 0$;
- a scalar function τ;
- a non-negative scalar function ρ;
- a vector field J.

Then one looks for a positive function u and a vector field W that solve the conformal constraint system, that, in the case of a Einstein-scalar field (see [**27, 26, 57**]) reads as

$$
\begin{cases} c_m\Delta u - \left(S - |\nabla\psi|^2\right)u + \left(\left|\sigma + \mathring{\mathcal{L}}W\right|^2 + \pi^2\right)u^{-N-1} \\ \qquad\qquad - \left(\frac{m-1}{m}\tau^2 - 2U(\psi)\right)u^{N-1} = 0 \\ \Delta_{\mathbb{L}}W + \frac{m-1}{m}u^N\nabla\tau - \pi\nabla\psi = 0\,. \end{cases} \tag{0.5}
$$

where ψ is the restriction on M of a scalar field Ψ defined in the whole spacetime, $U(t)$ its potential, and π the restriction of its normal time derivative on M. Here Δ, S, and $|\cdot|$ denote respectively the Laplace-Beltrami operator, the scalar curvature, and the norm in the metric $\langle\,,\,\rangle$. The operator $\mathring{\mathcal{L}}$ is the traceless Lie derivative, that is, in a local orthonormal coframe

$$
\left(\mathring{\mathcal{L}}W\right)_{ij} = W_{ij} - W_{ji} - \frac{2}{m}(\operatorname{div} W)\,\delta_{ij}
$$

and $\Delta_{\mathbb{L}} = \operatorname{div}\circ\mathring{\mathcal{L}}$ is the vector laplacian. The constants appearing in (0.5) are respectively given by

$$
N = \frac{2m}{m-2}\,, \qquad c_m = \frac{4(m-1)}{m-2}\,,
$$

in particular we note that N is the critical Sobolev exponent, observe that $(-N-1) < 1 < (N-1)$. If (u, W) is a solution of (0.5) then a suitable rescaling of functions, fields, and sources leads to a solution of the Einstein constraints (0.4), see [**27, 26**]. For further informations on the initial value problem for the Einstein equation we refer to the surveys [**17, 26, 29**], and the references therein.

The scalar equation of (0.5) is called the Lichnerowicz equation, since it is the main source of nonlinearity in the system (0.5), a good understanding of its solutions is a crucial step toward the resolution of the Einstein equations by the conformal method, see for instance [**26, 27, 30, 41, 42, 45, 52, 53, 57**].

This equation has the form of (0.2) and in recent years has been studied by many authors, see for instance [**52**], [**22**], [**53**], and [**31**]. Here we generalize many of the results obtained in the aforementioned papers, in particular from the point of view of the ambient manifold M. Indeed, to the best of our knowledge, most of the results about the Lichnerowicz equation are obtained on manifolds M which are compact or with *asymptotically simple* ends (euclidean, hyperbolic, cylindrical, periodic,...) (see for instance [**28, 30, 38**]), physically these cases correspond, respectively, to a cosmological solution or an isolated system. In our work we enlarge significantly the family of admissible Riemannian manifolds; from the point of view of General Relativity, this means a wider choice for the geometry of the initial data. We stress the fact that the knowledge of the precise behaviour at infinity (i.e. the *asympotically simple* ends), allows the study of the Lichnerowicz equation in a classical analytical setting by means of suitably weighted Sobolev spaces (see for instance the very recent [**32**]). In our general case we need to use other techniques to tackle the problem.

Another natural issue in this framework, inspired by the classical singularity theorems of Hawking and Penrose, is to understand the behaviour of an initial data set containing event horizons. The approach introduced in [**80**] and which has been highly developed in recent years, see for instance [**56, 45, 44**], consists in excising the regions containing black holes and coherently impose some boundary conditions on the conformal factor. The boundary conditions introduced in the aforementioned papers can be gathered into the following

$$\partial_\nu u + g_{H,j}u + g_{\theta,j}u^{e_j} + g_{\tau,j}u^{N/2} + g_{w,j}u^{-N/2} = 0 \qquad \text{on } \partial_j M\,, \tag{0.6}$$

for each boundary component $\partial_j M$; where the coefficients $g_{\cdot,j}$ and e_j, are related to the physical meaning of that boundary component, see [**45**] for a comprehensive exposition of the problem. This means that on a manifold with boundary $(M, \partial M, \langle\ ,\ \rangle)$ the Lichnerowicz equation of (0.5) has to be complemented with the boundary condition (0.6), that is

$$\begin{cases} c_m\Delta u + A(x)u - B(x)u^{N-1} + C(x)u^{-N-1} = 0 & \text{on int}\, M\,, \\ \partial_\nu u + g_{H,j}u + g_{\theta,j}u^{e_j} + g_{\tau,j}u^{N/2} + g_{w,j}u^{-N/2} = 0 & \text{on } \partial_j M\,, \end{cases}$$

where the coefficients are given by

$$A(x) = \Big(|\nabla\psi|^2 - S\Big),\ B(x) = \big(\tfrac{m-1}{m}\tau^2 - 2U(\psi)\big), \text{ and } C(x) = \bigg(\Big|\sigma + \mathring{\mathcal{L}}W\Big|^2 + \pi^2\bigg). \tag{0.7}$$

The discussion above motivates the study of positive solutions of the problem

$$\begin{cases} \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on int}\, M, \\ \partial_\nu u + \displaystyle\sum_{i=1}^N g_i(x)u^{q_i} = 0 & \text{on } \partial M; \end{cases}$$

on a general $(M, \partial M, \langle\ ,\ \rangle)$ with mild conditions on the coefficients $a(x)$, $b(x)$, $c(x)$, $g_i(x)$, σ, τ, and q_i, which is a special case of the problem (2.1).

Concerning the boundary conditions, the restrictions on $g_i(x)$ and q_i that we have in mind are

$$\min_{1\le i\le N} q_i < 1 < \max_{1\le i\le N} q_i$$

and such that

$$g_i(x)\,(q_i-1)\geq 0 \qquad \text{for all } i;$$

this last condition in the literature of mathematical General Relativity is known as the *defocusing case* and it is meaningful for the applications, see for instance the aforementioned [**45, 44**] and the references therein.

The other application that motivated us to start the study of problems like (0.1) comes from conformal geometry. We recall that a pointwise conformal deformation of $(M, \partial M, \langle\ ,\ \rangle)$ of dimension $m \geq 3$ is a Riemannian manifold $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$ where $\widetilde{\langle\ ,\ \rangle} = u^{\frac{4}{m-2}}\langle\ ,\ \rangle$ for some smooth positive function u, called the conformal factor of the deformation. Denoting with $(s,\, h)$ and $(\widetilde{s},\, \widetilde{h})$ the scalar curvature and the mean curvature of the boundary respectively of $(M, \partial M, \langle\ ,\ \rangle)$ and $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$, then, as it is well known (see for instance [**25, 36**]), these quantities are related by the equations

$$\begin{cases} \Delta u - c_m\left(s(x)u - \widetilde{s}(x)u^{\frac{m+2}{m-2}}\right) = 0 & \text{on } M \\ \partial_\nu u + d_m\left(h(x)u - \widetilde{h}(x)u^{\frac{m}{m-2}}\right) = 0 & \text{on } \partial M \end{cases} \tag{0.8}$$

where Δ and ν are respectively the Laplace-Beltrami operator of M and the outward unit normal of ∂M in the background metric $\langle\, ,\, \rangle$, while c_m and d_m are the constants given by

$$c_m = \frac{m-2}{4(m-1)}\,, \qquad d_m = \frac{m-2}{2}\,.$$

Clearly problem (0.8) is a very special case of (0.1), in particular we note that the first equation of (0.8) is the very well known Yamabe equation. Just to give a brief history of problem (0.8), we recall that the question of finding a pointwise conformal deformation of $(M, \partial M, \langle\ ,\ \rangle)$ with prescribed scalar curvature in M and prescribed mean curvature of ∂M has been first considered by Cherrier in [**25**]. At the end of the last century, in a series of remarkable papers (see for instance [**34**] and [**35**]), J.F. Escobar considered the Yamabe problem for compact manifolds with boundary. In recent years there has been interest in considering the case of noncompact manifolds, see for instance [**73**] and [**14**].

The plan of the work is the following.

In Chapter 1 we prove all the main analytical results; we have concentrated here the *technical* part of the work that is needed in the subsequent chapters. For this reason the chapter has to be intended as a mathematical *toolbox*. All the results of the chapter are new and not present in the mathematical literature, although the proofs of some results are just sketched when they are a modification of rather standard facts.
In particular. There are proved the various forms of the weak maximum principle that will

be used, the L^∞ estimate for subsolutions, and a sub/super solutions existence result for problems of the form (0.1). Moreover, in this chapter we fix the definitions, the geometric and spectral theory conventions that will be used in the rest of the work.

Chapter 2 is mainly devoted to the study of Lichnerowicz-type equations (0.2) with sign-chaging coefficient $b(x)$ on a complete manifold $(M, \partial M, \langle\ ,\ \rangle)$ with nonempty boundary ∂M. Indeed, it follows from (0.7) that our preferred sign for the coefficient $c(x)$ is $c(x) \geq 0$, while the coefficients $a(x)$ and $b(x)$ have no restrictions. We stress the fact that the presence of a sign-changing $b(x)$ constitutes a difficulty in finding positive solutions. Thus, denoting by $b_+(x)$ and $b_-(x)$ respectively the positive and negative part of $b(x)$ (that is $b(x) = b_+(x) - b_-(x)$), we introduce the function

$$b_\theta(x) = b_+(x) - \theta b_-(x)\,,$$

where $\theta \in (0, 1]$. The function $b_\theta(x)$ is perturbation of $b(x)$ that permits to *modulate* his negative part. Using a rather delicate technique, we prove that it is possible to find a small enough $\theta^* \in (0, 1]$ such that the perturbed equation

$$\Delta u + a(x)u - b_\theta(x)u^\sigma + c(x)u^\tau = 0$$

has a positive solution for each $\theta \in (0, \theta^*)$. We note that this kind of perturbation is coherent with other results appeared in the literature, see for instance [**42**] and [**57**]. Moreover, recalling (0.7), in the case of the Lichnerowicz equation for the Einstein-scalar field, the coefficient $b(x)$ is given by

$$b(x) = \frac{m-2}{4}\left(\frac{1}{m}\tau^2 - \frac{2}{m-1}U(\psi)\right)$$

thus, a simple computation yields

$$0 \leq \theta b_-(x) \leq \theta\frac{(m-2)}{2(m-1)}U_+(\psi)$$

so that the modulation $b_\theta(x)$ can be interpreted phisically as a smallness requirement for $U_+(\psi)$, the positive part of the potential. At the end of the chapter we prove two uniqueness results for solutions of a Dirichlet problem for (0.2) in the case of non negative $b(x)$, the first by assuming the validity of a suitable weak maximum principle while the second requires an integral condition at infinity.

In Chapter 3 we continue the study of Lichnerowicz-type equations, but we focus on the special case of equation (0.2) with a non negative $b(x)$ on a manifold $(M, \langle\ ,\ \rangle)$ without boundary. In this setting, under suitable spectral assumptions, we prove the existence of a *maximal* positive solution. Moreover, using a spectral technique developed in [**18, 19**], we prove another uniqueness theorem for positive solutions of (0.2). The last section of the chapter is devoted to some applications of the L^∞ estimates proved in Chapter 1. More precisely, using the peculiar simmetric structure of (0.2), under pointwise assumptions on the coefficients, we prove a bilateral *a priori* estimate for positive solutions of (0.2). This means that there exist two constants $0 < \mathcal{K} \leq \mathcal{H}$ such that any smooth positive solution u of (2.1) has to satisfy

$$\mathcal{K} \leq u \leq \mathcal{H}\,.$$

This estimate, together with the uniqueness result of Chapter 2, immediately yields a Liouville-type theorem for positive solutions of (0.2), under a volume growth assumption at infinity.

In Chapter 4 we deal with the problem of prescribing the scalar curvature on a complete manifold with boundary $(M, \partial M, \langle\ ,\ \rangle)$ via pointwise conformal deformations, that is, problem (0.8). We stress the fact that in our case ∂M is not supposed to be compact neither connected. Our main result is a generalization of the Schwarz Lemma in his geometric form, the so called Schwarz-Pick Lemma. We recall that a conformal diffeomorphism $f : (M, \partial M, \langle\ ,\ \rangle) \to (M, \partial M, \langle\ ,\ \rangle)$ with conformal factor u is said to be *weakly distance decreasing* if $u \leq 1$ on M, see [**69**]. Thus we prove that, assuming some natural conditions on the growth at infinity of the scalar curvatures $s(x)$, $\widetilde{s}(x)$, and of the volume of geodesic balls, a conformal deformation has to be weakly distance decreasing, provided that the technical condition

$$\widetilde{h}(x) \leq u^{-\frac{2}{m-2}} h(x) \quad \text{on } \partial M$$

holds. This result basically extends Theorem 3.3 of [**65**] to this new setting. The delicate issue in the present case is due to the above condition, which involves the conformal factor u. Thus we provide some natural geometric assumptions that imply the technical condition, providing some interesting rigidity results. We then turn to another rigidity geometric problem proposed by J.F. Escobar in the compact case [**36**]. The precise question is: given a conformal diffeomorphism of a Riemannian manifold with boundary $(M, \partial M, \langle\ ,\ \rangle)$ such that $\widetilde{s} = s$ on M and $\widetilde{h} = h$ on ∂M, when does it happens that $\widetilde{\langle\ ,\ \rangle} = \langle\ ,\ \rangle$? He proved that whenever $s(x)$ and $h(x)$ are non positive, then the diffeomoporhism is an isometry. We extend this result to the noncompact case by requiring a control of the growth of the volume of geodesic balls and of the scalar curvature $s(x)$ at infinity. It has to be stressed that this conditions are automatically satisfied in the compact case. The last two sections are devoted to prove the existence of non-trivial conformal deformations. Firstly we address the problem of finding nontrivial conformal deformations with possibily sign-changing scalar curvature $\widetilde{s}(x)$. In this case, supersolutions of (0.8) are constructed as we did in Chapter 2 for the Lichnerowicz-type equations, the construction of subsolutions is carried out using a delicate exhaustion with solutions of an associated Dirichlet problem.

Lastly we consider the Yamabe problem, namely the case of $\widetilde{s}(x)$ constant and $\widetilde{h}(x) \equiv 0$. In particular we prove that, under mild spectral and volume growth assumptions, a complete manifold $(M, \partial M, \langle\ ,\ \rangle)$ such that $h(x) \geq 0$ and $s(x) \in \mathrm{L}^{\infty}(M)$ can be deformed to a $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$ such that $\widetilde{s}(x) = C < 0$ and $\widetilde{h}(x) \equiv 0$.

CHAPTER 1

The analytical toolbox

This chapter is devoted to the main technical tools that will be repeatedly used in the next chapters.

1.1. Complete manifolds with boundary

In this section we fix notations and collect some useful facts on the geometry of complete Riemannian manifolds with boundary. From now on $(M, \partial M, \langle\ ,\ \rangle)$ will denote a smooth, complete Riemannian manifold of dimension $m \geq 2$, and smooth boundary ∂M. The topology on the manifold is understood to be the relative one, that is, a basis for the open sets is that of the metric balls $B_r(x_0)$ centered at any point of M, regardless of belonging or not to the boundary ∂M. To be clear, with this topology, the half ball

$$B_r^+(o) = \left\{ (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n x_i^2} < r,\ x_n \geq 0 \right\}$$

is open in the manifold with boundary $\left(\mathbb{R}^n_+, \mathbb{R}^{n-1}, \langle\, , \rangle_{eucl}\right)$. We will adopt the notation $\operatorname{int} M = M \setminus \partial M$.
Let $\Omega \subset M$ be a domain, in order to deal with boundary value problems on M we need to split the boundary, $\partial\Omega$, in two different parts

$$\partial_0\Omega = \partial\Omega \cap \operatorname{int} M\,; \qquad \partial_1\Omega = \partial\Omega \cap \partial M\,,$$

clearly $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$.
It is worth to spend some words on the notion of completeness for a Riemannian manifold with boundary. Indeed in this case the familiar Hopf-Rinow theorem does not hold, because the presence of the boundary prevents the infinite extendability of geodesics. Thus the completeness of $(M, \partial M, \langle\ ,\ \rangle)$ has to be understood in the sense of metric spaces. Here the distance between two points $p, q \in M$ is defined as usual as

$$\operatorname{dist}(p, q) = \inf_{\sigma \in \Sigma^1_{p,q}} l(\sigma)$$

where $\Sigma^1_{p,q}$ is the space of C^1 paths starting at p and ending at q, and $l(\sigma)$ is the lenght of σ with respect to the metric $\langle\, , \rangle$. A first unpleasant fact to be noted is that, differently to what happens when the boundary is empty, the optimal regularity of a geodesic connecting the points p and q is $C^{1,1}$ even if the boundary is smooth. For a deep analysis of the situation we refer to a series of papers by S.B. Alexander, I.D. Berg, and R.L. Bishop [**9, 10, 11**].

In the sequel we will assume that a reference point $o \in M$ has been fixed and we will denote by $r : M \to \mathbb{R}_0^+$ the distance function from o, that is,

$$r(x) = \mathrm{dist}(o, x)\,,$$

clearly $r \in \mathrm{Lip}(M)$. Moreover, for $t \in \mathbb{R}^+$ and $y \in M$, we let $B_t(y)$ be the geodesic ball of radius $t \in \mathbb{R}^+$ centered at $y \in M$, that is,

$$B_t(y) := \{x \in M : \mathrm{dist}(x, y) < t\}\ ,$$

in particular we set $B_t = B_t(o)$.

Let $\rho : M \to \mathbb{R}_0^+$ be the distance function from the boundary defined as

$$\rho(x) = \mathrm{dist}(x, \partial M) = \inf_{y \in \partial M} \mathrm{dist}(x, y)\,,$$

where the infimum is always attained since ∂M is a closed set of a complete metric space. Moreover $\rho \in \mathrm{Lip}(M)$ and it is smooth and minimizes the distance from ∂M out of his cut locus, which is a set of measure zero (see for instance [**54**]). For $\varepsilon > 0$ we set

$$M_\varepsilon = \{x \in M : \rho(x) < \varepsilon\}\ .$$

We introduce Fermi coordinates with respect to the boundary ∂M (see for instance Section 10 of [**62**] for a well written review on Fermi coordinates). Let us define, for $y \in \partial M$ and $t \in \mathbb{R}^+$,

$$\Phi_{\partial M}(y, t) := \exp_y(-t\nu_y),$$

where exp denotes the exponential map and ν_y the outward normal at the point y. From the properties of ρ (see [**54**]), for each $y \in \partial M$ there exist $\varepsilon_y > 0$ such that for $t \in [0, \varepsilon_y)$, $\Phi_{\partial M}(y, t)$ does not meet the cut locus of y, we define τ_y to be the sup of these ε_y. In general, if ∂M is noncompact, it can happen that $\inf_{y\in\partial M} \tau_y = 0$, this implies that it could not exist an ε such that there exist global Fermi coordinates on M_ε. Let $U = U_y \subset \partial M$ be an open bounded set (in the topology of ∂M), let $\tau_U = \inf_{y\in U} \tau_y > 0$ and choose $0 < \tau < \tau_U$. We define the *Fermi cylinder* of base U and height τ to be the subset of M given by

$$C_y(U, \tau) := \{\Phi_{\partial M}(y, t) : y \in U, t \in [0, \tau)\}\ .$$

1.2. Weak maximum principle

We note that recently some attention has been put on global properties of solutions (or subsolutions) to elliptic equations on complete manifolds with boundary (see for instance [**73, 46, 14**]). Motivated by this fact and by the main problem considered in this work, we introduce a version of the weak maximum principle for manifolds with boundary.

In what follows $q(x)$ will denote a continous and positive function on M. Let $\mathcal{F}(M)$ be a set of functions defined on M such that $\mathrm{C}^0(M) \supseteq \mathcal{F}(M)$. We start by stating the following

DEFINITION 1.1. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold with non empty boundary. We say that a function $u \in \mathcal{F}(M)$ such that $u^* = \sup_M u < +\infty$, satisfies the* q-boundary weak maximum principle*, for short q-∂WMP, on M for the operator L if for each $\gamma < u^*$ we have*

$$\inf_{\Omega_\gamma} q(x) Lu \leq 0\,,$$

where Ω_γ denotes the superlevel set

$$\Omega_\gamma = \{x \in M\ :\ u(x) > \gamma\}\,.$$

The definition above extends the weak maximum principle of Pigola, Rigoli, and Setti [**66**] (see also the very recent improvements in [**5, 13**] and the book [**12**]) to the case of manifolds with boundary. Here the point, to deal with ∂M, is on the choice of a suitable functional space $\mathcal{F}(M)$ to obtain the maximum principle. As it will be apparent in the sequel, the presence of a possibly nonempty boundary ∂M generates some subtleties.

The following example suggests the necessity of some *boundary conditions* for the validity of the weak maximum principle.

EXAMPLE 1.2. For some fixed $\varepsilon > 0$, we define the subset of $\mathbb{R}^m$

$$\Lambda = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m\ :\ x_m - \sum_{i=1}^{m-1} x_i^2 \geq \varepsilon^2 \right\}\,,$$

clearly Λ is a complete Riemannian manifold with boundary. Consider the function

$$u(x) = \varepsilon - \left(x_m - \sum_{i=1}^{m-1} x_i^2 \right)^{1/2}\,,$$

it is easy to see that $u \in \mathrm{C}^1(\Lambda) \cap \mathrm{C}^\infty(\operatorname{int}\Lambda)$. Furthermore $u \leq 0$ on Λ, the maximum $u^* = 0$ is attained at each point of $\partial\Lambda$ and only there. Indeed, for $\gamma < 0 = u^*$ the superlevel set

Ω_γ is given by

$$\Omega_\gamma = \left\{ x \in \mathbb{R}^m \, : \varepsilon^2 \le x_m - \sum_{i=1}^{m-1} x_i^2 \le (\varepsilon - \gamma)^2 \right\} .$$

A simple computation yields

$$\Delta u = \frac{1}{4\left(x_m - \sum_{i=1}^{m-1} x_i^2\right)^{3/2}} + \frac{(m-1)x_m + (2-m)\sum_{i=1}^{m-1} x_i^2}{\left(x_m - \sum_{i=1}^{m-1} x_i^2\right)^{3/2}} ,$$

from which it follows that

$$\inf_{\Omega_\gamma} \Delta u = \frac{1 + 4(m-1)\varepsilon^2}{(\varepsilon - \gamma)^3} > 0 .$$

We note that in the example above the function u is such that

$$\partial_\nu u > 0 \qquad \text{on } \partial\Lambda .$$

This shows that in general, we cannot expect to have the validity of the weak maximum principle if the outer normal derivative on the boundary is positive. On the other hand we will prove that, requiring a suitable relaxed form of the inequality

$$\partial_\nu u \le 0 \qquad \text{on } \partial\Lambda ,$$

the weak maximum principle holds true.

In what follows we shall deal with a large class of linear operators that we are now going to define. We let T be a symmetric, positive definite, covariant 2-tensor field on M. We define the operator $L = L_T$ acting on $u \in \mathrm{C}^2(M)$ as

$$\begin{aligned} Lu &= \operatorname{div}\left(T(\nabla u, \ )^\sharp\right) \\ &= \operatorname{tr}\left(T \circ \operatorname{Hess}(u)\right) + \operatorname{div} T^\sharp(\nabla u) \end{aligned} \tag{1.1}$$

where $^\sharp : T^*M \to TM$ denotes the musical isomorphism. On a manifold with boundary $(M, \partial M, \langle \ , \ \rangle)$ differential inequalities related to the above operator can be interpreted in the following weak sense: $u \in \mathrm{C}^1(M)$ is a solution of the differential inequality

$$Lu \ge f(u)$$

for some $f \in \mathrm{C}^0(\mathbb{R})$, if and only if $\forall\, \phi \in \mathrm{C}^\infty_c(M)$, $\phi \ge 0$

$$\int_M [T(\nabla u, \nabla\phi) + \phi f(u)] \le \int_{\partial M} \phi\, T(\nabla u, \nu) \tag{1.2}$$

where ν is the outward unit normal to ∂M. Moreover, the validity of the inequality

$$\int_M [T(\nabla u, \nabla\phi) + \phi f(u)] \le 0 \tag{1.3}$$

for all $\phi \in \mathrm{C}_c^\infty(M)$, $\phi \geq 0$ defines a weak solution of the Neumann problem

$$\begin{cases} Lu \geq f(u) & \text{on } M \\ T(\nabla u, \nu) \leq 0 & \text{on } \partial M\,. \end{cases} \tag{1.4}$$

The key point here is that we will exploit the weak form (1.4) to extend the action of (1.1) to classes of functions broader than $\mathrm{C}^1(M)$. Indeed we observe that Hölder's inequality implies that given $\phi \in \mathrm{C}_c^\infty(M)$, $\phi \geq 0$, the first equation of (1.4) is well defined for any $u \in \mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(M)$ (indeed $u \in \mathrm{L}^\infty_{\mathrm{loc}}(M) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(M)$ would be sufficient). When $\partial M \neq \emptyset$, the interpretation of the second equation of (1.4) requires a more subtle analysis. Here the issue is that the boundary ∂M is a set of measure zero in M and this means that the integral

$$\int_{\partial M} \phi\, T(\nabla u, \nu)$$

in general is not well defined for $u \in \mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(M)$.

A first way to solve the problem is suggested by Gagliardo trace theorem (see for instance Theorem 4.12 of [**2**]) which ensures that functions $u \in \mathrm{W}^{2,2}_{\mathrm{loc}}(M)$ have a well defined trace

$$\nabla u \in \mathrm{L}^2_{\mathrm{loc}}(\partial M)\,.$$

Another way is to restrict to test functions $\phi \in \mathrm{C}_c^\infty(M)$, $\phi \geq 0$, and such that $\phi|_{\partial M} \equiv 0$. In this way the boundary term vanishes identically.
By a standard density argument in the discussion above it is equivalent to consider as test functions those $\psi \in \mathrm{W}^{1,2}_0(M)$, such that $\psi \geq 0$. Here as usual $\mathrm{W}^{1,2}_0(M)$ denotes the closure of $\mathrm{C}_c^\infty(M)$ with respect to the $\mathrm{W}^{1,2}$-norm. This fact will be further discussed below.

The first result (Theorem 1.5 below) gives a useful criterion for the validity of q-∂WMP for the operator L_T under the assumption of a suitably controlled volume growth of geodesic balls at infinity.

Remark 1.3. The condition on the volume growth is very mild on a Riemannian manifold without boundary and, for instance, is implied (but much less demanding) by an appropriate corresponding conditions on the Ricci curvature of the manifold. In the case of a manifold with a nonempty boundary ∂M it is in general not possible to obtain informations about the volume of geodesic balls from curvature hypoteses. Indeed, as it is shown in [**9**] no Laplacian comparison theorem holds in this framework. Thus, the volume growth assumption seems to be quite adequate in this case.

We assume that T satisfies

$$0 < T_-(r) \leq T(X,X) \leq T_+(r)$$

for all $X \in T_xM$, $|X| = 1$, $x \in \partial B_r$, and some $T_\pm \in \mathrm{C}^0(\mathbb{R}_0^+)$. Furthermore, set

$$\Theta(r) = \max_{[0,r]} T_+(s)\,.$$

The following table defines our spaces of admissible functions.

Space	Regularity	Boundary behaviour	Test space
$\mathcal{B}_1(M)$	$\mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(M)$	$\forall x \in \partial M$, $\exists\, \varepsilon, \tau > 0$ s.t. $\forall\ 0 \le \psi \in \mathrm{L}^2_{\mathrm{loc}}(M)$, $\int_{C_x(B_\varepsilon(x),\tau)} \psi\, T(\nabla u, \nabla \rho) \ge 0$	$\phi \in \mathrm{W}^{1,2}_0(M)$, $\phi \ge 0$, $\phi\|_{\partial M} \equiv 0$
$\mathcal{B}_2(M)$	$\mathrm{C}^0(M) \cap \mathrm{W}^{2,2}_{\mathrm{loc}}(M)$	$T(\nabla u, \nu) \le 0$ on ∂M	$\phi \in \mathrm{W}^{1,2}_0(M)$, $\phi \ge 0$

Where the $C_x(B_\varepsilon(x), \tau)$ is the Fermi cylinder defined by (**??**). We also set the following.

Definition 1.4. *For $K \subseteq \partial M$ and $u \in \mathcal{B}_1(M)$,*

$$H_u(K) = \inf_{x \in K} \left\{ \tau(x) \ : \forall\ 0 \le \psi \in \mathrm{L}^2_{\mathrm{loc}}(M), \int_{C_x(B_\varepsilon(x),\tau(x))} \psi\, T(\nabla u, \nabla \rho) \ge 0 \right\}.$$

Clearly, if K is compact, then $H_u(K) > 0$.

We are now ready to prove the next result. Although stated in different terms, that is, as sufficient condition for the validity of the q-∂WMP, it is basically a generalization of Theorem A of [**65**] to the case of manifolds with boundary. Thus its proof follows the lines of the argument used in the proof of the aforementioned Theorem A. However, due to the very subtle technicalities involved in the reasoning, we feel necessary, for a better understanding and for the ease of the reader, to provide a complete and detailed proof exposition in this new setting.

Theorem 1.5. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete, noncompact, Riemannian manifold with boundary and denote with r the distance function from a fixed point $o \in M$. Let $q \in \mathrm{C}^0(M)$, $q \ge 0$, and such that $q(x) \le Q(r(x))$ where $Q(t)$ is positive, nondecreasing, satisfying*

$$\Theta(t)Q(t) = o(t^2) \qquad \text{as } t \to +\infty \tag{1.5}$$

and

$$\liminf_{t \to +\infty} \frac{\Theta(t)Q(t) \log \operatorname{vol} B_t}{t^2} < +\infty\,. \tag{1.6}$$

If $u \in \mathcal{B}_1(M)$ or $u \in \mathcal{B}_2(M)$ is such that $u^* = \sup_M u < +\infty$ then it satisfies the q-∂WMP on M for L.

PROOF. Assume, by way of contradiction, that the space $\mathcal{B}_1(M)$ (respectively $\mathcal{B}_2(M)$) is not L-admissible for the q-∂WMP on M. We may suppose that, for some $\gamma < u^*$ and $u \in \mathcal{B}_1(M)$ (respectively $\mathcal{B}_2(M)$) we have

$$Lu \geq \frac{B}{Q(r(x))} \quad \text{on } \Omega_\gamma$$

for some $B > 0$ that, without loss of generality we can suppose to be 1. Fix $0 < \eta < 1$. By choosing γ sufficiently close to u^*, we may suppose that

$$\Gamma = \gamma - u^* + \eta \geq \frac{\eta}{2} > 0,$$

so that, having defined $v = u - u^* + \eta$, we have

$$v^* = \sup v = \eta, \quad \Omega_\Gamma^v = \Omega_\gamma^u,$$

where Ω_Γ^v is defined as

$$\Omega_\Gamma^v = \{x \in M \, : \, v(x) > \Gamma\} \, .$$

Furthermore,

$$Lv \geq \frac{1}{Q(r(x))} \quad \text{on } \Omega_\Gamma^v. \tag{1.7}$$

Choose $R_0 > 0$ large enough that $B_{R_0} \cap \Omega_\Gamma^v \neq \emptyset$. For a fixed $R \geq R_0$ let $\psi_R : M \to [0,1]$ be a smooth cut-off function such that

$$\begin{array}{lll} i) & \psi_R \equiv 1 & \text{on } B_R; \\ ii) & \psi_R \equiv 0 & \text{on } M \setminus B_{2R}; \\ iii) & |\nabla \psi_R| \leq \frac{C_0}{R} \psi_R^{1/2}, & \end{array} \tag{1.8}$$

for some constant $C_0 > 0$. Note that requirement *iii)* is possible because the exponent $1/2$ is less than 1. Next, let $\lambda : \mathbb{R} \to \mathbb{R}_0^+$ be a $\mathcal{C}^1$ function such that

$$\begin{array}{lll} i) & \lambda \equiv 0 & \text{on } (-\infty, \Gamma]; \\ ii) & \lambda'(t) \geq 0 & \text{on } \mathbb{R}; \\ iii) & \lambda \leq 1. & \end{array} \tag{1.9}$$

Fix $\alpha > 2$ and $0 \leq \beta_R \in \mathrm{Lip}(B_{2R} \cap \Omega_\Gamma^v)$ to be determined later. Consider the function ϕ_R defined by

$$\phi_R = \beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha - 1} \quad \text{on } \Omega_\Gamma^v \tag{1.10}$$

and $\phi_R \equiv 0$ outside Ω_Γ^v. Note that $\phi_R \equiv 0$ off $B_{2R} \cap \Omega_\Gamma^v$ and moreover $\phi_R \in \mathrm{W}_0^{1,2}(M)$. For future use it can be checked that the weak gradient of ψ_R satisfies the following identity

$$\begin{aligned} \nabla \phi_R = {} & \psi_R^{2\alpha} \lambda(v) v^{\alpha-1} \nabla \beta_R + 2\alpha \beta_R \psi_R^{2\alpha - 1} \lambda(v) v^{\alpha - 1} \nabla \psi_R \\ & + \beta_R \psi_R^{2\alpha} \lambda'(v) v^{\alpha-1} \nabla v + (\alpha - 1) \beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha - 2} \nabla v \, . \end{aligned}$$

For the ease of notation we set

$$T_v = \frac{T(\nabla v, \nabla v)}{|\nabla v|^2}\,,$$

furthermore,

$$|T(\nabla v, \nabla \psi_R)| \leq \sqrt{\frac{T(\nabla v, \nabla v)}{|\nabla v|^2}}|\nabla v|\sqrt{\frac{T(\nabla \psi_R, \nabla \psi_R)}{|\nabla \psi_R|^2}}|\nabla \psi_R| \leq T_v^{1/2} T_+^{1/2}(R)|\nabla v||\nabla \psi_R|,$$

that is,

$$|T(\nabla v, \nabla \psi_R)| \leq T_v^{1/2} T_+^{1/2}(R)|\nabla v||\nabla \psi_R|\,. \tag{1.11}$$

Next, we consider two different cases.

