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# On the regularity of singular integral operators on complex domains

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# Contents

# Introduction

This thesis is divided into two parts, which deal with two similar problems, but with a different perspective.

The first part is devoted to the study of the Hardy spaces $H^p$, $p \in (1, \infty)$, on non-smooth *worm domains* (see below) and the mapping properties of the Szegő projection associated to these spaces. Naively, given a domain $\Omega$, the starting point is a space $H^p(\Omega)$ of holomorphic functions on $\Omega$ that respect some growth condition and that admit boundary values on $\partial\Omega$. To these spaces, we can associate an operator $S_\Omega$ that extends a function $F$ defined on $\partial\Omega$ to a a holomorphic function $S_\Omega F$ defined on $\Omega$. Our interest is to study the mapping properties of the operator $S_\Omega$ between the spaces $L^p(\partial\Omega)$ and $H^p(\Omega)$. Once we have proved that the range of $S_\Omega$ is contained in $H^p(\Omega)$, we can consider a new operator related to $S_\Omega$. Since every function $S_\Omega F$ in $H^p(\Omega)$ admits a boundary value function, say $\widetilde{S_\Omega F}$, defined on $\partial D$ we consider the operator $F \mapsto \widetilde{S_\Omega F}$, the so-called Szegő projection operator, and we are interested in studying its mapping properties. We remark that the Szegő projection associates to a function $F$ defined on $\partial\Omega$ another function $\widetilde{SF}_\Omega$ defined on $\partial\Omega$, but we also obtain an intermediate holomorphic function $S_\Omega F$ which belongs to a specific functional space and our interest in also in this space. In particular, we face this problem considering as domain $\Omega$ some non-smooth versions of the worm domain. It turns out that the Szegő projection of these domains has an integral representation and can be studied using the classical theory of Calderón-Zygmund operators.

In the second part we have again an operator of the kind $F \mapsto \widetilde{S_\Omega F}$, but we only require that the intermediate function $S_\Omega F$ is holomorphic on $\Omega$, without worrying if it belongs to some specific functional space $H^p$. More specifically, the domain we consider is the perturbed upper half space $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$ where $\mathbb{H}_{\Gamma_j}$ is the perturbed half plane $\mathbb{H}_{\Gamma_j} = \{x_j + iL_j(x_j) + it_j : x_j \in \mathbb{R},\ t_j > 0\}$ with $L_j$ Lipschitz functions. The operators that arise in this setting are biparameter singular integral operators whose studying of the mapping properties is delicate.

For this reason, the focus is more on the operators than on the properties of the holomorphic extension of the starting function.

We briefly describe here the main results we have obtained and illustrate the relationship with related results in the literature.

# Szegő kernel and projection on non-smooth worm domains

The smooth worm domain $\mathcal{W} = \mathcal{W}_\beta$ was first introduced by Diederich and Fornæss in [DF77] to provide counterexamples to certain classical conjectures about the geometry of pseudoconvex domains. For $\beta > \frac{\pi}{2}$, the worm domain is defined by

$$\mathcal{W} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - e^{i \log |z_2|^2}|^2 < 1 - \eta(\log |z_2|^2)\}, \tag{1}$$

where $\eta$ is a smooth, even, convex, non-negative function on the real line, chosen so that $\eta^{-1}(0) = [-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$ and so that $\mathcal{W}$ is bounded, smooth and pseudoconvex. See [KP08a] for a history of the study of the worm domain. Diederich and Fornæss introduced this domain to provide an example of a smooth, bounded and pseudoconvex domain whose closure does not have a Stein neighbourhood basis. Nearly 15 years after its introduction, the interest in the worm domain has been renewed since it turned out to be a counterexample to other important conjectures. Starting from ground-breaking works of Kiselman [Kis91] and Barrett [Bar92], Christ [Chr96a] finally proved that the worm domain does not satisfy the so-called Condition $R$. A domain $\Omega$ satisfies Condition $R$ if the Bergman projection $P_\Omega$ associated to the domain $\Omega$ maps $\mathcal{C}^\infty(\overline{\Omega})$ to $\mathcal{C}^\infty(\overline{\Omega})$.

We recall that, given a domain $\Omega$ in $\mathbb{C}^n$, the Bergman projection $P_\Omega$ of $\Omega$ is the Hilbert space projection $P : L^2(\Omega) \to A^2(\Omega)$ where $A^2(\Omega)$ is the closed subspace of $L^2(\Omega)$ consisting of holomorphic functions. It turns out that $P_\Omega$ has an integral representation, namely

$$P_\Omega f(z) = \int_\Omega K_\Omega(z, w) f(w) dA(w),$$

where $K_\Omega(z, w) = \overline{K_\Omega(w, z)}$ is the *Bergman Kernel.*

The interest in Condition $R$ dwells in the fact that it is closely related to the boundary regularity of biholomorphic mappings as it has been shown in works of Bell [Bel81] and

Bell and Ligocka [BL80]. Specifically, in [BL80] it is proved that given a biholomorphism $\Phi : \Omega_1 \to \Omega_2$ between smoothly bounded, Levi pseudoconvex domains of $\mathbb{C}^n$, one of which satisfies Condition $R$, then $\Phi$ extends to a $\mathcal{C}^\infty$ diffeomorphism $\Phi : \overline{\Omega_1} \to \overline{\Omega_2}$.

Due to the results of Christ's, it is natural to deeply investigate the Bergman kernel of the worm domain. This has been done extensively by Krantz and Peloso in [KP07],[KP08a] and [KP08b]. Following [Kis91] and [Bar92], they studied the $L^p$-mapping properties of the Bergman projection associated to two non-smooth versions of the original worm domain, namely

$$D_\beta = \Big\{(\zeta_1,\zeta_2) \in \mathbb{C}^2 : \mathrm{Re}(\zeta_1 e^{-i\log|\zeta_2|^2}) > 0, \big|\log|\zeta_2|^2\big| < \beta - \frac{\pi}{2}\Big\}$$

and

$$D'_\beta = \Big\{(z_1,z_2) \in \mathbb{C}^2 : \big|\,\mathrm{Im}\, z_1 - \log|z_2|^2\big| < \frac{\pi}{2}, \big|\log|z_2|^2\big| < \beta - \frac{\pi}{2}\Big\}.$$

Notice that the slices of $D_\beta$ for each fixed $z_2$ are half-planes in the variable $z_1$, while the slices of $D'_\beta$ for each fixed $z_2$ are strips in the variable $z_1$. These two domains, firstly introduced in [Kis91], are biholomorphically equivalent and, even if their geometry is rather different from the one of the original smooth worm, they are a model for $\mathcal{W}_\beta$ as it can be seen in [Bar92]. In [KP08b], the mapping properties of the Bergman projection are studied by giving an explicit computation of the kernels $K_{D'_\beta}$ and $K_{D_\beta}$. We remark that the actual computations are made for the domain $D'_\beta$ since it has an easier geometry and with the restriction that $\beta > \pi$. It is then possible to recover the Bergman kernel of $D_\beta$ thanks to the well-known transformation rule for the Bergman kernel under biholomorphism. The more complex geometry of $D_\beta$ is echoed also in the regularity of the Bergman projection. The $L^p$ mapping properties of the Bergman projection of $D'_\beta$ are better than the ones of the Bergman projection of $D_\beta$.

Recently, Krantz, Peloso and Stoppato studied in [KPS14] the mapping properties of the Bergman projection of another unbounded worm domain $\mathcal{W}_\infty$. This domain can be thought as a limit of the original smooth worm domain. Namely,

$$\mathcal{W}_\infty = \Big\{(z_1,z_2) \in \mathbb{C}^2 : |z_1 - e^{-i\log|z_2|^2}|^2 < 1, z_2 \neq 0\Big\}.$$

In [KPS14], the authors proved some results concerning the mapping properties of the Bergman projection of $\mathcal{W}_\infty$ in $L^p$ and Sobolev scale. The authors observe that the approach they use to deal with $\mathcal{W}_\infty$ may be useful to study the original worm domain of Diederich and Fornæss.

The first part of the thesis fits in this investigation of the geometry of the worm domain. We focus on the non-smooth worm domain $D'_\beta$ with the restriction that $\beta > \pi$ and we define the Hardy spaces $H^p(D'_\beta)$, $p \in (1,\infty)$, as follows. For every $p \in (1,\infty)$, we define the Hardy space $H^p(D'_\beta)$ as the functional space

$$H^p(D'_\beta) = \Big\{ F \text{ holomorphic in } D'_\beta : \|F\|^p_{H^p(D'_\beta)} = \sup_{(t,s)\in[0,\frac{\pi}{2})\times[0,\beta-\frac{\pi}{2})} \mathcal{L}_p F(t,s) < \infty \Big\},$$

where,

$$\begin{aligned}
&\mathcal{L}_p F(t,s) \\
&= \int_{\mathbb{R}} \int_0^1 \left| F\left(x + i(s+t), e^{\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx + \int_{\mathbb{R}} \int_0^1 \left| F\left(x - i(s+t), e^{-\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx \\
&\quad + \int_{\mathbb{R}} \int_0^1 \left| F\left(x + i(s-t), e^{\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p \, d\theta dx \, + \int_{\mathbb{R}} \int_0^1 \left| F\left(x - i(s-t), e^{-\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx.
\end{aligned}$$

We are then computing a growth condition on copies of the distinguished boundary

$$\partial D'_\beta = \partial_1 D'_\beta \cup \partial_2 D'_\beta \cup \partial_3 D'_\beta \cup \partial_4 D'_\beta$$

, where,

$$\begin{aligned}
\partial_1 D'_\beta &= \Big\{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = \beta, \log|z_2|^2 = \beta - \frac{\pi}{2}\Big\}; \\
\partial_2 D'_\beta &= \Big\{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = \beta - \pi, \log|z_2|^2 = \beta - \frac{\pi}{2}\Big\}; \\
\partial_3 D'_\beta &= \Big\{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = -\beta, \log|z_2|^2 = -\Big(\beta - \frac{\pi}{2}\Big)\Big\}; \\
\partial_4 D'_\beta &= \Big\{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = -(\beta - \pi), \log|z_2|^2 = -\Big(\beta - \frac{\pi}{2}\Big)\Big\}.
\end{aligned}$$

Notice that we can identify each $\partial_i D'_\beta$ with $\mathbb{R} \times \mathbb{T}$.

We prove that every function $F$ in $H^p(D'_\beta)$ admits a boundary value function $\widetilde{F}$ in $L^p(\partial D'_\beta)$ such that proper restrictions of $F$ converges to $\widetilde{F}$ in norm and pointwise (Theorems 2.37, 2.38, 2.45 and 2.46) and we denote with $H^p(\partial D'_\beta)$ the space of functions in $L^p(\partial D'_\beta)$ which are boundary values of function in $H^p(D'_\beta)$. Thus, we construct the operator $S_{D'_\beta}$ associated to $D'_\beta$ and we prove the following results (Theorem 2.32).

**Theorem.** *The operator $S_{D'_\beta}$ extends to a bounded linear operator*

$$S_{D'_\beta} : L^p(\partial D'_\beta) \to H^p(D'_\beta)$$

*for every $p$ in $(1,\infty)$.*

The operator $S_{D'_\beta}$ has an integral representation by means of the so-called Szegő kernel, that is,

$$S_{D'_\beta}F(z) = \int_{\partial D'_\beta} K_{D'_\beta}(z,\zeta)F(\zeta)\, d\zeta.$$

Following [KP08b], we perform an explicit computation of the principal singularities of the kernel $K_{D'_\beta}$ (Theorem 2.17). Then, we consider the Szegő projection $\widetilde{S}_{D'_\beta}$, the operator that associates to a function $F$ the boundary value function $\widetilde{S_{D'_\beta}F}$ of $S_{D'_\beta}F$. One can think of the Szegő projection as a boundary analogue of the Bergman projection. The regularity properties of the Szegő projection and its relationship with the Bergman projection have been intensively studied for a large class of domains in many papers. We cite [PS77, Boa85, Str86, Boa87, BCS88, BS89, NRSW89, Che91, BS91, MS97, Chr96b, LS04, CF11], among others. The worm domain is not included in any of the known situations, so we want to investigate if its pathological geometry affects the regularity of the Szegő projection as it does in the Bergman case.

The work presented here on the non-smooth worm domain $D'_\beta$ would like to be a starting point for this investigation. Thus, we focus on the $L^p$ and $W^{k,2}$ mapping properties of the operator $\widetilde{S}_{D'_\beta}$. Here $W^{k,2}$ denotes the classical Hilbert-Sobolev space of order $k$. The geometry of $D'_\beta$ allows to reduce our problem in two complex variables in a problem in one complex variable and the Szegő projection of $D'_\beta$ can be seen as an infinite sum of the Szegő projections of strips in the complex plane properly weighted. Using techniques of the theory of multipliers operators we prove the following result (Theorems 2.28 and 2.29).

**Theorem.** *The Szegő projection $\widetilde{S}_{D'_\beta}$ extends to a bounded linear operator*

$$\widetilde{S}_{D'_\beta} : L^p(\partial D'_\beta) \to H^p(\partial D'_\beta)$$

*for $1 < p < \infty$. Moreover, for all $k > 0$, $\widetilde{S}_{D'_\beta}$ extends to a bounded linear operator*

$$\widetilde{S}_{D'_\beta} : W^{k,2}(\partial D'_\beta) \to W^{k,2}(\partial D'_\beta).$$

Therefore, in analogy with the Bergman case, our theorem shows that $\widetilde{S}_{D'_\beta}$ has good mapping properties with respect to the $L^p$ and $W^{k,2}$ norms.

Unlike the Bergman case, in general, we do not have a transformation rule for Szegő kernels under biholomorphism, so the study of $S_{D_\beta}$ and $\widetilde{S}_{D_\beta}$ turns out to be more complicated and the research is still on-going. Our goal in the future is to investigate the Szegő projection of $D_\beta$ and, ultimately, of the original worm of Diederich and Fornæss.

# Holomorphic extension on product Lipschitz surfaces in two complex variables

The results presented here have been obtained in collaboration with Jarod Hart of Wayne State University [HM14]. Starting from a holomorphic extension problem we prove some results pertaining biparameter singular integrals operators and Littlewood-Paley-Stein theory. It is well known that the standard one parameter Hilbert transform is intrinsically related to the boundary behaviour of holomorphic functions in the half-plane $\mathbb{H} = \{x + it \in \mathbb{C} : x \in \mathbb{R}, t > 0\}$. Given a function $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, one can extend $f$ to a holomorphic function

$$F(x + it) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - (x + it)} dy; \text{ for } x \in \mathbb{R},\ t \neq 0.$$

This function $F$ is a holomorphic extension of $f$ in the the sense that $F$ is holomorphic on $\mathbb{C} \backslash \mathbb{R}$ and $f(x) = f_+(x) - f_-(x)$ for $x \in \mathbb{R}$, where

$$f_+(x) = \lim_{t \to 0^+} F(x + it) \quad \text{and} \quad f_-(x) = \lim_{t \to 0^+} F(x - it).$$

These limits hold almost everywhere in $\mathbb{R}$ and in $L^p(\mathbb{R})$. It also follows that $f_\pm = \frac{1}{2}(\pm I + iH)f$ where $I$ is the identity operator and $H$ is the Hilbert transform

$$Hf(x) = \lim_{t \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \frac{x - y}{(x - y)^2 + t^2} f(y) dy.$$

The setting we just described can be generalized to a Lipschitz perturbed upper half space of the form $\mathbb{H}_\Gamma = \{\gamma(x) + it \in \mathbb{C} : x \in \mathbb{R}, t > 0\}$ where $\gamma : \mathbb{R} \to \mathbb{C}$ is a Lipschitz graph. The holomorphic extension result corresponding to the one in the last paragraph is the following: given a function $g \in L^p(\Gamma)$ for $1 < p < \infty$, one can extend $g$ to a holomorphic function

$$G(z + it) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - (z + it)} d\xi; \text{ for } z \in \Gamma,\ t \neq 0,$$

which is a holomorphic extension of $g$ in the the sense that $G$ is holomorphic on $\mathbb{C} \backslash \Gamma$ and $g(z) = g_+(z) - g_-(z)$ for $z \in \Gamma$, where

$$g_+(z) = \lim_{t \to 0^+} G(z + it) \quad \text{and} \quad g_-(x) = \lim_{t \to 0^+} G(z - it)$$

and these limits exist in $L^p(\Gamma)$. The boundary values of $G$ can be realized in this setting as well by $g_{\pm}(z) = \frac{1}{2}(\pm I + iC_\Gamma)g(z)$, where $C_\Gamma$ is the Cauchy integral transform

$$C_\Gamma g(z) = \lim_{t \to 0^+} \frac{1}{\pi} \int_\Gamma \frac{z-\xi}{(z-\xi)^2+t^2} g(\xi) d\xi.$$

Progressing from the extension problem on $\mathbb{H}$ to the one on $\mathbb{H}_\Gamma$ was not an easy feat. It took more than 40 years from the proof of $L^p$ bounds for the Hilbert transform to prove the $L^p$ bounds for the Cauchy integral transform along Lipschitz curves with small constants, which was due to Calderón [Cal77]. The proof for a general Lipschitz constant appeared some years later in works of Coifman, McIntosh and Meyer [CMM82a, CMM82b]. Later, new proofs and generalizations appeared in the work of David, Journé and Semmes [DJS85], Jones [Jon89], and Christ [Chr90], among others.

There are results similar to the ones above in the product setting. Let us consider the product upper half plane $\mathbb{H} \times \mathbb{H}$ in $\mathbb{C}^2$. Then, one can extends a given function $f$ in $L^p(\mathbb{R}^2)$, $p \in (1, \infty)$, to a holomorphic function

$$F(x+it) = \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^2} \frac{f(y)}{(y_1 - (x_1+it_1))(y_2-(x_2+it_2))} dy; \text{ for } x \in \mathbb{R}^2,\ t = (t_1, t_2)$$

with $t_1, t_2 \neq 0$. This function $F$ is a holomorphic extension of $f$ in the the sense that $F$ is holomorphic on $(\mathbb{C} \backslash \mathbb{R}) \times (\mathbb{C} \backslash \mathbb{R})$ and $f(x) = f_{++}(x) - f_{+-}(x) - f_{-+}(x) + f_{--}(x)$ for $x \in \mathbb{R}$, where

$$f_{++}(x) = \lim_{t_1,t_2 \to 0^+} F(x_1+it_1, x_2+it_2), \qquad f_{+-}(x) = \lim_{t_1,t_2 \to 0^+} F(x_1+it_1, x_2-it_2),$$
$$f_{-+}(x) = \lim_{t_1,t_2 \to 0^+} F(x_1-it_1, x_2+it_2), \quad \text{and} \quad f_{--}(x) = \lim_{t_1,t_2 \to 0^+} F(x_1-it_1, x_2-it_2).$$

These limits hold almost everywhere in $\mathbb{R}^2$ and in $L^p(\mathbb{R}^2)$. In this situation, it follows that $f_{\pm,\pm} = \frac{1}{4}(\pm I + iH_1)(\pm I + iH_2)f(x)$ where $H_1 f$ and $H_2 f$ are the Hilbert transforms applied to the first and second variable of $f$ respectively. These operators $H_1$, $H_2$, and $H_1 H_2$ are sometimes called the partial and biparameter Hilbert transforms, which are bounded on $L^p(\mathbb{R}^2)$, see e.g. [Fef81, FS82]. These boundedness results are related to the biparameter Hardy space theory that is addressed in [MM77, GS79, Gun80, CF80, Fef81, FS82, Fef86, Fef87], among many others. Many of these articles work on the polydisk instead of products of upper half planes, but working in these two settings is essentially equivalent; look, for example, in [GS79].

The problem we deal with is a generalization of this situation since we work in a perturbed half-space.

Let $L_1, L_2 : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions and define $\gamma_1(x_1) = x_1 + iL_1(x_1)$, $\gamma_2(x_2) = x_2 + iL_2(x_2)$, and $\gamma(x) = (\gamma_1(x_1), \gamma_2(x_2)) \in \mathbb{C}^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then, we call $\Gamma = \Gamma_1 \times \Gamma_2 = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$ a product Lipschitz surface in $\mathbb{C}^2$. We say that $\Gamma$ is a product Lipschitz surface with small Lipschitz constants if the Lipschitz constants $\lambda_1$ and $\lambda_2$ of $L_1$ and $L_2$ respectively are both smaller than 1. The upper half space associated to $\Gamma$ is defined to be $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$, where $\mathbb{H}_{\Gamma_j} = \{\gamma_j(x_j) + it_j : x_j \in \mathbb{R},\ t_j > 0\}$. We also define $L^p(\Gamma)$ for a product Lipschitz surface $\Gamma$ as follows: given a product Lipschitz surface $\Gamma = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$, let $L^p(\Gamma)$ be the collection of measurable functions $g : \Gamma \to \mathbb{C}$ such that

$$\|g\|^p_{L^p(\Gamma)} = \int_{\mathbb{R}^2} |g(\gamma(x))|^p |\gamma_1'(x_1)\gamma_2'(x_2)| dx_1\, dx_2 < \infty.$$

Given a function $g : \Gamma \to \mathbb{C}$, we define for $\omega = (\omega_{t_1}, \omega_{t_2}) = (z_1 + it_1, z_2 + it_2)$ , where $(z_1, z_2) \in \Gamma$ and $t_1, t_2 \neq 0$, the function

$$G(\omega_{t_1}, \omega_{t_2}) := \frac{1}{(2\pi i)^2} \int_\Gamma \frac{g(\xi)d\xi}{(\xi_1 - \omega_{t_1})(\xi_2 - \omega_{t_2})}.$$

We prove the following result.

**Theorem.** *Let $\Gamma$ be a product Lipschitz surface with small Lipschitz constants in $\mathbb{C}^2$ defined by $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^2 \to \mathbb{C}^2$. Assume that*

$$\lim_{|x_1| \to \infty} \frac{\gamma_1(x_1)}{x_1} = c_1 \qquad \text{and} \qquad \lim_{|x_2| \to \infty} \frac{\gamma_2(x_2)}{x_2} = c_2,$$

*for some $c_1, c_2 \in \mathbb{C}$. If $g \in L^p(\Gamma)$ for some $1 < p < \infty$, then the function $G : (\mathbb{C} \backslash \Gamma_1) \times (\mathbb{C} \backslash \Gamma_2) \to \mathbb{C}$ is a holomorphic extension of $g$ such that, for $z = (z_1, z_2) \in \Gamma$,*

$$g(z) = g_{++}(z) - g_{+-}(z) - g_{-+}(z) + g_{--}(z),$$

*where,*

$$g_{++}(z) = \lim_{t_1, t_2 \to 0^+} G(z_1 + it_1, z_2 + it_2),$$

$$g_{+-}(z) = \lim_{t_1, t_2 \to 0^+} G(z_1 + it_1, z_2 - it_2),$$

$$g_{-+}(z) = \lim_{t_1, t_2 \to 0^+} G(z_1 - it_1, z_2 + it_2),$$

$$g_{--}(z) = \lim_{t_1, t_2 \to 0^+} G(z_1 - it_1, z_2 - it_2).$$

*and the limits hold in $L^p(\Gamma)$.*

We prove the above theorem using the approach David, Journè and Semmes used to apply their $Tb$ theorem to prove $L^p$ bounds for Cauchy integral transform in [DJS85]. For this, we prove the following reduced biparameter $Tb$ theorem.

**Theorem 0.1.** *Let $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions, and define $b(x) = b_1(x_1)b_2(x_2)$ and $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$. Also let $T$ be a biparameter operator of Calderón-Zygmund type associated to $b$ and $\tilde{b}$. If $T$ satisfies the weak boundedness property, mixed weak boundedness properties, and the $Tb = T^*\tilde{b} = 0$ conditions, then $T$ can be continuously extended to a bounded linear operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

There have been a number of results for biparameter singular integral operators of Calderón-Zygmund type, going back to R. Fefferman, Stein, and Journé, among others. There were different versions of $T1$ theorems proved in R. Fefferman-Stein [FS82], Journé [Jou85], Pott-Villaroya [PV11], Ou [Ou13], and [HLT]. In fact, [Ou13] includes a biparameter $Tb$ theorem as well. The formulation of the $Tb$ theorem in this work is different than the one in [Ou13], and even the definitions of biparameter Calderón-Zygmund operators are different. In Chapter 5, we define biparameter singular integral operators relying only on continuity in test function spaces, a full kernel representation, and testing conditions on normalized bumps, whereas in [Ou13] the singular integral operators addressed are required to have full and partial kernel representations as well as some a priori partial $L^2$ bounds. The $Tb$ theorem formulated in this work is a natural extension of the single parameter theory. Unfortunately, it is still not a full characterization of $L^p$ bounds for biparameter Calderón-Zygmund operators since difficulties of working with product $BMO$ persist, but this reduced $Tb = T^*b = 0$ theorem is sufficient to prove the holomorphic extension result we stated. Even though we will only apply $Tb$ theorem when $n_1 = n_2 = 1$, we prove it for general dimensions $n_1, n_2 \in \mathbb{N}$. Our strategy for the proof is to decompose the operator $T$,

$$\langle Tf, g \rangle = \sum_{\vec{k} \in \mathbb{Z}^2} \left\langle \Theta_{\vec{k}} f, g \right\rangle,$$

where $\Theta_{\vec{k}}$ are smooth truncations of $T$. These truncations $\Theta_{\vec{k}}$ are biparameter Littlewood-Paley-Stein operators, which have been studied extensively in the single parameter setting, see e.g. [DJ84, DJS85, Sem90, Han94]. There are a few results for biparameter Littlewood-Paley-Stein operators due to R. Fefferman, Stein, and Journé [Fef81, FS82, Fef86, Jou85], among others. All of these results are for operators of convolution type. We prove estimates for the square function associated to a larger class of operators including non-convolution operators, which we call biparameter Littlewood-Paley-Stein operators. In particular, we prove bounds for square function operators associated to biparameter Littlewood-Paley-Stein operators, defined by

$$Sf(x)^2 = \sum_{\vec{k}\in\mathbb{Z}} |\Theta_{\vec{k}} f(x)|^2 \tag{2}$$

for $x \in \mathbb{R}^n$ and appropriate $f : \mathbb{R}^n \to \mathbb{C}$.

**Theorem 0.2.** *Let $b_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions, and define $b(x) = b_1(x_1)b_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$. Also let $\Theta_{\vec{k}}$ for $\vec{k} \in \mathbb{Z}^2$ be a collection of biparameter Littlewood-Paley-Stein operators with kernels $\theta_{\vec{k}}$. If*

$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}(x,y)b_1(y_1)dy_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}(x,y)b_2(y_2)dy_2 = 0$$

*for all $\vec{k} \in \mathbb{R}^2$ and $x, y \in \mathbb{R}^n$, then $\|Sf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in L^p(\mathbb{R}^n)$ when $1 < p < \infty$. Note that $S$ is the square function operator defined in* (2)

The formulations and proofs of Theorems 0.1 and 0.2 were introduced by Hart, Lu and Torres [HLT] in a slightly different setting, where $b = \tilde{b} = 1$. Here, we reproduce the proofs from [HLT], and address the additional technical difficulties that arise when accretive functions $b$ and $\tilde{b}$ are used in place of 1.

Some other boundary value problems related to the ones we described can be found in [Boc44],[Wei69],[Ste67, Ste70, Ste73],[FKN81],[JK82],[KP87],[Jac73],[Kra80, Kra07] among others.

The thesis is organized in the following way. In Chapter 1 we recall and prove some results related to the Hardy space theory for the symmetric strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$. The boundedness results of the singular integrals which arise in this setting are a direct consequence of the standard theory of Calderón-Zygmund convolution operators, but we include most of the proofs since we perform some explicit computations which will be used in the later chapters. The Hardy spaces on the non-smooth worm domains are discussed in Chapter 2. In Chapter 3 we perform an explicit computation of the integral kernel of the Szegő projection studied in the previous chapter. In Chapter 4 we develop some biparameter Littlewood-Paley-Stein theory which will be used in Chapter 5 to prove our reduced biparameter $Tb$ theorem. Finally, in Chapter 6 we discuss the extension problem in the setting of the perturbed half-space.

Unless specified, we will use standard and self-explanatory notation. We will denote by $C$, possibly with subscripts, a constant that may change from place to place.

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# Part I

# Szegő kernel and projection on non-smooth worm domains

# Chapter 1

# Hardy Spaces on the symmetric strip

In the Introduction we mentioned that the non-smooth worm domain $D'_\beta$ can be sliced in strips. This feature of $D'_\beta$ will be fundamental in the development of the Hardy spaces $H^p(D'_\beta)$ since it will allow us to use the theory of Hardy spaces on a strip. Hence, we recall here some results concerning the $H^p(S_\beta)$ spaces where $S_\beta$ is the symmetric strip

$$S_\beta = \{x + iy \in \mathbb{C} : |y| < \beta\}.$$

The results contained in this chapter are well known. We refer also to [BK07]. The boundedness results of the singular integrals which arise in this context are consequence of the standard theory of Calderón-Zygmund convolution operators, but, to the best of the author's knowledge, they do not appear explicitly in the literature. Thus, we include most of the proofs since we perform some computations which will be used in the chapters that follow.

After defining the space $H^p(D'_\beta)$, in the first part of the chapter we focus on the Hilbert case $p = 2$; we prove that every function $F$ in $H^2(D'_\beta)$ admits a boundary value function $\widetilde{F}$ in $L^2(\partial D'_\beta)$ such that $\|F\|_{H^2(D'_\beta)} = \|\widetilde{F}\|_{L^2(D'_\beta)}$. This fact allows us to prove that $H^2(S_\beta)$ is a Reproducing Kernel Hilbert space (see e.g. [Aro50]) and we compute explicitly its reproducing kernel. A primary role in proving these results is played by the Paley–Wiener Theorem for the strip (Theorem 1.2). In the second part of the chapter, we extend the results obtained for the space $H^2(S_\beta)$ to the spaces $H^p(S_\beta)$, $p \in (1, \infty)$.

For every $p \in (1, \infty)$, the Hardy space for the strip $S_\beta$ is the functional space

$$H^p(S_\beta) = \left\{ f \text{ holomorphic in } S_\beta : \|f\|_{H^p(S_\beta)} < \infty \right\},$$

where

$$\|f\|^p_{H^p(S_\beta)} = \sup_{0\le y<\beta} \left[ \int_{\mathbb{R}} |f(x+iy)|^p + \int_{\mathbb{R}} |f(x-iy)|^p \, dy \right]. \tag{1.1}$$

The proof of the next proposition is elementary and we leave the details to the reader.

**Proposition 1.1.** *Let $K$ a compact subset of $S_\beta$. Then, for every $f \in H^p(S_\beta)$, it holds*

$$\sup_{z\in K} |f(z)| \le C_{p,K} \|f\|^p_{H^p(S_\beta)},$$

*where $C_{p,K}$ is a constant which depends only on $p$ and the compact set $K$.*

## 1.1 Case $p = 2$

We start stating the Paley–Wiener Theorem for a strip, which relates the growth of a holomorphic function in a strip with the growth of the Fourier transform of its restriction to the real line. Then, we study the boundary behaviour of functions in $H^2(S_\beta)$.

**Theorem 1.2.** ***(Paley–Wiener Theorem for a strip)*** *Let $f_0$ in $L^2(\mathbb{R})$. Then the following are equivalent:*

*(i) $f_0$ is the restriction to the real line of a function $F$ holomorphic in the strip $S_\beta$ such that*

$$\sup_{|y|<\beta} \int_{\mathbb{R}} |F(x+iy)|^2 \, dy < \infty;$$

*(ii) $e^{\beta|\xi|}\hat{f}_0 \in L^2(\mathbb{R})$.*

*Moreover, the following relationship holds*

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}_0(\xi) e^{iz\xi} \, d\xi \\ &= \mathcal{F}^{-1}[e^{-\operatorname{Im} z(\cdot)} \hat{f}_0](\operatorname{Re} z). \end{aligned} \tag{1.2}$$

*Proof.* See [PW87]. □

**Remark 1.3.** The notation used in the statement of the Paley–Wiener Theorem will be consistently used throughout this work.

The Paley–Wiener Theorem turns out to be extremely useful since it reduces the studying of holomorphic functions in a strip to the studying of some $L^2$ functions on the real line via the Fourier Transform. Using the Paley–Wiener Theorem we will prove that each function in $H^2(S_\beta)$ admits boundary values in $L^2(\partial S_\beta)$.

**Proposition 1.4.** *Let $F \in H^2(S_\beta)$ and for every $y$ in $[0, \beta)$ define*

$$\mathcal{L}(y) = \left[ \int_{\mathbb{R}} |F(x+iy)|^2 \, dy + \int_{\mathbb{R}} |F(x-iy)|^2 \, dy \right].$$

*Then*

$$\begin{aligned} \|F\|^2_{H^2(S_\beta)} &= \sup_{0 \le y < \beta} \mathcal{L}(y) \\ &= \lim_{y \to \beta^-} \mathcal{L}(y) \\ &= \frac{1}{\pi} \int_{\mathbb{R}} |\hat{f}_0(\xi)|^2 \, \mathrm{Ch}[2\beta\xi] \, d\xi. \end{aligned}$$

*Proof.* By the Paley–Wiener Theorem, we get

$$\begin{aligned} \mathcal{L}(y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-2y\xi} |\hat{f}_0(\xi)|^2 d\xi + \frac{1}{2\pi} \int_{\mathbb{R}} e^{2y\xi} |\hat{f}_0(\xi)|^2 \, d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} |\hat{f}_0(\xi)|^2 \, \mathrm{Ch}[2y\xi] \, d\xi \end{aligned}$$

and the conclusion follows. □

**Remark 1.5.** The previous proposition establishes an isometry between the Hardy space $H^2(S_\beta)$ and a weighted $L^2$ space of the real line.

Now we show that each $F$ in $H^2(S_\beta)$ admits boundary values in $L^2(\partial S_\beta)$. Since $\partial S_\beta$ has two components, when we consider a function, say $G$, defined on $\partial S_\beta$ we mean a couple of functions $(G_1, G_2)$ where $G_1$ is defined on $\partial_1 S_\beta := \{x + i\beta : x \in \mathbb{R}\}$ and $G_2$ is defined on $\partial_2 S_\beta := \{x - i\beta : x \in \mathbb{R}\}$. Hence, the norm $L^2(\partial S_\beta)$ is given by

$$\|G\|^2_{L^2(\partial S_\beta)} = \int_{\mathbb{R}} |G_1(x+i\beta)|^2 \, dx + \int_{\mathbb{R}} |G_2(x-i\beta)|^2 \, dx. \tag{1.3}$$

Notice that both $\partial_1 S_\beta$ and $\partial_2 S_\beta$ can be identified with the real line $\mathbb{R}$.

The Paley–Wiener Theorem guarantees that the following definition is meaningful.

**Definition 1.6.** Given a function $F$ in $H^2(S_\beta)$, we define on $\partial S_\beta$ the function $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2)$ where

$$\widetilde{F}_1(x+i\beta) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\beta\xi}\hat{f}_0 e^{ix\xi}\ d\xi \tag{1.4}$$

and

$$\widetilde{F}_2(x-i\beta) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{\beta\xi}\hat{f}_0 e^{ix\xi}\ d\xi. \tag{1.5}$$

**Proposition 1.7.** *Let $F$ be a function in $H^2(S_\beta)$ and consider $\widetilde{F}$ in $L^2(\partial S_\beta)$ defined as above. Then,*

$$\lim_{y\to\beta^-} \|F(\cdot+iy)-\widetilde{F}_1\|_{L^2(\partial_1 S_\beta)} = 0$$

*and*

$$\lim_{y\to\beta^-} \|F(\cdot-iy)-\widetilde{F}_2\|_{L^2(\partial_2 S_\beta)} = 0.$$

*Proof.* We only prove the convergence on the component $\partial_1 S_\beta$ of the boundary. The result for $\partial_2 S_\beta$ follows analogously. We have

$$\begin{aligned}\|F(\cdot+iy)-\widetilde{F}_1\|_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |F(x+iy)-\widetilde{F}_1(x+i\beta)|^2\ dx \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \left|e^{-y\xi}\hat{f}_0(\xi)-e^{-\beta\xi}\hat{f}_0(\xi)\right|^2\ d\xi \\ &\le \frac{1}{\pi}\int_0^\infty |\hat{f}_0(\xi)|^2\ d\xi + \frac{1}{\pi}\int_{-\infty}^0 e^{-2\beta\xi}|\hat{f}_0(\xi)|^2\ d\xi \\ &< \infty.\end{aligned}$$

By the Dominated Convergence Theorem we can conclude. □

**Remark 1.8.** We will constantly use the notation of Definition 1.6 to denote the boundary values of a function in $H^2(S_\beta)$.

From Proposition 1.7 and Proposition 1.4 we can deduce that $H^2(S_\beta)$ is a Reproducing Kernel Hilbert space. In fact, each function $F$ in $H^2(S_\beta)$ admits a boundary value function $\widetilde{F}$ such that

$$\|F\|^2_{H^2(S_\beta)} = \frac{1}{\pi}\int_{\mathbb{R}} |\hat{f}_0(\xi)|^2\,\mathrm{Ch}[2\beta\xi]\ d\xi$$

$$= \int_{\mathbb{R}} |\widetilde{F}_1(x+i\beta)|^2 \, dx + \int_{\mathbb{R}} |\widetilde{F}_2(x-i\beta)|^2 \, dx$$
$$= \|\widetilde{F}\|^2_{L^2(\partial S_\beta)}.$$

Therefore, we can endow $H^2(S_\beta)$ with a inner product; namely, given $F$ and $G$ in $H^2(S_\beta)$, we define

$$\begin{aligned}\langle F, G \rangle_{H^2(S_\beta)} &:= \left\langle \widetilde{F}, \widetilde{G} \right\rangle_{L^2(\partial S_\beta)} \\ &= \int_{\mathbb{R}} \widetilde{F}_1(x+i\beta)\overline{\widetilde{G}_1(x+i\beta)} \, dx + \int_{\mathbb{R}} \widetilde{F}_2(x-i\beta)\overline{\widetilde{G}_2(x-i\beta)}.\end{aligned}$$

Furthermore, Proposition 1.1 ensures that the point-evaluation functionals are bounded on $H^2(S_\beta)$. In conclusion, we have the following result.

