Internal Homs via extensions of dg functors ☆

Alberto Canonaco [a], Paolo Stellari [b,*]

[a] *Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy*
[b] *Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy*

A R T I C L E   I N F O

A B S T R A C T

We provide a simple proof of the existence of internal Homs in the localization of the category of dg categories with respect to all quasi-equivalences and of some of their main properties such as the so-called derived Morita theory. This was originally proved in a seminal paper by Toën.

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1. Introduction

The problem of characterizing exact functors between triangulated categories is certainly one of the major open questions in the theory of triangulated categories. As soon

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* Corresponding author.
E-mail addresses: alberto.canonaco@unipv.it (A. Canonaco), paolo.stellari@unimi.it (P. Stellari).
URL: http://users.unimi.it/stellari (P. Stellari).

as we deal with triangulated categories which are the bounded derived categories of coherent sheaves on smooth projective varieties, this challenge in the vague form above gets neater. More precisely, if X_1 and X_2 are smooth projective schemes and we denote by $\mathrm{D}^b(X_i)$ the bounded derived category of coherent sheaves on X_i, then one would expect that all exact functors $\mathsf{F} : \mathrm{D}^b(X_1) \longrightarrow \mathrm{D}^b(X_2)$ are of *Fourier–Mukai type* (see [2,15]). This means that there should exist $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$, where, denoting by $p_i : X_1 \times X_2 \to X_i$ the natural projections, $\Phi_{\mathcal{E}} : \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is the exact functor defined by $\Phi_{\mathcal{E}} := \mathbf{R}(p_2)_*(\mathcal{E} \otimes^{\mathbf{L}} p_1^*(-))$. Unfortunately, only partial results confirm the expectation above (see [5] for a survey about this).

If one changes perspective slightly and moves to higher categorical structures, the situation becomes amazingly beautiful. More precisely, one looks at the localization **Hqe** of the category **dgCat** of (small) dg categories over a commutative ring $\Bbbk$ with respect to all quasi-equivalences. Then one can take dg enhancements $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$ respectively. It turns out (see [2]) that all Fourier–Mukai functors at the triangulated level lift to morphisms between $\mathbf{D}_1$ and $\mathbf{D}_2$ in **Hqe**. More surprisingly, all morphisms in **Hqe** between these dg categories are of this type. This was observed in the seminal paper [18] and comes as a corollary of a very general and elegant result about the existence of internal Homs in **Hqe**. The statement can be formulated as follows.

Theorem 1.1 *(Toën, [18]). Let* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *be three dg categories over a commutative ring* $\Bbbk$. *Then there exists a natural bijection*

$$[\mathbf{A}, \mathbf{C}] \xleftarrow{\quad 1:1 \quad} \mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{C})^{\mathrm{rqr}})). \tag{1.1}$$

Moreover, the dg category $\mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}) := \text{h-proj}(\mathbf{B}^\circ \otimes^{\mathbf{L}} \mathbf{C})^{\mathrm{rqr}}$ *yields a natural bijection*

$$[\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B}, \mathbf{C}] \xleftarrow{\quad 1:1 \quad} [\mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C})] \tag{1.2}$$

proving that the symmetric monoidal category **Hqe** *is closed.*

Here $[-,-]$ denotes the set of morphisms in **Hqe**, while h-proj$(-)^{\mathrm{rqr}}$ denotes right quasi-representable h-projective dg modules, which will be precisely described in Section 3.1. In the original version [18] h-proj$(-)^{\mathrm{rqr}}$ is replaced by the dg category of right quasi-representable cofibrant dg modules, which is actually quasi-equivalent to h-proj$(-)^{\mathrm{rqr}}$. Recall that the monoidal structure provided by the derived tensor product $- \otimes^{\mathbf{L}} -$ is said to be *closed* if, for every $\mathbf{B}, \mathbf{C} \in$ **Hqe**, there exists $\mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}) \in$ **Hqe**, such that the functor $\mathbf{A} \mapsto [\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B}, \mathbf{C}]$ is isomorphic to $\mathbf{A} \mapsto [\mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C})]$.

Roughly speaking, Theorem 1.1 asserts that all dg (quasi-)functors are of Fourier–Mukai type. The reason is that, if one looks carefully at the proof of the bijection (1.1), one sees that all morphisms in **Hqe** are essentially provided by the tensorization by dg

bimodules, mimicking the definition of Fourier–Mukai functors given above in the triangulated setting. It is worth pointing out that the first part of Theorem 1.1 comes in [18] as a corollary of a much more general result involving substantially the simplicial structure on **dgCat**, seen as a model category (see [18, Thm. 4.2]).

The purpose of this paper is to provide a simple proof of Theorem 1.1 essentially based on the circle of ideas emerging from [6] and [7].

Comparing Toën's approach and ours

Following [18], one uses the model category structure on **dgCat** (see [16]) in such a way that any morphism in **Hqe** can be represented by a 'canonical' roof by means of the cofibrant replacements. At this point, the description of the morphisms between two dg categories in **Hqe** can be essentially carried out assuming that we are working with actual dg functors. In this way, Toën proves that, for two dg categories **A** and **B**, there is a bijection between [**A**, **B**] and the isomorphism classes of the homotopy category of right quasi-representable (fibrant and) cofibrant $\mathbf{A}^\circ \otimes \mathbf{B}$-dg modules (see Corollary 4.10 of [18]). This is essentially the first part of Theorem 1.1.

The existence of internal Homs follows from a characterization of the model category of dg functors between dg categories and a comparison between this and the presentation above [18, Thm. 6.1]. As an application of the existence of internal Homs, Toën deduces in [18, Thm. 7.2] a restriction theorem asserting in particular that, given two dg categories **A** and **B**, the Yoneda embedding of **A** into the dg category Int(**A**) of cofibrant **A**-dg modules yields a quasi-equivalence between the continuous internal Hom $\mathbb{R}\underline{Hom}_c(\mathrm{Int}(\mathbf{A}), \mathrm{Int}(\mathbf{B}))$ and $\mathbb{R}\underline{Hom}(\mathbf{A}, \mathrm{Int}(\mathbf{B}))$ (see Sections 2.1 and 4 for the precise definitions), which goes under the name of *derived Morita theory*. This is a very interesting result in itself with various geometric applications as explained in [18, Sect. 8]. One should keep in mind that, for a dg category **A**, the dg category Int(**A**) is contained in the dg category h-proj(**A**) of h-projective **A**-dg modules and the inclusion is a quasi-equivalence (see, for example, [11, Lemma 2.6] and the discussion in Section 3.2 of [8] for this standard fact).

In a sense, our argument starts from this sort of ending point in [18]. Indeed, we prove directly a weaker form of this restriction result (see Proposition 3.10) which provides a nice description of some morphisms in **Hqe** in terms of isomorphism classes of special dg bimodules. The existence of internal Homs and the proof of Theorem 1.1 follow then from a simple and purely formal argument explained in Section 4. All of this is achieved using the notion of extension of dg functors which is already contained in [7]. This is carried out in Sections 3.1 and further developed to deal with morphisms in **Hqe** in Section 3.2. As we will see, this is a conceptually very simple application of the notion of tensor product of dg modules (see [6] and [7]). In particular, it should be noted that the core and the really non-trivial part of this paper is the content of Section 3.

This slight change of perspective makes our proof easier also because we can forget about the model category structure and content ourselves with the fact that the category

of dg categories is a category of fibrant objects. This has certainly been known for a long time and is summarized in Section 2.2 (after a short introduction to dg categories in Section 2.1).

Once Theorem 1.1 is settled, some important properties of internal Homs which are proved in [18] can be deduced in a straightforward way. This is the case of Corollaries 4.1 and 4.2. The last one covers the dg Morita theory mentioned above.

It is probably worth pointing out here that Toën's result gave an input to further generalizations at the level of ∞-categories (see, for example, [1]). But for this one really needs the model and simplicial structures on **dgCat**. This is out of the scope of our paper.

The reader should be also aware that a different approach to the existence of internal Homs was proposed by Tabuada in [17]. In particular, he constructs a new model category structure for the homotopy category of dg categories, where the localization takes place with respect to the so-called *Morita equivalences* and not just the quasi-equivalences. Clearly, this means that [17] is not in the same generality as [18]. Moreover, Ref. [17] goes in a transversal direction with respect to the present work. Nevertheless, one can observe that in Tabuada's approach the internal Homs can be naturally interpreted as derived functors.

We conclude this summary going back to the triangulated setting presented at the beginning. It is important to observe that there is no hope that the beauty of Theorem 1.1 can appear in the triangulated context as well. Indeed, it has been shown in [4] that the object $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ realizing a Fourier–Mukai functor is by no means unique (up to isomorphism).

Notation and general assumptions

We denote by $\Bbbk$ a commutative ring. By a $\Bbbk$-linear category we mean a category whose Hom spaces are $\Bbbk$-modules and such that the compositions are $\Bbbk$-bilinear, not assuming that finite direct sums exist. For a category **A**, we denote by Iso(**A**) the set of isomorphism classes of objects in **A**.

Throughout the paper, we assume that a universe $\mathbb{U}$ containing an infinite set is fixed. Anticipating some definitions that will be explained in Section 2, let us spend some words to clarify the context. We will consider $\mathbb{U}$-*small* dg categories, meaning dg categories **D** such that $\mathrm{Hom}_{\mathbf{D}}(D_1, D_2)$ is a complex of $\Bbbk$-modules which are isomorphic to objects of $\mathbb{U}$ and such that the collection of objects of **D** is isomorphic to an object of $\mathbb{U}$ as well. Analogously one can speak about $\mathbb{U}$-small sets. We will then define the dg categories of **D**-dg modules and of h-projective **D**-dg modules. Again, with this we tacitly mean the dg categories of $\mathbb{U}$-small **D**-dg modules and h-projective **D**-dg modules. This simply refers to the dg modules that take values in the dg category of complexes of $\mathbb{U}$-small $\Bbbk$-modules. These are no longer $\mathbb{U}$-small dg categories. Nevertheless, due to the results in [10, Appendix A], they are dg equivalent to $\mathbb{V}$-small dg categories, for some universe $\mathbb{U} \in \mathbb{V}$. By its definition and main properties (see Section 4), the dg category h-proj$(-)^{\mathrm{rqr}}$

mentioned above is *essentially* $\mathbb{U}$*-small*. This means that the isomorphism classes of its objects form a $\mathbb{U}$-small set. After these warnings and to simplify the notation, we will not mention explicitly the universe where we are working any longer in the paper, as it should be clear from the context. We refer to [10, Appendix A] for all possible subtle logical issues.

2. Basic properties of dg categories

This section collects some very well-known facts concerning dg categories. The emphasis is on the properties of morphisms in the localization of the category of dg categories by quasi-equivalences.

2.1. Dg categories and tensor product of dg modules

A *dg category* is a $\Bbbk$-linear category $\mathbf{A}$ such that, for all $A, B \in \mathrm{Ob}(\mathbf{A})$, the morphism spaces $\mathbf{A}(A, B)$ are $\mathbb{Z}$-graded $\Bbbk$-modules with a differential $d \colon \mathbf{A}(A, B) \to \mathbf{A}(A, B)$ of degree 1 and the composition maps are morphisms of complexes. A *dg functor* $\mathsf{F} \colon \mathbf{A} \to \mathbf{B}$ between two dg categories is the datum of a map $\mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and of morphisms of complexes of $\Bbbk$-modules $\mathbf{A}(A, B) \to \mathbf{B}(\mathsf{F}(A), \mathsf{F}(B))$, for all $A, B \in \mathrm{Ob}(\mathbf{A})$, which are compatible with the compositions and the units (i.e. the identity maps which are, automatically, closed morphisms in degree 0). The category with objects dg categories and morphisms dg functors will be denoted by $\mathbf{dgCat}$. Recalled that a dg functor $\mathsf{F} \colon \mathbf{A} \to \mathbf{B}$ is *full* if the morphisms of complexes of $\Bbbk$-modules $\mathbf{A}(A, B) \to \mathbf{B}(\mathsf{F}(A), \mathsf{F}(B))$ are surjective, for all $A, B \in \mathrm{Ob}(\mathbf{A})$. If such maps are injective, then F is *faithful*.

Example 2.1. (i) Every $\Bbbk$-linear category can be regarded as a dg category in which the morphism spaces are concentrated in degree 0.

(ii) Every dg algebra A over $\Bbbk$ defines a dg category with one object and A as its space of endomorphisms. Notice that an ordinary $\Bbbk$-algebra (in particular, $\Bbbk$ itself) can be regarded as a dg algebra in degree 0, hence as a dg category with one object.

(iii) We denote by $\mathrm{C}_{\mathrm{dg}}(\Bbbk)$ the dg category whose objects are complexes of $\Bbbk$-modules. We refer to [8, Sect. 2.2] for the precise definition.