Case I: $u \in \mathcal{B}_1(M)$.
In this case for $R \geq R_0$ we consider the function $0 \leq \beta_R \in \mathrm{Lip}(B_{2R} \cap \Omega_\Gamma^v)$ defined by

$$\beta_R(x) = \begin{cases} \frac{1}{\varepsilon}\rho(x) & \text{on } M_\varepsilon \cap B_{2R} \cap \Omega_\Gamma^v \\ 1 & \text{on } (M \setminus M_\varepsilon) \cap B_{2R} \cap \Omega_\Gamma^v \end{cases} \tag{1.12}$$

where

$$\varepsilon = \varepsilon(R) = \min\left\{\mathrm{inj}_\rho(\partial M \cap B_{2R}),\, H_u(\partial M \cap B_{2R})\right\}\,, \tag{1.13}$$

with

$$\mathrm{inj}_\rho(U) = \sup\left\{\tau \in \mathbb{R}^+ \ : C_x(U, \tau) \cap \mathrm{cut}_\rho(\partial M) = \emptyset\right\},$$

and $H_u(\partial M \cap B_{2R})$ as in Definition 3.4. Since $\partial M \cap B_{2R} \subset\subset \partial M$, it follows that $\varepsilon(R) > 0$ for $R > R_0$ (see for instance [**54**]), and β_R is well defined. We note that for $S \geq R$ we have the trivial inclusion $B_{2R} \subseteq B_{2S}$, thus, from (1.13) it follows that $\varepsilon(S) \leq \varepsilon(R)$. In particular this implies that, for $S \geq R$, $0 \leq \beta_S \in \mathrm{Lip}(B_{2R} \cap \Omega_\Gamma^v)$ and moreover

$$\beta_S \geq \beta_R \quad \text{on } B_{2R}. \tag{1.14}$$

With this choice of β_R we have that $0 \leq \phi_R \in \mathrm{W}_0^{1,2}(M)$ and $\phi|_{\partial M} \equiv 0$. Thus ϕ_R is an admissible test function for $u \in \mathcal{B}_1(M)$. Recalling that $\lambda' \geq 0$ and using ϕ_R to test inequality (1.7) we get

$$\begin{aligned} 0 \geq \int_{B_{2R}} & \psi_R^{2\alpha}\lambda(v)v^{\alpha-1}T(\nabla v, \nabla \beta_R) + 2\alpha\beta_R\psi_R^{2\alpha-1}\lambda(v)v^{\alpha-1}T(\nabla v, \nabla \psi_R) \\ &+ \int_{B_{2R}} \beta_R\psi_R^{2\alpha}\lambda(v)v^{\alpha-1}\frac{1}{Q(r(x))} + (\alpha-1)\beta_R\psi_R^{2\alpha}\lambda(v)v^{\alpha-2}T_v\,|\nabla v|^2\,. \end{aligned}$$

If we set

$$I_R(\alpha) = \int_{B_{2R}} \psi_R^{2\alpha}\lambda(v)v^{\alpha-1}T(\nabla v, \nabla \beta_R)\,,$$

then using (1.11) and rearranging, we obtain

$$\int_{B_{2R}} \frac{\beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha-1}}{Q(r(x))} \le -I_R(\alpha) - (\alpha - 1) \int_{B_{2R}} \beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha-2} T_v \left|\nabla v\right|^2$$
$$+ 2\alpha \int_{B_{2R}} \beta_R \psi_R^{2\alpha-1} \lambda(v) v^{\alpha-1} T_v^{1/2} T_+^{1/2} \left|\nabla v\right| \left|\nabla \psi_R\right| .$$

We apply to the second integral on the right hand side the inequality

$$ab \le \sigma \frac{a^2}{2} + \frac{b^2}{2\sigma}$$

with

$$a = \psi_R^{\alpha} v^{\alpha/2-1} T_v^{1/2} \left|\nabla v\right| ,$$
$$b = \psi_R^{\alpha-1} v^{\alpha/2} T_+^{1/2} \left|\nabla \psi_R\right| ,$$

and $\sigma = \frac{\alpha-1}{\alpha}$ so that the first integral on the right hand side cancels out. Indeed, we have

$$\int_{B_{2R}} \frac{\beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha-1}}{Q(r(x))} \le -I_R(\alpha) + \frac{\alpha^2}{\alpha-1} \int_{B_{2R}} \beta_R \psi_R^{2\alpha-2} \lambda(v) v^{\alpha} T_+ \left|\nabla \psi_R\right|^2 . \tag{1.15}$$

Now, in order to control the first term on the right hand side, we note that from the definition of β_R it follows that

$$I_R(\alpha) = \frac{1}{\varepsilon} \int_{M_\varepsilon \cap B_{2R} \cap \Omega_\Gamma^v} \psi_R^{2\alpha} \lambda(v) v^{\alpha-1} T(\nabla v, \nabla \rho) ,$$

thus, since $v \in \mathcal{B}_1(M)$, $\psi_R^{2\alpha} \lambda(v) v^{\alpha-1}$ is locally bounded (indeed continuous), from the choice (1.13) we conclude that

$$I_R(\alpha) \ge 0 , \tag{1.16}$$

for $R \ge R_0$.

Now, since Q is non-decreasing, $Q(r(x)) \le Q(2R)$ on the support of ψ and the left hand side of (1.15) is bounded from below by

$$\frac{1}{Q(2R)} \int_{B_{2R}} \beta_R \psi_R^{2\alpha} \lambda(v) v^{\alpha-1} . \tag{1.17}$$

On the other hand

$$\frac{\alpha}{\alpha-1} \le 2 \quad \text{for } \alpha \ge 2 ,$$

and furthermore, using (1.8) *iii)*, we may write

$$\psi_R^{2\alpha-2} |\nabla \psi_R|^2 = \psi_R^{2\alpha-1} (\psi_R^{-1/2} |\nabla \psi_R|)^2 \le \psi_R^{2\alpha-1} \frac{C_0^2}{R^2} .$$

Finally, we recall that

$$T_+(r(x)) \le \Theta(2R) \quad \text{on } B_{2R}.$$

Thus, the right hand side of (1.15) can be estimated from above by

$$2\alpha\,\Theta(2R)\frac{C_0^2}{R^2}\int \beta_R\psi_R^{2\alpha-1}\lambda(v)v^{\alpha}\,.$$

Now, we apply Hölder's inequality with conjugate exponents $\alpha/(\alpha-1)$ and α to estimate from above this last expression with

$$2\alpha\,\Theta(2R)\frac{C_0^2}{R^2}\left(\int \beta_R\psi_R^{2\alpha}\lambda(v)v^{\alpha-1}\right)^{\frac{\alpha-1}{\alpha}}\left(\int \beta_R\psi_R^{\alpha}\lambda(v)v^{2\alpha-1}\right)^{\frac{1}{\alpha}}. \tag{1.18}$$

Using (1.16), (1.17), and (1.18) into (1.15), after a rearrangement we have

$$\int \beta_R\psi_R^{2\alpha}\lambda(v)v^{\alpha-1} \le \left(2\alpha\,\Theta(2R)Q(2R)\frac{C_0^2}{R^2}\right)^{\alpha}\int \beta_R\psi_R^{\alpha}\lambda(v)v^{2\alpha-1}.$$

Recalling that $\psi_R \equiv 1$ on B_R, $\psi_R \equiv 0$ on $M\setminus B_{2R}$ and that $\eta/2 \le v \le \eta$ on Ω_Γ^v when $\lambda(v) > 0$, we deduce that

$$\int_{B_R} \beta_R\lambda(v) \le \left(\eta\alpha\, 2^{(2\alpha-1)/\alpha}\Theta(2R)Q(2R)\frac{C_0^2}{R^2}\right)^{\alpha}\int_{B_{2R}} \beta_R\lambda(v).$$

Moreover, using (1.14) with $S = 2R$, we get

$$\begin{aligned}\int_{B_R} \beta_R\lambda(v) &\le \frac{1}{2}\left(\eta\alpha\,\Theta(2R)Q(2R)\frac{C_1}{R^2}\right)^{\alpha}\int_{B_{2R}} \beta_{2R}\lambda(v)\\ &\le \left(\eta\alpha\,\Theta(2R)Q(2R)\frac{C_1}{R^2}\right)^{\alpha}\int_{B_{2R}} \beta_{2R}\lambda(v).\end{aligned} \tag{1.19}$$

with

$$C_1 = 4C_0^2$$

We now set

$$\alpha = \alpha(R) = \frac{1}{2\eta C_1}\frac{R^2}{\Theta(2R)Q(2R)}$$

(which, as follows from (1.5), is ≥ 2 for R sufficiently large) so that we can rewrite (1.19) as

$$\int_{B_R} \beta_R\lambda(v) \le \left(\frac{1}{2}\right)^{\frac{1}{2\eta C_1}\frac{R^2}{\Theta(2R)Q(2R)}}\int_{B_{2R}} \beta_{2R}\lambda(v), \tag{1.20}$$

for each $R \ge R_0$. Note that C_1 is independent of R_0 and η. We now need the following result proved in [**65**] (see Lemma 1.1).

Lemma 1.6. *Let $G, F : [R_0, +\infty)\to\mathbb{R}_0^+$ be non-decreasing functions such that for some constants $0 < \Lambda < 1$ and $B, \theta > 0$*

$$G(R) \le \Lambda^{B\frac{R^\theta}{F(2R)}}G(2R), \text{ for each } R \ge R_0. \tag{1.21}$$

Then there exists a constant $S = S(\theta) > 0$ *such that for each* $R \geq 2R_0$

$$\frac{F(R)}{R^\theta}\log G(R) \geq \frac{F(R)}{R^\theta}\log G(R_0) + SB\log(\frac{1}{\Lambda}). \tag{1.22}$$

We set $G(R) = \int_{B_R}\beta_R\lambda(v)$. G is non-decreasing, indeed, using the monotonicity of integral and (1.14), for $S \geq R$

$$G(S) = \int_{B_S}\beta_S\lambda(v) \geq \int_{B_R}\beta_S\lambda(v) \geq \int_{B_R}\beta_R\lambda(v) = G(R)\,.$$

Thus we can apply Lemma 1.6 with $G(R)$ as above, $\theta = 2$, $\Lambda = 1/2$, $B = \frac{1}{2\eta C_1}$, $F(R) = Q(R)\Theta(R)$ to deduce that for each $R \geq 2R_0$

$$\frac{Q(R)\Theta(R)}{R^2}\log\int_{B_R}\beta_R\lambda(v) \geq \frac{Q(R)\Theta(R)}{r^2}\log\int_{B_R}\beta_R\lambda(v) + \frac{1}{24\eta\, C_1}\log 2. \tag{1.23}$$

Now since $\sup\beta_R = \sup\lambda = 1$, letting $R \to +\infty$ in (1.23) and using (1.5) we obtain

$$\begin{aligned}\liminf_{R\to+\infty}\frac{Q(R)\Theta(R)}{R^2}\log\operatorname{vol}B_R &\geq \liminf_{R\to+\infty}\frac{Q(R)\Theta(R)}{R^2}\log\int_{B_R}\beta_R\lambda(v)\\ &\geq \frac{1}{24\eta\, C_1}\log 2\,,\end{aligned}$$

with C_1 independent of η. Letting $\eta \to 0^+$ we contradict (1.6). This completes the proof of the theorem.

Case II: $u \in \mathcal{B}_2(M)$.
In this case the proof is simpler, indeed we take $\beta_R \equiv 1$ for each R, then the boundary behaviour of $\mathcal{B}_2(M)$ permits to estimate immediately the boundary term (the $I_R(\alpha)$ term of the previous case), obtaining inequality (1.19).
Then the proof follows that of Case I. □

From the theorem above we deduce easily the following result which extends Theorem A of [**65**].

Theorem 1.7. *Let* $(M, \partial M, \langle\ ,\ \rangle)$ *be a Riemannian manifold with boundary. Let* $f \in C^0(\mathbb{R})$ *and assume that* $u \in \mathcal{B}_1(M)$ *(or* $\mathcal{B}_2(M)$*) satisfy* $u^* = sup_M u < +\infty$*, and*

$$Lu \geq b(x)f(u) \quad \text{on } \Omega_\gamma \tag{1.24}$$

where as usual

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}\ ,$$

for some $\gamma < u^*$*, where* $b(x)$ *is a continuous positive function on* Ω_γ *satisfying*

$$b(x) \geq \frac{1}{Q(r(x))} \quad \text{outside a compact set} \tag{1.25}$$

and $Q(x)$ is as Theorem 1.5. If Q satisfies (1.5) and (1.6) then $f(u^*) \leq 0$.

PROOF. Assume by contradiction that $f(u^*) = 2\varepsilon > 0$, then by the continuity of f and u, there exists a $\gamma < \gamma_\varepsilon < u^*$ such that

$$f(u) > \varepsilon \qquad \text{on } \Omega_{\gamma_\varepsilon},$$

thus, from (1.24) it follows that

$$\inf_{\Omega_{\gamma_\varepsilon}} \frac{1}{b(x)} Lu \geq \inf_{\Omega_{\gamma_\varepsilon}} f(u) > \varepsilon > 0\,,$$

which is impossible, since by Theorem 1.5 it follows that u satisfies the $\frac{1}{b}$-∂WMP on M. $\square$

We conclude the section with the following observation, providing a sufficient condition for the validity of the weak maximum principle in the case of $L = \Delta$.

REMARK 1.8. Let $L = \Delta$, the Laplace-Beltrami operator on M. In this case $\Theta(r) \equiv 1$ and choosing $\mu < 2$, we have that the result of Theorem 1.5 holds true for any $q(x) \leq 1 + r(x)^\mu$ if conditions (1.5) and (1.6) are substituted by the single condition

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{2-\mu}}\,. \tag{1.26}$$

Analogously, condition (1.25) of Theorem 1.7 reads as

$$b(x) \geq \frac{C}{r(x)^\mu}\,, \tag{1.27}$$

for some $C > 0$ and μ as above.

1.3. An equivalent form of the weak maximum principle

The aim of this section is to localize the weak maximum principle to the family of open sets Ω whose boundary intersects non trivially with $\operatorname{int} M$. Here is the appropriate

DEFINITION 1.9. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold with non empty boundary ∂M. We say that the* open boundary q-weak maximum principle*, for short q-$O\partial WMP$, holds on M for the operator L if for each $f \in C^0(\mathbb{R})$, for each open set $\Omega \in M$ with $\partial_0 \Omega \neq \emptyset$, and for each $v \in C^0(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ satisfying*

$$\begin{cases} q(x)Lv \geq f(v) & \text{on } \Omega \\ T(\nabla v, \nu) \leq 0 & \text{on } \partial_1\Omega \\ \sup_\Omega v < +\infty \end{cases} \tag{1.28}$$

we have that either

$$\sup_{\Omega} v = \sup_{\partial_0\Omega} v \ , \tag{1.29}$$

or

$$f(\sup_{\Omega} v) \leq 0. \tag{1.30}$$

REMARK 1.10. If $\partial M = \emptyset$ in the above definition the request $T(\nabla v, \nu) \leq 0$ on $\partial_1\Omega$ becomes vacuous and the definition coincides with that of the open q-weak maximum principle introduced in [**13**], that is, for each open set $\Omega \Subset M$ with $\partial\Omega \neq \emptyset$ and for each $v \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ satisfying

$$\begin{cases} q(x)Lv \geq f(v) & \text{on } \Omega \\ \sup_\Omega v < +\infty \end{cases}$$

we have that either

$$\sup_{\Omega} v = \sup_{\partial\Omega} v \ ,$$

or

$$f(\sup_{\Omega} v) \leq 0.$$

The next result shows that the *q-∂WMP* and the *q-$O\partial WMP$* are equivalent. The same result holds for the q-WMP and its open form, see [**13**]. However, in the present case we need a different proof to deal with the boundary condition.

THEOREM 1.11. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold with boundary ∂M. Then the q-∂WMP holds on M for L if and only if the q-$O\partial WMP$ holds on M for L.*

PROOF. We begin by assuming the validity of *q-∂WMP* on M for L. Let $\Omega \Subset M$ with $\partial_0\Omega \neq \emptyset$ and for $v \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ satisfying (1.28), suppose that $f(\sup_\Omega v) = 2\alpha > 0$. Then, there exists $\eta < \sup_\Omega v$ such that

$$f(v(x)) > \alpha \quad \text{on } \Lambda_\eta, \tag{1.31}$$

with

$$\Lambda_\eta = \{x \in \Omega \,:\, v(x) > \eta\} \,.$$

We claim that, $\forall\, \eta \leq \gamma < \sup_\Omega v$, $\Lambda_\gamma \cap \partial_0\Omega \neq \emptyset$. This clearly implies

$$\sup_{\Omega} v = \sup_{\partial_0\Omega} v \,,$$

that is, (1.29). We reason by contradiction and we suppose that for some $\eta \leq \overline{\gamma} < \sup_\Omega v$

$$\Lambda_{\overline{\gamma}} \cap \partial_0\Omega = \emptyset \,.$$

This implies that the function

$$u(x) = u_{\overline{\gamma}}(x) = \begin{cases} \max\{\overline{\gamma},\, v(x)\} & \text{on } \Omega; \\ \overline{\gamma} & \text{on } M\backslash\Omega, \end{cases}$$

belongs to $\mathrm{C}^0(M) \cap \mathrm{Lip}_{\mathrm{loc}}(\operatorname{int} M)$, $u^* < +\infty$, and $T(\nabla u, \nu) \le 0$ on ∂M. From the q-∂WMP

$$\inf_{\Omega_{\overline{\gamma}}} q(x)Lu \le 0 \tag{1.32}$$

with

$$\Omega_{\overline{\gamma}} = \{x \in M \,:\, u(x) > \overline{\gamma}\} = \Lambda_{\overline{\gamma}}\,.$$

But $\overline{\gamma} \ge \eta$ and thus $\Lambda_{\overline{\gamma}} \subseteq \Lambda_\eta$ so that, by (1.31)

$$q(x)Lu(x) \ge f(v(x)) > \alpha > 0 \quad \text{on } \Omega_{\overline{\gamma}}$$

contradicting (1.32).

Now let the q-$O\partial WMP$ hold on M for L. Let $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$ be such that $u^* < +\infty$ and $T(\nabla u, \nu) \le 0$ on ∂M. Assume by contradiction that there exists $\gamma < u^*$ such that

$$\inf_{\Omega_\gamma} q(x)Lu \ge \alpha > 0\,. \tag{1.33}$$

Note that, by (1.33) u is non constant so that we can possibly choose a larger $\gamma < u^*$ so that (1.33) still holds and $\partial_0\Omega_\gamma \ne \emptyset$.
We set $\Omega = \Omega_\gamma$ and $v = u|_{\overline{\Omega}}$. Then we have

$$\begin{cases} q(x)Lv \ge \alpha & \text{on } \Omega; \\ T(\nabla v, \nu) \le 0 & \text{on } \partial_1\Omega; \\ \sup_\Omega v < +\infty\,. \end{cases}$$

Now, since $\alpha > 0$, (1.30) cannot occur hence the validity of q-$O\partial WMP$ yields

$$\sup_\Omega v = \sup_{\partial_0\Omega} v = \gamma$$

which is impossible since

$$\sup_\Omega v = \sup_{\Omega_\gamma} u = u^* > \gamma\,.$$

This completes the proof of the theorem. $\square$

A careful reading of the above proof yields the validity of the following form of the theorem useful in applications.

THEOREM 1.12. *Let* $(M, \partial M, \langle\ ,\ \rangle)$ *be a Riemannian manifold with boundary* ∂M. *Then the* q-∂WMP *holds on* M *for the operator* L *if and only if for each open set* $\Omega \subset M$ *with* $\partial_0\Omega \neq \emptyset$, $\forall\, \beta \in \mathbb{R}^+$ *and for each* $v \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ *satisfying*

$$\begin{cases} q(x)Lv \geq \beta & \text{on } \Omega \\ T(\nabla v, \nu) \leq 0 & \text{on } \partial_1\Omega, \text{ if not empty} \\ \sup_\Omega v < +\infty \end{cases}$$

we have

$$\sup_\Omega v = \sup_{\partial_0\Omega} v\,. \tag{1.34}$$

Note that from here we also deduce the next result.

THEOREM 1.13. *Let* $(M, \partial M, \langle\ ,\ \rangle)$ *be a Riemannian manifold with boundary* $\partial M \neq \emptyset$. *Assume that the* q-∂WMP *holds on* M *for the operator* L. *Then,* $\forall\, \beta \in \mathbb{R}^+$ *and for each* $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$ *with* $u^* < +\infty$ *and satisfying*

$$Lu \geq \beta \quad \text{on } M \tag{1.35}$$

we have

$$\sup_M u = \sup_{\partial M} u\,. \tag{1.36}$$

PROOF. We reason by contradiction and we suppose

$$\sup_M u > \sup_{\partial M} u\,.$$

Next we fix $\varepsilon > 0$ sufficiently small that

$$\sup_M u > \sup_{\partial M} u + 2\varepsilon$$

and we define

$$U_\varepsilon = \left\{ x \in M\ : u(x) > \sup_M u - \varepsilon \right\}\,.$$

Clearly $\overline{U}_\varepsilon \subseteq \operatorname{int} M$, in particular $\partial_1 U_\varepsilon = \emptyset$. Hence, on $\Omega = U_\varepsilon$, $v = u|_{\overline{U}_\varepsilon}$ satisfies $v \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and

$$\begin{cases} q(x)Lv \geq \beta > 0 & \text{on } \Omega \\ \sup_\Omega v = u^* < +\infty\,. \end{cases}$$

The validity of the q-∂WMP, hence that of its open form, implies

$$u^* = \sup_\Omega v = \sup_{\partial_0\Omega} v = u^* - \varepsilon\,,$$

contradiction. □

1.4. A Functional Theoretic approach to the WMP

In this section we explore the possibility of developing a *functional theoretic* approach to the weak maximum principle on manifolds with boundary. By functional theoretic approach, we mean that the validity of the ∂WMP descends from the existence of a suitable *barrier function* γ on the manifold. The result we are going to present generalizes Theorem A in the very recent [**5**].
We recall that in the previous section the proof of the weak maximum principle relied on the divergence structure of the operator. The main tool of this section will be the strong maximum principle for Lipschitz subsolutions of elliptic inequalities proved in [**68**] (Theorem 5.6).
In what follows we shall deal with functions $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$, note that the classical Rademacher's theorem implies the existence of the gradient ∇u almost everywhere on M. We underline that some of the most important natural functions defined on a Riemannian manifold $(M, \partial M, \langle\ ,\ \rangle)$, namely the distance functions from closed sets, belong to this class. We begin with the following

Definition 1.14. *Let $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$ and $g \in \mathrm{C}^0(\partial M)$. We say that*

$$\partial_\nu u \le g \quad \text{on } \partial M,$$

if, for all $x \in \partial M$,

$$\limsup_{t\to 0^+} \frac{u(\zeta_x(t))}{t} \le g(x)\,,$$

where, for ε sufficiently small, $\zeta_x : [0, \varepsilon) \to M$ is a geodesic segment starting from x and normal to ∂M, that is $\zeta_x(0) = x$ and $\zeta_x'(0) = \nu_x$.
Similarly,

$$\partial_\nu u \ge g \quad \text{on } \partial M,$$

if, for all $x \in \partial M$,

$$\liminf_{t\to 0^+} \frac{u(\zeta_x(t))}{t} \ge g(x)\,.$$

Remark 1.15. It is clear that the above definition coincides with the classical one whenever the normal derivative exists. Moreover, and this will be important for us, if $\partial_\nu u_1 \ge g_1$ and $\partial_\nu u_2 \le g_2$ on ∂M, then

$$\partial_\nu(u_1 - u_2) \ge g_1 - g_2 \quad \text{on } \partial M.$$

The core of the section is the following result, analogous to Theorem A in [**5**], but presently for manifolds with boundary. Although the idea of the proof is the same, the presence of the boundary introduces technical difficulties that we have to treat with more care. They are encoded in a Hopf's lemma type result included in the proof of Theorem 5.6 of [**68**].

THEOREM 1.16. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold with boundary and let L be as in (1.1). Let $q(x) \in \mathrm{C}^0(M)$, $q(x) \geq 0$, and suppose that*

$$q(x) > 0 \text{ outside a compact set.} \tag{1.37}$$

Suppose that there exists $\gamma \in \mathrm{Lip}_{\mathrm{loc}}(M)$ such that

$$\begin{cases} i) \quad \gamma(x) \to +\infty & \text{as } x \to \infty, \\ ii) \quad q(x)L\gamma(x) \leq B & \text{outside a compact set} \\ iii) \quad \partial_\nu \gamma(x) \geq 0 & \text{on } \partial M \end{cases} \tag{1.38}$$

for some constant $B > 0$. If $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$ is such that

$$\partial_\nu u \leq 0 \quad \text{on } \partial M \tag{1.39}$$

and $u^* < +\infty$, then it satisfies the q-boundary weak maximum principle.

PROOF. We fix $\eta > 0$ and let

$$A_\eta = \{x \in M : \ u(x) > u^* - \eta\}. \tag{1.40}$$

We claim that

$$\inf_{A_\eta} \{q(x)Lu(x)\} \leq 0, \tag{1.41}$$

in the weak-Lip sense. We reason by contradiction and we suppose that for some $\eta > 0$

$$q(x)Lu(x) \geq \sigma_0 > 0 \quad \text{on } A_\eta. \tag{1.42}$$

First we observe that u^* cannot be attained at any point $x_0 \in M$. Indeed if $x_0 \in \mathrm{int}\, M$ then it would violate the strong maximum principle of Theorem 5.6 of [**68**]. While if we suppose that the maximum is attained at $x_0 \in \partial M$, then the Hopf boundary lemma implicitly stated in the proof of Theorem 5.6 of [**68**] would imply that

$$\liminf_{t \to 0^+} \frac{u(\zeta(t))}{t} > 0$$

where $\zeta : [0, \varepsilon) \to M$ is a geodesic segment normal to the boundary starting at x_0, contradicting assumption (1.39). Next we let

$$\Lambda_t = \{x \in M : \ \gamma(x) > t\},$$

and define

$$u_t^* = \sup_{x \in \Lambda_t^c} u(x).$$

Clearly Λ_t^c is closed; we show that it is also compact. In fact, by (1.38) *i)* there exists a compact set K_t such that $\gamma(x) > t$ for every $x \notin K_t$. In other words, $\Lambda_t^c \subset K_t$ and hence it is also compact. In particular, $u_t^* = \max_{x \in \Lambda_t^c} u(x)$.

Since u^* is not attained in M and $\{\Lambda_t^c\}$ is a nested family exhausting M, we find a divergent sequence $\{t_j\} \subset \mathbb{R}_0^+$ such that

$$u_{t_j}^* \to u^* \quad \text{as } j \to +\infty, \tag{1.43}$$

and we can choose $T_1 > 0$ sufficiently large in such a way that

$$u_{T_1}^* > u^* - \frac{\eta}{2}. \tag{1.44}$$

Furthermore we can also suppose to have chosen T_1 sufficiently large that $q(x) > 0$ and (1.38) *ii)* holds on Λ_{T_1}. We choose α satisfying $u_{T_1}^* < \alpha < u^*$. Because of (1.43) we can find j sufficiently large that

$$T_2 = t_j > T_1 \quad \text{and} \quad u_{T_2}^* > \alpha.$$

Next, we select $\overline{\eta} > 0$ small enough that

$$\alpha + \overline{\eta} < u_{T_2}^*. \tag{1.45}$$

For $\sigma \in (0, \sigma_0)$ we define

$$\gamma_\sigma(x) = \alpha + \sigma(\gamma(x) - T_1). \tag{1.46}$$

We note that

$$\gamma_\sigma(x) = \alpha \quad \text{for every } x \in \partial\Lambda_{T_1},$$

and

$$q(x)L\gamma_\sigma(x) = \sigma q(x)L\gamma(x) \le \sigma B < \sigma_0 \quad \text{on } \Lambda_{T_1}, \tag{1.47}$$

up to have chosen σ sufficiently small. Since on $\Lambda_{T_1} \setminus \Lambda_{T_2}$ we have

$$\alpha < \gamma_\sigma(x) \le \alpha + \sigma(T_2 - T_1)$$

we can choose $\sigma \in (0, \sigma_0)$ sufficiently small, so that

$$\sigma(T_2 - T_1) < \overline{\eta} \tag{1.48}$$

and then

$$\alpha \le \gamma_\sigma(x) < \alpha + \overline{\eta} \quad \text{on} \quad \Lambda_{T_1} \setminus \Lambda_{T_2}.$$

For any such σ on $\partial\Lambda_{T_1}$, we have

$$\gamma_\sigma(x) = \alpha > u_{T_1}^* \ge u(x),$$

so that

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \partial\Lambda_{T_1}. \tag{1.49}$$

Furthermore, if $\overline{x} \in \Lambda_{T_1} \setminus \Lambda_{T_2}$ is such that

$$u(\overline{x}) = u^*_{T_2} > \alpha + \overline{\eta}$$

then

$$(u - \gamma_\sigma)(\overline{x}) \geq u^*_{T_2} - \alpha - \sigma(T_2 - T_1) > u^*_{T_2} - \alpha - \overline{\eta} > 0$$

by (1.45) and (1.48). Finally, (1.38) $i)$ and the fact that $u^* < +\infty$ imply

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \Lambda_{T_3} \tag{1.50}$$

for $T_3 > T_2$ sufficiently large. Therefore,

$$m = \sup_{x \in \overline{\Lambda}_{T_1}} (u - \gamma_\sigma)(x) > 0,$$

and it is in fact a positive maximum attained at a certain point z_0 in the compact set $\overline{\Lambda}_{T_1} \setminus \Lambda_{T_3}$. Moreover by (1.49) we know that $\gamma(z_0) > T_1$. Therefore, at z_0 we have

$$u(z_0) = \gamma_\sigma(z_0) + m > \gamma_\sigma(z_0) > \alpha > u^*_{T_1} > u^* - \frac{\eta}{2},$$

and hence $z_0 \in A_\eta \cap \Lambda_{T_1}$. In particular $q(x) > 0$ and (1.38) $ii)$ holds in a neighborhood of z_0. From (1.42) and (1.47) we have

$$q(x)L(u - \gamma_\sigma)(x) > 0 \quad \text{on } A_\eta \cap \Lambda_{T_1},$$

and furthermore from (1.39) and (1.38) iii) it follows that

$$\partial_\nu(u - \gamma_\sigma)(x) \leq 0 \quad \text{on } \partial M.$$

Thus, reasoning as before, we can use Theorem 5.6 of [**68**] to conclude that there cannot occur a positive maximum at z_0, contradicting (1.42). $\square$

Remark 1.17. This result extends genuinely Theorem A of [**5**], since if $\partial M = \emptyset$ then the additional condition (1.38) iii) on γ is trivially satisfied.

1.5. L^∞ estimates

The following *a priori* estimate (in fact its consequence Corollary **??** below) extends Theorem B of [**65**] to the case of manifolds with boundary. Analogously to Theorem 1.5, the proof of the result follows the lines of the aforementioned Theorem B of [**65**] but we feel necessary to provide a complete and detailed proof for the ease of the reader.

THEOREM 1.18. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold with boundary. Let b, Q, T, and Θ be as above. Assume that $u \in \mathcal{B}_1(M)$ (or $\mathcal{B}_2(M)$) satisfies*

$$Lu \geq b(x)f(u) \quad \text{on } \Omega_\gamma \tag{1.51}$$

for some $\gamma < u^* \leq +\infty$, where f is a continous function on $\mathbb{R}$ such that

$$\liminf_{t\to+\infty} \frac{f(t)}{t^\sigma} > 0 \tag{1.52}$$

for some $\sigma > 1$. If (1.5) and (1.6) hold true, then u is bounded above.

PROOF. Assume, by way of contradiction, that u is not bounded above, so that the set

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}$$

is nonempty for each $\gamma > 0$. By increasing γ, if necessary, we may assume that $f(t) \geq Bt^\sigma$ if $t \geq \gamma$. For the ease of notation, we let $B = 1$ so that on Ω_γ

$$\operatorname{div}\left(\widetilde{T}(\nabla u)\right) \geq b(x)u^\sigma, \tag{1.53}$$

weakly.