**Proposition 1.9.** *The Hardy space $H^2(S_\beta)$ is a Reproducing Kernel Hilbert space with the inner product $\langle F, G \rangle_{H^2(S_\beta)} = \left\langle \widetilde{F}, \widetilde{G} \right\rangle_{L^2(\partial S_\beta)}$. Thus, there exists a function $K : S_\beta \times S_\beta \to \mathbb{C}$ such that*

*(i) for all $z, w$ in $S_\beta$, $K(z,w) = \overline{K(w,z)}$;*

*(ii) for all $z$ in $S_\beta$, $K(\cdot, z)$ belongs to $H^2(S_\beta)$;*

*(iii) for all $F$ in $H^2(S_\beta)$ and $z$ in $S_\beta$ we have $f(z) = \langle f, K(\cdot, z) \rangle_{H^2(S_\beta)}$.*

*Such a function $K$ is called reproducing kernel of $H^2(S_\beta)$.*

In general it is very hard to find an explicit formula for the reproducing kernel of a space, but, in our setting, the Paley–Wiener Theorem helps us once again.

**Theorem 1.10.** *The reproducing kernel of $H^2(S_\beta)$ is the function*

$$K(z,w) = \frac{1}{2\beta} \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}(w - \overline{z})]}. \tag{1.6}$$

*Proof.* Let $F$ be a function in $H^2(S_\beta)$. Then,

$$\begin{aligned}F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}_0(\xi) e^{iz\xi} \, d\xi \\ &= \int_{\mathbb{R}} \widetilde{F}_1(x+i\beta)\overline{\widetilde{K}_1(x+i\beta, z)} \, dx + \int_{\mathbb{R}} \widetilde{F}_2(x-i\beta)\overline{\widetilde{K}_2(x-i\beta, z)} \, dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-2\beta\xi} \widehat{f}_0(\xi)\overline{\widehat{k}_0(\xi, z)} \, d\xi + \frac{1}{2\pi} \int_{\mathbb{R}} e^{2\beta\xi} \widehat{f}_0(\xi)\overline{\widehat{k}_0(\xi, z)} d\xi\end{aligned}$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \widehat{f_0}(\xi)\overline{\widehat{k_0}(\xi,z)}(z,\xi)\,\mathrm{Ch}[2\beta\xi]\ d\xi,$$

where $k_0(\cdot,z)$ is the restriction to the real line of $K(\cdot,z)$. We deduce that

$$\widehat{k_0}(\xi,z) = \frac{1}{2}\frac{e^{-i\bar{z}\xi}}{\mathrm{Ch}[2\beta\xi]},$$

therefore

$$\begin{aligned} K(w,z) &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{k_0}(\xi,z)e^{iw\xi}\ d\xi \\ &= \frac{1}{4\pi}\int_{\mathbb{R}} \frac{e^{i(w-\bar{z})\xi}}{\mathrm{Ch}[2\beta\xi]}\ d\xi. \qquad (1.7) \end{aligned}$$

Notice that for $(w,z)$ in $\overline{S_\beta}\times S_\beta$ the above integral is absolutely convergent. So, our problem is now to compute the integral

$$\int_{\mathbb{R}} \frac{e^{i\tau\xi}}{\mathrm{Ch}[2\beta\xi]}\ d\xi,$$

where $|\,\mathrm{Im}\,\tau| < 2\beta$. We conclude this computation using the Residue Theorem. Suppose $\tau$ is such that $\mathrm{Re}\,\tau > 0$. Consider the function $g_\tau(\zeta) = \frac{e^{i\tau\zeta}}{\mathrm{Ch}[2\beta\xi]}$ and, chosen a number $R > 0$, the rectangle $\mathcal{R}$ of vertices $(-R,0),(0,R),(R,i\frac{\pi}{2\beta})$ and $(-R,i\frac{\pi}{2\beta})$. Then

$$\int_{\partial\mathcal{R}} g_\tau(\zeta)\ d\zeta = 2\pi i \mathrm{Res}[g_\tau, i\frac{\pi}{4\beta}],$$

where

$$\mathrm{Res}[g_\tau, i\frac{\pi}{4\beta}] = \lim_{\zeta\to i\frac{\pi}{4\beta}} \frac{(\zeta - i\frac{\pi}{4\beta})e^{i\tau\xi}}{\mathrm{Ch}[2\beta\xi]} = \frac{1}{2\beta i}e^{-\tau\frac{\pi}{4\beta}}$$

and

$$\int_{\partial\mathcal{R}} g_\tau(\zeta)\ d\zeta = \int_{-R}^{R} [g_\tau(\xi) - g_\tau(\xi + i\frac{\pi}{2\beta})]\ d\xi + \int_0^{\frac{\pi}{2\beta}} i[g_\tau(R+i\xi) - g_\tau(-R+i\xi)]\ d\xi.$$

Now

$$\begin{aligned} \int_{-R}^{R} [g_\tau(\xi) - g_\tau(\xi + i\frac{\pi}{2\beta})]\ d\xi &= \int_{-R}^{R} \Big[\frac{e^{i\tau\xi}}{\mathrm{Ch}[2\beta\xi]} - \frac{e^{i\tau\xi}e^{-\frac{\tau\pi}{2\beta}}}{\mathrm{Ch}[2\beta\xi + i\pi]}\Big]\ d\xi \\ &= (1 + e^{-\frac{\tau\pi}{2\beta}})\int_{-R}^{R} \frac{e^{i\tau\xi}}{\mathrm{Ch}[2\beta\xi]}\ d\xi, \end{aligned}$$

while

$$\Big|\int_0^{\frac{\pi}{2\beta}} i[g_\tau(R+i\xi) - g_\tau(-R+i\xi)]\ d\xi\Big| \le 2\,\mathrm{Sh}[\mathrm{Im}\,\tau R]\int_0^{\frac{\pi}{2\beta}} \frac{e^{-\xi Re\tau}}{[\mathrm{Sh}^2[2\beta R] + \cos^2(2\beta\xi)]^{\frac{1}{2}}}$$

$$\leq C \frac{\mathrm{Sh}[\operatorname{Im} \tau R]}{\mathrm{Sh}[2\beta R]}.$$

Hence, when $R$ tends to 0, we have

$$\int_0^{\frac{\pi}{2\beta}} i[g_\tau(R + i\xi) - g_\tau(-R + i\xi)] \, d\xi \to 0$$

uniformly in $\tau$ when $\tau$ varies in a compact subset of $S_{2\beta}$. In conclusion

$$\begin{aligned} K(w, z) &= \frac{1}{4\pi} \int_{\mathbb{R}} \frac{e^{i(w-\overline{z})\xi}}{\mathrm{Ch}[2\beta\xi]} \, d\xi \\ &= \frac{1}{4\pi} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i(w-\overline{z})\xi}}{\mathrm{Ch}[2\beta\xi]} \, d\xi \\ &= \frac{1}{2\beta} \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}(w - \overline{z})]} \end{aligned}$$

as we wished. Notice that we performed the computation for $\tau = w - \overline{z}$ such that $\operatorname{Re} \tau > 0$; the computation when $\operatorname{Re} \tau < 0$ follows analogously integrating along the rectangle of vertices $(-R, 0), (0, R), (R, -i\frac{\pi}{2\beta})$ and $(-R, -i\frac{\pi}{2\beta})$. □

**Remark 1.11.** For every fixed $z \in S_\beta$ the function $K(\cdot, z)$ is well-defined on $\overline{S_\beta}$. Hence, the boundary value functions $\widetilde{K}_1(\cdot, z)$ and $\widetilde{K}_2(\cdot, z)$ are simply the restrictions of $K(\cdot, z)$ to $\partial_1 S_\beta$ and $\partial_2 S_\beta$ respectively.

Having proved that $H^2(S_\beta)$ is a Reproducing Kernel Hilbert space, we have that every function $F$ of $H^2(S_\beta)$ is reproduced by integration against $K$, that is,

$$\begin{aligned} F(z) &= \langle F, K(\cdot, z)\rangle_{H^2(S_\beta)} \\ &= \int_{\mathbb{R}} \widetilde{F}_1(x + i\beta) K(z, x + i\beta) \, dx + \int_{\mathbb{R}} \widetilde{F}_2(x - i\beta) K(z, x - i\beta) \, dx. \end{aligned}$$

We show that we can actually *produce* functions in $H^2(S_\beta)$ via integration against the kernel $K$. We have the following proposition.

**Proposition 1.12.** *Let be $F = (F_1, F_2)$ a function in $L^2(\partial S_\beta)$, that is, $F_1$ and $F_2$ are functions in $L^2(\mathbb{R})$. Let us define the function*

$$SF(z) := \int_{\mathbb{R}} F_1(x) K(z, x + i\beta) \, dx + \int_{\mathbb{R}} F_2(x) K(z, x - i\beta) \, dx.$$

*Then, the operator* $F \mapsto SF$ *is a bounded linear operator*

$$S : L^2(\partial S_\beta) \to H^2(S_\beta)$$

*such that*

$$\|SF\|_{H^2(S_\beta)} \le \|F\|_{L^2(\partial S_\beta)}.$$

*Moreover*

$$\widetilde{SF}_1(x+i\beta) = \mathcal{F}^{-1}\Big[\frac{\widehat{F}_1(\cdot)e^{-2\beta(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](x) + \mathcal{F}^{-1}\Big[\frac{\widehat{F}_2(\cdot)}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](x) \tag{1.8}$$

$$\widetilde{SF}_2(x-i\beta) = \mathcal{F}^{-1}\Big[\frac{\widehat{F}_1(\cdot)}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](x) + \mathcal{F}^{-1}\Big[\frac{\widehat{F}_2(\cdot)e^{2\beta(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](x). \tag{1.9}$$

*Proof.* First of all we must prove that the function $SF$ is holomorphic in $S_\beta$. In order to do so we prove that

$$\int_{\mathbb{R}} |F_i(x+i\beta)K(z,x+i\beta)|\ dx$$

is uniformly bounded in $z$ varying in a compact subset $K$ of $S_\beta$ and $i=1,2$. If we prove this, then, for every closed curve $\gamma$ in $S_\beta$, Fubini's theorem and the holomorphicity of $K(z,x+i\beta)$ for every $x$ in $\mathbb{R}$ would assure that

$$\int_\gamma\int_{\mathbb{R}} F_i(x+i\beta)K(\zeta,x+i\beta)\ dxd\zeta = \int_{\mathbb{R}}\int_\gamma F_i(x+i\beta)K(\zeta,x+i\beta)\ d\zeta dx = 0,$$

hence, the holomorphicity of $SF$. So, for every $z$ in a compact subset $K$ of $S_\beta$,

$$\begin{aligned}
|K[z,x+i\beta]| &= \frac{1}{2\beta}\frac{1}{|\,\mathrm{Ch}[\frac{\pi}{4\beta}(z-x+i\beta)]|}\\
&= \frac{1}{2\beta}\frac{1}{\big[\,\mathrm{Sh}^2[\frac{\pi}{4\beta}(\mathrm{Re}\,z-x)]+\cos^2[\frac{\pi}{4\beta}(\mathrm{Im}\,z+\beta)]\big]^{\frac12}}\\
&\le \frac{1}{2\beta}\frac{1}{\big[\,\mathrm{Sh}^2[\frac{\pi}{4\beta}(\mathrm{Re}\,z-x)]+C_K\big]^{\frac12}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{z\in K}\int_{\mathbb{R}} |F_1(x+i\beta)K(z,x+i\beta)|\ dx &\le \|K[z,\cdot+i\beta]\|_{L^2(\mathbb{R})}\|F_1\|_{L^2(\mathbb{R})}\\
&\le C_{\beta,K}\|F_1\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Analogusly we obtain the estimate

$$\sup_{z\in K}\int_{\mathbb{R}} |F_2(x-i\beta)K(z,x-i\beta)|\ dx \le C_{\beta,K}\|F_2\|_{L^2(\mathbb{R})}$$

and thus the holomorphicity of $SF$ on $S_\beta$. Using Theorem 1.10 and Parseval's identity we obtain that

$$\begin{aligned} SF(z) &= \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\widehat{F}_1(\xi)e^{-\beta\xi}e^{iz\xi}}{\mathrm{Ch}[2\beta\xi]}\ d\xi + \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\widehat{F}_2(\xi)e^{\beta\xi}e^{iz\xi}}{\mathrm{Ch}[2\beta\xi]}\ d\xi \\ &= \mathcal{F}^{-1}\Big[\frac{\widehat{F}_1(\cdot)e^{-(\operatorname{Im} z+\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](\operatorname{Re} z) + \mathcal{F}^{-1}\Big[\frac{\widehat{F}_2(\cdot)e^{-(\operatorname{Im} z-\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\Big](\operatorname{Re} z). \end{aligned} \tag{1.10}$$

Plancherel's theorem leads now to the estimate

$$\|SF\|^2_{H^2(S_\beta)} \le \|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})}.$$

Another application of Parseval's identity shows that

$$\lim_{y\to\beta}\|SF(\cdot+iy) - \widetilde{SF}_1\|_{L^2(\mathbb{R})} = 0$$
$$\lim_{y\to\beta}\|SF(\cdot-iy) - \widetilde{SF}_2\|_{L^2(\mathbb{R})} = 0$$

as we wished. □

**Remark 1.13.** We stress that if we start with a couple of functions $(F_1, F_2)$ which already are boundary values of a functions $F$ in $H^2(S_\beta)$, then (1.8) and (1.9) coincide with (1.4) and (1.5), respectively, as expected. This can easily seen by means of the Paley–Wiener Theorem once again.

So far we have defined a space of holomorphic functions $H^2(S_\beta)$ and proved that every function $F$ in this space admits boundary values $(\widetilde{F}_1, \widetilde{F}_2)$ in $L^2(\partial S_\beta)$. These functions $\widetilde{F}_1$ and $\widetilde{F}_2$ are defined by (1.4) and (1.5) and they are boundary values of $F$ in the sense of Theorem 1.7. Moreover, we proved that $H^2(S_\beta)$ is a Reproducing Kernel Hilbert space and we showed in Theorem 1.12 how to obtain a function in $H^2(S_\beta)$ given any couple of fuctions $(F_1, F_2)$ in $L^2(\partial S_\beta)$. Since to every function in $F$ in $H^2(S_\beta)$ we can associate a function $\widetilde{F}$ in $L^2(\partial S_\beta)$, we can see $H^2(S_\beta)$ as a subspace of $L^2(\partial S_\beta)$. To remark this point of view we introduce the notation

$$H^2(\partial S_\beta) := \{\widetilde{G} = (G_1, G_2) \in L^2(\partial S_\beta) : \exists F \in H^2(S_\beta) \text{ s.t. } (G_1, G_2) = (\widetilde{F}_1, \widetilde{F}_2)\}. \tag{1.11}$$

Notice that Proposition 1.1 assures that $H^2(\partial S_\beta)$ is a closed subspace of $L^2(\partial S_\beta)$. Thus, we can summarize what we have seen so far in the following theorem.

**Theorem 1.14.** *The operator*

$$\begin{aligned} \widetilde{S} : L^2(\partial S_\beta) &\to H^2(\partial S_\beta) \\ (F_1, F_2) &\to (\widetilde{SF}_1, \widetilde{SF}_2) \end{aligned} \tag{1.12}$$

*defined by (1.8) and (1.9) is a Hilbert space orthogonal projection operator.*

**Remark 1.15.** The operator $\widetilde{S}$ associates to a couple of functions $(F_1, F_2)$ in a new couple of functions $(\widetilde{SF}_1, \widetilde{SF}_2)$. We will costantly use the compact notation $F \to \widetilde{SF}$ meaning (1.12). If we need to be more specific and indicate which component of the boundary $\partial S_\beta$ we are interested in, we will use the notation with subscripts.

**Definition 1.16.** The operator $\widetilde{S} : L^2(\partial S_\beta) \to H^2(\partial S_\beta)$ is called *Szegő projection.*

## 1.2 Case $1 < p < \infty$

In this section we prove the validity of Theorem 1.14 for every $p$ in $(1, +\infty)$, that is

**Theorem 1.17.** *The Szegő projection $S$ extends to a bounded linear operator*

$$\begin{aligned} \widetilde{S} : L^p(\partial S_\beta) &\to H^p(\partial S_\beta) \\ (F_1, F_1) &\to (\widetilde{SF}_1, \widetilde{SF}_2) \end{aligned}$$

*for every $p \in (1, \infty)$,*

The operator $\widetilde{S}$ acts on a couple of functions $(F_1, F_2)$, but, by linearity, it suffices to prove our boundedness results for initial data of the form $(F_1, 0_2)$ or $(0_1, F_2)$ where the functions $0_i$, $i = 1, 2$, are the constant functions zero.

Let us focus now on $F = (F_1, 0_2)$ and suppose that $F_1$ is in $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$. The situation for the couple $(0_1, F_2)$ is analogue. Then, for every $x + iy$ in $S_\beta$, we have

$$SF(x + iy) = \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)e^{-(y+\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta\cdot]}\right](x) = \frac{1}{2\beta}\int_{\mathbb{R}} \frac{F_1(u)}{\mathrm{Ch}[\frac{\pi}{4\beta}(x - u + i(y + \beta))]}\, du \tag{1.13}$$

and

$$\widetilde{SF}_1(x + i\beta) = \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)e^{-2\beta(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\right](x); \tag{1.14}$$

$$\widetilde{SF}_2(x+i\beta) = \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)}{2\,\mathrm{Ch}[2\beta(\cdot)]}\right](x) = \frac{1}{2\beta}\int_{\mathbb{R}} \frac{F_1(y)}{\mathrm{Ch}[\frac{\pi}{4\beta}(x-y)]}\, dy. \tag{1.15}$$

All the above operators are multipliers operators, therefore they admit a representation via a convolution kernel. The convolution kernels of the operators $F \mapsto SF$ and $F \mapsto \widetilde{SF}_2$ are immediately obtained by Theorem 1.10 and 1.12. Later we will find explicitly also the convolution kernel of the operator $F \mapsto \widetilde{SF}_1$.

**Proposition 1.18.** *Let $F = (F_1, 0_2)$ be in $L^p(\partial S_\beta)$, $p \in (1,\infty)$. Then,*

$$SF(z) = \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)e^{-(\mathrm{Im}z+\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta\cdot]}\right](\mathrm{Re}z) = \frac{1}{2\beta}\int_{\mathbb{R}} \frac{F_1(y)}{\mathrm{Ch}[\frac{\pi}{4\beta}(\mathrm{Re}z - y + i(\mathrm{Im}z+\beta))]}\, dy$$

*is in $H^p(S_\beta)$. Moreover, there exists a constant $C_p$ such that*

$$\|SF\|_{H^p(S_\beta)} \le C_p \|F\|_{L^p(\partial S_\beta)}.$$

*Hence, $S$ is a bounded linear operator $S : L^p(\partial S_\beta) \to H^p(S_\beta)$.*

*Proof.* The holomorphicity of $SF$ on $S_\beta$ follows as in the proof of Proposition 1.12. It remains to prove that $\|SF\|_{H^p(S_\beta)} < \infty$. For every fixed $y$ such that $|y| < \beta$, the operator

$$F \mapsto SF(\cdot + iy) = \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)e^{-(y+\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta\cdot]}\right]$$

is a multiplier operator trivially bounded on $L^p(\mathbb{R})$ since the multiplier

$$m_y(\xi) = \frac{e^{-(y+\beta)\xi}}{2\,\mathrm{Ch}[2\beta\xi]}$$

is a Schwartz function for every fixed $y$ with $|y| < \beta$. We prove that the norm of this multiplier operator is bounded by a constant which does not depend on $y$. We do it showing that $m_y$ satisfies Mihlin's multiplier condition uniformly in $y$ (see, for instance, [Gra08, Thm. 5.2.7]). We have

$$\begin{aligned}
\left|\frac{d}{d\xi}m_y(\xi)\right| &= \left|\frac{-(y+\beta)e^{-(y+\beta)\xi}\,\mathrm{Ch}[2\beta\xi] - 2\beta e^{-(y+\beta)\xi}\,\mathrm{Sh}[2\beta\xi]}{\mathrm{Ch}^2[2\beta\xi]}\right| \\
&= \frac{e^{-(y+\beta)\xi}}{\mathrm{Ch}^2[2\beta\xi]}\left|y\Big[\mathrm{Ch}[2\beta\xi] + \mathrm{Sh}[2\beta\xi]\Big] + [\beta - y]\Big[\mathrm{Sh}[2\beta\xi]\Big] + \beta\Big[\mathrm{Ch}[2\beta\xi] + \mathrm{Sh}[2\beta\xi]\Big]\right| \\
&\le \frac{e^{-(y+\beta)\xi}}{\mathrm{Ch}^2[2\beta\xi]}\left[2\beta\Big|\mathrm{Ch}[2\beta\xi] + \mathrm{Sh}[2\beta\xi]\Big| + [\beta - y]\Big|\mathrm{Sh}[2\beta\xi]\Big|\right]
\end{aligned}$$

$$= I + II.$$

It is easily seen that

$$I = 2\beta \frac{e^{-(y+\beta)\xi}}{\mathrm{Ch}^2[2\beta\xi]} \Big| \mathrm{Ch}[2\beta\xi] + \mathrm{Sh}[2\beta\xi] \Big|$$

is bounded and decays exponentially when $|\xi| \to \infty$ for every $y$ such that $|y| \le |\beta|$. About $II$ we have

$$\begin{aligned} II &= \frac{e^{-(y+\beta)\xi}}{\mathrm{Ch}^2[2\beta\xi]} [\beta - y] \Big| \mathrm{Sh}[2\beta\xi] \Big| \\ &\le [\beta - y] e^{-(y+\beta)\xi} e^{-2\beta|\xi|}. \end{aligned}$$

Hence, we have exponential decay for every $y$ such that $|y| \le \beta$ when $\xi \to +\infty$. If $\xi \to -\infty$, we obtain

$$\begin{aligned} II &= [\beta - y] e^{(\beta - y)\xi} \\ &= \frac{|\xi| [\beta - y] e^{\xi(\beta - y)}}{|\xi|} \\ &\le \frac{C_\beta}{|\xi|}. \end{aligned}$$

Hence, we can conclude that $\sup_{|y|<\beta} \left| \frac{d}{d\xi} m_y(\xi) \right| =\le \frac{C_\beta}{|\xi|}$. Moreover,

$$\sup_{|y|<\beta} \|m_y\|_{L^\infty} = 1.$$

Thus, Mihlin's multiplier theorem implies

$$\begin{aligned} \int_{\mathbb{R}} |SF(x+iy)|^p \, dx &\le C_p \int_{\mathbb{R}} |F_1(x)|^p \, dx \\ &= C_p \int_{\mathbb{R}} |F(x)|^p \, dx. \end{aligned} \tag{1.16}$$

The proof is complete. □

It remains to prove that $SF(\cdot + iy) \to \widetilde{SF}_1(\cdot + i\beta)$ and $SF(\cdot + iy) \to \widetilde{SF}_2(\cdot - i\beta)$ in $L^p(\mathbb{R})$, $p \in (1, \infty)$. The latter limit is easily obtained using (1.13) and (1.15). In fact,

$$\int_{\mathbb{R}} |SF(x+iy) - \widetilde{SF}_2(x - i\beta)|^p \, dx = \frac{1}{2\beta} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \Big[ \frac{F_1(x-u)}{\mathrm{Ch}[\frac{\pi}{4\beta}(u + i(y+\beta))]} - \frac{F_1(x-u)}{\mathrm{Ch}[\frac{\pi}{4\beta} u]} \, du \Big] \right|^p dx$$

$$\leq \|F\|^p_{L^p(\mathbb{R})} \Big[ \int_{\mathbb{R}} \Big| \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}(u+i(y+\beta))]} - \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}u]} \, du \Big| \Big]^p$$
$$\to 0 \qquad (1.17)$$

as $y$ tends to $-\beta^+$.

To show that $SF(\cdot + iy) \to \widetilde{SF}_1(\cdot + i\beta)$ as $y$ tends to $\beta^-$ is more complicated to prove and we start proving that it holds in a weak sense when $F$ is regular.

**Proposition 1.19.** *Let $F = (F_1, 0_2)$ be in $L^p(\partial S_\beta)$ where $F_1$ is a Schwartz function. Then, for every $y$ such that $|y| < \beta$, $SF(\cdot + iy)$ converges weakly $L^p$ to $\widetilde{SF}_1(\cdot + i\beta)$ as $y$ tends to $\beta^-$, $p \in (1, \infty)$.*

*Proof.* For every Schwartz function $G$ we have

$$\begin{aligned}
\lim_{y\to\beta^-} \int_{\mathbb{R}} SF(x+iy)\overline{G(x)} \, dx &= \lim_{y\to\beta^-} \int_{\mathbb{R}} \frac{e^{-(y+\beta)\xi}\widehat{F_1}(\xi)}{2\,\mathrm{Ch}[2\beta\xi]}\overline{\widehat{G}(\xi)} \, d\xi \\
&= \int_{\mathbb{R}} \frac{e^{-2\beta\xi}\widehat{F_1}(\xi)}{2\,\mathrm{Ch}[2\beta\xi]}\overline{\widehat{G}(\xi)} \, d\xi \\
&= \int_{\mathbb{R}} \widetilde{SF}_1(x)\overline{G(x)} \, dx,
\end{aligned}$$

where we can to switch the limit and the integral by Dominated Convergence Theorem. □

Therefore, if we prove that $SF(\cdot + iy)$ admits a limit in a stronger sense, the limit has to be $\widetilde{SF}_1$, at least when $F$ is regular. To make our notation lighter, instead of computing the limit $\lim_{y\to\beta^-} SF(\cdot + iy)$, we compute the equivalent limit $\lim_{\varepsilon\to 0^+} SF[\cdot + i(\beta - \varepsilon)]$, where

$$\begin{aligned}
SF[x+i(\beta-\varepsilon)] &= \int_{\mathbb{R}} \frac{e^{-(2\beta-\varepsilon)\xi}\widehat{F_1}(\xi)}{2\,\mathrm{Ch}[2\beta\xi]} \, d\xi \\
&= \frac{1}{2\beta}\int_{\mathbb{R}} \frac{F(x-y)}{\mathrm{Ch}[\frac{\pi}{4\beta}(y+i(2\beta-\varepsilon))]} \, dy \\
&= \frac{1}{2\beta i}\int_{\mathbb{R}} \frac{F(x-y)}{\mathrm{Sh}[\frac{\pi}{4\beta}(y-i\varepsilon)]} \, dy \\
&= \frac{1}{2\beta}\int_{\mathbb{R}} F(x-y)\frac{\mathrm{Ch}[\frac{\pi y}{4\beta}]\sin[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi u}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} dy - \frac{i}{2\beta}\int_{\mathbb{R}} F(x-y)\frac{\mathrm{Sh}[\frac{\pi y}{4\beta}]\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi u}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} dy \\
&= [K_\varepsilon * F](x) - i[\widetilde{K}_\varepsilon * F](x). \qquad (1.18)
\end{aligned}$$

Thus, we can study the convolution kernels $K_\varepsilon$ and $\widetilde{K}_\varepsilon$ separately.

**Proposition 1.20.** *The family of functions*

$$\frac{1}{2}K_\varepsilon(x) = \frac{1}{\beta}\frac{\mathrm{Ch}[\frac{\pi x}{4\beta}]\sin[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi x}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]}$$

*is a summability kernel for* $\varepsilon \to 0^+$.

*Proof.* We have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}|K_\varepsilon(x)|\,dx =& \frac{1}{2}\int_{\mathbb{R}}K_\varepsilon(x)\,dx = \frac{1}{4\beta}\int_{\mathbb{R}}\frac{\mathrm{Ch}[\frac{\pi x}{4\beta}]\sin[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi x}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]}dx\\ &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{1+x^2}\,dx\\ &= 1.\end{aligned}$$

Moreover, for every $\delta > 0$, we have

$$\begin{aligned}\int_{|x|>\delta}\frac{\mathrm{Ch}[\frac{\pi x}{4\beta}]\sin[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi x}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]}dx &\le \int_{|x|>\delta}\frac{\mathrm{Ch}[\frac{\pi x}{4\beta}]\sin[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi x}{4\beta}]}dx\\ &\to 0\end{aligned}$$

as $\varepsilon$ tends to 0. Hence, the proposition is proved. □

Now, for suitable functions $\varphi$, we define

$$\big\langle \varphi, p.v.\frac{1}{\mathrm{Sh}[\frac{\pi x}{4\beta}]}\big\rangle := \frac{1}{2\beta}\lim_{\varepsilon\to 0^+}\int_{|\frac{\pi x}{4\beta}|>\varepsilon}\frac{\varphi(x)}{\mathrm{Sh}[\frac{\pi x}{4\beta}]}\,dx$$

where $\varphi$ is a Schwartz function. It is not hard to prove that $p.v.\frac{1}{\mathrm{Sh}[\frac{\pi x}{4\beta}]}$ is a well-defined tempered distribution.

**Theorem 1.21.** *The operator*

$$\begin{aligned}T : \mathcal{S}(\mathbb{R}) &\to \mathcal{S}(\mathbb{R})\\ \varphi &\mapsto p.v.\frac{1}{\mathrm{Sh}[\frac{\pi x}{4\beta}]} * \varphi\end{aligned} \tag{1.19}$$

*extends to a bounded linear operator*

$$T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$$

*for every* $1 < p < \infty$.

*Proof.* This is a standard fact; see e.g. [Gra08, Thm. 4.4.1]. □

Let us denote by $T_\varepsilon$ the truncated operator associated to $T$, that is,

$$T_\varepsilon F(x) = \int_{|\frac{\pi x}{4\beta}|>\varepsilon} \frac{F(x-y)}{\mathrm{Sh}[\frac{\pi y}{4\beta}]}\, dy.$$

We compare the operator $T_\varepsilon$ and the convolution with the kernel $\widetilde{K}_\varepsilon$ obtaining the following result.

**Proposition 1.22.** *Let $F$ be a function in $L^p(\mathbb{R})$. Then,*

$$\lim_{\varepsilon\to 0^+} \|\widetilde{K}_\varepsilon * F - T_\varepsilon F\|_{L^p(\mathbb{R})} \to 0$$

*for every $p$ in $(1,\infty)$.*

*Proof.* It holds

$$\widetilde{K}_\varepsilon * F(x) - T_\varepsilon F(x) = \mathscr{K}_\varepsilon * F(x),$$

where

$$\mathscr{K}_\varepsilon(y) = \begin{cases} \frac{\mathrm{Sh}[\frac{\pi y}{4\beta}]\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} & \text{if } |\frac{\pi y}{4\beta}| < \varepsilon \\ \\ \frac{\mathrm{Sh}[\frac{\pi y}{4\beta}]\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} - \frac{1}{\mathrm{Sh}[\frac{\pi y}{4\beta}]} & \text{if } |\frac{\pi y}{4\beta}| > \varepsilon. \end{cases}$$

We show that the family of functions $\mathscr{K}_\varepsilon$ is a multiple of a summability kernel. We have

$$\begin{aligned} \int_{|\frac{\pi y}{4\beta}|<\varepsilon} \frac{|\mathrm{Sh}[\frac{\pi y}{4\beta}]|\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]}\, dy &= \frac{8\beta}{\pi}\int_0^\varepsilon \frac{\mathrm{Sh}[t]\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[t]+\sin^2[\frac{\pi\varepsilon}{4\beta}]}\, dt \\ &\le \frac{8\beta}{\pi}\int_0^\varepsilon \frac{\mathrm{Sh}[t]}{\mathrm{Ch}^2[t]-\cos^2[\frac{\pi\varepsilon}{4\beta}]}\, dt \\ &= \frac{8\beta}{\pi}\int_1^{\mathrm{Ch}[\varepsilon]} \frac{1}{t^2-\cos^2[\frac{\pi\varepsilon}{4\beta}]}\, dt \\ &\to C_\beta, \end{aligned}$$

with $C_\beta < \infty$, as $\varepsilon$ tends to $0^+$.

Moreover,

$$\int_{|\frac{\pi y}{4\beta}|>\varepsilon} \left| \frac{\mathrm{Sh}[\frac{\pi y}{4\beta}]\cos[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} - \frac{1}{\mathrm{Sh}[\frac{\pi y}{4\beta}]} \right| dy$$

$$\begin{aligned}
&\leq \int_{|\frac{\pi y}{4\beta}|>\varepsilon} \left| \frac{\mathrm{Sh}[\frac{\pi y}{4\beta}](\cos[\frac{\pi\varepsilon}{4\beta}]-1)}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]} \right| \, dy + \int_{|\frac{\pi y}{4\beta}|>\varepsilon} \left| \frac{\sin^2[\frac{\pi\varepsilon}{4\beta}]}{\mathrm{Sh}[\frac{\pi y}{4\beta}][\mathrm{Sh}^2[\frac{\pi y}{4\beta}]+\sin^2[\frac{\pi\varepsilon}{4\beta}]]} \right| \, dy \\
&\leq (1-\cos\frac{\pi y}{4\beta}) \int_{|\frac{\pi y}{4\beta}|>\varepsilon} \frac{1}{\mathrm{Sh}[\frac{\pi y}{4\beta}]} \, dy + \sin^2[\frac{\pi\varepsilon}{4\beta}] \int_{|\frac{\pi y}{4\beta}|>\varepsilon} \frac{1}{\mathrm{Sh}^2[\frac{\pi y}{4\beta}]} \, dy \\
&\to C_\beta
\end{aligned}$$

where again $C_\beta < \infty$, as $\varepsilon$ tends to $0^+$. Thanks to these estimates we can conclude that

$$\sup_{\varepsilon} \int_{\mathbb{R}} |\mathscr{K}_\varepsilon(y)| \, dy < \infty.$$

Using analogue estimates, it is easy to see that, for every $\delta > 0$,

$$\lim_{\varepsilon\to 0^+} \int_{|y|>\delta} |\mathscr{K}_\varepsilon(y)| \, dy = 0.$$

At last,

$$\int_{\mathbb{R}} \mathscr{K}\varepsilon(y) \, dy = 0$$

since $\mathscr{K}_\varepsilon$ is odd. Therefore, we can conclude that for every $F$ in $L^p(\mathbb{R})$

$$\|\mathscr{K}_\varepsilon * F - 0\cdot F\|_{L^p} = \|\widetilde{K}_\varepsilon * F - T_\varepsilon F\|_{L^p} \to 0$$

as $\varepsilon$ tends to $0^+$. □

Proposition 1.20 and Proposition 1.22 together prove that

$$\lim_{\varepsilon\to 0^+} SF[x+i(\beta-\varepsilon)] = 2F - iTF, \tag{1.20}$$

where the limit is in $L^p$. Moreover, by density, by the uniqueness of the limit and by Proposition 1.19, we can conclude that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} SF[\cdot+i(\beta-\varepsilon)] &= 2F - iTF \\
&= \widetilde{SF}_1(\cdot+i\beta) \\
&= \mathcal{F}^{-1}\left[\frac{\widehat{F_1}(\cdot)e^{-2\beta(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\right].
\end{aligned} \tag{1.21}$$

We sum up everything in the following theorem.

**Theorem 1.23.** *The Szegő projection $S$ extends to a bounded linear operator*

$$\begin{aligned} S : L^p(\partial S_\beta) &\to H^p(\partial S_\beta) \\ (F_1, F_2) &\mapsto (\widetilde{SF}_1, \widetilde{SF}_2) \end{aligned}$$

*for every $p \in (1, \infty)$.*

*Proof.* As we already remarked, it is enough to prove the theorem for $F = (F_1, 0_2)$ and $G = (0_1, F_2)$ and then use linearity. We prove the result for $F$, the case of G being completely analogous. Thus, (1.17) and (1.21) guarantees that $\widetilde{SF}_2$ and $\widetilde{SF}_1$ are boundary values of the $H^p(S_\beta)$ function $SF$ defined in Proposition 1.18. Moreover, by (1.16),

$$\begin{aligned} \|\widetilde{SF}\|^p_{H^p(\partial S_\beta)} &= \int_{\mathbb{R}} |\widetilde{SF}_1(x + i\beta)|^p \, dx + \int_{\mathbb{R}} |\widetilde{S0_2}(x - i\beta)|^p \, dx \\ &\leq C_p \int_{\mathbb{R}} |F_1(x)|^p \, dx \\ &= C_p \int_{\partial S_\beta} |F(x)|^p \, dx \\ &= C_p \|F\|^p_{L^p(\partial S_\beta)}. \end{aligned}$$

The proof is complete. □

We conclude the chapter with a theorem which states that $SF$ in $H^p(S_\beta)$ converges to its boundary value function $\widetilde{SF}$ not only in norm, but also pointwise almost everywhere.

**Theorem 1.24.** *Let $F = (F_1, F_2)$ be a function in $L^p(\partial S_\beta)$, $p \in (1, \infty)$. Then,*

$$\lim_{y \to \beta^-} SF(x + iy) = \widetilde{SF}_1(x + i\beta) \qquad \lim_{y \to -\beta^+} SF(x + iy) = \widetilde{SF}_2(x - i\beta)$$

*for almost every $x$ in $\mathbb{R}$.*

*Proof.* As usual, we work with $F = (F_1, 0_2)$. By (1.17),

$$\begin{aligned} |SF(x + iy) - \widetilde{SF}_2(x - i\beta)| &= \left| \int_{\mathbb{R}} F_1[x - u] \left[ \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}(u + i(y + \beta))]} - \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}\pi]} \right] du \right| \\ &\leq \|F_1\|_{L^p(\mathbb{R})} \left[ \int_{\mathbb{R}} \left| \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}(u + i(y + \beta))]} - \frac{1}{\mathrm{Ch}[\frac{\pi}{4\beta}\pi]} \right|^{p'} du \right]^{\frac{1}{p'}} \\ &\to 0 \end{aligned}$$

as $y \to -\beta^{-1}$. By (1.18) and Proposition 1.22,

$$\begin{aligned}\lim_{y\to\beta^-} SF(x+iy) &= \lim_{\varepsilon\to 0^+} SF(x+i(\beta-\varepsilon)\\ &= \lim_{\varepsilon\to 0^+} \big[K_\varepsilon * F - i\widetilde{K}_\varepsilon * F\big](x)\\ &= \lim_{\varepsilon\to 0^+} \big[K_\varepsilon * F - iT_\varepsilon F\big](x).\end{aligned}$$

Now,

$$\lim_{\varepsilon\to 0^+} K_\varepsilon * F(x) = 2F(x)$$

almost everywhere thanks to Proposition 1.20. The pointwise convergence of $\widetilde{K}_\varepsilon * F$ is a consequence of the boundedness of the maximal truncated operator associated to $T_\varepsilon F$. We do not report the proof, but we refer to [Gra08, Theorems 2.1.14 and 4.4.5]. □

# Chapter 2

# Hardy Spaces on the non-smooth worm domain $D'_\beta$

In this chapter we develop the theory of Hardy spaces on the domain $D'_\beta$. We refer to the Introduction and the bibliographic references therein for background results on worm domains.