(iv) Given two dg categories $\mathbf{A}$ and $\mathbf{B}$, one can construct the dg categories $\underline{Hom}(\mathbf{A}, \mathbf{B})$ and $\mathbf{A} \otimes \mathbf{B}$ (see [8, Sect. 2.3] for the precise definitions). The objects of $\underline{Hom}(\mathbf{A}, \mathbf{B})$ are dg functors from $\mathbf{A}$ to $\mathbf{B}$ and morphisms are given by (dg) natural transformations. On the other hand, the objects of $\mathbf{A} \otimes \mathbf{B}$ are pairs (A, B) with $A \in \mathbf{A}$ and $B \in \mathbf{B}$, while the morphisms are defined by

$$\mathbf{A} \otimes \mathbf{B}((A_1, B_1), (A_2, B_2)) = \mathbf{A}(A_1, A_2) \otimes_{\Bbbk} \mathbf{B}(B_1, B_2),$$

for all $(A_i, B_i) \in \mathbf{A} \otimes \mathbf{B}$ and $i = 1, 2$. Notice that the tensor product defines a symmetric monoidal structure on $\mathbf{dgCat}$. Namely, up to isomorphism, the tensor product is asso-

ciative, commutative and $\Bbbk$ acts as the identity. It is also easy to see that two dg functors $\mathsf{F} \colon \mathbf{A} \to \mathbf{B}$ and $\mathsf{G} \colon \mathbf{C} \to \mathbf{D}$ naturally induce a dg functor $\mathsf{F} \otimes \mathsf{G} \colon \mathbf{A} \otimes \mathbf{C} \to \mathbf{B} \otimes \mathbf{D}$.

(v) If $\mathbf{A}$ is a dg category, $\mathbf{A}^\circ$ denotes the opposite dg category. The objects of $\mathbf{A}^\circ$ are the same as those of $\mathbf{A}$, while $\mathbf{A}^\circ(A, B) := \mathbf{A}(B, A)$ and the compositions in $\mathbf{A}^\circ$ are defined as in $\mathbf{A}$, up to a sign (see [8, Sect. 2.2] for details).

If $\mathbf{A}$ is a dg category, we denote by $Z^0(\mathbf{A})$ (respectively $H^0(\mathbf{A})$) the ($\Bbbk$-linear) category with the same objects as $\mathbf{A}$ and whose morphisms from A to B are given by $Z^0(\mathbf{A}(A, B))$ (respectively $H^0(\mathbf{A}(A, B))$). The category $H^0(\mathbf{A})$ is called the *homotopy category* of $\mathbf{A}$, and it has a natural structure of triangulated category if $\mathbf{A}$ is pretriangulated (see [8, Sect. 4.5] for the precise definition). A morphism of $Z^0(\mathbf{A})$ is a *dg isomorphism* (respectively a *homotopy equivalence*) if it is an isomorphism (respectively if its image in $H^0(\mathbf{A})$ is an isomorphism). Accordingly, two objects A and B of $\mathbf{A}$ are *dg isomorphic* (respectively *homotopy equivalent*) if $A \cong B$ in $Z^0(\mathbf{A})$ (respectively in $H^0(\mathbf{A})$). If $\mathbf{B}$ is another dg category, two dg functors from $\mathbf{A}$ to $\mathbf{B}$ will be said to be dg isomorphic (respectively homotopy equivalent) if they are dg isomorphic (respectively homotopy equivalent) in $\underline{Hom}(\mathbf{A}, \mathbf{B})$.

For all dg categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, there is a natural isomorphism in **dgCat**

$$\underline{Hom}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \cong \underline{Hom}(\mathbf{A}, \underline{Hom}(\mathbf{B}, \mathbf{C})). \tag{2.1}$$

In particular, there is a natural bijection between the objects of the two dg categories above, which shows that the functor $- \otimes \mathbf{B} \colon \mathbf{dgCat} \to \mathbf{dgCat}$ is left adjoint to $\underline{Hom}(\mathbf{B}, -) \colon \mathbf{dgCat} \to \mathbf{dgCat}$. Hence the symmetric monoidal structure on **dgCat** discussed in Example 2.1 (iv) is closed.

For a dg category $\mathbf{A}$, we set $\mathrm{dgMod}(\mathbf{A}) := \underline{Hom}(\mathbf{A}^\circ, \mathrm{C}_{\mathrm{dg}}(\Bbbk))$. The objects in $\mathrm{dgMod}(\mathbf{A})$ are called $\mathbf{A}$-*dg modules*. Denote by $\mathrm{h\text{-}proj}(\mathbf{A})$ the full dg subcategory of $\mathrm{dgMod}(\mathbf{A})$ with objects the *h-projective* $\mathbf{A}$-*dg modules*. Recall that $M \in \mathrm{dgMod}(\mathbf{A})$ is h-projective if $H^0(\mathrm{dgMod}(\mathbf{A}))(M, N) = 0$, for all $N \in \mathrm{dgMod}(\mathbf{A})$ which are acyclic (meaning that $N(A)$ is an acyclic complex, for all $A \in \mathbf{A}$).

Remark 2.2. For every dg category $\mathbf{A}$, both $\mathrm{dgMod}(\mathbf{A})$ and $\mathrm{h\text{-}proj}(\mathbf{A})$ are pretriangulated dg categories, and their homotopy categories are closed under arbitrary direct sums (see [7, Sect. 2.2]).

At the same time we denote by $\mathrm{Perf}(\mathbf{A})$ the full dg subcategory of $\mathrm{h\text{-}proj}(\mathbf{A})$ consisting of *perfect* $\mathbf{A}$-*dg modules*, i.e. the compact objects in the triangulated category $H^0(\mathrm{h\text{-}proj}(\mathbf{A}))$. Recall that an object C in a triangulated category $\mathbf{D}$ is *compact* if, given $\{D_i\}_{i \in I} \subset \mathbf{D}$ such that I is a set and $\bigoplus_i D_i$ exists in $\mathbf{D}$, the canonical map $\bigoplus_i \mathbf{D}(C, D_i) \longrightarrow \mathbf{D}(C, \oplus_i D_i)$ is an isomorphism. Moreover, a set of compact objects $\{C_j\}_{j \in J} \subset \mathbf{D}$ is a set of *compact generators* for $\mathbf{D}$ if, given $D \in \mathbf{D}$ with $\mathbf{D}(C_j, D[i]) = 0$ for all $j \in J$ and all $i \in \mathbb{Z}$, then $D \cong 0$.

The Yoneda embedding of $\mathbf{A}$ is the fully faithful (and injective on objects) dg functor $\mathsf{Y_A} \colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{A})$ defined on objects by $\mathsf{Y_A}(A) := \mathbf{A}(-, A)$. The image of $\mathsf{Y_A}$ is always contained in h-proj($\mathbf{A}$). We denote by $\overline{\mathbf{A}} \subseteq \mathrm{h\text{-}proj}(\mathbf{A})$ the full dg subcategory of $\mathrm{dgMod}(\mathbf{A})$ with objects the dg modules which are homotopy equivalent to objects in the image of $\mathsf{Y_A}$. Notice that $\mathsf{Y_A}$ factors through the dg category Perf($\overline{\mathbf{A}}$) which, in turn, contains $\overline{\mathbf{A}}$ (see, for example, [8, Sect. 3.5]).

Recall that a natural transformation θ between two $\mathbf{A}$-dg modules M and N is a *quasi-isomorphism* if it is closed of degree 0 and $\theta(A) \colon M(A) \to N(A)$ is a quasi-isomorphism, for every $A \in \mathbf{A}$. It can be proved that for every $M \in \mathrm{dgMod}(\mathbf{A})$ there exists an *h-projective resolution* of M, namely a quasi-isomorphism $N \to M$ with $N \in \mathrm{h\text{-}proj}(\mathbf{A})$ (see [7, Sect. 3.1] and [6, Sect. 14.8]). Moreover, a quasi-isomorphism between two h-projective dg modules is a homotopy equivalence (see, for example, [9, Thm. 3.4]).

A dg functor $\mathsf{F} \colon \mathbf{A} \to \mathbf{B}$ induces a functor $H^0(\mathsf{F}) \colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$, which is exact (between triangulated categories) if $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated. A dg functor $\mathsf{F} \colon \mathbf{A} \to \mathbf{B}$ is a *quasi-equivalence*, if the maps $\mathbf{A}(A, B) \to \mathbf{B}(\mathsf{F}(A), \mathsf{F}(B))$ are quasi-isomorphisms, for every $A, B \in \mathbf{A}$, and $H^0(\mathsf{F})$ is an equivalence of categories. One can consider the localization $\mathbf{Hqe}$ of $\mathbf{dgCat}$ with respect to quasi-equivalences (which is denoted by $Ho(\mathbf{dgCat})$ in [18]).

For a dg functor F, we write $[\mathsf{F}]$ for its image in $\mathbf{Hqe}$ and, given two dg categories $\mathbf{A}$ and $\mathbf{B}$, we denote by $[\mathbf{A}, \mathbf{B}]$ the morphisms in $\mathbf{Hqe}$ between $\mathbf{A}$ and $\mathbf{B}$. Notice that any $f \in [\mathbf{A}, \mathbf{B}]$ induces a ($\Bbbk$-linear) functor $H^0(f) \colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$, well defined up to isomorphism. We say that $H^0(f)$ is *continuous* if it commutes with arbitrary direct sums in $H^0(\mathbf{A})$. For simplicity, we sometimes say that f is continuous if $H^0(f)$ is. We denote by $[\mathbf{A}, \mathbf{B}]_c$ the set of continuous morphisms in $[\mathbf{A}, \mathbf{B}]$.

Definition 2.3. Let $\mathsf{G_1}, \mathsf{G_2} \colon \mathbf{A} \to \mathbf{B}$ be two dg functors. A natural transformation $\theta \colon \mathsf{G_1} \to \mathsf{G_2}$ is a *termwise homotopy equivalence* if it is closed of degree 0 and $\theta(A) \colon \mathsf{G_1}(A) \to \mathsf{G_2}(A)$ is a homotopy equivalence, for all $A \in \mathbf{A}$.

Remark 2.4. First of all, one observes that the natural transformation in Definition 2.3 induces a natural transformation θ' between $H^0(\mathsf{G_1})$ and $H^0(\mathsf{G_2})$ and θ is a termwise homotopy equivalence if and only if θ' is an isomorphism. If $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated dg categories, the functors $H^0(\mathsf{G_1})$ and $H^0(\mathsf{G_2})$ are exact and the check that θ is a termwise homotopy equivalence is just a question involving standard techniques in the theory of triangulated categories.

More precisely, there is indeed a general principle that will be applied later on. Namely, assume that α is a natural transformation between two exact continuous functors $\mathsf{F_1}, \mathsf{F_2} \colon \mathbf{D} \to \mathbf{D}'$, where $\mathbf{D}$ and $\mathbf{D}'$ are triangulated categories with arbitrary direct sums. Let $\mathbf{D_1}$ be a full triangulated subcategory of $\mathbf{D}$ consisting of compact generators of $\mathbf{D}$. Suppose further that, for all $D \in \mathbf{D_1}$, we have that $\alpha(D)$ is an isomorphism. Then one observes that the full subcategory $\mathbf{D_2}$ of $\mathbf{D}$ consisting of all objects D such that $\alpha(D)$ is an isomorphism is obviously triangulated and contains $\mathbf{D_1}$. On the other hand,

it is easy to see that α is automatically compatible with arbitrary direct sums, since F_1 and F_2 are continuous. Hence $\mathbf{D}_2$ is closed under arbitrary direct sums and so $\mathbf{D}_2 = \mathbf{D}$ (see [14]), which proves that α is an isomorphism.

Tensor product of dg modules

Following [6], if $\mathbf{A}$ is a dg category, the *tensor product* of $M \in \mathrm{dgMod}(\mathbf{A})$ and $N \in \mathrm{dgMod}(\mathbf{A}^\circ)$ is defined as

$$M \otimes_{\mathbf{A}} N := \mathrm{coker}\left(\Xi \colon \bigoplus_{A,B \in \mathbf{A}} M(B) \otimes_{\Bbbk} \mathbf{A}(A,B) \otimes_{\Bbbk} N(A) \longrightarrow \bigoplus_{C \in \mathbf{A}} M(C) \otimes_{\Bbbk} N(C) \right), \tag{2.2}$$

where, given $v_1 \in M(B)$ homogeneous of degree m, $f \colon A \to B$, homogeneous of degree n, and $v_2 \in N(A)$, we have

$$\Xi((v_1, f, v_2)) := M(f)(v_1) \otimes v_2 - (-1)^{mn} v_1 \otimes N(f)(v_2)$$
$$\in M(A) \otimes_{\Bbbk} N(A) \oplus M(B) \otimes_{\Bbbk} N(B). \tag{2.3}$$

In this version, $M \otimes_{\mathbf{A}} N$ is a complex of $\Bbbk$-modules, hence an object of $\mathrm{C}_{\mathrm{dg}}(\Bbbk) \cong \mathrm{dgMod}(\Bbbk)$. Clearly, one can repeat the same definition taking $M \in \mathrm{dgMod}(\mathbf{A} \otimes \mathbf{B})$ and $N \in \mathrm{dgMod}(\mathbf{B}^\circ \otimes \mathbf{C})$. In this case, $M \otimes_{\mathbf{B}} N$ is an object in $\mathrm{dgMod}(\mathbf{A} \otimes \mathbf{C})$.