Clearly we may also assume that $b(x)$ is bounded above. Let $R_0 > 0$ be large enough that $\Omega_\gamma \cap B_{R_0} \neq \emptyset$. Now we will proceed as in the proof of Theorem 1.5, that is, we are going to define a suitable family of test functions in order to get a contradiction. Fix $\xi > 1$ satisfying

$$1 - \frac{2}{\sigma - 1}\left(1 - \frac{1}{\xi}\right) > 0 \tag{1.54}$$

For each $R \geq R_0$ let $\psi = \psi_R : M \to [0, 1]$ be a smooth cut-off function such that

$$\begin{array}{lll} \textit{i)} & \psi_R \equiv 1 & \text{on } B_R; \\ \textit{ii)} & \psi_R \equiv 0 & \text{on } M \setminus B_{2R}; \\ \textit{iii)} & |\nabla\psi_R| \leq \frac{C_0}{R}\psi_R^{1/\xi}, & \end{array} \tag{1.55}$$

for some constant $C_0 > 0$. Note that this latter requirement *iii)* is possible since $\xi > 1$. Next, let $\lambda : \mathbb{R} \to \mathbb{R}_0^+$ be a C^1 function such that

$$\begin{array}{lll} \textit{i)} & \lambda \equiv 0 & \text{on } (-\infty, \gamma]; \\ \textit{ii)} & \lambda'(t) \geq 0 & \text{on } \mathbb{R}; \\ \textit{iii)} & \sup \lambda = \frac{1}{\sup_M b} > 0. & \end{array}$$

Finally, fix $\alpha > 2\sigma$, $\mu > 0$, and $0 \leq \beta_R \in \operatorname{Lip}(B_{2R} \cap \Omega_\Gamma^v)$ to be determined later. Consider the function ϕ_R defined by

$$\phi_R = \beta_R\psi_R^\alpha\lambda(u)u^\mu \quad \text{on } \Omega_\gamma \tag{1.56}$$

and $\phi_R \equiv 0$ outside Ω_γ. Note that $\phi_R \equiv 0$ off $B_{2R} \cap \Omega_\gamma$ and moreover $\phi_R \in W_0^{1,2}(M)$. It can be checked that the weak gradient of ϕ_R satisfies

$$\begin{aligned}\nabla \phi_R = \psi_R^\alpha \lambda(u) u^\mu \nabla \beta_R + \alpha \beta_R \psi_R^{\alpha-1} \lambda(u) u^\mu \nabla \psi_R \\ + \beta_R \psi_R^\alpha \lambda'(u) u^{\mu-1} \nabla u + \mu \beta_R \psi_R^\alpha \lambda(u) u^{\mu-1} \nabla u\,.\end{aligned}$$

Now we proceed as in the proof of Theorem 1.5 using the function ϕ_R to test the inequality (1.53). We recall that $\lambda' > 0$, use (1.11), and furthermore choose β_R according to the function space of u as above, in order to get rid of the boundary term. Thus we obtain

$$\begin{aligned}\int_{B_{2R}} \beta_R \psi_R^\alpha \lambda(u) u^{\mu+\sigma} b(x) \leq -\mu \int_{B_{2R}} \beta_R \psi_R^\alpha \lambda(u) u^{\mu-1} T_u \left|\nabla u\right|^2 \\ + \alpha \int_{B_{2R}} \beta_R \psi_R^{\alpha-1} \lambda(u) u^\mu T_u^{1/2} T_+^{1/2} \left|\nabla u\right| \left|\nabla \psi_R\right| \,.\end{aligned}$$

We apply to the second integral on the right hand side the inequality

$$ab \leq \varepsilon \frac{a^2}{2} + \frac{b^2}{2\varepsilon}$$

with

$$\begin{aligned}a &= \psi_R^{\alpha/2} u^{(\mu-1)/2} T_u^{1/2} \left|\nabla u\right| \,, \\ b &= \psi_R^{\alpha/2-1} u^{(\mu+1)/2} T_+^{1/2} \left|\nabla \psi_R\right| \,,\end{aligned}$$

and $\varepsilon = \frac{2\mu}{\alpha}$ so that the first integral on the right hand side cancels out and we obtain

$$\int_{B_{2R}} \beta_R \psi_R^\alpha \lambda(u) u^{\mu+\sigma} b(x) \leq \frac{\alpha^2}{4\mu} \int_{B_{2R}} \beta_R \psi_R^{\alpha-2} \lambda(u) u^{\mu+1} T_+ \left|\nabla \psi_R\right|^2 \,. \tag{1.57}$$

Multiplying and dividing by $b(x)^{1/p}$ in the integral on the right hand side, and applying Hölder's inequality with conjugate exponents p and q, yields

$$\begin{aligned}\int \beta_R \psi_R^{\alpha-2} \lambda(u) u^{\mu+1} T_+ |\nabla \psi|^2 \leq \left(\int \beta_R \psi_R^\alpha b(x) \lambda(u) u^{p(\mu+1)} \right)^{1/p} \\ \times \left(\int \beta_R \psi_R^{\alpha-2q(1-1/\xi)} \lambda(u) b(x)^{1-q} T_+^q \left(\frac{|\nabla \psi_R|}{\psi_R^{1/\xi}} \right)^{2q} \right)^{1/q} ,\end{aligned}$$

provided

$$\alpha - 2q(1 - 1/\xi) > 0. \tag{1.58}$$

Choosing $p = \dfrac{\mu+\sigma}{\mu+1} > 1$ since $\sigma > 1$, the first integral on the right hand side of the above inequality is equal to the integral on the left hand side of (1.57). Thus, inserting into this latter and simplifying, we obtain

$$\int \beta_R \psi_R^\alpha b(x)\lambda(u)u^{\mu+\sigma} \le \left(\frac{\alpha^2}{4\mu}\right)^q \int \beta_R \psi_R^{\alpha-2q(1-1/\xi)} \lambda(u) b(x)^{1-q} T_+^q \left(\frac{|\nabla \psi_R|}{\psi_R^{1/\xi}}\right)^{2q}.$$

Since $u > \gamma$ on Ω_γ and $\psi \equiv 1$ on B_R,

$$\gamma^{\mu+\sigma} \int_{B_R} \beta_R b(x)\lambda(u) \le \int \beta_R \psi_R^\alpha b(x)\lambda(u)u^{\mu+\sigma}.$$

On the other hand, using (1.55) ii), iii), the fact that ψ_R is supported on B_{2R}, and the monotonicity of β_S with respect to S, we have

$$\begin{aligned}\left(\frac{\alpha^2}{4\mu}\right)^q \int \beta_R \psi_R^{\alpha-2q(1-1/\xi)} \lambda(u) b(x)^{1-q} T_+^q \left(\frac{|\nabla \psi_R|}{\psi_R^{1/\xi}}\right)^{2q} \\ \le \left(\frac{\alpha^2}{4\mu}\frac{C_0^2}{R^2}\sup_{B_{2R}}\frac{T_+}{b(x)}\right)^q \int_{B_{2R}} \beta_{2R} b(x)\lambda(u).\end{aligned}$$

We use these two latter inequalities, the fact that $b(x) \ge Q(r(x))^{-1}$ with q non-decreasing, the validity of

$$T_+(r(x)) \le \Theta(2R) \tag{1.59}$$

on B_{2R}, and

$$q = \frac{\mu+\sigma}{\sigma-1}$$

to obtain

$$\int_{B_R} \beta_R b(x)\lambda(u) \le \left(\frac{C_0^2}{4\gamma^{\sigma-1}}\frac{\Theta(2R)Q(2R)}{R^2}\left(\frac{\alpha}{\mu}\right)\alpha\right)^{\frac{\mu+\sigma}{\sigma-1}} \int_{B_{2R}} \beta_{2R} b(x)\lambda(u)\,. \tag{1.60}$$

Now we choose

$$\alpha = \mu+\sigma = \frac{1}{C_0^2}\gamma^{\sigma-1}\frac{R^2}{\Theta(2R)Q(2R)}$$

so that (1.54) implies that (1.58) holds. Moreover, because of (1.5), $\alpha \to +\infty$ as $R \to +\infty$. Hence, for R sufficiently large $\frac{\alpha}{\mu} \le 2$. It follows that, for such values of R, (1.60) gives

$$\int_{B_R} \beta_R b(x)\lambda(u) \le \left(\frac{1}{2}\right)^{\frac{\gamma^{\sigma-1}}{C_0^2(\sigma-1)}\frac{R^2}{\Theta(2R)Q(2R)}} \int_{B_{2R}} \beta_{2R} b(x)\lambda(u). \tag{1.61}$$

We let

$$G(R) = \int_{B_R} \beta_R b(x)\lambda(u)$$

and

$$F(R) = \Theta(R)Q(R)$$

be defined on $[R_0, +\infty)$ for some R_0 sufficiently large such that (1.61) holds for $R \geq R_0$. Then

$$G(R) \leq \left(\frac{1}{2}\right)^{B\frac{R^2}{F(2R)}} G(2R)$$

with $B = \frac{\gamma^{\sigma-1}}{C_0^2(\sigma-1)} > 0$. Then by Lemma 1.6, there exists a constant $S > 0$ such that, for each $R \geq 2R_0$

$$\frac{Q(R)\Theta(R)}{R^2} \log \int_{B_R} \beta_R b(x)\lambda(u) \geq \frac{Q(R)\Theta(R)}{R^2} \log \int_{B_R} \beta_R b(x)\lambda(u) + SB \log 2,$$

To reach the desired contradiction, we recall that $\sup \lambda = \frac{1}{\sup_M b} > 0$ so that $b(x)\lambda(u) \leq 1$. Taking R going to $+\infty$ in the above and using (1.5) we deduce

$$\liminf_{R \to +\infty} \frac{Q(R)\Theta(R)}{R^2} \log \operatorname{vol} B_R \geq SB \log 2 = \frac{\gamma^{\sigma-1}}{C_0^2(\sigma-1)} S \log 2.$$

This contradicts (1.6) by choosing γ sufficiently large. $\square$

As a consequence of Theorem 1.18 we have the following *a priori* estimate for solutions of the differential inequality (1.62) below. The importance of this type of results can be hardly overestimated in PDE's Theory and it will be used in the last chapter.

COROLLARY 1.19. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold with boundary. Let $a(x)$, $b(x) \in C^0(M)$, and assume that $\|a_-\|_\infty < +\infty$. Let $u \in \mathcal{B}_1(M)$ (or $\mathcal{B}_2(M)$) be a non-negative solution of*

$$Lu \geq b(x)u^\sigma + a(x)u \quad on\ \Omega_\gamma \tag{1.62}$$

for some $\gamma < u^* \leq +\infty$, and for some $\sigma > 1$. Assume furthermore that $b(x) > 0$ on Ω_γ satisfies (1.25) and that, for some $H > 0$,

$$\frac{a_-(x)}{b(x)} \leq H \quad on\ \Omega_\gamma.$$

If Q satisfies (1.5) and (1.6), then u satisfies

$$u(x) \leq H^{1/(\sigma-1)} \quad on\ \Omega_\gamma. \tag{1.63}$$

PROOF. The assumptions on $a(x)$ and $b(x)$ imply that

$$Lu \geq b(x)\,(u^\sigma - Hu) \qquad \text{on } \Omega_\gamma\,,$$

thus, since

$$\liminf_{t \to +\infty} \frac{t^\sigma - Ht}{t^\sigma} = 1\,,$$

it follows from Theorem 1.18 that u is bounded above. Furthermore, by Theorem 1.7 it follows that $(u^*)^\sigma - Hu^* \leq 0$ on Ω_γ, which implies that

$$u(x) \leq H^{1/(\sigma-1)} \quad \text{on } \Omega_\gamma.$$

□

We conclude the section with the following remark, that will be helpful in the applications, in particular it will be crucial in Chapter 4.

REMARK 1.20. A careful inspection of the proofs of this section shows that Theorem 1.18 and Corollary 1.19 remain valid if the condition $u \in \mathcal{B}_1(M)$ (or $\mathcal{B}_2(M)$) is substituted with $u \in \mathrm{C}^1(\operatorname{int} M) \cap \mathrm{C}^0(M)$ such that

$$\partial_\nu u \leq 0 \qquad \text{on } \partial_1\Omega_\gamma\,.$$

REMARK 1.21. Analogously to what we observed in Remark 1.8, if $L = \Delta$, the results of Theorem 1.18 and Corollary 1.19 are true if we assume (1.26) and (1.27) for some $C > 0$ and $0 \leq \mu < 2$.

1.6. Method of sub/super solutions

We are interested in positive solutions of semilinear equations on $(M, \partial M, \langle\ ,\ \rangle)$, a complete manifold with boundary, with a possibly nonlinear boundary condition. More precisely we are interested in the following problem

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{on } \operatorname{int} M \\ \partial_\nu u - g(x,u) = 0 & \text{on } \partial M. \end{cases} \tag{1.64}$$

Since our solutions will be obtained as limits of solutions defined on subsets of M, we need to introduce an adequate family of subsets, that is, it has to be at the same time an exhaustion of $\operatorname{int} M$ and of ∂M. We set the following

DEFINITION 1.22. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete manifold with boundary and $\{\Omega_n\}_{n\in\mathbb{N}}$ a family of relatively compact open subsets of M with smooth boundary. Then we say that $\{\Omega_n\}_{n\in\mathbb{N}}$ is a* ∂-regular *exhaustion of M if it satisfies the following conditions:*

- $\Omega_n \subset\subset \Omega_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\Omega_n \nearrow M$;
- $\partial_1\Omega_n \subset\subset \partial_1\Omega_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\partial_1\Omega_n \nearrow \partial M$;

REMARK 1.23. On a complete manifold $(M, \partial M, \langle\ ,\ \rangle)$ there exists a ∂-regular exhaustion. Indeed, fix an origin $o \in M$ and define for each $n \in \mathbb{N}$

$$B_n = B_n(o)\,,$$

since $(M, \partial M, \langle\ ,\ \rangle)$ is complete as a metric space, B_n is an exhaustion satisfying the conditions of the definition above. The problem is that ∂B_n need not to be smooth, thus we need to *regularize* it. We claim that for all $n \in \mathbb{N}$ there exists a set Ω_n such that $B_n \subset \Omega_n \subset\subset B_{n+1}$ and $\partial\Omega_n$ is smooth. This follows from the approximation theory of Lipschitz submanifolds by smooth ones (see for instance [**43**]) since $B_n \subset\subset B_{n+1}$, and $\partial B_n \in \mathrm{Lip}$.

To solve such nonlinear boundary value problems, we shall need a generalization of the monotone iteration scheme (see for instance [**15**] or [**72**]) for semilinear elliptic equations with nonlinear boundary conditions. It is a generalization of Theorem 6.19 of [**55**].
Let Ω be a bounded open domain, and let $f(x,s) \in \mathrm{C}^0(\Omega \times \mathbb{R})$, $g(x,s) \in \mathrm{C}^0(\partial\Omega \times \mathbb{R})$. Let $\beta(x) \in \mathrm{C}^{1,\alpha}(\partial\Omega)$ and define the boundary operator B acting on elements of $\mathrm{C}^1(\overline{\Omega})$ as

$$Bu = \partial_\nu u + \beta(x)u\,.$$

Then $u_+ \in \mathrm{C}^1(\overline{\Omega})$ is a supersolution of

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{in } \Omega\,, \\ Bu = g(x,u) & \text{on } \partial\Omega\,. \end{cases} \tag{1.65}$$

if

$$\begin{cases} \Delta u_+ + f(x,u_+) \leq 0 & \text{in } \Omega\,, \\ Bu_+ \geq g(x,u_+) & \text{on } \partial\Omega\,. \end{cases}$$

where the first differential inequality has to be understood in the weak sense. The definition of a subsolution is obtained reversing the inequalities.

THEOREM 1.24. *Let Ω be a relatively compact open domain with smooth boundary $\partial\Omega$. Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a locally Hölder function such that $s \to f(x,s)$ is locally Lipschitz with respect to s uniformly with respect to x and $g(x,s) \in \mathrm{C}^{2,\alpha}(\partial\Omega \times \mathbb{R})$. Suppose that φ and $\psi \in \mathrm{C}^1(\overline{\Omega})$ are respectively a subsolution and a supersolution of (1.65) satisfying*

$$\varphi \leq \psi \quad \text{on } \overline{\Omega}\,.$$

Then (1.65) has a solution $u \in \mathrm{C}^{2,\alpha}(\overline{\Omega})$ satisfying $\varphi \leq u \leq \psi$.

PROOF. There exist positive constants H and K such that the functions

$$s \to f(x,s) + Hs = F(x,s)\,,$$

and

$$s \to g(x,s) + Ks = G(x,s)\,,$$

are monotone increasing for every fixed x. For every $w \in \mathrm{C}^{\alpha}(\overline{\Omega})$ we let $v = Tw \in \mathrm{C}^{2,\alpha}(\overline{\Omega})$ be the solution of the boundary value problem

$$\begin{cases} (\Delta - H)v = -F(x,w) & \text{in } \Omega\,, \\ (B+K)v = G(x,w) & \text{on } \partial\Omega\,. \end{cases} \tag{1.66}$$

which exists and is unique by classical elliptic theory (see for instance [**39**], Theorem 6.31). By the monotonicity of $F(x,s)$ and $G(x,s)$ it follows that the operator T is monotone, that is, if $w_1 \le w_2$ on Ω then $Tw_1 \le Tw_2$. Indeed the function $\widetilde{v} = Tw_2 - Tw_1$ satisfies

$$\begin{cases} (\Delta - H)\widetilde{v} = -\left[F(x,w_2) - F(x,w_1)\right] \le 0 & \text{in } \Omega\,, \\ (B+K)\widetilde{v} = \left[G(x,w_2) - G(x,w_1)\right] \ge 0 & \text{on } \partial\Omega\,. \end{cases}$$

and therefore, by the strong maximum principle $\widetilde{v} \ge 0$ on Ω.

Now we set $u_1^- = T\varphi$, $u_1^+ = T\psi$, and for every $k \ge 1$, $u_{k+1}^{\pm} = Tu_k^{\pm}$. Reasoning inductively as above we obtain

$$\varphi \le u_1^- \le u_2^- \le \cdots \le u_k^- \le \cdots \le u_k^+ \le \cdots \le u_2^+ \le u_1^+ \le \psi\,.$$

Thus there exist u^- and u^+ such that $u_k^{\pm} \to u^{\pm}$. The regularity of $u^{\pm}$ and the fact that they are solutions of (1.65) follow as in the proof of Theorem 6.19 of [**55**]. □

Thus we have the following result

Proposition 1.25. *Let $f : M \times \mathbb{R} \to \mathbb{R}$ be a locally Hölder function such that $s \to f(x,s)$ is locally Lipschitz with respect to s uniformly with respect to x and $g(x,s) \in \mathrm{C}^{2,\alpha}(\partial M \times \mathbb{R}^+)$. Suppose that u^- and $u^+ \in \mathrm{C}^1(M)$ are respectively a subsolution and a supersolution of (1.64) satisfying*

$$0 \le u^- < u^+ \quad \text{on } M\,.$$

Then (1.64) has a solution $u \in \mathrm{C}^2(M)$ satisfying $u^- \le u \le u^+$.

Proof. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a ∂-regular exhaustion of M. Then there exists a family $\{\Gamma_n\}_{n\in\mathbb{N}}$ of relatively compact open subsets of ∂M such that $\Gamma_n \subset\subset \partial_1\Omega_n$ and $\Gamma_n \nearrow \partial M$. Now, consider a family of cutoff functions $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\psi_n \in \mathrm{C}^{\infty}(\partial\Omega_n)$ and

- $\operatorname{supp}\psi_n \subset\subset \partial_1\Omega_n$;
- $0 \le \psi_n \le 1$;
- $\psi_n \equiv 1$ on Γ_n.

Now, for each $n \in \mathbb{N}$ we introduce the following family of problems

$$\begin{cases} \Delta w + f(x,w) = 0 & \text{on } \operatorname{int}\Omega_n \\ \partial_\nu w - g_n(x,w) = 0 & \text{on } \partial\Omega_n\,, \end{cases} \tag{1.67}$$

here $g_n(x,w)$ is

$$g_n(x,w) = \psi_n(x)g(x,w) + A_n\,(1-\psi_n(x))\left(\frac{m(x)-w}{\delta(x)}\right)$$

where $m(x) = \frac{1}{2}\,(u^+ + u^-)$, $\delta(x) = \frac{1}{2}\,(u^+ - u^-) > 0$, and

$$A_n = \max\left\{\max_{x\in\partial_1\Omega_n} g(x,u^-), \max_{x\in\partial\Omega_n}\partial_\nu u^-, \max_{x\in\partial_1\Omega_n}\big(g(x,u^+)\big)_-, \max_{x\in\partial\Omega_n}\big(\partial_\nu u^+\big)_-\right\}.$$

We claim that u^+ is a supersolution of (1.67) for each $n\in\mathbb{N}$, since the first inequality is trivially satisfied we are left to check the boundary condition, that is

$$\begin{aligned}\partial_\nu u^+ - g_n(x,u^+) &= \psi_n(x)\left[\partial_\nu u^+ - g(x,u^+)\right] + (1-\psi_n(x))\left[\partial_\nu u^+ + A_n\right]\\ &\geq (1-\psi_n(x))\left[\partial_\nu u^+ + A_n\right]\\ &\geq (1-\psi_n(x))\left[\big(g(x,u^+)+A_n\big)\,\chi_{\partial_1\Omega_n} + \big(\partial_\nu u^+ + A_n\big)\,\chi_{\partial_0\Omega_n}\right]\\ &\geq 0\,,\end{aligned}$$

where, the first and the second inequalities follow from the facts that $\operatorname{supp}\psi_n \subset\subset \partial_1\Omega_n$, and that u^+ is a supersolution, the last inequality is a consequence of the definition of A_n. Analougusly it can be shown that u^- is a subsolution of (1.67). From the monotone iteration scheme (see for instance Theorem 1.24) it follows that there exist a family $\{u_n\}_{n\in\mathbb{N}}$ of positive solutions of the problems (1.67). By the standard elliptic regularity theory (see for instance [**39**], Theorem 6.31) and the fact that $\Gamma_n \subset\subset \Gamma_{n+1}$, it follows that for $m\geq n$, $u_m \in \mathrm{C}^{2,\alpha}(\operatorname{int}\Omega_n\cup\Gamma_n)$ and are uniformly bounded there, thus, up to a subsequence, $u_m \to u_n^* \in \mathrm{C}^2(\operatorname{int}\Omega_n\cup\Gamma_n)$. Now, since $\operatorname{int}\Omega_n\cup\Gamma_n \nearrow M$, we can arrange a diagonal subsequence such that $u_m \to u\in\mathrm{C}^2(M)$ positive solution of (1.64). □

1.7. Some spectral considerations

Let $a(x)\in\mathrm{C}^0(M)$ and $L=\Delta + a(x)$. If Ω is a non-empty open set, the first Dirichlet eigenvalue $\lambda_1^L(\Omega)$ is variationally characterized by means of the formula

$$\lambda_1^L(\Omega) = \inf\left\{\int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2 \,:\, \varphi\in\mathrm{W}_0^{1,2}(\Omega)\,, \int_\Omega\varphi^2 = 1\right\}, \tag{1.68}$$

similarly, it is possible to define the so-called Zaremba eigenvalue $\zeta_1^L(\Omega)$, that is variationally characterized as

$$\zeta_1^L(\Omega) = \inf\left\{\int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2 \,:\, \varphi\in\mathrm{W}_0^{1,2}(\Omega\cup\partial_1\Omega)\,, \int_\Omega\varphi^2 = 1\right\}, \tag{1.69}$$

where $\mathrm{W}_0^{1,2}(\Omega\cup\partial_1\Omega)$ denotes the closure of $\mathrm{C}_0^1(\Omega\cup\partial_1\Omega)$ in $\mathrm{W}^{1,2}(\Omega)$. It is clear that the two definitions above coincide if $\partial_1\Omega=\emptyset$, in general, since $\mathrm{W}_0^{1,2}(\Omega)\subset\mathrm{W}_0^{1,2}(\Omega\cup\partial_1\Omega)$, it holds that

$$\lambda_1^L(\Omega) \geq \zeta_1^L(\Omega)\,. \tag{1.70}$$

We recall that, if $\Omega \subset M$ is bounded, the infimum $\lambda_1^L(\Omega)$ is attained by the unique positive eigenfunction v on Ω satisfying

$$\begin{cases} \Delta v + a(x)v + \lambda_1^L(\Omega)v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \\ \|v\|_{\mathrm{L}^2(\Omega)} = 1. \end{cases} \tag{1.71}$$

In the general case, for a Lipschitz $\partial\Omega$, the infimum is attained by the unique positive eigenfunction v on Ω of the so-called Zaremba problem (or mixed Dirichlet-Neumann problem) that is

$$\begin{cases} \Delta v + a(x)v + \zeta_1^L(\Omega)v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \partial_0\Omega \\ \partial_\nu v = 0 & \text{on } \partial_1\Omega \\ \|v\|_{\mathrm{L}^2(\Omega)} = 1. \end{cases} \tag{1.72}$$

While the existence theory for (1.71) is a well known standard fact descending from the direct method of calculus of variations, the existence theory for (1.72) seems to be absent in the mathematical literature. For the interested reader, the existence of solutions of (1.72) can be obtained by adapting the proof of Theorem 8.37 of [**39**]. The only difference here is that the Kondrakov compactness theorem of the embedding $\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ (in the Dirichlet case it has to be considered $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$) requires the Lipschitz regularity of the boundary $\partial\Omega$, see Theorem 7.26 of [**39**]. The positivity of the eigenfunction v is a standard fact and follows from the fact that also $|v|$ is a solution of (1.72) and from Harnack inequality.

We then define two different notions of first eigenvalue of L on M, the Dirichlet one is given by

$$\lambda_1^L(M) = \inf_\Omega \lambda_1^L(\Omega)$$

where Ω runs over all bounded domains of M, and the Neumann one, that is

$$\zeta_1^L(M) = \inf_\Omega \zeta_1^L(\Omega)$$

where Ω runs over all bounded domains of M. Observe that, due to the monotonicity of λ_1^L and ζ_1^L with respect to the domain, that is

$$\Omega_1 \subseteq \Omega_2 \quad \text{implies} \quad \lambda_1^L(\Omega_1) \geq \lambda_1^L(\Omega_2) \text{ and } \zeta_1^L(\Omega_1) \geq \zeta_1^L(\Omega_2)\,, \tag{1.73}$$

we have

$$\lambda_1^L(M) = \lim_{r\to+\infty} \lambda_1^L(B_r) \tag{1.74}$$

and

$$\zeta_1^L(M) = \lim_{r\to+\infty} \zeta_1^L(B_r) \tag{1.75}$$

where, as usual, B_r denotes the geodesic ball in the complete manifold $(M, \partial M, \langle\ ,\ \rangle)$ of radius r centered at the fixed origin $o \in M$.
Note that in case $\Omega_2 \setminus \Omega_1$ has non-empty interior the inequality in (1.73) becomes strict.
We need to extend definitions (1.68) and (1.69) to an arbitrary bounded subset B of M. We do this by setting

$$\lambda_1^L(B) = \sup_{\Omega} \lambda_1^L(\Omega) \tag{1.76}$$

and

$$\zeta_1^L(B) = \sup_{\Omega} \zeta_1^L(\Omega) \tag{1.77}$$

where the supremum is taken over all open bounded sets $\Omega \subset M$, $B \subset \Omega$ (see for instance Section 6.1.2 of [**55**]). Observe that, by definition, if $B = \emptyset$ then $\lambda_1^L(B) = +\infty$. We note that, in case $B \subset\subset \operatorname{int} M$ (that is always true in case $\partial M = \emptyset$), then $\zeta_1^L(B) = \lambda_1^L(B)$.

We would like to remark that since the first Dirichlet eigenvalue for the Laplacian of a ball $B_r(x_0)$ centered at a point $x_0 \in \operatorname{int} M$ grows like r^{-2} as $r \to 0^+$, $\lambda_1^L(B_r(x_0)) \geq 0$ provided r is sufficiently small and one may think of $\lambda_1^L(B) > 0$ as a condition expressing the fact that B is small in a spectral sense. Of course, this condition also depends on the behaviour of $a(x)$ therefore *small in a spectral sense* does not necessarily mean, for instance, *small in a Lebesgue measure sense.* This is clear if $a(x) \leq 0$ because $\lambda_1^\Delta(M) \geq 0$ in any complete manifold $(M, \langle\ ,\ \rangle)$ with $\partial M = \emptyset$ so that in this case $\lambda_1^L(M) \geq 0$ and thus $\lambda_1^L(B) > 0$ on any bounded set $B \subset M$.

CHAPTER 2

Lichnerowicz-type equations

In this chapter we consider Lichnerowicz-type equations on complete manifolds with boundary and nonlinear Neumann conditions, that is problems of the form

$$\begin{cases} \Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0 & \text{on } \operatorname{int} M \\ \partial_{\nu} u - g(x,u) = 0 & \text{on } \partial M. \end{cases} \tag{2.1}$$

As already mentioned in the Introduction, this kind of nonlinear problems arise quite naturally in the study of solutions for the Einstein-scalar field equations of General Relativity in the framework of the so called Conformal Method.

The main result of the chapter is an existence theorem for positive solutions of (2.1) with a possibly sign-changing $b(x)$, under appropriate spectral assumptions, namely Theorem 2.6 below. The proof is carried out via the sub/supersolution method presented in Section 1.6. The chapter is organized as follows.
In Section 2.1 we produce positive supersolutions for a suitable perturbation of (2.1) (see also [**42, 57**]).
In Section 2.2 we exploit a simmetry property of (2.1) to obtain a subsolution from a supersolution of an appropriate dual problem.
In Section 2.3 the existence of a positive solution of (2.1) is proved by showing that among the sub/supersolutions constructed before it is possible to find an ordered pair.
Section 2.4 is devoted to the discussion of the boundary nonlinearity $g(x,t)$. Indeed the results of the previous sections are stated in term of general properties of the function $g(x,t)$, while, in view of the applications (cf. the Introduction) it is worth to specialize our analysis to some particular nonlinearities.
To conclude the Chapter, in Section 2.5 we prove some uniqueness results for positive solutions of (2.1), this time it will be assumed that $b(x)$ is non-negative, since the proofs are based on comparison techniques relying in the monotonicity of the equation with respect to u.

2.1. Construction of a supersolution

We prove in this section that under suitable spectral assumptions and assuming a control on the coefficients, we can find a positive supersolution of a perturbation of the equation in (2.1). Namely, by setting $b_\theta(x) = b_+(x) - \theta b_-(x)$, the main result of this section is that, under suitable spectral assumptions and a control on the coefficients, there exists a positive supersolution of

$$\begin{cases} \Delta u + a(x)u - b_\theta(x)u^\sigma + c(x)u^\tau = 0 & \text{on } \operatorname{int} M \\ \partial_\nu u - g(x,u) = 0 & \text{on } \partial M, \end{cases} \tag{2.2}$$

up to choosing $\theta > 0$ small enough. This means that, it is possible to *modulate* the negative part of $b(x)$ in order to ensure existence of supersolutions. This idea is coherent with the results obtained in the case of compact manifolds in the very recent papers [**42**] and [**57**]. In what follows we set $B_0 = \{x \in M \,:\, b(x) \leq 0\}$.

THEOREM 2.1. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be complete $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \leq 1$, $c(x) \geq 0$, and suppose that B_0 is compact. Let σ, $\tau \in \mathbb{R}$ be such that $\tau < 1 < \sigma$. Let $g(x,t) \in \mathrm{C}^0(\partial M \times \mathbb{R}^+)$ be such that*

$$\begin{cases} i) \qquad \exists\, \overline{\gamma} > 0 \,:\, \sup_{x \in \partial M} g(x,\gamma) \leq 0 \quad \textit{for all } \gamma \geq \overline{\gamma} \\ ii) \qquad \lim_{t \to \infty} \dfrac{g(x,t)}{t} = -\infty\,. \end{cases} \tag{2.3}$$

Assume that

$$\begin{cases} i) \qquad \zeta_1^L(B_0) > 0 \\ ii) \qquad \limsup_{x \to \infty} \dfrac{a_+(x) + c_+(x)}{b_+(x)} < +\infty\,. \end{cases} \tag{2.4}$$

Then there exists $\theta_0 \in (0,1]$ such that for each $\theta \in (0, \theta_0]$ there exists $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of

$$\begin{cases} \Delta u + a(x)u - b_\theta(x)u^\sigma + c(x)u^\tau \leq 0 & \textit{on } \operatorname{int} M \\ \partial_\nu u - g(x,u) \geq 0 & \textit{on } \partial M. \end{cases} \tag{2.5}$$

PROOF. Since $(M, \partial M, \langle\ ,\ \rangle)$ is a complete metric space, it follows that B_0 is bounded. By the definition of ζ_1^L and assumption (2.4) i) we can choose two bounded open sets D' and D such that $\partial D \cap \operatorname{int} M$ is smooth,

$$B_0 \subset\subset D' \subset\subset D \qquad \text{and} \qquad \zeta_1^L(D) > 0\,.$$

Let u_1 be the positive solution of

$$\begin{cases} \Delta u_1 + a(x)u_1 + \zeta_1^L(D)u_1 = 0 & \text{on } D \\ u_1 = 0 & \text{on } \partial_0 D \\ \partial_\nu u_1 = 0 & \text{on } \partial_1 D\,, \end{cases}$$

such that $\|u_1\|_{\mathrm{L}^\infty(D)} = 1$. Now we choose a cut-off function $\psi \in \mathrm{C}_0^\infty(M)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on D', and $\operatorname{supp}\psi \subset D$. For positive constants η, μ, let define

$$u = \eta\left(\psi u_1 + (1-\psi)\mu\right)\,. \tag{2.6}$$

We start with the case $\|c\|_{\mathrm{L}^\infty(D')} \neq 0$, the other one is simpler and will be considered later. First of all we consider the behaviour of u in $\operatorname{int} M$. We define $\xi = \left(\inf_{D'} u_1\right)^{\tau-1} > 0$. Since $\zeta_1^L(D) > 0$, on $\overline{D'}$ and using the fact that $\|u_1\|_{\mathrm{L}^\infty(D)} = 1$, on $\overline{D}$ we have

$$\begin{aligned} Lu - b_\theta(x)u^\sigma + c(x)u^\tau &= L\left(\eta u_1\right) - b_\theta(x)\left(\eta u_1\right)^\sigma + c(x)\left(\eta u_1\right)^\tau \\ &\le \left(\eta u_1\right)\left[-\zeta_1^L(D) + \theta b_-(x)\left(\eta u_1\right)^{\sigma-1} + c(x)\left(\eta u_1\right)^{\tau-1}\right] \\ &\le \left(\eta u_1\right)\left[-\zeta_1^L(D) + \theta\left\|b_-\right\|_{\mathrm{L}^\infty(M)}\eta^{\sigma-1} + \xi\left\|c\right\|_{\mathrm{L}^\infty(D')}\eta^{\tau-1}\right] \end{aligned}$$

To study the sign of the RHS of the above inequality we need the following elementary calculus lemma.