We focus attention on the non-smooth worm domain

$$D'_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log|z_2|^2| < \frac{\pi}{2}, \big|\log|z_2|^2\big| < \beta - \frac{\pi}{2}\Big\}, \tag{2.1}$$

where $\beta$ is assumed to be $\beta > \pi$. This domain is biholomorphically equivalent to the domain

$$D_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_1 e^{-i\log|z_2|^2}) > 0, \big|\log|z_2|^2\big| < \beta - \frac{\pi}{2}\Big\} \tag{2.2}$$

via the map

$$\begin{aligned} \Psi : D'_\beta &\to D_\beta \\ (z_1, z_2) &\mapsto (e^{z_1}, z_2) \end{aligned}$$

and

$$\begin{aligned} \Psi^{-1} : D_\beta &\to D'_\beta \\ (\zeta_1, \zeta_2) &\mapsto (\operatorname{Log}[\zeta_1 e^{-i\log|\zeta_2|^2}] + i\log|\zeta_2|^2, \zeta_2), \end{aligned}$$

where $\operatorname{Log}(\zeta)$ is the Principal Logarithm.

Inspired by [Kis91], Barrett used the domains $D_\beta$ and $D'_\beta$ as models to study the (ir)regularity in Sobolev scale of the Bergman projection of the smooth worm $\mathcal{W}$. Later, in [KP07], [KP08a] and [KP08b], Krantz and Peloso studied the $L^p$ mapping properties of the Bergman projection of $D_\beta$ and $D'_\beta$ by an explicit computation of the Bergman kernel. The actual computation of the kernel are made for the domain $D'_\beta$ since it has an easier geometry. It is then possible to recover the kernel of $D_\beta$ by the transformation rule of the Bergman kernel under biholomorphism. Unlike the Bergman case, in general, we do not have a transformation rule for the Szegő kernel. At the moment, we are able to study the mapping properties of the Szegő projection of $D'_\beta$ only. The research on $D_\beta$ is on-going and our goal in the future is to study the Szegő projection of the smooth worm $\mathcal{W}$.

This chapter is organized as follows. After defining the spaces $H^p(D'_\beta)$ , we focus on the case $p = 2$. Using Fourier analysis, we see that $H^2(D'_\beta)$ can be decomposed in orthogonal subspaces and we see which relationship exists between these subspaces and the space $H^2(S_\beta)$ of the previous chapter. We prove that every function of $H^2(D'_\beta)$ admits boundary values and that $H^2(D'_\beta)$ is a Reproducing Kernel Hilbert space. Thus, we define the Szegő projection of $H^2(D'_\beta)$ and, following [KP08b], we provide an explicit formula for the reproducing kernel $K_{D'_\beta}$. We conclude the first part of the chapter proving a Paley–Wiener theorem in this setting and proving a regularity result for the Szegő projection in Sobolev scale.

In the second part of the chapter we extend the results to the case $p \in (1,\infty)$. We study the mapping properties of the Szegő projection in $L^p$-scale and we show that the spaces $H^p(D'_\beta)$ admit a decomposition similarly to the Hilbert case. In addition, we conclude the chapter proving a Fatou type theorem, that is, we prove that an appropriate restriction of a function in $H^p(D'_\beta)$ converges to its boundary value function pointwise almost everywhere.

## 2.1 The spaces $H^p(D'_\beta)$

Let us focus on

$$D'_\beta = \Big\{(z_1,z_2)\in\mathbb{C}^2 : \big|\operatorname{Im} z_1 - \log|z_2|^2\big| < \frac{\pi}{2}, \big|\log|z_2|^2\big| < \beta - \frac{\pi}{2}\Big\}$$

where $\beta > \pi$. This domain is rotationally invariant in the $z_2$ variable and we can represent it in the plane $(\operatorname{Im} z_1, \log|z_2|)$ as in Figure 2.1.

**Remark 2.1.** Notice that definition of $D'_\beta$ (as well as the one of $D_\beta$ and $\mathcal{W}$) requires only that $\beta > \frac{\pi}{2}$. For simplicity of the arguments, we restrict ourselves to the case $\beta > \pi$. This is not a serious constraint since, at least in the case of Bergman spaces and Bergman projection, the most interesting situations occur when $\beta$ tends to $+\infty$.

The feature which makes the analysis on $D'_\beta$ easier than on $D_\beta$ is the following. Let $z_1 \in \mathbb{C}$ such that $|\operatorname{Im} z_1| < \beta$ be fixed; then, as it is elementary to check, the set

$$D'_\beta(z_1) := \{z_2 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\}$$

is connected. This is not the case for the domain $D_\beta$ and the main difference between the two domains.

As we already mentioned, the domain $D'_\beta$ can be sliced in strips. More in detail, let us fix $z_2 \in \mathbb{C}$ such that $|\log|z_2|^2| < \beta - \frac{\pi}{2}$; then, the set

$$D'_\beta(z_2) = \{z_1 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\} = \{z_1 \in \mathbb{C} : |\operatorname{Im} z_1 - \log|z_2|^2| < \frac{\pi}{2}\}$$

can be identified with a strip centered in $\log|z_2|^2$ and width equals to $\pi$. All these characteristics will be reflected in our results. The rotationally invariance in the $z_2$-variable will allow us to use the theory of Fourier series, while the "strip-like" geometry in the $z_1$-variable will make the results of Chapter 1 available.

In order to define Hardy spaces on $D'_\beta$ we need to establish a $H^p$-type growth condition for holomorphic functions on $D'_\beta$. We do this by restricting the functions to copies of the distinguished boundary $\partial D'_\beta$ of $D'_\beta$. In detail, the distinguished boundary $\partial D'_\beta$ is the set

$$\partial D'_\beta = \partial_1 D'_\beta \cup \partial_2 D'_\beta \cup \partial_3 D'_\beta \cup \partial_4 D'_\beta.$$

where

$$\partial_1 D'_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = \beta, \log|z_2|^2 = \beta - \frac{\pi}{2}\Big\};$$
$$\partial_2 D'_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = \beta - \pi, \log|z_2|^2 = \beta - \frac{\pi}{2}\Big\};$$
$$\partial_3 D'_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = -\beta, \log|z_2|^2 = -\Big(\beta - \frac{\pi}{2}\Big)\Big\};$$
$$\partial_4 D'_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = -(\beta - \pi), \log|z_2|^2 = -\Big(\beta - \frac{\pi}{2}\Big)\Big\}.$$

For every $p \in (1, \infty)$, we define the Hardy spaces $H^p(D'_\beta)$ as the functional space

$$H^p(D'_\beta) = \Big\{F \text{ holomorphic in } D'_\beta : \|F\|^p_{H^p(D'_\beta)} = \sup_{(t,s)\in[0,\frac{\pi}{2})\times[0,\beta-\frac{\pi}{2})} \mathcal{L}_p F(t, s) < \infty\Big\},$$

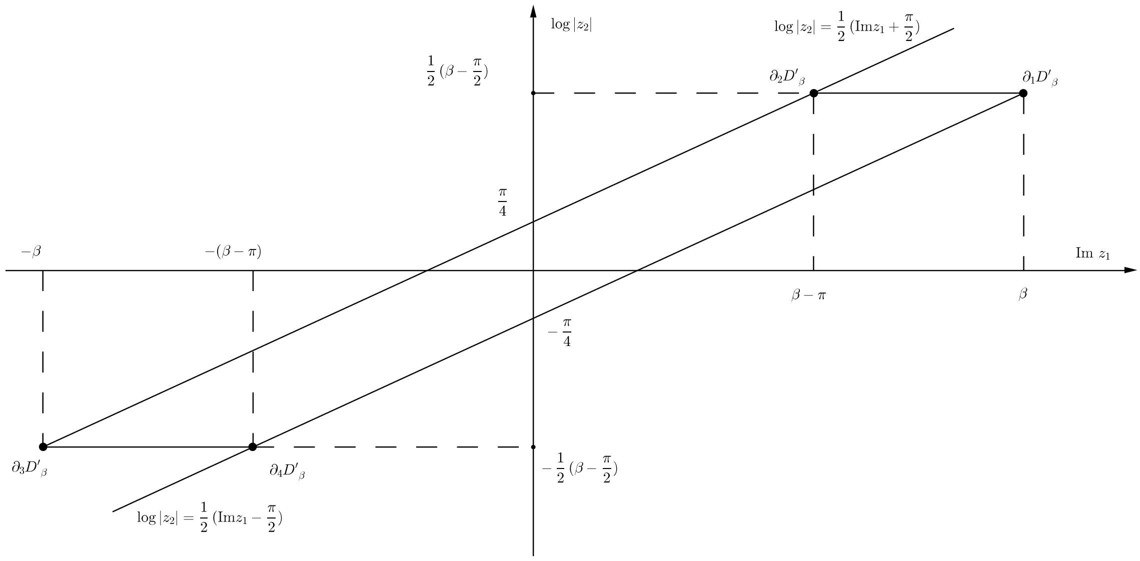


Figure 2.1: A representation of the domain $D'_\beta$ in the $(\operatorname{Im} z_1, \log|z_2|)$-plane.

where

$$
\begin{aligned}
&\mathcal{L}_p F(t,s) \\
&\quad = \int_{\mathbb{R}}\int_0^1 \left|F\left(x+i(s+t), e^{\frac{s}{2}}e^{2\pi i\theta}\right)\right|^p d\theta dx + \int_{\mathbb{R}}\int_0^1 \left|F\left(x-i(s+t), e^{-\frac{s}{2}}e^{2\pi i\theta}\right)\right|^p d\theta dx \\
&\qquad + \int_{\mathbb{R}}\int_0^1 \left|F\left(x+i(s-t), e^{\frac{s}{2}}e^{2\pi i\theta}\right)\right|^p \; d\theta dx \; + \int_{\mathbb{R}}\int_0^1 \left|F\left(x-i(s-t), e^{-\frac{s}{2}}e^{2\pi i\theta}\right)\right|^p d\theta dx.
\end{aligned}
$$

We emphasize that the domain $D'_\beta$ is not a product domain, while, on the other hand, every component $\partial_i D'_\beta$ of the distinguished boundary is and it can be identified with $\mathbb{R}\times\mathbb{T}$.

**Remark 2.2.** Since $\partial D'_\beta$ has four different components, we can think of a function $F \in L^p(\partial D'_\beta)$ as a vector $F = (F_1, F_2, F_3, F_4)$ where each function $F_k$ is thought as defined on $\partial_k D'_\beta = \mathbb{R}\times\mathbb{T}$, $k = 1,2,3,4$ and

$$\|F\|^p_{L^p(\partial D'_\beta)} = \sum_{k=1}^4 \|F_k\|^p_{L^p(\partial_k D'_\beta)} = \sum_{k=1}^4 \|F_k\|^p_{L^p(\mathbb{R}\times\mathbb{T})}.$$

As in the case of the strip, it is not hard to prove that convergence in $H^p(D'_\beta)$ implies uniform convergence on compact subsets of $D'_\beta$.

**Proposition 2.3.** *Let $K$ a compact subset of $D'_\beta$ and $F$ a function of $H^p(D'_\beta)$. Then*

$$\sup_{(z_1,z_2)\in K} |F(z_1,z_2)| \le C_K \|F\|^p_{H^p}.$$

Using only the definition of $H^p(D'_\beta)$, we can immediately prove that every function $F$ in $H^p(D'_\beta)$ admits a boundary value function $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4)$ in $L^p(\partial D'_\beta)$ at least in a weak-$*$ sense.

We need to restrict the holomorphic function $F$ to copies of the distinguished boundary $\partial D'_\beta$ inside the domain. Since $\partial D'_\beta$ is union of four disjoint components, we denote by $F_k^{(t,s)}$ such restrictions, $k = 1, \dots, 4$. They are defined as follows.

**Definition 2.4.** Let $F$ be a function in $H^p(D'_\beta)$, $p \in (1, \infty)$. For every $(t, s) \in [0, \frac{\pi}{2}) \times [0, \beta - \frac{\pi}{2})$, we define

$$\begin{aligned}
F_1^{(t,s)}(\zeta_1,\zeta_2) &:= F\big(\operatorname{Re}\zeta_1 + i\frac{s+t}{\beta}\operatorname{Im}\zeta_1, e^{-\frac12(\beta-\frac{\pi}{2}-s)}\zeta_2\big);\\
F_2^{(t,s)}(\zeta_1,\zeta_2) &:= F\big(\operatorname{Re}\zeta_1 + i\frac{s-t}{\beta-\pi}\operatorname{Im}\zeta_1, e^{-\frac12(\beta-\frac{\pi}{2}-s_2)}\zeta_2\big);\\
F_3^{(t,s)}(\zeta_1,\zeta_2) &:= F\big(\operatorname{Re}\zeta_1 + i\frac{s+t}{\beta}\operatorname{Im}\zeta_1, e^{\frac12(\beta-\frac{\pi}{2}+s)}\zeta_2\big);\\
F_4^{(t,s)}(\zeta_1,\zeta_2) &:= F\big(\operatorname{Re}\zeta_1 + i\frac{s-t}{\beta-\pi}\operatorname{Im}\zeta_1, e^{\frac12(\beta-\frac{\pi}{2}+s)}\zeta_2\big).
\end{aligned}$$

Each function $F_k^{(t,s)}$ is a well-defined function in $L^p(\partial_k D'_\beta)$, $k = 1, 2, 3, 4$.

**Proposition 2.5.** *Let $F$ be a function in $H^p(D'_\beta)$, $p \in (1, \infty)$. Then, there exist a function $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4)$ in $L^p(\partial D'_\beta)$ and a subsequence $(t, s)_n$, such that, for every function $G$ in $L^{p'}(\mathbb{R} \times \mathbb{T})$,*

$$\int_{\partial_k D'_\beta} F^{(t,s)_n}(\zeta_1,\zeta_2)G(\zeta_1,\zeta_2)\ d\zeta_1 d\zeta_2 \to \int_{\partial_k D'_\beta} \widetilde{F}_1(\zeta_1,\zeta_2)G(\zeta_1,\zeta_2)\ d\zeta_1 d\zeta_2$$

*as $n$ tends to $+\infty$ and $k = 1, \dots, 4$.*

*Proof.* Let $(t, s)_m$, a sequence such that $(t, s)_m \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$ as $m \to +\infty$. Then, $\{F_k^{(t,s)_m}\}$ is a bounded set in $L^p(\mathbb{R} \times \mathbb{T})$. By the Banach-Alaoglu Theorem, there exists a subsequence $\{F_k^{(t,s)_{m_n}}\}$ converging in the weak-$*$ topology to a function $\widetilde{F}_i$ in $L^p(\mathbb{R} \times \mathbb{T})$. The conclusion follows from the definition of weak-$*$ topology. □

## 2.2 Case $p = 2$

In this section we study the Hardy space $H^2(D'_\beta)$ according to this plan:

- we first decompose $H^2(D'_\beta)$ as direct sum of subspaces $\mathcal{H}^2_j$ using the rotational invariance in the second variable and the theory of Fourier series (Proposition 2.6 and Remark 2.15) ;
- using such a decomposition we show that each $F \in H^2(D'_\beta)$ admits boundary values in $L^2(\partial D'_\beta)$ (Proposition 2.13);
- we show that $H^2(D'_\beta)$ is a Reproducing Kernel Hilbert space by identifying the inner product in $H^2(D'_\beta)$ as an $L^2$ inner product on the distinguished boundary (Theorem 2.14);
- we describe the Reproducing Kernel of $H^2(D'_\beta)$ (Theorem 2.17);
- we define the Szegő projection operator (Theorem 2.26) and we formulate a Paley–Wiener Theorem for the domain $D'_\beta$ (Theorem 2.27);
- we study the Sobolev regularity of the Szegő projection (Theorem 2.28).

We adapt a decomposition introduced by Barrett [Bar92], while providing some details for the reader's convenience.

**Proposition 2.6.** *Let $F$ be a function in $H^2(D'_\beta)$. Then $F(z_1, z_2) = \sum_{j\in\mathbb{Z}} f_j(z_1)z_2^j$ where the series converges pointwise and each function $f_j$ belongs to the Hardy space $H^2(S_\beta)$.*

*Proof.* If $F$ is a function in $H^2(D'_\beta)$ and $(z_1, z_2)$ is a point of $D'_\beta$, it is immediate that

$$\int_0^1 |F(z_1, |z_2|e^{2\pi i\theta})|^2 \, d\theta < \infty.$$

Thus, by the theory of Fourier series in $L^2(\mathbb{T})$, we get

$$\begin{aligned} F(z_1, z_2) = F(z_1, |z_2|e^{2\pi i\gamma_2}) &= \sum_{j\in\mathbb{Z}} \Big[ \int_0^1 F(z_1, |z_2|e^{2\pi i\theta})e^{-2\pi ij\theta} \, d\theta \Big] e^{2\pi ij\gamma_2} \\ &= \sum_{j\in\mathbb{Z}} \Big[ \int_0^1 F(z_1, e^{i\theta}z_2)e^{-2\pi ij\theta} \, d\theta \Big] \end{aligned}$$

$$= \sum_{j \in \mathbb{Z}} F_j(z_1, z_2),$$

where the convergence is pointwise for every $(z_1, z_2)$ in $D'_\beta$.

Notice that the function

$$G_j(z_1, z_2) = \Big[ \int_0^1 F(z_1, e^{i\theta} z_2) e^{-2\pi i j \theta} \, d\theta \Big] z_2^{-j}$$

is holomorphic in $D'_\beta$ and depends only in $|z_2|$. Hence, it must be locally constant in $z_2$. As we already stressed, for all $z_1$, the set $D'_\beta(z_1) = \{z_2 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\}$ is connected, therefore $G_j(z_1, z_2) \equiv f_j(z_1)$. Since $F_j(z_1, z_2) = f_j(z_1) z_2^j$ is holomorphic on $D'_\beta$, it follows that $f_j$ is holomorphic on the strip $S_\beta = \{x + iy \in \mathbb{C} : |y| < \beta\}$.

Finally, we have

$$\begin{aligned}
\infty &> \sup_{(t,s)} \mathcal{L}_2 F(t, s) \\
&= \sup_{(t,s)} \sum_{j \in \mathbb{Z}} \mathcal{L}_2 F_j(t, s) \\
&\geq \sup_{(t,s)} \min\{e^{js}, e^{-js}\} \left\{ \int_{\mathbb{R}} \left| f_j[x \pm i(s+t)] \right|^2 dx \right\} \\
&\geq c_j \|f_j\|^2_{H^2(S_\beta)},
\end{aligned}$$

therefore each $f_j$ belongs to $H^2(S_\beta)$. □

**Remark 2.7.** A few comments on the last proposition:

- we proved that $F(z_1, z_2) = \sum_{j \in \mathbb{Z}} F_j(z_1, z_2) = \sum_{j \in \mathbb{Z}} f_j(z_1) z_2^j$. Notice that each function $F_j$ satisfies the equality

$$F_j(z_1, e^{i\theta} z_2) = e^{ij\theta} F_j(z_1, z_2).$$

  Thus, we define the following subspaces of $H^2(D'_\beta)$. For every $j$ in $\mathbb{Z}$,

$$\mathcal{H}^2_j = \{F \in H^2(D'_\beta) : F(z_1, e^{i\theta} z_2) = e^{ij\theta} F(z_1, z_2)\}; \tag{2.3}$$

- since each function $f_j$ belongs to the Hardy space $H^2(S_\beta)$, all the results contained in the previous chapter are available. In particular, we know that the each function $f_j$ admits a boundary value function $\widetilde{f_j}$ in $L^2(\partial S_\beta)$.

**Remark 2.8.** The connectedness of the set $D'_\beta(z_1) = \{z_2 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\}$ for every fixed $z_1$ has a primary role since it permits to split the variables in each function $F_j$.

We now use the Paley–Wiener Theorem for the strip to compute the $H^2(D'_\beta)$ norm of each function $F_j$.

**Proposition 2.9.** *Let $F_j(z_1, z_2) = f_j(z_1)z_2^j$ be a function in $\mathcal{H}_j$ and $\widetilde{f}_j$ in $L^2(\partial S_\beta)$ a boundary value function for $f_j$. Then,*

$$\begin{aligned}\|F_j\|^2_{H^2(D'_\beta)} = &\Big[e^{j(\beta-\frac{\pi}{2})}\|f_j[\cdot + i(\beta - \frac{\pi}{2})]\|^2_{H^2(S_{\frac{\pi}{2}})} + \\ &+ e^{-j(\beta-\frac{\pi}{2})}\|f_j[\cdot - i(\beta - \frac{\pi}{2})]\|^2_{H^2(S_{\frac{\pi}{2}})}\Big] \\ = &\frac{2}{\pi}\int_{\mathbb{R}} |\hat{f}_{j,0}(\xi|^2 \operatorname{Ch}(\pi\xi) \operatorname{Ch}[(2\beta - \pi)(\xi - \frac{j}{2})]\ d\xi.\end{aligned}$$

*In particular,*

$$\sup_{(t,s)} \mathcal{L}_2 F_j(t, s) = \lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \mathcal{L}_2 F_j(t, s).$$

*Proof.* By the Paley–Wiener Theorem we get

$$\begin{aligned}\|F_j\|^2_2 = &\sup_{(t,s)}\Big[\Big(\int_{\mathbb{R}} |f_j(x + i(s + t))|^2 e^{js} + |f_j(x + i(s - t))|^2 e^{js} + \\ &+ |f_j(x - i(s - t))|^2 e^{-js} + |f_j(x - i(s + t))|^2 e^{-js}\Big) dx\Big] \\ = &\sup_{(t,s)} \frac{2}{\pi}\int_{\mathbb{R}} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[2t\xi] \operatorname{Ch}[s(2\xi - j)]\ d\xi, \qquad (2.4)\end{aligned}$$

where $f_{j,0} = f_{j|_{\mathbb{R}}}$, that is, the restriction of $f_j$ to the real line, and $\hat{f}_{j,0}$ is its Fourier transform. The Paley–Wiener Theorem assures that $e^{\beta|\cdot|}\hat{f}_{j,0}$ is in $L^2(\mathbb{R})$. Hence, using the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\sup_{(t,s)} \mathcal{L}_2 F_j(t, s) &\le \frac{2}{\pi}\int_{\mathbb{R}} \sup_{(t,s)} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[2t\xi] \operatorname{Ch}[s(2\xi - j)]\ d\xi \\ &= \frac{2}{\pi}\int_{\mathbb{R}} \lim_{(t,s)} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[2t\xi] \operatorname{Ch}[s(2\xi - j)]\ d\xi \\ &= \lim_{(t,s)} \frac{2}{\pi}\int_{\mathbb{R}} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[2t\xi] \operatorname{Ch}[s(2\xi - j)]\ d\xi\end{aligned}$$

$$= \frac{2}{\pi} \int_{\mathbb{R}} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[\pi\xi] \operatorname{Ch}[(2\beta - \pi)(\xi - \frac{j}{2})].$$

In conclusion,

$$\sup_{(t,s)} \mathcal{L}_2(t,s) = \lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \mathcal{L}_2(t,s).$$

Since now we know that the supremum is obtained for $(t,s) = (\frac{\pi}{2}, \beta - \frac{\pi}{2})$, we do not have to see the norm of $F_j$ necessarily from the Fourier transform side. Therefore, we have

$$\begin{aligned} \|F_j\|_2^2 &= \int_{\mathbb{R}} \Big( |\widetilde{f}_j(x+i\beta)|^2 e^{j(\beta-\frac{\pi}{2})} + |\widetilde{f}_j(x+i(\beta-\pi))|^2 e^{j(\beta-\frac{\pi}{2})} + \\ &\qquad + |\widetilde{f}_j(x-i(\beta-\pi))|^2 e^{-j(\beta-\frac{\pi}{2})} + |\widetilde{f}_j(x-i\beta)|^2 e^{-j(\beta-\frac{\pi}{2})} \Big) dx \\ &= e^{j(\beta-\frac{\pi}{2})} \|f_j(\cdot + i(\beta - \frac{\pi}{2}))\|^2_{H^2(S_{\frac{\pi}{2}})} + e^{-j(\beta-\frac{\pi}{2})} \|f_j(\cdot - i(\beta - \frac{\pi}{2}))\|^2_{H^2(S_{\frac{\pi}{2}})}. \end{aligned} \tag{2.5}$$

□

**Remark 2.10.** Notice that

$$\frac{2}{\pi} \int_{\mathbb{R}} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[\pi\xi] \operatorname{Ch}[(2\beta - \pi)(\xi - \frac{j}{2})]\ d\xi \tag{2.6}$$

can be thought as a weighted norm of the Hardy space of the strip $H^2(S_\beta)$. We denote with $H^2_j(S_\beta)$ the Hardy space of the strip equipped with this weighted norm. We remark that $H^2_0(S_\beta)$ is the standard unweighted Hardy space $H^2(S_\beta)$ and the different norms of the spaces $H^2_j(D'_\beta)$ are all equivalent when $j$ varies. In conclusion, the previous proposition shows that $F_j \mapsto \widetilde{f}_j$ is an isometry between $H^2_j(D'_\beta)$ and $L^2_j(\partial S_\beta)$ where

$$\begin{aligned} \|\widetilde{f}_j\|^2_{L^2_j(\partial S_\beta)} &= \int_{\mathbb{R}} |\hat{f}_{j,0}(\xi)|^2 \operatorname{Ch}[\pi\xi] \operatorname{Ch}[(2\beta - \pi)(\xi - \frac{j}{2})]\ d\xi \\ &= \int_{\mathbb{R}} \Big( |\widetilde{f}_j(x \pm i\beta)|^2 e^{\pm j(\beta-\frac{\pi}{2})} + |\widetilde{f}_j(x \pm i(\beta-\pi))|^2 e^{\pm j(\beta-\frac{\pi}{2})} \Big)\ dx. \end{aligned}$$

We stress that in the above norm the function $\widetilde{f}$ appears evaluated at heights $\pm i\beta$ and $\pm i(\beta - \pi)$; while $\widetilde{f}(x \pm i\beta)$ truly are boundary values, it trivially holds that $\widetilde{f}(x \pm i(\beta-\pi)) = f(x \pm i(\beta - \pi))$.

**Proposition 2.11.** *Let be $F$ a function in $H^2(D'_\beta)$. Then*

$$\|F\|^2_{H^2(D'_\beta)} = \sup_{(t,s)} \sum_{j\in\mathbb{Z}} \mathcal{L}_2 F_j(t,s) = \sum_{j\in\mathbb{Z}} \sup_{(t,s)} \mathcal{L}_2 F_j(t,s) = \sum_{j\in\mathbb{Z}} \|F_j\|^2_{H^2(D'_\beta)},$$

*where the supremum is taken for $(t,s)$ varying in $[0, \frac{\pi}{2}) \times [0, \beta - \frac{\pi}{2})$.*

*Proof.* We already know that $\|F\|^2_{H^2(D'_\beta)} = \sup\limits_{(t,s)} \sum_{j\in\mathbb{Z}} \mathcal{L}_2 F_j(t,s)$; it trivially follows from the orhogonality of trigonometric monomials. We would like to prove that it is possible to switch the supremum with the sum, i.e.

$$\sup_{(t,s)} \sum_{j\in\mathbb{Z}} \mathcal{L}_2 F_j(t,s) = \sum_{j\in\mathbb{Z}} \sup_{(t,s)} \mathcal{L}_2 F_j(t,s).$$

Since we know from Proposition 2.9 that $\sup\limits_{(t,s)} \mathcal{L}_2 F_j(t,s) = \lim\limits_{(t,s)} \mathcal{L}_2 F_j(t,s)$, we can conclude using the Monotone Convergence Theorem. □

We sum up everything we have seen so far in the following theorem.

**Theorem 2.12.** *Every function $F(z_1, z_2)$ in $H^2(D'_\beta)$ admits a decomposition*

$$F(z_1,z_2) = \sum_{j\in\mathbb{Z}} F_j(z_1,z_2) = \sum_{j\in\mathbb{Z}} f_j(z_1) z_2^j$$

*where each $f_j$ belongs to $H^2_j(S_\beta)$ and*

$$\|F\|^2_{H^2(D'_\beta)} = \sum_{j\in\mathbb{Z}} \|F_j\|^2_{H^2(D'_\beta)} = \sum_{j\in\mathbb{Z}} \|\widetilde{f}_j\|^2_{L^2_j(\partial S_\beta)}.$$

*Moreover,*

$$\|F - S_N F\|_{H^2(D'_\beta)} = \|F - \sum_{j=-N}^{N} F_j\|_{H^2(D'_\beta)} \to 0$$

*as $N$ tends to $+\infty$.*

*Proof.* The only thing we still have to prove is the norm convergence of $S_N F$. From Proposition 2.6 we have the pointwise convergence, while the previous proposition assures that $\{S_N F\}$ is a Cauchy sequence in $H^2(D'_\beta)$. Hence, the conclusion follows. □

Finally, we are able to prove that a function $F \in H^2(D'_\beta)$ admits boundary values in $L^2(\partial D'_\beta)$. We denote with $F_k^{(t,s)}$, $k = 1,2,3,4$, the functions defined in Definition 2.4.

**Proposition 2.13.** *Let $F(z_1,z_2) = \sum\limits_{j\in\mathbb{Z}} f_j(z_1) z_2^j$ a function in $H^2(D'_\beta)$. For $(\zeta_1,\zeta_2) \in \partial D'_\beta$ define*

$$\widetilde{F}(\zeta_1,\zeta_2) := \sum_{j\in\mathbb{Z}} \widetilde{f}_j(\zeta_1)\zeta_2^j.$$

*Then $F_k^{(t,s)} \to \widetilde{F}$ in $L^2(\partial_k D'_\beta)$ as $(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$, $k = 1,2,3,4$.*

*Proof.* Theorem 2.12 guarantees that $\widetilde{F}$ is well defined. We prove the proposition only $\partial_1 D'_\beta$, thus $(\zeta_1, \zeta_2) = (x + i\beta, e^{\frac{1}{2}(\beta - \frac{\pi}{2})} e^{i\theta})$. The proof for $k = 2, 3, 4$ is similar and we omit it. We want to prove that

$$\int_{\mathbb{R}\times\mathbb{T}} \left| \widetilde{F}(x + i\beta, e^{\frac{1}{2}(\beta - \frac{\pi}{2})} e^{i\theta}) - F_1^{(t,s)}(x + i\beta, e^{\frac{1}{2}(\beta - \frac{\pi}{2})} e^{i\theta}) \right|^2 dx d\theta \to 0$$

as $(t, s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$. Since $F$ is in $H^2(D'_\beta)$, it holds

$$\|\widetilde{F} - F_1^{(t,s)}\|^2_{L^2(\partial_1 D'_\beta)} = \sum_{j\in\mathbb{Z}} \|\widetilde{F}_j - F_{1,j}^{(t,s)}\|^2_{L^2(\partial_1 D'_\beta)} < \infty.$$

Moreover, $\|\widetilde{F}_j - F_{1,j}^{(t,s)}\|^2_{L^2(\partial_1 D'_\beta)} \to 0$ as $(t, s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$. By Monotone Convergence Theorem for decreasing sequences, we can switch the sum and the limit obtaining

$$\begin{aligned} \lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|\widetilde{F} - F_1^{(t,s)}\|^2_{L^2(\partial_1 D'_\beta)} &= \sum_{j\in\mathbb{Z}} \lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|\widetilde{F}_j - F_{1,j}^{(t,s)}\|^2_{L^2(\partial_1 D'_\beta)} \\ &= 0. \end{aligned}$$

The conclusion follows. □

Thus, we proved that a given function $F(z_1, z_2) = \sum_{j\in\mathbb{Z}} f_j(z_1) z_2^j$ admits a boundary value function $\widetilde{F}(\zeta_1, \zeta_2) = \sum_{j\in\mathbb{Z}} \widetilde{f}_j(\zeta_1)\zeta_2^j$ in $L^2(\partial D'_\beta)$. Moreover, as expected, it holds the identity

$$\|F\|_{H^2(D'_\beta)} = \|\widetilde{F}\|_{L^2(\partial H^2_\beta)}. \tag{2.7}$$

This fact allows to prove that $H^2(D'_\beta)$ is a Reproducing Kernel Hilbert space by identifying the inner product in $H^2(D'_\beta)$ as an $L^2$ inner product on the distinguished boundary.

**Theorem 2.14.** *The Hardy space $H^2(D'_\beta)$ is a Reproducing Kernel Hilbert space with the inner product*

$$\begin{aligned} \langle F, G\rangle_{H^2(D'_\beta)} &= \langle \widetilde{F}, \widetilde{G}\rangle_{L^2(\partial D'_\beta)} \\ &= \sum_{k=1}^{4} \int_{\partial_k D'_\beta} \widetilde{F}(\zeta_1, \zeta_2)\overline{\widetilde{G}(\zeta_1, \zeta_2)}\, d\zeta_1 d\zeta_2. \end{aligned} \tag{2.8}$$

*Proof.* It follows from (2.7) and Proposition 2.3. □

**Remark 2.15.** In conclusion, we proved that the space $H^2(D'_\beta)$ admits an orthogonal decomposition

$$H^2(D'_\beta) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}^2_j, \tag{2.9}$$

where the $\mathcal{H}^2_j$'s are the subspaces of $H^2(D'_\beta)$ defined in (2.3)

Before investigating the reproducing kernel $K_{D'_\beta}$ of $H^2(D'_\beta)$, we investigate the reproducing kernels of the subspaces $\mathcal{H}^2_j$. The particular structure of each $\mathcal{H}^2_j$ and Proposition 2.9 allow us to look for the kernels of the spaces $H^2_j(S_\beta)$.

**Proposition 2.16.** *The reproducing kernel of $H^2_j(S_\beta)$ is the function*

$$k_j(z_1,z_2) = \frac{1}{8\pi}\int_{\mathbb{R}} \frac{e^{i(z_1-\overline{z_2})\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\,d\xi.$$

*Proof.* Given $z_2$ in $S_\beta$, by Remark 2.6, we have

$$\begin{aligned} f(z_2) &= \frac{2}{\pi}\int_{\mathbb{R}} \hat{f}_0(\xi)\overline{\hat{k}_{j,0}(\xi,z_2)}\operatorname{Ch}(\pi\xi)\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_0(\xi)e^{iz_2\xi}\,d\xi, \end{aligned}$$

where the last equality holds since $f$ belongs to $H^2(S_\beta)$. It follows

$$\hat{k}_{j,0}(\xi,z_2) = \frac{1}{4}\frac{e^{-i\overline{z_2}\xi}}{\operatorname{Ch}(\pi\xi)\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}.$$

Using the inverse Fourier transform we finally obtain

$$k_j(z_1,z_2) = \frac{1}{8\pi}\int_{\mathbb{R}} \frac{e^{i(z_1-\overline{z_2})\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\,d\xi.$$

□

The reproducing kernel of $H^2(D'_\beta)$ is then given by

$$\begin{aligned} K_{D'_\beta}[(w_1,w_2),(z_1,z_2)] &= \sum_{j\in\mathbb{Z}} K_{j,D'_\beta}[(w_1,w_2),(z_1,z_2)] \\ &= \sum_{j\in\mathbb{Z}} {w_2}^j\overline{z_2}^j k_j(w_1,z_2) \\ &= \sum_{j\in\mathbb{Z}} \frac{{w_2}^j\overline{z_2}^j}{8\pi}\int_{\mathbb{R}} \frac{e^{i(w_1-\overline{z_1})\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\,d\xi. \end{aligned} \tag{2.10}$$

### 2.2.1 Asymptotic expansion of the kernel $K_{D'_\beta}$

Following [KP08b], we obtain an asymptotic expansion for the kernel $K_{D'_\beta}$. Since the proof is long and technical, we do not report it here, but we refer to Chapter 3.