Remark 2.5. (i) The definition in (2.2) is dg functorial. More precisely, assume we have $M_1, M_2 \in \mathrm{dgMod}(\mathbf{A} \otimes \mathbf{B})$, $N \in \mathrm{dgMod}(\mathbf{B}^\circ \otimes \mathbf{C})$ and a natural transformation $f \colon M_1 \to M_2$ of dg functors. Then it is straightforward that this induces a natural transformation $M_1 \otimes_{\mathbf{B}} N \to M_2 \otimes_{\mathbf{B}} N$ of dg functors.

(ii) Given $M \in \mathrm{dgMod}(\mathbf{A})$ and $N \in \mathrm{dgMod}(\mathbf{B})$, we can think of them as objects in $\mathrm{dgMod}(\mathbf{A} \otimes \Bbbk)$ and $N \in \mathrm{dgMod}(\Bbbk^\circ \otimes \mathbf{B})$. Now take the object $M \otimes_{\Bbbk} N \in \mathrm{dgMod}(\mathbf{A} \otimes \mathbf{B})$. We claim that, for all $(A, B) \in \mathbf{A} \otimes \mathbf{B}$, we get $(M \otimes_{\Bbbk} N)((A, B)) = M(A) \otimes_{\Bbbk} N(B)$, where the tensor product on the right-hand side is the usual tensor product of complexes of $\Bbbk$-modules. Indeed, as $\Bbbk$ consists of only one object and every morphism is a scalar multiple of the identity, the map Ξ in (2.3) is trivial in this case, and thus we get the result from (2.2). In the rest of the paper, we write $M \otimes N$ for $M \otimes_{\Bbbk} N$.

Remark 2.6. The tensor product of dg modules over a dg category looks very much like the ordinary tensor product of modules over a (not necessarily commutative) ring (see, for example, [12, Sect. VI.7] for the latter case). In particular, they share many properties, such as the associativity. Of course they can be proved directly using the definition. The easy computations are left to the reader, who, on the other hand, can have a look at [13, Sect. 6] for the proof of associativity in the case of tensor product of modules over a $\mathbb{Z}$-linear category.

For a dg category $\mathbf{A}$, an $\mathbf{A}$-dg module M is *h-flat* if, for any $N \in \mathrm{dgMod}(\mathbf{A}^\circ)$ which is acyclic, the tensor product $M \otimes_{\mathbf{A}} N$ is acyclic. One can check that any h-projective dg module is h-flat and that a dg module which is homotopy equivalent to an h-flat dg module is h-flat itself (see [9, Sect. 3.5], for some more details).

Derived tensor product of dg categories

We say that a dg category $\mathbf{A}$ is *h-projective* if $\mathbf{A}(A, B)$ is in h-proj($\Bbbk$), for all $A, B \in \mathbf{A}$. It is clear that if $\mathbf{A}$ if h-projective, then $\mathbf{A}^\circ$ is h-projective as well and all dg categories are h-projective if $\Bbbk$ is a field. It is also easy to see that if $\mathbf{A}$ and $\mathbf{B}$ are h-projective, then $\mathbf{A} \otimes \mathbf{B}$ is h-projective, too. We denote by $\mathbf{hp\text{-}dgCat}$ the full subcategory of $\mathbf{dgCat}$ consisting of h-projective dg categories.

Remark 2.7. Using [6, Sect. 13.5] (or the fact that $\mathbf{dgCat}$ is a model category [16]), for any dg category $\mathbf{A}$, one can construct an explicit h-projective dg category $\mathbf{A}^{\mathrm{hp}}$ with a quasi-equivalence $\mathsf{Q_A} \colon \mathbf{A}^{\mathrm{hp}} \to \mathbf{A}$, which will be fixed once and for all. In particular, if $\mathbf{A}$ is h-projective we assume that $\mathsf{Q_A} = \mathrm{id}_{\mathbf{A}}$. If not, following [6], $\mathbf{A}^{\mathrm{hp}}$ can be constructed as a semi-free resolution of $\mathbf{A}$, namely a semi-free dg category $\mathbf{B}$ with a quasi-equivalence $\mathbf{B} \to \mathbf{A}$. Although it is not needed in the rest of the paper, let us briefly recall that, following [6, Sect. 13.4], a dg category $\mathbf{A}$ is *semi-free* if it can be represented as the union of an increasing sequence of dg subcategories $\mathbf{A}_i$, where $i \in \mathbb{N}$, such that $\mathbf{A}_0$ is a discrete dg category and, for $i > 0$, each $\mathbf{A}_i$ is freely generated over $\mathbf{A}_{i-1}$, as a graded category, by a family of homogeneous morphisms f_j whose differentials $d(f_j)$ are morphisms in $\mathbf{A}_{i-1}$. It is then a simple calculation to see that $\mathbf{A}^{\mathrm{hp}}(A, B)$ is h-projective for all $A, B \in \mathbf{A}^{\mathrm{hp}}$ (one can also combine [6, Lemma 13.6], [18, Prop. 2.3 (3)] and the fact that a cofibrant complex of $\Bbbk$-modules is h-projective).

Denoting by $\mathbf{hp\text{-}Hqe}$ the localization of $\mathbf{hp\text{-}dgCat}$ by all quasi-equivalences, it is then easy to verify that the natural functor $\mathbf{hp\text{-}Hqe} \to \mathbf{Hqe}$ is an equivalence (one can use [6, Lemma 13.5] for the faithfulness).

Hence, given two dg categories $\mathbf{A}$ and $\mathbf{B}$, we define the *derived tensor product* as

$$\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B} := \mathbf{A}^{\mathrm{hp}} \otimes \mathbf{B}.$$

Remark 2.8. Given an h-projective dg category $\mathbf{A}$ and a quasi-equivalence $\mathsf{F} \colon \mathbf{B} \to \mathbf{B}'$, the induced dg functor $\mathsf{G} := \mathrm{id}_{\mathbf{A}} \otimes \mathsf{F} \colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{A} \otimes \mathbf{B}'$ is again a quasi-equivalence. Indeed, the complex $\mathbf{A}(A_1, A_2)$ is h-flat (being h-projective), for all $A_1, A_2 \in \mathbf{A}$, and thus the maps

$$\mathbf{A} \otimes \mathbf{B}((A_1, B_1), (A_2, B_2)) \to \mathbf{A} \otimes \mathbf{B}'(\mathsf{G}((A_1, B_1)), \mathsf{G}((A_2, B_2)))$$

are quasi-isomorphisms, for all $A_1, A_2 \in \mathbf{A}$ and all $B_1, B_2 \in \mathbf{B}$, since the maps $\mathbf{B}(B_1, B_2) \to \mathbf{B}'(\mathsf{F}(B_1), \mathsf{F}(B_2))$ are quasi-isomorphisms. Hence it remains to show that $H^0(\mathrm{id}_{\mathbf{A}} \otimes \mathsf{F})$ is essentially surjective. This is clear from the definition of tensor product of dg categories, being $H^0(\mathsf{F})$ essentially surjective.

It follows from the universal property of the localization of $\mathbf{dgCat}$ by quasi-equivalences that the functor $\mathbf{A} \otimes -\colon \mathbf{dgCat} \to \mathbf{dgCat}$ naturally induces a functor $\mathbf{A} \otimes -\colon \mathbf{Hqe} \to \mathbf{Hqe}$.

Similarly, given a dg category $\mathbf{B}$ and a quasi-equivalence $\mathsf{G}\colon \mathbf{A} \to \mathbf{A}'$, with $\mathbf{A}$ and $\mathbf{A}'$ h-projective, the induced dg functor $\mathsf{G} \otimes \mathrm{id}_{\mathbf{B}}\colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{A}' \otimes \mathbf{B}$ is again a quasi-equivalence. Hence the functor $- \otimes \mathbf{B}\colon \mathbf{hp\text{-}dgCat} \to \mathbf{dgCat}$ naturally induces a functor $- \otimes \mathbf{B}\colon \mathbf{hp\text{-}Hqe} \to \mathbf{Hqe}$.

Putting Remarks 2.7 and 2.8 together, we get a well-defined functor $- \otimes^{\mathbf{L}} -\colon \mathbf{Hqe} \times \mathbf{Hqe} \longrightarrow \mathbf{Hqe}$ which endows $\mathbf{Hqe}$ with a symmetric monoidal structure.

2.2. Some properties of morphisms in $\mathbf{Hqe}$

The content of this section is probably well known to experts. Moreover, most of the properties of $\mathbf{dgCat}$ and $\mathbf{Hqe}$ mentioned here have trivial proofs when we regard $\mathbf{dgCat}$ as a model category (see [16]). Nonetheless, to achieve a proof of Theorem 1.1 much less is needed and we sketch in this section the minimal amount of information which is required. Trying to keep the paper as much self-contained as possible, we outline the proofs of the results of this section.

Recall from [3, Sect. 1] the notion of *category of fibrant objects*. Let $\mathbf{C}$ be a category with finite products and assume that $\mathbf{C}$ has two distinguished classes of morphisms, called weak equivalences and fibrations. A morphism which is both a weak equivalence and a fibration will be called a trivial fibration. A path object for $C \in \mathbf{C}$ is an object C^I of $\mathbf{C}$ together with a weak equivalence $C \to C^I$ and a fibration $C^I \to C \times C$ whose composition is the diagonal $C \to C \times C$. We say that $\mathbf{C}$, together with its weak equivalences and fibrations, is a category of fibrant objects if the following axioms are satisfied:

(A) Let f and g be morphisms of $\mathbf{C}$ such that $g \circ f$ is defined. If two of the morphisms f, g and $g \circ f$ are weak equivalences, then so is the third. Any isomorphism is a weak equivalence.
(B) The composition of two fibrations is a fibration. Any isomorphism is a fibration.
(C) Fibrations and trivial fibrations are preserved by base extension.
(D) For every $C \in \mathbf{C}$ there exists at least one path object C^I.
(E) For every $C \in \mathbf{C}$ the morphism from C to a terminal object is a fibration.

Notice that if $\mathbf{C}$ is a model category, then the full subcategory of $\mathbf{C}$ consisting of all fibrant objects is a category of fibrant objects.

As in [3, Sect. 2], we say that two morphisms $f_0, f_1 \colon C \to D$ in a category of fibrant objects are *homotopic* if there exist a path object $D \to D^I \xrightarrow{(p_0, p_1)} D \times D$ and a morphism $h \colon C \to D^I$ (called *homotopy*) such that $f_i = p_i \circ h$, for $i = 0, 1$.

Remark 2.9. If $g \colon B \to C$ is a weak equivalence and $f_0, f_1 \colon C \to D$ are morphisms such that $f_0 \circ g$ and $f_1 \circ g$ are homotopic, it follows from [3, Prop. 1] that for any path object D^I for D there exists a weak equivalence $g' \colon B' \to C$ such that $f_0 \circ g'$ and $f_1 \circ g'$ are homotopic by a homotopy $B' \to D^I$.

Now we want to prove that **dgCat** is a category of fibrant objects, if one takes as weak equivalences the quasi-equivalences and as fibrations the full dg functors whose H^0 is an isofibration (a functor $\mathsf{F} \colon \mathbf{C} \to \mathbf{D}$ is an isofibration if for every $C \in \mathbf{C}$ and every isomorphism $f \colon \mathsf{F}(C) \xrightarrow{\sim} D$ in $\mathbf{D}$ there exists an isomorphism $g \colon C \xrightarrow{\sim} C'$ in $\mathbf{C}$ such that $\mathsf{F}(g) = f$). To this purpose, we need to fix some notation.

As in [6, Sect. 2.9], for every dg category $\mathbf{A}$ we denote by $\mathrm{Mor}(\mathbf{A})$ the dg category whose objects are triples (A, B, f) with $f \in Z^0(\mathbf{A}(A, B))$, and whose morphisms are given by

$$\mathrm{Mor}(\mathbf{A})((A, B, f), (A', B', f'))^n := \mathbf{A}(A, A')^n \oplus \mathbf{A}(B, B')^n \oplus \mathbf{A}(A, B')^{n-1}$$

for every $n \in \mathbb{Z}$. If $(a, b, h) \in \mathrm{Mor}(\mathbf{A})((A, B, f), (A', B', f'))^n$, the differential is defined by

$$d(a, b, h) := (d(a), d(b), d(h) + (-1)^n(f' \circ a - b \circ f))$$

and the composition by $(a', b', h') \circ (a, b, h) := (a' \circ a, b' \circ b, b' \circ h + (-1)^n h' \circ a)$. Actually we will be interested in the full dg subcategory $P(\mathbf{A})$ of $\mathrm{Mor}(\mathbf{A})$ with objects the triples (A, B, f) such that f is a homotopy equivalence (see [17, Sect. 3]).