Lemma 2.2. *Let A, B, C be positive constants. For $t \in \mathbb{R}^+$ consider the function*

$$f(t) = At^p + Bt^q - C$$

where $q < 0 < p$. Define the positive constant

$$M(p,q) = \left(-\frac{q}{p}\right)^{p/(p-q)} + \left(-\frac{q}{p}\right)^{q/(p-q)}\,.$$

If

$$A^{-q}B^p < \left(\frac{C}{M(p,q)}\right)^{p-q} \tag{2.7}$$

then $f(t)$ attains a negative minimum at the point

$$\bar{t} = \left(-\frac{qB}{pA}\right)^{1/(p-q)} \tag{2.8}$$

Proof of Lemma 2.2. Since $\lim\limits_{t\to 0^+} f(t) = \lim\limits_{t\to+\infty} f(t) = +\infty$, it follows that $f(t)$ attains an absolute minimum at some point $\bar{t} \in \mathbb{R}^+$ such that $f'(\bar{t}) = 0$, where

$$f'(t) = pAt^{p-1} + qBt^{q-1}\,.$$

It follows that

$$\bar{t} = \left(-\frac{qB}{pA}\right)^{1/(p-q)} .$$

Condition (2.7) thus guarantees that $f(\bar{t}) < 0$. □

Going back to the proof of the Theorem we choose $A = \theta \|b_-\|_{L^\infty(M)}$, $B = \xi \|c\|_{L^\infty(D')}$, $C = \zeta_1^L(D)$, $p = \sigma - 1$, and $q = \tau - 1$. In this case, (2.7) and (2.8) read as functions of θ respectively as

$$\theta < M_1$$

and

$$\bar{t} = \bar{t}(\theta) = \theta^{1/(\tau-\sigma)} M_2 ,$$

where M_1 and M_2 are positive constants depending only on $(b(x), c(x), D, D', \sigma, \tau)$. We note that $\bar{t}(\theta) \nearrow +\infty$ as $\theta \downarrow 0$, thus, there exists $0 < \theta_* < M_1$ such that $\bar{t}(\theta) > 1$ for $\theta < \theta_*$. Setting

$$\eta = \bar{t}(\theta) \tag{2.9}$$

in (2.6), we deduce that

$$Lu - b_\theta(x)u^\sigma + c(x)u^\tau \le 0 \qquad \text{on } \overline{D'} \cap \operatorname{int} M , \tag{2.10}$$

for each $\theta \le \theta_*$.
If $\|c\|_{L^\infty(D')} = 0$ we proceed as above, deducing that on $\overline{D}$ it holds

$$Lu - b_\theta(x)u^\sigma + c(x)u^\tau \le (\eta u_1)\left[-\zeta_1^L(D) + \theta \|b_-\|_{L^\infty(M)} \eta^{\sigma-1}\right] .$$

In this case it is easier to analyze the RHS, indeed it is apparent that it has an absolute minimum at $\eta = 0$ and is negative for

$$\eta < \tilde{t}(\theta) = \left(\frac{\zeta_1^L(D)}{\theta \|b_-\|_{L^\infty(M)}}\right)^{1/(\sigma-1)} ,$$

since we are interested in a positive u, we set, to fix the ideas,

$$\eta = \frac{1}{2}\tilde{t}(\theta) .$$

Next we consider $M \setminus D$. Since $\operatorname{supp} \psi \subset D$, it follows that $u = \eta\mu$ there. Thus, setting $\alpha = \|a(x)\|_{\infty,\Omega\setminus D}$, $\beta = \inf_{\Omega\setminus D}(b(x))$, $\gamma = \|c(x)\|_{\infty,\Omega\setminus D}$, for $\theta < \theta_*$ we have that

$$\begin{aligned} Lu - b_\theta(x)u^\sigma + c(x)u^\tau &\le \eta\mu\left[\alpha - \beta(\eta\mu)^{\sigma-1} + \gamma(\eta\mu)^{\tau-1}\right] \\ &\le \eta\mu\left[\alpha - \beta\mu^{\sigma-1} + \gamma\mu^{\tau-1}\right] \end{aligned}$$

by monotonicity, there exists a positive constant Λ_0 (not depending on θ) such that the RHS is negative for $\mu > \Lambda_0$.

It remains to analyze the situation on $D \setminus \overline{D'}$. First of all we note that, by standard elliptic regularity theory (see [**39**]), $u_1 \in \mathrm{C}^2(D)$. Thus, since $\operatorname{supp}\psi \subset D$, it follows that $u \in \mathrm{C}^2(\Omega)$, in particular this implies that there exist positive constants H, K such that

$$\begin{cases} Lu \leq \eta H & \text{on } D \setminus \overline{D'}\,, \\ \partial_\nu u \geq \eta K & \text{on } \partial_1\left(D \setminus \overline{D'}\right), \end{cases} \tag{2.11}$$

Thus on $D \setminus \overline{D'}$ we have

$$\begin{aligned} Lu - b_\theta(x)u^\sigma + c(x)u^\tau \leq \eta H &- b(x)\eta^\sigma\left(\psi u_1 + (1-\psi)\,\mu\right)^\sigma \\ &+ c(x)\eta^\tau\left(\psi u_1 + (1-\psi)\,\mu\right)^\tau \,. \end{aligned}$$

Because of our choices of ψ and D, there exist positive constants ε and E such that

$$\begin{aligned} &\inf_{D\setminus\overline{D'}} b(x)\left(\psi u_1 + (1-\psi)\,\mu\right)^\sigma \geq \varepsilon\,, \\ &\sup_{D\setminus\overline{D'}} c(x)\left(\psi u_1 + (1-\psi)\,\mu\right)^\tau \leq E\,. \end{aligned}$$

Therefore, on $D \setminus \overline{D'}$

$$Lu - b_\theta(x)u^\sigma + c(x)u^\tau \leq \eta\left(H - \varepsilon\eta^{\sigma-1} + E\eta^{\tau-1}\right)\,.$$

Since $\sigma > 1$ and $\tau < 1$, it follows that there exists a constant $\Lambda_1 > 0$ depending only on D and D' such that

$$H - \varepsilon\eta^{\sigma-1} + E\eta^{\tau-1} \leq 0$$

for $\eta \geq \Lambda_1$.

Now, it follows from (2.6) and (2.11) that

$$\partial_\nu u - g(x,u) = -g(x,u) \qquad \text{on } \partial M \setminus \partial_1\left(D \setminus \overline{D'}\right)$$

and

$$\partial_\nu u - g(x,u) \geq -g(x,u) - \eta K \qquad \text{on } \partial_1\left(D \setminus \overline{D'}\right).$$

Since $u_1 > 0$ on D, $\operatorname{supp}\psi \subset D$, and $\mu > 0$, it holds that

$$\rho = \inf_{x\in\partial M}\left(\psi u_1 + (1-\psi)\mu\right) > 0$$

thus, it follows from (2.3) i) that

$$g(x,u) \leq 0 \qquad \text{on } \partial M \tag{2.12}$$

for $\eta \geq \overline{\gamma}/\rho$. Moreover

$$-g(x,u) - \eta K \geq -\eta\rho\left[\frac{g(x,u)}{u} + \frac{K}{\rho}\right] \qquad \text{on } \partial M\,,$$

since $\partial_1\left(D\setminus\overline{D'}\right)$ is bounded, $g(x,t)$ is uniformly continous there (as a function of x), thus it follows from (2.3) ii) that there exists a constant $\Lambda_2>0$ such that the RHS of the inequality above is non-negative for $\eta\geq\Lambda_2$.

Since $\overline{t}(\theta)\nearrow+\infty$ monotonically as $\theta\searrow 0^+$, there exists a $0<\theta_0<\theta_*$ such that

$$\eta=t_*(\theta)\geq\max\{1,\Lambda_1,\Lambda_2,\overline{\gamma}/\rho\}$$

for $\theta\leq\theta_0$. With this choice of θ and consequently of η, and with the previous choice of μ, u is the desidered solution of (2.5). $\square$

Remark 2.3. The solution u of (2.5) constructed above is strictly positive, that is $u_*=\inf\limits_{x\in M}u>0$.

Remark 2.4. If $B_0\subset\operatorname{int}M$ (that is $\partial B_0\subset\operatorname{int}M$), the mixed spectral condition (2.4) i) can be substituted with the usual Dirichlet spectral condition, that is

$$\lambda_1^L(B_0)>0\,.$$

2.2. Construction of a subsolution

In this section we are going to find positive (and bounded) subsolutions to equation (2.2). The proof is based on two elementary observations. The first is that since $b_+(x)\geq b_\theta(x)$ for any $\theta>0$, then any positive subsolution of

$$\begin{cases}\Delta u+a(x)u-b_+(x)u^\sigma+c(x)u^\tau=0 & \text{on }\operatorname{int}M\\ \partial_\nu u-g(x,u)=0 & \text{on }\partial M,\end{cases}\tag{2.13}$$

is also a subsolution for (2.2).
The second is that equation (2.13) has an interesting symmetry property, indeed, let $a(x)$, $b(x)$, $c(x)\in C^{0,\alpha}_{loc}(M)$ for some $0<\alpha\leq 1$, and σ, $\tau\in\mathbb{R}$ satisfying $\tau<1<\sigma$. Setting

$$\begin{cases}\overline{a}(x)=-a(x)\\ \overline{b}(x)=c(x)\\ \overline{c}(x)=b_+(x)\\ \overline{\tau}=2-\sigma\\ \overline{\sigma}=2-\tau\end{cases}$$

it follows that also $\overline{a}(x)$, $\overline{b}(x)$, $\overline{c}(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$, and $\overline{\tau} < 1 < \overline{\sigma}$. Now, suppose that $v \in \mathrm{L}^\infty(M)$ is a positive supersolution of

$$\begin{cases} \Delta u + \overline{a}(x)u - \overline{b}(x)u^{\overline{\sigma}} + \overline{c}(x)u^{\overline{\tau}} = 0 & \text{on } \operatorname{int} M \\ \partial_\nu u - \overline{g}(x,u) = 0 & \text{on } \partial M, \end{cases} \tag{2.14}$$

where

$$\overline{g}(x,t) = -t^2 g\left(x, \frac{1}{t}\right) \tag{2.15}$$

then a simple computation shows that $u^- = \dfrac{1}{v}$ is a positive subsolution of (2.13).
In what follows we set $C_0 = \{x \in M \,:\, c(x) \le 0\}$.

THEOREM 2.5. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be complete $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \le 1$, $c(x) \ge 0$, and suppose that C_0 is compact. Let σ, $\tau \in \mathbb{R}$ be such that $\tau < 1 < \sigma$. Let $g(x,t) \in \mathrm{C}^0(\partial M \times \mathbb{R}^+)$ be such that*

$$\begin{cases} i) & \exists\, \overline{\omega} > 0 \,:\, \inf\limits_{x \in \partial M} g(x,\omega) \ge 0 \quad \textit{for all } 0 < \omega \le \overline{\omega} \\ ii) & \lim\limits_{s \to 0^+} \dfrac{g(x,s)}{s} = +\infty\,. \end{cases} \tag{2.16}$$

Assume that

$$\begin{cases} i) & \zeta_1^{\overline{L}}(C_0) > 0 \\ ii) & \limsup\limits_{x \to \infty} \dfrac{a_-(x) + b_+(x)}{c_+(x)} < +\infty\,, \end{cases} \tag{2.17}$$

where $\overline{L} = \Delta - a(x)$. Then there exists $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of

$$\begin{cases} \Delta u + a(x)u - b_+(x)u^\sigma + c(x)u^\tau \ge 0 & \textit{on } \operatorname{int} M \\ \partial_\nu u - g(x,u) \le 0 & \textit{on } \partial M. \end{cases} \tag{2.18}$$

PROOF. It follows easily from the observations made above, simply noting that (2.16) implies the validity of (2.3) for $\overline{g}(x,t)$ defined in (2.15), while (2.17) corresponds to conditions (2.4) for equation (2.14). Thus there exists $v \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of (2.14) from which we obtain $u = \dfrac{1}{v} \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of (2.18). $\square$

2.3. Existence of a positive solution

Now we put together the results of the previous sections to get a positive solution $u \in \mathrm{C}^2(\mathrm{int}\, M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ of (2.1). Our main result is the following

THEOREM 2.6. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be complete and suppose that $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \leq 1$, $c(x) \geq 0$, and assume that B_0, C_0 are compact. Let σ, $\tau \in \mathbb{R}$ be such that $\tau < 1 < \sigma$. Let $g(x,t) \in \mathrm{C}^0(\partial M \times \mathbb{R}^+)$ satisfy*

$$(2.19)\qquad \begin{cases} i) & \exists\, \overline{\omega} > 0 : \inf_{x \in \partial M} g(x, \omega) \geq 0 \quad \textit{for all } 0 < \omega \leq \overline{\omega}\,; \\ ii) & \exists\, \overline{\gamma} > 0 : \sup_{x \in \partial M} g(x, \gamma) \leq 0 \quad \textit{for all } \gamma \geq \overline{\gamma}\,; \\ iii) & \lim_{s \to 0^+} \dfrac{g(x,s)}{s} = +\infty\,; \\ iv) & \lim_{t \to \infty} \dfrac{g(x,t)}{t} = -\infty\,, \end{cases}$$

and let $\dfrac{g(x,t)}{t}$ be non-increasing. Assume that

$$(2.20)\qquad \begin{cases} i) & \zeta_1^L(B_0) > 0\,; \\ ii) & \zeta_1^{\overline{L}}(C_0) > 0\,; \\ iii) & \limsup_{x \to \infty} \dfrac{a_+(x) + c_+(x)}{b_+(x)} < +\infty\,; \\ iv) & \limsup_{x \to \infty} \dfrac{a_-(x) + b_+(x)}{c_+(x)} < +\infty\,, \end{cases}$$

where $L = \Delta + a(x)$ and $\overline{L} = \Delta - a(x)$. Then there exists $\theta_0 \in (0,1]$ such that for each $\theta \in (0, \theta_0]$ there exists $u \in \mathrm{C}^2(\mathrm{int}\, M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of

$$(2.21)\qquad \begin{cases} \Delta u + a(x)u - b_\theta(x)u^\sigma + c(x)u^\tau = 0 & \textit{on } \mathrm{int}\, M \\ \partial_\nu u - g(x,u) = 0 & \textit{on } \partial M. \end{cases}$$

PROOF. By (2.19) and (2.20) it follows that the hypoteses of Theorem 2.1 and Theorem 2.5 are satisfied, thus there exist u^+, $u^- \in \mathrm{C}^2(\mathrm{int}\, M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ respectively a supersolution of (2.21) and a subsolution of (2.18) (and thus also a subsolution of (2.21)). Morever by Remark 2.3 we can assume that there exists a $m > 0$ such that $u^+ \geq m$. Now, for $s \in (0,1)$, set $u_s = su^-$. We claim that u_s is still a subsolution of (2.18), indeed

$$\begin{aligned} \Delta u_s + a(x)u_s - b_+(x)u_s^\sigma + c(x)u_s^\tau &= s\left(\Delta u + a(x)u - b_+(x)s^{\sigma-1}u^\sigma + c(x)s^{\tau-1}u^\tau\right) \\ &\geq s\left(\Delta u + a(x)u - b_+(x)u^\sigma + c(x)u^\tau\right) \\ &\geq 0\,, \end{aligned}$$

on M, and since $u_s < u^-$, it follows from the monotonicity of $\dfrac{g(x,t)}{t}$ that

$$\begin{aligned}\partial_\nu u_s - g(x,u_s) &= s\partial_\nu u - g(x,u_s)\\ &\le sg(x,u) - g(x,u_s)\\ &= u_s\left(\frac{g(x,u)}{u} - \frac{g(x,u_s)}{u_s}\right)\\ &\le 0,\end{aligned}$$

on ∂M. Thus u_s is still a subsolution of (2.18) for any $s \in (0,1)$. In particular, choosing $s < \dfrac{m}{\sup_{x\in M} u^-}$, we have that $u_s \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ is a subsolution of (2.21) such that $0 < u^- < u^+$, thus we can apply Proposition 1.25 to get the desired positive solution $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ of (2.21). $\square$

2.4. Boundary conditions and applications

Here we present a class of nonlinearities $g(x,t)$ that we have in mind in view of applications. In particular we will obtain more explicit expressions for conditions (2.3) and (2.16). We consider the following *simple* nonlinearity

$$g(x,t) = \sum_{i=1}^{N} g_i(x)t^{q_i} \tag{2.22}$$

where $g_i \in \mathrm{C}^0(\partial M)$ and $q_i \in \mathbb{R}$ are such that $q_1 < q_2 < \cdots < q_N$. We say that $g(x,t)$ in (2.22) is a *strongly defocusing nonlinearity* if the following conditions are fullfilled

$$\begin{cases} i) & (q_i - 1)g_i(x) \le 0 \text{ for all } 1 \le i \le N;\\[2ex] ii) & \begin{array}{l}\exists\, k \text{ s.t. } q_k > 1,\ g_k(x) \neq 0,\\ \text{and } \frac{g_i(x)}{g_k(x)} \in \mathrm{L}^\infty(\partial M) \text{ for all } i \text{ s.t. } q_i \le 1;\end{array}\\[2ex] iii) & \begin{array}{l}\exists\, h \text{ s.t. } q_h < 1,\ g_h(x) \neq 0,\\ \text{and } \frac{g_i(x)}{g_h(x)} \in \mathrm{L}^\infty(\partial M) \text{ for all } i \text{ s.t. } q_i \ge 1.\end{array}\end{cases} \tag{2.23}$$

The following corollary of Theorem 2.6 is a straightforward application.

Corollary 2.7. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be complete and suppose that $a(x)$, $b(x)$, $c(x)$, σ, and τ are as in Theorem 2.6. Let $g(x,t)$ be as in (2.22), satisfying conditions (2.23). Furthermore assume that the conditions in (2.20) are satisfied.*

Then there exists $\theta_0 \in (0,1]$ *such that for each* $\theta \in (0,\theta_0]$ *there exists* $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ *positive solution of (2.21).*

Proof. The corollary follows from Theorem 2.6 once we show that conditions (2.23) implies the validity of (2.19) and the monotonicity of $\frac{g(x,t)}{t}$. First of all, condition (2.23) i) means that, for any fixed $x \in M$, $\frac{g(x,t)}{t}$ is a non-increasing function of $t \in \mathbb{R}^+$, indeed

$$\frac{\partial}{\partial t}\left(\frac{g(x,t)}{t}\right) = \frac{\partial}{\partial t}\sum_{i=1}^{N} g_i(x)t^{q_i-1} = \sum_{i=1}^{N}(q_i-1)g_i(x)t^{q_i-2} \leq 0\,.$$

Now, for all $x \in M$

$$\frac{g(x,t)}{t} = g_k(x)t^{q_k-1} + \sum_{i\neq k} g_i(x)t^{q_i-1}\,,$$

but from (2.23) i), ii) it follows that the second summand is bounded above by a positive constant, while

$$\lim_{t\to+\infty} g_k(x)t^{q_k-1} = -\infty\,,$$

that is, (2.19) iv) is satisfied. Moreover for $\gamma \in \mathbb{R}^+$

$$\begin{aligned} g(x,\gamma) &= g_k(x)\gamma\left[\sum_{q_i>1}\frac{g_i(x)}{g_k(x)}\gamma^{q_i-1} + \sum_{q_i\leq 1}\frac{g_i(x)}{g_k(x)}\gamma^{q_i-1}\right] \\ &\leq g_k(x)\gamma\left[\gamma^{q_k-1} - \sum_{q_i\leq 1}\left\|\frac{g_i(x)}{g_k(x)}\right\|_{\mathrm{L}^\infty(\partial M)}\gamma^{q_i-1}\right] \end{aligned}$$

and there exists a $\overline{\gamma} > 0$ such that the quantity in square brackets is positive for $\gamma > \overline{\gamma}$, thus (2.19) ii) follows. Conditions (2.19) i), and iii) can be derived similarly. □

2.5. Uniqueness results for $b(x) \geq 0$

The aim of this section is to prove uniqueness of positive solutions of equation (2.1) on M. To avoid technicalities we suppose $u \in \mathrm{C}^2(M)$ but this assumption can be relaxed as it will become clear from the arguments we are going to present. We begin by proving a global comparison result with the aid of the open form of the *q-Weak Maximum Principle* of Section 1.3. For the present purposes we let L be a linear operator of the form

$$Lu = \Delta u - \langle X, \nabla u\rangle \tag{2.24}$$

for some vector field X on M.

THEOREM 2.8. *Let* $a(x), b(x), c(x) \in \mathrm{C}^0(M)$ *and* $\sigma, \tau \in \mathbb{R}$ *be such that* $\tau < 1 < \sigma$. *Assume*

$$
\begin{aligned}
&i)\ b(x) > 0 \quad \text{on } M \\
&ii)\ c(x) \geq 0 \quad \text{on } M \\
&iii)\ \sup_M \frac{a_-(x)}{b(x)} < +\infty \\
&iv)\ \sup_M \frac{c(x)}{b(x)} < +\infty
\end{aligned}
\tag{2.25}
$$

where, a_- *denotes the negative part of* a. *Let* $u, v \in \mathrm{C}^0(M) \cap \mathrm{C}^2(\operatorname{int} M)$ *be positive solutions of*

$$
\begin{cases}
Lu + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0 \\
Lv + a(x)v - b(x)v^\sigma + c(x)v^\tau \leq 0
\end{cases}
\tag{2.26}
$$

on $\operatorname{int} M$ *satisfying*

$$\liminf_{x \to +\infty} v(x) > 0\,, \tag{2.27}$$

$$\limsup_{x \to +\infty} u(x) < +\infty\,, \tag{2.28}$$

and

$$0 < \delta < u(x) \leq v(x) \qquad \text{on } \partial M\,. \tag{2.29}$$

Then

$$u(x) \leq v(x) \tag{2.30}$$

on M *provided that the* $1/b$*-WMP holds on* M *for* L.

REMARK 2.9. As it will be observed in the proof of the theorem, in case $0 \leq \tau < 1$ assumption (2.25) iv) can be dropped.

PROOF. Without loss of generality we can suppose that $\operatorname{int} M$ is connected. From positivity of v, (2.27), (2.28), and (2.29) there exist positive constants C_1, C_2 such that

$$v(x) \geq C_1 \qquad u(x) \leq C_2 \qquad \text{on } \operatorname{int} M\,. \tag{2.31}$$

We set $\zeta = \sup_M \left(\frac{u}{v}\right)$, from the assumptions on v, u, and (2.31) it follows that ζ satisfies

$$0 < \zeta < +\infty. \tag{2.32}$$

Note that if $\zeta \leq 1$ then $u \leq v$ on M. Thus, assume by contradiction that $\zeta > 1$ and define

$$\varphi = u - \zeta v\,,$$

then $\varphi \leq 0$ on M. It is not hard to realize, using (2.32) and the definition of ζ, that

$$\sup_M \varphi = 0. \tag{2.33}$$

We now use (2.26) to compute

$$\begin{aligned} L\varphi \geq & -a(x)\varphi + b(x)\left[u^\sigma - (\zeta v)^\sigma\right] - c(x)\left[u^\tau - (\zeta v)^\tau\right] \\ & + b(x)\zeta v\left[(\zeta v)^{\sigma-1} - v^{\sigma-1}\right] + c(x)\zeta v\left[v^{\tau-1} - (\zeta v)^{\tau-1}\right]. \end{aligned} \tag{2.34}$$

We let

$$h(x) = \begin{cases} \sigma u^{\sigma-1}(x) & \text{if} \quad u(x) = \zeta v(x) \\ \\ \dfrac{\sigma}{u(x) - \zeta v(x)} \displaystyle\int_{\zeta v(x)}^{u(x)} t^{\sigma-1} dt & \text{if} \quad u(x) < \zeta v(x). \end{cases}$$

and, similarly, for $\tau \neq 0$,

$$j(x) = \begin{cases} -\tau u^{\tau-1}(x) & \text{if} \quad u(x) = \zeta v(x) \\ \\ \dfrac{\tau}{\zeta v(x) - u(x)} \displaystyle\int_{\zeta v(x)}^{u(x)} t^{\tau-1} dt & \text{if} \quad u(x) < \zeta v(x). \end{cases}$$

In case $\tau = 0$ choose $j(x) \equiv 0$. Note that h and j are continous on M and h is non-negative. Using h and j, and observing that $-a(x)\varphi \geq a_-(x)\varphi$, from (2.34) we obtain

$$\begin{aligned} L\varphi \geq & \left[a_-(x) + b(x)h(x) + c(x)j(x)\right]\varphi \\ & + b(x)\zeta v\left[(\zeta v)^{\sigma-1} - v^{\sigma-1}\right] + c(x)\zeta v\left[v^{\tau-1} - (\zeta v)^{\tau-1}\right]. \end{aligned} \tag{2.35}$$

Let

$$\Omega_{-1} = \{x \in M : \varphi(x) > -1\}.$$

Since u is bounded above on M, there exists a constant $C > 0$ such that

$$v(x) = \frac{1}{\zeta}(u(x) - \varphi(x)) \leq \frac{1}{\zeta}(C+1) \tag{2.36}$$

on Ω_{-1}. Using the definitions of h and j, from the mean value theorem for integrals, we deduce

$$h(x) = \sigma y_h^{\sigma-1}, \qquad j(x) = -\tau y_j^{\tau-1}$$

for some $y_h = y_h(x)$ and $y_j = y_j(x)$ in the range $[u(x), \zeta v(x)]$. Since $u(x)$ and $v(x)$ are bounded above on Ω_{-1}

$$\max\{h(x), j(x\}) \leq C \tag{2.37}$$

on Ω_{-1} for some constant $C > 0$. Next we recall that $b(x) > 0$ on M to rewrite (2.35) in the form

$$\frac{1}{b(x)} L\varphi \geq \left[\frac{a_-(x)}{b(x)} + h(x) + \frac{c(x)}{b(x)} j_+(x)\right]\varphi$$
$$+ \zeta v\left[(\zeta v)^{\sigma-1} - v^{\sigma-1}\right] + \frac{c(x)}{b(x)}\zeta v\left[v^{\tau-1} - (\zeta v)^{\tau-1}\right].$$

Since $\varphi \leq 0$, (2.25) and (2.37) imply

$$\left[\frac{a_-(x)}{b(x)} + h(x) + \frac{c(x)}{b(x)} j_+(x)\right]\varphi \geq C\varphi \tag{2.38}$$

for some constant $C > 0$ on Ω_{-1}. For further use we observe here that when $0 \leq \tau < 1$, $j_+(x) \equiv 0$ so that in this case assumption (2.25) iv) is not needed to obtain (2.38). Thus

$$\frac{1}{b(x)} L\varphi \geq C\varphi + \zeta v\left[(\zeta v)^{\sigma-1} - v^{\sigma-1}\right] + \frac{c(x)}{b(x)}\zeta v\left[v^{\tau-1} - (\zeta v)^{\tau-1}\right]$$

on Ω_{-1}. Recalling the elementary inequalities

$$\begin{cases} a^s - b^s \geq s b^{s-1}(a-b) & \text{for } s < 0 \text{ and } s > 1\,; \\ a^s - b^s \geq s a^{s-1}(a-b) & \text{for } 0 \leq s \leq 1\,, \end{cases} \tag{2.39}$$

$a, b \in \mathbb{R}^+$, coming from the mean value theorem for integrals (see Theorem 41 in [**40**]) we conclude

$$\frac{1}{b(x)} L\varphi \geq C\varphi + (\sigma-1)\zeta^{\min\{1,\sigma-1\}}(\zeta-1)v^\sigma + (1-\tau)\,\frac{c(x)}{b(x)}\frac{\zeta-1}{\zeta^{1-\tau}}v^\tau\,,$$

on Ω_{-1}. Now we use the fact that $\tau < 1$, v is bounded from below by a positive constant, (2.25) i), ii), iv) to get (again if $0 \leq \tau < 1$ we do not need (2.25) iv))

$$\frac{1}{b(x)} L\varphi \geq C\varphi + B \quad \text{on } \Omega_{-1},$$

for some positive constants B, C. Finally, we choose $0 < \varepsilon < 1$ sufficiently small that

$$C\varphi > -\frac{1}{2}B$$

on

$$\Omega_{-\varepsilon} = \{x \in M : \varphi(x) > -\varepsilon\} \subset \Omega_{-1}.$$

Therefore

$$\frac{1}{b(x)} L\varphi \geq \frac{1}{2}B > 0 \quad \textit{on} \quad \Omega_{-\varepsilon}. \tag{2.40}$$

Furthermore, note that

$$\varphi(x) \leq \max\left\{-\varepsilon,\ (1-\zeta)\min_{\partial M} v\right\} < 0 \qquad \text{on } \partial\Omega_{-\varepsilon}\,.$$

As a consequence $\sup_{\partial\Omega_{-\varepsilon}} \varphi < 0$ while $\sup_{\Omega_{-\varepsilon}} \varphi = 0$. By Theorem 1.13, (2.40) and the above fact, we obtain the required contradiction, proving that $\zeta \leq 1$. $\square$

As an immediate consequence of Theorem 2.8 we obtain the following uniqueness result

Corollary 2.10. *In the assumptions of Theorem 2.8 the problem*

$$\begin{cases} Lu + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on } \operatorname{int} M \\ u = g(x) & \text{on } \partial M \end{cases}$$

admits at most one solution $u \in \mathrm{C}^0(M) \cap \mathrm{C}^2(\operatorname{int} M)$ satisfying

$$C_1 \leq u(x) \leq C_2 \qquad \text{on } M \tag{2.41}$$

for some positive constants C_1, C_2, provided that the $1/b$-WMP holds on M for the operator L.

We conclude the section with a second uniqueness result whose proof is based on that of Theorem 4.1 of [**21**], see also Theorem 5.1 in [**55**].

Theorem 2.11. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete manifold, $a(x), b(x), c(x) \in \mathrm{C}^0(M)$, $\tau < 1 < \sigma$, and assume*

$$b(x) \geq 0, \qquad c(x) \geq 0\,, \tag{2.42}$$

and

$$b(x) + c(x) \not\equiv 0 \quad \text{on } M\,. \tag{2.43}$$

Let u, $v \in \mathrm{C}^2(M)$ be positive solutions of

$$\begin{cases} \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on } \operatorname{int} M \\ u = g(x) & \text{on } \partial M \end{cases}$$

such that

$$\left\{ \int_{\partial B_r} (u-v)^2 \right\}^{-1} \notin \mathrm{L}^1(+\infty)\,. \tag{2.44}$$

Then $u \equiv v$ on M.

Remark 2.12. Note that condition (2.44) is implied by $u - v \in \mathrm{L}^2(M)$ or even by the weaker request

$$\int_{B_r} (u-v)^2 = o(r^2) \qquad \text{as } r \to +\infty\,.$$

See for instance Proposition 1.3 in [**70**].