**Theorem 2.17.** *Let $\beta > \pi$ and define $\nu_\beta = \frac{\pi}{2\beta-\pi}$ Let $h$ be fixed such that*

$$\frac{\nu_\beta}{2} < h < \min\left(\frac{1}{2}, \frac{3\nu_\beta}{2}\right).$$

*Then there exist functions $F_1, F_2, G_1, \dots, G_8, E$ and $\widetilde{E}$ that are holomorphic in $w$ and anti-holomorphic in $z$, for $w = (w_1, w_2)$ and $z = (z_1, z_2)$ varying in a neighborhood of $D'_\beta$, and remain bounded, together with all their derivatives, for $w, z \in \overline{D'_\beta}$, as $|\operatorname{Re}(w_1 - \overline{z_1})| \to +\infty$. Then,*

$$K_{D'_\beta}(w, z) = e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))\frac{(w_1-\overline{z_1})\nu_\beta}{2}} K(w, z) + e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))(w_1-\overline{z_1})h}\widetilde{K}(w, z),$$

*where*

$$\begin{aligned} K(w, z) &= \frac{F_1(w, z)}{1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}} + \frac{F_2(w, z)}{1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}} + E(w, z) \\ &= K_1(w, z) + K_2(w, z) + E(w, z) \end{aligned}$$

*and*

$$\begin{aligned} \widetilde{K}(w, z) = &\frac{G_1(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}]} + \frac{G_2(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}]} + \\ &+ \frac{G_3(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}][1 - (w_2\overline{z_2})e^{\beta-\frac{\pi}{2}}]} + \\ &+ \frac{G_4(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}][i(w_1 - \overline{z_1}) + 2\beta]} + \\ &+ \frac{G_5(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}][i(w_1 - \overline{z_1}) - 2\beta]} + \\ &+ \frac{G_6(w, z)}{[1 - (w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}][1 - (w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})}]} + \\ &+ \frac{G_7(w, z)}{[i(w_1 - \overline{z_1}) + 2\beta][1 - (w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})}]} + \end{aligned}$$

$$
\begin{aligned}
&+\frac{G_8(w,z)}{[i(w_1-\overline{z_1})-2\beta][1-(w_2\overline{z_2})e^{\beta-\frac{\pi}{2}}]}+\widetilde{E}(w,z)\\
&=\widetilde{K}_1(w,z)+\ldots+\widetilde{K}_8(w,z)+\widetilde{E}(w,z).
\end{aligned}
$$

**Remark 2.18.** A comment about the singularities of $K_{D'_\beta}$ is required. We have the following facts:

- for $w, z \in D'_\beta$ the terms $K_1$ and $\widetilde{K}_1$ become singular only if

$$w_2\overline{z_2} \to e^{-\frac{i(w_1-\overline{z_1})+\pi}{2}}.$$

  This can happen only if $\log|w_2|^2 \to \mathrm{Im}(w_1)-\frac{\pi}{2}$ and $\log|z_2| \to \mathrm{Im}(z_1)-\frac{\pi}{2}$. Thus, $K_1$ and $\widetilde{K}_1$ are singular only when both $w$ and $z$ tend to the right oblique boundary line of the domain in Figure 2.1;

- the terms $K_2$ and $\widetilde{K}_2$ are similar to $K_1$ and $\widetilde{K}_1$ and they are singular on the left oblique boundary line of the domain in Figure 2.1;

- the term $\widetilde{K}_3$ is singular when

$$w_2\overline{z}_2 \to e^{-\frac{i(w_1-\overline{z_1})+\pi}{2}} \qquad \text{or} \qquad w_2\overline{z}_2 \to e^{-(\beta-\frac{\pi}{2})}.$$

  Thus, $\widetilde{K}_3$ is singular when both $w$ and $z$ tend either to the lower horizontal or the right oblique boundary line on of the domain in Figure 2.1. Notice that the term is more singular when $w_2\overline{z}_2 \to e^{-(\beta-\frac{\pi}{2})}$ and $(w_1-\overline{z_1}) \to 2(\beta-\pi)$ since the singularities add up. Therefore, $\widetilde{K}_3$ is more singular on the component of the distinguished boundary $\partial_4 D'_\beta$;

- the term $\widetilde{K}_4$ is singular when

$$w_2\overline{z}_2 \to e^{-\frac{i(w_1-\overline{z_1})+\pi}{2}} \qquad \text{or} \qquad \mathrm{Im}(w_1-\overline{z_1}) \to 2\beta.$$

  Therefore, $\widetilde{K}_4$ is singular when both $w, z$ tend to the right oblique boundary line of the domain of Figure 2.1. The term becomes more singular on the component of the distinguished boundary $\partial_1 D'_\beta$;

- the singularities of $\widetilde{K}_5$ are similar to the ones of $\widetilde{K}_4$ and the worst situation is when both $w, z$ tend to $\partial_3 D'_\beta$;

– the singularities of $\widetilde{K_6}$ are similar to the ones of $\widetilde{K}_3$. The term in singular both $w$ and $z$ tend to left oblique or the upper boundary line of the domain in Figure 2.1and it becomes more singular on $\partial_2 D'_\beta$;

– the term $\widetilde{K}_7$ becomes singular when

$$w_2\overline{z}_2 \to e^{\beta-\frac{\pi}{2}} \qquad \text{or} \qquad \operatorname{Im}(w_1 - \overline{z_1}) \to 2\beta.$$

Therefore, the term becomes singular when both $w$ and $z$ tend to the upper boundary line of the domain in Figure 2.1 and, like $\widetilde{K}_4$ it is more singular when $w, z$ tend to $\partial_1 D'_\beta$:

– the last term $\widetilde{K}_8$ is symmetric to $\widetilde{K}_7$. It is singular when $w, z$ tends to the lower boundary line of the domain in Figure 2.1 and it is more singular when $w, z$ tend to $\partial_3 D'\beta$ .

## 2.3 The Szegő projection of $D'_\beta$

To conclude the study of $H^2(D'_\beta)$ it remains to prove that the integration against the kernel $K_{D'_\beta}$ actually produces functions in $H^2(D'_\beta)$.

We start the section proving two propositions on the convergence of the series which defines the kernel $K_{D'_\beta}$.

**Proposition 2.19.** *Let us consider* $K_{D'_\beta}(z,\zeta) = K_{D'_\beta}[(z_1, z_2),(\zeta_1,\zeta_2)]$ *where* $(\zeta_1,\zeta_2) \in \partial D'_\beta$ *and* $(z_1, z_2)$ *varies in a compact set* $K \subseteq D'_\beta$. *Then,*

$$\sum_{j\in\mathbb{Z}} \sup_{(z,\zeta)\in K\times\partial D'_\beta} \left|k_j(z_1,\zeta_1) z_2^j \overline{\zeta}_2^j\right| < \infty$$

*Proof.* We prove the proposition supposing that $(\zeta_1, \zeta_2)$ is in $\partial_1 D'_\beta$. The general case will follow analogously. In order to estimate the size of $k_j$, suppose for the moment that $j < 0$. Then,

$$|k_j(z_1,\zeta_1)| = |k_j(z_1, x+i\beta)| \le \int_{\mathbb{R}} \frac{e^{-[\operatorname{Im} z_1+\beta]\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi$$

$$= \left( \int_{-\infty}^{\frac{j}{2}} + \int_{\frac{j}{2}}^{0} + \int_{0}^{+\infty} \right) \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi.$$

It follows that

$$\begin{aligned}
\int_{-\infty}^{\frac{j}{2}} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi &\approx \int_{-\infty}^{\frac{j}{2}} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{e^{-\pi\xi}e^{-(2\beta-\pi)(\xi-\frac{j}{2})}}\, d\xi \\
&= C\frac{e^{-j(\beta-\frac{\pi}{2})}e^{\frac{j}{2}(\beta-\operatorname{Im} z_1)}}{\beta-\operatorname{Im} z_1}; \\
\int_{\frac{j}{2}}^{0} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi &\approx \int_{0}^{\frac{j}{2}} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{e^{-\pi\xi}e^{(2\beta-\pi)(\xi-\frac{j}{2})}}\, d\xi \\
&= Ce^{j(\beta-\frac{\pi}{2})}\frac{e^{-\frac{j}{2}[\operatorname{Im} z_1+3\beta-2\pi]}-1}{\operatorname{Im} z_1+3\beta-2\pi}; \\
\int_{0}^{+\infty} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi &\approx \int_{0}^{+\infty} \frac{e^{-[\operatorname{Im} z_1 + \beta]\xi}}{e^{\pi\xi}e^{(2\beta-\pi)(\xi-\frac{j}{2})}}\, d\xi \\
&= C\frac{e^{j(\beta-\frac{\pi}{2})}}{\operatorname{Im} z_1+3\beta}.
\end{aligned}$$

Notice that all these estimates do not depend on $\operatorname{Re}\zeta_1$ and the term

$$\frac{e^{-\frac{j}{2}[\operatorname{Im} z_1+3\beta-2\pi]}-1}{\operatorname{Im} z_1+3\beta-2\pi}$$

is not singular when $\operatorname{Im} z_1 + 3\beta - 2\pi \to 0$. Finally,

$$\begin{aligned}
&\sum_{j<0} |z_2|^j e^{\frac{j}{2}(\beta-\frac{\pi}{2})}|k_j(z_1, x+i\beta)| \le \\
&\quad \le C\sum_{j<0}\left[\frac{e^{\frac{j}{2}[\log|z_2|^2+\frac{\pi}{2}-\operatorname{Im} z_1]}}{\beta-\operatorname{Im} z_1} + \frac{e^{\frac{j}{2}[\log|z_2|^2-\operatorname{Im} z_1+\frac{\pi}{2}]} - e^{\frac{j}{2}[\log|z_2|^2+3\beta-\frac{3\pi}{2}]}}{\operatorname{Im} z_1+3\beta-2\pi} + \frac{e^{\frac{j}{2}[3\beta-\frac{3}{2}\pi+\log|z_2|^2]}}{\operatorname{Im} z_1+3\beta}\right]
\end{aligned}$$

and it is immediate to see that we get a uniform bound for $(z_1, z_2) \in K$. Analogous computations prove the estimate for the sum over positive $j$'s. □

By the property of Reproducing Kernel Hilbert spaces, we know that $K_{D'_\beta}[(z_1,z_2),(\cdot,\cdot)]$ is in $H^2(D'_\beta)$ for every fixed $(z_1,z_2)$ in $D'_\beta$. In particular, $K_{D'_\beta}[(z_1,z_2),(\cdot,\cdot)]$ admits boundary values in $L^2(\partial D'_\beta)$. Notice that for $(z_1,z_2)$ fixed in $D'_\beta$, the kernel $K_{D'_\beta}[(z_1,z_2),(\cdot,\cdot)]$ is well-defined on $\partial D'_\beta$, thus its boundary value function is just its extension to $\overline{D'_\beta}$. Regarding the $L^2(\partial D'_\beta)$ norm of $K_{D'_\beta}[(z_1,z_2),(\cdot,\cdot)]$ we have the following estimate.

**Proposition 2.20.** *Let $K$ be a compact subset of $D'_\beta$. Then,*

$$\sup_{(z_1,z_2)\in K} \|K_{D'_\beta}[(z_1,z_2),(\cdot,\cdot)]\|_{L^2(\partial D'_\beta)} \le C_K, \tag{2.11}$$

*where $C_K$ is a constant which depends on $K$.*

*Proof.* We prove the proposition for only one of the component of the distinguished boundary, say $\partial_1 D'_\beta$. The computation for the other three components is analogue. Therefore, by Proposition 2.12, we get

$$\begin{aligned}\int_{\partial_1 D'_\beta} \left|K_{D'_\beta}[(z_1,z_2),(\zeta_1,\zeta_2)]\right|^2 d\zeta_1 d\zeta_2 &= C\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}}\int_0^1 \left|K_{j,D'_\beta}[(z_1,z_2),(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma})]\right|^2 d\gamma dx\\ &= C\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}}\int_0^1 \left|z_2^j e^{\frac{j}{2}(\beta-\frac{\pi}{2})} e^{-2\pi ij\gamma} k_j[z_1, x+i\beta]\right|^2 d\gamma dx\\ &= C\sum_{j\in\mathbb{Z}} |z_2|^{2j} e^{j(\beta-\frac{\pi}{2})} \int_{\mathbb{R}} \left|\frac{e^{-(\operatorname{Im} z_1+\beta)\xi}}{\operatorname{Ch}[2\beta\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\right|^2 d\xi\end{aligned}$$

where in the last equality we used Plancherel's theorem. The computation continues similarly to the computation in the proof of Proposition 2.19. □

**Remark 2.21.** In order to have more readable proof, as we did in the previous chapter for the strip $S_\beta$, we can think to have a function $F$ in $L^2(\partial D'_\beta)$ such that $F = (F_1, 0_2, 0_3, 0_4)$ where $0_i$'s are constant zero fuctions. The results for a general $F$ will follow by linearity, since, as element of $L^2(\partial D'_\beta)$,

$$(F_1,F_2,F_3,F_4) = (F_1,0_2,0_3,0_4) + (0_1,F_2,0_3,0_4) + (0_1,0_2,F_3,0_4) + (0_1,0_2,0_3,F_4).$$

**Notation.** Given a function $F$ in $C_0^\infty(\mathbb{R}\times\mathbb{T})$, we denote with $\mathcal{F}_{\mathbb{R}}F(\xi,\hat{j})$ the Fourier transform of $F$ in the first variable and the $j$th Fourier coefficient in the second, i.e.

$$\mathcal{F}_{\mathbb{R}}F(\xi,\hat{j}) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^1 F(x,\gamma)e^{-ix\xi}e^{-2\pi ij\gamma}\, d\gamma dx.$$

**Proposition 2.22.** *Let $F = (F_1,F_2,F_3,F_4)$ a function in $L^2(\partial D'_\beta)$. Then, the function*

$$SF(z_1,z_2) := \langle F, K[(\cdot,\cdot),(z_1,z_2)]\rangle_{L^2(\partial D'_\beta)} = \sum_{k=1}^4 \langle F_k, K[(\cdot,\cdot),(z_1,z_2)]\rangle_{L^2(\partial_k D'_\beta)}$$

*is in $H^2(D'_\beta)$. Moreover,*

$$\|SF\|_{H^2(D'_\beta)} \le \|F\|_{L^2(\partial D'_\beta)}.$$

*Proof.* We prove the proposition for a function $F$ in $L^2(\partial D'_\beta)$ of the form $F = (F_1, 0_2, 0_3, 0_4)$. Therefore, by Plancherel's theorem,

$$\begin{aligned}\|F\|^2_{L^2(\partial D'_\beta)} &= \int_{\partial D'_\beta} |F(\zeta_1,\zeta_2)|^2 \, d\zeta_1 d\zeta_2 \\ &= \int_0^1 \int_{\mathbb{R}} |F_1(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})} e^{2\pi i\theta})|^2 \, dx d\theta \\ &= \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}} F_1(\xi, \widehat{j})|^2 \, d\xi.\end{aligned}$$

The holomorphicity of $SF$ follows using the estimate in Proposition 2.20 and an argument analogue to the one used in Proposition 1.12. It remains to prove that $SF$ satisfies the $H^2$ growth condition. To simplify notation, we set $F_1(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\theta}) := F_1(x,\theta)$. Thus,

$$\begin{aligned}SF(u+iv, re^{2\pi i\gamma}) &= \big\langle F, K[(\cdot,\cdot),(u+iv, re^{2\pi i\gamma})]\big\rangle_{L^2(\partial D'_\beta)} \\ &= \int_{\mathbb{R}} \int_0^1 F_1(x,\theta) \sum_{j\in\mathbb{Z}} k_j(u+iv, x+i\beta) r^j e^{2\pi i j\gamma} e^{\frac{j}{2}(\beta-\frac{\pi}{2})} e^{-2\pi i j\theta} \, d\theta dx \\ &= \frac{1}{4} \sum_{j\in\mathbb{Z}} r^j e^{\frac{j}{2}(\beta-\frac{\pi}{2})} e^{2\pi i j\gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ \frac{e^{-(v+\beta)(\cdot)} \mathcal{F}_{\mathbb{R}} F_1(\cdot, \hat{j})}{\operatorname{Ch}[\pi\cdot] \operatorname{Ch}[(2\beta-\pi)(\cdot - \frac{j}{2})]} \right](u).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}} \int_0^1 \left| SF[u+i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}] \right|^2 \, d\gamma du \\ &\qquad = \frac{1}{8\pi} \sum_{j\in\mathbb{Z}} e^{j(s+\beta-\frac{\pi}{2})} \int_{\mathbb{R}} \Big| \frac{e^{-(s+t+\beta)\xi} \mathcal{F}_{\mathbb{R}} F_1(\cdot, \widehat{j})}{\operatorname{Ch}[\pi\xi] \operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]} \Big|^2 \, d\xi \\ &\qquad = \frac{1}{8\pi} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \left| \frac{e^{-(s+\beta-\frac{\pi}{2})(\xi-\frac{j}{2})} e^{-(\frac{\pi}{2}+t)\xi} \mathcal{F}_{\mathbb{R}} F_1(\xi, \widehat{j})}{\operatorname{Ch}[\pi\xi] \operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]} \right|^2 \, d\xi \\ &\qquad \le \frac{1}{8\pi} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}_{\mathbb{R}} F_1(\xi, \hat{j}) \right|^2 \, d\xi. \qquad (2.12)\end{aligned}$$

Taking the supremum for $(t,s) \in [,\frac{\pi}{2}) \times [0, \beta - \frac{\pi}{2})$ we obtain

$$\|SF\|_{H^2(D'_\beta)} \le \frac{1}{4} \|F_1\|_{L^2(\mathbb{R}\times\mathbb{T})} \le \|F\|_{L^2(\partial D'_\beta)} \qquad (2.13)$$

and the conclusion follows. □

**Remark 2.23.** We report for completeness the explicit expression of $SF$ given a general initial data $F = (F_1, F_2, F_3, F_4)$ in $L^2(\partial D'_\beta)$. Let $(u + iv, r^{2\pi i\gamma})$ in $D'_\beta$, then

$$\begin{aligned} SF(u+iv, re^{2\pi i\gamma}) = \frac{1}{4}\sum_{j\in\mathbb{Z}} r^j e^{\frac{j}{2}(\beta-\frac{\pi}{2})} e^{2\pi ij\gamma} \mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(v+\beta)(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](u) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} r^j e^{\frac{j}{2}(\beta-\frac{\pi}{2})} e^{2\pi ij\gamma} \mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(v+\beta-\pi)(\cdot)}\mathcal{F}_{\mathbb{R}}F_2(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](u) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} r^j e^{-\frac{j}{2}(\beta-\frac{\pi}{2})} e^{2\pi ij\gamma} \mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(v-\beta)(\cdot)}\mathcal{F}_{\mathbb{R}}F_3(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]+}\right](u) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} r^j e^{-\frac{j}{2}(\beta-\frac{\pi}{2})} e^{2\pi ij\gamma} \mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(v-\beta+\pi)(\cdot)}\mathcal{F}_{\mathbb{R}}F_4(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](u). \end{aligned} \tag{2.14}$$

Since $SF$ is a function in $H^2(D'_\beta)$, we know it admits a boundary value function $\widetilde{SF}$. We show an explicit formula of $\widetilde{SF}$.

**Definition 2.24.** Given $(F_1, F_2, F_3, F_4)$ in $L^2(\partial D'_\beta)$, we define

$$\begin{aligned} \widetilde{SF}_1(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) := \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}F_2(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{\mathcal{F}_{\mathbb{R}}F_3(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_4(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x); \end{aligned}$$

$$\begin{aligned} \widetilde{SF}_2[x+i(\beta-\pi), e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] := \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\widehat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}e^{\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_2(\cdot,\widehat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ + \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_3(\cdot,\widehat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \end{aligned}$$

$$+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{\mathcal{F}_{\mathbb{R}}F_4(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x);$$

$$\begin{aligned}\widetilde{SF}_3[x-i\beta, e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] := &\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{\mathcal{F}_{\mathbb{R}}F_1(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_2(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{(2\beta-\pi)(\cdot-\frac{j}{2})}e^{\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_3(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}F_4(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x);\end{aligned}$$

$$\begin{aligned}\widetilde{SF}_4[x-i(\beta-\pi), e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] := &\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{\mathcal{F}_{\mathbb{R}}F_2(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}F_3(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x)\\
&+\frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{(2\beta-\pi)(\cdot-\frac{j}{2})}e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_4(\cdot,\widehat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x).\end{aligned}$$

**Proposition 2.25.** *Let $F=(F_1,F_2,F_3,F_4)$ a function in $L^2(\partial D'_\beta)$. Then,*

$$\begin{aligned}&\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})}\|SF(\cdot+i(s+t),e^{\frac{s}{2}}e^{2\pi i(\cdot)})-\widetilde{SF}_1(\cdot+i\beta,e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i(\cdot)})\|_{L^2(\mathbb{R}\times\mathbb{T})}=0;\\
&\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})}\|SF(\cdot+i(s-t),e^{\frac{s}{2}}e^{2\pi i(\cdot)})-\widetilde{SF}_2[\cdot+i(\beta-\pi),e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i(\cdot)}]\|_{L^2(\mathbb{R}\times\mathbb{T})}=0;\\
&\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})}\|SF(\cdot-i(s+t),e^{-\frac{s}{2}}e^{2\pi i(\cdot)})-\widetilde{SF}_3(\cdot-i\beta,e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i(\cdot)})\|_{L^2(\mathbb{R}\times\mathbb{T})}=0;\\
&\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})}\|SF(\cdot-i(s-t),e^{-\frac{s}{2}}e^{2\pi i(\cdot)})-\widetilde{SF}_4[\cdot-i(\beta-\pi),e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i(\cdot)}]\|_{L^2(\mathbb{R}\times\mathbb{T})}=0;\end{aligned}$$

*Proof.* We compute only one of the four limits for a simpler function $F$ of the form $F=(F_1,0_2,0_3,0_4)$. The other limits follow analogously. We have

$$\|SF(\cdot+i(s+t),e^{\frac{s}{2}}e^{2\pi i(\cdot)})-\widetilde{SF}_1(\cdot+i\beta,e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i(\cdot)})\|_{L^2(\mathbb{R}\times\mathbb{T})}=$$

$$
\begin{aligned}
&= \frac{1}{8\pi}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\mathcal{F}_{\mathbb{R}}F_1(\xi,\widehat{j})\frac{e^{-(s+\beta-\frac{\pi}{2})(\xi-\frac{j}{2})}e^{-(\frac{\pi}{2}+t)\xi}-e^{-(2\beta-\pi)\xi}e^{-\pi\xi}}{\mathrm{Ch}[\pi\xi]\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\right|^2 d\xi\\
&\le \frac{1}{8\pi}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\mathcal{F}_{\mathbb{R}}F_1(\xi,\widehat{j})\right|^2 d\xi\\
&< \infty.
\end{aligned}
$$

By the Dominated Convergence Theorem, we can conclude. The conclusion for a general function $F$ follows by linearity as explained in Remark 2.2. □

Let us define

$$H^2(\partial D'_\beta) := \{G = (G_1, G_2, G_3, G_4) \in L^2(\partial D'_\beta) : \exists F \in H^2(D'_\beta)\ s.t.\ G = \widetilde{F}\}.$$

From Proposition 2.3 we deduce that $H^2(\partial D'_\beta)$ is a closed subspace of $L^2(\partial D'_\beta)$.

Everything we proved so far can be summarized in the following theorem.

**Theorem 2.26.** *The operator*

$$
\begin{aligned}
\widetilde{S} : L^2(\partial D'_\beta) &\to H^2(\partial D'_\beta)\\
(F_1, F_2, F_3, F_4) &\mapsto (\widetilde{SF}_1, \widetilde{SF}_2, \widetilde{SF}_3, \widetilde{SF}_4)
\end{aligned}
$$

*is a Hilbert space orthogonal projection operator. We call $\widetilde{S} : L^2(\partial D'_\beta) \to H^2(\partial D'_\beta)$ the Szegő projection operator.*

We conclude this section with a Paley–Wiener type of result.

**Theorem 2.27.** ***(Paley–Wiener Theorem for $D'_\beta$)*** *Let $F = (F_1, F_2, F_3, F_4)$ be a function in $L^2(\partial D'_\beta)$. Then, $F$ is in $H^2(\partial D'_\beta)$ if and only if there exists a sequence of functions $\{g_j\}$ such that*

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} |\hat{g}_j(\xi)|^2\,\mathrm{Ch}[\pi\xi]\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2}]\ d\xi < \infty$$

*and*

$$
\begin{aligned}
F_1(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) &= \sum_{j\in\mathbb{Z}} f_{1,j}(x+i\beta)e^{\frac{j}{2}(\beta-\frac{\pi}{2})}e^{2\pi ij\gamma};\\
F_2[x+i(\beta-\pi), e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] &= \sum_{j\in\mathbb{Z}} f_{2,j}[x+i(\beta-\frac{\pi}{2})]e^{\frac{j}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma};
\end{aligned}
$$

$$F_3(x - i\beta, e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) = \sum_{j\in\mathbb{Z}} f_{3,j}(x - i\beta)e^{-\frac{j}{2}(\beta-\frac{\pi}{2})}e^{2\pi ij\gamma};$$
$$F_4[x - i(\beta - \pi), e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] = \sum_{j\in\mathbb{Z}} f_{4,j}[x - i(\beta - \frac{\pi}{2})]e^{-\frac{j}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma},$$

*where, for every $j \in \mathbb{Z}$,*

$$f_{1,j}[x + i\beta] = \mathcal{F}_{\mathbb{R}}^{-1}\Big[e^{-\beta(\cdot)}g_j(\cdot)\Big](x);$$
$$f_{2,j}[x + i(\beta - \pi)] = \mathcal{F}_{\mathbb{R}}^{-1}\Big[e^{-(\beta-\pi)(\cdot)}g_j(\cdot)\Big](x);$$
$$f_{3,j}(x + i\beta) = \mathcal{F}_{\mathbb{R}}^{-1}\Big[e^{\beta(\cdot)}g_j(\cdot)\Big](x);$$
$$f_{4,j}[x - i(\beta - \pi)] = \mathcal{F}_{\mathbb{R}}^{-1}\Big[e^{(\beta-\pi)(\cdot)}g_j(\cdot)\Big](x).$$

*Proof.* Suppose that $F$ belongs to $H^2(\partial D'_\beta)$. Then, the conclusion follows from Proposition 2.12. Conversely, let $\{g_j\}$ be a sequence which defines $F = (F_1, F_2, F_3, F_4)$ as in the hypothesis. It follows that $SF$ belongs to $H^2(D'_\beta)$ and the formulas in Definition 2.24 guarantee that $\widetilde{SF}_k = F_k$, $k = 1, 2, 3, 4$. The proof is complete. □

### 2.3.1 Sobolev regularity

We conclude this section on $H^2(D'_\beta)$ studying the regularity of the Szegő projection $\widetilde{S}$ in Sobolev norm. For every $k > 0$, let us consider the Sobolev space

$$W^k(\partial D'_\beta) = \left\{ F = (F_1, F_2, F_3, F_4) : \|F\|^2_{W^k(\partial D'_\beta)} = \sum_{i=1}^4 \|F_i\|^2_{W^k(\partial_i D'_\beta)} \right\},$$

where

$$\|F_i\|^2_{W^k(\partial_i D'_\beta)} = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} (1 + j^2 + \xi^2)^k \big|\mathcal{F}_{\mathbb{R}}F(\xi, \hat{j})\big|^2 \, d\xi.$$

We prove that the Szegő projection $\widetilde{S}$ preserves the regularity of functions.

**Theorem 2.28.** *The Szegő projection $\widetilde{S}$ is a bounded linear operator*

$$\widetilde{S} : W^k(\partial D'_\beta) \to W^k(\partial D'_\beta)$$
$$(F_1, F_2, F_3, F_4) \mapsto (\widetilde{SF}_1, \widetilde{SF}_2, \widetilde{SF}_3, \widetilde{SF}_4)$$

*for every $k > 0$.*

*Proof.* We only show explicitely that $\|\widetilde{S}F_1\|_{W^k(\partial_1 D'_\beta)} \leq \|F_1\|_{W^k(\partial_q D'_\beta)}$; the computation for the other term is similar. Moreover, by Remark 2.2, it is enough to prove the theorem for $F=(F_1,0_2,0_3,0_4)$. For such a function $F$, it holds

$$\widetilde{S}F_1(x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) := \frac{1}{4}\sum_{j\in\mathbb{Z}} e^{2\pi i j\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x).$$

Thus,

$$\begin{aligned}
\|\widetilde{S}F_1\|^2_{W^k(\partial_1 D'_\beta)} &= \frac{1}{16}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}(1+j^2+\xi^2)^k\left|\mathcal{F}_{\mathbb{R}}\widetilde{S}F_1(\xi,\hat{j})\right|^2\,d\xi \\
&= \frac{1}{16}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})}e^{-\pi\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\right|^2(1+j^2+\xi^2)^k\left|\mathcal{F}_{\mathbb{R}}F_1(\xi,\hat{j})\right|^2\,d\xi \\
&= \|\widetilde{SG^k_1}\|^2_{L^2(\partial D'_\beta)},
\end{aligned}$$

where

$$G^k(x,\gamma)=\sum_{j\in\mathbb{Z}}e^{2\pi i j\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[[1+j^2+(\cdot)^2]^{\frac{k}{2}}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\right](x).$$

By hypothesis, the function $G^k$ is in $L^2(\mathbb{R}\times\mathbb{T})$, therefore,

$$\begin{aligned}
\|\widetilde{S}F_1\|^2_{W^k(\partial_1 D'_\beta)} &= \|\widetilde{SG^k_1}\|^2_{L^2(\partial D'_\beta)} \\
&\leq \|G^k\|^2_{L^2(\mathbb{R}\times\mathbb{T})} \\
&= \|F_1\|^2_{W^k(\partial_1 D'_\beta)}.
\end{aligned}$$

The conclusion follows. □

## 2.4 Case $1<p<\infty$

In this section we extend the results we have seen so far to the case $p\in(1,\infty)$. In detail,

- we show that the Szegő projection can be realized as a composition of simpler operators we are able to study and we extend Theorem 2.26;
- we prove that the space $H^p(D'_\beta)$, $p\in(1,\infty)$, admits a decomposition analogous to (2.9) for the case $p=2$ (Proposition 2.39);

– we prove a Fatou-type theorem; that is, we prove that an approrpariate restriction of a function $F$ in $H^p(D'_\beta)$, $p \in (1,\infty)$, converges to its boundary value function $\widetilde{F}$ pointwise almost everywhere (Theorem 2.45).

One of the goals of this section is to prove the following boundedness result the for Szegő porjection.

**Theorem 2.29.** *The Szegő projection $\widetilde{S}$ extends to a bounded linear operator*

$$\begin{aligned}\widetilde{S} : L^p(\partial D'_\beta) &\to H^p(\partial D'_\beta)\\ (F_1,F_2,F_3,F_4) &\mapsto (\widetilde{SF}_1,\widetilde{SF}_2,\widetilde{SF}_3,\widetilde{SF}_4)\end{aligned}$$

*for every $p \in (1,\infty)$.*

As already pointed out in Remark 2.2, it is enough to prove the theorem for $F$ in $L^p(\partial D'_\beta)$ of the form $F = (F_1, 0_2, 0_3, 0_4)$, where $F_1$ is a function in $L^p(\mathbb{R}\times\mathbb{T})$. From now on we will always think to work with a function $F$ in $L^p(\mathbb{R}\times\mathbb{T})$ of such a form unless specified. Keeping this in mind, the formulas in Definition 2.24 reduce to

$$\widetilde{SF}_1(x+i\beta, e^{\frac12(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) = \frac14\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x); \tag{2.15}$$

$$\widetilde{SF}_2[x+i(\beta-\pi), e^{\frac12(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] = \frac14\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x); \tag{2.16}$$

$$\widetilde{SF}_3(x-i\beta, e^{-\frac12(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}) = \frac14\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x); \tag{2.17}$$

$$\widetilde{SF}_4[x-i(\beta-\pi), e^{-\frac12(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] = \frac14\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-\pi(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x). \tag{2.18}$$

Moreover, if $(x+iy, re^{2\pi i\gamma})$ is in $D'_\beta$, the formula (2.14) reduces to

$$S_{y,s}F(x,\gamma) := SF(x+iy, e^{\frac{s}{2}}e^{2\pi i\gamma}) = \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(\beta-\frac{\pi}{2}+s)(\cdot-\frac{j}{2})}e^{-(\frac{\pi}{2}-s+y)(\cdot)}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})}{4\,\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x). \tag{2.19}$$

We observe that the operators $F \mapsto \widetilde{SF}_i, i = 1,2,3,4$, and $F \mapsto SF_{y,s}$ are well defined on the set

$$\left\{F(x,\gamma) = \sum_{\sharp j<\infty} F(x,j)e^{2\pi ij\gamma} : F(\cdot,j) \in C_0^\infty(\mathbb{R})\right\},$$

where the sum is on a finite number of $j$'s. Moreover,

**Proposition 2.30.** *For every* $p \in (1, \infty)$*,*

$$\overline{\left\{ \sum_{\sharp j < \infty} F(x,j) e^{2\pi i j \gamma} : F(\cdot, j) \in C_0^\infty(\mathbb{R}) \right\}}^{\|\cdot\|_{L^p(\mathbb{R}\times\mathbb{T})}} = L^p(\mathbb{R} \times \mathbb{T}).$$

*Proof.* Let $F \in L^p(\mathbb{R} \times \mathbb{T}))$, then $F(x, \cdot)$ is in $L^p(\mathbb{T})$ for almost every $x \in \mathbb{R}$. Therefore,

$$\lim_{N\to\infty} \int_0^1 \left| F(x,\gamma) - \sum_{j=-N}^{N} F(x,\hat{j}) e^{2\pi i j \gamma} \right|^p d\gamma = 0.$$

Since the partial sum operator is uniformly bounded for 1-dimensional Fourier series, by Dominated Convergence Theorem, it follows

$$\lim_{N\to+\infty} \int_{\mathbb{R}} \int_0^1 \left| F(x,\gamma) - \sum_{j=-N}^{N} F(x,\hat{j}) e^{2\pi i j \gamma} \right|^p d\gamma dx = 0.$$

Now, fix $\varepsilon > 0$ and let $N(\varepsilon)$ such that

$$\int_{\mathbb{R}} \int_0^1 \left| F(x,\gamma) - \sum_{j=-N(\varepsilon)}^{N(\varepsilon)} F(x,\hat{j}) e^{2\pi i j \gamma} \right|^p d\gamma dx < \varepsilon^p.$$

For every function $F(\cdot, \hat{j})$ there exists a function $\widetilde{F}(\cdot, j)$ in $C_0^\infty(\mathbb{R})$ such that

$$\left[ \int_{\mathbb{R}} \left| F(x,\hat{j}) - \widetilde{F}(x,j) \right|^p dx \right]^{\frac{1}{p}} < \frac{\varepsilon}{2N(\varepsilon)}.$$

Thus,

$$\begin{aligned} &\left[ \int_{\mathbb{R}} \int_0^1 \left| F(x,\gamma) - \sum_{j=-N(\varepsilon)}^{N(\varepsilon)} \widetilde{F}(x,j) e^{2\pi i j \gamma} \right|^p d\gamma dx \right]^{\frac{1}{p}} \\ &\leq \left[ \int_{\mathbb{R}} \int_0^1 \left| F(x,\gamma) - \sum_{j=-N(\varepsilon)}^{N(\varepsilon)} F(x,\hat{j}) e^{2\pi i j \gamma} \right|^p d\gamma dx \right]^{\frac{1}{p}} \\ &\qquad + \left[ \int_{\mathbb{R}} \int_0^1 \left| \sum_{j=-N(\varepsilon)}^{N(\varepsilon)} F(x,\hat{j}) e^{2\pi i j \gamma} - \widetilde{F}(x,j) e^{2\pi i j \gamma} \right|^p d\gamma dx \right]^{\frac{1}{p}} \\ &\leq 2\varepsilon, \end{aligned}$$

where we used triangle inequality and the hypothesis on $\widetilde{F}(\cdot, j)$ to estimate the sum in $j$. The proof is complete. □

**Proposition 2.31.** *Let $F = (F_1, 0_2, 0_3, 0_4)$ be a function in $L^p(\partial D'_\beta)$. Then, for every $p \in (1, \infty)$,*

$$\|S_{y,s}F\|_{L^p(\partial D'_\beta)} \le C_p \|F\|_{L^p(\partial D'_\beta)},$$

*where the constant $C_p$ does not depend on $y$ and $s$.*

*Proof.* Let be $F_1(x,\gamma) = \sum_{j=-N}^{N} F_1(x,j)e^{2\pi i j\gamma}$ as in Proposition 2.30. Then,

$$S_{y,s}F(x,\gamma) = [\lambda'_{y,s} \circ \lambda_s]F(x,\gamma),$$

where

$$\begin{aligned} \lambda_s F(x,\gamma) &= \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{j=-N}^{N} e^{2\pi i j\gamma} \frac{e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{4\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]} \mathcal{F}_{\mathbb{R}} F_1(\xi,j) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{j=-N}^{N} e^{2\pi i j\gamma} m_s(\xi - \frac{j}{2}) \mathcal{F}_{\mathbb{R}} F_1(\xi,j) e^{ix\xi}\, d\xi \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \lambda'_{y,s} F(x,\gamma) &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-(\frac{\pi}{2}-s+y)\xi}}{\mathrm{Ch}[\pi\xi]} \mathcal{F}_{\mathbb{R}} F_1 \xi,\gamma) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} m'_{y,s}(\xi) \mathcal{F}_{\mathbb{R}} F_1(\xi,\gamma) e^{ix\xi}\, d\xi. \end{aligned} \tag{2.21}$$

We recall that $y$ and $s$ are such that $(x+iy, e^{\frac{s}{2}}e^{2\pi i\gamma})$ is in $D'_\beta$, thus $|s| \in (0, \beta - \frac{\pi}{2})$ and $|y-s| \in (0, \frac{\pi}{2})$. Following the proof of Proposition 1.18, we obtain that $m'_{y,s}$ is a multiplier of $L^p(\mathbb{R})$ for every $p \in (1,\infty)$ with norm independent of $y$ and $s$. Thus the operator $\lambda'_{y,s}$ extends to a bounded linear operator $L^p(\mathbb{R}\times\mathbb{T}) \to L^p(\mathbb{R}\times\mathbb{T})$ for every $p \in (1,\infty)$. About $\lambda_s$ we have

$$\begin{aligned} \lambda_s F(x,\gamma) &= \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{j=-N}^{N} e^{2\pi i j(\gamma+\frac{x}{4\pi})} m_s(\xi) \mathcal{F}_{\mathbb{R}} F_1(\xi + \frac{j}{2}, j) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{j=-N}^{N} e^{2\pi i j(\gamma+\frac{x}{4\pi})} m_s(\xi) \mathcal{F}_{\mathbb{R}} [e^{-i\frac{j}{2}(\cdot)} F_1(\cdot, j)](\xi) e^{ix\xi}\, d\xi. \end{aligned}$$

Therefore, by a change of variables and the periodicity of the exponential function,

$$\int_{\mathbb{R}\times\mathbb{T}} |\lambda_s F(x,\gamma)|^p dx d\gamma = \int_{\mathbb{R}}\int_0^1 \left| \frac{1}{2\pi} \sum_{j=-N}^{N} \int_{\mathbb{R}} e^{2\pi i j(\gamma)} m_s(\xi) \mathcal{F}_{\mathbb{R}} [e^{-i\frac{j}{2}(\cdot)} F_1(\cdot, j)](\xi) e^{ix\xi}\, d\xi \right|^p d\gamma dx$$

$$= \int_0^1 \int_{\mathbb{R}} \left| \frac{1}{2\pi} \int_{\mathbb{R}} m_s(\xi) \sum_{j=-N}^{N} e^{2\pi i j\gamma} \mathcal{F}_{\mathbb{R}}[e^{-i\frac{j}{2}(\cdot)} F_1(\cdot, j)](\xi) e^{ix\xi}\, d\xi \right|^p dx d\gamma$$

$$= \int_0^1 \int_{\mathbb{R}} \left| \frac{1}{2\pi} \int_{\mathbb{R}} m_s(\xi) \mathcal{F}_{\mathbb{R}} \left[ \sum_{j=-N}^{N} e^{-i\frac{j}{2}(\cdot)} F_1(\cdot, j) e^{2\pi i j\gamma} \right](\xi) e^{ix\xi}\, d\xi \right|^p dx d\gamma.$$

Following again the proof of Proposition 1.18 we obtain that $m_s$ is a multipliers of $L^p(\mathbb{R})$ for every $p \in (1, \infty)$ with norm indepedent of $s$. Therefore, if we prove that the function $\sum_{j=-N}^{N} e^{-i\frac{j}{2}t} F_1(t, j) e^{2\pi i j\gamma}$ is in $L^p(\mathbb{R} \times \mathbb{T})$, we will obtain the $L^p$ boundedness of the operator $\lambda_s$. By a change of variables and the periodicity of the exponential function, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 \left| \sum_{j=-N}^{N} e^{-i\frac{j}{2}t} F_1(t, j) e^{2\pi i j\gamma} \right|^p d\gamma dt &= \int_{\mathbb{R}} \int_0^1 \left| \sum_{j=-N}^{N} F_1(t, j) e^{2\pi i j\gamma} \right|^p d\gamma dt \\ &= \|F_1\|^p_{L^p(\mathbb{R} \times \mathbb{T})} \\ &< \infty. \end{aligned}$$

Finally,

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 \left| S_{y,s} F(x, \gamma) \right|^p d\gamma dx &= \int_{\mathbb{R}} \int_0^1 \left| [\lambda_{y,s} \circ \lambda_s] F(x, \gamma) \right|^p d\gamma dx \\ &\le C_p \int_{\mathbb{R}} \int_0^1 \left| \lambda_s F(x, \gamma) \right|^p d\gamma dx \\ &\le C_p \int_{\mathbb{R}} \int_0^1 \left| F(x, \gamma) \right|^p d\gamma dx, \end{aligned}$$

as we wished.