Notice that there is a natural dg functor $\mathsf{I}_{\mathbf{A}} \colon \mathbf{A} \to P(\mathbf{A})$, defined on objects by $A \mapsto (A, A, \mathrm{id}_A)$ and on morphisms by $f \mapsto (f, f, 0)$. Similarly, there are obvious dg functors $\mathsf{S}_{\mathbf{A}}, \mathsf{T}_{\mathbf{A}} \colon P(\mathbf{A}) \to \mathbf{A}$ ("source" and "target") defined both on objects and on morphisms as the projection respectively on the first and on the second component.

Lemma 2.10. *With the above defined weak equivalences and fibrations, **dgCat** is a category of fibrant objects.*

Proof. The result depends on some elementary (but tedious) checks which are left to the reader. We simply outline the main ingredients in the proof.

First of all observe that finite products exist in **dgCat**, and they are given by the corresponding products both on objects and on morphisms (with differentials and compositions defined componentwise). In particular, a terminal object is the dg category with one object and 0 as the space of morphisms.

Axioms (A), (B) and (E) are straightforward to check from the definitions. As for axiom (C), note that for every dg functors $\mathsf{F} \colon \mathbf{A} \to \mathbf{C}$ and $\mathsf{G} \colon \mathbf{B} \to \mathbf{C}$ the fibre product $\mathbf{D} := \mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ along F and G exists in $\mathbf{dgCat}$, and it is given by the full subcategory of $\mathbf{A} \times \mathbf{B}$ with objects those $(A, B) \in \mathbf{A} \times \mathbf{B}$ such that $\mathsf{F}(A) = \mathsf{G}(B)$ and morphisms those morphisms (f, g) of $\mathbf{A} \times \mathbf{B}$ such that $\mathsf{F}(f) = \mathsf{G}(g)$. It is easy to show that, if G is a fibration (resp. a trivial fibration), then the projection dg functor $\mathbf{D} \to \mathbf{A}$ is a fibration (resp. a trivial fibration), too.

Passing to axiom (D), one shows that the dg functors $\mathbf{A} \xrightarrow{\mathsf{I_A}} P(\mathbf{A}) \xrightarrow{(\mathsf{S_A}, \mathsf{T_A})} \mathbf{A} \times \mathbf{A}$ define a path object for any dg category $\mathbf{A}$. More precisely, as the composition is clearly the diagonal $\mathbf{A} \to \mathbf{A} \times \mathbf{A}$, one just shows that $\mathsf{I_A}$ is a quasi-equivalence and $(\mathsf{S_A}, \mathsf{T_A})$ is a fibration. $\square$

Two dg functors $\mathsf{F}, \mathsf{G} \colon \mathbf{A} \to \mathbf{B}$ will be called *standard homotopic* if there exists a dg functor $\mathsf{H} \colon \mathbf{A} \to P(\mathbf{B})$ such that $\mathsf{F} = \mathsf{S_B} \circ \mathsf{H}$ and $\mathsf{G} = \mathsf{T_B} \circ \mathsf{H}$. From the results proved in [3, Sect. 2] for the localization of an arbitrary category of fibrant objects with respect to weak equivalences, we obtain the following properties of morphisms in $\mathbf{Hqe}$.

Proposition 2.11.

(1) *Given two dg functors $\mathsf{F}, \mathsf{G} \colon \mathbf{A} \to \mathbf{B}$, we have $[\mathsf{F}] = [\mathsf{G}]$ if and only if there exists a quasi-equivalence $\mathsf{I} \colon \mathbf{A}' \to \mathbf{A}$ such that $\mathsf{F} \circ \mathsf{I}$ and $\mathsf{G} \circ \mathsf{I}$ are (standard) homotopic.*
(2) *Given two dg functors $\mathsf{F} \colon \mathbf{A} \to \mathbf{C}$ and $\mathsf{G} \colon \mathbf{B} \to \mathbf{C}$ with G a quasi-equivalence, there exist dg functors $\mathsf{F}' \colon \mathbf{D} \to \mathbf{B}$ and $\mathsf{G}' \colon \mathbf{D} \to \mathbf{A}$ with G' a quasi-equivalence such that $\mathsf{F} \circ \mathsf{G}'$ and $\mathsf{G} \circ \mathsf{F}'$ are homotopic (hence $[\mathsf{F} \circ \mathsf{G}'] = [\mathsf{G} \circ \mathsf{F}']$ by part (1)). If moreover F is a quasi-equivalence, then F' is a quasi-equivalence, too.*
(3) *Given morphisms $f_i \colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{Hqe}$, for $i = 1, \ldots, n$, there exist dg functors $\mathsf{I} \colon \mathbf{A}' \to \mathbf{A}$ and $\mathsf{F}_i \colon \mathbf{A}' \to \mathbf{B}$ with I a quasi-equivalence such that $f_i = [\mathsf{F}_i] \circ [\mathsf{I}]^{-1}$, for $i = 1, \ldots, n$.*

Proof. Taking into account Remark 2.9, (1) follows from part (ii) of [3, Thm. 1]. The first part of (2) follows from [3, Prop. 2]. The last statement in (2) is then an easy consequence of axiom (A), using the fact that a morphism homotopic to a weak equivalence is also a weak equivalence (to see this, one uses again axiom (A)). For $n = 1$, (3) follows from part (i) of [3, Thm. 1]. In general, one can easily reduce by induction to $n = 2$. Then, by the case $n = 1$, for $i = 1, 2$ there exist dg functors $\mathsf{I}_i \colon \mathbf{A}_i \to \mathbf{A}$ and $\mathsf{G}_i \colon \mathbf{A}_i \to \mathbf{B}$, with I_i a quasi-equivalence such that $f_i = [\mathsf{G}_i] \circ [\mathsf{I}_i]^{-1}$. On the other hand, by part (2), there exist two quasi-equivalences $\mathsf{J}_i \colon \mathbf{A}' \to \mathbf{A}_i$ such that $[\mathsf{I}_1] \circ [\mathsf{J}_1] = [\mathsf{I}_2] \circ [\mathsf{J}_2]$. Set $\mathsf{F}_i := \mathsf{G}_i \circ \mathsf{J}_i$ and $\mathsf{I} := \mathsf{I}_1 \circ \mathsf{J}_1$. It is then clear that $f_i = [\mathsf{F}_i] \circ [\mathsf{I}]^{-1}$, for $i = 1, 2$. $\square$

Corollary 2.12. *Let $\mathsf{F}, \mathsf{G} \colon \mathbf{A} \to \mathbf{B}$ be dg functors and assume that there exists a termwise homotopy equivalence $\alpha \colon \mathsf{F} \to \mathsf{G}$ (in particular, this is the case if F and G are homotopy equivalent). Then $[\mathsf{F}] = [\mathsf{G}] \in \mathbf{Hqe}(\mathbf{A}, \mathbf{B})$.*

Proof. The assumption on α implies that there is a dg functor $\mathsf{H}\colon \mathbf{A} \to P(\mathbf{B})$ defined on objects by $A \mapsto (\mathsf{F}(A), \mathsf{G}(A), \alpha(A))$ and on morphisms by $a \mapsto (\mathsf{F}(a), \mathsf{G}(a), 0)$. As H clearly gives a standard homotopy between F and G, we conclude that $[\mathsf{F}] = [\mathsf{G}]$ by part (1) of Proposition 2.11. $\square$

3. Extensions of morphisms in Hqe and bimodules

In this section we develop the key ingredients in our proof of Theorem 1.1. As it turns out, they rely on some natural properties of extension of dg functors. Finally, we provide an interpretation of the morphisms in **Hqe** in terms of dg modules over tensor dg categories.

3.1. Extensions of dg functors

Given two dg categories $\mathbf{A}$ and $\mathbf{B}$, by (2.1) there is an isomorphism of dg categories $\mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B}) \cong \underline{Hom}(\mathbf{A}, \mathrm{dgMod}(\mathbf{B}))$, so in particular an object $E \in \mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B})$ corresponds to a dg functor $\Phi_E\colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{B})$. Conversely, for every dg functor $\mathsf{F}\colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{B})$ there exists a unique $E \in \mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B})$ such that $\Phi_E = \mathsf{F}$. An object $E \in \mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes \mathbf{B})$ is called *right quasi-representable* if $\Phi_E(\mathbf{A}) \subset \overline{\mathbf{B}}$. The full dg subcategory of $\mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes \mathbf{B})$ consisting of all right quasi-representable dg modules will be denoted by $\mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$. Notice that $\mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ is always closed under homotopy equivalences and, if $\mathbf{A}$ is the dg category $\Bbbk$, then it is isomorphic to $\overline{\mathbf{B}}$.

Let $\mathsf{F}\colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{B})$ be a dg functor corresponding to $E \in \mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B})$. Following [7, Sect. 6.1], we define the *extension* of F to be the dg functor

$$\widehat{\mathsf{F}}\colon \mathrm{dgMod}(\mathbf{A}) \longrightarrow \mathrm{dgMod}(\mathbf{B}) \qquad \widehat{\mathsf{F}}(-) := - \otimes_{\mathbf{A}} E.$$

Notice that the definition of $\widehat{\mathsf{F}}$ is a reformulation of the usual notion of Kan extension in the context of dg functors. There is also a natural dg functor

$$\widetilde{\mathsf{F}}\colon \mathrm{dgMod}(\mathbf{B}) \to \mathrm{dgMod}(\mathbf{A}) \qquad \widetilde{\mathsf{F}}(M) := \mathrm{dgMod}(\mathbf{B})(\mathsf{F}(-), M),$$

for every $M \in \mathrm{dgMod}(\mathbf{B})$.

Remark 3.1. Notice that, in [7], the dg functors $\widehat{\mathsf{F}}$ and $\widetilde{\mathsf{F}}$ above are denoted by T_E and H_E respectively.

If $\mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ is a dg functor, $\widehat{\mathsf{Y}_{\mathbf{B}} \circ \mathsf{G}}$ is usually denoted by $\mathrm{Ind}_{\mathsf{G}}\colon \mathrm{dgMod}(\mathbf{A}) \to \mathrm{dgMod}(\mathbf{B})$, whereas (due to the dg version of Yoneda's lemma) $\widetilde{\mathsf{Y}_{\mathbf{B}} \circ \mathsf{G}}$ is dg isomorphic to the dg functor $\mathrm{Res}_{\mathsf{G}}\colon \mathrm{dgMod}(\mathbf{B}) \to \mathrm{dgMod}(\mathbf{A})$ defined by $\mathrm{Res}_{\mathsf{G}}(M) := M(\mathsf{G}(-))$.

Proposition 3.2. *Let* $\mathsf{F}\colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{B})$ *and* $\mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ *be dg functors.*

(1) $\widehat{\mathsf{F}}$ *is left adjoint to* $\widetilde{\mathsf{F}}$ *(hence* $\mathrm{Ind}_{\mathsf{G}}$ *is left adjoint to* $\mathrm{Res}_{\mathsf{G}}$*).*
(2) $\widehat{\mathsf{F}} \circ \mathsf{Y}_{\mathbf{A}}$ *is dg isomorphic to* F *and* $H^0(\widehat{\mathsf{F}})$ *is continuous (hence* $\mathrm{Ind}_{\mathsf{G}} \circ \mathsf{Y}_{\mathbf{A}}$ *is dg isomorphic to* $\mathsf{Y}_{\mathbf{B}} \circ \mathsf{G}$ *and* $H^0(\mathrm{Ind}_{\mathsf{G}})$ *is continuous).*
(3) $\widehat{\mathsf{F}}(\mathrm{h\text{-}proj}(\mathbf{A})) \subseteq \mathrm{h\text{-}proj}(\mathbf{B})$ *if and only if* $\mathsf{F}(\mathbf{A}) \subseteq \mathrm{h\text{-}proj}(\mathbf{B})$ *(hence* $\mathrm{Ind}_{\mathsf{G}}(\mathrm{h\text{-}proj}(\mathbf{A}))$ $\subseteq \mathrm{h\text{-}proj}(\mathbf{B})$*).*
(4) $\mathrm{Res}_{\mathsf{G}}(\mathrm{h\text{-}proj}(\mathbf{B})) \subseteq \mathrm{h\text{-}proj}(\mathbf{A})$ *if and only if* $\mathrm{Res}_{\mathsf{G}}(\overline{\mathbf{B}}) \subseteq \mathrm{h\text{-}proj}(\mathbf{A})$*; moreover,* $H^0(\mathrm{Res}_{\mathsf{G}})$ *is always continuous.*
(5) $\mathrm{Ind}_{\mathsf{G}}\colon \mathrm{h\text{-}proj}(\mathbf{A}) \to \mathrm{h\text{-}proj}(\mathbf{B})$ *is a quasi-equivalence if* G *is a quasi-equivalence.*

Proof. All the statements above are probably well known (see [7]). Thus we simply sketch the main ingredients in the proofs. The proof of (1) uses exactly the same argument as in [6, Sect. 14.9] for the adjunction between $\mathrm{Ind}_{\mathsf{G}}$ and $\mathrm{Res}_{\mathsf{G}}$. The first part of (2) follows from Eq. (14.2) in [6], and $H^0(\widehat{\mathsf{F}})$ is continuous because it is left adjoint to $H^0(\widetilde{\mathsf{F}})$ by (1).