PROOF. The proof follows, mutatis mutandis, that reported in Theorem 5.1 of [**55**], we report it here for convenience of the reader. First of all we introduce the vector field

$$W = \left(v^2 - u^2\right) \nabla \left(\log \frac{v}{u}\right) ,$$

noting that $W \equiv 0$ on ∂M since $u \equiv v$ there. Applying the divergence theorem to W on the geodesic ball B_R and using the equation, we get

$$\begin{aligned} &\int_{\partial_1 B_R} \langle W, \nabla r\rangle - \int_{B_R} \left\{ \left|\nabla v - \frac{v}{u}\nabla u\right|^2 + \left|\nabla u - \frac{u}{v}\nabla v\right|^2 \right\} \\ &= \int_{B_R} \left\{ b(x)\left(v^2 - u^2\right)\left(v^{\sigma-1} - u^{\sigma-1}\right) + c(x)\left(v^2 - u^2\right)\left(u^{\tau-1} - v^{\tau-1}\right)\right\} . \end{aligned} \tag{2.45}$$

The RHS is a nonnegative and nondecreasing function of R, thus it tends to a nonnegative limit $B \leq +\infty$ as $r \to +\infty$. Now, by Gauss' lemma and Schwarz's inequality

$$|\langle W, \nabla r\rangle| \leq |v - u| \left(\left|\nabla v - \frac{v}{u}\nabla u\right| + \left|\nabla u - \frac{u}{v}\nabla v\right|\right)$$

and therefore

$$\begin{aligned} \left[\int_{\partial_0 B_R} \langle W, \nabla r\rangle\right]^2 &\leq 2\left[\int_{\partial_0 B_R} (v-u)^2\right] \int_{\partial_0 B_R} \left\{ \left|\nabla v - \frac{v}{u}\nabla u\right|^2 + \left|\nabla u - \frac{u}{v}\nabla v\right|^2 \right\} \\ &\leq 2\left[\int_{S_R} (v-u)^2\right] \int_{S_R} \left\{ \left|\nabla v - \frac{v}{u}\nabla u\right|^2 + \left|\nabla u - \frac{u}{v}\nabla v\right|^2 \right\} , \end{aligned}$$

where $S_R \supseteq \partial_0 B_R$ is the geodesic sphere or radius R. We claim that

$$\left|\nabla v - \frac{v}{u}\nabla u\right|^2 + \left|\nabla u - \frac{u}{v}\nabla v\right|^2 \in \mathrm{L}^1(M) . \tag{2.46}$$

Indeed, set

$$G(R) = \int_{B_R} \left|\nabla v - \frac{v}{u}\nabla u\right|^2 + \left|\nabla u - \frac{u}{v}\nabla v\right|^2$$

and assume by contradiction that $G(R) \to +\infty$ as $R \to +\infty$, since

$$\int_{\partial_1 B_R} \langle W, \nabla r\rangle - G(R) \to B \geq 0 \qquad \text{as } R \to +\infty,$$

it follows that

$$\int_{\partial_1 B_R} \langle W, \nabla r\rangle \geq \frac{1}{2} G(R) \geq 0$$

for R sufficiently large. Then, we have the following

$$\left[\frac{1}{2} G(R)\right]^2 \leq 2G'(R) \left[\int_{S_R} (v-u)^2\right] .$$

Now the claim follows by a standard application of the coarea formula, together with (2.44) (see for instance the Step 1 of the proof of Theorem 3.2 of [**55**]). Beacause of (2.44) and (2.46), there exists an increasing sequence $\{r_k\} \nearrow +\infty$ such that

$$\lim_{k\to+\infty}\left[\int_{S_{r_k}}(v-u)^2\right]\int_{S_{r_k}}\left\{\left|\nabla v-\frac{v}{u}\nabla u\right|^2+\left|\nabla u-\frac{u}{v}\nabla v\right|^2\right\}=0\,,$$

and consequently

$$\lim_{k\to+\infty}\int_{\partial_0 B_{r_k}}\langle W,\nabla r\rangle=0\,.$$

Now, evaluating (2.45) along the sequence $\{r_k\}$ and letting $k\to+\infty$ we get

$$\begin{aligned}\int_M b(x)\left(v^2-u^2\right)\left(v^{\sigma-1}-u^{\sigma-1}\right)+\int_M c(x)\left(v^2-u^2\right)\left(u^{\tau-1}-v^{\tau-1}\right)+\\ +\int_M\left\{\left|\nabla v-\frac{v}{u}\nabla u\right|^2+\left|\nabla u-\frac{u}{v}\nabla v\right|^2\right\}=0\,.\end{aligned}\tag{2.47}$$

Because of (2.42) we deduce $\left|\nabla u-\frac{u}{v}\nabla v\right|\equiv 0$ on M so that $u=Av$ for some constant $A>0$. Substituting into (2.47) yields

$$\left(1-A^2\right)\left(1-A^{\sigma-1}\right)\int_M b(x)v^{\sigma+1}=0$$

and

$$\left(1-A^2\right)\left(A^{\tau-1}-1\right)\int_M c(x)v^{\tau+1}=0\,.$$

Since $v>0$, (2.43) implies $A=1$, that is, $u=v$ on M. $\square$

Remark 2.13. The exponent 2 in (2.44) is sharp, see the discussion after Theorem 5.1 in [**55**].

CHAPTER 3

Further results in the boundaryless case

In this chapter we continue the study of positive solutions of the Lichnerowicz-type equation

$$\Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0 \tag{3.1}$$

on a complete manifold $(M, \partial M, \langle\ ,\ \rangle)$, restricting to the special case $\partial M = \emptyset$. Moreover, in this chapter, we will restrict mainly to the case of nonnegative $b(x)$, that is, we will assume the following sign restrictions

$$b(x) \geq 0\,, \qquad c(x) \geq 0\,. \tag{3.2}$$

3.1. Maximal solutions

In this section we will deal with the problem of finding maximal positive solutions to problem (3.1). In this case *maximal* means that if $0 < u \in \mathrm{C}^2(M)$ is a solution of (3.1) and $0 < v \in \mathrm{C}^2(M)$ is a second solution of (3.1), then $v \leq u$.

THEOREM 3.1. *Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \leq 1$. Assume (3.2), suppose that $b(x)$ is strictly positive outside some compact set, and*

$$\lambda_1^L(B_0) > 0 \tag{3.3}$$

with $L = \Delta + a(x)$. Assume that

$$\lambda_1^L(M) < 0\,, \tag{3.4}$$

then (3.1) has a maximal positive solution $u \in \mathrm{C}^2(M)$.

This existence result should be compared with those obtained at the end of the very recent paper [**53**].

The main result of the paper is in fact the following proposition, whose proof will be the content of the section.

Proposition 3.2. *Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \le 1$. Assume (3.2) and suppose that $b(x)$ is strictly positive outside some compact set. Furthermore, suppose*

$$\lambda_1^L(B_0) > 0 \tag{3.5}$$

with $L = \Delta + a(x)$. If $0 < u_- \in \mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{loc}(M)$ is a global subsolution of (3.1) on M, then (3.1) has a maximal positive solution $u \in \mathrm{C}^2(M)$.

The proof of Proposition 3.2 is divided into several steps. In what follows we keep the notations of the Proposition.

Lemma 3.3. *Let $\overline{a}(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \le 1$, and let (3.2) hold. Suppose that B_0 is bounded and*

$$\lambda_1^{\overline{L}}(B_0) > 0 \tag{3.6}$$

where $\overline{L} = \Delta + \overline{a}(x)$. If Ω is a bounded open set such that $B_0 \subset \Omega$, then there exists v_+ solution of

$$\begin{cases} \Delta v_+ + \overline{a}(x) v_+ - b(x) v_+^\sigma + c(x) v_+^\tau \le 0 & \text{on } \Omega \\ v_+ > 0 & \text{on } \overline{\Omega}. \end{cases} \tag{3.7}$$

Proof. Let D and D' be bounded open domains such that

$$B_0 \subset\subset D' \subset\subset D \subset\subset \Omega\,,$$

and $\lambda_1^{\overline{L}}(D) > 0$. Let u_1 be a positive solution of

$$\begin{cases} \Delta u_1 + \overline{a}(x) u_1 + \lambda_1^{\overline{L}}(D) u_1 = 0 & \text{on } D \\ u_1 = 0 & \text{on } \partial D. \end{cases}$$

Since $b(x) > 0$ on $M \setminus B_0$ and $\overline{\Omega} \setminus D' \subset\subset M \setminus B_0$,

$$\beta = \inf_{\overline{\Omega} \setminus D'} b(x) > 0\,.$$

Define

$$\alpha = \sup_{\overline{\Omega} \setminus D'} \overline{a}(x)\,, \qquad \delta = \sup_{\overline{\Omega} \setminus D'} c(x)\,,$$

and note that α, $\delta < +\infty$ since Ω is bounded. Let U be a positive constant. Then

$$\begin{aligned} \Delta U + \overline{a}(x) U - b(x) U^\sigma + c(x) U^\tau &= U \left(\overline{a}(x) - b(x) U^{\sigma-1} + c(x) U^{\tau-1} \right) \\ &\le U \left(\alpha - \beta U^{\sigma-1} + \delta U^{\tau-1} \right) \end{aligned}$$

on $\Omega \setminus D'$. We observe that the RHS of the above is non-positive for U sufficiently large, say

$$U \geq \Lambda_0 > 0\,. \tag{3.8}$$

Next we choose a cut-off function $\psi \in C_0^\infty(M)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on D', and $\operatorname{supp}\psi \subset D$. Fix a positive constant γ and define

$$u = \gamma\left(\psi u_1 + (1-\psi)\Lambda_0\right)\,. \tag{3.9}$$

Since $b(x) \geq 0$ and $\lambda_1^L(D) > 0$, on $\overline{D'}$ we have

$$\begin{aligned}\overline{L}u - b(x)u^\sigma + c(x)u^\tau &= \overline{L}\,(\gamma u_1) - b(x)\,(\gamma u_1)^\sigma + c(x)\,(\gamma u_1)^\tau\\ &= -\left[\lambda_1^L(D)\,(\gamma u_1) + b(x)\,(\gamma u_1)^\sigma - c(x)\,(\gamma u_1)^\tau\right]\\ &= -\,(\gamma u_1)\left[\lambda_1^L(D) + b(x)\,(\gamma u_1)^{\sigma-1} - c(x)\,(\gamma u_1)^{\tau-1}\right].\end{aligned}$$

For the RHS of the above to be non-positive on $\overline{D'}$ it is sufficient to have

$$c(x)\,(\gamma u_1)^{\tau-1} \leq \lambda_1^L(D) \qquad \text{on } \overline{D'}. \tag{3.10}$$

Towards this aim we note that, since $\overline{D'}$ is compact, $u_1 > 0$ on $\overline{D'}$, and $\tau < 1$, then (3.10) is satisfied for

$$\gamma \geq \Gamma_0 = \Gamma_0(u_1) > 0\,, \tag{3.11}$$

sufficiently large. We now consider $\Omega \setminus D$, since $\operatorname{supp}\psi \subset D$, it follows that $u = \gamma\Lambda_0$ there. Thus, using $\Omega \setminus D \subset \Omega \setminus D'$, from (3.8) it follows that

$$\Delta u + \overline{a}(x)u - b(x)u^\sigma + c(x)u^\tau \leq 0 \qquad \text{on } \Omega \setminus D\,,$$

is satisfied if we choose $\gamma \geq 1$; indeed in this case

$$\gamma\Lambda_0 \geq \Lambda_0\,.$$

It remains to analyze the situation on $D \setminus \overline{D'}$. First of all we note that, by standard elliptic regularity theory, $u_1 \in C^2(D)$. Thus, since $\operatorname{supp}\psi \subset D$, it follows that $u \in C^2(\Omega)$, in particular this implies that there exists a positive constant C_0 such that

$$\left(\Delta + \overline{a}(x)\right)u \leq \gamma C_0 \qquad \text{on } D \setminus \overline{D'}\,.$$

Thus on $D \setminus \overline{D'}$ we have

$$\begin{aligned}\Delta u + \overline{a}(x)u - b(x)u^\sigma + c(x)u^\tau \leq \gamma C_0 &- b(x)\gamma^\sigma\left(\psi u_1 + (1-\psi)\,\Lambda_0\right)^\sigma\\ &+ c(x)\gamma^\tau\left(\psi u_1 + (1-\psi)\,\Lambda_0\right)^\tau\,.\end{aligned}$$

Now there exists constants ε and E such that

$$\inf_{D\setminus\overline{D'}} b(x)\left(\psi u_1 + (1-\psi)\,\Lambda_0\right)^{\sigma} = \varepsilon > 0\,,$$
$$\sup_{D\setminus\overline{D'}} c(x)\left(\psi u_1 + (1-\psi)\,\Lambda_0\right)^{\tau} = E < +\infty\,.$$

Therefore, on $D\setminus\overline{D'}$

$$\Delta u + \overline{a}(x)u - b(x)u^{\sigma} + c(x)u^{\tau} \le \gamma\left(C_0 - \varepsilon\gamma^{\sigma-1} + E\gamma^{\tau-1}\right)\,.$$

Since $\sigma > 1$ and $\tau < 1$, it follows that there exists a positive constant Γ_1 depending only on D and D' such that

$$C_0 - \varepsilon\gamma^{\sigma-1} + E\gamma^{\tau-1} \le 0$$

for $\gamma \ge \Gamma_1$.
Thus, by choosing

$$\gamma \ge \max\{1, \Gamma_0, \Gamma_1\}$$

u is the desidered supersolution v_+ of (3.7) on Ω. $\square$

Definition 3.4. *We say that* property (Σ) *holds on M (for equation (3.1)) if there exists $R_o \in \mathbb{R}^+$ such that for all $R \ge R_o$ there exists a solution $\varphi \in \mathrm{C}^0(\overline{B_R}) \cap \mathrm{W}^{1,2}_{loc}(B_R)$ of*

$$\begin{cases} \Delta\varphi + a(x)\varphi - b(x)\varphi^{\sigma} + c(x)\varphi^{\tau} \ge 0 & \text{on } B_R \\ \varphi > 0 & \text{on } B_R\,. \end{cases} \tag{3.12}$$

In Proposition 3.6 below we shall give some sufficient conditions for the validity of property (Σ).

Lemma 3.5. *Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \le 1$, and let (3.2) hold. Suppose that B_0 is bounded and $\lambda_1^L(B_0) > 0$ with $L = \Delta + a(x)$. Furthermore assume that property (Σ) holds on M. Let Ω be a bounded domain such that $B_0 \subset \Omega$. Then, for each $n \in \mathbb{N}$, there exists a solution u of the problem*

$$\begin{cases} \Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0 & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = n & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Proof. By the definition of $\lambda_1^L(B_0)$ and the assumption of positivity, there exists an open domain D with smooth boundary such that $B_0 \subset D \subset\subset \Omega$ and $\lambda_1^L(D) > 0$. Let $\rho \in \mathrm{C}^{\infty}(M)$ be a cut-off function satisfying $0 \le \rho \le 1$, $\rho \equiv 1$ on D, $\rho \equiv 0$ on $M\setminus\overline{\Omega}$. Fix $N \ge \max\left\{\sup_{\overline{\Omega}}|a(x)| + 1\,, \lambda_1^{\Delta}(M\setminus\overline{\Omega}) + 1\right\}$. Define

$$\overline{a}(x) = \rho(x)a(x) + N\,(1-\rho(x))$$

and consider the operator $\overline{L} = \Delta + \overline{a}(x)$. Since $\overline{a}(x) = a(x)$ on D,

$$\lambda_1^{\overline{L}}(B_0) = \lambda_1^{L}(B_0) > 0\,. \tag{3.14}$$

Furthermore, from $N \geq \lambda_1^{\Delta}(M \setminus \overline{\Omega}) + 1$, we deduce $\lambda_1^{\overline{L}}(M \setminus \overline{\Omega}) \leq -1$ and it follows that there exists $R > 0$ sufficiently large such that

$$\overline{\Omega} \subset B_R \qquad \text{and} \qquad \lambda_1^{\overline{L}}(B_R) < 0\,. \tag{3.15}$$

Fix $\varepsilon > 0$. Then $\lambda_1^{\overline{L}}(B_{R+\varepsilon}) < 0$. Let φ be the normalized eigenfunction of $\overline{L}$ on $B_{R+\varepsilon}$ relative to the eigenvalue $\lambda_1^{\overline{L}}(B_{R+\varepsilon})$ (here, without loss of generality, that is, possibly substituting $B_{R+\varepsilon}$ with a slighly larger open set with smooth boundary, we are supposing $\partial B_{R+\varepsilon}$ smooth) so that

$$\begin{cases} \overline{L}\varphi + \lambda_1^{L}(B_{R+\varepsilon})\varphi = 0 & \text{on } B_{R+\varepsilon} \\ \varphi \equiv 0 & \text{on } \partial B_{R+\varepsilon} \\ \varphi > 0 & \text{on } B_{R+\varepsilon} \\ \|\varphi\|_{\mathrm{L}^2(B_{R+\varepsilon})} = 1 \end{cases} \tag{3.16}$$

We fix $\gamma > 0$ sufficiently small that

$$\int_{B_{R+\varepsilon}} |\nabla\varphi|^2 - \overline{a}(x) + \gamma\,[b(x) - c(x)]\,\varphi^2 = \lambda_1^{L}(B_{R+\varepsilon}) + \int_{B_{R+\varepsilon}} \gamma\,[b(x) - c(x)]\,\varphi^2 < 0\,.$$

This shows that the operator $\widetilde{L} = \Delta + \overline{a}(x) - \gamma\,[b(x) - c(x)]$ satisfies $\lambda_1^{\widetilde{L}}(B_{R+\varepsilon}) < 0$. Let ψ be a positive eigenfunction corresponding to $\lambda_1^{\widetilde{L}}(B_{R+\varepsilon})$. The ψ satisfies

$$\begin{cases} \Delta\psi + \overline{a}(x)\psi - \gamma b(x)\psi + \gamma c(x)\psi \geq 0 & \text{on } B_{R+\varepsilon} \\ \psi \equiv 0 & \text{on } \partial B_{R+\varepsilon} \\ \psi > 0 & \text{on } B_{R+\varepsilon}\,. \end{cases}$$

Thus

$$\begin{cases} \Delta\psi + \overline{a}(x)\psi - \gamma b(x)\psi + \gamma c(x)\psi \geq 0 & \text{on } B_R \\ \psi > 0 & \text{on } \overline{B_R}\,. \end{cases} \tag{3.17}$$

Let $\mu > 0$ and define $v_- = \mu\psi$ on B_R. Choosing

$$\mu \leq \min\left\{ \gamma^{\frac{1}{\sigma-1}} \left(\sup_{B_R}\psi\right)^{-1}, \gamma^{\frac{1}{\tau-1}} \left(\sup_{B_R}\psi\right)^{-1} \right\}$$

we have

$$\text{i) } \gamma\mu^{1-\sigma}\psi^{1-\sigma} \geq 1 \qquad \text{and} \qquad \text{ii) } \gamma\mu^{1-\tau}\psi^{1-\tau} \leq 1 \tag{3.18}$$

on B_R. Hence, using (3.17) and (3.18) we deduce

$$\begin{aligned} 0 &\leq \Delta v_- + \overline{a}(x)v_- - \gamma b(x)v_-^\sigma \left(\mu^{1-\sigma}\psi^{1-\sigma}\right) + \gamma c(x)v_-^\tau \left(\mu^{1-\tau}\psi^{1-\tau}\right) \\ &\leq \Delta v_- + \overline{a}(x)v_- - b(x)v_-^\sigma + c(x)v_-^\tau \,, \end{aligned}$$

that is,

$$\begin{cases} \Delta v_- + \overline{a}(x)v_- - b(x)v_-^\sigma + c(x)v_-^\tau \geq 0 & \text{on } B_R \\ v_- > 0 & \text{on } \overline{B_R}\,. \end{cases} \tag{3.19}$$

Because of the validity of (3.14), Lemma 3.3 yields the existence of v_+ satisfying

$$\begin{cases} \Delta v_+ + \overline{a}(x)v_+ - b(x)v_+^\sigma + c(x)v_+^\tau \leq 0 & \text{on } B_R \\ v_+ > 0 & \text{on } \overline{B_R}\,. \end{cases} \tag{3.20}$$

Note that if $0 < \gamma \leq 1$, γv_- still satisfies (3.19); hence up to choosing a suitable γ we can suppose that

$$\sup_{\overline{B_R}} v_- \leq \inf_{\overline{B_R}} v_+ \qquad \text{on } \overline{B_R}\,. \tag{3.21}$$

Let

$$\alpha_+ = \inf_{\partial B_R} v_+\,, \qquad \alpha_- = \sup_{\partial B_R} v_-\,,$$

and fix $\alpha \in [\alpha_-, \alpha_+]$. Then, by the monotone interation scheme, there exists a solution w of

$$\begin{cases} \Delta w + \overline{a}(x)w - b(x)w^\sigma + c(x)w^\tau = 0 & \text{on } B_R \\ w \equiv \alpha > 0 & \text{on } \partial B_R\,, \end{cases}$$

and with the further property that

$$0 < v_- \leq w \leq v_+ \qquad \text{on } \overline{B_R}\,.$$

Therefore, since $\overline{a}(x) \geq a(x)$ on $\overline{B_R}$ and $w > 0$ we have

$$\begin{cases} \Delta w + a(x)w - b(x)w^\sigma + c(x)w^\tau \leq 0 & \text{on } B_R \\ w \equiv \alpha > 0 & \text{on } \partial B_R \\ w > 0 & \text{on } B_R\,. \end{cases} \tag{3.22}$$

Fix any $n \in \mathbb{N}$. Let $\zeta \in \mathbb{R}$ be such that

$$\zeta \geq \max\left\{1, \frac{n}{\sup_{\partial\Omega} w}\right\}\,,$$

and define $w_+ = \zeta w$. Then, because of (3.22), the fact that $\Omega \subset\subset B_R$, and the signs of $b(x)$ and $c(x)$, w_+ satisfies

$$\begin{cases} \Delta w_+ + a(x)w_+ - b(x)w_+^\sigma + c(x)w_+^\tau \leq 0 & \text{on } \Omega \\ w_+ \geq n & \text{on } \partial\Omega \\ w_+ > 0 & \text{on } \Omega\,. \end{cases}$$

We can suppose that the R chosen above is such that $R \geq R_o$, where R_o is that of the property (Σ) in Definition 3.4. This choice implies that there exists a solution ψ of

$$\begin{cases} \Delta\psi + a(x)\psi - b(x)\psi^\sigma + c(x)\psi^\tau \geq 0 & \text{on } B_R \\ \psi \geq 0 & \text{on } \overline{B_R}\,. \end{cases}$$

If we define $w_- = \beta\psi$ where $0 < \beta \leq 1$, reasoning as above we can find β so small that

$$\begin{cases} \Delta w_- + a(x)w_- - b(x)w_-^\sigma + c(x)w_-^\tau \geq 0 & \text{on } \Omega \\ 0 \leq w_- \leq w_+ & \text{on } \Omega \\ w_- \leq n & \text{on } \partial\Omega\,. \end{cases}$$

Using the monotone iteration scheme we easily arrange a solution w of (3.13) between w_- and w_+. We note that the positivity of w is obvious. □

We note that the corresponding result for Yamabe-type equations, namely equations of the type (3.1) with $c(x) \equiv 0$, can be proved without the additional assumption of property (Σ). In this case we consider the global subsolution $w_- = 0$. Then the solution w obtained by the monotone interation scheme satisfies

$$\begin{cases} \Delta w + a(x)w - b(x)w^\sigma = 0 & \text{on } \Omega \\ w \geq 0 & \text{on } \Omega \\ w = n & \text{on } \partial\Omega\,. \end{cases}$$

so that, being *sigma* > 1, by the strong maximum principle (see [**39**]) $w > 0$ on Ω.
The next proposition provides some sufficient conditions for the validity of property (Σ) on M.

PROPOSITION 3.6. *Assume the validity of one of the following*

i) for some $\Lambda > 0$

$$\lambda_1^{\Delta+a-b+\Lambda c}(M) < 0\,;$$

ii) let $C_0 = \{x \in M : c(x) = 0\}$ *be bounded and such that*

$$\lambda_1^{\Delta-a}(C_0) > 0\,;$$

iii) there exists a positive subsolution $\varphi_- \in \mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{loc}(M)$ *of (3.1).*

Then property (Σ) *holds on* M*.*

PROOF. If i) holds true, there exists $R_o > 0$ sufficiently large such that

$$\lambda_1^{\Delta+a-b+\Lambda c}(B_R) < 0\,,$$

for $R \geq R_o$. Accordingly there exists a corresponding positive eigenfunction ψ on $B_{R+\varepsilon}$, $\varepsilon > 0$ small say, for which

$$\begin{cases} \Delta\psi + a(x)\psi - b(x)\psi + \Lambda c(x)\psi \geq 0 & \text{on } B_R \\ \psi > 0 & \text{on } \overline{B_R}\,. \end{cases} \tag{3.23}$$

We let

$$0 < \mu \leq \min\left\{ \Lambda^{\frac{1}{\tau-1}}, \left(\sup_{B_R} \psi\right)^{-1} \right\} \tag{3.24}$$

and we define

$$\varphi = \mu\psi\,.$$

Note that from (3.23)

$$0 \leq \Delta\varphi + a(x)\varphi - b(x)\varphi^\sigma \left(\mu\psi\right)^{1-\sigma} + \Lambda c(x)\varphi^\tau \left(\mu\psi\right)^{1-\tau}\,.$$

Now, because of (3.24)

$$\left(\mu\psi\right)^{1-\sigma} \geq 1 \qquad \text{and} \qquad \left(\mu\psi\right)^{1-\tau} \leq 1$$

on $\overline{B_R}$. Hence the above inequality implies the validity of (3.12) with φ strictly positive on $\overline{B_R}$.

If ii) holds true, then by Lemma 3.3, there exists $R_o > 0$ sufficiently large such that $C_0 \subset B_{R_o}$ and for $R \geq R_o$ there exists a solution ψ of

$$\begin{cases} \Delta\psi - a(x)\psi - c(x)\psi^{2-\tau} + b(x)\psi^{2-\sigma} \leq 0 & \text{on } B_R \\ \psi > 0 & \text{on } \overline{B_R}. \end{cases}$$

Thus, defining $\varphi = \dfrac{1}{\psi}$, we have

$$\Delta\varphi = -\varphi^2\Delta\psi + 2\varphi^3 \left|\nabla\psi\right|^2 \geq -\varphi^2\Delta\psi\,,$$

which implies (3.12) always with $\varphi > 0$ on $\overline{B_R}$. Case iii) is obvious. $\square$

In the sequel we shall need the following *a priori* estimate. Here $B_T(q)$ denotes the geodesic ball of radius T centered at q.

LEMMA 3.7. *Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, $\tau < 1 < \sigma$, $0 < \widetilde{T} < T$, and $\Omega \subset\subset B_{\widetilde{T}}(q) \subset M$. Assume $b(x) > 0$ on $\overline{B_T(q)}$. Then there exists an absolute contant $C > 0$ such that any positive solution $u \in \mathrm{C}^2(\overline{B_T(q)})$ of*

$$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0 \tag{3.25}$$

satisfies

$$\sup_\Omega u \leq C\,. \tag{3.26}$$

PROOF. We let $\rho(x) = \operatorname{dist}(x, q)$ and, on the compact ball $\overline{B_T(q)}$ we consider the continous function

$$F(x) = \left[T^2 - \rho(x)^2\right]^{\frac{2}{\sigma-1}} u(x)$$

where $u(x)$ is any nonnegative C^2 solution of (3.25). Note that $F(x)|_{\partial B_T(q)} = 0$, thus, unless $u \equiv 0$ and in this case there is nothing to prove, F has a positive absolute maximum at some point $\overline{x} \in B_T(q)$. In particular $u(\overline{x}) > 0$. Now, proceeding as in the proof of Lemma 2.6 in [**66**], we conclude that, at $\overline{x}$,

$$bu^{\sigma-1} \leq \frac{8\,(\sigma+1)}{(\sigma-1)^2}\frac{\rho^2}{(T^2-\rho^2)^2} + \frac{4}{\sigma-1}\frac{m+(m-1)\,A\rho}{T^2-\rho^2} + a_+ + cu^{\tau-1}\,,$$

for some constant $A \geq 0$, independent of u. We now state an elementary lemma postponing its proof.

LEMMA 3.8. *Let α, $\beta \in [0, +\infty)$, and μ, $\nu \in (0, +\infty)$. If $t \in \mathbb{R}^+$ satisfies*

$$t^\mu \leq \alpha + \frac{\beta}{t^\nu}\,,$$

then

$$t \leq \left(\alpha + \beta^{\frac{\mu}{\mu+\nu}}\right)^{\frac{1}{\mu}}\,. \tag{3.27}$$

Since $\sigma > 1$ and $\tau < 1$, from the lemma we conclude that, at $\overline{x}$,

$$u \leq b^{-\frac{1}{\sigma-1}}\left(\frac{8\,(\sigma+1)}{(\sigma-1)^2}\frac{\rho^2}{(T^2-\rho^2)^2} + \frac{4}{\sigma-1}\frac{m+(m-1)\,A\rho}{T^2-\rho^2} + a_+ + c^{\frac{\sigma-1}{\sigma-\tau}}\right)^{\frac{1}{\sigma-1}}\,.$$

Now the proof proceeds exactly as in Lemma 2.6 of [**66**] by substituting the a_+ there with $a_+ + c^{\frac{\sigma-1}{\sigma-\tau}}$. □

PROOF OF LEMMA 3.8. If $t^\mu \leq \alpha$ we are done, since $\mu > 0$ and $\beta \geq 0$. In the other case set $s = t^\mu$, then $s > \alpha$ and thus

$$s \leq \alpha + \frac{\beta}{s^{\frac{\nu}{\mu}}} < \alpha + \frac{\beta}{(s-\alpha)^{\frac{\nu}{\mu}}}\,.$$

Setting $r = s - \alpha$ we conclude that

$$r^{\frac{\mu+\nu}{\mu}} < \beta$$

and (3.27) follows. □

The next simple comparison result reveals quite useful.

LEMMA 3.9. *Let $\Omega \subseteq M$ be a bounded open set. Assume that $f_i : M \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are measurable functions such that for a.e. $x \in M$*

$$\frac{f_2(x,s)}{s} \geq \frac{f_1(x,t)}{t}\,, \tag{3.28}$$

for $s \leq t$. Let $u, v \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$ be solutions on Ω respectively of

$$\begin{cases} \Delta u + f_1(x,u) \geq 0\,; \\ \Delta v + f_2(x,v) \leq 0\,, \end{cases} \tag{3.29}$$

with $u \geq 0$, $v > 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.

PROOF. Set $\psi(x) = \frac{u(x)}{v(x)} \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{C}^2(\Omega)$, from (3.29) a standard computation yields

$$\Delta\psi \geq \frac{u}{v^2} f_2(x,v) - \frac{1}{v} f_1(x,u) - 2\langle \nabla\psi, \nabla \log v\rangle\,. \tag{3.30}$$

Now, by contradiction we assume that $u > v$ somewhere in Ω. Then there exists $\varepsilon > 0$ such that

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \psi(x) > 1 + \varepsilon\} \neq \emptyset$$

and $\partial\Omega_\varepsilon \subset \Omega$. Thus from (3.30) it follows that the following inequality holds on Ω_ε

$$\Delta\psi + 2\langle \nabla\psi, \nabla \log v\rangle \geq \psi \left[\frac{f_2(x,v)}{v} - \frac{f_1(x,u)}{u}\right] \geq 0\,,$$

moreover $\psi \equiv 1 + \varepsilon$ on $\partial\Omega_\varepsilon$, and thus by the maximum principle $\psi \leq 1 + \varepsilon$ on Ω_ε contradicting the definition of Ω_ε. □

REMARK 3.10. We note that the hypohteses on f_i of Lemma 3.9 are satisfied, for instance, if $f_1 = f_2 : M \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for all $x \in M$

$$s \mapsto \frac{f_i(x,s)}{s}$$

is a non increasing function and that the lemma can also be stated for $f_1 = f_2 : M \times \mathbb{R}^+ \to \mathbb{R}$ with u, $v > 0$. In particular this is the case for the Lichnerowicz-type nonlinearities

$$f(x,s) = a(x)s - b(x)s^\sigma + c(x)s^\tau\,,$$

with $b(x)$, $c(x)$ non negative and $\tau < 1 < \sigma$. Indeed, for any fixed $x \in M$ the function

$$g_x(s) = \frac{f(x,s)}{s} = a(x) - b(x)s^{\sigma-1} + c(x)s^{\tau-1}$$

is smooth on $\mathbb{R}^+$ and its derivative is given by

$$g'_x(s) = -(\sigma-1)b(x)s^{\sigma-2} + c(x)(\tau-1)s^{\tau-2}\,,$$

which is non positive by our assumptions on $b(x)$, $c(x)$, σ, and τ.