□

The last proposition allows us to prove that the operator $S$ extends to a continuous operator with respect to the $L^p$ norm.

**Theorem 2.32.** *Let $F = (F_1, 0_2, 0_3, 0_4)$ a function in $L^p(\partial D'_\beta)$. Then, for every $p \in (1, \infty)$, the operator $S$ extends to a bounded linear operator*

$$S : L^p(\partial D'_\beta) \to H^p(D'_\beta).$$

*Proof.* Suppose that $F = (F_1, 0_2, 0_3, 0_4)$ is a function in $L^p(\partial D'_\beta) \cap L^2(\partial D'_\beta)$. Then, Proposition 2.22 assures that $SF$ is holomorphic on $D'_\beta$. Moreover,

$$\|SF\|^p_{H^p(D'_\beta)} = \sup_{(t,s) \in [0, \frac{\pi}{2}) \times [0, \beta - \frac{\pi}{2})} \mathcal{L}_p SF(t, s)$$

$$
\begin{aligned}
&= \sup_{(t,s)} \Big[ \|S_{s+t,s}F\|^p_{L^p(\mathbb{R}\times\mathbb{T})} + \|S_{s-t,s}F\|^p_{L^p(\mathbb{R}\times\mathbb{T})} \\
&\qquad + \|S_{-(s+t),-s}F\|^p_{L^p(\mathbb{R}\times\mathbb{T})} + \|S_{-(s-t),-s}F\|^p_{L^p(\mathbb{R}\times\mathbb{T})} \Big] \\
&\le C_p \|F\|^p_{L^p(\mathbb{R}\times\mathbb{T})} \qquad (2.22)
\end{aligned}
$$

with $C_p$ independent of $t$ and $s$ thanks to Proposition 2.31. Thus, we proved the theorem when $F$ is in $L^p(\partial D'_\beta)\cap L^2(\partial D'_\beta)$. Suppose now that $G$ is a general function in $L^p(\partial D'_\beta)$. Then, there exists a sequence $\{G_n\}\subseteq L^p(\partial D'_\beta)\cap L^2(\partial D'_\beta)$ such that $\|G-G_n\|_{L^p(\partial D'_\beta)}\to 0$ as $n$ tends to $\infty$. From Proposition 2.31 we obtain

$$\|S[G-G_n]\|_{H^p(D'_\beta)} \le C_p \|G-G_n\|_{L^p(\partial D'_\beta)},$$

thus $SG_n \to SG$ in $H^p(D'_\beta)$. It remains to prove that the function $SG$ is holomorphic on $D'_\beta$. From the first part of the proof and Proposition 2.3 we know that the functions $SG_n$'s are holomorphic on $D'_\beta$ and

$$\sup_{(z_1,z_2)\in K} |S[G_n-G_m](z_1,z_2)| \le C_k \|G_n-G_m\|^p_{H^p(D'_\beta)}$$

for every compact set $K\subseteq D'_\beta$. It then follows that $SG$ is holomorphic on $D'_\beta$. □

It remains to prove that (2.15), (2.16), (2.17) and (2.18) are boundary values for $SF$. At the moment, we focus on (2.15) and we fix some notation. We have

$$
\begin{aligned}
T^1_{t,s}F(x,\gamma) &:= [\widetilde{SF}_1 - S_{s+t,s}F](x,\gamma) \\
&= \sum_{j\in\mathbb{Z}} e^{2\pi i j\gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ \frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}e^{-\pi(\cdot)} - e^{-(\beta-\frac{\pi}{2}+s)(\cdot-\frac{j}{2})}e^{-(\frac{\pi}{2}+t)(\cdot)}}{4\,\mathrm{Ch}[\pi(\cdot)]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]} \mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j}) \right](x) \\
&= \sum_{j\in\mathbb{Z}} e^{2\pi i j\gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ m^{1,I}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\right](x) + \sum_{j\in\mathbb{Z}} e^{2\pi i j\gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ m^{1,II}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\right](x) \\
&= T^{1,I}_{t,s}F(x,\gamma) + T^{1,II}_{t,s}F(x,\gamma), \qquad (2.23)
\end{aligned}
$$

where

$$
\begin{aligned}
m^{1,I}_{t,s}(\xi,j) &= \frac{1}{8}\left[\frac{e^{-\pi\xi}-e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]}\right]\left[\frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})}+e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\right] = \frac{1}{8}\big[m^{1,I}_t(\xi)\big]\big[m^{2,I}_s(\xi-\frac{j}{2})\big]; \\
m^{1,II}_{t,s}(\xi,j) &= \frac{1}{8}\left[\frac{e^{-\pi\xi}+e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]}\right]\left[\frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})}-e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\right] = \frac{1}{8}\big[m^{1,II}_t(\xi)\big]\big[m^{2,II}_s(\xi-\frac{j}{2})\big].
\end{aligned}
$$

Thus, the operator $T^{1,I}_{t,s}$ can be seen as a composition of two operators, that is,

$$T^{1,I}_{t,s}F(x,\gamma) = [\Lambda_s \circ \Xi_t]F(x,\gamma), \tag{2.24}$$

where, $\Lambda_s$ and $\Xi_t$, acting on a suitable function $G$, are defined by

$$\Lambda_s G(x,\gamma) := \sum_{j\in\mathbb{Z}} \frac{e^{2\pi ij\gamma}}{2\pi} \int_{\mathbb{R}} \frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})} + e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]} \mathcal{F}_{\mathbb{R}}G(\xi,\hat{j})e^{ix\xi}\, d\xi;$$
$$\Xi_t G(x,\gamma) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-\pi\xi} - e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]} \mathcal{F}_{\mathbb{R}}G(\xi,\gamma)e^{ix\xi} d\xi.$$

The situation for the operator $T^{1,II}_{t,s}$ is analogue. We have

$$T^{1,II}_{t,s}F(x,\gamma) = [\Lambda'_s \circ \Xi'_t]F(x,\gamma), \tag{2.25}$$

where the operators $\Lambda'_s$ and $\Xi'_t$ are defined by

$$\Lambda'_s G(x,\gamma) := \sum_{j\in\mathbb{Z}} \frac{e^{2\pi ij\gamma}}{2\pi} \int_{\mathbb{R}} \frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})} - e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]} \mathcal{F}_{\mathbb{R}}G(\xi,\hat{j})e^{ix\xi}\, d\xi;;$$
$$\Xi'_t G(x,\gamma) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-\pi\xi} + e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]} \mathcal{F}_{\mathbb{R}}G(\xi,\gamma)e^{ix\xi} d\xi.$$

So, in order to obtain information on the mapping properties of the operator $T^1_{t,s}$, we study the operators $\Lambda_s, \Xi_t, \Lambda'_s$ and $\Xi'_t$ separately. The realization of $T^1_{t,s}$ as composition of these operators is particularly effective since the parameters $t$ and $s$ become, in some sense, independent.

**Proposition 2.33.** *The operator $\Lambda_s$ extends to a bounded linear operator*

$$\Lambda_s : L^p(\mathbb{R}\times\mathbb{T}) \to L^p(\mathbb{R}\times\mathbb{T})$$

*for every $p \in (1,\infty)$. Moreover,*

$$\sup_{s\in[0,\beta-\frac{\pi}{2})} \|\!|\Lambda_s|\!\|_p < \infty. \tag{2.26}$$

*Proof.* Let $G(x,\gamma) = \sum_{j=-N}^{N} G(x,j)e^{2\pi ij\gamma}$ be a function as in Proposition 2.30. Then, similarly to the proof of Proposition 2.31 for the operator $\lambda_s$, we obtain

$$\int_{\mathbb{R}}\int_0^1 \left|\Lambda_s G(x,\gamma)\right|^p d\gamma dx = \int_{\mathbb{R}}\int_0^1 \left|\int_{\mathbb{R}} \frac{m^{2,I}_s(\xi)}{2\pi} \mathcal{F}_{\mathbb{R}}\left[\sum_{j=-N}^{N} e^{i\frac{j}{2}(\cdot)}(\cdot)G(\cdot,j)e^{2\pi ij\gamma}\right](\xi)e^{ix\xi}d\xi\right|^p d\gamma dx.$$

By Mihlin's condition (see, for instance, [Gra08, Thm. 5.2.7]), we obtain that the function

$$m_s^{2,I}(\xi) = \frac{e^{-(2\beta-\pi)\xi} + e^{-(\beta-\frac{\pi}{2}+s)\xi}}{\mathrm{Ch}[(2\beta-\pi)\xi]} \tag{2.27}$$

identifies a multiplier operator that is bounded on $L^p(\mathbb{R})$ for every $p \in (1,\infty)$ and that satisfies (2.26). Notice also that the function $\sum_{j=-N}^{N} e^{i\frac{j}{2}x} G(x,j) e^{2\pi i j\gamma}$ is in $L^p(\mathbb{R}\times\mathbb{T})$. In fact,

$$\int_{\mathbb{R}}\int_0^1 \left| \sum_{j=-N}^{N} e^{i\frac{j}{2}x} G(x,j) e^{2\pi i j\gamma} \right|^p d\gamma dx = \int_{\mathbb{R}}\int_0^1 \left| \sum_{j=-N}^{N} G(x,j) e^{2\pi i j\gamma} \right|^p d\gamma dx < \infty,$$

where we performed a change of variables and used the periodicity of the exponential function. Finally, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}}\int_0^1 \left|\Lambda_s G(x,\gamma)\right|^p d\gamma dx &= \int_0^1\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{m_s^{2,I}(\xi)}{2\pi} \mathcal{F}_{\mathbb{R}}\left[ \sum_{j=-N}^{N} e^{i\frac{j}{2}}(\cdot) G(\cdot,j) e^{2\pi i j\gamma}\right](\xi) e^{ix\xi} d\xi \right|^p dx d\gamma \\ &\le C_p \int_0^1\int_{\mathbb{R}} \left| \sum_{j=-N}^{N} e^{i\frac{j}{2}x} G(x,j) e^{2\pi i j\gamma} \right|^p dx d\gamma \\ &= C_p \int_0^1\int_{\mathbb{R}} \left| \sum_{j=-N}^{N} G(x,j) e^{2\pi i j\gamma} \right|^p dx d\gamma.\end{aligned}$$

By Proposition 2.30, the proof is complete. □

**Proposition 2.34.** *The operator $\Xi'_t$ extends to a bounded linear operator*

$$\Xi'_t : L^p(\mathbb{R}\times\mathbb{T}) \to L^p(\mathbb{R}\times\mathbb{T})$$

*for every $p \in (1,\infty)$. Moreover,*

$$\sup_{t\in[0,\frac{\pi}{2})} \|\!|\Xi'_t\|\!|_p < \infty. \tag{2.28}$$

*Proof.* By Mihlin's condition we obtain that the function $m_t^{1,II}(\xi)$ is a $L^p(\mathbb{R})$ multipliers for every $p \in (1,\infty)$ which satisfies (2.28). By Fubini's theorem we conclude. □

**Proposition 2.35.** *The operator $\Xi_t$ extends to a bounded linear operator*

$$\Xi_t : L^p(\mathbb{R}\times\mathbb{T}) \to L^p(\mathbb{R}\times\mathbb{T})$$

*for every $p \in (1,\infty)$. Moreover,*

$$\sup_{t\in[0,\frac{\pi}{2})} \|\!|\Xi_t\|\!|_p < \infty$$

*and*

$$\lim_{t \to \frac{\pi}{2}} \|\Xi_t G\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0$$

*for every function* $G$ *in* $L^p(\mathbb{R}\times\mathbb{T})$*.*

*Proof.* The boundedness of $\Xi_t$ follows once again by Mihlin's condition, while the limit is computed as in (1.20) for the strip $S_{\frac{\pi}{2}}$. □

**Proposition 2.36.** *The operator* $\Lambda_s'$ *extends to a bounded linear operator*

$$\Lambda_s' : L^p(\mathbb{R}\times\mathbb{T}) \to L^p(\mathbb{R}\times\mathbb{T})$$

*for every* $p \in (1,\infty)$*. Moreover,*

$$\sup_{s\in[0,\beta-\frac{\pi}{2}} \|\Lambda_s'\|_p < \infty.$$

*and*

$$\lim_{s \to \beta-\frac{\pi}{2}} \|\Lambda_s' G\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0$$

*for every function* $G$ *in* $L^p(\mathbb{R}\times\mathbb{T})$*.*

*Proof.* The proof follows similarly as the proofs of Proposition 2.33 and Proposition 2.35 □

Thanks to the last proposition, we can finally prove the norm convergence of a function in $H^p(D_\beta')$ to its boundary value function.

**Theorem 2.37.** *Let* $F = (F_1, 0_2, 0_3, 0_4)$ *be a function in* $L^p(\partial D_\beta')$*. Then, for every* $p \in (1,\infty)$*,*

$$\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|S_{s+t,s}F - \widetilde{SF}_1\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0. \tag{2.29}$$

*Proof.* From (2.23), it is enough to prove that $\|T_{t,s}^{1,I}\|_p$ and $\|T_{t,s}^{1,II}\|_p$ tends to 0 as $(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$. Thus, using Proposition 2.33 and Proposition 2.35,

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}} |T_{t,s}^{1,I} F(x,\gamma)|^p \, dx d\gamma &= \int_{\mathbb{R}\times\mathbb{T}} \left|[\Lambda_s \circ \Xi_t] F(x,\gamma)\right|^p \, dx d\gamma \\ &\le C \int_{\mathbb{R}\times\mathbb{T}} \left|\Xi_t F(x,\gamma)\right|^p \, dx d\gamma \\ &\to 0\end{aligned}$$

as $(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$. Similarly, using Proposition 2.34 and Proposition 2.36, we get

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}} \left|T^{1,II}_{t,s}F(x,\gamma)\right|^p \, dxd\gamma &= \int_{\mathbb{R}\times\mathbb{T}} \left|[\Xi'_t \circ \Lambda'_s]F(x,\gamma)\right|^p \, dxd\gamma \\ &\le C \int_{\mathbb{R}\times\mathbb{T}} \left|\Lambda'_s F(x,\gamma)\right|^p \, dxd\gamma \\ &\to 0\end{aligned}$$

as $(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})$. The proof is complete. □

Following the same scheme, we can prove that we have convergence in norm to the boundary values also on the other components of the distinguished boundary.

**Theorem 2.38.** *Let $F = (F_1, 0_2, 0_3, 0_4)$ be a function in $L^p(\partial D'_\beta)$. Then, for every $p \in (1,\infty)$,*

$$\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|S_{s-t,s}F - \widetilde{SF}_2\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0; \tag{2.30}$$

$$\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|S_{-(s+t),-s}F - \widetilde{SF}_3\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0; \tag{2.31}$$

$$\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \|S_{-(s-t),-s}F - \widetilde{SF}_4\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0. \tag{2.32}$$

*Proof.* We have

$$\begin{aligned}T^2_{t,s}F(x,\gamma) &:= [\widetilde{SF}_2 - S_{s-t,s}F](x,\gamma) \\ &= \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{2,I}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) + \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{2,II}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) \\ &= T^{2,I}_{t,s}F(x,\gamma) + T^{2,II}_{t,s}(x,\gamma); \\ T^3_{t,s}F(x,\gamma) &:= [\widetilde{SF}_3 - S_{-(s+t),-s}F](x,\gamma) \\ &= \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{3,I}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) + \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{3,II}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) \\ &= T^{3,I}_{t,s}F(x,\gamma) + T^{3,II}_{t,s}(x,\gamma); \\ T^4_{t,s}F(x,\gamma) &:= [\widetilde{SF}_3 - S_{-(s-t),-s}F](x,\gamma) \\ &= \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{4,I}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) + \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}^{-1}_{\mathbb{R}}\Big[m^{4,II}_{t,s}(\cdot,j)\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\Big](x) \\ &= T^{4,I}_{t,s}F(x,\gamma) + T^{4,II}_{t,s}(x,\gamma),\end{aligned}$$

where

$$m^{2,I}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{1-e^{-(\frac{\pi}{2}-t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})}+e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big];$$
$$m^{2,II}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{1+e^{-(\frac{\pi}{2}-t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{e^{-(2\beta-\pi)(\xi-\frac{j}{2})}-e^{-(\beta-\frac{\pi}{2}+s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big];$$
$$m^{3,I}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{1-e^{-(\frac{\pi}{2}-t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{1+e^{-(\beta-\frac{\pi}{2}-s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big];$$
$$m^{3,II}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{1+e^{-(\frac{\pi}{2}-t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{1-e^{-(\beta-\frac{\pi}{2}-s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big];$$
$$m^{4,I}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{e^{-\pi\xi}-e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{1+e^{-(\beta-\frac{\pi}{2}-s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big];$$
$$m^{4,II}_{t,s}(\xi,j) = \frac{1}{8}\Big[\frac{e^{-\pi\xi}+e^{-(\frac{\pi}{2}+t)\xi}}{\mathrm{Ch}[\pi\xi]}\Big]\Big[\frac{1-e^{-(\beta-\frac{\pi}{2}-s)(\xi-\frac{j}{2})}}{\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\Big].$$

The conclusion follows by an argument similar to the proof of Theorem 2.37. □

Finally, we are able to prove Theorem 2.29.

*Proof. (Theorem 2.29)* As pointed out in Remark 2.2, it is enough to prove the theorem for $F = (F_1, 0_2, 0_3, 0_4)$. For such a function $F$, the thesis follows combining (2.22), Theorem 2.37 and Theorem 2.38.

□

### 2.4.1 A decomposition of $H^p(D'_\beta)$

In this section we prove that the the space $H^p(D'_\beta)$ admits for every $p \in (1,\infty)$ a decomposition

$$H^p(D'_\beta) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}^p_j \tag{2.33}$$

analogously to (2.9) for the case $p = 2$. We recall that, for every $j \in \mathbb{Z}$,

$$\mathcal{H}^p_j = \left\{F \in H^p(D'_\beta) : F(z_1, e^{2\pi i\theta z_2}) = e^{2\pi ij\theta} F(z_1, z_2)\right\}.$$

Thus, we will prove that given a function $F$ in $H^p(D'_\beta)$, there exist functions $F_j$'s such that

$$\lim_{N\to\infty} \Big\| F - \sum_{j=-N}^{N} F_j \Big\|_{H^p(D'_\beta)} = 0,$$

where each function $F_j$ belongs to $\mathcal{H}^p_j$.

We begin proving this result for functions which belong to the range of the operator $S$. As usual, without losing generality, we work using simplified initial data. Given a function $F = (F_1, 0_2, 0_3, 0_4)$ in $L^p(\partial D'_\beta)$, we define

$$S_N F(x+iy, e^{\frac{s}{2}} e^{2\pi i j\gamma}) := \sum_{j=-N}^{N} e^{2\pi i j \gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ \frac{e^{-(\beta - \frac{\pi}{2} + s)(\cdot - \frac{j}{2})} e^{-(\frac{\pi}{2} - s + y)(\cdot)} \mathcal{F}_{\mathbb{R}} F_1(\cdot, \hat{j})}{4 \operatorname{Ch}[\pi \cdot] \operatorname{Ch}[(2\beta - \pi)(\cdot - \frac{j}{2})]} \right](x)$$
$$= \sum_{j=-N}^{N} S_j F(x+iy, e^{\frac{s}{2}} e^{2\pi i j \gamma}).$$

Notice that each function $S_j F$ trivially belongs to $\mathcal{H}^p_j$.

**Proposition 2.39.** *Let $F = (F_1, 0_2, 0_3, 0_4)$ be a function in $L^p(D'_\beta)$, $p \in (1, \infty)$. Then,*

$$\lim_{N \to \infty} \| SF - S_N F \|_{H^p(D'_\beta)} = 0.$$

*Proof.* For almost every function $x \in \mathbb{R}$, the function $F_1(x, \cdot)$ is in $L^p(\mathbb{R})$. Thus, the $L^p$ convergence of one-dimensional Fourier series guarantees that

$$\lim_{N \to \infty} \int_0^1 |F_1(x, y) - F_1^{(N)}(x, \gamma)|^p \, d\gamma = 0,$$

where $F_1^{(N)}(x, \gamma) = \sum_{j=-N}^{N} F_1(x, \hat{j}) e^{2\pi i j \gamma}$. By the Dominated Convergence Theorem we can conclude that

$$\lim_{N \to \infty} \int_{\mathbb{R}} \int_0^1 |F_1(x, \gamma) - F_1^{(N)}(x, \gamma)|^p \, d\gamma dx = \int_{\mathbb{R}} \lim_{N \to \infty} \int_0^1 |F_1(x, \gamma) - F_1^{(N)}(x, \gamma)|^p \, d\gamma dx$$
$$= 0.$$

Thus we can conclude that

$$\lim_{N \to \infty} \| F - F^{(N)} \|_{L^p(\partial D'_\beta)} \to 0,$$

where $F^{(N)} = (F_1^{(N)}, 0_2, 0_3, 0_4)$. By definition, it holds

$$S[F^{(N)}](x+iy, e^{\frac{s}{2}} e^{2\pi i \gamma}) = \sum_{j=-N}^{N} e^{2\pi i j \gamma} \mathcal{F}^{-1}_{\mathbb{R}} \left[ \frac{e^{-(\beta - \frac{\pi}{2} + s)(\cdot - \frac{j}{2})} e^{-(\frac{\pi}{2} - s + y)(\cdot)} \mathcal{F}_{\mathbb{R}} F_1(\cdot, \hat{j})}{4 \operatorname{Ch}[\pi \cdot] \operatorname{Ch}[(2\beta - \pi)(\cdot - \frac{j}{2})]} \right](x)$$
$$= S_N F(x+iy, e^{\frac{s}{2}} e^{2\pi i \gamma})$$

$$= \sum_{j=-N}^{N} S_j F(x+iy, e^{\frac{s}{2}} e^{2\pi i\gamma})$$

and it is easily seen that $S_j F \in H^p_j(D'_\beta)$. Finally, using estimates (2.22), we get

$$\begin{aligned}\lim_{N\to\infty} \|SF - S_N F\|_{H^p(D'_\beta)} &= \lim_{N\to\infty} \|SF - S[F^{(N)}]\|_{H^p(D'_\beta)} \\ &\le C_p \lim_{N\to\infty} \|F - F^{(N)}\|_{L^p(\partial D'_\beta)} \\ &= C_p \lim_{N\to\infty} \|F_1 - F_1^{(N)}\|_{L^p(\mathbb{R}\times\mathbb{T})} \\ &= 0.\end{aligned}$$

The proof is complete. □

So far we proved that every function which is in the range of $S$ admits a decomposition

$$SF = \sum_{j\in\mathbb{Z}} S_j F$$

where the equality is meant in $H^p(D'_\beta)$ and each $S_j F$ belongs to $\mathcal{H}^p_j(D'_\beta)$. To obtain (2.33) it remains to prove that the operator $S$ is surjective on $H^p(D'_\beta)$. We already know this the case for the case $p = 2$, therefore the following result will be useful.

**Proposition 2.40.** *For every $p$ in $(1,\infty)$, we have*

$$\overline{H^2(D'_\beta) \cap H^p(D'_\beta)}^{\|\cdot\|_{H^p}} = H^p(D'_\beta).$$

*Proof.* For every $\varepsilon > 0$ and $z_1 \in \mathcal{S}_\beta$ consider the function

$$G^\varepsilon(z_1) = \frac{1}{1+\varepsilon[2\beta + iz_1]}.$$

Since $G^\varepsilon$ is bounded, it follows that $F\cdot G^\varepsilon$ is in $H^p(D'_\beta)$ for every function $F \in H^p(D'_\beta)$, $p \in (1,\infty)$. Moreover, $F\cdot G^\varepsilon$ belongs to $H^2(D'_\beta)\cap H^p(D'_\beta)$. In fact, let $(t,s) \in [0,\frac{\pi}{2})\times[0,\beta-\frac{\pi}{2})$, then

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}} \left|F[x+i(s+t), e^{\frac{s}{2}}e^{2\pi i\gamma}]G^\varepsilon[x+i(s+t)]\right|^2 dxd\gamma \le \int_0^1 \left[\int_{|F|^2<1} + \int_{|F|^2>1}\right] d\gamma \\ &\le \int_0^1\int_{\mathbb{R}} |G^\varepsilon[x+i(s+t)]|^2\, dxd\gamma + \int_0^1\int_{\mathbb{R}} |F[x+i(s+t), e^{\frac{s}{2}}e^{2\pi i\gamma}]|^p\, dxd\gamma\end{aligned}$$

$$\leq C(\varepsilon) + \|F\|^p_{H^p(D'_\beta)}.$$

Analogue estimates hold for the other terms of the norm $H^p(D'_\beta)$. Thus, for every fixed $\varepsilon > 0$, the function $F \cdot G^\varepsilon$ is in $H^2(D'_\beta) \cap H^p(D'_\beta)$ and $FG^\varepsilon = S[\widetilde{FG^\varepsilon}]$. Notice that $G^\varepsilon$ admits a continous extension to $\overline{D'_\beta}$, therefore $\widetilde{FG^\varepsilon} = \widetilde{F}G^\varepsilon$, where $\widetilde{F}$ is the weak-$*$ limit of $F$ (see Proposition 2.5). Now,

$$\lim_{\varepsilon\to 0^+} \|F - FG^\varepsilon\|^p_{H^p(D'_\beta)} \leq \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \int_0^1 \int_{\mathbb{R}} |(F - FG^\varepsilon)[x \pm i(s+t), e^{\pm\frac{s}{2}} e^{2\pi i\gamma}]|^p \, dx d\gamma +$$
$$+ \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \int_0^1 \int_{\mathbb{R}} |(F - FG^\varepsilon)[x \pm i(s-t), e^{\pm\frac{s}{2}} e^{2\pi i\gamma}]|^p \, dx d\gamma.$$

We focus on one of these term; the computation for the other terms is similar. Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \sup_{(t,s)} \int_0^1 \int_{\mathbb{R}} |(F - FG^\varepsilon)[x + i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}]|^p \, dx d\gamma = \\
&= \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \int_0^1 \int_{\mathbb{R}} \left| F[x + i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}] \left[1 - G^\varepsilon[x + i(s+t)]\right] \right|^p dx d\gamma \\
&\leq \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \liminf_{\delta\to 0^+} \int_0^1 \int_{\mathbb{R}} \left| F[x + i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}] \left[(G^\delta - G^\varepsilon)[x + i(s+t)]\right] \right|^p dx d\gamma \\
&= \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \liminf_{\delta\to 0^+} \int_0^1 \int_{\mathbb{R}} \left| S[\widetilde{F}(G^\delta - G^\varepsilon)][x + i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}] \right|^p dx d\gamma \\
&\leq \lim_{\varepsilon\to 0^+} \sup_{(t,s)} \liminf_{\delta\to 0^+} \|S[\widetilde{F}(G^\delta - G^\varepsilon)]\|^p_{H^p(D'_\beta)} \\
&\leq C_p \lim_{\varepsilon\to 0^+} \liminf_{\delta\to 0^+} \|\widetilde{F}(G^\delta - G^\varepsilon)\|_{L^p(\partial D'_\beta)} \\
&= 0,
\end{aligned}$$

where in the last two lines we used the boundedness of the operator $S$ and the Dominated Convergence Theorem. The proof is complete. □

Finally, we can now prove that the operator $S$ is surjective on $H^p(D'_\beta)$.

**Proposition 2.41.** *Let $F$ be a function in $H^p(D'_\beta)$, $p \in (1, \infty)$. Then, there exists $\widetilde{F}$ in $L^p(\partial D'_\beta)$ such that $F = S\widetilde{F}$.*

*Proof.* From the previous proposition we know there exists a sequence $\{G_n\}$ of functions in $H^2(D'_\beta) \cap H^p(D'_\beta)$ such that $\|F - G_n\|_{H^p(D'_\beta)} \to 0$. Since $G_n$ is in $H^p(D'_\beta)$, there exists

$\widetilde{G_n} = (\widetilde{G_{n,1}}, \widetilde{G_{n,2}}, \widetilde{G_{n,3}}, \widetilde{G_{n,4}})$ in $L^p(\partial D'_\beta)$ such that, with the notation of Proposition 2.12, for $k = 1, \ldots, 4$,

$$G^{(t,s)}_{n,k}(\zeta_1, \zeta_2) \rightharpoonup^* \widetilde{G_{n,k}}(\zeta_1, \zeta_2)$$

where the convergence is weak-$*$ in $L^p(\mathbb{R} \times \mathbb{T})$, $k = 1, 2, 3, 4$. Now

$$\begin{aligned}
\|\widetilde{G_n} - \widetilde{G_m}\|_{L^p(\partial D'_\beta)} &= \sup_{\substack{H \in L^{p'}(\partial D'_\beta) \\ \|H\|_{p'}=1}} \int_{\partial D'_\beta} [\widetilde{G_n} - \widetilde{G_m}] H(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2 \\
&= \sup_{\substack{H \in L^{p'}(\partial D'_\beta) \\ \|H\|_{p'}=1}} \sum_{k=1}^{4} \int_{\partial_i D'_\beta} [\widetilde{G_n} - \widetilde{G_m}] H(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2 \\
&= \sup_{\substack{H \in L^{p'}(\partial D'_\beta) \\ \|H\|_{p'}=1}} \sum_{k=1}^{4} \lim_{(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})} \int_{\partial_k D'_\beta} [G^{(t,s)}_n - G^{(t,s)}_m] H(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2 \\
&\leq C \sum_{i=1}^{4} \lim_{(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})} \|G^{(t,s)}_{n,k} - G^{(t,s)}_m\|_{L^p(\partial_k D'_\beta)} \\
&\leq C \|G_n - G_m\|_{H^p(D'_\beta)}.
\end{aligned}$$

Thus, the sequence $\{\widetilde{G_n}\}$ is a Cauchy sequence in $L^p(\partial D'_\beta)$ which admits a limit $\widetilde{G}$ in $L^p(\partial D'_\beta)$. We recall that, since $G_n$ is a function in $H^2(D'_\beta) \cap H^p(D'_\beta)$, then $G_n = S\widetilde{G_n}$. Now, for every fixed $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for every $n > N(\varepsilon)$, it holds

$$\|F - G_n\|_{H^p(D'_\beta)} < \varepsilon \qquad \text{and} \qquad \|\widetilde{G_n} - \widetilde{G}\|_{L^p(\partial D'_\beta)} < \varepsilon.$$

Therefore,

$$\begin{aligned}
\|F - S\widetilde{G}\|_{H^p(D'_\beta)} &\leq \|F - G_n\|_{H^p(D'_\beta)} + \|G_n - S\widetilde{G}\|_{H^p(D'_\beta)} \\
&\leq \varepsilon + \|S\widetilde{G_n} - S\widetilde{G}\|_{H^p(D'_\beta)} \\
&\leq \varepsilon + \|\widetilde{G_n} - \widetilde{G}\|_{L^p(\partial D'_\beta)} \\
&\leq 2\varepsilon,
\end{aligned}$$

where we used the boundedness of the operator $S$. Since this holds for every $\varepsilon > 0$, we can conclude that $F = S\widetilde{G}$ and the proposition is proved. □

**Remark 2.42.** Theorem 2.37 and Theorem 2.38 show that every function in the range of $S$ tends to its boundary values in norm. The previous proposition allows to conclude that this is true for every element of $H^p(D'_\beta)$, $p \in (1, \infty)$.

**Remark 2.43.** Proposition 2.39 and Proposition 2.41 together prove the decomposition (2.33).

### 2.4.2 Pointwise convergence

We conclude this chapter proving a Fatou-type theorem. We prove that an appropriate restriction of a function $F$ in $H^p(D'_\beta)$, $p \in (1, \infty)$, converges to its boundary value function $\widetilde{F}$ also pointwise almost everywhere . As usual, we prove our results in a simplified situation. The general case follows by linearity. Let $F = (F_1, 0_2, 0_3, 0_4)$ be a function in $L^p(\partial D'_\beta)$, then we proved that, for example,

$$\lim_{(t,s)\to(\frac{\pi}{2},\beta-\frac{\pi}{2})} \int_{\mathbb{R}} \int_0^1 \left| SF[x+i(s+t), e^{\frac{s}{2}} e^{2\pi i\gamma}] - SF[x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})} e^{2\pi i\gamma}] \right|^p d\gamma dx = 0.$$

In general, to prove a pointwise convergence result, we expect that we need to put some restrictions on the parameters $t$ and $s$. For example, also in the simpler case of the polydisc $D^2(0,1) = D(0,1) \times D(0,1)$, we are able to prove the almost everywhere existence of the pointwise radial limit

$$\lim_{(r_1,r_2)\to(1,1)} G(r_1 e^{2\pi i\theta}, r_2 e^{2\pi i\gamma})$$

for a function $G$ in $H^p(D^2)$ under the hypothesis that the ratio $\frac{1-r_1}{1-r_2}$ is bounded (see, for example, [Rud69, Chapter 2, Section 2.3]).

At the moment, we are able to prove a pointwise convergence result which depends only on one parameter. It would be interesting to determine a larger approach region to the distinguished boundary $\partial D'_\beta$.

We need the following lemma.

**Lemma 2.44.** *Let $S_\beta$ be the strip $S_\beta = \{z = x + iy \in \mathbb{C} : |y| < \beta\}$. Let $G = (G_+, G_-)$ be a function in $L^p(\partial S_\beta)$, $p \in (1, \infty)$. Then the function*

$$SG(x+iy) = \mathcal{F}^{-1}\left[ \frac{\widehat{G_+}(\cdot) e^{-(y+\beta)(\cdot)} + \widehat{G_-}(\cdot) e^{-(y-\beta)(\cdot)}}{4 \operatorname{Ch}[\pi(\cdot)] \operatorname{Ch}[(2\beta - \pi(\cdot - \frac{j}{2})]} \right](x)$$

*belogs to $H^p(S_\beta)$ for every integer $j$.*

*Proof.* Without losing generality and to simplify notation we suppose $G = (G, 0)$. Thus,

$$SG(x+iy) = \mathcal{F}^{-1}\left[\frac{\widehat{G_1}(\cdot)e^{-(y+\beta)(\cdot)}}{4\,\mathrm{Ch}[\pi(\cdot)]\,\mathrm{Ch}[(2\beta-\pi(\cdot-\frac{j}{2})]}\right](x).$$

If $G$ is in $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\begin{aligned} SG(x+iy) &= \mathcal{F}^{-1}\left[\frac{\widehat{G}(\cdot)e^{-(y+\beta)(\cdot)}}{4\,\mathrm{Ch}[\pi(\cdot)]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\frac{\mathrm{Ch}[2\beta(\cdot)]}{\mathrm{Ch}[2\beta(\cdot)]}\right](x) \\ &= \mathcal{F}^{-1}\left[\frac{\widehat{\mathscr{G}}(\cdot)e^{-(y+\beta)(\cdot)}}{2\,\mathrm{Ch}[2\beta(\cdot)]}\right](x), \end{aligned}$$

where

$$\begin{aligned} \mathscr{G}(x) &= \mathcal{F}^{-1}\left[\frac{\mathrm{Ch}[2\beta(\cdot)]\widehat{G}(\cdot)}{2\,\mathrm{Ch}[\pi(\cdot)]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) \\ &= \mathcal{F}^{-1}\left[m(\cdot)\widehat{G}(\cdot)\right](x). \end{aligned}$$

Since $m$ is bounded and $G$ belongs to $L^p \cap L^2$ the function $\mathscr{G}$ is well defined. From Proposition 1.18 we deduce that $SG$ is $H^p(S_\beta)$. The conclusion for a general function $G$ in $L^p(\mathbb{R})$ follows by density. □

**Theorem 2.45.** *Let $F = (F_1, 0_2, 0_3, 0_4)$ be a function in $L^p(\partial D'_\beta)$, $p \in (1, \infty)$. Then,*

$$\lim_{t\to\beta^-} SF[x+it, e^{\frac{t}{2\beta}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] = \widetilde{SF}_1[x+i\beta, e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] \tag{2.34}$$

*for almost every $(x, \gamma) \in \mathbb{R} \times \mathbb{T}$.*

*Proof.* By (2.23), we want to prove that

$$\begin{aligned} L_t(x,\gamma) &= \left|\sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-2\beta(\cdot)}e^{j(\beta-\frac{\pi}{2})} - e^{-(\beta+t)(\cdot)}e^{\frac{j}{2}(\beta-\frac{\pi}{2})(1+\frac{t}{\beta})}}{4\,\mathrm{Ch}[\pi(\cdot)]\,\mathrm{Ch}[(2\beta-\frac{\pi}{2})(\cdot-\frac{j}{2}]}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\right](x)\right| \\ &= \left|\sum_{j\in\mathbb{Z}} S_j^t F(x,\gamma)\right| \to 0 \end{aligned}$$

for almost every $(x, \gamma) \in \mathbb{R} \times \mathbb{T}$ when $t$ tends to $\beta^-$. Let $\varepsilon > 0$ be fixed. Then,

$$\left|\left\{(x,\gamma) \in \mathbb{R}\times\mathbb{T} : \limsup_{t\to\beta^-} L_t(x,\gamma) > \varepsilon\right\}\right|$$

$$\leq \sum_{j\in\mathbb{Z}} \left| \left\{ (x,\gamma) \in \mathbb{R}\times\mathbb{T} : \limsup_{t\to\beta^-} |S_j^t F(x,\gamma)| > \alpha_j \right\} \right|,$$

where the $\alpha_j$'s are positive and $\sum_{j\in\mathbb{Z}} \alpha_j = \varepsilon$. We claim that the sets in the right-hand side of the previous inequality are all of measure zero. Following the proof of Theorem 2.37 we obtain that

$$\lim_{t\to\beta^-} \|S_j^t(F)\|_{L^p(\mathbb{R}\times\mathbb{T})} = 0. \tag{2.35}$$

Therefore, it is enough to prove the existence of the pointwise limit

$$\lim_{t\to\beta^-} e^{2\pi i j\gamma}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(\beta+t)(\cdot)}e^{\frac{j}{2}(\beta-\frac{\pi}{2})(1+\frac{t}{\beta})}}{4\operatorname{Ch}[\pi(\cdot)]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\mathcal{F}_{\mathbb{R}}F_1(\cdot,\hat{j})\right](x)$$

for almost every $(x,\gamma)\in\mathbb{R}\times\mathbb{T}$.