The non-trivial implication in (3) is a consequence of (2) and of $\mathsf{F}(\mathbf{A}) \subseteq \mathrm{h\text{-}proj}(\mathbf{B})$. Indeed, we use here that the objects of $\mathbf{A}$ form a set of compact generators for $\mathrm{h\text{-}proj}(\mathbf{A})$ (see [7, Sect. 4.2]) and that $H^0(\mathsf{F})$ is continuous by (2), arguing exactly as at the end of Remark 2.4.

A similar argument applies to the first part of (4). For the fact that $H^0(\mathrm{Res}_{\mathsf{G}})$ is continuous, we use that $\mathrm{Res}_{\mathsf{G}}$ has a right adjoint (see, for example, [10, Sect. 1]). Finally, (5) is observed in [6, Remark 4.3]. $\square$

Let $\mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^{\circ} \otimes \mathbf{B})$ be the full dg subcategory of $\mathrm{dgMod}(\mathbf{A}^{\circ} \otimes \mathbf{B})$ with objects E such that $\Phi_E(\mathbf{A}) \subseteq \mathrm{h\text{-}proj}(\mathbf{B})$. Denoting by $\Delta_{\mathbf{A}} \in \mathrm{dgMod}(\mathbf{A}^{\circ} \otimes \mathbf{A})$ the object such that $\Phi_{\Delta_{\mathbf{A}}} = \mathsf{Y}_A\colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{A})$, obviously $\Delta_{\mathbf{A}} \in \mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^{\circ} \otimes \mathbf{A})$.

Remark 3.3. If $\alpha\colon E \to E'$ is a quasi-isomorphism in $\mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^{\circ} \otimes \mathbf{B})$, then clearly $\Phi_\alpha\colon \Phi_E \to \Phi_{E'}$ is such that $\Phi_\alpha(A)$ is a quasi-isomorphism in $\mathrm{h\text{-}proj}(\mathbf{B})$, and hence a homotopy equivalence, for every $A \in \mathbf{A}$. In other words, Φ_α is a termwise homotopy equivalence.

Lemma 3.4. *The inclusion* $\mathrm{h\text{-}proj}(\mathbf{A}^{\circ} \otimes^{\mathbf{L}} \mathbf{B}) \subseteq \mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^{\circ} \otimes^{\mathbf{L}} \mathbf{B})$ *holds.*

Proof. First, we can assume, without loss of generality, that $\mathbf{A}$ is h-projective and work with $\mathbf{A}^{\circ} \otimes \mathbf{B}$. Then, for an object A of $\mathbf{A}$, consider the inclusion $\mathsf{I}_A\colon \mathbf{B} \to \mathbf{A}^{\circ} \otimes \mathbf{B}$ defined as $\mathsf{I}_A(B) := (A, B)$, for all B in $\mathbf{B}$. Now, given $E \in \mathrm{dgMod}(\mathbf{A}^{\circ} \otimes \mathbf{B})$ and A in $\mathbf{A}$, we have $\Phi_E(A) = \mathrm{Res}_{\mathsf{I}_A}(E)$. Thus, it is enough to observe that $\mathrm{Res}_{\mathsf{I}_A}(\mathrm{h\text{-}proj}(\mathbf{A}^{\circ} \otimes \mathbf{B})) \subseteq \mathrm{h\text{-}proj}(\mathbf{B})$. For this, one applies part (4) of Proposition 3.2, since $\mathrm{Res}_{\mathsf{I}_A}(\mathsf{Y}_{\mathbf{A}^{\circ} \otimes \mathbf{B}}(A', B)) \cong \mathbf{A}(A', A) \otimes \mathsf{Y}_{\mathbf{B}}(B) \in \mathrm{h\text{-}proj}(\mathbf{B})$ (thanks to the fact that $\mathbf{A}(A', A) \in \mathrm{h\text{-}proj}(\Bbbk)$, $\mathbf{A}$ being h-projective), for all (A', B) in $\mathbf{A}^{\circ} \otimes \mathbf{B}$. $\square$

Lemma 3.5. *The map $E \mapsto \widehat{\Phi}_E$ extends to a dg functor*

$$\mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B}) \longrightarrow \underline{Hom}(\mathrm{dgMod}(\mathbf{A}), \mathrm{dgMod}(\mathbf{B})),$$

which restricts to a dg functor $\mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^\circ \otimes \mathbf{B}) \to \underline{Hom}(\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B}))$.

Proof. The first part of the statement is a simple consequence of the fact that, by definition, the tensor product of dg modules is functorial (see Remark 2.5 (i)). For the second part, observe that, by definition, $\Phi_E(\mathbf{A}) \subseteq \text{h-proj}(\mathbf{B})$, when $E \in \mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^\circ \otimes \mathbf{B})$. Hence we apply part (3) of Proposition 3.2. $\quad\square$

Lemma 3.5 implies that, given a dg functor $\mathsf{F} \colon \mathbf{A} \to \text{h-proj}(\mathbf{B})$, we can think of the extension of F as a dg functor $\widehat{\mathsf{F}} \colon \text{h-proj}(\mathbf{A}) \to \text{h-proj}(\mathbf{B})$.

Lemma 3.6. *Given two dg functors* $\mathsf{F} \colon \mathbf{A} \to \mathrm{dgMod}(\mathbf{B})$ *and* $\mathsf{G} \colon \mathbf{B} \to \mathrm{dgMod}(\mathbf{C})$, *there is a dg isomorphism of dg functors*

$$\widehat{\mathsf{G} \circ \mathsf{F}} \cong \widehat{\mathsf{G}} \circ \widehat{\mathsf{F}} \colon \mathrm{dgMod}(\mathbf{A}) \to \mathrm{dgMod}(\mathbf{C}).$$

Moreover, if $\mathsf{F}' \colon \mathbf{A} \to \mathbf{B}$ *and* $\mathsf{G}' \colon \mathbf{B} \to \mathbf{C}$ *are two other dg functors, then there are also dg isomorphisms* $\widehat{\mathsf{G} \circ \mathsf{F}'} \cong \widehat{\mathsf{G}} \circ \mathrm{Ind}_{\mathsf{F}'}$ *and* $\widehat{\mathrm{Ind}_{\mathsf{G}'} \circ \mathsf{F}} \cong \mathrm{Ind}_{\mathsf{G}'} \circ \widehat{\mathsf{F}}$.

Proof. Let $F \in \mathrm{dgMod}(\mathbf{A}^\circ \otimes \mathbf{B})$ and $G \in \mathrm{dgMod}(\mathbf{B}^\circ \otimes \mathbf{C})$ be such that $\mathsf{F} = \Phi_F$ and $\mathsf{G} = \Phi_G$. Then $\widehat{\mathsf{G}} \circ \widehat{\mathsf{F}} \cong \widehat{\mathsf{H}}$, where $\mathsf{H} := \Phi_{F \otimes_{\mathbf{B}} G}$, by the associativity of the tensor product. Using part (2) of Proposition 3.2, it follows that

$$\mathsf{H} \cong \widehat{\mathsf{H}} \circ \mathsf{Y}_{\mathbf{A}} \cong \widehat{\mathsf{G}} \circ \widehat{\mathsf{F}} \circ \mathsf{Y}_{\mathbf{A}} \cong \widehat{\mathsf{G}} \circ \mathsf{F},$$

which proves the first part. The last statement then follows taking $\mathsf{F} = \mathsf{Y}_{\mathbf{B}} \circ \mathsf{F}'$ or $\mathsf{G} = \mathsf{Y}_{\mathbf{C}} \circ \mathsf{G}'$ and using again part (2) of Proposition 3.2. $\quad\square$

Lemma 3.7. *Given dg categories* $\mathbf{A}_i$, $\mathbf{B}_i$ *and objects* $E_i \in \mathrm{dgMod}(\mathbf{A}_i^\circ \otimes \mathbf{B}_i)$ *for* $i = 1, 2$, *the diagram*

$$
\begin{array}{ccc}
\mathbf{A}_1 \otimes \mathbf{A}_2 & \xrightarrow{\ \Phi_{E_1} \otimes \Phi_{E_2}\ } & \mathrm{dgMod}(\mathbf{B}_1) \otimes \mathrm{dgMod}(\mathbf{B}_2) \\
& {\scriptstyle \Phi_{E_1 \otimes E_2}} \searrow & \downarrow {\scriptstyle -\otimes-} \\
& & \mathrm{dgMod}(\mathbf{B}_1 \otimes \mathbf{B}_2)
\end{array}
$$

(where $E_1 \otimes E_2 \in \mathrm{dgMod}(\mathbf{A}_1^\circ \otimes \mathbf{B}_1 \otimes \mathbf{A}_2^\circ \otimes \mathbf{B}_2) \cong \mathrm{dgMod}((\mathbf{A}_1 \otimes \mathbf{A}_2)^\circ \otimes (\mathbf{B}_1 \otimes \mathbf{B}_2)))$ *commutes in* **dgCat** *up to dg isomorphism. Moreover,* $\widehat{\Phi}_{E_1 \otimes E_2}(-) \cong E_1 \otimes_{\mathbf{A}_1^\circ} - \otimes_{\mathbf{A}_2} E_2$.

Proof. The commutativity of the diagram follows directly from the fact that $\Phi_{E_i}(A) = E_i((A, -))$ and Remark 2.5 (ii). The second part amounts to showing that, for all $M \in$ dgMod$(\mathbf{A}_1 \otimes \mathbf{A}_2)$, we have the isomorphism $M \otimes_{\mathbf{A}_1 \otimes \mathbf{A}_2} (E_1 \otimes E_2) \cong E_1 \otimes_{\mathbf{A}_1^\circ} M \otimes_{\mathbf{A}_2} E_2$. This is an easy exercise using the definition (2.2). $\square$

Proposition 3.8. *Let* $\mathsf{F} \colon \mathbf{A} \to \mathbf{A}'$ *and* $\mathsf{G} \colon \mathbf{B} \to \mathbf{B}'$ *be dg functors with* $\mathbf{A}$ *and* $\mathbf{A}'$ *h-projective.*

(1) *The dg functor* F *induces a natural map* $\mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})^{\mathrm{rqr}})) \to \mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}))$; *if moreover* F *is a quasi-equivalence, then this map is bijective and* $\mathrm{Ind}_{\mathsf{F}^\circ \otimes \mathrm{id}_{\mathbf{B}}}$ *restricts to a quasi-equivalence* $\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}} \to$ $\text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$.

(2) *The dg functor* $\mathrm{Ind}_{\mathrm{id}_{\mathbf{A}^\circ} \otimes \mathsf{G}}$ *restricts to a dg functor* $\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}} \to \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}')^{\mathrm{rqr}}$, *which is a quasi-equivalence if* G *is such.*

Proof. As for (1), notice that, setting $\mathsf{F}_1 := \mathsf{F}^\circ \otimes \mathrm{id}_{\mathbf{B}}$, the dg functor $\mathrm{Ind}_{\mathsf{F}_1} \colon \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}) \to \text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})$ clearly induces a natural map $\mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}))) \to \mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})))$. On the other hand, one can also define a natural map $\mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B}))) \to \mathrm{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})))$ by $[E']_{\mathrm{iso}} \mapsto [E]_{\mathrm{iso}}$, where E is an h-projective resolution of $\mathrm{Res}_{\mathsf{F}_1}(E')$. It is not difficult to show that, if F (hence $\mathrm{Ind}_{\mathsf{F}_1}$, by Remark 2.8 and part (5) of Proposition 3.2) is a quasi-equivalence, then these two maps are bijective and inverse to each other (see, for example, [6, Sect. 14.12]). Therefore it is enough to prove that $E' \in \text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ implies $E \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$, and that the converse is true if F is a quasi-equivalence. To see this, observe that the quasi-isomorphism $E \to \mathrm{Res}_{\mathsf{F}_1}(E')$ induces, for every $A \in \mathbf{A}$, a quasi-isomorphism $\Phi_E(A) \to \Phi_{\mathrm{Res}_{\mathsf{F}_1}(E')}(A) = \Phi_{E'}(\mathsf{F}(A))$, which is in fact a homotopy equivalence (due to the fact that both the source and the target are in h-proj$(\mathbf{B})$ by Lemma 3.4). It follows that $\Phi_E(A) \in \overline{\mathbf{B}}$ if and only if $\Phi_{E'}(\mathsf{F}(A)) \in \overline{\mathbf{B}}$, which is enough to conclude. Indeed, by the definition of Φ_E, we have that $E \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ if and only if $\Phi_E(A) \in \overline{\mathbf{B}}$, for all $A \in \mathbf{A}$. On the other hand, it is clear that $\Phi_E(A) \in \overline{\mathbf{B}}$ because $E' \in \text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ and thus $\Phi_{E'}(\mathsf{F}(A)) \in \overline{\mathbf{B}}$, for all $A \in \mathbf{A}$. Clearly, if F is a quasi-equivalence, the same argument shows that $E' \in \text{h-proj}(\mathbf{A}'^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ if $E \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$.