A reasoning similar to that in the proof of Lemma 3.9 will be used at the end of the argument in the proof of the next

LEMMA 3.11. *In the assumptions of Lemma 3.5 there exists a positive solution u of the problem*

$$\begin{cases} \Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0 & \text{on } \Omega \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

PROOF. For $n \in \mathbb{N}$, let $u_n > 0$ on Ω be the solution of (3.32) obtained in Lemma 3.5 so that

$$\begin{cases} \Delta u_n + a(x)u_n - b(x)u_n^{\sigma} + c(x)u_n^{\tau} = 0 & \text{on } \Omega \\ u_n > 0 & \text{on } \Omega \\ u_n = n & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

First of all we claim that

$$u_n \le u_{n+1}\,. \tag{3.33}$$

Indeed, $u_n = n < n+1 = u_{n+1}$ on $\partial\Omega$. We then apply Lemma 3.9 with the choice $f_1 = f_2 = f$ and recalling Remark 3.10 we obtain the validity of (3.33).
If we show the convergence of the monotone sequence u_n to a function u solution of (3.31) we are done, indeed u will certainly be positive. Towards this aim, by standard regularity theory, it is enough to show that the sequence $\{u_n\}$ is uniformly bounded on any compact subset K of Ω. If $K \subset \Omega \setminus B_0$, then we can find a finite covering of balls $\{B_i\}$ for K such that $b(x) > 0$ on each B_i. Applying Lemma 3.7 we deduce the existence of a constant $C_1 > 0$ such that

$$u_n(x) \le C_1 \qquad \forall\, x \in K\,,\ \forall n \in \mathbb{N}\,. \tag{3.34}$$

It remains to find an upper bound on a neighborhood of B_0. Towards this end, for $\eta > 0$ we let

$$N_\eta = \{x \in M : d(x, B_0) < \eta\}$$

where η is small enough that $\overline{N_\eta} \subset \Omega$. Furthermore, by the definition of $\lambda_1^L(B_0)$ and the fact that $\lambda_1^L(B_0) > 0$, we can also suppose to have chosen η so small that

$$\lambda_1^L(N_\eta) > 0\,.$$

Now $\partial N_{\eta/2}$ is closed and bounded (because B_0 is so), hence compact by the completeness of M, this implies the existence of a constant C_2 such that

$$u_n(x) \le C_2 \qquad \forall\, x \in \partial N_{\eta/2}\,,\, \forall n \in \mathbb{N}\,;$$

this follows from Lemma 3.7 by considering a finite covering of $\partial N_{\eta/2}$ with balls of radii less than $\eta/2$.
Next we let φ be a positive eigenfunction corresponding to $\lambda_1^L(N_\eta)$. Then, since $\inf_{N_{\eta/2}} \varphi > 0$, it follows that there exists a constant $\mu_o > 0$ such that

$$\mu\varphi(x) > C_2 \qquad \forall\, x \in \partial N_{\eta/2}\,,\, \forall \mu \ge \mu_o\,.$$

On $N_{\eta/2}$ we have

$$\Delta\,(\mu\varphi) + a(x)\,(\mu\varphi) = -\lambda_1^L(N_\eta)\,(\mu\varphi) < 0\,. \tag{3.35}$$

We now choose $\mu \ge \mu_o$ sufficiently large that

$$\mu^{\tau-1}\left(\inf_{N_{\eta/2}} \varphi\right)^{\tau-1}\left(\sup_{N_{\eta/2}} c(x)\right) < \lambda_1^L(N_\eta)\,,$$

this is possible since $\tau < 1$ and $\inf_{N_{\eta/2}} \varphi > 0$. Then, for each $\varepsilon > 0$,

$$\mu^{\tau-1}\left(\inf_{N_{\eta/2}} \varphi\right)^{\tau-1}\left(\sup_{N_{\eta/2}} c(x)\right)(1+\varepsilon)^{\tau-1} < \lambda_1^L(N_\eta)\,. \tag{3.36}$$

We let $\psi = \frac{u}{\mu\varphi}$ on $N_{\eta/2}$, where u is any of the functions of the sequence $\{u_n\}$. The same computations as in the proof of Lemma 3.9 using (3.32) and (3.35) yields

$$\Delta\psi + 2\langle\nabla\psi, \nabla\log\,(\mu\varphi)\rangle \ge \left(\lambda_1^L(N_\eta) + b(x)u^{\sigma-1} - c(x)u^{\tau-1}\right)\psi\,. \tag{3.37}$$

Note that, accordingly to our choice of μ,

$$\mu\varphi > C_2 > u \qquad \text{on } \partial N_{\eta/2}\,.$$

We claim that $\psi \le 1$ on $\partial N_{\eta/2}$. By contradiction suppose the contrary. Then, for some $\varepsilon_1 > 0$, the open set

$$\Omega_{\varepsilon_1} = \left\{x \in N_{\eta/2} : \psi(x) > 1 + \varepsilon_1\right\} \neq \emptyset$$

and $\Omega_{\varepsilon_1} \subset\subset N_{\eta/2}$. On Ω_{ε_1}

$$u > (1+\varepsilon_1)\,\mu\varphi\,.$$

Therefore, since $\tau < 1$

$$u^{\tau-1} \leq (1+\varepsilon_1)^{\tau-1} (\mu\varphi)^{\tau-1} \ .$$

Then, inserting this into (3.37), using (3.36), we deduce

$$\Delta\psi + 2\langle \nabla\psi, \nabla \log(\mu\varphi)\rangle \geq \left(\lambda_1^L(N_\eta) - \left(\sup_{N_{\eta/2}} c(x) \right) (1+\varepsilon_1)^{\tau-1} \mu^{\tau-1}\varphi^{\tau-1} \right) \psi \geq 0 \, .$$

By the maximum principle it follows that ψ attains its maximum on $\partial\Omega_{\varepsilon_1}$ but there $\psi(x) = 1 + \varepsilon_1$ contradicting the assumption $\Omega_{\varepsilon_1} \neq \emptyset$.
Thus $\psi \leq 1$ on $N_{\eta/2}$, that is, $u \leq \mu\psi$ on $N_{\eta/2}$. Hence, for all $n \in \mathbb{N}$

$$u_n \leq \max \left\{ C_2, \sup_{N_{\eta/2}} \mu\varphi \right\} \, .$$

This completes the proof of the lemma. □

We are now ready to prove Proposition 3.2. The proof, very similar to that of Theorem 6.5 of [**55**], follows a standard argument and it is reported here for the sake of completeness.

Proof of Proposition 3.2. First of all we note that, by part iii) of Proposition 3.6, the existence of the global positive subsolution u_- implies that the Σ-property holds on M. We fix an exhausting sequence $\{D_k\}$ of open, precompact sets with smooth boundaries such that

$$B_0 \subset D_k \subset \overline{D_k} \subset D_{k+1} \, ,$$

and for each k we denote by u_k^∞ the solution of the problem

$$\begin{cases} \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on } D_k \, ; \\ u = +\infty & \text{on } \partial D_k \, , \end{cases}$$

which exists by Lemma 3.11. It follows from Lemma 3.9 that

$$u_- \leq u_{k+1}^\infty \leq u_k^\infty \qquad \text{on } \overline{D_k} \, . \tag{3.38}$$

Thus $\{u_k^\infty\}$ converges monotonically to a function u solving (3.1), and satisfying, because of (3.38), $u \geq u_- > 0$. Let now $u_1 > 0$ be a second solution of (3.1) on M. By Lemma 3.9, $u_1 \leq u_k^\infty$ on D_k for all k, and therefore $u_1 \leq u$, thus u is a maximal positive solution. □

We can now prove Theorem 3.1 using an existence result for solutions of Yamabe-type equations contained in [**55**].

Proof of Theorem 3.1. By Proposition 3.2 it follows that to prove the theorem is sufficient to show that assumption (3.4) implies the existence of a global subsolution u_- of (3.1). Toward this aim we consider the following Yamabe-type equation

$$\Delta v + a(x)v - b(x)v^\sigma = 0 \qquad \text{on } M\,, \tag{3.39}$$

with σ, $a(x)$, and $b(x)$ as in Theorem 3.1. Then, by the sign assumptions (3.2) it follows that a global subsolution v of (3.39) is also a global subsolution of (3.1). Now we recall Theorem 6.7 of [**55**] which provides a positive solution v of (3.39) under assumptions (3.3) and (3.4) to conclude the proof. $\square$

In the same vein we have the following result, where the spectral condition (3.4) is substituted by a spectral smallness requirement on the zero set of the coefficient $c(x)$ and a pointwise control on the coefficients, this is similar to what we have done in Theorem 2.6.

Theorem 3.12. *Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \leq 1$. Assume the validity of (3.2); let*

$$B_0 = \{x \in M : b(x) = 0\}\,, \qquad C_0 = \{x \in M : c(x) = 0\}\,,$$

and assume

$$\lambda_1^{\Delta + a}(B_0) > 0\,. \tag{3.40}$$

Suppose that there exist two bounded open sets Ω_1, Ω_2 such that $C_0 \subset \Omega_1 \subset\subset \Omega_2$,

$$\sup_{M \setminus \overline{\Omega}_1} \frac{a_-(x) + b(x)}{c(x)} < +\infty\,, \tag{3.41}$$

and

$$\lambda_1^{\Delta - a}(\Omega_2) > 0\,. \tag{3.42}$$

Then (3.1) has a maximal positive solution $u \in \mathrm{C}^2(M)$.

The technique of the proof is the same of Theorem 3.1: we provide a global subsolution and then we apply Proposition 3.2. In this case the subsolution is obtained by pasting a subsolution defined inside a compact set with a second one defined in the complement of a compact set.

PROOF. Reasoning as in Lemma 3.3, assumption (3.40) implies the existence of a solution $\psi \in \mathrm{C}^2(\Omega_2)$ of the following problem

$$\begin{cases} \Delta\psi + a(x)\psi - b(x)\psi^\sigma + c(x)\psi^\tau \geq 0 & \text{on } \Omega_2 \\ \psi > 0 & \text{on } \Omega_2 \\ \psi = 0 & \text{on } \partial\Omega_2\,, \end{cases}$$

thus $u_1 = \psi$ is a subsolution of (3.1) in Ω_2. In particular, since $\partial\Omega_1 \subset \Omega_2$, if we set $\nu = \min_{\partial\Omega_1} \psi$, we have that $\nu > 0$.
Now we note that (3.41) implies that there exists $\mu \in \mathbb{R}^+$ such that

$$\sup_{M\setminus\overline{\Omega}_1} \frac{a_-(x) + b(x)}{c(x)} = \mu\,.$$

Let us define $\mu_* = \min\left\{1, \mu^{\frac{1}{\tau-1}}, \nu/2\right\}$. Then on $M \setminus \overline{\Omega}_1$ we have that

$$\begin{aligned} \Delta\mu_* + a(x)\mu_* - b(x)\mu_*^\sigma + c(x)\mu_*^\tau &= a(x)\mu_* - b(x)\mu_*^\sigma + c(x)\mu_*^\tau \\ &\geq -a_-(x)\mu_* - b(x)\mu_* + c(x)\mu_*^\tau \\ &= c(x)\mu_* \left[\mu_*^{\tau-1} - \frac{a_-(x) + b(x)}{c(x)}\right] \\ &\geq c(x)\mu_* \left[\mu_*^{\tau-1} - \mu\right] \\ &\geq 0\,. \end{aligned}$$

Thus $u_2 = \mu_*$ is a subsolution of (3.1) in $M \setminus \overline{\Omega}_1$. Set

$$u_- = \begin{cases} u_1 & \text{on } \overline{\Omega}_1 \\ \max\{u_1, u_2\} & \text{on } \Omega_2 \setminus \overline{\Omega}_1 \\ u_2 & \text{on } M \setminus \Omega_2\,. \end{cases}$$

We claim that u_- is the required global subsolution. To prove the claim, we start by noting the fact that $0 < \mu_* < \nu/2$ implies $0 < u_- \in \mathrm{C}^0(M) \cap \mathrm{W}^{1,2}_{loc}(M)$. For the same reason it is clear that there exists $\varepsilon > 0$ such that u_- is a subsolution of (3.1) on $\left(\overline{\Omega}_1\right)_\varepsilon \cup (M \setminus \Omega_2)_\varepsilon$, where

$$(U)_\varepsilon = \bigcup_{x\in U} B_\varepsilon(x)$$

for any set $U \subset M$ ($B_\varepsilon(x)$ denotes the geodesic ball of radius ε centered in x). Thus we are left to show that u_- is a subsolution of (3.1) on $\Omega_2 \setminus \overline{\Omega}_1$, this is a rather standard fact but we sketch the proof here for the sake of completeness. First of all we set

$$f(x, v) = a(x)v - b(x)v^\sigma + c(x)v^\tau\,,$$

then we note that for any test function $\varphi \in \mathrm{W}_0^{1,2}(\Omega_2 \setminus \overline{\Omega}_1)$, $\varphi \geq 0$ we have

$$\int_{\Omega_2 \setminus \overline{\Omega}_1} \langle \nabla u_1, \nabla \varphi \rangle - \varphi f(x, u_1) \leq 0 \,,$$

and

$$f(x, u_2) \geq 0 \qquad \text{on } \Omega_2 \setminus \overline{\Omega}_1 \,.$$

Now, for any $\varphi \in \mathrm{W}_0^{1,2}(\Omega_2 \setminus \overline{\Omega}_1)$ and $w \in \mathrm{W}^{1,2}(\Omega_2 \setminus \overline{\Omega}_1)$ consider

$$H(w, \varphi) = \int_{\Omega_2 \setminus \overline{\Omega}_1} \langle \nabla w, \nabla \varphi \rangle - \varphi f(x, u_-) \,,$$

It is clear that $H(\cdot, \varphi) : \mathrm{W}^{1,2}(\Omega_2 \setminus \overline{\Omega}_1) \to \mathbb{R}$ is a continous functional, for any φ. We want to show that $H(u_-, \varphi) \leq 0$, for any test function $\varphi \geq 0$ on $\Omega_2 \setminus \overline{\Omega}_1$.
For $\varepsilon > 0$, let

$$u_\varepsilon = \frac{1}{2} \left(u_1 + u_2 + \sqrt{(u_1 - u_2)^2 + \varepsilon^2} \right) \,,$$

then u_ε is smooth with

$$\nabla u_\varepsilon = \frac{1}{2} \left(1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) \nabla u_1$$

moreover, by the definition of u_-, $u_\varepsilon \xrightarrow{\mathrm{W}^{1,2}} u_-$ as $\varepsilon \to 0$. If $\varphi \in \mathrm{W}_0^{1,2}(\Omega_2 \setminus \overline{\Omega}_1)$ and $\varepsilon > 0$, then

$$\varphi_\varepsilon = \frac{1}{2} \left(1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) \varphi$$

belongs to $\mathrm{W}_0^{1,2}(\Omega_2 \setminus \overline{\Omega}_1)$ and its gradient is given by

$$\nabla \varphi_\varepsilon = \frac{1}{2} \left(1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) \nabla \varphi + \frac{\varepsilon^2}{2 \left(\sqrt{(u_1 - u_2)^2 + \varepsilon^2} \right)^3} \varphi \nabla u_1 \,.$$

The following computation uses the properties of u_ε, φ_ε, and the fact that u_1 and u_2 are subsolutions

$$
\begin{aligned}
H(u_\varepsilon,\varphi) &= \int_{\Omega_2\setminus\overline{\Omega}_1} \frac{1}{2}\left(1+\frac{u_1-u_2}{\sqrt{(u_1-u_2)^2+\varepsilon^2}}\right)\langle\nabla u_1,\nabla\varphi\rangle - \varphi f(x,u_-)\\
&= \int_{\Omega_2\setminus\overline{\Omega}_1} \langle\nabla u_1,\nabla\varphi_\varepsilon\rangle - \frac{\varepsilon^2\varphi\,|\nabla u_1|^2}{2\left(\sqrt{(u_1-u_2)^2+\varepsilon^2}\right)^3} - \varphi f(x,u_-)\\
&\le \int_{\Omega_2\setminus\overline{\Omega}_1} \langle\nabla u_1,\nabla\varphi_\varepsilon\rangle - \varphi f(x,u_-)\\
&\le \int_{\Omega_2\setminus\overline{\Omega}_1} \varphi_\varepsilon f(x,u_1) - \varphi f(x,u_-)\,.
\end{aligned}
$$

Now, since

$$
\varphi_\varepsilon \xrightarrow{L^2} \begin{cases} \varphi & \text{if } u_1>u_2\\ 0 & \text{if } u_1\le u_2\end{cases}
$$

from the continuity of $H(\cdot,\varphi)$ we conclude that

$$
H(u_-,\varphi) = \lim_{\varepsilon\to 0} H(u_\varepsilon,\varphi) \le -\int_{\{u_1\le u_2\}\cap\Omega_2\setminus\overline{\Omega}_1} \varphi f(x,u_2) \le 0,
$$

for any test function φ. □

3.2. A further comparison and uniqueness result

In this section we prove a comparison result and a corresponding uniqueness result based on a spectral property of the operator $L=\Delta+a(x)$. As we have seen, the request $\lambda_1^L(M)<0$ facilitates the search of solutions of equation (3.1). Somehow the opposite request seems to limitate the existence of solutions. The results of this section are in the spirit of the very recent [**18**] and [**19**].

We recall that L has finite index if and only if there exists a positive solution u of the differential inequality

$$
Lu \le 0\,, \tag{3.43}
$$

outside a compact set K. In what follows we shall denote with $(M,\langle\,,\,\rangle,G)$ a triple with the following properties: $(M,\langle\;,\;\rangle)$ is a complete manifold with a preferred origin o and

$G \in \mathrm{C}^2(M \setminus \{o\})$, $G : M \setminus \{o\} \to \mathbb{R}^+$ is such that

$$\begin{cases} i)\ \ \Delta G \leq 0 & \text{on } M \setminus \{o\}\,; \\ ii)\ G(x) \to +\infty & \text{as } x \to o\,; \\ iii)\ G(x) \to 0 & \text{as } x \to +\infty\,, \end{cases} \tag{3.44}$$

Clearly a good candidate for G is the (positive) Green kernel at o on a non-parabolic complete manifold, which, however, might not satisfy (3.44) iii). Observe that, for instance by the work of Li and Yau, [**50**], iii) is satisfied by the Green kernel if $\mathrm{Ric} \geq 0$. Other examples always concerning the Green kernel, are given by non-parabolic complete manifolds supporting a Sobolev inequality of the type

$$S(\alpha)^{-1}\left(\int_M v^{\frac{2}{1-\alpha}}\right)^{1-\alpha} \leq \int_M |\nabla v|^2 \qquad \text{for all } v \in \mathrm{C}^\infty_c(M) \tag{3.45}$$

for some $\alpha \in (0,1)$, $S(\alpha) > 0$, and for all $v \in \mathrm{C}^\infty_c(M)$. For further examples see [**58**] and the references therein. Note that, in these results, the authors also describe the behavior of $G(x)$ at infinity from above and below. That of $|\nabla G(x)|$ from above can often be obtained by classical gradient estimates. This is helpful for instance in Theorem 3.17 below.
However, since we only require superharmonicity of G, under a curvature assumption we can use *transplantation* from a non-parabolic model. The argument is as follow. Assume $(M, \langle\ ,\ \rangle)$ is a m-dimensional manifold with a pole o and with radial sectional curvature (with respect to o) K_{rad} satisfying

$$\mathrm{K}_{rad} \leq -F(r(x)) \qquad \text{on } M,$$

with $r(x) = \mathrm{dist}_M(x,o)$, $F \in \mathrm{C}^0(\mathbb{R}^+_0)$. Let g be a C^2-solution of the problem

$$\begin{cases} g'' - F(r)g \leq 0 \\ g(0) = 0\,, \quad g'(0) = 1\,, \end{cases} \tag{3.46}$$

and suppose that $g > 0$ on $\mathbb{R}^+$. Note that this request is easily achieved by bounding appropriately F from above. See for instance [**18**]. Then by the Laplacian comparison theorem

$$\Delta r \geq (m-1)\frac{g'(r)}{g(r)} \qquad \text{on } M \setminus \{o\}, \tag{3.47}$$

and weakly on M. Consider the C^2-model M_g defined by g with the metric

$$\langle\, ,\, \rangle_g = dr^2 + g^2(r)d\theta^2$$

on $M \setminus \{o\} = \mathbb{R}^+ \times \mathrm{S}^{m-1}$, S^{m-1} the unit sphere, $d\theta^2$ its canonical metric. Then $M \setminus \{o\}$ is non-parabolic if and only if $\frac{1}{g^{m-1}} \in \mathrm{L}^1(+\infty)$. Now we transplant the positive Green function on $M \setminus \{o\}$ evaluated at (y, o) to M, that is, we let

$$G(x) = \int_{r(x)}^{+\infty} \frac{ds}{g(s)^{m-1}} > 0 \qquad \text{on } M \setminus \{o\}.$$

An immediate computation yields

$$\Delta G(x) = -\frac{1}{g(r(x))^{m-1}} \left\{ \Delta r(x) - (m-1)\frac{g'(r(x))}{g(r(x))} \right\} \qquad \text{on } M \setminus \{o\}.$$

Hence, (3.47) implies (3.44) i). The remaining of (3.44) be trivially satisfied.
Thus we solve the problem by looking for a solution of (3.46) satisfying $\frac{1}{g^{m-1}} \in \mathrm{L}^1(+\infty)$. For a detailed analysis we refer also to [**19**]. On $M \setminus \{o\}$ we define

$$t(x) = -\frac{1}{2} \log G(x) \tag{3.48}$$

and, for $s \in \mathbb{R}$, we set

$$\Lambda_s = \{x \in M \setminus \{o\} : t(x) < s\} \cup \{o\} \ , \tag{3.49}$$

so that

$$\partial\Lambda_s = \{x \in M \setminus \{o\} : t(x) = s\} \ . \tag{3.50}$$

Note that, because of (3.44) ii), Λ_s is open and $\{\Lambda_s\}_{s\in\mathbb{R}}$ is an exhausting family of open sets. Property (3.44) iii) and completeness of $(M, \langle\ ,\ \rangle)$ implies that $\overline{\Lambda_s}$ is compact for each $s \in \mathbb{R}$.

We are now ready to prove the following

THEOREM 3.13. *Let $(M, \langle\,,\,\rangle, G)$ be as above and suppose that $a(x) \in \mathrm{C}^0(M)$ satisfy*

$$a(x) \leq \left\{ 1 + \frac{1}{\log^2 G(x)} \left[1 + \frac{1}{\log^2 \left(-\log\sqrt{G(x)} \right)} \right] \right\} \frac{|\nabla \log G(x)|^2}{4} \ , \tag{3.51}$$

outside a compact set K. Then the operator $L = \Delta + a(x)$ has finite index.

REMARK 3.14. Observe that condition (3.51) is meaningful outside a sufficiently large compact set K because of (3.44) iii).

PROOF. On $\mathbb{R}^+$ we define the function

$$\kappa(s) = 1 + \frac{1}{4s^2}\left[1 + \frac{1}{\log^2 s} \right] , \tag{3.52}$$

so that inequality (3.51) can be rewritten as

$$a(x) \leq \kappa(t(x)) \frac{|\nabla \log G(x)|^2}{4} \qquad \text{on } M \setminus K \, . \tag{3.53}$$

To prove the theorem we need to provide a positive solution u of (3.43) on $M \setminus \widehat{K}$ for some compact $\widehat{K}$. Towards this aim we look for u of the form

$$u(x) = \sqrt{G(x)}\beta(t(x)) = \mathrm{e}^{-t(x)}\beta(t(x))\,, \tag{3.54}$$

on $M \setminus \Lambda_T$ for some $T > 0$ sufficiently large and with $\beta : [T, +\infty) \to \mathbb{R}^+$. Now a simple computation shows that u satisfies

$$\Delta u + \left[1 - \frac{\ddot{\beta}}{\beta}(t(x))\right] \frac{|\nabla \log G(x)|^2}{4} u = \frac{1}{2}\frac{\Delta G}{\sqrt{G}}(x) \left[\beta(t(x)) - \dot{\beta}(t(x))\right] \tag{3.55}$$

on $M \setminus \Lambda_T$, where $\dot{}$ means the derivative with respect to t. Thus, using (3.53) and (3.55) we obtain

$$\begin{aligned}\Delta u + a(x)u \le &\left[\kappa(t(x)) - 1 + \frac{\ddot{\beta}}{\beta}(t(x))\right] \frac{|\nabla \log G(x)|^2}{4} u \\ &+ \frac{\Delta G}{2\sqrt{G}}(x) \left[\beta(t(x)) - \dot{\beta}(t(x))\right].\end{aligned}$$

Hence using (3.44) i) we have that (3.43) is satisfied on $M \setminus \Lambda_T$ for u as in (3.54) if we show the existence of a positive solution β of

$$\ddot{\beta} + [\kappa(t) - 1]\,\beta = 0 \tag{3.56}$$

satisfying the further requirement

$$\beta - \dot{\beta} \ge 0 \tag{3.57}$$

on $[T, +\infty)$ for some $T > 0$; in other words we have to show that (3.56) is non-oscillatory and that (3.57) holds at least in a neighborhood of $+\infty$. As for non oscillation, applying Theorem 6.44 of [**18**], we see that this is the case if

$$\kappa(t) - 1 \le \frac{1}{4t^2}\left[1 + \frac{1}{\log^2 t}\right]$$

on $[T, +\infty)$ for some $T > 0$ sufficiently large. This is guaranteed by the definition (3.52) of κ. To show the validity of (3.57) we use the following trick. Fix $n \ge 3$ and define $\rho \in \mathbb{R}^+$ via the prescription

$$t = t(\rho) = \log\left(\sqrt{n-2}\rho^{\frac{n-2}{2}}\right). \tag{3.58}$$

Note that

$$t(0^+) = -\infty\,, \qquad t(+\infty) = +\infty\,, \qquad t'(\rho) = \frac{n-2}{2}\frac{1}{\rho} \qquad \text{on } \mathbb{R}^+. \tag{3.59}$$

We then define

$$z(\rho) = \mathrm{e}^{-t(\rho)}\beta(t(\rho))\,. \tag{3.60}$$

If β is a solution of (3.56) on $[T,+\infty)$, having set $R=\rho(T)>0$ with $\rho(t)$ the inverse function of $t(\rho)$, z satisfies

$$\left(\rho^{n-1}z'\right)'+\kappa(t(\rho))\frac{(n-2)^2}{4\rho^2}\rho^{n-1}z=0 \qquad \text{on } [R,+\infty). \tag{3.61}$$

we can also fix the initial conditions

$$z(R)=1\,, \qquad z'(R)=0\,. \tag{3.62}$$

Hence, since $\kappa\geq 0$ on $[R,+\infty)$ a first integration of the solution z of the above Cauchy problem yields

$$z'(\rho)\leq 0 \qquad \text{on } [R,+\infty).$$

But

$$z'(\rho)=\frac{\sqrt{n-2}}{2}\frac{1}{\rho^{\frac{n}{2}}}\left\{\dot{\beta}(t(\rho))-\beta(t(\rho))\right\}$$

and therefore (3.57) is satisfied.
This completes the proof of the Theorem. □

Remark 3.15. We have just proved that the equation

$$\ddot{\beta}+\frac{1}{4t^2}\left[1+\frac{1}{\log^2 t}\right]\beta=0 \qquad \text{on } [T,+\infty) \tag{3.63}$$

(say $T\geq \mathrm{e}$) is non-oscillatory. This is not a consequence of the usual Hille-Nehari criterion (see [**74**]). Indeed, setting $h(t)$ to denote the coefficient of the linear term in (3.63), the condition of the classical criterion to guarantee the non-oscillatory character of the equation is that $h(t)\geq 0$ for $t>>1$ and

$$\limsup_{t\to+\infty} t\int_t^{+\infty}h(s)ds<\frac{1}{4}\,. \tag{3.64}$$

However, in this case we have

$$\frac{1}{4}<t\int_t^{+\infty}\frac{ds}{4s^2}<t\int_t^{+\infty}h(s)ds<\frac{1}{4}+\frac{1}{4}\int_t^{+\infty}\frac{ds}{s\log^2 s}=\frac{1}{4}+\frac{1}{4\log t}$$

so that (3.64) is not satisfied.

We shall now see how to get non-oscillation of (3.63) following the idea in the proof of the mentioned Theorem 6.44 of [**18**]. This will enable us to determine the asymptotic behavior of a solution β of (3.63) at $+\infty$ and therefore of u defined in (3.54) and solution of (3.43). This will be later used in Theorem 3.17.

Towards this aim we consider the function

$$w(t)=\sqrt{t}\log t \tag{3.65}$$

solution of Euler equation

$$\ddot{w} + \frac{1}{4t^2} w = 0 \tag{3.66}$$

on $[T, +\infty)$, $T > 0$, and positive on $[T, +\infty)$ for $T > 1$. Then the function

$$z = \frac{\beta}{w} \tag{3.67}$$

satisfies

$$\left(w^2 \dot{z}\right)^{\cdot} + \left(\kappa(t) - 1 - \frac{1}{4t^2}\right) w^2 z = 0 \qquad \text{on } [T, +\infty) \tag{3.68}$$

for $T >> 1$. Since $\frac{1}{w^2} \in \mathrm{L}^1(+\infty)$ we can define the critical curve χ_{w^2} relative to w^2 as in (4.21) of [**18**]. A computation yields

$$\chi_{w^2}(t) = \frac{1}{4} \frac{1}{t^2 \log^2 t} \qquad \text{for } t >> 1\,,$$

so that

$$\kappa(t) - 1 - \frac{1}{4t^2} = \chi_{w^2}(t)\,.$$

Hence from Theorem 5.1 and Proposition 5.7 of [**18**] we deduce that the solution $z(t)$ of (3.68) satisfies

$$z(t) \sim \frac{C}{\sqrt{\log t}} \log\log t \qquad \text{as } t \to +\infty\,,$$

for some constant $C > 0$ and therefore

$$\beta(t) \sim C\sqrt{t \log t}\, \log\log t \qquad \text{as } t \to +\infty\,. \tag{3.69}$$

Using the above, we finally obtain the asymptotic behavior of u in (3.54), that is,

$$u(x) \sim \varphi(x) \qquad \text{as } x \to \infty \text{ in } M\,, \tag{3.70}$$

with

$$\varphi(x) = C\sqrt{G(x)} \sqrt{-\log\sqrt{G(x)}\, \log\left(-\log\sqrt{G(x)}\right)}\, \log\log\left(-\log\sqrt{G(x)}\right) \tag{3.71}$$

as $x \to \infty$ on M.
In particular the behavior of u at infinity is known once that of $G(x)$ is known.
Next we prove a version of Theorem 5.20 of [**18**] for equation (3.1).

THEOREM 3.16. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold, $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, $\sigma > 1$, $\tau < 1$, and assume (3.2) and (2.43). Let Ω be a relatively compact open set and assume the existence of $w \in \mathrm{C}^2(M \setminus \overline{\Omega})$ positive solution of*

$$Lw = \Delta w + a(x)w \leq 0 \qquad \text{on } M \setminus \overline{\Omega}. \tag{3.72}$$

Suppose that u and v are positive C^2 solutions on M of

$$\begin{cases} \Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} \leq 0 \\ \Delta v + a(x)v - b(x)v^{\sigma} + c(x)v^{\tau} \geq 0\,. \end{cases} \tag{3.73}$$

If

$$u - v = o(w) \qquad \text{as } x \to \infty\,, \tag{3.74}$$

then $v \leq u$ on M.