To prove this, it is sufficient to prove that

$$\lim_{t\to\beta^-} \mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(\beta+t)(\cdot)}\mathcal{F}_{\mathbb{R}}G(\cdot)}{4\operatorname{Ch}[\pi(\cdot)]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2}]}\right](x)$$

exists for almost every $x$ in $\mathbb{R}$ and for every function $G$ in $L^p(\mathbb{R})$, $p\in(1,\infty)$. The existence of this last limit follows immediately from the lemma and Theorem 1.24. □

Analogously we can prove the pointwise convergence of $SF$ to the other components of $\partial D'_\beta$.

**Theorem 2.46.** *Let $F=(F_1,0_2,0_3,0_4)$ be a function in $L^p(\partial D'_\beta)$, $p\in(1,\infty)$. Then,*

$$\lim_{t\to(\beta-\frac{\pi}{2})^-} SF[x+it, e^{\frac{t}{2}}e^{2\pi i\gamma}] = \widetilde{SF}_2[x+i(\beta-\frac{\pi}{2}), e^{\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}];$$
$$\lim_{t\to-\beta^+} SF[x+it, e^{\frac{t}{2\beta}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}] = \widetilde{SF}_3[x-i\beta, e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}];$$
$$\lim_{t\to-(\beta-\frac{\pi}{2})^-} SF[x+it, e^{\frac{t}{2}}e^{2\pi i\gamma}] = \widetilde{SF}_4[x-i(\beta-\frac{\pi}{2}), e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}]$$

*for almost every $(x,\gamma)$ in $\mathbb{R}\times[0,1)$.*

**Remark 2.47.** We proved the previous theorems for functions that belong to the range of the operator $S$. From Proposition 2.41 we can conclude that the results are true for every function in $H^p(D'_\beta), p\in(1,\infty)$.

# Chapter 3

# The reproducing kernel of $H^2(D'_\beta)$

We report here the proof of Theorem 2.17. The proof is obtained following the arguments in [KP08b]. We recall that

$$K_{D'_\beta}[(z_1,z_2),(w_1,w_2)] = \sum_{j\in\mathbb{Z}} \frac{w_2^j \overline{z}_2^j}{8\pi} \int_{\mathbb{R}} \frac{e^{i(w_1-\overline{z_1})\xi}}{\mathrm{Ch}[\pi\xi]\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi.$$

The proof is based on a direct computation of the sum which defines $K_{D'_\beta}$. To simplify the notation we define

$$I_j(\tau) = \int_{\mathbb{R}} \frac{e^{i\tau\xi}}{\mathrm{Ch}[\pi\xi]\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi.$$

Then, we would like to compute the sum

$$\sum_{j\in\mathbb{Z}} I_j(\tau)\lambda^j, \tag{3.1}$$

where the couple $(\tau,\lambda)$ belongs to the set

$$D' = \{(\tau,\lambda)\in\mathbb{C}^2 : \left|\operatorname{Im}\tau - \log|\lambda|^2\right| < \pi, e^{-(\beta-\frac{\pi}{2})} < |\lambda| < e^{\beta-\frac{\pi}{2}}\}.$$

To compute $I_j(\tau)$ we use the Residue Theorem. We denote $g_j(\zeta)$ the holomorphic function

$$g_j(\zeta) := \frac{e^{i\tau\zeta}}{\mathrm{Ch}[\pi\zeta]\,\mathrm{Ch}[(2\beta-\pi)(\zeta-\frac{j}{2})]}.$$

About the function $g_j$, we have the following result whose easy proof we do not report.

**Proposition 3.1.** *The function $g_j$ is holomorphic in the plane except at the points*

$$\zeta = i\left(\frac{1}{2}+k\right),\ k\in\mathbb{Z}, \qquad \zeta = i\nu_\beta\left(\frac{1}{2}+k\right)+\frac{j}{2},\ k\in\mathbb{Z},$$

*where* $\nu_\beta = \frac{\pi}{2\beta-\pi}$. *Moreover*

$$Res\left(g_j, \frac{j}{2}\pm i\frac{\nu_\beta}{2}\right) = \pm\frac{e^{i\tau(\frac{j}{2}\pm i\frac{\nu_\beta}{2})}}{i(2\beta-\pi)\operatorname{Ch}\left[\pi\left(\frac{j}{2}\pm i\frac{\nu_\beta}{2}\right)\right]}.$$

To compute $I_j(\tau)$ we shall distinguish two cases according to whether $\operatorname{Re}\tau\geq 0$ or $\operatorname{Re}\tau < 0$. Let us focus now on the case $\operatorname{Re}\tau > 0$. We shall use the method of contour integrals. As contour of integration we choose the rectangular box $\gamma_N$ centered on the imaginary axis with corners $N+i0$, $-N+i0$, $N+ih$ and $N-ih$ where $h$ is chosen so that

$$\frac{\nu_\beta}{2} < h < \min\left(\frac{1}{2}, \frac{3\nu_\beta}{2}\right).$$

By the Residue Theorem we have the following result.

**Proposition 3.2.** *Let* $\beta > \pi$ *and fix* $h$ *as above. We define*

$$R_j(\tau) = 2\pi i\cdot Res\left(g_j, \frac{j}{2}+i\frac{\nu_\beta}{2}\right), \qquad J_j(\tau) = \int_{\mathbb{R}} g_j(\xi+ih)\ d\xi.$$

*Then, for all* $j$ *in* $\mathbb{Z}$,

$$I_j(\tau) = R_j(\tau) + J_j(\tau).$$

*Proof.* By the Residue Theorem, we have

$$\int_{-N}^{N} g_j(\xi)\ d\xi = R_j(\tau) + \int_{-N}^{N} g_j(\xi+ih)\ d\xi - i\int_0^h g_j(N+i\xi)\ d\xi - i\int_h^0 g_j(-N+i\xi)\ d\xi.$$

Thus, we want to show that the integrals along the vertical sides go to zero. It holds

$$i\int_0^h g_j(N+i\xi)\ d\xi = i\int_0^h \frac{e^{i\tau[N+iy]}}{\operatorname{Ch}[\pi(N+iy)]\operatorname{Ch}[(2\beta-\pi)(N+iy-\frac{j}{2})]}\ d\xi.$$

Therefore,

$$\left|\int_0^h g_j(N+i\xi)\ d\xi\right| \leq \int_0^h \frac{e^{-\operatorname{Re}(\tau)y-\operatorname{Im}\tau N}}{e^{2\beta N}e^{(2\beta-\pi)\frac{j}{2}}}\ dy$$

$$\leq \int_0^\infty \frac{e^{-\operatorname{Re}(\tau)y-\operatorname{Im}\tau N}}{e^{2\beta N}e^{(2\beta-\pi)\frac{j}{2}}}\, dy$$
$$\to 0$$

uniformly when $\tau$ varies in a compact subset of $S_{2\beta}$ and $N$ goes to infinity. The proof when $\operatorname{Re}\tau < 0$ is completely analogous, but we integrate along the analogue rectangular box in the bottom half-plane. □

So, we have splitted the sum (3.1) into two different sums. Namely,

$$\sum_{j\in\mathbb{Z}} I_j(\tau)\lambda^j = \sum_{j\in\mathbb{Z}} R_j(\tau)\lambda^j + \sum_{j\in\mathbb{Z}} J_j(\tau)\lambda^j.$$

where the couple $(\tau,\lambda)$ belongs to the domain

$$\mathcal{D} = \left\{(\tau,\lambda)\in\mathbb{C}^2 : \left|\operatorname{Im}\tau - \log|\lambda|^2\right| < \pi, e^{-(\beta-\frac{\pi}{2})} < |\lambda| < e^{(\beta-\frac{\pi}{2})}\right\}.$$

We focus on the sum of the $R_j$. Unless specified, we are always supposing to work with $\tau$ such that $\operatorname{Re}\tau \geq 0$.

Before stating a result concerning the sum of the $R_j$, we remark that the following equality will have a prominent role in our computation. Let $a, b$ in $\mathbb{R}$ such that $a \neq 0$, then

$$\frac{e^{|a|}}{\operatorname{Ch}(a+ib)} = 2e^{-i\operatorname{sgn}(a)b}\left(1 - \frac{e^{-2\operatorname{sgn}(a)(a+ib)}}{1+e^{-2\operatorname{sgn}(a)(a+ib)}}\right). \tag{3.2}$$

**Proposition 3.3.** *There exists a function $E(\tau,\lambda)$ which is smooth in a neighborhood of $\overline{\mathcal{D}}$ such that*

$$\begin{aligned}\mathcal{R}(\tau,\lambda) &= \sum_{j\in\mathbb{Z}} R_j(\tau)\lambda^j \qquad (3.3)\\ &= \frac{4\nu_\beta}{e^{\frac{\tau\nu_\beta}{2}}}\left\{\left[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + \left[\frac{e^{-\frac{i\pi\nu_\beta}{2}}\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}}\right] + E(\tau,\lambda) + \frac{1}{\operatorname{Ch}[i\frac{\nu_\beta}{2}]}\right\}.\end{aligned}$$

*The convergence of the series is uniform on compact subsets of $\mathcal{D}$.*

*Proof.* From the previous results we have

$$R_j(\tau) = 2\pi i \cdot Res\left(g_j, \frac{j}{2} + i\frac{\nu_\beta}{2}\right) = \frac{2\pi e^{i\tau\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)}}{(2\beta-\pi)\operatorname{Ch}\left[\pi\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)\right]}$$

$$= \frac{2\nu_\beta e^{i\tau\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)}}{\mathrm{Ch}\left[\pi\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)\right]}.$$

Our problem is then to compute the sum

$$\sum_{j\in\mathbb{Z}} \frac{2\nu_\beta e^{i\tau\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)}}{\mathrm{Ch}\left[\pi\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)\right]}\lambda^j = 2\nu_\beta e^{-\frac{\tau\nu_\beta}{2}} \sum_{j\in\mathbb{Z}} \frac{e^{\frac{ij\tau}{2}}\lambda^j}{\mathrm{Ch}\left[\pi\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)\right]}. \tag{3.4}$$

If we consider only the sum on the right-hand side of the previous equation, from (3.2), it follows

$$\sum_{j\in\mathbb{Z}} \frac{e^{\frac{ij\tau}{2}}\lambda^j}{\mathrm{Ch}\left[\pi\left(\frac{j}{2}+i\frac{\nu_\beta}{2}\right)\right]} = 2\sum_{j\in\mathbb{Z}} e^{\frac{ij\tau}{2}}\lambda^j \frac{e^{-i\sigma(j)\frac{\pi\nu_\beta}{2}}}{e^{\frac{|j|\pi}{2}}}\left[1-\frac{e^{-2\sigma(j)\left(\frac{j\pi}{2}+i\frac{\pi\nu_\beta}{2}\right)}}{1+e^{-2\sigma(j)\left(\frac{j\pi}{2}+i\frac{\pi\nu_\beta}{2}\right)}}\right]$$
$$= 2\left(F-E+G\right),$$

where

- $\sigma(j) = \mathrm{sgn}(j)$;
- $F = F(\tau,\lambda) = \sum_{j\neq 0} e^{\frac{ij\tau}{2}}\lambda^j e^{-\frac{\pi}{2}\left(|j|+i\nu_\beta\sigma(j)\right)}$;
- $E = E(\tau,\lambda) = \sum_{j\neq 0} \frac{e^{\frac{ij\tau}{2}}\lambda^j e^{-\frac{\pi}{2}\left(|j|+i\nu_\beta\sigma(j)\right)}e^{-\pi\sigma(j)(j+i\nu_\beta)}}{1+e^{-\pi\sigma(j)(j+i\nu_\beta)}}$;
- $G = \frac{1}{\mathrm{Ch}\left(i\frac{\pi\nu_\beta}{2}\right)}$.

About $F$, we have

$$F = e^{\frac{i\pi\nu_\beta}{2}}\sum_{j<0} e^{j\left(\frac{i\tau}{2}+\frac{\pi}{2}\right)}\lambda^j + e^{-\frac{i\pi\nu_\beta}{2}}\sum_{j>0} e^{j\left(\frac{i\tau}{2}-\frac{\pi}{2}\right)}\lambda^j$$
$$= e^{\frac{i\pi\nu_\beta}{2}}\left[\frac{1}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + e^{-\frac{i\pi\nu_\beta}{2}}\left[\frac{\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}}\right] \tag{3.5}$$

and the convergence of the two series is guaranteed exactly when $e^{\frac{\mathrm{Im}\tau-\pi}{2}} < |\lambda| < e^{\frac{\mathrm{Im}\tau+\pi}{2}}$.

We analyze now the error term $E$. It results

$$E = e^{\frac{3i\pi\nu_\beta}{2}}\sum_{j<0}\frac{\lambda^j e^{j\frac{i\tau+3\pi}{2}}}{1+e^{\pi(j+i\nu_\beta)}} + e^{-\frac{3i\pi\nu_\beta}{2}}\sum_{j>0}\frac{\lambda^j e^{j\frac{i\tau-3\pi}{2}}}{1+e^{-\pi(j+i\nu_\beta)}}.$$

It is easy to prove that there exists a constant $c > 0$ such that $|1 + e^{-\pi\sigma(j)+i\nu_\beta}| > c > 0$ for every $j$. Hence the series which define $E$ converge when $e^{\frac{\mathrm{Im}\tau-3\pi}{2}} < |\lambda| < e^{\frac{\mathrm{Im}\tau+3\pi}{2}}$ which is an annulus strictly larger than $e^{\frac{\mathrm{Im}\tau-\pi}{2}} < |\lambda| < e^{\frac{\mathrm{Im}\tau+\pi}{2}}$. Thus the sums of the two series are smooth and bounded, with all derivatives smooth and bounded, on a neighborhood of the closure $\overline{\mathcal{D}'}$ of $\mathcal{D}'$.

In conclusion, we have

$$\mathcal{R}(\tau,\lambda) = \frac{4\nu_\beta}{e^{\frac{\tau\nu_\beta}{2}}}\left\{\left[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + \left[\frac{e^{-\frac{i\pi\nu_\beta}{2}}\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}}\right] + E(\tau,\lambda) + \frac{1}{\mathrm{Ch}\left(i\frac{\pi\nu_\beta}{2}\right)}\right\},$$

as we wished. □

It remains to compute $J_j(\tau)$ and then $\sum J_j(\tau)\lambda^j$. We recall that

$$J_j(\tau) = \int_{\mathbb{R}} \frac{e^{i\tau(\xi+ih)}}{\mathrm{Ch}[\pi(\xi+ih)]\,\mathrm{Ch}[(2\beta-\pi)(\xi+ih-j/2)]}\,d\xi.$$

From equation (3.2) we obtain

$$\begin{aligned}\frac{1}{\mathrm{Ch}[\pi(\xi+ih)]\,\mathrm{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\\ = 4\frac{e^{-i\,\mathrm{sgn}(\xi)\pi h - i\,\mathrm{sgn}(\xi-\frac{j}{2})(2\beta-\pi)h}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}} - \frac{e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}}{1+e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}} - \frac{e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}}{1+e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}}.\end{aligned}$$

Let us define $\sigma(\xi) = e^{-i\,\mathrm{sgn}(\xi)\pi h - i\,\mathrm{sgn}(\xi-\frac{j}{2})(2\beta-\pi)h}$. Then we have

$$J_j(\tau) = 4e^{-\tau h}\left(M_j(\tau) - E_j^{(1)}(\tau) - E_j^{(2)}(\tau) + E_j^{(3)}(\tau)\right), \tag{3.6}$$

where

$$M_j(\tau) = \int_{\mathbb{R}} \sigma(\xi)\frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}}\,d\xi; \tag{3.7}$$

$$E_j^{(1)}(\tau) = \int_{\mathbb{R}} \sigma(\xi)\frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}}\left[\frac{e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}}{1+e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}}\right]d\xi; \tag{3.8}$$

$$E_j^{(2)}(\tau) = \int_{\mathbb{R}} \sigma(\xi)\frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}}\left[\frac{e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}}{1+e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}}\right]d\xi; \tag{3.9}$$

$$E_j^{(3)}(\tau) = \int_{\mathbb{R}} \sigma(\xi)\frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}}\left[\frac{e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}}{1+e^{-2\,\mathrm{sgn}(\xi)[\pi(\xi+ih)]}}\right]$$

$$\times \left[ \frac{e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}}{1+ e^{-2\,\mathrm{sgn}(\xi-\frac{j}{2})[(2\beta-\pi)(\xi-\frac{j}{2}+ih)]}} \right] d\xi. \tag{3.10}$$

Our problem has become the computation of the sum

$$\sum_{j\in\mathbb{Z}} J_j(\tau)\lambda^j = 4e^{-\tau h}\left[ \sum_{j\in\mathbb{Z}} M_j(\tau)\lambda^j + \sum_{k=1}^{3}\sum_{j\in\mathbb{Z}} E_j^{(k)}(\tau)\lambda^j \right]. \tag{3.11}$$

To compute the integrals (3.7)-(3.10) we will use the following scheme. If $j > 0$, we choose a positive $\delta$ such that $0 < \delta < j/2$ and we consider

$$\begin{aligned} \int_{\mathbb{R}} f &= \int_{-\infty}^{-\delta} f + \int_{-\delta}^{\delta} f + \int_{\delta}^{\frac{j}{2}-\delta} f + \int_{\frac{j}{2}-\delta}^{\frac{j}{2}+\delta} f + \int_{\frac{j}{2}+\delta}^{+\infty} f \\ &= I + \mathcal{E}_1 + II + \mathcal{E}_2 + III. \end{aligned} \tag{3.12}$$

Analogously, for negative $j$'s, we choose a positive $\delta$ such that $j/2 < -\delta < 0$ and we will consider

$$\begin{aligned} \int_{\mathbb{R}} f &= \int_{-\infty}^{\frac{j}{2}-\delta} f + \int_{\frac{j}{2}-\delta}^{\frac{j}{2}+\delta} + \int_{\frac{j}{2}+\delta}^{-\delta} f + \int_{-\delta}^{\delta} f + \int_{\delta}^{+\infty} f \\ &= I^* + \mathcal{E}_1^* + II^* + \mathcal{E}_2^* + III^*. \end{aligned} \tag{3.13}$$

We remark that the case $j = 0$ is somehow special, but it could be treated in a similar way. Also, notice that the decomposition of the integrals above make sense even for $\delta = 0$; this choice of $\delta$ will be the case in the computation of the sum of the $M_j$'s as we immediately see.

**Proposition 3.4.** *There exist entire functions $\psi_i(\tau, \lambda), i = 1, 2, 3, 4,$ such that*

$$\begin{aligned} 4e^{-\tau h}\sum_{j\in\mathbb{Z}} M_j(\tau)\lambda^j = 4e^{-\tau h}\Bigg[ &\frac{e^{2\beta ih}}{i\tau+2\beta} + \frac{-e^{-2\beta ih}}{i\tau-2\beta}\frac{-e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})} \\ &+ \frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})} + \frac{\psi_1(\lambda)}{(i\tau+2\beta)(1-\lambda e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_2(\tau,\lambda)}{(i\tau-2\beta)(1-\lambda e^{\frac{i\tau-\pi}{2}})} \\ &+ \frac{\psi_3(\tau,\lambda)}{(1-\lambda e^{\frac{i\tau-\pi}{2}})(1-\lambda e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_4(\tau,\lambda)}{(1-\lambda e^{\beta-\frac{\pi}{2}})(1-\lambda e^{\frac{i\tau+\pi}{2}})}\Bigg], \end{aligned} \tag{3.14}$$

*where*

$$\psi_1(\lambda) = \lambda e^{2\beta ih}e^{-(\beta-\frac{\pi}{2})};$$

$$\psi_2(\tau,\lambda) = -\lambda e^{-2\beta ih} e^{\frac{i\tau-\pi}{2}};$$

$$\psi_3(\tau,\lambda) = \lambda e^{-(\beta-\frac{\pi}{2})} e^{2(\beta-\pi)ih} \left[\frac{e^{\frac{i\tau}{2}+\beta-\pi}-1}{i\tau+2\beta-2\pi}\right];$$

$$\psi_4(\tau,\lambda) = \lambda e^{-2(\beta-\pi)ih} e^{\beta-\frac{\pi}{2}} \left[\frac{e^{\frac{i\tau}{2}-\beta+\pi}-1}{i\tau-2\beta+2\pi}\right].$$

*Proof.* First of all, we have to compute each single $M_j(\tau)$. In this case we choose $\delta = 0$ in (3.12) and (3.13) so that we do not have the error terms $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_1^*$ and $\mathcal{E}_2^*$. We begin focusing on positive $j$'s. Therefore,

$$I = e^{2\beta ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{-\infty}^{-\delta} e^{(i\tau+2\beta)\xi}\, d\xi; \tag{3.15}$$

$$II = e^{2(\beta-\pi)ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{\delta}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta-2\pi)\xi}\, d\xi; \tag{3.16}$$

$$III = e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{\frac{j}{2}+\delta}^{+\infty} e^{(i\tau-2\beta)\xi}\, d\xi; \tag{3.17}$$

With some easy computations we obtain

$$I = e^{2\beta ih}\left[\frac{e^{-\delta(i\tau+2\beta)}}{i\tau+2\beta}\right] e^{-(2\beta-\pi)\frac{j}{2}};$$

$$II = \frac{e^{2(\beta-\pi)ih}}{i\tau+2\beta-2\pi}\left[e^{-\delta(i\tau+2\beta-2\pi)} e^{(i\tau-\pi)\frac{j}{2}} - e^{\delta(i\tau+2\beta-2\pi)} e^{-(2\beta-\pi)\frac{j}{2}}\right];$$

$$III = -e^{-2\beta ih}\left[\frac{e^{\delta(i\tau-2\beta)}}{i\tau-2\beta}\right] e^{(i\tau-\pi)\frac{j}{2}};$$

Finally, taking $\delta = 0$, it results

$$I = \frac{e^{2\beta ih}}{i\tau+2\beta} e^{-(2\beta-\pi)\frac{j}{2}};$$

$$II = \frac{e^{2(\beta-\pi)ih}}{i\tau+2\beta-2\pi}\left(e^{(i\tau-\pi)\frac{j}{2}} - e^{-(2\beta-\pi)\frac{j}{2}}\right);$$

$$III = -\frac{e^{-2\beta ih}}{i\tau-2\beta} e^{(i\tau-\pi)\frac{j}{2}}.$$

Summing up over the positive $j$'s we obtain

$$\sum_{j>0} M_j(\tau)\lambda^j = \sum_{j>0}(I+II+III)\lambda^j \tag{3.18}$$

$$= \frac{e^{2\beta ih}}{i\tau+2\beta}\left[\frac{\lambda}{e^{\beta-\frac{\pi}{2}}-\lambda}\right] + \frac{e^{2(\beta-\pi)ih}}{i\tau+2\beta-2\pi}\left[\frac{\lambda}{e^{-\frac{i\tau-\pi}{2}}-\lambda} - \frac{\lambda}{e^{\beta-\frac{\pi}{2}}-\lambda}\right] - \frac{e^{-2\beta ih}}{i\tau-2\beta}\left[\frac{\lambda}{e^{-\frac{i\tau-\pi}{2}}-\lambda}\right]$$

$$= \frac{e^{2\beta ih}}{i\tau+2\beta}\left[\frac{\lambda}{e^{\beta-\frac{\pi}{2}}-\lambda}\right] + \frac{\lambda e^{-(\beta-\frac{\pi}{2})}e^{2(\beta-\pi)ih}}{\left(1-\lambda e^{\frac{i\tau-\pi}{2}}\right)\left(1-\lambda e^{-(\beta-\frac{\pi}{2})}\right)}\left[\frac{e^{\frac{i\tau}{2}+\beta-\pi}-1}{i\tau+2\beta-2\pi}\right] - \frac{e^{-2\beta ih}}{i\tau-2\beta}\left[\frac{\lambda}{e^{-\frac{i\tau-\pi}{2}}-\lambda}\right]$$

Notice that we do not have a singularity when $\tau \to 2\beta - 2\pi$.

This is the computation only for the positive $j$'s. Analogously, using (3.13), we obtain a result for negative $j$'s. Remembering that we have chosen $\delta = 0$, we have

$$I^* = \frac{e^{2\beta ih}}{i\tau+2\beta}e^{(i\tau+\pi)\frac{j}{2}};$$

$$II^* = \frac{e^{-2(\beta-\pi)ih}}{i\tau-2\beta+2\pi}\left(e^{(2\beta-\pi)\frac{j}{2}} - e^{(i\tau+\pi)\frac{j}{2}}\right);$$

$$III^* = -\frac{e^{-2\beta ih}}{i\tau-2\beta}e^{(2\beta-\pi)\frac{j}{2}}.$$

Then, it results

$$\sum_{j<0} M_j(\tau)\lambda^j$$

$$= \sum_{j<0}(I^* + II^* + III^*)\lambda^j \tag{3.19}$$

$$= \frac{e^{2\beta ih}}{i\tau+2\beta}\left[\frac{1}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + \frac{e^{-2(\beta-\pi)ih}}{i\tau-2\beta+2\pi}\left[\frac{1}{\lambda e^{\beta-\frac{\pi}{2}}-1} - \frac{1}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] - \frac{e^{-2\beta ih}}{i\tau-2\beta}\frac{1}{\lambda e^{\beta-\frac{\pi}{2}}-1}$$

$$= \frac{e^{2\beta ih}}{i\tau+2\beta}\left[\frac{1}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + \frac{\lambda e^{-2(\beta-\pi)ih}e^{\beta-\frac{\pi}{2}}}{\left(\lambda e^{\beta-\frac{\pi}{2}}-1\right)\left(\lambda e^{\frac{i\tau+\pi}{2}}-1\right)}\left[\frac{e^{\frac{i\tau}{2}-\beta+\pi}-1}{i\tau-2\beta+2\pi}\right] - \frac{e^{-2\beta ih}}{i\tau-2\beta}\frac{1}{\lambda e^{\beta-\frac{\pi}{2}}-1}.$$

Notice that we do not have a singularity when $\tau \to -2\beta + 2\pi$. It remains to compute $M_0(\tau)$; it is easy to verify that

$$M_0(\tau) = \frac{e^{2\beta ih}}{i\tau+2\beta} - \frac{e^{-2\beta ih}}{i\tau-2\beta}. \tag{3.20}$$

In conclusion, we found that

$$\sum_{j\in\mathbb{Z}} M_j(\tau)\lambda^j = \frac{e^{2\beta ih}}{i\tau+2\beta}\left[\frac{\lambda e^{-(\beta-\frac{\pi}{2})}}{1-\lambda e^{-(\beta-\frac{\pi}{2})}}\right] - \frac{e^{-2\beta ih}e^{\frac{i\tau-\pi}{2}}}{i\tau-2\beta}\left[\frac{\lambda}{1-\lambda e^{\frac{i\tau-\pi}{2}}}\right]$$
$$+ \frac{\lambda e^{-(\beta-\frac{\pi}{2})}e^{2(\beta-\pi)ih}}{\left(1-\lambda e^{\frac{i\tau-\pi}{2}}\right)\left(1-\lambda e^{-(\beta-\frac{\pi}{2})}\right)}\left[\frac{e^{\frac{i\tau}{2}+\beta-\pi}-1}{i\tau+2\beta-2\pi}\right]$$

$$+\frac{e^{2\beta ih}}{i\tau+2\beta}-\frac{e^{-2\beta ih}}{i\tau-2\beta}-\frac{e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})}$$
$$+\frac{\lambda e^{-2(\beta-\pi)ih}e^{\beta-\frac{\pi}{2}}}{\left(1-\lambda e^{\beta-\frac{\pi}{2}}\right)\left(1-\lambda e^{\frac{i\tau+\pi}{2}}\right)}\left[\frac{e^{\frac{i\tau}{2}-\beta+\pi}-1}{i\tau-2\beta+2\pi}\right]$$
$$+\frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})}. \tag{3.21}$$

Simplifying the notation a little bit,

$$4e^{-\tau h}\sum_{j\in\mathbb{Z}}M_j(\tau)\lambda^j=4e^{-\tau h}\left[\frac{e^{2\beta ih}}{i\tau+2\beta}+\frac{-e^{-2\beta ih}}{i\tau-2\beta}\right.$$
$$+\frac{-e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})}+\frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})}$$
$$+\frac{\psi_1(\lambda)}{(i\tau+2\beta)(1-\lambda e^{-(\beta-\frac{\pi}{2})})}+\frac{\psi_2(\tau,\lambda)}{(i\tau-2\beta)(1-\lambda e^{\frac{i\tau-\pi}{2}})}$$
$$\left.+\frac{\psi_3(\tau,\lambda)}{(1-\lambda e^{\frac{i\tau-\pi}{2}})(1-\lambda e^{-(\beta-\frac{\pi}{2})})}+\frac{\psi_4(\tau,\lambda)}{(1-\lambda e^{\beta-\frac{\pi}{2}})(1-\lambda e^{\frac{i\tau+\pi}{2}})}\right],$$

where

$$\psi_1(\lambda)=\lambda e^{2\beta ih}e^{-(\beta-\frac{\pi}{2})}; \tag{3.22}$$
$$\psi_2(\tau,\lambda)=-\lambda e^{-2\beta ih}e^{\frac{i\tau-\pi}{2}}; \tag{3.23}$$
$$\psi_3(\tau,\lambda)=\lambda e^{-(\beta-\frac{\pi}{2})}e^{2(\beta-\pi)ih}\left[\frac{e^{\frac{i\tau}{2}+\beta-\pi}-1}{i\tau+2\beta-2\pi}\right]; \tag{3.24}$$
$$\psi_4(\tau,\lambda)=\lambda e^{-2(\beta-\pi)ih}e^{\beta-\frac{\pi}{2}}\left[\frac{e^{\frac{i\tau}{2}-\beta+\pi}-1}{i\tau-2\beta+2\pi}\right]. \tag{3.25}$$

This concludes the proof.

□

We wish to evaluate the sums $\sum_{j\in\mathbb{Z}}E_j^{(k)}(\tau)\lambda^j$ for $k=1,2,3$. We recall that we are still supposing that $\operatorname{Re}\tau\geq 0$. We first introduce the following domains

$$\mathcal{D}'=\left\{(\tau,\lambda)\in\mathbb{C}^2:\left|\operatorname{Im}\tau-\log|\lambda|^2\right|<2\pi,\left|\log|\lambda|^2\right|<2\beta-\frac{\pi}{2}\right\}; \tag{3.26}$$
$$\mathcal{D}''=\left\{(\tau,\lambda)\in\mathbb{C}^2:\left|\operatorname{Im}\tau-\log|\lambda|^2\right|<2\pi,|\log|\lambda|^2|<3(2\beta-\frac{4}{3}\pi)\right\}; \tag{3.27}$$

$$\mathcal{D}_{\infty,2\pi} = \left\{(\tau,\lambda)\in\mathbb{C}^2 : \left|\operatorname{Im}\tau - \log|\lambda|^2\right| < 2\pi, |\lambda|>0\right\}; \tag{3.28}$$

$$S_{2\beta+\frac{3}{2}\pi} = \left\{\tau\in\mathbb{C} : |\operatorname{Im}\tau| < 2\beta+\frac{3}{2}\pi\right\}. \tag{3.29}$$

We notice that $\mathcal{D}'$,$\mathcal{D}''$ and $\mathcal{D}_{\infty,2\pi}$ are all neighborhood of $\mathcal{D}$.

**Proposition 3.5.** *Let $E_j^{(1)}(\tau)$ be defined as in (3.8), that is,*

$$E_j^{(1)}(\tau) = \int_{\mathbb{R}} \sigma(\xi)\frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}}\left[\frac{e^{-2\operatorname{sgn}(\xi)[\pi(\xi+ih)]}}{1+e^{-2\operatorname{sgn}(\xi)[\pi(\xi+ih)]}}\right]\,d\xi,$$

*where $\sigma(\xi) = e^{-i\operatorname{sgn}(\xi)\pi h - i\operatorname{sgn}(\xi-j/2)(2\beta-\pi)h}$. Then*

$$e^{-\tau h}\sum_{j\in\mathbb{Z}} E_j^{(1)}(\tau)\lambda^j = e^{-\tau h}\left[\frac{\mathbf{\Psi}_1^{(1)}(\tau,\lambda)}{e^{\beta-\frac{\pi}{2}}-\lambda} + \frac{\mathbf{\Psi}_1^{(2)}(\tau,\lambda)}{e^{-(\beta-\frac{\pi}{2})}-\lambda} + \mathbf{\Psi}_1^{(3)}(\tau,\lambda)\right], \tag{3.30}$$

*where $\mathbf{\Psi}_{\mathbf{k}}^{(\mathbf{j})}$ are holomorphic functions in a neighborhood of $\overline{\mathcal{D}}$, bounded together with all their derivatives as $|\operatorname{Re}\tau|\to\infty$.*

*Proof.* Notice that choosing $h$ as we do it results that $1+e^{-2\operatorname{sgn}(\xi)[\pi(\xi+ih)]}\neq 0$ for every $\xi$. We decompose the integral defining $E_j^{(1)}$ as in (3.12) and (3.13), according to whether $j$ is positive or negative. So, we recal that, for a fixed $\delta>0$,

$$E_j^{(1)}(\tau) = I + \mathcal{E}_1 + II + \mathcal{E}_2 + III$$

when $j$ is positive, and

$$E_j^{(1)}(\tau) = I^* + \mathcal{E}_1^* + II^* + \mathcal{E}_2^* + III^*$$

when $j$ is negative. We start analyzing the error terms $\mathcal{E}_1$ and $\mathcal{E}_2$. We have

$$\begin{aligned}\mathcal{E}_1 = {} & e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{-\delta}^{0} e^{(i\tau+2\beta)\xi}\frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\,d\xi \\ & + e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{0}^{\delta} e^{(i\tau+2\beta-2\pi)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\,d\xi,\end{aligned}$$

from which we deduce

$$\sum_{j>0}\mathcal{E}_1\lambda^j = \left[\frac{\lambda e^{-(\beta-\frac{\pi}{2})}}{1-\lambda e^{\beta-\frac{\pi}{2}}}\right]\left[e^{2\beta ih}\int_{-\delta}^{0} e^{(i\tau+2\beta)\xi}\frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\,d\xi\right.$$

$$+\, e^{2(\beta-\pi)ih}\int_0^{\delta} e^{(i\tau+2\beta-2\pi)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi\Bigg]. \tag{3.31}$$

We conclude that

$$e^{-\tau h}\sum_{j>0}\mathcal{E}_1\lambda^j = \left[\frac{e^{-(\beta-\frac{\pi}{2})}}{1-\lambda e^{\beta-\frac{\pi}{2}}}\right] e^{-\tau h}\Psi_{\mathcal{E}_1}(\tau,\lambda), \tag{3.32}$$

where $\Psi_{\mathcal{E}_1}(\tau,\lambda)$ is entire, bounded together with all its derivatives as $|\operatorname{Re}\tau|\to\infty$ and $\operatorname{Im}\tau$ remains bounded.