As for (2), we just need to show that, setting $\mathsf{G}_1 := \mathrm{id}_{\mathbf{A}^\circ} \otimes \mathsf{G}$, the dg functor $\mathrm{Ind}_{\mathsf{G}_1} \colon \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}) \to \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}')$ sends h-proj$(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$ to h-proj$(\mathbf{A}^\circ \otimes \mathbf{B}')^{\mathrm{rqr}}$ (because then the second part of the statement can be proved with an argument which is completely similar to the one used in (1)). Given $E \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})$ and setting $E' := \mathrm{Ind}_{\mathsf{G}_1}(E) \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}')$, we claim that $\Phi_{E'} \cong \mathrm{Ind}_{\mathsf{G}} \circ \Phi_E$. Notice that this is enough to conclude that $E' \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}')^{\mathrm{rqr}}$ if $E \in \text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B})^{\mathrm{rqr}}$, as clearly $\mathrm{Ind}_{\mathsf{G}}(\overline{\mathbf{B}}) \subseteq \overline{\mathbf{B}'}$. Now, denoting by $G_1 \in \mathrm{dgMod}((\mathbf{A}^\circ \otimes \mathbf{B})^\circ \otimes (\mathbf{A}^\circ \otimes \mathbf{B}')) \cong \mathrm{dgMod}(\mathbf{A} \otimes \mathbf{A}^\circ \otimes \mathbf{B}^\circ \otimes \mathbf{B}')$ the dg module such that $\Phi_{G_1} = \mathsf{Y}_{\mathbf{A}^\circ \otimes \mathbf{B}'} \circ \mathsf{G}_1$, it is easy to see that $G_1 \cong \Delta_{\mathbf{A}^\circ} \otimes G$, where $G \in \mathrm{dgMod}(\mathbf{B}^\circ \otimes \mathbf{B}')$ denotes the dg module such that $\Phi_G = \mathsf{Y}_{\mathbf{B}'} \circ \mathsf{G}$. As $\mathrm{Ind}_{\mathsf{G}_1} = \widehat{\Phi_{G_1}}$, by Lemma 3.7 we get

$$E' = \widehat{\Phi_{G_1}}(E) \cong \widehat{\Phi_{\Delta_{\mathbf{A}\circ}\otimes G}}(E) \cong \Delta_{\mathbf{A}\circ} \otimes_{\mathbf{A}} E \otimes_{\mathbf{B}} G \cong E \otimes_{\mathbf{B}} G,$$
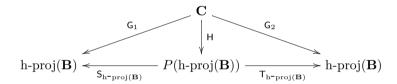
where the last isomorphism is due to [6, Sect. 14.6]. It follows from the associativity of the tensor product that $\widehat{\Phi_{E'}} \cong \widehat{\Phi_G} \circ \widehat{\Phi_E} = \mathrm{Ind}_{\mathsf{G}} \circ \widehat{\Phi_E}$, which implies that $\Phi_{E'} \cong \mathrm{Ind}_{\mathsf{G}} \circ \Phi_E$ by part (2) of Proposition 3.2. $\quad\square$

3.2. Extending morphisms in **Hqe**

Let **A** and **B** be dg categories.

Lemma 3.9. *If* $\mathsf{F}_1, \mathsf{F}_2 \colon \mathbf{A} \to \mathrm{h\text{-}proj}(\mathbf{B})$ *are dg functors such that* $[\mathsf{F}_1] = [\mathsf{F}_2]$, *then* $[\widehat{\mathsf{F}_1}] = [\widehat{\mathsf{F}_2}]$ *in* $[\mathrm{h\text{-}proj}(\mathbf{A}), \mathrm{h\text{-}proj}(\mathbf{B})]$.

Proof. As $[\mathsf{F}_1] = [\mathsf{F}_2]$, by part (1) of Proposition 2.11 there exists a quasi-equivalence $\mathsf{I} \colon \mathbf{C} \to \mathbf{A}$ such that $\mathsf{G}_i := \mathsf{F}_i \circ \mathsf{I}$ for $i = 1,2$ sit in the commutative diagram

$$
\begin{array}{ccccc}
 & & \mathbf{C} & & \\
 & \overset{\mathsf{G}_1}{\swarrow} & \downarrow{\scriptstyle\mathsf{H}} & \overset{\mathsf{G}_2}{\searrow} & \\
\mathrm{h\text{-}proj}(\mathbf{B}) & \underset{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{B})}}{\longleftarrow} & P(\mathrm{h\text{-}proj}(\mathbf{B})) & \underset{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{B})}}{\longrightarrow} & \mathrm{h\text{-}proj}(\mathbf{B})
\end{array}
$$

for some dg functor $\mathsf{H} \colon \mathbf{C} \to P(\mathrm{h\text{-}proj}(\mathbf{B}))$. Thus, by Lemma 3.6, we have

$$\widehat{\mathsf{G}_1} = \widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{B})}} \circ \mathsf{H} \cong \widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{B})}} \circ \mathrm{Ind}_{\mathsf{H}} \qquad \widehat{\mathsf{G}_2} = \widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{B})}} \circ \mathsf{H} \cong \widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{B})}} \circ \mathrm{Ind}_{\mathsf{H}}.$$
(3.1)

Now observe that, by definition, $\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{A})} \circ \mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})} = \mathrm{id}_{\mathrm{h\text{-}proj}(\mathbf{A})} = \mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{A})} \circ \mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}$. Hence

$$\widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{A})} \circ \mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}} \cong \widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{A})}} \circ \mathrm{Ind}_{\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}} \cong \widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{A})} \circ \mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}}$$
$$\cong \widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{A})}} \circ \mathrm{Ind}_{\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}}$$

where the first and the last isomorphisms are again due to Lemma 3.6. Then $[\widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{A})}}] \circ [\mathrm{Ind}_{\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}}] = [\widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{A})}}] \circ [\mathrm{Ind}_{\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}}]$. Since $\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}$ (and thus, by part (5) of Proposition 3.2, $\mathrm{Ind}_{\mathsf{I}_{\mathrm{h\text{-}proj}(\mathbf{A})}}$) is a quasi-equivalence, we get $[\widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{A})}}] = [\widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{A})}}]$.

Using this and (3.1), we obtain

$$[\widehat{\mathsf{G}_1}] = [\widehat{\mathsf{S}_{\mathrm{h\text{-}proj}(\mathbf{B})}}] \circ [\mathrm{Ind}_{\mathsf{H}}] = [\widehat{\mathsf{T}_{\mathrm{h\text{-}proj}(\mathbf{B})}}] \circ [\mathrm{Ind}_{\mathsf{H}}] = [\widehat{\mathsf{G}_2}].$$

Again by Lemma 3.6, we have $[\widehat{\mathsf{G}_i}] = [\widehat{\mathsf{F}_i \circ \mathsf{I}}] = [\widehat{\mathsf{F}_i} \circ \mathrm{Ind}_{\mathsf{I}}] = [\widehat{\mathsf{F}_i}] \circ [\mathrm{Ind}_{\mathsf{I}}]$, for $i = 1,2$. As $\mathrm{Ind}_{\mathsf{I}}$ is a quasi-equivalence by part (5) of Proposition 3.2, the identity $[\widehat{\mathsf{G}_1}] = [\widehat{\mathsf{G}_2}]$ implies $[\widehat{\mathsf{F}_1}] = [\widehat{\mathsf{F}_2}]$. $\quad\square$
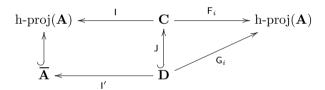
Proposition 3.10. *If* **A** *and* **B** *are dg categories, the natural map of sets*

$$[\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})]_c \longrightarrow [\mathbf{A}, \text{h-proj}(\mathbf{B})] \qquad f \mapsto f \circ [\mathsf{Y}_\mathbf{A}]$$

(where $\mathsf{Y}_\mathbf{A}$ *denotes the Yoneda embedding* $\mathbf{A} \to \text{h-proj}(\mathbf{A})$*) is a bijection.*

Proof. Given $f \in [\mathbf{A}, \text{h-proj}(\mathbf{B})]$, by part (3) of Proposition 2.11 there exist a quasi-equivalence $\mathsf{I} \colon \mathbf{C} \to \mathbf{A}$ and a dg functor $\mathsf{F} \colon \mathbf{C} \to \text{h-proj}(\mathbf{B})$ such that $f = [\mathsf{F}] \circ [\mathsf{I}]^{-1}$. As Ind_I is a quasi-equivalence by part (5) of Proposition 3.2, we can define $f' := [\widehat{\mathsf{F}}] \circ [\text{Ind}_\mathsf{I}]^{-1} \in [\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})]$. By part (2) of Proposition 3.2 we see that $f' \in [\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})]_c$ and $f' \circ [\mathsf{Y}_\mathbf{A}] = f$, thereby proving that the map is surjective.

We now want to show that the map is injective. To this end, let $f_1, f_2 \in [\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})]_c$ be such that $f_1 \circ [\mathsf{Y}_\mathbf{A}] = f_2 \circ [\mathsf{Y}_\mathbf{A}]$. By part (3) of Proposition 2.11, there exist a quasi-equivalence $\mathsf{I} \colon \mathbf{C} \to \text{h-proj}(\mathbf{A})$ and two dg functors $\mathsf{F}_1, \mathsf{F}_2 \colon \mathbf{C} \to \text{h-proj}(\mathbf{B})$ such that $f_i = [\mathsf{F}_i] \circ [\mathsf{I}]^{-1}$, for $i = 1, 2$. Let $\mathbf{D}$ be the full dg subcategory of $\mathbf{C}$ such that $\mathsf{I}' := \mathsf{I}|_\mathbf{D} \colon \mathbf{D} \to \overline{\mathbf{A}}$ is a quasi-equivalence and let $\mathsf{J} \colon \mathbf{D} \to \mathbf{C}$ be the inclusion. If we set $\mathsf{G}_i := \mathsf{F}_i \circ \mathsf{J}$, for $i = 1, 2$, then the diagram

commutes in **dgCat**, and it is easy to see that $f_i \circ [\mathsf{Y}_\mathbf{A}] = [\mathsf{G}_i] \circ [\mathsf{I}']^{-1} \circ [\mathsf{Y}_\mathbf{A}]$, for $i = 1, 2$, where on the right-hand side we regard $\mathsf{Y}_\mathbf{A}$ as the quasi-equivalence $\mathbf{A} \to \overline{\mathbf{A}}$. As $f_1 \circ [\mathsf{Y}_\mathbf{A}] = f_2 \circ [\mathsf{Y}_\mathbf{A}]$, we deduce $[\mathsf{G}_1] \circ [\mathsf{I}']^{-1} = [\mathsf{G}_2] \circ [\mathsf{I}']^{-1}$. Thus $[\mathsf{G}_1] = [\mathsf{G}_2]$ and, by Lemma 3.9, we get $[\widehat{\mathsf{G}_1}] = [\widehat{\mathsf{G}_2}]$.

Take now the dg functor $\mathsf{K} := \text{Res}_\mathsf{J} \circ \mathsf{Y}_\mathbf{C} \colon \mathbf{C} \to \text{dgMod}(\mathbf{D})$. Observe that J is fully faithful and $H^0(\mathsf{J}(\mathbf{D}))$ is a set of compact generators for the triangulated category with arbitrary direct sums $H^0(\mathbf{C})$ (use that $\mathbf{C}$ is quasi-equivalent to $\text{h-proj}(\mathbf{A})$ under a quasi-equivalence inducing a quasi-equivalence between $\mathbf{D}$ and $\overline{\mathbf{A}}$). Observe that $H^0(\mathsf{K})$ is continuous and the image of K is contained in $\text{h-proj}(\mathbf{D})$. Indeed, the fact that $H^0(\mathsf{K})$ is continuous follows along the same lines as in the proof of [10, Prop. 1.17]. This, together with the simple fact that $\mathsf{K}(\mathsf{J}(\mathbf{D})) \cong \mathsf{Y}_\mathbf{D}(\mathbf{D}) \subseteq \text{h-proj}(\mathbf{D})$, gives that the image of K is contained in $\text{h-proj}(\mathbf{D})$.