Proof. The idea of the proof is the same as that of Theorem 5.20 of [**18**]. We report it here for the sake of completeness and for some minor differences. First we extend w to a positive function $\widetilde{w}$ on M. Towards this end let Ω' be a relatively compact open set such that $\overline{\Omega} \subset \Omega'$ Fix a cut-off function ψ, $0 \leq \psi \leq 1$ such that $\psi \equiv 1$ on Ω and $\psi \equiv 0$ on $M \setminus \Omega'$. Define $\widetilde{w} = \psi + (1-\psi)w$. Note that $\widetilde{w} > 0$ on M and $\widetilde{w} = w$ on $M \setminus \overline{\Omega'}$ so that $L\widetilde{w} \leq 0$ on $M \setminus \overline{\Omega'}$. For notational convenience we write again w and Ω in place of $\widetilde{w}$ and Ω', but this time $w > 0$ on M.
Let $\varepsilon > 0$ and define $u_\varepsilon = u + \varepsilon w$ on M. Then u_ε is a solution on M of

$$\Delta u_\varepsilon + a(x)u_\varepsilon \leq b(x)u^{\sigma} - c(x)u^{\tau} + \varepsilon L w\,.$$

Therefore, interpreting the differential inequality in the weak sense, we have that for each $\varphi \in \mathrm{Lip}_{loc}(M)$, $\varphi \geq 0$

$$-\int_M \langle \nabla u_\varepsilon, \nabla\varphi \rangle + \int_M a(x)u_\varepsilon\varphi \leq \int_M b(x)u^{\sigma}\varphi - \int_M c(x)u^{\tau}\varphi + \varepsilon\int_M \varphi L w\,.$$

Now, by the second Green formula

$$\int_M \varphi L w = \int_M a(x)w\varphi + \int_M w\Delta\varphi = \int_M wL\varphi$$

and therefore we can rewrite the above inequality as

$$-\int_M \langle \nabla u_\varepsilon, \nabla\varphi \rangle + \int_M a(x)u_\varepsilon\varphi \leq \int_M b(x)u^{\sigma}\varphi - \int_M c(x)u^{\tau}\varphi + \varepsilon\int_M wL\varphi\,. \tag{3.75}$$

Similarly, interpreting the second differential inequalitty of (3.73) in the weak sense

$$-\int_M \langle \nabla v, \nabla\varphi \rangle + \int_M a(x)v\varphi \geq \int_M b(x)v^{\sigma}\varphi - \int_M c(x)v^{\tau}\varphi\,, \tag{3.76}$$

with φ as above.
Next, by contradiction suppose that

$$\Gamma = \{x \in M : v(x) > u(x)\} \neq \emptyset\,.$$

Then, for $\varepsilon > 0$ sufficiently small

$$\Gamma_\varepsilon = \{x \in M : v(x) > u_\varepsilon(x)\} \neq \emptyset\,. \tag{3.77}$$

We now consider the Lipschitz function $\gamma_\varepsilon = \left(v^2 - u_\varepsilon^2\right)_+$. Condition (3.74) implies that γ_ε has compact support in M and it is not identically zero because of (3.77). Thus the functions $\varphi_1 = \frac{\gamma_\varepsilon}{u_\varepsilon}$ and $\varphi_2 = \frac{\gamma_\varepsilon}{v}$ are admissible, respectively for (3.75) and (3.76). Substituting we have

$$-\int_M \langle \tfrac{\nabla u_\varepsilon}{u_\varepsilon}, \nabla\gamma_\varepsilon\rangle - \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2}\gamma_\varepsilon - a(x)\gamma_\varepsilon \leq \int_M b(x)\frac{u^\sigma}{u_\varepsilon}\gamma_\varepsilon - c(x)\frac{u^\tau}{u_\varepsilon}\gamma_\varepsilon + \varepsilon w L\left(\frac{\gamma_\varepsilon}{u_\varepsilon}\right),$$

and

$$-\int_M \langle \tfrac{\nabla v}{v}, \nabla\gamma_\varepsilon\rangle - \frac{|\nabla v|^2}{v^2}\gamma_\varepsilon - a(x)\gamma_\varepsilon \geq \int_M b(x)v^{\sigma-1}\gamma_\varepsilon - c(x)v^{\tau-1}\gamma_\varepsilon\,.$$

Thus, subtracting the second from the first we deduce

$$\begin{aligned} -\int_{\Gamma_\varepsilon} \langle \tfrac{\nabla u_\varepsilon}{u_\varepsilon} - \tfrac{\nabla v}{v}, \nabla\gamma_\varepsilon\rangle + \int_{\Gamma_\varepsilon}\left(\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} - \frac{|\nabla v|^2}{v^2}\right)\gamma_\varepsilon \leq \\ \leq \int_{\Gamma_\varepsilon} b(x)\left(\frac{u^\sigma}{u_\varepsilon} - v^{\sigma-1}\right)\gamma_\varepsilon - \int_{\Gamma_\varepsilon} c(x)\left(\frac{u^\tau}{u_\varepsilon} - v^{\tau-1}\right)\gamma_\varepsilon + \varepsilon\int_M wL\left(\frac{\gamma_\varepsilon}{u_\varepsilon}\right). \end{aligned}$$

Inserting the expression for γ_ε and rearranging, we finally have

$$\begin{aligned} \int_{\Gamma_\varepsilon}\left|\nabla u_\varepsilon - \frac{u_\varepsilon}{v}\nabla v\right|^2 - \left|\nabla v - \frac{v}{u_\varepsilon}\nabla u_\varepsilon\right|^2 \leq \int_{\Gamma_\varepsilon} b(x)\left(\frac{u^\sigma}{u_\varepsilon} - v^{\sigma-1}\right)\gamma_\varepsilon - \\ - \int_{\Gamma_\varepsilon} c(x)\left(\frac{u^\tau}{u_\varepsilon} - v^{\tau-1}\right)\gamma_\varepsilon + \varepsilon\int_M wL\left(\frac{\gamma_\varepsilon}{u_\varepsilon}\right). \end{aligned} \tag{3.78}$$

Let V be a relatively compact open set with smooth boundary such that $\overline{\Omega} \subset V$ and let ψ, $0 \leq \psi \leq 1$ be a cut-off function such that $\psi \equiv 1$ on Ω and $\psi \equiv 0$ on $M \setminus \overline{V}$. Then, using again the second Green formula and (3.72) we have

$$\begin{aligned} \int_M wL\left(\frac{\gamma_\varepsilon}{u_\varepsilon}\right) &= \int_M wL\left(\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right) + \int_M wL\left((1-\psi)\frac{\gamma_\varepsilon}{u_\varepsilon}\right) \\ &= \int_M wL\left(\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right) + \int_M (1-\psi)\frac{\gamma_\varepsilon}{u_\varepsilon}Lw \\ &\leq \int_M wL\left(\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right). \end{aligned}$$

Now since u_ε is bounded from below by a positive constant on $\overline{V}$, by applying the dominated convergence theorem we deduce that

$$\lim_{\varepsilon\to 0}\varepsilon\left|\int_M wL\left(\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right)\right| \leq \lim_{\varepsilon\to 0}\varepsilon\left[\int_V |\nabla w|\left|\nabla\left(\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right)\right| + \left|a(x)w\psi\frac{\gamma_\varepsilon}{u_\varepsilon}\right|\right] = 0\,.$$

Therefore, letting $\varepsilon \to 0$ in (3.78), using Fatou's lemma and the last two inequalities, we get

$$
\begin{aligned}
0 &\leq \int_\Gamma \left|\nabla u - \frac{u}{v}\nabla v\right|^2 + \int_\Gamma \left|\nabla v - \frac{v}{u}\nabla u\right|^2 \\
&\leq \int_\Gamma b(x)\left(u^{\sigma-1} - v^{\sigma-1}\right)\left(v^2 - u^2\right) - \int_\Gamma c(x)\left(u^{\tau-1} - v^{\tau-1}\right)\left(v^2 - u^2\right) \\
&\leq 0\,.
\end{aligned} \tag{3.79}
$$

Therefore $\dfrac{v}{u}$ is constant on any connected component of Γ. Clearly Γ must have no boundary because otherwise letting $x \to \partial\Gamma$ we would deduce $u = v$ on Γ which is a contradiction. By connectedness $v = Au$ on M for some $A > 1$ and inserting into (3.79) we deduce

$$
\int_\Gamma b(x)\left(1 - A^{\sigma-1}\right)\left(1 - A^2\right)u^{\sigma+1} + \int_\Gamma c(x)\left(A^{\tau-1} - 1\right)\left(1 - A^2\right)u^{\tau+1} \equiv 0\,.
$$

Since $u > 0$, this contradicts assumptions (3.2) and (2.43). Hence $\Gamma = \emptyset$ that is, $v \leq u$ on M. $\square$

Thus, considering φ defined in (3.71), as a consequence of Theorem 3.16, Theorem 3.13, and the subsequent discussion we have

THEOREM 3.17. *Let $(M, \langle\,,\,\rangle, G)$ as in (3.44), $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, $\sigma > 1$, $\tau < 1$, and assume (3.2), (2.43), and*

$$
a(x) \leq \left\{1 + \frac{1}{\log^2 G(x)}\left[1 + \frac{1}{\log^2\left(-\log\sqrt{G(x)}\right)}\right]\right\}\frac{|\nabla \log G(x)|^2}{4}
$$

outside a compact set. If u and v are positive C^2 solutions of (3.1) such that

$$
u(x) - v(x) = o(\varphi(x)) \qquad \text{as } x \to \infty
$$

with $\varphi(x)$ as in (3.71), then $u \equiv v$ on M.

It is reasonable that if we strenghten the upper bound (3.53) on $a(x)$ the growth of u defined in (3.54) should improve in (3.71).
For the sake of simplicity let us suppose

$$
a(x) \leq \lambda\frac{|\nabla \log G(x)|^2}{4} \tag{3.80}
$$

on $M \setminus K$ for some constant $\lambda \in (-\infty, 1]$. We proceed as in the proof of Theorem 3.13 to arrive to (3.56) that now reads

$$
\ddot{\beta} + [\lambda - 1]\,\beta = 0 \tag{3.81}
$$

on $[T, +\infty)$ for some $T > 0$. Positive solutions of the above are immediately obtained. Indeed, for $\lambda = 1$ we let $\beta(t) = Ct$ for some constant $C > 0$ while for $\lambda \in (-\infty, 1)$ we let $\beta(t) = C\mathrm{e}^{\sqrt{1-\lambda}t}$, $C > 0$. Thus the positive solution $u(x)$ of $Lu \leq 0$ given in (3.54) satisfies

$$u(x) \sim \begin{cases} C\sqrt{G(x)}\log \frac{1}{G(x)} & \text{if } \lambda = 1 \\ CG(x)^{\frac{1-\sqrt{1-\lambda}}{2}} & \text{if } \lambda \in (-\infty, 1) \end{cases} \tag{3.82}$$

as $x \to \infty$ for some constant $C > 0$.
Thus, going back to Theorem 3.17 we obtain the following version

THEOREM 3.18. *Let $(M, \langle\,,\,\rangle, G)$ as in (3.44), $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, $\sigma > 1$, $\tau < 1$, and assume (3.2), (2.43), and*

$$a(x) \leq \lambda \frac{|\nabla \log G(x)|^2}{4}$$

outside a compact set, for some constant $\lambda \in (-\infty, 1]$. If u and v are positive C^2 solutions of (3.1) such that

$$u(x) - v(x) = \begin{cases} o\left(\sqrt{G(x)}\log \frac{1}{G(x)}\right) & \text{if } \lambda = 1 \\ o\left(G(x)^{\frac{1-\sqrt{1-\lambda}}{2}}\right) & \text{if } \lambda \in (-\infty, 1) \end{cases} \qquad \text{as } x \to \infty\,,$$

then $u \equiv v$ on M.

As a final remark we observe that finiteness of the index of $L = \Delta + a(x)$ can be also deduced by the validity of a Sobolev-type inequality on M. Indeed, according to Lemma 7.33 of [**67**], the validity of (3.45) and the assumption

$$a_+(x) \in \mathrm{L}^{1/\alpha}(M)$$

imply that L has finite index.

3.3. A Liouville-type theorem

In this section we will apply the results of Section 1.5 to get a*Liouville-type theorem* for positive solutions of (3.1) that should be compared with those obtained in [**52**], [**22**], and [**31**]. The main differences with previous work in the literature is that our geometric requirement on the manifold consist only in a mild volume growth assumption for geodesic balls and in the fact that we allow for non constant coefficients $a(x)$, $b(x)$, $c(x)$ in equation (3.1). In this last setting in general there are no *trivial* solutions at hand. Thus, to provide a complete analysis of the problem in this case, we need to find an *a priori* estimate and use it to detect a *trivial* solution of (3.1). In particular, Corollary 3.23 is our main *Liouville-type result.*

The first result is a direct application of Corollary 1.19 and yields a bound from above for subsolutions of (3.1). Then, exploiting the simmetric structure of the equation (as we already made in Section 2.2), we will obtain also an estimate from below. From these estimates we will get a pair of ordered sub/super solutions and thus we will conclude that there exists a positive solution between them.

PROPOSITION 3.19. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold. Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, and assume $\|a_+ + c_+\|_\infty < +\infty$, that $b(x) > 0$ on M and that it satisfies (1.27) for some $\mu < 2$ outside a compact set. Suppose the validity of (1.26) and of*

$$\sup_M \frac{a_+(x) + c_+(x)}{b(x)} < +\infty\,. \tag{3.83}$$

Let $\sigma > 1$, $\tau < 1$, and $u \in \mathrm{C}^2(M)$ be a positive solution of

$$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0 \tag{3.84}$$

on

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}\ , \tag{3.85}$$

for some $\gamma < u^* \leq +\infty$. Then $u^* < +\infty$ and indeed

$$u^* \leq \max\left\{\gamma^*,\, H_{\gamma^*}^{1/(\sigma-1)}\right\} \tag{3.86}$$

where $\gamma^* = \max\{1, \gamma\}$ and

$$H_{\gamma^*} = \sup_{\Omega_{\gamma^*}} \frac{a_+(x) + c_+(x)}{b(x)}\,.$$

PROOF. First we show that $u^* < +\infty$. We can suppose $u^* > 1$. If $\gamma < 1$ we let $\widetilde{\gamma}$ be such that $1 \leq \widetilde{\gamma} < u^*$ and note that $\Omega_{\widetilde{\gamma}} \subset \Omega_\gamma$. It follows that (3.84) holds on $\Omega_{\widetilde{\gamma}}$. Thus, without loss of generality, we can suppose $\gamma \geq 1$. Since $u^\tau \leq u$ on Ω_γ, from (3.84) we have

$$\begin{aligned}
\Delta u + a_+(x)u - b(x)u^\sigma + c_+(x)u \geq& \Delta u + a_+(x)u - b(x)u^\sigma + c_+(x)u^\tau \\
\geq& \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \\
\geq& 0\,,
\end{aligned}$$

on Ω_γ; in other words

$$\Delta u + [a_+(x) + c_+(x)]\, u - b(x)u^\sigma \geq 0 \quad \text{on } \Omega_\gamma.$$

Applying Corollary 1.19 and recalling Remark 1.8 we deduce that $u^* < +\infty$. To prove (3.86) first let $\gamma \geq 1$ so that $\gamma^* = \gamma$, $\Omega_{\gamma^*} = \Omega_\gamma$, $H_{\gamma^*} = H_\gamma$ and (3.86) follows directly from (1.63) of Corollary 1.19. Suppose now $\gamma < 1$. Then $\gamma^* = 1$ and $\Omega_{\gamma^*} = \Omega_1 \subset \Omega_\gamma$. If $\Omega_1 = \emptyset$ then $u^* \leq \gamma^*$. If $\Omega_1 \neq \emptyset$ then (3.84) holds on Ω_1 and applying again Corollary 1.19 we deduce the validity of (3.86). $\square$

Now, as we made for Theorem 2.5 (the same trick that we used for Proposition 3.6), we are going to exploit the simmetry of equation (2.1) to obtain a bilateral *a priori* estimate. This is the content of the next crucial

THEOREM 3.20. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold. Let $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^0(M)$, $\|a_+ + c\|_\infty < +\infty$, $\|a_- + b\|_\infty < +\infty$. Moreover assume that $b(x) > 0$ and $c(x) > 0$ on M, and that both satisfy (1.27) for some $\mu < 2$. Suppose the validity of (1.26) and of*

$$\sup_M \frac{a_+(x) + c(x)}{b(x)} = H < +\infty\,, \tag{3.87}$$

and

$$\sup_M \frac{a_-(x) + b(x)}{c(x)} = K < +\infty\,. \tag{3.88}$$

Let $\sigma > 1$, $\tau < 1$. Then any positive, C^2 solution of

$$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on } M, \tag{3.89}$$

satisfies

$$\mathcal{K} \le u(x) \le \mathcal{H} \quad \text{on } M, \tag{3.90}$$

where

$$\mathcal{K} = \min\left\{1,\ K^{1/(\tau-1)}\right\}, \qquad \mathcal{H} = \max\left\{1,\ H^{1/(\sigma-1)}\right\}. \tag{3.91}$$

PROOF. Suppose $\Omega_1 = \{x \in M : u(x) > 1\} \neq \emptyset$, then the validity of (3.89) implies that of

$$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \ge 0 \quad \text{on } \Omega_1,$$

thus the estimate from above in (3.90) follows from Proposition 3.19. In case $\Omega_1 = \emptyset$ the same estimate is trivially true because of the definition (3.91) of $\mathcal{H}$. For the estimate from below we consider the function $v = \frac{1}{u} \in \mathrm{C}^2(M)$, since $u > 0$ on M. Since $\Delta v = -v^2\Delta u + 2v^3\,|\nabla u|^2$, using (3.89) we have

$$\Delta v + \widetilde{a}(x)v - \widetilde{b}(x)v^{\widetilde{\sigma}} + \widetilde{c}(x)v^{\widetilde{\tau}} \ge 0 \quad \text{on } M,$$

where we have set $\widetilde{a}(x) = -a(x)$, $\widetilde{b}(x) = c(x)$, $\widetilde{c}(x) = b(x)$, $\widetilde{\sigma} = 2-\tau > 1$, and $\widetilde{\tau} = 2-\sigma < 1$. Now, since

$$\frac{a_-(x) + b_+(x)}{c(x)} = \frac{\widetilde{a}_+(x) + \widetilde{c}_+(x)}{\widetilde{b}(x)}\,,$$

we can reason as above and deduce

$$v \le \max\left\{1, K^{1/(\widetilde{\sigma}-1)}\right\} = \max\left\{1, K^{1/(1-\tau)}\right\}$$

and the lower bound in (3.90) follows immediately from the definition of v. $\square$

We note that the existence of solutions for equation (3.1) can be easily obtained under the hypoteses of Theorem 3.20 by direct application of the monotone iteration scheme of Section 1.6. Indeed in this case it is relatively easy to find an ordered pair of global sub/super solutions.

LEMMA 3.21. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold. Let $a(x)$, $b(x)$, $c(x)$, σ, τ, H, K, $\mathcal{H}$, and $\mathcal{K}$ be as in Theorem 3.20. Then $u^+ \equiv \mathcal{H}$ and $u^- \equiv \mathcal{K}$ are respectively a global supersolution and a global subsolution of (3.89). Moreover $u^- \le u^+$.*

PROOF. First of all we note that since $\mathcal{H} \ge 1$ and $\tau < 1$, then it follows that $\mathcal{H}^{\tau-1} \le 1$. This implies that

$$\begin{aligned} \Delta u^+ + a(x)u^+ - b(x)(u^+)^\sigma + c(x)(u^+)^\tau &= \mathcal{H}\left[a(x) - b(x)\mathcal{H}^{\sigma-1} + c(x)\mathcal{H}^{\tau-1}\right] \\ &\le b(x)\mathcal{H}\left[\frac{a_+(x)+c(x)}{b(x)} - \mathcal{H}^{\sigma-1}\right] \\ &\le 0 \end{aligned}$$

where in the last passage we have used (3.87) and the fact that $\mathcal{H} \ge H^{\frac{1}{\sigma-1}}$, thus u^+ is a global supersolution. The proof of the fact that u^- is a subsolution is analogous and $u^- \le u^+$ follows from the definitions of $\mathcal{H}$ and $\mathcal{K}$. $\square$

From this we immediately deduce the next existence result (see also [**69**] for a similar result).

PROPOSITION 3.22. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold. Let $a(x)$, $b(x)$, $c(x)$, σ, τ be as in Theorem 3.20 and assume that $a(x)$, $b(x)$, $c(x) \in \mathrm{C}^{0,\alpha}(M)$ for some $\alpha > 0$. Then (3.1) has a positive solution $u \in \mathrm{C}^2(M)$.*

PROOF. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a family of bounded open sets with smooth boundaries such that

$$\Omega_k \subset\subset \Omega_{k+1}\,; \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = M\,.$$

For each $k \in \mathbb{N}$ consider the Dirichlet problem

$$\begin{cases} \Delta v + a(x)v - b(x)v^\sigma + c(x)v^\tau = 0 & \text{on } \Omega_k\,; \\ v = u^+ & \text{on } \partial\Omega_k\,, \end{cases} \tag{3.92}$$

where $u^+ = \mathcal{H}$ is the global supersolution of Lemma 3.21. Since u^+ and u^- of Lemma 3.21 are respectively a supersolution and a subsolution of (3.92) for any $k \in \mathbb{N}$, it follows from the monotone iteration scheme that for any k there exists a solution $v_k \in C^{2,\alpha}(\Omega_k)$ of (3.92) such that $u^- \le v_k \le u^+$. From Lemma 3.9 it follows that

$$u^- \le v_i \le v_j \le u^+ \quad \text{on } \Omega_k$$

for all $i, j \in \mathbb{N}$ such that $i \ge j \ge k$. Thus, from the Schauder interior estimates and the compactness of the embedding $C^{2,\alpha}(\Omega_k) \subset C^2(\Omega_k)$ it follows that the v_k converge uniformly on compact sets to a solution $u \in C^2(M)$ of (2.1). Moreover $u(x) \ge u^- > 0$. □

Now we are ready to state and prove the following *Liouville-type theorem* that is the main result of the section.

THEOREM 3.23. *In the assumptions of Theorem 3.20 with* $0 \le \mu < 2$ *the equation*

$$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on } M. \tag{3.93}$$

admits a unique positive solution $u \in C^2(M)$.

PROOF. The proof is a straightforward application of the results above. Indeed, by Proposition 3.22, equation (3.93) admits a positive solution $u \in C^2(M)$. Now by Theorem 3.20 any positive solution $u \in C^2(M)$ of (3.93) is such that

$$0 < \mathcal{K} \le u \le \mathcal{H}\,, \tag{3.94}$$

where $\mathcal{K}$ and $\mathcal{H}$ are those of (3.91). By Remark 1.8, conditions (1.27) and (1.26) yield the validity of the $1/b$-WMP for the Laplacian, this, together with (3.94), imply that we can apply Corollary 2.10 (in this case $\partial M = \emptyset$, thus the boundary condition is trivially satisfied) to conclude that u found above is the unique solution of (3.93). □

The next corollary deals with the special case where $a(x)$, $b(x)$, and $c(x)$ are of the form $\zeta f(x)$ where $0 < f(x) \in C^0(M) \cap L^\infty(M)$ and $\zeta \in \mathbb{R}$. It generalizes Theorem 2 of [**52**] and Theorem 7 of [**53**]. Furthermore it should be compared with Theorem 3.15 and Example 3.18 of [**31**].

COROLLARY 3.24. *Let* $(M, \langle\ ,\ \rangle)$ *be a complete Riemannian manifold. Let* α, β, $\gamma \in \mathbb{R}$ *such that* β, $\gamma > 0$. *Let* $\sigma > 1$, $\tau < 0$, $0 < f(x) \in C^0(M) \cap L^\infty(M)$ *satisfying (1.27) outside a compact set for some* $\mu < 2$, *and assume the validity of (1.26). Then the unique positive solution of*

$$\Delta u + f(x)\,(\alpha u - \beta u^\sigma + \gamma u^\tau) = 0 \quad \text{on } M$$

is given by $u \equiv \lambda$, *where* $\lambda \in \mathbb{R}^+$ *satisfies* $p(\lambda) = 0$, *with*

$$p(t) = \alpha + \beta t^{\sigma-1} - \gamma t^{\tau-1}\,.$$

CHAPTER 4

Yamabe-type equations and applications to Geometry

We recall that a pointwise conformal deformation of $(M, \partial M, \langle\ ,\ \rangle)$, $\dim M = m \geq 3$, is a Riemannian manifold $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$ where $\widetilde{\langle\ ,\ \rangle} = u^{\frac{4}{m-2}} \langle\ ,\ \rangle$ for some smooth positive function u called the conformal factor of the deformation. We denote with $(s,\, h)$ and $(\widetilde{s},\, \widetilde{h})$ the scalar curvature and the mean curvature of the boundary, respectively of $(M, \partial M, \langle\ ,\ \rangle)$ and $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$. Then, as it is well known (see for instance [**25, 34**]), these quantities are related by the equations

$$
\begin{cases} \Delta u - c_m \left(s(x)u - \widetilde{s}(x)u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ \partial_\nu u + d_m \left(h(x)u - \widetilde{h}(x)u^{\frac{m}{m-2}} \right) = 0 & \text{on } \partial M \end{cases} \tag{4.1}
$$

where Δ and ν are the Laplace-Beltrami operator of M and the outward unit normal to ∂M in the background metric $\langle\ ,\ \rangle$, while c_m and d_m are constants respectively given by

$$
c_m = \frac{m-2}{4(m-1)}\,, \qquad d_m = \frac{m-2}{2}\,.
$$

The first equation of (4.1) is a Yamabe-type equation, in fact the original Yamabe equation, and the exponent $\dfrac{m+2}{m-2} > 1$ is the well known critical exponent for the Sobolev embedding. The problem of finding a pointwise conformal deformation of $(M, \partial M, \langle\ ,\ \rangle)$ with prescribed scalar curvature on M and prescribed mean curvature of ∂M has been first considered by Cherrier in [**25**]. A few years later in two cornerstone papers [**34, 35**], Escobar considered the related Yamabe problem on compact manifolds with boundary. Since then, many efforts have been made towards a complete solution of the boundary Yamabe problem in the compact case. For the case of noncompact manifolds with boundary, we quote the recent work [**73**] by F. Schwartz. In this paper he considers the problem of finding a conformal diffeomormorphism with $\widetilde{s} \equiv 0$ and prescribed $\widetilde{h}$ on a noncompact manifold with compact boundary and a controlled volume growth on each end. A related work is the even more recent paper by Almaraz at al. [**14**] where they consider a positive mass theorem for asymptotically flat manifolds with noncompact boundary. We tackle the problem of prescribing the scalar curvature in the more general case of a noncompact manifold with possibly noncompact boundary.

Most of the results of this chapter are proved with the aid the weak maximum principle for manifolds with boundary and the related L^∞ estimate developed in Sections 1.2 and 1.5. We start by noting that on a smooth Riemannian manifold with smooth boundary the scalar curvature s and the mean curvature of the boundary h are smooth functions, namely $s \in C^\infty(M)$ and $h \in C^\infty(\partial M)$. Thus, by standard elliptic regularity theory (see [**39**]), solutions u of (4.1) are smooth, indeed $u \in C^\infty(M)$.

4.1. A Schwarz-type Lemma

Schwarz Lemma (see III.3.I in [**24**]) is a basic tool in complex analysis whose importance can be hardly overstimated; its use for a one-line-proof of Liouville's theorem on constancy of entire holomorphic functions is an enlightening example of its strength. As reported in detail by R. Osserman in his survey [**59**], beside its use in complex analysis, Schwarz Lemma turns out to be a fundamental tool in studying properties of conformal deformations of manifolds of negative curvature. The main observation that led to this use of the result is the *geometric* formulation of the Lemma, namely the so called Schwarz-Pick Lemma proved by G. Pick in [**63**]. We recall that the Schwarz-Pick Lemma states that if $f(z)$ is a holomorphic map from the unit disk D into itself, then

$$\operatorname{dist}_{\mathbb{H}}(f(z_1), f(z_2)) \leq \operatorname{dist}_{\mathbb{H}}(z_1, z_2) \qquad \text{for all } z_1, z_2 \in D, \tag{4.2}$$

where $\operatorname{dist}_{\mathbb{H}}$ denotes the hyperbolic distance in D. In other words, a holomorphic map from the unit disk into itself decreases the hyperbolic distance.

Next step was tkaen in 1938 by L.V. Ahlfors that generalized the Schwarz-Pick Lemma considering holomorphic maps from the unit disk D into a general Riemann surface of negative curvature [**4**]. After this seminal paper, many efforts have been made to deal with maps from general Riemann surfaces and, more generally, with maps between higher dimensional complex manifolds. A further step in extending the result is to not consider just a holomorphic map from a complex manifold to another, but instead deal with conformal mappings between Riemannian manifolds (holomorphic maps are conformal in dimension 2). In these directions the literature is wide and we only cite the cornerstone papers by S.T. Yau [**75, 77**], the well known book of S. Kobayashi on hyperbolic complex spaces [**47**], and a more recent paper by A. Ratto, M. Rigoli, L. Véron [**69**].

In this section we deal with the case of pointwise conformal deformations of noncompact Riemannian manifolds with boundary. It seems that this case has not been considered previously in the literature, indeed, the research that stemmed from the Schwarz-Pick-Ahlfors Lemma focused on the complete and boundaryless case. An intriguing feature of considering the case of manifolds with boundary is that it resembles the classical results of complex analysis. For instance we recall the boundary Schwarz lemmas by D. Burns and S. Krantz [**23**], and R. Osserman [**61**]. We refer to the recent surveys by H. Boas [**20**] and S. Krantz [**48**] for a comprehensive treatment of the boundary Schwarz Lemma.

Following the philosophy of Pick, our generalization of Schwarz Lemma is stated, as in (4.2), in terms of contraction of distances. Let us recall that a conformal diffeomorphism $f : (M, \partial M, \langle\ ,\ \rangle) \to (M, \partial M, \langle\ ,\ \rangle)$ with conformal factor u is said to be *weakly distance decreasing* if $u \leq 1$ on M, see [**69**]. The main result of the sections is the following

THEOREM 4.1. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete, noncompact, Riemannian manifold with boundary ∂M and dimension $m \geq 3$. Assume that*

$$\liminf_{r \to +\infty} \frac{Q(r) \log \operatorname{vol} B_r}{r^2} < +\infty \tag{4.3}$$

where $Q(t)$ is a nondecreasing function satisfying $Q(r) = o(r^2)$ as $r \to +\infty$. Let f be a conformal diffeomorphism of $(M, \partial M, \langle\ ,\ \rangle)$ into itself such that, for some constant $c > 0$, the scalar curvature $\widetilde{s}(x)$ of the new metric $\widetilde{\langle\, ,\rangle} = f^*\langle\, ,\rangle = u^{\frac{4}{m-2}}\langle\, ,\rangle$ satisfies

$$-c \leq \widetilde{s}(x) < \min\{0, s(x)\} \quad \textit{on } M$$

and

$$\widetilde{s}(x) \leq -\frac{1}{Q(r(x))} \quad \textit{outside a compact set.}.$$

Furthermore, for $\gamma \in \mathbb{R}$ let

$$\Omega_\gamma = \{x \in M \,:\, u(x) > \gamma\}$$

and assume that

$$\widetilde{h}(x) \leq u^{-\frac{2}{m-2}} h(x) \quad \textit{on } \partial_1\Omega_\gamma \tag{4.4}$$

for some $\gamma < u^* \leq +\infty$. Then f is weakly distance decreasing.

Recall $\partial_1\Omega_\gamma = \overline{\Omega}_\gamma \cap \partial M$. that We stress that, although the result is stated when the domain and target manifolds coincide, it can be easily generalized to the case of a conformal map between different manifolds with boundary. This result basically extends Theorem 3.3 of [**65**] to this new setting. The delicate issue in the present case is due to condition (4.4) which involves the conformal factor u; however (4.4) is satisfied with no reference to u whenever the geometric request

$$\widetilde{h}(x) \leq 0 \leq h(x) \quad \text{on } \partial_1\Omega_\gamma\,,$$

holds. In view of applications it is useful to introduce the next

DEFINITION 4.2. *Let* $(M, \partial M, \langle\ ,\ \rangle)$ *be a Riemannian manifold with boundary and dimension* $m \geq 3$. *We say that a conformal diffeomorphism* f *of* M *into itself with conformal factor* u *is* ∂-rigid *if*

$$\partial_\nu u = 0 \quad on\ \partial M.$$

With this definition in mind we obtain the following corollary of Theorem 4.1 characterizing isometries in the group of conformal diffeomorphisms of $(M, \partial M, \langle\ ,\ \rangle)$ onto itself preserving the scalar curvature.

COROLLARY 4.3. *Let* $(M, \partial M, \langle\ ,\ \rangle)$ *be a complete, noncompact, manifold with boundary, dimension* $m \geq 3$ *and scalar curvature* $s(x)$ *satisfying*

$$(4.5) \qquad i)\ -c \leq s(x) < 0\,, \quad ii)\ s(x) \leq -\frac{1}{Q(r(x))} \quad for\ r(x) >> 1$$

for some positive constant c *and with* $Q(r)$ *as in the statement of Theorem 4.1. Assume that (4.3) holds. Then, any conformal diffeomorphism* f *of* $(M, \partial M, \langle\ ,\ \rangle)$ *onto itself which is* ∂*-rigid and preserves the scalar curvature is an isometry.*

PROOF OF THEOREM 4.1. From (4.1) and (4.4) we have that u satisfies

$$\begin{cases} \Delta u - c_m \left(s(x)u - \widetilde{s}(x)u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } \Omega_\gamma \\ \partial_\nu u \leq 0 & \text{on } \partial_1\Omega_\gamma\,, \end{cases}$$

moreover, since Ω_γ is a superlevel set, it holds that $\partial_\nu u \leq 0$ on $\partial_0\Omega_\gamma$, thus

$$\begin{cases} \Delta u - c_m \left(s(x)u - \widetilde{s}(x)u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } \Omega_\gamma \\ \partial_\nu u \leq 0 & \text{on } \partial\Omega_\gamma\,. \end{cases}$$

Now, recalling Remark 1.20, we can apply Corollary 1.19 to conclude the proof. $\square$

PROOF OF COROLLARY 4.3. First note that for $\widetilde{\langle\, ,\, \rangle} = f^*\langle\, ,\, \rangle = u^{\frac{4}{m-2}}\langle\, ,\, \rangle$; ∂-rigidity assumption on f implies $\widetilde{h}(x) = u^{-\frac{2}{m-2}}h(x)$ on ∂M so that (4.5) and $\widetilde{s}(x) = s(x)$ imply that the assumptions of Corollary 1.19 are satisfied. Hence $u \leq 1$.
We need to prove $u \geq 1$. Toward this aim we observe that for the inverse diffeomorphism $\left(f^{-1}\right)^* \langle\, ,\, \rangle = w^{\frac{4}{m-2}}\langle\, ,\, \rangle$ with $w(y) = \frac{1}{u(f^{-1}(y))}$ we have that w satisfies

$$\begin{cases} \Delta w - c_m s(y) \left(w - w^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ \partial_\nu w + d_m \left(\widetilde{h}(y)w - h(y)w^{\frac{m}{m-2}} \right) = 0 & \text{on } \partial M. \end{cases}$$

The proof follows from Corollary 1.19 if we show that $\partial_\nu w = 0$ on ∂M. Toward this aim we compute

$$\begin{aligned} \partial_\nu w(y) &= -\frac{d_y\left(u\circ f^{-1}\right)[\nu_y]}{\left(u\circ f^{-1}\right)^2(y)} \\ &= -\frac{\left(d_{f^{-1}(y)}u\right)[(f^{-1})_*\nu_y]}{\left(u\circ f^{-1}\right)^2(y)} \end{aligned} \tag{4.6}$$

where $(f^{-1})_*\nu_y \in T_{f^{-1}(y)}M$ (see Chapter 3 of [**49**] for the definition of the tangent space at points $x \in \partial M$), and since f^{-1} is a conformal diffeomorphism it preserves the normal vectors at boundary, that is $(f^{-1})_*\nu_y = \mu(y)\nu_{f^{-1}(y)}$ for some positive function μ. Set $x = f^{-1}(y)$, then from (4.6) and $\partial_\nu u = 0$

$$\begin{aligned} \partial_\nu w(f(x)) &= -\mu(f(x))\frac{d_x u[\nu_x]}{u^2(x)} \\ &= -\mu(f(x))\frac{\partial_\nu u(x)}{u^2(x)} \\ &= 0. \end{aligned}$$

Now, reasoning as above we conclude that $w \le 1$, and therefore $u \ge 1$. □

4.2. More on ∂-rigidity

This section is devoted to clarify the concept of ∂-rigidity by means of geometric examples. Indeed, it will be shown that the condition of being ∂-rigid for a conformal diffeomorphism is automatically satisfied under requirements on the curvature of the boundary. We start with the following

Remark 4.4. From (4.1) it follows immediately that, for a conformal diffeomorphism, the condition of being ∂-rigid is equivalent to requiring

$$\widetilde{h}(x) = u^{-\frac{2}{m-2}}h(x) \quad \text{on } \partial M\,. \tag{4.7}$$

From this equation we observe that a ∂-rigid diffeomorphism preserves pointwise the sign of the mean curvature.
We observe that condition (4.7) is automatically satisfied whenever the boundary ∂M is minimal with respect to the metric $\langle\,,\,\rangle$ and we look for diffeomorphisms preserving this property, that is, minimality of the boundary in the conformally deformed metric. Furthermore we have that if the mean curvatures h and $\widetilde{h}$ have the same sign and never vanish on ∂M, then the diffeomorphism is ∂-rigid if and only if u is a solution of the

overdetermined problem

$$\begin{cases} \Delta u - c_m \left(s(x)u - \widetilde{s}(x)u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ u = \left(\dfrac{h(x)}{\widetilde{h}(x)} \right)^{\frac{m-2}{2}} & \text{on } \partial M \\ \partial_\nu u = 0 & \text{on } \partial M\,. \end{cases}$$

In particular the conformal factor of a conformal diffeomorphism such that $\widetilde{s} = s$ and $\widetilde{h} = h$ on ∂M is ∂-rigid if and only if it is a solution of the problem

$$\begin{cases} \Delta u - c_m s(x) \left(u - u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ u = 1 & \text{on } \partial M \\ \partial_\nu u = 0 & \text{on } \partial M\,. \end{cases} \tag{4.8}$$

Other sufficient conditions for the ∂-rigidity of a conformal diffeomorphism can be deduced by imposing some restrictions on higher order extrinsic curvatures.
Toward this aim we recall some definitions. Let $\varphi : \Sigma^{m-1} \to M^m$ denote an immersion of a connected, $(m-1)$-dimensional Riemannian manifold and assume that it is oriented by a globally defined unit normal vector field N.
Let A denote the second fundamental form of the immersion in the direction of N. Then, the k-mean curvatures of the hypersurface are defined by

$$H_k = \binom{m}{k}^{-1} S_k,$$

where $S_0 = 1$ and, for $k = 1, \dots, m$, S_k is the k-th elementary symmetric function of the eigenvalues of A, the principal curvatures of the hypersurface. In particular, $H_1 = h$ is the mean curvature and H_m is the Gauss-Kronecker curvature of Σ.