To deal wih $\mathcal{E}_2$ is a little more complicated since the integration extremes depend on $j$, but we cannot compute explicitly the integral in order to proceed with the sum in $j$. In fact, we have

$$\begin{aligned}\mathcal{E}_2 &= e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}-\delta}^{\frac{j}{2}} e^{(i\tau+2\beta-2\pi)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi\\ &\quad + e^{-2\beta ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}}^{\frac{j}{2}+\delta} e^{(i\tau-2\beta)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi\\ &= \mathfrak{I}+\mathfrak{I}\mathfrak{I}.\end{aligned}$$

We notice that

$$\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}} = -\sum_{k>0}\left[-e^{-2\pi(\xi+ih)}\right]^k,\ \ \xi>0,$$

where the series converges uniformly on compact sets with bounds uniform in $j>0$. This allows to interchange the order of integration and summation over $k$. Then

$$\begin{aligned}\mathfrak{I} &= -e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}-\delta}^{\frac{j}{2}} e^{(i\tau+2\beta-2\pi)\xi}\sum_{k>0}\left[-e^{-2\pi(\xi+ih)}\right]^k\, d\xi\\ &= -e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\int_{\frac{j}{2}-\delta}^{\frac{j}{2}} e^{(i\tau+2\beta-2\pi-2\pi k)\xi}\, d\xi.\end{aligned}$$

Summing up on positive $j$'s, we obtain

$$\begin{aligned}\sum_{j>0}\mathfrak{I}\lambda^j &= -e^{2(\beta-\pi)ih}\sum_{j>0}\lambda^j e^{-(2\beta-\pi)\frac{j}{2}}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\int_{\frac{j}{2}-\delta}^{\frac{j}{2}} e^{(i\tau+2\beta-2\pi-2\pi k)\xi}d\xi\\ &= -e^{2(\beta-\pi)ih}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\sum_{j>0}\lambda^j e^{(i\tau-\pi)\frac{j}{2}}\int_{-\delta}^{0} e^{(i\tau+2\beta-2\pi)\xi}e^{-2\pi k\left(\xi+\frac{j}{2}\right)}d\xi\end{aligned}$$

$$
\begin{aligned}
&= -e^{2(\beta-\pi)ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \left[\frac{\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}{1-\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}\right] \int_{-\delta}^{0} e^{(i\tau+2\beta-2\pi-2\pi k)\xi} d\xi \\
&= -e^{2(\beta-\pi)ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \left[\frac{\lambda}{e^{\frac{\pi+2\pi k-i\tau}{2}}-\lambda}\right] \int_{-\delta}^{0} e^{(i\tau+2\beta-2\pi-2\pi k)\xi} d\xi \\
&= -e^{2(\beta-\pi)ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \left[\frac{\lambda}{e^{\frac{\pi+2\pi k-i\tau}{2}}-\lambda}\right] h_1^{(k)}(\tau), \qquad (3.33)
\end{aligned}
$$

where $h_1^{(k)}(\tau)$ is an entire function such that

$$
\left|h_1^{(k)}(\tau)\right| \le c_\delta e^{2\pi k\delta}\left[\frac{1-e^{-\delta(2\beta-\operatorname{Im}\tau)}}{2\beta-\operatorname{Im}\tau}\right].
$$

Notice that we do not have a singularity when $\operatorname{Im}\tau \to 2\beta$. The convergence of the sum in $j$ is guaranteed when $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$. This last condition is satisfied for every positive $k$ when the pair $(\tau,\lambda)$ belongs to $\mathcal{D}_{\infty,2\pi}$ .

We still have to study $\sum_{j>0} \mathfrak{I}\mathfrak{I}\lambda^j$. We have

$$
\mathfrak{I}\mathfrak{I} = -e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \int_{\frac{j}{2}}^{\frac{j}{2}+\delta} e^{(i\tau-2\beta-2\pi k)\xi}\, d\xi.
$$

Then,

$$
\begin{aligned}
\sum_{j>0}\mathfrak{I}\mathfrak{I}\lambda^j &= -e^{-2\beta ih}\sum_{j>0}\lambda^j e^{(2\beta-\pi)\frac{j}{2}}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k \int_{\frac{j}{2}}^{\frac{j}{2}+\delta} e^{(i\tau-2\beta-2\pi k)\xi}\, d\xi \\
&= -e^{-2\beta ih}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k \sum_{j>0}\lambda^j e^{(i\tau-\pi-2\pi k)\frac{j}{2}} \int_0^\delta e^{(i\tau-2\beta-2\pi k)\xi}\, d\xi \\
&= -e^{-2\beta ih}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k \left[\frac{\lambda}{e^{\frac{\pi+2\pi k-i\tau}{2}}-\lambda}\right] h_2^{(k)}(\tau). \qquad (3.34)
\end{aligned}
$$

Here $h_2^{(k)}(\tau)$ is an entire function such $\left|h_2^{(k)}\right| < \left[\frac{1-e^{-\delta(\operatorname{Im}\tau+2\beta)}}{\operatorname{Im}\tau+2\beta}\right]$ and we use the fact that $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$ for every positive $k$. In conclusion,

$$
\begin{aligned}
\sum_{j>0}\mathcal{E}_2\lambda^j = -\sum_{k>0}[-e^{-2\pi ih}]^k &\left[\frac{\lambda}{e^{\frac{\pi+2\pi k-i\tau}{2}}-\lambda}\right] \\
&\times\left[e^{2(\beta-\pi)ih}h_1^{(k)}(\tau)+e^{-2\beta ih}h_2^{(k)}(\tau)\right]. \qquad (3.35)
\end{aligned}
$$

We want to prove that this sum on $k$ converges to a function holomorphic on the domain $\mathcal{D}_{\infty,2\pi}$. To prove this is enough to assume $\delta < 1/2$ and to notice that, for fixed $M > 0$, it is possible to select $k_0$ large enough so that for all $k \geq k_0$, when $(\tau,\lambda) \in \mathcal{D}_{\infty,2\pi}$ with $\operatorname{Im}\tau \leq M$ and $|\lambda| \leq e^M$, we have that

$$\left| e^{\frac{\pi+2\pi k - i\tau}{2}} - \lambda \right| \geq c e^{\pi k}.$$

Thus, the sum in $k$ is uniform on the fixed compact set. In conclusion, we have

$$e^{-\tau h} \sum_{j>0} \mathcal{E}_2 \lambda^j = e^{-\tau h} \left[ e^{2(\beta-\pi)ih} \Psi^{(1)}_{\mathcal{E}_2}(\tau,\lambda) + e^{-2\beta ih} \Psi^{(2)}_{\mathcal{E}_2}(\tau,\lambda) \right], \tag{3.36}$$

where $\Psi^{(i)}_{\mathcal{E}_2}(\tau,\lambda)$ are holomorphic on $\mathcal{D}_{\infty,2\pi}$, bounded together with their derivatives as $|\operatorname{Re}\tau| \to \infty$ and $\operatorname{Im}\tau$ and $\lambda$ remain bounded. We took care of the error terms $\mathcal{E}_1$ and $\mathcal{E}_2$. With the same strategy, we now study $I, II$ and $III$. We have

$$\begin{aligned}
I &= e^{2\beta ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{-\infty}^{-\delta} e^{(i\tau+2\beta)\xi} \frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\, d\xi; \\
II &= e^{2(\beta-\pi)ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{\delta}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta-2\pi)\xi} \frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi \\
&= -e^{2(\beta-\pi)ih} e^{-(2\beta-\pi)\frac{j}{2}} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \int_{\delta}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta-2\pi-2\pi k)\xi}\, d\xi; \\
III &= e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{\frac{j}{2}+\delta}^{+\infty} e^{(i\tau-2\beta)\xi} \frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi \\
&= -e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \int_{\frac{j}{2}+\delta}^{+\infty} e^{(i\tau-2\beta-2\pi k)\xi}\, d\xi.
\end{aligned}$$

Then, if $\left|\lambda e^{-(\beta-\frac{\pi}{2})}\right| < 1$, we obtain

$$\begin{aligned}
\sum_{j>0} I\lambda_j &= e^{2\beta ih} \left[ \frac{\lambda e^{-\frac{2\beta-\pi}{2}}}{1-\lambda e^{-\frac{2\beta-\pi}{2}}} \right] \left[ \int_{-\infty}^{-\delta} e^{(i\tau+2\beta)\xi} \frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\, d\xi \right] \\
&= e^{2\beta ih} \left[ \frac{\lambda}{e^{(\beta-\frac{\pi}{2})}-\lambda} \right] \sum_{k>0} \left[-e^{2\pi ih}\right]^k \left[ \frac{e^{-\delta(i\tau+2\beta+2\pi k)}}{i\tau+2\beta+2\pi k} \right] \\
&= e^{2\beta ih} \left[ \frac{\Psi_I(\tau,\lambda)}{e^{(\beta-\frac{\pi}{2})}-\lambda} \right]
\end{aligned} \tag{3.37}$$

where $\Psi_I$ is holomorphic in a neighborhood of $\overline{\mathcal{D}}$. In fact, let us consider

$$\mathcal{D}' = \left\{ (\tau,\lambda) \in \mathbb{C}^2 : \left|\operatorname{Im}\tau - \log|\lambda|^2\right| < 2\pi, \left|\log|\lambda|^2\right| < 2\beta - \frac{\pi}{2} \right\}. \tag{3.38}$$

Then, on this set, the holomorphicity of $\Psi_I(\tau,\cdot)$ as a function of $\lambda$ for every $\tau$ fixed is obvious, while for the holomorphicity of $\Psi_I(\cdot,\lambda)$ we have

$$\left|\left[-e^{2\pi ih}\right]^k \frac{e^{-\delta(i\tau+2\beta+2\pi k)}}{i\tau+2\beta+2\pi k}\right| \le e^{\delta(\operatorname{Im}\tau-2\beta)} \frac{e^{-2\pi k\delta}}{[(\operatorname{Re}\tau)^2+(2\beta+2\pi k-\operatorname{Im}\tau)^2]^{\frac{1}{2}}}$$
$$\le Ce^{\delta(\operatorname{Im}\tau-2\beta)}e^{-2\pi k\delta}.$$

This is true because $\operatorname{Im}\tau < 2\beta+\frac{3}{2}\pi < 2\beta+2\pi k$ for all $k\ge 1$. Thus, we have uniform convergence. So, we can conclude that

$$e^{-\tau h}\sum_{j>0} I\lambda^j = e^{-\tau h}e^{2\beta ih}\left[\frac{\Psi_I(\tau,\lambda)}{e^{(\beta-\frac{\pi}{2})}-\lambda}\right], \tag{3.39}$$

where $\Psi_I(\tau,\lambda)$ is holomorphic in $\mathcal{D}'$.

About $II$, notice that

$$II = -e^{2(\beta-\pi)ih-(2\beta-\pi)\frac{j}{2}}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k e^{(i\tau+2\beta-2\pi-2\pi k)\delta}\left\{\frac{e^{(i\tau+2\beta-2\pi-2\pi k)(\frac{j}{2}-2\delta)}-1}{i\tau+2\beta-2\pi-2\pi k}\right\}$$

and we do not have a singularity when $i\tau+2\beta-2\pi-2\pi k$ tends to 0.

If we suppose again $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$, we get

$$\sum_{j>0} II\lambda^j = \frac{-e^{2(\beta-\pi)ih}}{e^{\beta-\frac{\pi}{2}}-\lambda}\sum_{k>0}\frac{\lambda\left[-e^{-2\pi ih}\right]^k e^{(i\tau+2\beta-2\pi-2\pi k)\delta}}{i\tau+2\beta-2\pi-2\pi k}\left[\frac{e^{\beta-\frac{\pi}{2}}-e^{\frac{2\pi k+\pi-i\tau}{2}}}{e^{\frac{2\pi k+\pi-i\tau}{2}}-\lambda}\right].$$

Again, of course, notice that we do not have a singularity when $i\tau+2\beta-2\pi-2\pi k$ tends to 0. We want to say something more about the sum in $k$. For each $M>0$ we can select $k_0$ such that for every $k>k_0$ and $(\tau,\lambda)$ with $|\operatorname{Im}\tau|<M$ and $|\lambda|<e^M$, we have

$$\left|e^{\frac{2\pi k+2\pi-i\tau}{2}}-\lambda\right| \ge e^{\frac{2\pi(k+1)-M}{2}}-e^M \ge \frac{1}{2}e^{\pi k},$$

so that the series in $k$ converges uniformly on the fixed compact set. We conclude that

$$e^{-\tau h}\sum_{j>0} II\lambda^j = -e^{2(\beta-\pi)ih}e^{-\tau h}\left[\frac{\Psi_{II}(\tau,\lambda)}{e^{\beta-\frac{\pi}{2}}-\lambda}\right] \tag{3.40}$$

where $\Psi_{II}$ is a function holomorphic in $\mathcal{D}_{\infty,2\pi}$.

Finally, for $(\tau,\lambda)$ in $\mathcal{D}'$, it holds for every positive $k$ that $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$, so the sum of the $III$'s results to be

$$\sum_{j>0} III\lambda^j = e^{-2\beta ih}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\left[\frac{e^{\delta(i\tau-2\beta-2\pi k)}}{i\tau-2\beta-2\pi k}\right]\left[\frac{\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}{1-\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}\right]. \tag{3.41}$$

We have to discuss the sum over $k$. We notice that

$$\begin{aligned}\left|\frac{\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}{1-\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}}\right| &\leq \frac{|\lambda| e^{\frac{-\operatorname{Im}\tau-\pi-2\pi k}{2}}}{e^{\frac{-\operatorname{Im}\tau-\pi-2\pi k}{2}}\left|\operatorname{Im}\left[\lambda e^{\frac{i\operatorname{Re}\tau}{2}}\right]\right|} \\ &= \frac{|\lambda| e^{\frac{-\operatorname{Im}\tau-\pi}{2}}}{e^{\frac{-\operatorname{Im}\tau-\pi}{2}}\left|\operatorname{Im}\left[\lambda e^{\frac{i\operatorname{Re}\tau}{2}}\right]\right|}.\end{aligned}$$

So $\left|\frac{\lambda e^{-\frac{i\tau-\pi-2\pi k}{2}}}{1-\lambda e^{-\frac{i\tau-\pi-2\pi k}{2}}}\right|$ is uniforlmy bounded in k. Moreover, since $\operatorname{Im}\tau > -2\beta - \frac{3}{2}\pi > -2\beta - 2\pi k$ for every positive $k$, the series $\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\left[\frac{e^{\delta(i\tau-2\beta-2\pi k)}}{i\tau-2\beta-2\pi k}\right]$ converges uniformly in $\tau$. We then conclude that

$$e^{-\tau h}\sum_{j>0} III\lambda^j = e^{-2\beta ih}e^{-\tau h}\Psi_{III}(\tau,\lambda), \tag{3.42}$$

where $\Psi_{III}(\tau,\lambda)$ is holomorphic in $D'$. We remark that the functions $\Psi_I, \Psi_{II}$ and $\Psi_{III}$ are bounded together with all their derivative as $|\operatorname{Re}\tau| \to \infty$ and $\operatorname{Im}\tau$ and $\lambda$ remain bounded.

We now focus on the sum over negative $j$'s. Again, we start analyzing the error terms $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$. We have

$$\begin{aligned}\mathcal{E}_1^* &= e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}-\delta}^{\frac{j}{2}}\frac{e^{(i\tau+2\beta)\xi}e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\,d\xi + e^{-2(\beta-\pi)ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}}^{\frac{j}{2}+\delta}\frac{e^{(i\tau-2\beta+2\pi)\xi}e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}d\xi \\ &= \mathfrak{I} + \mathfrak{II}.\end{aligned}$$

If we suppose $(\tau,\lambda) \in \mathcal{D}_{\infty,2\pi}$, then $\left|\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}\right| > 1$ for every positive $k$, so we obtain

$$\sum_{j<0}\mathfrak{I}\lambda^j = -e^{2\beta ih}\sum_{k>0}\frac{\left[-e^{2\pi ih}\right]^k}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\int_{-\delta}^{0}e^{(i\tau+2\beta+2\pi k)\xi}\,d\xi.$$

Now, since $(\tau,\lambda)$ is in $D_{\infty,2\pi}$, it holds $|\lambda| > e^{\frac{\operatorname{Im}\tau}{2}-\pi} > e^{\frac{\operatorname{Im}\tau}{2}-\frac{3}{2}\pi} \geq e^{\frac{\operatorname{Im}\tau}{2}-\frac{\pi}{2}-\pi k}$ for every positive $k$, so

$$\begin{aligned}\left|\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1\right| &= e^{\frac{\pi+2\pi k-\operatorname{Im}\tau}{2}}\left|\lambda - e^{-\frac{i\tau+\pi+2\pi k}{2}}\right| \\ &\geq e^{\frac{\pi+2\pi k-\operatorname{Im}\tau}{2}}\left(|\lambda| - e^{\frac{\operatorname{Im}\tau-\pi-2\pi k}{2}}\right) \\ &\geq e^{\frac{\pi+2\pi k-\operatorname{Im}\tau}{2}}\left(|\lambda| - e^{\frac{\operatorname{Im}\tau-3\pi}{2}}\right) \\ &\geq e^{\frac{\pi+2\pi k-\operatorname{Im}\tau}{2}}\left(e^{\frac{\operatorname{Im}\tau}{2}-\pi} - e^{\frac{\operatorname{Im}\tau-3\pi}{2}}\right)\end{aligned}$$

$$= ce^{\frac{\pi+2\pi k}{2}}$$
$$> 0.$$

Using this estimates and the fact that $\left|\int_{-\delta}^{0} e^{(i\tau+2\beta+2\pi k)\xi} d\xi\right|$ is uniformly bounded in $k$, we can conclude that

$$\sum_{j<0} \mathfrak{I}\lambda^j = e^{2\beta ih}\Psi^{(1)}_{\mathcal{E}_1^*}(\tau,\lambda),$$

where $\Psi^{(1)}_{\mathcal{E}_1^*}$ is holomorphic in $\mathcal{D}_{\infty,2\pi}$. Similarly,

$$\sum_{j<0} \mathfrak{I}\mathfrak{I}\lambda^j = -e^{-2(\beta-\pi)ih}\sum_{k>0}\frac{\left[-e^{2\pi ih}\right]^k}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\int_0^\delta e^{(i\tau-2\beta+2\pi+2\pi k)\xi}\, d\xi,$$

Arguing as before, if in addition we suppose $\delta < \frac{1}{2}$, we obtain

$$\sum_{j<0} \mathfrak{I}\mathfrak{I}\lambda^j = e^{-2(\beta-\pi)ih}\Psi^{(2)}_{\mathcal{E}_1^*}(\tau,\lambda),$$

where $\Psi^{(2)}_{\mathcal{E}_1^*}$ is holomorphic in $\mathcal{D}_{\infty,2\pi}$.

In conclusion we obtain

$$e^{-\tau h}\sum_{j<0}\mathcal{E}_1^*\lambda^j = e^{-\tau h}\left[e^{2\beta ih}\Psi^{(1)}_{\mathcal{E}_1^*}(\tau,\lambda) + e^{-2(\beta-\pi)ih}\Psi^{(2)}_{\mathcal{E}_1^*}(\tau,\lambda)\right], \tag{3.43}$$

where $\Psi^{(i)}_{\mathcal{E}_1^*}(\tau,\lambda)$ are holomorphic on $D_{\infty,2\pi}$.

For $\mathcal{E}_2^*$ it results

$$\mathcal{E}_2^* = e^{-2(\beta-\pi)ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{-\delta}^0 \frac{e^{(i\tau-2\beta+2\pi)\xi}e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\, d\xi + e^{-2\beta ih}e^{(2\beta-\pi)\frac{j}{2}}\int_0^\delta \frac{e^{(i\tau-2\beta)\xi}e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi.$$

It follows, for $\left|\lambda e^{\beta-\frac{\pi}{2}}\right| > 1$,

$$\begin{aligned}\sum_{j<0}\mathcal{E}_2^*\lambda^j = \left[\frac{1}{\lambda e^{\frac{2\beta-\pi}{2}}-1}\right]\Bigg[& e^{-2(\beta-\pi)ih}\int_{-\delta}^0 e^{(i\tau-2\beta+2\pi)\xi}\frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\, d\xi \\ &+ e^{-2\beta ih}\int_0^\delta e^{(i\tau-2\beta)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\, d\xi\Bigg].\end{aligned} \tag{3.44}$$

We conclude that

$$e^{-\tau h}\sum_{j<0}\mathcal{E}_2^*\lambda^j = e^{-\tau h}\left[\frac{\Psi_{\mathcal{E}_2^*}(\tau)}{\lambda e^{\beta-\frac{\pi}{2}}-1}\right], \tag{3.45}$$

where $\Psi_2^*(\tau)$ is entire.

Let us see what happens with the principal terms $I^*, II^*$ and $III^*$. We have

$$I^* = e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{-\infty}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta)\xi}\frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}\ d\xi$$
$$= -e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{-\infty}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta)\xi}\sum_{k>0}\left[-e^{2\pi(\xi+ih)}\right]^k\ d\xi;$$
$$II^* = e^{-2(\beta-\pi)ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}+\delta}^{-\delta} e^{(i\tau-2\beta+2\pi)\xi}\frac{e^{2\pi(\xi+ih)}}{1+e^{2\pi(\xi+ih)}}d\xi$$
$$= -e^{-2(\beta-\pi)ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}+\delta}^{-\delta} e^{(i\tau+2\pi-2\beta)\xi}\sum_{k>0}\left[-e^{2\pi(\xi+ih)}\right]^k\ d\xi;$$
$$III^* = e^{-2\beta ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\delta}^{+\infty} e^{(i\tau-2\beta)\xi}\frac{e^{-2\pi(\xi+ih)}}{1+e^{-2\pi(\xi+ih)}}\ d\xi.$$

Then, if we suppose $(\tau,\lambda)$ in $\mathcal{D}'$, it holds $\left|\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}\right| > 1$ for every positive $k$, so

$$\sum_{j<0} I^*\lambda^j = -e^{2\beta ih}\sum_{k>0}\left[-e^{2\pi ih}\right]^k\frac{e^{-\delta(i\tau+2\beta+2\pi k)}}{i\tau+2\beta+2\pi k}\left[\frac{1}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\right]; \tag{3.46}$$

$$\sum_{j<0} II^*\lambda^j = -e^{2(\beta-\pi)ih}\sum_{k>0}\frac{\left[-e^{2\pi ih}\right]^k}{i\tau-2\beta+2\pi+2\pi k}\times\left[\frac{e^{-\delta(i\tau-2\beta+2\pi+2\pi k)}}{\lambda e^{\frac{2\beta-\pi}{2}}-1}-\frac{e^{\delta(i\tau-2\beta+2\pi+2\pi k)}}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\right]; \tag{3.47}$$

$$\sum_{j<0} III^*\lambda^j = e^{-2\beta ih}\left[\frac{1}{\lambda e^{\frac{2\beta-\pi}{2}}-1}\right]\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\frac{e^{\delta(i\tau-2\beta-2\pi k)}}{i\tau-2\beta-2\pi k}. \tag{3.48}$$

Notice that $(\tau,\lambda)\in\mathcal{D}'$ implies that $i\tau+2\beta+2\pi k\neq 0$ for every positive $k$, so we can conclude that

$$e^{-\tau h}\sum_{j<0} I^*\lambda^j = -e^{2\beta ih}e^{-\tau h}\Psi_{I^*}(\tau,\lambda), \tag{3.49}$$

where $\Psi_{I^*}(\tau,\lambda)$ is holomorphic in $D'$. Analogously, for $\sum_{j<0} III^*\lambda^j$ we have

$$\frac{e^{\delta(i\tau-2\beta-2\pi k)}}{i\tau-2\beta-2\pi l} \leq \frac{e^{\delta(-\operatorname{Im}\tau-2\beta)}e^{-2\pi\delta k}}{[(\operatorname{Re}\tau)^2-(\operatorname{Im}\tau+2\beta+2\pi k)^2]^{\frac{1}{2}}} \leq Ce^{-2\pi\delta k},$$

where the last inequality is true since $\operatorname{Im}\tau > -2\beta-\frac{3}{2}\pi > -2\beta-2\pi k$ for every $k\geq 1$. So

$$e^{-\tau h}\sum_{j<0} III^*\lambda^j = e^{2\beta ih}\left[\frac{e^{-\tau h}\Psi_{III^*}(\tau)}{\lambda e^{\beta-\frac{\pi}{2}}-1}\right], \tag{3.50}$$

where $\Psi_{III^*}(\tau)$ is holomorphic in $S_{2\beta+\frac{3}{2}\pi}$. About (3.47) we notice that we do not have a singularity when $i\tau - 2\beta + 2\pi + 2\pi k \to 0$. Then, for every $M > 0$ and $(\tau, \lambda) \in D_{\infty,2\pi}$ such that $e^M > |\lambda| > e^{-M}$ and $|\operatorname{Im}\tau| < M$ we can choose $k_0$ such that for every $k > k_0$ it holds

$$\left|\lambda e^{\frac{i\tau+\pi+2\pi k}{2}} - 1\right| \geq e^{-M} e^{\frac{-M+\pi+2\pi k}{2}} - 1 \geq \frac{1}{2} e^{\pi k}.$$

Using this last estimate we can conclude that

$$e^{-\tau h} \sum_{j<0} II^* \lambda^j = -e^{2(\beta-\pi)ih} \left[\frac{e^{-\tau h}\Psi_{II^*}(\tau,\lambda)}{\lambda e^{\beta-\frac{\pi}{2}} - 1}\right] \tag{3.51}$$

where $\Psi_{II^*}(\tau, \lambda)$ is holomorphic on $D_{2\pi,\infty}$. It remains to study the term $E_0^{(1)}(\tau)$. Using some of the same arguments we used before it is possible to conclude that $E_0^{(1)}(\tau)$ is an holomorphic function in $S_{2\beta+\frac{3}{2}\pi}$. We remark that all the functions $\Psi_*$ are bounded together with all their derivatives as $|\operatorname{Re}\tau| \to \infty$ and $\operatorname{Im}\tau$ and $\lambda$ remain bounded. □

**Proposition 3.6.** *Let*

$$E_j^{(2)}(\tau) = \int_{\mathbb{R}} \sigma(\xi) \frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}} \frac{e^{-2\operatorname{sgn}\left(\xi-\frac{j}{2}\right)\left[(2\beta-\pi)(\xi-\frac{j}{2}+ih)\right]}}{1+e^{-2\operatorname{sgn}\left(\xi-\frac{j}{2}\right)\left[(2\beta-\pi)(\xi-\frac{j}{2}+ih)\right]}} d\xi,$$

*where*

$$\sigma(\xi) = e^{-i\operatorname{sgn}(\xi)\pi h} e^{-i\operatorname{sgn}(\xi-\frac{j}{2})(2\beta-\pi)h}.$$

*Then*

$$e^{-\tau h} \sum_{j\in\mathbb{Z}} E_j^{(2)}(\tau)\lambda^j = e^{-\tau h}\left[\frac{\mathbf{\Phi}_2^{(1)}(\tau,\lambda)}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + \frac{\mathbf{\Phi}_2^{(2)}(\tau,\lambda)}{\lambda e^{\frac{i\tau+\pi}{2}} - 1} + \mathbf{\Phi}_2^{(3)}(\tau,\lambda)\right], \tag{3.52}$$

*where $\mathbf{\Phi}_{\mathbf{k}}^{(\mathbf{j})}$ are holomorphic functions in a neighborhood of $\overline{\mathcal{D}}$, bounded together with all their derivatives as $|\operatorname{Re}\tau| \to \infty$.*

*Proof.* We divide the integral as before. We have

$$I = -e^{2\beta ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{-\infty}^{-\delta} e^{(i\tau+2\beta)\xi} \sum_{k>0} \left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi.$$

Now, if $(\tau, \lambda)$ belongs to $\mathcal{D}''$, we have that $|\lambda| < e^{3\left(\beta-\frac{2}{3}\pi\right)} < e^{(\beta-\frac{\pi}{2})(1+2k)}$ and $\operatorname{Im}\tau < 3(2\beta - \frac{2}{3}\pi) \leq 2\beta + 2k(2\beta-\pi)$ for every $k \geq 1$. This allows us to have

$$\sum_{j>0} I\lambda^j = -e^{2\beta ih} \sum_{k>0} \left[-e^{2(2\beta-\pi)ih}\right]^k \left[\int_{-\infty}^{-\delta} e^{[i\tau+2\beta+2k(2\beta-\pi)]\xi}\right] d\xi \sum_{j>0} \left[\lambda e^{-(\beta-\frac{\pi}{2})(1+2k)}\right]^j$$

$$= -e^{2\beta ih}\sum_{k>0}\left[-e^{2(2\beta-\pi)ih}\right]^k\left[\int_{-\infty}^{-\delta}e^{[i\tau+2\beta+2k(2\beta-\pi)]\xi}\right]d\xi\left[\frac{\lambda e^{-\frac{1}{2}(2\beta-\pi)(1+2k)}}{1-\lambda e^{-\frac{1}{2}(2\beta-\pi)(1+2k)}}\right]$$

$$= -e^{2\beta ih}\sum_{k>0}\left[-e^{2(2\beta-\pi)ih}\right]^k\left[\frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2k)}-\lambda}\right]\left[\frac{e^{-\delta(i\tau+2\beta+2k(2\beta-\pi))}}{i\tau+2\beta+2k(2\beta-\pi)}\right].$$

We can conclude that

$$e^{-\tau h}\sum_{j>0} I\lambda^j = -e^{-\tau h}\Phi_I(\tau,\lambda), \tag{3.53}$$

where $\Phi_I$ is holomorphic on $\mathcal{D}''$, bounded together with all its derivative as $|\operatorname{Re}\tau|\to\infty$ and $\operatorname{Im}\tau$ and $\lambda$ remain bounded.

About $II$ we have

$$II = -e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{\delta}^{\frac{j}{2}-\delta}e^{(i\tau+2\beta-2\pi)\xi}\sum_{k>0}\left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k\, d\xi$$

$$= -e^{2(\beta-\pi)ih}e^{-(2\beta-\pi)\frac{j}{2}}\sum_{k>0}\left[-e^{2(2\beta-\pi)(ih-\frac{j}{2})}\right]^k$$

$$\times\, e^{\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}\int_0^{\frac{j}{2}-2\delta}e^{[i\tau+2\beta-2\pi+2k(2\beta-\pi)]\xi}\, d\xi$$

$$= -e^{2(\beta-\pi)ih}\sum_{k>0}\left[-e^{2(2\beta-\pi)ih}\right]^k\frac{e^{\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}}{i\tau+2\beta-2\pi+2k(2\beta-\pi)}$$

$$\times\left[e^{-2\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}e^{(i\tau-\pi)\frac{j}{2}}-e^{-(2\beta-\pi)(2k+1)\frac{j}{2}}\right]$$

$$= A+B,$$

where

$$A = -e^{2(\beta-\pi)ih}\sum_{k>0}\frac{\left[-e^{2(2\beta-\pi)ih}\right]^k e^{-\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}}{i\tau+2\beta-2\pi+2k(2\beta-\pi)}e^{(i\tau-\pi)\frac{j}{2}}$$

and

$$B = e^{2(\beta-\pi)ih}\sum_{k>0}\frac{\left[-e^{2(2\beta-\pi)ih}\right]^k e^{\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}}{i\tau+2\beta-2\pi+2k(2\beta-\pi)}e^{(2\beta-\pi)(1+2k)\frac{j}{2}}.$$

Then, if $\left|\lambda e^{\frac{i\tau-\pi}{2}}\right|<1$, it results

$$\sum_{j>0}A\lambda^j = -\left[\frac{e^{2(\beta-\pi)ih}\lambda}{e^{-\frac{i\tau-\pi}{2}}-\lambda}\right]\sum_{k>0}\frac{\left[-e^{2(2\beta-\pi)ih}\right]^k e^{-\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}}{i\tau+2\beta-2\pi+2k(2\beta-\pi)}$$

$$= \frac{\Phi^{(1)}_{II}(\tau,\lambda)}{e^{-\frac{i\tau-\pi}{2}}-\lambda},$$

where $\Phi^{(1)}_{II}$ is holomorphic in $\mathcal{D}'$. Notice that, if $(\tau,\lambda)\in\mathcal{D}'$, it holds $i\tau+2\beta-2\pi+2k(2\beta-\pi)$ for every positive $k$.

About $B$, if we suppose,$|\lambda|<e^{3(\beta-\frac{\pi}{2})}$ and $\delta<\frac{1}{2}$, we can conclude that

$$\begin{aligned}\sum_{j>0} B\lambda^j &= e^{2(\beta-\pi)ih}\sum_{k>0}\frac{\left[-e^{2(2\beta-\pi)ih}\right]^k}{i\tau+2\beta-2\pi+2k(2\beta-\pi)}\frac{e^{\delta[i\tau+2\beta-2\pi+2k(2\beta-\pi)]}\lambda}{e^{(\beta-\frac{\pi}{2})(1+2k)}-\lambda}\\ &= \Phi^{(2)}_{II}(\tau,\lambda),\end{aligned}$$

where $\Phi^{(2)}_{II}$ is holomorphic on $\mathcal{D}'$. So

$$e^{-\tau h}\sum_{j>0} II\lambda^j = e^{-\tau h}\left[\frac{\Phi^{(1)}_{II}(\tau,\lambda)}{e^{-\frac{i\tau-\pi}{2}}-\lambda}+\Phi^{(2)}_{II}(\tau,\lambda)\right] \tag{3.54}$$

About $III$ we have

$$III - e^{-2\beta ih}e^{(2\beta-\pi)\frac{j}{2}}\int_{\frac{j}{2}+\delta}^{+\infty} e^{(i\tau-2\beta)\xi}\sum_{k>0}\left[-e^{-2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k\, d\xi.$$

So, if $\left|\lambda e^{\frac{i\tau-\pi}{2}}\right|<1$, it holds,

$$\begin{aligned}e^{-\tau h}\sum_{j>0} III\lambda^j &= -e^{-\tau h}e^{-2\beta ih}\sum_{k>0}\left[-e^{-2(2\beta-\pi)ih}\right]^k\sum_{j>0}\lambda^j e^{(2\beta-\pi)(1+2k)\frac{j}{2}}\int_{\frac{j}{2}+\delta}^{+\infty} e^{[i\tau-2\beta-2(2\beta-\pi)k]\xi}\, d\xi\\ &= -e^{-\tau h}e^{-2\beta ih}\sum_{k>0}\left[-e^{-2(2\beta-\pi)ih}\right]^k\sum_{j>0}\lambda^j e^{(i\tau-\pi)\frac{j}{2}}\int_{\delta}^{+\infty} e^{[i\tau-2\beta-2(2\beta-\pi)k]\xi}\, d\xi\\ &= \left[\frac{e^{-\tau h}e^{-2\beta ih}\lambda}{e^{-\frac{i\tau-\pi}{2}}-\lambda}\right]\sum_{k>0}\left[-e^{-2(2\beta-\pi)ih}\right]^k\int_{\delta}^{+\infty} e^{[i\tau-2\beta-2(2\beta-\pi)k]\xi}\, d\xi. \end{aligned}\tag{3.55}$$

We then conclude that

$$e^{-\tau h}\sum_{j>0}\lambda^j = e^{-\tau h}\frac{\Phi_{III}(\tau,\lambda)}{e^{-\frac{i\tau-\pi}{2}}-\lambda}, \tag{3.56}$$

where $\Phi_{III}$ is holomorphic in $\mathcal{D}''$.

Let us see the error terms. We have

$$\mathcal{E}_1 = -e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{-\delta}^{0} e^{(i\tau+2\beta)\xi}\sum_{k>0}\left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k\, d\xi$$

$$- e^{2(\beta-\pi)ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_0^{\delta} e^{(i\tau+2\beta-2\pi)\xi} \sum_{k>0} \left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi.$$

Then, if $\left|\lambda e^{-(\beta-\frac{\pi}{2})(1+2k)}\right| < 1$,

$$\sum_{j>0} \mathcal{E}_1 \lambda^j = -\sum_{k>0} \left[\frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2k)} - \lambda}\right] \left[-e^{2(2\beta-\pi)ih}\right]^k$$
$$\times \left[e^{2\beta ih} \int_{-\delta}^{0} e^{[i\tau+2\beta+2(2\beta-\pi)k]\xi} \, d\xi + e^{2(\beta-\pi)ih} \int_0^{\delta} e^{[i\tau+2\beta-2\pi+2k(2\beta-\pi)]\xi}\right] d\xi. \quad (3.57)$$

If we suppose $\delta < \frac{1}{2}$ and $(\tau, \lambda) \in \mathcal{D}'$, we get

$$e^{-\tau h} \sum_{j>0} \mathcal{E}_1 \lambda^j = e^{-\tau h} \left[\Phi_{\mathcal{E}_1}^{(1)}(\tau, \lambda) + \Phi_{\mathcal{E}_1}^{(2)}(\tau, \lambda)\right], \quad (3.58)$$

where $\Phi_{\mathcal{E}_1}^{(i)}$ are holomorphic on $D'$.

About $\mathcal{E}_2$, after an obvious change of variables, we have

$$\mathcal{E}_2 = -e^{2(\beta-\pi)ih} e^{(i\tau-\pi)\frac{j}{2}} \int_{-\delta}^{0} e^{(i\tau+2\beta-2\pi)\xi} \frac{e^{2(2\beta-\pi)(\xi+ih)}}{1+e^{2(2\beta-\pi)(\xi+ih)}} \, d\xi$$
$$- e^{-2\beta ih} e^{(i\tau-\pi)\frac{j}{2}} \int_0^{\delta} e^{(i\tau-2\beta)\xi} \frac{e^{-2(2\beta-\pi)(\xi+ih)}}{1+e^{-2(2\beta-\pi)(\xi+ih)}} \, d\xi.$$

Then , if $\left|\lambda e^{\frac{i\tau-\pi}{2}}\right| < 1$,

$$\sum_{j>0} \mathcal{E}_2 \lambda^j = -\left[\frac{\Phi_{\mathcal{E}_2}(\tau, \lambda)}{e^{-\frac{i\tau-\pi}{2}} - \lambda}\right], \quad (3.59)$$

where the function $\Phi_{\mathcal{E}_2}$ is entire.

Let us see the negative $j$'s. We have

$$I^* = -e^{2\beta ih} e^{-(2\beta-\pi)\frac{j}{2}} \int_{-\infty}^{\frac{j}{2}-\delta} e^{(i\tau+2\beta)\xi} \sum_{k>0} \left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k.$$

So, if $\left|\lambda e^{\frac{i\tau+\pi}{2}}\right| > 1$,

$$e^{-\tau h} \sum_{j<0} I^* \lambda^j = -\left[\frac{e^{-\tau h} e^{2\beta ih}}{\lambda e^{\frac{i\tau+\pi}{2}} - 1}\right] \sum_{k>0} \left[-e^{2(2\beta-\pi)ih}\right]^k \int_{-\infty}^{-\delta} e^{[i\tau+2\beta+2(2\beta-\pi)k]\xi} \, d\xi$$
$$= -e^{-\tau h} \frac{\Phi_{I^*}(\tau)}{\lambda e^{\frac{i\tau+\pi}{2}} - 1}, \quad (3.60)$$

where $\Phi_{I^*}$ is a function holomorphic in $S_{2\beta+\frac{3}{2}\pi}$.