Now observe that, for $i = 1, 2$, we have $[\mathsf{F}_i] = [\widehat{\mathsf{G}_i}] \circ [\mathsf{K}]$. Indeed, by part (1) of Proposition 3.2, $\widehat{\mathsf{G}_i}$ has a right adjoint $\widetilde{\mathsf{G}}_i$. Notice that, for all $C \in \mathbf{C}$, we have $\widetilde{\mathsf{G}}_i \circ \mathsf{F}_i(C) = \text{h-proj}(\mathbf{B})(\mathsf{G}_i(-), \mathsf{F}_i(C))$. Thus the composition with F_i yields a natural map

$$\mathsf{K}(C) = \mathbf{C}(\mathsf{J}(-), C) \longrightarrow \text{h-proj}(\mathbf{B})(\mathsf{G}_i(-), \mathsf{F}_i(C)) = \widetilde{\mathsf{G}}_i \circ \mathsf{F}_i(C)$$

and hence, by adjunction, a natural transformation $\theta \colon \widehat{\mathsf{G}}_i \circ \mathsf{K} \to \mathsf{F}_i$ with the property that $H^0(\theta)|_{\mathsf{J}(\mathbf{D})}$ is an isomorphism. Since $H^0(\widehat{\mathsf{G}}_i \circ \mathsf{K})$ and $H^0(\mathsf{F}_i)$ are continuous, Remark 2.4 yields that θ is a termwise homotopy equivalence. By Corollary 2.12, we have $[\mathsf{F}_i] = [\widehat{\mathsf{G}}_i] \circ [\mathsf{K}]$, for $i = 1, 2$. As $[\widehat{\mathsf{G}}_1] = [\widehat{\mathsf{G}}_2]$, we obtain $[\mathsf{F}_1] = [\mathsf{F}_2]$, which obviously implies $f_1 = f_2$. $\quad\square$

On the other hand, given three dg categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ with a fully faithful dg functor $\mathsf{J} \colon \mathbf{B} \to \mathbf{C}$, we have another natural map of sets

$$\Sigma_{\mathbf{A}, \mathsf{J}} \colon [\mathbf{A}, \mathbf{B}] \to [\mathbf{A}, \mathbf{C}] \qquad f \mapsto [\mathsf{J}] \circ f.$$

Proposition 3.11. *The natural map of sets $\Sigma_{\mathbf{A}, \mathsf{J}}$ is injective.*

Proof. Set $\mathbf{C}'$ to be the full dg subcategory of $\mathbf{C}$ consisting of all objects in the essential image of $H^0(\mathsf{J})$ and denote by $\mathsf{J}_1 \colon \mathbf{B} \to \mathbf{C}'$ the natural quasi-equivalence. Let $\mathsf{J}_2 \colon \mathbf{C}' \to \mathbf{C}$ be the natural inclusion inducing a natural dg functor $\mathsf{J}_3 \colon P(\mathbf{C}') \to P(\mathbf{C})$ such that $\mathsf{J}_3((C_1, C_2, f)) = (\mathsf{J}_2(C_1), \mathsf{J}_2(C_2), \mathsf{J}_2(f))$ for every $(C_1, C_2, f) \in P(\mathbf{C}')$. It is easy to verify that J_3 is fully faithful since J_2 is.

Given $f_1, f_2 \in [\mathbf{A}, \mathbf{B}]$ such that $\Sigma_{\mathbf{A}, \mathsf{J}}(f_1) = \Sigma_{\mathbf{A}, \mathsf{J}}(f_2)$, by part (3) of Proposition 2.11 there exist a quasi-equivalence $\mathsf{I} \colon \mathbf{D} \to \mathbf{A}$ and dg functors $\mathsf{F}_i \colon \mathbf{D} \to \mathbf{B}$ such that $f_i = [\mathsf{F}_i] \circ [\mathsf{I}]^{-1}$, for $i = 1, 2$. As $[\mathsf{J} \circ \mathsf{F}_1] = [\mathsf{J} \circ \mathsf{F}_2]$, by part (1) of Proposition 2.11 there exist a quasi-equivalence $\mathsf{I}' \colon \mathbf{D}' \to \mathbf{D}$ and a dg functor $\mathsf{H} \colon \mathbf{D}' \to P(\mathbf{C})$ such that, setting $\mathsf{G}_i := \mathsf{J} \circ \mathsf{F}_i \circ \mathsf{I}' \colon \mathbf{D}' \to \mathbf{C}$ for $i = 1, 2$, $\mathsf{G}_1 = \mathsf{S}_{\mathbf{C}} \circ \mathsf{H}$ and $\mathsf{G}_2 = \mathsf{T}_{\mathbf{C}} \circ \mathsf{H}$. Observe that, by definition, $\mathsf{H}(D) = (\mathsf{G}_1(D), \mathsf{G}_2(D), f)$, where $f \colon \mathsf{G}_1(D) \to \mathsf{G}_2(D)$ is a homotopy equivalence, for all $D \in \mathbf{D}$. It is easy to see that H factors through J_3. This means that there exists a dg functor $\mathsf{H}' \colon \mathbf{D}' \to P(\mathbf{C}')$ such that $\mathsf{H} = \mathsf{J}_3 \circ \mathsf{H}'$. Thus, if we set $\mathsf{G}_i' = \mathsf{J}_1 \circ \mathsf{F}_i \circ \mathsf{I}'$, we get $\mathsf{G}_1' = \mathsf{S}_{\mathbf{C}'} \circ \mathsf{H}'$ and $\mathsf{G}_2' = \mathsf{T}_{\mathbf{C}'} \circ \mathsf{H}'$, so that $[\mathsf{G}_1'] = [\mathsf{G}_2']$.

Therefore $[\mathsf{J}_1] \circ [\mathsf{F}_1] \circ [\mathsf{I}'] = [\mathsf{G}_1'] = [\mathsf{G}_2'] = [\mathsf{J}_1] \circ [\mathsf{F}_2] \circ [\mathsf{I}']$ and, using that J_1 and I' are quasi-equivalences, we conclude that $[\mathsf{F}_1] = [\mathsf{F}_2]$, whence $f_1 = f_2$. $\quad\square$

3.3. Morphisms in **Hqe** as dg modules

If $E, E' \in \mathrm{dgMod}_{\mathrm{hp}}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B})$ are quasi-isomorphic, then it follows from Remark 3.3 and Corollary 2.12 that $[\Phi_E] = [\Phi_{E'}] \colon \mathbf{A}^{\mathrm{hp}} \to \mathrm{h\text{-}proj}(\mathbf{B})$. In particular, denoting by $[E]_{\mathrm{iso}} \in \mathrm{Iso}(H^0(\mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B})))$ the homotopy equivalence class of $E \in \mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B})$ and composing with the natural bijection between $[\mathbf{A}, \mathbf{B}]$ and $[\mathbf{A}^{\mathrm{hp}}, \mathbf{B}]$ induced by the quasi-equivalence $\mathsf{Q}_{\mathbf{A}} \colon \mathbf{A}^{\mathrm{hp}} \to \mathbf{A}$, we get a well-defined map:

$$\Lambda_{\mathbf{A}, \mathbf{B}} \colon \mathrm{Iso}(H^0(\mathrm{h\text{-}proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B}))) \longrightarrow [\mathbf{A}, \mathrm{h\text{-}proj}(\mathbf{B})] \qquad [E]_{\mathrm{iso}} \mapsto [\Phi_E] \circ [\mathsf{Q}_{\mathbf{A}}]^{-1}$$

Proposition 3.12. *For all dg categories $\mathbf{A}$ and $\mathbf{B}$, the map $\Lambda_{\mathbf{A}, \mathbf{B}}$ is bijective.*

Proof. First of all, we can assume, without loss of generality, that $\mathbf{A}$ is h-projective. Given $f\colon \mathbf{A} \to \text{h-proj}(\mathbf{B})$ in $\mathbf{Hqe}$, by part (3) of Proposition 2.11 there exist dg functors $\mathsf{I}\colon \mathbf{A}' \to \mathbf{A}$ and $\mathsf{F}\colon \mathbf{A}' \to \text{h-proj}(\mathbf{B})$ with I a quasi-equivalence such that $f = [\mathsf{F}] \circ [\mathsf{I}]^{-1}$. Notice that f corresponds to $\widehat{f} = [\widehat{\mathsf{F}}] \circ [\text{Ind}_{\mathsf{I}}]^{-1}\colon \text{h-proj}(\mathbf{A}) \to \text{h-proj}(\mathbf{B})$ under the bijection given by Proposition 3.10. We denote by $I \in \text{dgMod}_{\text{hp}}(\mathbf{A}'^{\circ} \otimes \mathbf{A})$ and $F \in \text{dgMod}_{\text{hp}}(\mathbf{A}'^{\circ} \otimes \mathbf{B})$ the objects such that $\Phi_I = \mathsf{Y}_{\mathbf{A}} \circ \mathsf{I}$ and $\Phi_F = \mathsf{F}$. Since $\text{Ind}_{\text{id}_{\mathbf{A}^{\circ}} \otimes \mathsf{I}}\colon \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{A}') \to \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{A})$ is a quasi-equivalence (by Remark 2.8 and part (5) of Proposition 3.2), there exists $D' \in \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{A}')$ such that $D := \text{Ind}_{\text{id}_{\mathbf{A}^{\circ}} \otimes \mathsf{I}}(D')$ is an h-projective resolution of $\Delta_{\mathbf{A}}$. It is easy to see, using Lemma 3.7, that $\text{Ind}_{\text{id}_{\mathbf{A}^{\circ}} \otimes \mathsf{I}} \cong \widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes I}}$ and we obtain

$$D \cong \widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes I}}(D') \cong \Delta_{\mathbf{A}^{\circ}} \otimes_{\mathbf{A}} D' \otimes_{\mathbf{A}'} I \cong D' \otimes_{\mathbf{A}'} I,$$

whence $\widehat{\Phi_D} \cong \widehat{\Phi_I} \circ \widehat{\Phi_{D'}}$ by the associativity of tensor product. Notice that the last isomorphism above is proved in [6, Sect. 14.6]. Taking into account that $[\widehat{\Phi_D}] = [\text{id}_{\text{h-proj}(\mathbf{A})}]$, it follows that $[\widehat{\Phi_{D'}}] = [\widehat{\Phi_I}]^{-1} = [\text{Ind}_{\mathsf{I}}]^{-1}$. Setting $E' := D' \otimes_{\mathbf{A}'} F \in \text{dgMod}_{\text{hp}}(\mathbf{A}^{\circ} \otimes \mathbf{B})$, we have $\widehat{\Phi_{E'}} \cong \widehat{\Phi_F} \circ \widehat{\Phi_{D'}}$. Therefore $[\widehat{\Phi_{E'}}] = [\widehat{\Phi_F}] \circ [\widehat{\Phi_{D'}}] = [\widehat{\mathsf{F}}] \circ [\text{Ind}_{\mathsf{I}}]^{-1} = \widehat{f}$. Taking $E \in \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{B})$ an h-projective resolution of E', this proves that $[\Phi_E] = [\Phi_{E'}] = f$, whence $\Lambda_{\mathbf{A},\mathbf{B}}$ is surjective.

As for injectivity, let $E, E' \in \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{B})$ be such that $[\Phi_E] = [\Phi_{E'}]$. Then $[\text{id}_{\mathbf{A}^{\circ}} \otimes \Phi_E] = [\text{id}_{\mathbf{A}^{\circ}} \otimes \Phi_{E'}]$ by Remark 2.8. Hence also $[\Phi_{\Delta_{\mathbf{A}^{\circ}}} \otimes \Phi_E] = [\Phi_{\Delta_{\mathbf{A}^{\circ}}} \otimes \Phi_{E'}]\colon \mathbf{A}^{\circ} \otimes \mathbf{A} \to \text{h-proj}(\mathbf{A}^{\circ}) \otimes \text{h-proj}(\mathbf{B})$. From Lemma 3.7 we deduce that $[\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E}] = [\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E'}]$, and so (by Lemma 3.9)

$$[\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E}}] = [\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E'}}]\colon \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{A}) \to \text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{B}). \tag{3.2}$$

Denoting as before by D an h-projective resolution of $\Delta_{\mathbf{A}}$, again by Lemma 3.7 we have

$$\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E}}(D) \cong \Delta_{\mathbf{A}^{\circ}} \otimes_{\mathbf{A}} D \otimes_{\mathbf{A}} E \cong D \otimes_{\mathbf{A}} E.$$

As $D \to \Delta_{\mathbf{A}}$ is a quasi-isomorphism between dg modules which are h-flat over $\mathbf{A}$, also the induced map $D \otimes_{\mathbf{A}} E \to \Delta_{\mathbf{A}} \otimes_{\mathbf{A}} E \cong E$ is a quasi-isomorphism, hence a homotopy equivalence, since both the source and the target are in $\text{h-proj}(\mathbf{A}^{\circ} \otimes \mathbf{B})$. This proves that $[\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E}}(D)]_{\text{iso}} = [E]_{\text{iso}}$, and similarly $[\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E'}}(D)]_{\text{iso}} = [E']_{\text{iso}}$. As $[\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E}}(D)]_{\text{iso}} = [\widehat{\Phi_{\Delta_{\mathbf{A}^{\circ}} \otimes E'}}(D)]_{\text{iso}}$ by (3.2), we conclude that $[E]_{\text{iso}} = [E']_{\text{iso}}$. $\square$