The Newton tensors $P_k : TM \to TM$ associated to the oriented immersion are defined inductively by $P_0 = I$ and

$$P_k = S_k I - A P_{k-1}, \quad 1 \le k \le m.$$

Note, for further use, that

$$\mathrm{Tr} P_k = (m-k) S_k \quad \text{and} \quad \mathrm{Tr}\,(A P_k) = (k+1) S_{k+1}.$$

In the case of a Riemannian manifold with boundary we can consider the k-mean curvatures of the immersion $\varphi : \partial M \to M$.

In the following discussion we modify the previous notation for the ease of the reader. Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold of dimension m with boundary and, for a smooth function f on M, consider the pointwise conformal change of metric $\widetilde{\langle,\rangle} = \mathrm{e}^{2f}\langle,\rangle$. In the previous notation it was $\mathrm{e}^f = u^{\frac{2}{m-2}}$ for a positive smooth function. We know from equation (1.3) of [**34**], that under the above conformal transformation, the second fundamental form (in the direction of the outward unit normal) of the boundary changes in the following way

$$\widetilde{A} = e^f \left(A + \partial_\nu f \langle,\rangle\right) .$$

Componentwise

$$\widetilde{A}_{ij} = e^f \left(A_{ij} + \partial_\nu f g_{ij}\right)$$

where g_{ij} are the components of the metric tensor $\langle,\rangle$. We also note that the components of the inverse of the metric tensor change according to the rule

$$\widetilde{g}^{ij} = \mathrm{e}^{-2f} g^{ij} .$$

The following lemma is probably well known (see for instance [**1**]); we present a simple proof using classical tensor formalism for the sake of completeness.

LEMMA 4.5. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a Riemannian manifold with boundary of dimension $m \geq 3$. On ∂M define*

$$\Lambda = m^2(m-1)\left(H_2 - H_1^2\right) \tag{4.9}$$

where H_2 and $H_1 = h$ are the second and first mean curvatures of ∂M. Then, under the pointwise conformal change of metric $\widetilde{\langle,\rangle} = \mathrm{e}^{2f}\langle,\rangle$, with the obvious meaning of the notation, we have

$$\widetilde{\Lambda} = \mathrm{e}^{-2f}\Lambda . \tag{4.10}$$

PROOF. First of all we use the Newton tensors to express Λ in terms of the second fundamental form A, that is

$$\begin{aligned}
\Lambda &= m^2(m-1)\left(H_2 - H_1^2\right) \\
&= 2mS_2 - (m-1)S_1^2 \\
&= m \operatorname{tr}(AP_1) - (m-1)(\operatorname{tr} A)^2 \\
&= (\operatorname{tr} A)^2 - m \operatorname{tr} A^2 .
\end{aligned}$$

Thus, to find the expression of $\widetilde{\Lambda}$ we have to compute $\left(\operatorname{tr}\widetilde{A}\right)^2$ and $\operatorname{tr}\left(\widetilde{A}^2\right)$. We start with the trace of the second fundamental form

$$\begin{aligned}
\operatorname{tr}\widetilde{A} &= \widetilde{g}^{ij}\widetilde{A}_{ij} \\
&= \mathrm{e}^{-f} g^{ij}\left(A_{ij} + \partial_\nu g_{ij}\right) \\
&= \mathrm{e}^{-f}\left(\operatorname{tr} A + m\partial_\nu f\right)
\end{aligned}$$

from which follows that

$$\left(\operatorname{tr}\widetilde{A}\right)^2 = e^{-2f}\left[(\operatorname{tr}A)^2 + 2m\,(\operatorname{tr}A)\,\partial_\nu f + m^2\,(\partial_\nu f)^2\right].$$

Similarly we have

$$\begin{aligned}\operatorname{tr}\widetilde{A}^2 &= \widetilde{A}^j_i\widetilde{A}^i_j\\ &= \mathrm{e}^{-2f}\left[A^j_iA^i_j + 2\partial_\nu f A^i_i + m\,(\partial_\nu f)^2\right]\\ &= \mathrm{e}^{-2f}\left[\operatorname{tr}A^2 + 2\,(\operatorname{tr}A)\,\partial_\nu f + m\,(\partial_\nu f)^2\right].\end{aligned}$$

Summing up we obtain

$$\left(\operatorname{tr}\widetilde{A}\right)^2 - m\operatorname{tr}\widetilde{A}^2 = \mathrm{e}^{-2f}\left[(\operatorname{tr}A)^2 - m\operatorname{tr}A^2\right]$$

concluding the proof of the lemma. □

We note that the quantity Λ is the conformal Willmore integrand for surfaces immersed in 3-manifolds, indeed its integral is a conformal invariant for immersed surfaces.

In the next result we exploit the formal similarity between equations (4.7) and (4.10) to find sufficient conditions for a conformal deformation to be ∂-rigid. We have the following further consequence of Theorem 4.1

Corollary 4.6. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete manifold with boundary, dimension $m \geq 3$ and scalar curvature $s(x)$ such that (4.5) and (4.3) hold. Then, any conformal diffeomorphism of $(M, \partial M, \langle\ ,\ \rangle)$ into itself which preserves the scalar curvature, the sign of the mean curvature, and such that $\widetilde{H}_2 = H_2 \equiv 0$, is an isometry.*

Proof. The idea is to show that any conformal transformation preserving the sign of the mean curvature and such that $\widetilde{H}_2 = H_2 \equiv 0$ is indeed ∂-rigid, so that we can apply Corollary 4.3. From equations (4.9) and (4.10)

$$\left(\widetilde{H}_2 - \widetilde{h}^2\right) = u^{-\frac{4}{m-2}}\left(H_2 - h^2\right) \quad \text{on } \partial M\,.$$

Since $\widetilde{H}_2 = H_2 \equiv 0$ and $h(x)$ has the sign of $\widetilde{h}(x)$, it follows that

$$\widetilde{h} = u^{-\frac{2}{m-2}}h$$

that is, the transformation is ∂-rigid. □

4.3. A generalization of a result by Escobar

We turn our attention to a slightly different geometric problem proposed by J.F. Escobar in the compact case [**36**]. The question is: given a conformal diffeomorphism of a Riemannian manifold with boundary $(M, \partial M, \langle\ ,\ \rangle)$ such that $\widetilde{s} = s$ on M and $\widetilde{h} = h$ on ∂M, when is it true that $\widetilde{\langle\ ,\ \rangle} = \langle\ ,\ \rangle$? He proved the following

Theorem (Corollary 2 in [**36**]). *Let $(M, \langle\ ,\ \rangle)$ be a compact Riemannian manifold with boundary. Assume that $\widetilde{\langle\ ,\ \rangle} = u^{\frac{4}{m-2}} \langle\ ,\ \rangle$, $\widetilde{s} = s \leq 0$ on M and $\widetilde{h} = h \leq 0$ on ∂M. Then $\widetilde{\langle\ ,\ \rangle} = \langle\ ,\ \rangle$.*

For the noncompact case we have an analogous rigidity result, namely

Theorem 4.7. *Let $(M, \partial M, \langle\ ,\ \rangle)$ be a complete, noncompact, manifold with boundary, dimension $m \geq 3$ and scalar curvature $s(x)$ satisfying (4.5) for a positive constant c. Assume that (4.3) holds. Then the identity is the only conformal diffeomorphism of $(M, \partial M, \langle\ ,\ \rangle)$ onto itself such that $\widetilde{s} = s$ on M and $\widetilde{h} = h \leq 0$ on ∂M.*

We stress the fact that Theorem 4.7 has the same hypoteses of the result by Escobar, without any other technical assumption but a control of the growth of geodesc balls at infinity.

Proof of Theorem 4.7. The case $\widetilde{h} = h \equiv 0$ on ∂M follows from Corollary 4.3 and Remark 4.4. In the general case assume by contradiction that $1 < u^* \leq +\infty$, choosing $1 < \gamma < u^*$ we have

$$\begin{cases} \Delta u = c_m s(x) \left(u - u^{\frac{m+2}{m-2}} \right) & \text{on } \Omega_\gamma \\ \partial_\nu u \leq 0 & \text{on } \partial_0 \Omega_\gamma \\ \partial_\nu u = d_m h(x) \left(u^{\frac{2}{m-2}} - 1 \right) u & \text{on } \partial_1 \Omega_\gamma \,. \end{cases}$$

Since $\gamma > 1$, and $h \leq 0$ we deduce that

$$\begin{cases} \Delta u = c_m s(x) \left(1 - u^{\frac{4}{m-2}} \right) u & \text{on } \Omega_\gamma \\ \partial_\nu u \leq 0 & \text{on } \partial \Omega_\gamma \end{cases}$$

Corollary 1.19 and Remark 1.20 imply that $u \leq 1$ on Ω_γ, contradicting the assumption that $u^* > 1$. This shows that $u \leq 1$ on M. To conclude the proof we recall that the conformal factor of the inverse deformation f^{-1} is $w(y) = \frac{1}{u(f^{-1}(y))}$ which satisfies

$$\begin{cases} \Delta w = c_m s(y) \left(1 - w^{\frac{4}{m-2}} \right) w & \text{on } M \\ \partial_\nu w = d_m h(y) \left(w^{\frac{2}{m-2}} - 1 \right) w & \text{on } \partial M \,. \end{cases}$$

Then, reasoning as for u, we conclude that $w \leq 1$, completing the proof. $\square$

4.4. Existence of conformal deformations

In this section we construct positive solutions of problem (4.1) for a possibly sign-changing $\widetilde{s}(x)$. As in the previous chapters, the strategy to find such solutions consists in providing an ordered pair of sub/supersolutions and then applying the monotone iteration scheme of Section 1.6.

The existence of the supersolution u_+ is a delicate issue because of the possible change of sign of $\widetilde{s}(x)$ but it can be proved with the aid of the previously proved Theorem 2.1 in the particular case $c(x) \equiv 0$. To find an adequate subsolution, the key observation, analogously to what has be done in Section 2.2, is that any subsolution of the modified problem

$$\begin{cases} \Delta u - c_m \left(s(x)u + \widetilde{s}_-(x)u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ \partial_\nu u + d_m \left(h(x)u - \widetilde{h}(x)u^{\frac{m}{m-2}} \right) = 0 & \text{on } \partial M, \end{cases} \tag{4.11}$$

is also a subsolution of (4.1), where $\widetilde{s}_-(x) = \max\{0, -\widetilde{s}(x)\}$ as usual. Moreover, since in this case the nonlinear Neumann condition is non-singular (the power $m/(m-2)$ of the nonlinearity is positive), we will see that a positive solution of an associated Dirichlet problem will provide a positive solution of (4.11). We start with the following result, which is a simple extension of Theorem 6.7 of [**55**] to the case of a manifold with non empty boundary.

Proposition 4.8. *Let $a(x)$, $b(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0 < \alpha \leq 1$. Assume that $b(x) \geq 0$, $b(x) > 0$ outside a compact set, and that $\lambda_1^L(B_0) > 0$, where $L = \Delta + a(x)$. Assume that*

$$\lambda_1^L(M) < 0\,, \tag{4.12}$$

then the Dirichlet problem

$$\begin{cases} \Delta u + a(x)u - b(x)u^\sigma = 0 & \text{on } M \\ u = 0 & \text{on } \partial M \end{cases} \tag{4.13}$$

has a minimal positive solution $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M)$.

Proof. Since $\lambda_1^L(M) < 0$, there exists an open bounded domain with smooth boundary $\Omega_0 \supset B_0$ with $\partial_1\Omega_0 \supset\supset \partial_1 B_0$ and such that $\lambda_1^L(\Omega_0) < 0$. Now, arguing exactly as in the proof of Lemma 6.3 of [**55**] we deduce the existence of a positive solution $\overline{u} \in \mathrm{C}^2(\Omega_0)$ of

$$\begin{cases} \Delta\overline{u} + a(x)\overline{u} - b(x)\overline{u}^\sigma \geq 0 & \text{on } \Omega_0 \\ \overline{u} = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{4.14}$$

Moreover, since $\partial\Omega_0$ is smooth, by elliptic regularity we have that

$$\sup_{\partial\Omega_0} |\partial_\nu \overline{u}| < +\infty\,. \tag{4.15}$$

Let D and D' be bounded open domains such that

$$\begin{cases} B_0 \subset D' \subset D \subset \Omega_0 \\ \partial_1 D' \subset\subset \partial_1 D \\ \partial_0 D' \subset \operatorname{int} D \end{cases}$$

and $\lambda_1^L(D) > 0$. Let v_1 be the positive solution of

$$\begin{cases} \Delta v_1 + a(x)v_1 + \lambda_1^L(D)v_1 = 0 & \text{on } D \\ v_1 = 0 & \text{on } \partial D \\ \|v_1\|_{\mathrm{L}^\infty(D)} = 1. \end{cases}$$

Since $b(x) > 0$ on $M \setminus B_0$ and $\overline{\Omega_0} \setminus D' \subset\subset M \setminus B_0$,

$$\beta = \inf_{\overline{\Omega_0}\setminus D'} b(x) > 0\,.$$

Define

$$\alpha = \sup_{\overline{\Omega}\setminus D'} a(x)\,,$$

and note that $\alpha < +\infty$ since Ω_0 is bounded. Let U be a positive constant. Then

$$\begin{aligned} \Delta U + a(x)U - b(x)U^\sigma &= U\left(a(x) - b(x)U^{\sigma-1}\right) \\ &\leq U\left(\alpha - \beta U^{\sigma-1}\right) \end{aligned}$$

on $\Omega_0 \setminus D'$. We observe that the RHS of the above is non-positive for U sufficiently large, say

$$U \geq \Lambda_0 > 0\,. \tag{4.16}$$

Next we choose a cut-off function $\psi \in \mathrm{C}_0^\infty(M)$ such that $0 \leq \psi \leq 1$, and $\psi \equiv 1$ on D', $\operatorname{supp}\psi \subset D$. Fix a positive constant γ and define

$$v = \gamma\left(\psi v_1 + (1-\psi)\Lambda_0\right)\,.$$

Since $b(x) \geq 0$ and $\lambda_1^L(D) > 0$, on $\operatorname{int} D'$ we have

$$\begin{aligned}\Delta v + a(x)v - b(x)v^\sigma &= L(\gamma v_1) - b(x)(\gamma v_1)^\sigma \\ &= -\left[\lambda_1^L(D)(\gamma v_1) + b(x)(\gamma v_1)^\sigma\right] \\ &\leq 0 .\end{aligned}$$

We now consider $\Omega_0 \setminus D$, since $\operatorname{supp}\psi \subset D$, it follows that $v = \gamma\Lambda_0$ there. Thus, using $\Omega_0 \setminus D \subset \Omega_0 \setminus D'$, from (4.16) it follows that

$$\Delta v + a(x)v - b(x)v^\sigma \leq 0 \qquad \text{on } \Omega_0 \setminus D ,$$

is satisfied if we choose $\gamma \geq 1$; indeed in this case

$$\gamma\Lambda_0 \geq \Lambda_0 .$$

It remains to analyze the situation on $D \setminus \overline{D'}$. First of all we note that, by standard elliptic regularity theory, $v_1 \in \mathrm{C}^2(\operatorname{int} D)$. Thus, since $\operatorname{supp}\psi \subset D$, it follows that $v \in \mathrm{C}^2(\operatorname{int}\Omega_0)$, in particular this implies that there exists a positive constant C_0 such that

$$(\Delta + \overline{a}(x))\, v \leq \gamma C_0 \qquad \text{on int}\left(D \setminus \overline{D'}\right) .$$

Thus on $\operatorname{int}\left(D \setminus \overline{D'}\right)$ we have

$$\Delta v + a(x)v - b(x)v^\sigma \leq \gamma C_0 - b(x)\gamma^\sigma \left(\psi v_1 + (1-\psi)\Lambda_0\right)^\sigma .$$

Now there exists a constant ε such that

$$\inf_{D \setminus \overline{D'}} b(x)\left(\psi u_1 + (1-\psi)\Lambda_0\right)^\sigma = \varepsilon > 0 ,$$

Therefore, on $\operatorname{int}\left(D \setminus \overline{D'}\right)$

$$\Delta v + a(x)v - b(x)v^\sigma \leq \gamma\left(C_0 - \varepsilon\gamma^{\sigma-1}\right) .$$

Since $\sigma > 1$, it follows that there exists a positive constant Γ_1 depending only on D and D' such that

$$C_0 - \varepsilon\gamma^{\sigma-1} \leq 0$$

for $\gamma \geq \Gamma_1$.
Thus, by choosing

$$\gamma \geq \max\{1, \Gamma_0, \Gamma_1\}$$

v solves

$$\begin{cases} \Delta v + a(x)v - b(x)v^\sigma \leq 0 & \text{on } \Omega_0 \\ v > 0 & \text{on } \partial_0\Omega_0 \\ v \geq 0 & \text{on } \partial_1\Omega_0. \end{cases} \tag{4.17}$$

Moreover $v|_{\partial_1\Omega_0} = \gamma\Lambda_0\,(1-\psi)$, thus $v=0$ on $\partial_1\Omega_0$ if and only if $\psi=1$, and furthermore in that case

$$\partial_\nu v = \gamma\partial_\nu v_1 < \partial_\nu v_1 < 0$$

since $\gamma \geq 1$. Thus on $\partial\Omega_0$ we have that at least $v>0$ or $\partial_\nu v < 0$ holds. Thus, recalling (4.15) and noting that for each $0<s<1$ the function $\overline{u}_s = s\overline{u}$ is still a solution of (4.14), there exists a $0 < s_0 < 1$ such that $\overline{u}_{s_0} \leq v$. Using (4.17), (4.14), and the monotone iteration scheme we deduce the existence of a positive solution $u_0 \in \mathrm{C}^2(\Omega_0)$ of

$$\begin{cases} \Delta u_0 + a(x)u_0 - b(x)u_0^\sigma = 0 & \text{on } \Omega_0 \\ u_0 = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{4.18}$$

Now let $\Omega_1 \supset \Omega_0$. By domain monotonicity

$$\lambda_1^L(\Omega_1) \leq \lambda_1^L(\Omega_0) < 0$$

and proceeding as above we obtain a positive solution u_1 of (4.18) on Ω_1. Since $\partial\Omega_0 \subset \overline{\Omega_1}$, $u_0 = 0 \leq u_1$ on $\partial\Omega_0$, it follows by Lemma 3.9 and Remark 3.10 that $u_1 \geq u_0$ on Ω_0. Choosing $\{\Omega_i\}_{i=0}^\infty$, a ∂-regular exhaustion of M as in Definition 1.22, the above procedure yields a sequence of functions $\{u_i\}$ on Ω_i satisfying

$$\begin{cases} \Delta u + a(x)u - b(x)u^\sigma = 0 & \text{on } \Omega_i \\ u = 0 & \text{on } \partial\Omega_i \\ u_i \leq u_{i+1} & \text{on } \Omega_i. \end{cases}$$

Arguing as in the proof of Lemma 3.11 we see that $\{u_i\}$ is uniformly bounded on $\overline{\Omega_k}$ for $i \geq k+1$; therefore, since $\{\partial\Omega_i\}$ is an exhaustion of ∂M, the sequence u_i converges to a positive solution $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(\partial M)$ of (4.13). The minimality of the solution is a consequence of Lemma 3.9, indeed, let $\widetilde{u} \in \mathrm{C}^2(\operatorname{int} M)\cap \mathrm{C}^0(\partial M)$ be another positive solution of 4.13, then $\widetilde{u} \geq u_i$ on Ω_i, for each $i \geq 1$. Thus $\widetilde{u} \geq u$ and u is minimal. $\square$

Remark 4.9. Since the solution of (4.13) is such that $u>0$ on $\operatorname{int} M$ and $u=0$ on ∂M, it follows that $\partial_\nu u \leq 0$ on ∂M.

The next result is a L^∞ estimate for positive solutions of (4.13) under a volume growth assumption, in the spirit of Theorem 1.18. The main novelty in this case is that we explicitly allow an amount of negativity for the function $b(x)$.

Proposition 4.10. *Let $a(x)$, $b(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$ for some $0<\alpha\leq 1$. Assume that $b(x)>0$ outside a compact set B_0, and*

$$\sup_{M\setminus B_0} \frac{a_-(x)}{b(x)} \leq H$$

for some $H > 0$. *Suppose that*

$$b(x) \geq \frac{C}{r(x)^{\mu}}$$

outside a compact set for some constants $C > 0$, $\mu < 2$, *and*

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{2-\mu}} < +\infty \,.$$

Then, any positive solution $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M)$ *of (4.13) is bounded.*

PROOF. Set $\beta = \sup_{B_0} u$, since u is continous and B_0 is compact, it follows that $\beta < +\infty$. Now choose $\gamma > \beta$ and consider the set

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} \ ,$$

clearly $\Omega_\gamma \subset M \setminus B_0$, thus $b(x) > 0$ there. If Ω_γ is empty then $u \leq \gamma < +\infty$ and we have finished. Otherwise we have

$$\begin{cases} \Delta u + a(x)u - b(x)u^{\sigma} = 0 & \text{on } \Omega_\gamma \\ \partial_\nu u \leq 0 & \text{on } \partial\Omega_\gamma \,, \end{cases}$$

thus, by Corollary 1.19 and Remark 1.21 it follows that $u \leq H^{1/(\sigma-1)} < +\infty$ on Ω_γ, and therefore u is bounded on M. $\square$

PROPOSITION 4.11. *Let* $s(x)$, $\widetilde{s}(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$, $h(x)$, $\widetilde{h}(x) \in \mathrm{C}^{0,\alpha}_{loc}(\partial M)$ *for some* $0 < \alpha \leq 1$. *Assume that* $\widetilde{s}(x) < 0$ *outside a compact set, and that* $\lambda_1^L(S_0) > 0$, *where*

$$S_0 = \{x \in M \ : \ \widetilde{s}(x) \geq 0\}$$

and $L = \Delta - s(x)$. *Assume that*

$$\lambda_1^L(M) < 0 \,. \tag{4.19}$$

Then (4.1) has a positive subsolution $u_- \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M)$.

PROOF. Setting $a(x) = -c_m s(x)$, $b(x) = c_m \widetilde{s}_-(x)$, and $\sigma = \dfrac{m+2}{m-2} > 1$, the hypoteses of Proposition 4.8 are fullfilled, thus there exists a positive solution $u_- \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M)$ of (4.13). Now, elementary computations show that

$$\Delta u_- - c_m \left(s(x)u_- - \widetilde{s}(x)u_-^{\frac{m+2}{m-2}} \right) \geq \Delta u_- - c_m \left(s(x)u_- + \widetilde{s}_-(x)u_-^{\frac{m+2}{m-2}} \right) = 0$$

on $\operatorname{int} M$, and by Remark 4.9 and the fact that $u_- \equiv 0$ on ∂M, we also have that

$$\partial_\nu u_- + d_m \left(h(x)u_- - \widetilde{h}(x)u_-^{\frac{m}{m-2}} \right) = \partial_\nu u_- \leq 0$$

on ∂M. Thus u_- is a subsolution of (4.1). $\square$

In analogy with what has be done in Section 2.1, we define the quantity

$$\widetilde{s}_\theta(x) = \theta\widetilde{s}_+(x) - \widetilde{s}_-(x)\,,$$

for $\theta \in \mathbb{R}^+$. Now we are ready to state and prove the main result of the section.

THEOREM 4.12. *Let $s(x)$, $\widetilde{s}(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$, $h(x)$, $\widetilde{h}(x) \in \mathrm{C}^{0,\alpha}_{loc}(\partial M)$ for some $0 < \alpha \leq 1$. Suppose that*

$$\widetilde{h}(x) < 0\,, \qquad \frac{h(x)}{\widetilde{h}(x)} \in \mathrm{L}^\infty(\partial M)\,, \tag{4.20}$$

and

$$\limsup_{x\to\infty} \frac{s_-(x)}{\widetilde{s}_-(x)} < +\infty\,. \tag{4.21}$$

Assume that $\widetilde{s}(x) < 0$ outside the compact set $S_0 = \{x \in M\,:\, \widetilde{s}(x) \geq 0\}$, that

$$\zeta_1^L(S_0) > 0$$

and

$$\lambda_1^L(M) < 0\,.$$

where $L = \Delta - s(x)$. Assume that

$$\sup_{M\setminus S_0} \frac{s_+(x)}{\widetilde{s}_-(x)} \leq H$$

for some $H > 0$ and suppose that

$$\widetilde{s}_-(x) \geq \frac{C}{r(x)^\mu}$$

outside a compact set for some constants $C > 0$, $\mu < 2$, and

$$\liminf_{r\to+\infty} \frac{\log \mathrm{vol}\, B_r}{r^{2-\mu}} < +\infty\,.$$

Then there exists $\theta_0 \in (0,1]$ such that for each $\theta \in (0,\theta_0]$ there exists $u \in \mathrm{C}^2(\mathrm{int}\, M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ positive solution of

$$\begin{cases} \Delta u - c_m\left(s(x)u - \widetilde{s}_\theta(x)u^{\frac{m+2}{m-2}}\right) = 0 & \text{on } M \\ \partial_\nu u + d_m\left(h(x)u - \widetilde{h}(x)u^{\frac{m}{m-2}}\right) = 0 & \text{on } \partial M. \end{cases} \tag{4.22}$$

PROOF. First of all, from Theorem 2.1, there exists $\theta_0 \in (0,1]$ such that for each $\theta \in (0,\theta_0]$ there exists a $u_+ \in \mathrm{C}^2(\mathrm{int}\, M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ supersolution of (4.22) satisfying

$$\inf_M u_+ \geq \Lambda$$

for a positive constant Λ.
Next we recall that, by (1.70) it follows that also $\lambda_1^L(S_0) > 0$, thus, by Proposition 4.11, there exists a $u_- \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M)$ subsolution of (4.11) (and thus also of (4.22)) such that $u_- > 0$ on $\operatorname{int} M$ and $u \equiv 0$ on ∂M. Furthermore, from Proposition 4.10 it follows that such subsolution is also bounded by above. For $\sigma \in (0,1)$ we set

$$u_\sigma = \sigma u_- \, .$$

Since $\widetilde{s}_- > 0$, reasoning as in the proof of Theorem 2.6, u_σ is again a subsolution of (4.11) and moreover there exists a $\sigma_0 > 0$ such that $0 \le u_\sigma < u_+$ on M.
Now it follows from Proposition 1.25 that there exists $u \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ solution of 4.22 and such that $0 \le u_\sigma \le u \le u_+$. To conclude the proof we are left to show that u is positive on M. This is clearly true on $\operatorname{int} M$, since $u_\sigma > 0$ there, so we only have to check the positivity on ∂M. Since u solves (4.22), we have that

$$\begin{cases} \Delta u + \beta(x) u = 0 & \text{on } M \\ u \ge 0 & \text{on } M \end{cases} \tag{4.23}$$

where we set

$$\beta(x) = c_m \left(\widetilde{s}_\theta(x) u^{\frac{4}{m-2}} - s(x) \right) \, .$$

Suppose that there exists a $x_0 \in \partial M$ such that $u(x_0) = 0$, thus $\partial_\nu u(x_0) = 0$ by (4.22). By the Hopf boundary point lemma (Lemma 3.4 of [**39**]), since u is a solution of (4.23), we should heve $\partial_\nu u(x_0) < 0$, contradicting the assumption $u(x_0) = 0$. □

4.5. The Yamabe problem

In the aforementioned paper [**34**] J.F. Escobar introduced the Yamabe problem for a manifold with boundary, namely he asked when it was possible to find a conformal deformation $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$ of a complete manifold $(M, \partial M, \langle\ ,\ \rangle)$ such that the new scalar curvature $\widetilde{s}(x)$ is constant and the boundary is minimal, that is, $\widetilde{h}(x) \equiv 0$ on ∂M. This is equivalent to ask when it is possible to find a constant C such that the problem

$$\begin{cases} \Delta u - c_m \left(s(x) u - C u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ \partial_\nu u + d_m h(x) u = 0 & \text{on } \partial M \end{cases} \tag{4.24}$$

admits a positive solution. As remarked at the beginning of the chapter, the original formulation of the problem deals with the case of $(M, \partial M, \langle\ ,\ \rangle)$ compact, where the existence of solutions is treated in a variational way, as for the classical Yamabe problem on a compact manifold $(M, \langle\ ,\ \rangle)$. In this section we show how the sub/super solution techniques developed in the previous chapters can give a partial answer to the Yamabe problem on a complete manifold with boundary $(M, \partial M, \langle\ ,\ \rangle)$, at least in the case of negative C.

THEOREM 4.13. *Let $s(x) \in \mathrm{C}^{0,\alpha}_{loc}(M)$, $h(x) \in \mathrm{C}^{0,\alpha}_{loc}(\partial M)$ for some $0 < \alpha \leq 1$. Suppose that $h(x) \geq 0$, $s(x) \in \mathrm{L}^{\infty}(M)$, and*

$$\lambda_1^L(M) < 0\,.$$

where $L = \Delta - s(x)$. Assume that

$$\liminf_{r\to+\infty} \frac{\log \operatorname{vol} B_r}{r^{2-\mu}} < +\infty$$

for a constant $0 \leq \mu < 2$. Then there exists a conformal deformation of $(M, \partial M, \langle\ ,\ \rangle)$ to a manifold $(M, \partial M, \widetilde{\langle\ ,\ \rangle})$ with constant negative scalar curvature and minimal boundary.

PROOF. We have to show that there exists a constant $C < 0$ such that (4.24) admits a positive solution. Note that the non-homogeneous structure of (4.24) implies that this is equivalent to finding a positive solution of

$$\begin{cases} \Delta u - c_m \left(s(x)u + u^{\frac{m+2}{m-2}} \right) = 0 & \text{on } M \\ \partial_\nu u + d_m h(x) u = 0 & \text{on } \partial M. \end{cases} \tag{4.25}$$

Indeed suppose that v is a positive solution of (4.24) then, setting $v_\sigma = \sigma v$ for $\sigma \in \mathbb{R}^+$, we have that

$$\begin{aligned} \Delta v_\sigma - c_m \left(s(x) v_\sigma + v_\sigma^{\frac{m+2}{m-2}} \right) &= \sigma \left[\Delta v - c_m \left(s(x) v + \sigma^{\frac{4}{m-2}} v^{\frac{m+2}{m-2}} \right) \right] \\ &= \sigma \left(C + \sigma^{\frac{4}{m-2}} \right) v^{\frac{m+2}{m-2}}\,, \end{aligned}$$

on M, thus, choosing $\sigma = (-C)^{\frac{m-2}{4}}$, we have that v_σ satisfies (4.25).

The hypoteses of Proposition 4.11 and Proposition 4.10 are satisfied, thus there exists $u_- \in \mathrm{C}^2(\operatorname{int} M) \cap \mathrm{C}^0(M) \cap \mathrm{L}^\infty(M)$ subsolution of (4.25) such that $u > 0$ on $\operatorname{int} M$ and $u \equiv 0$ on ∂M.

Choosing a positive constant $H > \max \left\{ \|s_-\|^{\frac{m-2}{m+2}}_{\mathrm{L}^\infty(M)}, \|u_-\|_{\mathrm{L}^\infty(M)} \right\}$, it follows that $u_+ = H$ is a supersolution of (4.25) such that $u_+ \geq u_-$, thus the theorem follows from the sub/super solution method together with the Hopf lemma, as in the proof of Theorem 4.12.

$\square$

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