4. The new proof of Theorem 1.1

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be dg categories. In view of the basic properties of the derived tensor product, of Remark 2.8 and of Proposition 3.8, we can assume in the proof of Theorem 1.1, without loss of generality, that these three dg-categories are h-projective. In this way the tensor product does not need to be derived. Putting the results in the previous section together, we get the following maps:

$$[\mathbf{A} \otimes \mathbf{B}, \mathbf{C}] \xleftarrow[1:1]{\Psi} [\mathbf{A} \otimes \mathbf{B}, \overline{\mathbf{C}}] \xhookrightarrow{\Sigma} [\mathbf{A} \otimes \mathbf{B}, \text{h-proj}(\mathbf{C})] \xleftarrow[1:1]{\Lambda} \text{Iso}(H^0(\text{h-proj}((\mathbf{A} \otimes \mathbf{B})^\circ \otimes \mathbf{C})))$$

where Ψ is induced by the quasi-equivalence $\mathbf{C} \hookrightarrow \overline{\mathbf{C}}$ while $\Sigma := \Sigma_{\mathbf{A} \otimes \mathbf{B}, \overline{\mathbf{C}} \hookrightarrow \text{h-proj}(\mathbf{C})}$ and $\Lambda := \Lambda_{\mathbf{A} \otimes \mathbf{B}, \mathbf{C}}$ are the maps with the properties discussed in Propositions 3.11 and 3.12. Obviously, by definition, $\text{Im}(\Sigma)$ consists of all $f \in [\mathbf{A} \otimes \mathbf{B}, \text{h-proj}(\mathbf{C})]$ such that $\text{Im}(H^0(f)) \subseteq H^0(\overline{\mathbf{C}})$. Using Λ, we get a bijection between $\text{Im}(\Sigma)$ and the set of isomorphism classes of the objects $E \in H^0(\text{h-proj}((\mathbf{A} \otimes \mathbf{B})^\circ \otimes \mathbf{C}))$ such that $H^0(\Phi_E): H^0(\mathbf{A} \otimes \mathbf{B}) \to H^0(\text{h-proj}(\mathbf{C}))$ factors through $H^0(\overline{\mathbf{C}})$. Thus, by definition, we have a natural bijection between the sets

$$[\mathbf{A} \otimes \mathbf{B}, \mathbf{C}] \xleftarrow{\quad 1:1 \quad} \text{Iso}(H^0(\text{h-proj}((\mathbf{A}^\circ \otimes \mathbf{B}^\circ) \otimes \mathbf{C})^{\text{rqr}})). \qquad (4.1)$$

On the other hand, we have the following sequence of natural maps of sets:

$$[\mathbf{A}, \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}}] \xhookrightarrow{\Sigma} [\mathbf{A}, \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})] \xleftarrow[1:1]{\Lambda} \text{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes (\mathbf{B}^\circ \otimes \mathbf{C})))),$$

where $\Sigma := \Sigma_{\mathbf{A}, \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}} \hookrightarrow \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})}$ and $\Lambda := \Lambda_{\mathbf{A}, \mathbf{B}^\circ \otimes \mathbf{C}}$ have the properties discussed in Propositions 3.11 and 3.12. In analogy with the previous case, $\text{Im}(\Sigma)$ consists of all $f \in [\mathbf{A}, \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})]$ such that $\text{Im}(H^0(f)) \subseteq H^0(\text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}})$ (here we use that $\text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}}$ is by definition closed under homotopy equivalences in $\text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})$). The map Λ yields a natural bijection between $\text{Im}(\Sigma)$ and the set of isomorphism classes of objects $F \in H^0(\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{B}^\circ \otimes \mathbf{C}))$ such that $H^0(\Phi_F): H^0(\mathbf{A}) \to H^0(\text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C}))$ factors through $H^0(\text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}})$. Again by definition, this provides a natural bijection of sets

$$[\mathbf{A}, \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}}] \xleftarrow{\quad 1:1 \quad} \text{Iso}(H^0(\text{h-proj}((\mathbf{A}^\circ \otimes \mathbf{B}^\circ) \otimes \mathbf{C})^{\text{rqr}})). \qquad (4.2)$$

If $\mathbf{B}$ is the (h-projective) dg category $\Bbbk$, then we observed that $\text{h-proj}(\mathbf{B}^\circ \otimes^{\mathbf{L}} \mathbf{C})^{\text{rqr}} \cong \overline{\mathbf{C}}$. Thus, we get the natural bijection between the sets $[\mathbf{A}, \mathbf{C}]$ and $\text{Iso}(H^0(\text{h-proj}(\mathbf{A}^\circ \otimes \mathbf{C})^{\text{rqr}}))$, which is (1.1).

As in the statement of Theorem 1.1, set $\mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}) := \text{h-proj}(\mathbf{B}^\circ \otimes^{\mathbf{L}} \mathbf{C})^{\text{rqr}}$, for two dg categories $\mathbf{B}$ and $\mathbf{C}$. If $\mathbf{B}$ is an h-projective dg category, we have $\mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}) = \text{h-proj}(\mathbf{B}^\circ \otimes \mathbf{C})^{\text{rqr}}$ because we do not need to derive the tensor product (see Remark 2.7). Due to Proposition 3.8 and the naturality of the bijections in (4.1) and (4.2), we get a natural bijection between the sets $[\mathbf{A} \otimes \mathbf{B}, \mathbf{C}]$ and $[\mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C})]$, which is (1.2). So **Hqe** is a closed symmetric monoidal category, and this concludes the proof of Theorem 1.1.

Corollary 4.1. *Given three dg categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the dg categories $\mathbb{R}\underline{Hom}(\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B}, \mathbf{C})$ and $\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}))$ are isomorphic in **Hqe**.*

Proof. For any dg category $\mathbf{D}$ and using (1.2) and the associativity of the derived tensor product, we get the following natural bijections:

$$
\begin{array}{ccc}
[\mathbf{D} \otimes^{\mathbf{L}} (\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B}), \mathbf{C}] & \xleftarrow{\ 1:1\ } & [\mathbf{D} \otimes^{\mathbf{L}} \mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C})] \\
{\scriptstyle 1:1} \updownarrow & & \updownarrow {\scriptstyle 1:1} \\
[\mathbf{D}, \mathbb{R}\underline{Hom}(\mathbf{A} \otimes^{\mathbf{L}} \mathbf{B}, \mathbf{C})] & & [\mathbf{D}, \mathbb{R}\underline{Hom}(\mathbf{A}, \mathbb{R}\underline{Hom}(\mathbf{B}, \mathbf{C}))],
\end{array}
$$

for every dg category $\mathbf{D}$. By Yoneda's lemma, we conclude. $\quad\square$

For two dg categories $\mathbf{A}$ and $\mathbf{B}$, we denote by $\mathbb{R}\underline{Hom}_c(\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B}))$ the full dg subcategory of $\mathbb{R}\underline{Hom}(\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B}))$ consisting of all $F \in \mathbb{R}\underline{Hom}(\text{h-proj}(\mathbf{A}),$ $\text{h-proj}(\mathbf{B}))$ such that $[F]_{\text{iso}} \in [\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})]_c$ under (1.1). We can prove the following, which is Theorem 7.2 in [18] and is usually referred to as derived Morita theory.

Corollary 4.2. *Given two dg categories* $\mathbf{A}$ *and* $\mathbf{B}$, $\mathbb{R}\underline{Hom}(\mathbf{A}, \text{h-proj}(\mathbf{B}))$ *and* $\text{h-proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B})$ *are isomorphic in* $\mathbf{Hqe}$. *Moreover, there exist quasi-equivalences* $\mathbb{R}\underline{Hom}_c(\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{B})) \to \mathbb{R}\underline{Hom}(\mathbf{A}, \text{h-proj}(\mathbf{B}))$ *and* $\mathbb{R}\underline{Hom}(\text{Perf}(\mathbf{A}), \text{Perf}(\mathbf{B})) \to$ $\mathbb{R}\underline{Hom}(\mathbf{A}, \text{Perf}(\mathbf{B}))$ *induced by the Yoneda embedding* $\mathsf{Y}_{\mathbf{A}} : \mathbf{A} \to \text{Perf}(\mathbf{A}) \subset \text{h-proj}(\mathbf{A})$.

Proof. As for the first part of the statement, observe that, in view of Proposition 3.12, we have natural bijections

$$
[\mathbf{C}, \text{h-proj}(\mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B})] \xleftarrow{\ 1:1\ } \text{Iso}(H^0(\text{h-proj}((\mathbf{C}^\circ \otimes^{\mathbf{L}} \mathbf{A}^\circ) \otimes^{\mathbf{L}} \mathbf{B}))) \xleftarrow{\ 1:1\ } [\mathbf{C} \otimes^{\mathbf{L}} \mathbf{A}, \text{h-proj}(\mathbf{B})],
$$

$$\tag{4.3}$$

for every dg category $\mathbf{C}$. Using (1.2), we get the result by Yoneda's lemma.

For the second part, we argue as at the beginning of the proof of Theorem 7.2 of [18]. So we have to show that, for any dg category $\mathbf{C}$, the Yoneda embedding $\mathsf{Y}_{\mathbf{A}}$ induces a bijection $[\text{h-proj}(\mathbf{A}) \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})]'_c \longrightarrow [\mathbf{A} \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})]$, where $[\text{h-proj}(\mathbf{A}) \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})]'_c$ is the subset of $[\text{h-proj}(\mathbf{A}) \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})]$ containing all morphisms f such that $H^0(f)((-, C))$ is continuous for all $C \in \mathbf{C}$. Indeed, by (4.3), we get the natural bijection $[\mathbf{A} \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})] \to [\mathbf{A}, \text{h-proj}(\mathbf{C}^\circ \otimes^{\mathbf{L}} \mathbf{B})]$. Similarly, one deduces the natural bijection $[\text{h-proj}(\mathbf{A}) \otimes^{\mathbf{L}} \mathbf{C}, \text{h-proj}(\mathbf{B})]'_c \to [\text{h-proj}(\mathbf{A}), \text{h-proj}(\mathbf{C}^\circ \otimes^{\mathbf{L}} \mathbf{B})]_c$. Now we simply apply Proposition 3.10.

As for perfect dg modules, given a dg category $\mathbf{C}$, we consider the dg functor $\mathsf{F} :=$ $\text{id}_{\mathbf{C}^\circ} \otimes \mathsf{Y}_{\mathbf{A}}^\circ \otimes \text{id}_{\mathbf{B}} : \mathbf{D}_1 \to \mathbf{D}_2$, where $\mathbf{D}_1 := \mathbf{C}^\circ \otimes^{\mathbf{L}} \mathbf{A}^\circ \otimes^{\mathbf{L}} \mathbf{B}$ and $\mathbf{D}_2 := \mathbf{C}^\circ \otimes^{\mathbf{L}} \text{Perf}(\mathbf{A})^\circ \otimes^{\mathbf{L}} \mathbf{B}$. By [10, Prop. 1.15], we have that $\text{Ind}_{\mathsf{F}} : \text{h-proj}(\mathbf{D}_1) \to \text{h-proj}(\mathbf{D}_2)$ is a quasi-equivalence.

Thus, by Proposition 3.12 and the same computations as in the proof of Theorem 1.1 above, we get a commutative diagram

$$
\begin{array}{ccccc}
[\mathbf{C} \otimes^{\mathbf{L}} \mathbf{A}, \mathrm{Perf}(\mathbf{B})] & \hookrightarrow & [\mathbf{C} \otimes^{\mathbf{L}} \mathbf{A}, \mathrm{h\text{-}proj}(\mathbf{B})] & \xleftrightarrow[1:1]{\Lambda} & \mathrm{Iso}(H^0(\mathrm{h\text{-}proj}(\mathbf{D}_1))) \\
\Big\uparrow {\scriptstyle \mathsf{Y}_{\mathbf{C}\otimes^{\mathbf{L}}\mathbf{A}} \circ -} & & & & \Big\updownarrow {\scriptstyle 1:1} \\
[\mathbf{C} \otimes^{\mathbf{L}} \mathrm{Perf}(\mathbf{A}), \mathrm{Perf}(\mathbf{B})] & \hookrightarrow & [\mathbf{C} \otimes^{\mathbf{L}} \mathrm{Perf}(\mathbf{A}), \mathrm{h\text{-}proj}(\mathbf{B})] & \xleftrightarrow[1:1]{\Lambda} & \mathrm{Iso}(H^0(\mathrm{h\text{-}proj}(\mathbf{D}_2))),
\end{array}
$$

where the right vertical bijection is induced by the quasi-equivalence $\mathrm{Ind}_{\mathsf{F}}$. Thus the Yoneda embedding induces a natural bijection between $[\mathbf{C} \otimes^{\mathbf{L}} \mathrm{Perf}(\mathbf{A}), \mathrm{Perf}(\mathbf{B})]$ and $[\mathbf{C} \otimes^{\mathbf{L}} \mathbf{A}, \mathrm{Perf}(\mathbf{B})]$, for all dg categories $\mathbf{C}$. By (1.2), we conclude using Yoneda's lemma. $\square$

Acknowledgments

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