About $II^*$ we have

$$\begin{aligned} II^* &= -e^{-2(\beta-\pi)ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{\frac{j}{2}+\delta}^{-\delta} e^{(i\tau-2\beta+2\pi)\xi} \sum_{k>0} \left[-e^{-2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi. \\ &= A+B, \end{aligned}$$

where

$$A = -e^{-2(\beta-\pi)ih} \sum_{k>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^k e^{-\delta[i\tau-2\beta+2\pi-2k(2\beta-\pi)]}}{i\tau-2\beta+2\pi-2k(2\beta-\pi)} e^{(2\beta-\pi)\frac{j}{2}(1+2k)};$$

$$B = e^{-2(\beta-\pi)ih} \sum_{k>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^k e^{\delta[i\tau-2\beta+2\pi-2k(2\beta-\pi)]}}{i\tau-2\beta+2\pi-2k(2\beta-\pi)} e^{(i\tau+\pi)\frac{j}{2}}$$

So, choosing $|\lambda| > e^{-3(\beta-\frac{\pi}{2})}$ in order to have $\left|\lambda e^{(\beta-\frac{\pi}{2})(1+2k)}\right| > 1$ for every $k$ and if $\left|\lambda e^{\frac{i\tau+\pi}{2}}\right| > 1$ and $\delta < \frac{1}{2}$, it holds

$$\sum_{j<0} A\lambda^j = -e^{-2(2\beta-\pi)ih} \sum_{k>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^k e^{-\delta[i\tau-2\beta+2\pi-2k(2\beta-\pi)]}}{[i\tau-(2\beta-\pi)(1+2k)+\pi]\left(\lambda e^{(\beta-\frac{\pi}{2})(1+2k)}-1\right)};$$

$$\sum_{j<0} B\lambda^j = \frac{e^{2(\beta-\pi)ih}}{\lambda e^{\frac{i\tau+\pi}{2}}-1} \sum_{k>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^k e^{\delta[i\tau-2\beta+2\pi-2k(2\beta-\pi)]}}{i\tau-2\beta+2\pi-2k(2\beta-\pi)}.$$

We conclude that

$$e^{-\tau h} \sum_{j<0} II^* \lambda^j = e^{-\tau h} \left[\Phi^{(1)}_{II^*}(\tau,\lambda) + \frac{\Phi^{(2)}_{II^*}(\tau)}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right], \tag{3.61}$$

where $\Phi^{(1)}_{II^*}$ is holomorphic on $\mathcal{D}'$ and $\Phi^{(2)}_{II^*}$ is holomorphic in $S_{2\beta+\frac{3}{2}\pi}$.

About $III^*$,

$$III^* = -e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{\delta}^{+\infty} e^{(i\tau-2\beta)\xi} \sum_{k>0} \left[-e^{-2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi.$$

If $(\tau,\lambda)$ is in $\mathcal{D}''$, we get

$$\begin{aligned} e^{-\tau h} \sum_{j<0} III^* \lambda^j &= -e^{-2\beta ih} \sum_{k>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^k}{\lambda e^{(\beta-\frac{\pi}{2})(1+2k)}-1} \left[\int_{\delta}^{+\infty} e^{[i\tau-2\beta-2(2\beta-\pi)k]\xi}\right] d\xi \\ &= e^{-\tau h} \Phi_{III^*}(\tau,\lambda), \end{aligned} \tag{3.62}$$

where $\Phi_{III*}$ is holomorphic on $\mathcal{D}''$.

About the error terms, we have

$$\mathcal{E}_1^* = -e^{2\beta ih} e^{(i\tau+\pi)\frac{j}{2}} \int_{-\delta}^{0} e^{(i\tau+2\beta)\xi} \sum_{k>0} \left[-e^{2(2\beta-\pi)(\xi+ih)}\right]^k \, d\xi$$
$$- e^{-2(\beta-\pi)ih} e^{(i\tau+\pi)\frac{j}{2}} \int_{0}^{\delta} e^{(i\tau-2\beta+2\pi)\xi} \sum_{k>0} \left[-e^{-2(2\beta-\pi)(\xi+ih)}\right]^k \, d\xi.$$

So, if $\left|\lambda e^{\frac{i\tau+\pi}{2}}\right| > 1$,

$$\sum_{j<0} \mathcal{E}_1^* = -\frac{e^{2\beta ih}}{\lambda e^{\frac{i\tau+\pi}{2}} - 1} \sum_{k>0} \left[-e^{2(2\beta-\pi)ih}\right]^k \int_{-\delta}^{0} e^{[i\tau+2\beta+2(2\beta-\pi)k]\xi} \, d\xi$$
$$- \frac{e^{-2(\beta-\pi)ih}}{\lambda e^{\frac{i\tau+\pi}{2}} - 1} \sum_{k>0} \left[-e^{-2(2\beta-\pi)ih}\right]^k \int_{0}^{\delta} e^{[i\tau-2\beta+2\pi-2(2\beta-\pi)]\xi} \, d\xi. \tag{3.63}$$

So,

$$e^{-\tau h} \sum_{j<0} \mathcal{E}_1^* \lambda^j = -e^{-\tau h} \left[\frac{\Phi_{\mathcal{E}_1^*}(\tau)}{\lambda e^{\frac{i\tau+\pi}{2}} - 1}\right], \tag{3.64}$$

where $\Phi_{\mathcal{E}_1^*}$ is entire.

Finally,

$$\mathcal{E}_2^* = -e^{-2(\beta-\pi)ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{-\delta}^{0} e^{(i\tau-2\beta+2\pi)\xi} \sum_{k>0} \left[-e^{-2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi$$
$$- e^{-2\beta ih} e^{(2\beta-\pi)\frac{j}{2}} \int_{0}^{\delta} e^{(i\tau-2\beta)\xi} \sum_{k>0} \left[-e^{-2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^k \, d\xi.$$

Arguing as for (3.58), we can conclude that

$$e^{-\tau h} \sum_{j<0} \mathcal{E}_2^* \lambda^j = -e^{-\tau h} \Phi_{\mathcal{E}_2^*}(\tau, \lambda) \tag{3.65}$$

where $\Phi_{\mathcal{E}_2^*}$ is holomorphic on $\mathcal{D}'$. It remains to consider the term $E_0^{(2)}(\tau)$; we notice that $E_0^{(2)}(\tau) = E_0^{(1)}(\tau)$, therefore we already know that $E_0^{(2)}(\tau)$ is an holomorphic function in $S_{2\beta+\frac{3}{2}\pi}$. We remark that all the functions $\Phi_*$ are bounded together with all their derivatives as $|\operatorname{Re}\tau| \to \infty$ and $\operatorname{Im}\tau$ and $\lambda$ remain bounded. □

**Proposition 3.7.** *Let*

$$E_j^{(3)}(\tau) = \int_{\mathbb{R}} \sigma(\xi) \frac{e^{i\tau\xi}}{e^{\pi|\xi|+(2\beta-\pi)|\xi-\frac{j}{2}|}} \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1+e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \frac{e^{-2\operatorname{sgn}\left(\xi-\frac{j}{2}\right)\left[(2\beta-\pi)(\xi-\frac{j}{2}+ih)\right]}}{1+e^{-2\operatorname{sgn}\left(\xi-\frac{j}{2}\right)\left[(2\beta-\pi)(\xi-\frac{j}{2}+ih)\right]}} d\xi,$$

*where*

$$\sigma(\xi) = e^{-i\,\mathrm{sgn}(\xi)\pi h} e^{-i\,\mathrm{sgn}(\xi-\frac{j}{2})(2\beta-\pi)h}.$$

*Then*

$$e^{-\tau h}\sum_{j\in\mathbb{Z}} E_j^{(3)}\lambda^j = e^{-\tau h}\boldsymbol{\Theta}(\tau,\lambda), \tag{3.66}$$

*where* $\boldsymbol{\Theta}$ *is a holomorphic function in a neighborhood of* $\overline{\mathcal{D}}$*, bounded together with all its derivatives as* $|\operatorname{Re}\tau|\to\infty$*.*

*Proof.* We divide the integral as before. We have

$$\begin{aligned}
I &= e^{2\beta ih}e^{-(2\beta-\pi)\frac{j}{2}}\int_{-\infty}^{-\delta} e^{(i\tau+2\beta)\xi}\sum_{k>0}\left[-e^{2\pi(\xi+ih)}\right]^k\sum_{l>0}\left[-e^{2(2\beta-\pi)(\xi-\frac{j}{2}+ih)}\right]^l\\
&= e^{2\beta ih}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\sum_{k>0}\left[-e^{2\pi ih}\right]^k e^{-\frac{j}{2}(2\beta-\pi)(1+2l)}\int_{-\infty}^{-\delta} e^{[i\tau+2\beta+2\pi k+2l(2\beta-\pi)]\xi}\,d\xi\\
&= e^{2\beta ih}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\sum_{k>0}\left[-e^{2\pi ih}\right]^k e^{-\frac{j}{2}(2\beta-\pi)(1+2l)}\frac{e^{-\delta[i\tau+2\beta+2(2\beta-\pi)l+2\pi k]}}{i\tau+2\beta+2\pi k+2l(2\beta-\pi)},
\end{aligned}$$

where we have supposed $(\tau,\lambda)$ in $\mathcal{D}''$, so that $\operatorname{Im}\tau < 2\beta+2\pi k+2l(2\beta-\pi)$ for every $k,l$ and $\left|\lambda e^{-(\beta-\frac{\pi}{2})(1+2l)}\right| < 1$ for every positive $l$. Summing up on positive $j$'s, we obtain

$$\begin{aligned}
\sum_{j>0} I\lambda^j &= e^{2\beta ih}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\sum_{k>0}\left[-e^{2\pi ih}\right]^k\sum_{j>0}\lambda^j e^{-\frac{j}{2}(2\beta-\pi)(1+2l)}\frac{e^{-\delta[i\tau+2\beta+2(2\beta-\pi)l+2\pi k]}}{i\tau+2\beta+2\pi k+2l(2\beta-\pi)}\\
&= e^{2\beta ih}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2l)}-\lambda}\sum_{k>0}\left[-e^{2\pi ih}\right]^k\frac{e^{-\delta[i\tau+2\beta+2(2\beta-\pi)l+2\pi k]}}{i\tau+2\beta+2\pi k+2l(2\beta-\pi)}
\end{aligned} \tag{3.67}$$

We can conclude that

$$e^{-\tau h}\sum_{j>0} I\lambda^j = e^{-\tau h}\Theta_I(\tau,\lambda), \tag{3.68}$$

where $\Theta_I$ is a function holomorphic on $\mathcal{D}''$.

About $II$ we have

$$\begin{aligned}
II = e^{2(\beta-\pi)ih}&\sum_{l>0}\left[-e^{2(2\beta-\pi)ih)}\right]^l e^{-\frac{j}{2}(2\beta-\pi)(1+2l)}\\
&\times\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\int_{\delta}^{\frac{j}{2}-\delta} e^{[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]\xi}\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&= e^{2(\beta-\pi)ih} \sum_{l>0} \left[-e^{2(2\beta-\pi)ih)}\right]^l e^{-\frac{j}{2}(2\beta-\pi)(1+2l)} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \\
&\quad \times \left[ \frac{e^{[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)](\frac{j}{2}-\delta)} - e^{[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]\delta}}{i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)} \right] \\
&= e^{2(\beta-\pi)ih} \sum_{l>0} \left[-e^{2(2\beta-\pi)ih)}\right]^l \sum_{k>0} \left[-e^{-2\pi ih}\right]^k [C+D],
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \frac{e^{-\delta[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]} e^{[i\tau-\pi-2\pi k]\frac{j}{2}}}{i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)}; \\
D &= \frac{e^{\delta[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]} e^{-(2\beta-\pi)(1+2l)\frac{j}{2}}}{i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)}
\end{aligned}
$$

Notice that we do not have a singularity when $i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi) \to 0$. Now, if $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$,

$$
\sum_{j>0} C\lambda^j = \left[ \frac{\lambda}{e^{-\frac{i\tau-\pi-2\pi k}{2}} - \lambda} \right] \frac{e^{-\delta[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]}}{i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)},
$$

and, if $\left|\lambda e^{-(\beta-\frac{\pi}{2})(1+2l)}\right| < 1$,

$$
\sum_{j>0} D\lambda^j = \left[ \frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2l)} - \lambda} \right] \frac{e^{\delta[i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)]}}{i\tau+2\beta-2\pi-2\pi k+2l(2\beta-\pi)}.
$$

In conclusion

$$
e^{-\tau h} \sum_{j>0} II\lambda^j = e^{-\tau h} \left[ \Theta_{II}^{(1)}(\tau,\lambda) + \Theta_{II}^{(2)}(\tau,\lambda) \right], \tag{3.69}
$$

where $\Theta_{II}^{(1)}$ is holomorphic on $\mathcal{D}_{\infty,2\pi}$ and $\Theta_{II}^{(2)}$ is holomorphic on $\mathcal{D}''$.

For $III$, supposing that $(\tau,\lambda)$ is in $\mathcal{D}''$, it holds

$$
\begin{aligned}
III &= e^{-2\beta ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k e^{(i\tau-\pi-2\pi k)\frac{j}{2}} \sum_{l>0} \left[-e^{-2(2\beta-\pi)ih}\right]^l \int_\delta^{+\infty} e^{[i\tau-2\beta-2\pi k-2l(2\beta-\pi)]\xi}\, d\xi \\
&= -e^{-2\beta ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k e^{(i\tau-\pi-2\pi k)\frac{j}{2}} \sum_{l>0} \left[-e^{-2(2\beta-\pi)ih}\right]^l \frac{e^{\delta[i\tau-2\beta-2\pi k-2l(2\beta-\pi)]}}{i\tau-2\beta-2\pi k-2l(2\beta-\pi)},
\end{aligned}
$$

Summing up, if $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$, we obtain

$$
\sum_{j>0} III\lambda^j = e^{-2\beta ih} \sum_{k>0} \left[-e^{-2\pi ih}\right]^k \left[ \frac{\lambda}{e^{-\frac{i\tau-\pi-2\pi k}{2}} - \lambda} \right] \times
$$

$$\times \sum_{l>0} \left[-e^{-2(2\beta-\pi)ih}\right]^l \frac{e^{\delta[i\tau-2\beta-2\pi k-2l(2\beta-\pi)]}}{i\tau-2\beta-2\pi k-2l(2\beta-\pi)}.$$

We conclude,

$$e^{-\tau h}\sum_{j>0} III\lambda^j = e^{-\tau h}\Theta_{III}(\tau,\lambda), \tag{3.70}$$

where $\Theta_{III}$ is holomorphic on $\mathcal{D}''$.

Let us study the first error term $\mathcal{E}_1$. We set $m(\xi) = \frac{e^{-2\,\mathrm{sgn}(\xi)\pi(\xi+ih)}}{1+e^{-2\,\mathrm{sgn}(\xi)\pi(\xi+ih)}}$. Then, we have

$$\mathcal{E}_1 = A + B,$$

where

$$A = -e^{2\beta ih}\int_{-\delta}^{0}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l e^{-\frac{j}{2}(2\beta-\pi)(1+2l)} e^{[i\tau+2\beta+2l(2\beta-\pi)]\xi} m(\xi)\ d\xi;$$

$$B = -e^{2(\beta-\pi)ih}\int_{0}^{\delta}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l e^{-\frac{j}{2}(2\beta-\pi)(1+2l)+[i\tau+2\beta-2\pi+2(2\beta-\pi)l]\xi} m(\xi) d\xi.$$

So, if $\left|\lambda e^{-(\beta-\frac{\pi}{2})(1+2l)}\right| < 1$,

$$\sum_{j>0} A\lambda^j = e^{2\beta ih}\int_{-\delta}^{0}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l \left[\frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2l)}-\lambda}\right] e^{[i\tau+2\beta+2l(2\beta-\pi)]\xi} m(\xi)\ d\xi;$$

$$\sum_{j>0} B\lambda^j = e^{2(\beta-\pi)ih}\int_{0}^{\delta}\sum_{l>0}\left[-e^{2(\beta-\pi)ih}\right]^l \left[\frac{\lambda}{e^{(\beta-\frac{\pi}{2})(1+2l)}-\lambda}\right] \times e^{[i\tau+2\beta-2\pi+2(2\beta-\pi)l]\xi} m(\xi)\ d\xi.$$

Now, the inner sums on $l$ converge uniformly for $\xi$ in $[-\delta,\delta]$ to a smooth a function in $\xi$ and holomorphic in $\lambda$ for $|\lambda| < e^{3(\beta-\frac{\pi}{2})}$. We can conclude that

$$e^{-\tau h}\sum_{j>0}\mathcal{E}_1\lambda^j = e^{-\tau h}\left[\Theta^{(1)}_{\mathcal{E}_1}(\tau,\lambda) + \Theta^{(2)}_{\mathcal{E}_1}(\tau,\lambda)\right], \tag{3.71}$$

where the functions $\Theta^{(i)}_{\mathcal{E}_1}$ are holomorphic on $\mathcal{D}''$.

Now we analyze the second error term. We set $m_1(\xi) = \frac{e^{-2\,\mathrm{sgn}(\xi)(2\beta-\pi)(\xi+ih)}}{1+e^{-2\,\mathrm{sgn}(\xi)(2\beta-\pi)(\xi+ih)}}$. Then, we have

$$\mathcal{E}_2 = \int_{\frac{j}{2}-\delta}^{\frac{j}{2}} + \int_{\frac{j}{2}}^{\frac{j}{2}+\delta} = E + F,$$

where

$$E = -e^{2(\beta-\pi)ih}\int_{-\delta}^{0}\sum_{l>0}\left[-e^{-2\pi ih}\right]^l e^{\frac{j}{2}(i\tau-\pi-2\pi l)} e^{[i\tau+2\beta-2\pi-2\pi l]\xi} m_1(\xi)\ d\xi;$$

$$F = -e^{-2\beta ih}\int_0^{\delta}\sum_{l>0}\left[-e^{-2\pi ih}\right]^l e^{\frac{j}{2}(i\tau-\pi-2\pi l)}e^{[i\tau-2\beta-2\pi l]\xi}m_1(\xi)\ d\xi.$$

So, if $\left|\lambda e^{\frac{i\tau-\pi-2\pi k}{2}}\right| < 1$,

$$\sum_{j>0} E\lambda^j = -e^{2(\beta-\pi)ih}\int_{-\delta}^{0}\sum_{l>0}\left[-e^{-2\pi ih}\right]^l\left[\frac{\lambda}{e^{-\frac{i\tau-\pi-2\pi l}{2}}-\lambda}\right]e^{[i\tau+2\beta-2\pi-2\pi l]\xi}m_1(\xi)\ d\xi;$$

$$\sum_{j>0} F\lambda^j = -e^{-2\beta ih}\int_{0}^{\delta}\sum_{l>0}\left[-e^{-2\pi ih}\right]^l\left[\frac{\lambda}{e^{-\frac{i\tau-\pi-2\pi l}{2}}-\lambda}\right]e^{[i\tau-2\beta-2\pi l]\xi}m_1(\xi)\ d\xi.$$

Now, the inner sums on $l$ converge uniformly in $\xi$ in $[-\delta,\delta]$ to a smooth function in $\xi$ and holomorphic in $(\tau,\lambda)$ if $\operatorname{Im}\tau - \log|\lambda|^2 > -3\pi$.

We conclude

$$e^{-\tau h}\sum_{j>0}\mathcal{E}_2\lambda^j = e^{-\tau h}\left[\Theta^{(1)}_{\mathcal{E}_2}(\tau,\lambda)+\Theta^{(2)}_{\mathcal{E}_2}(\tau,\lambda)\right], \tag{3.72}$$

where $\Theta^{(i)}_{\mathcal{E}_2}$ are holomorphic on $\mathcal{D}_{\infty,2\pi}$.

It remains to compute the sum on negative indices. Suppose that $(\tau,\lambda)$ is in $\mathcal{D}''$, then

$$\begin{aligned} I^* &= e^{2\beta ih}\sum_{k>0}\left[-e^{2\pi ih}\right]^k e^{(i\tau+\pi+2\pi k)\frac{j}{2}}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\int_{-\infty}^{-\delta}e^{[i\tau+2\beta+2k\pi+2l(2\beta-\pi)]\xi}\ d\xi \\ &= e^{2\beta ih}\sum_{k>0}\left[-e^{2\pi ih}\right]^k e^{(i\tau+\pi+2\pi k)\frac{j}{2}}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\frac{e^{-\delta[i\tau+2\beta+2k\pi+2l(2\beta-\pi)]}}{i\tau+2\beta+2k\pi+2l(2\beta-\pi)}. \end{aligned}$$

Summing up on $j$, we obtain

$$\sum_{j<0} I^*\lambda^j = e^{2\beta ih}\sum_{k>0}\frac{\left[-e^{2\pi ih}\right]^k}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\sum_{l>0}\left[-e^{2(2\beta-\pi)ih}\right]^l\frac{e^{-\delta[i\tau+2\beta+2k\pi+2l(2\beta-\pi)]}}{i\tau+2\beta+2k\pi+2l(2\beta-\pi)}.$$

In conclusion

$$e^{-\tau h}\sum_{j<0} I^*\lambda^j = e^{-\tau h}\Theta_{I^*}(\tau,\lambda), \tag{3.73}$$

where $\Theta_{I^*}$ is holomorphic on $\mathcal{D}''$.

About $II^*$, it holds

$$\begin{aligned} II^* &= e^{-2(\beta-\pi)ih}\sum_{l>0}\left[-e^{-2(2\beta-\pi)ih}\right]^l e^{\frac{j}{2}(2\beta-\pi)(1+2l)}\sum_{k>0}\left[-e^{2\pi ih}\right]^k\int_{\frac{j}{2}+\delta}^{-\delta}e^{[i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)]\xi}\ d\xi \\ &= e^{-2(\beta-\pi)ih}\sum_{l>0}\left[-e^{-2(2\beta-\pi)ih}\right]^l\sum_{k>0}\left[-e^{2\pi ih}\right]^k\left[C+D\right], \end{aligned}$$

where

$$C = \frac{e^{-\delta[i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)]}e^{\frac{j}{2}(2\beta-\pi)(1+2l)}}{i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)};$$
$$D = \frac{e^{\delta[i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)]}e^{(i\tau+\pi+2\pi k)\frac{j}{2}}}{i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)}.$$

Notice that $C+D$ is not singular when $i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)\to 0$. So, if $\left|\lambda e^{(\beta-\frac{\pi}{2})(1+2l)}\right|>1$ and $\left|\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}\right|>1$, we obtain

$$\sum_{j<0} C\lambda^j = \left[\frac{1}{\lambda e^{(\beta-\frac{\pi}{2})(1+2l)}-1}\right]\frac{e^{-\delta[i\tau-2\beta+2\pi+2k\pi-2l(2\beta-\pi)]}}{i\tau-2\beta+2\pi+2k\pi-2l(2\beta-\pi)};$$
$$\sum_{j<0} D\lambda^j = \left[\frac{1}{\lambda e^{\frac{i\tau+\pi+2\pi k}{2}}-1}\right]\frac{e^{\delta[i\tau-2\beta+2\pi+2k\pi-2l(2\beta-\pi)]}}{i\tau-2\beta+2\pi+2\pi k-2l(2\beta-\pi)}.$$

We conclude that

$$e^{-\tau h}\sum_{j<0} II^*\lambda^j = e^{-\tau h}\left[\Theta^{(1)}_{II^*}(\tau,\lambda)+\Theta^{(2)}_{II^*}(\tau,\lambda)\right], \tag{3.74}$$

where $\Theta^{(1)}_{II^*}$ is holomorphic on $\mathcal{D}''$ and $\Theta^{(2)}_{II^*}$ is holomoprhic on $\mathcal{D}_{\infty,2\pi}$.

The last term $III^*$ is given by

$$\begin{aligned} III^* &= e^{-2\beta ih}\sum_{l>0}\left[-e^{-2(2\beta-\pi)ih}\right]^l e^{\frac{j}{2}(2\beta-\pi)(1+2l)}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\int_\delta^{+\infty} e^{[i\tau-2\beta-2\pi k-2l(2\beta-\pi)]\xi}d\xi \\ &= -e^{-2\beta ih}\sum_{l>0}\left[-e^{-2(2\beta-\pi)ih}\right]^l e^{\frac{j}{2}(2\beta-\pi)(1+2l)}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\frac{e^{\delta[i\tau-2\beta-2k\pi-2l(2\beta-\pi)]}}{i\tau-2\beta-2k\pi-2l(2\beta-\pi)}, \end{aligned}$$

where we are supposing $(\tau,\lambda)\in\mathcal{D}''$. Then

$$\sum_{j<0} III^*\lambda^j = e^{-2\beta ih}\sum_{l>0}\frac{\left[-e^{-2(2\beta-\pi)ih}\right]^l}{\lambda e^{(\beta-\frac{\pi}{2})(1+2l)}-1}\sum_{k>0}\left[-e^{-2\pi ih}\right]^k\left[\frac{e^{\delta[i\tau-2\beta-2k\pi-2l(2\beta-\pi)]}}{i\tau-2\beta-2k\pi-2l(2\beta-\pi)}\right].$$

We conclude

$$e^{-\tau h}\sum_{j<0} III^*\lambda^j = e^{-\tau h}\Theta_{III^*}(\tau,\lambda), \tag{3.75}$$

where $\Theta_{III^*}$ is holomorphic in $\mathcal{D}''$.

We now study the first error term $\mathcal{E}^*_1$. We set $n(\xi)=\frac{e^{-2\operatorname{sgn}(\xi)(2\beta-\pi)(\xi+ih)}}{1+e^{-2(2\operatorname{sgn}(\xi)\beta-\pi)(\xi+ih)}}$. Then, we have

$$\mathcal{E}^*_1 = \int_{\frac{j}{2}-\delta}^{\frac{j}{2}} + \int_{\frac{j}{2}}^{\frac{j}{2}+\delta} = A+B,$$

where

$$A = -e^{2\beta ih} \int_{-\delta}^{0} \sum_{l>0} \left[-e^{2\pi ih}\right]^l e^{[i\tau+2\beta+2\pi l]} e^{[i\tau+\pi+2\pi l]\frac{j}{2}} n(\xi)\ d\xi;$$

$$B = -e^{-2(\beta-\pi)ih} \int_{0}^{\delta} \sum_{l>0} \left[-e^{2\pi ih}\right]^l e^{[i\tau-2\beta+2\pi+2\pi l]\xi} e^{(i\tau+\pi+2\pi l)\frac{j}{2}} n(\xi)\ d\xi.$$

Thus, if $\left|\lambda e^{\frac{i\tau+\pi+2\pi l}{2}}\right| > 1$, we obtain

$$\sum_{j<0} A\lambda^j = -e^{2\beta ih} \int_{-\delta}^{0} \sum_{l>0} \frac{\left[-e^{2\pi ih}\right]^l}{\lambda e^{\frac{i\tau+\pi+2\pi l}{2}} - 1} e^{[i\tau+2\beta+2\pi l]\xi} n(\xi)\ d\xi;$$

$$\sum_{j<0} B\lambda^j = -e^{-2(\beta-\pi)ih} \int_{0}^{\delta} \sum_{l>0} \frac{\left[-e^{2\pi ih}\right]^l}{\lambda e^{\frac{i\tau+\pi+2l\pi}{2}} - 1} e^{[i\tau-2\beta+2\pi+2\pi l]\xi} n(\xi)\ d\xi.$$

The inner sums converge uniformly for $\xi \in [-\delta, \delta]$ to a smooth function in $\xi$ and holomorphic in $\lambda$ for $|\lambda| > e^{-3(\beta-\frac{\pi}{2})}$. So, we conclude

$$e^{-\tau h} \sum_{j<0} \mathcal{E}_1^* \lambda^j = e^{-\tau h} \left[\Theta^{(1)}_{\mathcal{E}_1^*}(\tau,\lambda) + \Theta^{(2)}_{\mathcal{E}_1^*}(\tau,\lambda)\right], \tag{3.76}$$

where $\Theta^{(i)}_{\mathcal{E}_1^*}$ are holomorphic functions on $\mathcal{D}''$. Now the second error term $\mathcal{E}_2*$. We set $n_1(\xi) = \frac{e^{-2\,\mathrm{sgn}(\xi)\pi(\xi+ih)}}{1+e^{-2\,\mathrm{sgn}(\xi)\pi(\xi+ih)}}$. Then, we have

$$\mathcal{E}_2^* = \int_{-\delta}^{0} + \int_{0}^{\delta} = E + F,$$

where

$$E = -e^{-2(\beta-\pi)ih} \int_{-\delta}^{0} \sum_{l>0} \left[-e^{-2(2\beta-\pi)ih}\right]^l e^{\frac{j}{2}(2\beta-\pi)(1+2l)} e^{[i\tau-2\beta+2\pi-2l(2\beta-\pi)]\xi} n_1(\xi)\ d\xi;$$

$$F = -e^{-2\beta ih} \int_{0}^{\delta} \sum_{l>0} \left[-e^{-2(2\beta-\pi)ih}\right]^l e^{\frac{j}{2}(2\beta-\pi)(1+2l)} e^{[i\tau-2\beta-2l(2\beta-\pi)]\xi} n_1(\xi)\ d\xi.$$

Supposing that $\left|\lambda e^{(\beta-\frac{\pi}{2})(1+2l)}\right| > 1$, we get

$$\sum_{j<0} E\lambda^j = -e^{-2(\beta-\pi)ih} \int_{-\delta}^{0} \sum_{l>0} \frac{\left[-e^{2\pi ih}\right]^l}{\lambda e^{(\beta-\frac{\pi}{2})(1+2l)} - 1} e^{[i\tau-2\beta+2\pi-2l(2\beta-\pi)]\xi} n_1(\xi)\ d\xi;$$

$$\sum_{j<0} F\lambda^j = -e^{-2\beta ih} \int_{0}^{\delta} \sum_{l>0} \frac{\left[-e^{-2(2\beta-\pi)ih}\right]^l}{\lambda e^{(\beta-\frac{\pi}{2})(1+2l)} - 1} e^{[i\tau-2\beta-2l(2\beta-\pi)]\xi} n_1(\xi)\ d\xi.$$

The inner sums converge uniformly for $\xi$ in $[-\delta, \delta]$ to a smooth function in $\xi$ and holomorphic in $\lambda$ for $|\lambda| > e^{-3(\beta-\frac{\pi}{2})}$. We conclude that

$$e^{-\tau h}\sum_{j<0}\mathcal{E}_2^*\lambda^j = e^{-\tau h}\left[\Theta^{(1)}_{\mathcal{E}_2^*}(\tau,\lambda) + \Theta^{(2)}_{\mathcal{E}_2^*}(\tau,\lambda)\right], \tag{3.77}$$

where $\Theta^{(i)}_{\mathcal{E}_2^*}$ are holomorphic functions on $\mathcal{D}''$. □

In conclusion,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} J_j(\tau)\lambda^j &= 4e^{-\tau h}\Bigg[\sum_{j\in\mathbb{Z}} M_j(\tau)\lambda^j + \sum_{k=1}^{3}\sum_{j\in\mathbb{Z}} E_j^{(k)}(\tau)\lambda^j\Bigg] \\ &= 4e^{-\tau h}\Bigg[\frac{e^{2\beta ih}}{i\tau+2\beta} + \frac{-e^{-2\beta ih}}{i\tau-2\beta} + \frac{-e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})} \\ &\quad + \frac{\psi_1(\lambda)}{(i\tau+2\beta)(1-\lambda e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_2(\tau,\lambda)}{(i\tau-2\beta)(1-\lambda e^{\frac{i\tau-\pi}{2}})} + \frac{\psi_3(\tau,\lambda)}{(1-\lambda e^{\frac{i\tau-\pi}{2}})(1-\lambda e^{-(\beta-\frac{\pi}{2})})} \\ &\quad + \frac{\psi_4(\tau,\lambda)}{(1-\lambda e^{\beta-\frac{\pi}{2}})(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{\mathbf{\Psi}_1^{(1)}(\tau,\lambda)}{e^{\beta-\frac{\pi}{2}}-\lambda} + \frac{\mathbf{\Psi}_1^{(2)}(\tau,\lambda)}{e^{-(\beta-\frac{\pi}{2})}-\lambda} + \mathbf{\Psi}_1^{(3)}(\tau,\lambda) \\ &\quad + \frac{\mathbf{\Phi}_2^{(1)}(\tau,\lambda)}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + \frac{\mathbf{\Phi}_2^{(2)}(\tau,\lambda)}{\lambda e^{\frac{i\tau+\pi}{2}}-1} + \mathbf{\Phi}_2^{(3)}(\tau,\lambda) + \mathbf{\Theta}(\tau,\lambda)\Bigg].\end{aligned}$$

Moreover, we know

$$\mathcal{R}(\tau,\lambda) = \frac{4\nu_\beta}{e^{\frac{\tau\nu_\beta}{2}}}\left\{\left[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{\lambda e^{\frac{i\tau+\pi}{2}}-1}\right] + \left[\frac{e^{-\frac{i\pi\nu_\beta}{2}}\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}}\right] + E(\tau,\lambda) + \frac{1}{\mathrm{Ch}\left(i\frac{\pi\nu_\beta}{2}\right)}\right\}$$

where $E(\tau,\lambda)$ is a smooth and bounded function with all derivatives smooth and bounded in a neighborhood of $\overline{\mathcal{D}}$. In conclusion,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} I_j(\tau)\lambda^j &= 4\nu_\beta e^{-\frac{\tau\nu_\beta}{2}}\left[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{\lambda e^{\frac{i\tau+\pi}{2}}-1} + \frac{e^{-\frac{i\pi\nu_\beta}{2}}\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + E(\tau,\lambda)\right] \\ &\quad + 4e^{-\tau h}\Bigg[\frac{e^{2\beta ih}}{i\tau+2\beta} + \frac{-e^{-2\beta ih}}{i\tau-2\beta} + \frac{-e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})} \\ &\quad + \frac{\psi_1(\lambda)}{(i\tau+2\beta)(1-\lambda e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_2(\tau,\lambda)}{(i\tau-2\beta)(1-\lambda e^{\frac{i\tau-\pi}{2}})} + \frac{\psi_3(\tau,\lambda)}{(1-\lambda e^{\frac{i\tau-\pi}{2}})(1-\lambda e^{-(\beta-\frac{\pi}{2})})} \\ &\quad + \frac{\psi_4(\tau,\lambda)}{(1-\lambda e^{\beta-\frac{\pi}{2}})(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{\mathbf{\Psi}_1^{(1)}(\tau,\lambda)}{e^{\beta-\frac{\pi}{2}}-\lambda} + \frac{\mathbf{\Psi}_1^{(2)}(\tau,\lambda)}{e^{-(\beta-\frac{\pi}{2})}-\lambda} + \mathbf{\Psi}_1^{(3)}(\tau,\lambda) + \\ &\quad + \frac{\mathbf{\Phi}_2^{(1)}(\tau,\lambda)}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + \frac{\mathbf{\Phi}_2^{(2)}(\tau,\lambda)}{\lambda e^{\frac{i\tau+\pi}{2}}-1} + \mathbf{\Phi}_2^{(3)}(\tau,\lambda) + \mathbf{\Theta}(\tau,\lambda)\Bigg].\end{aligned}$$

We recall that the above formula holds for $\operatorname{Re}\tau > 0$. For general $\tau$ we have

$$\sum_{j\in\mathbb{Z}} I_j(\tau)\lambda^j = 4\nu_\beta e^{-\operatorname{sgn}(\operatorname{Re}\tau)\frac{\tau\nu_\beta}{2}}\left[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{\lambda e^{\frac{i\tau+\pi}{2}}-1} + \frac{e^{-\frac{i\pi\nu_\beta}{2}}\lambda e^{\frac{i\tau-\pi}{2}}}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + E(\tau,\lambda)\right] +$$
$$+ 4e^{-\operatorname{sgn}(\operatorname{Re}\tau)\tau h}\Bigg[\frac{e^{2\beta ih}}{i\tau+2\beta} + \frac{-e^{-2\beta ih}}{i\tau-2\beta} + \frac{-e^{2\beta ih}}{(i\tau+2\beta)(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{e^{-2\beta ih}}{(i\tau-2\beta)(1-\lambda e^{\beta-\frac{\pi}{2}})} +$$
$$+ \frac{\psi_1(\lambda)}{(i\tau+2\beta)(1-\lambda e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_2(\tau,\lambda)}{(i\tau-2\beta)(1-\lambda e^{\frac{i\tau-\pi}{2}})} + \frac{\psi_3(\tau,\lambda)}{(1-\lambda e^{\frac{i\tau-\pi}{2}})(1-\lambda e^{-(\beta-\frac{\pi}{2})})}$$
$$+ \frac{\psi_4(\tau,\lambda)}{(1-\lambda e^{\beta-\frac{\pi}{2}})(1-\lambda e^{\frac{i\tau+\pi}{2}})} + \frac{\mathbf{\Psi}_1^{(1)}(\tau,\lambda)}{e^{\beta-\frac{\pi}{2}}-\lambda} + \frac{\mathbf{\Psi}_1^{(2)}(\tau,\lambda)}{e^{-(\beta-\frac{\pi}{2})}-\lambda} + \mathbf{\Psi}_1^{(3)}(\tau,\lambda)$$
$$+ \frac{\mathbf{\Phi}_{\mathbf{2}}^{(\mathbf{1})}(\tau,\lambda)}{1-\lambda e^{\frac{i\tau-\pi}{2}}} + \frac{\mathbf{\Phi}_2^{(2)}(\tau,\lambda)}{\lambda e^{\frac{i\tau+\pi}{2}}-1} + \mathbf{\Phi}_2^{(3)}(\tau,\lambda) + \mathbf{\Theta}(\tau,\lambda)\Bigg].$$

Finally, recalling that $\tau = w_1 - \overline{z_1}$ and $\lambda = w_2 - \overline{z_2}$, we obtain

$$K_{D'_\beta}[(w_1,w_2),(z_1,z_2)] = \sum_{j\in\mathbb{Z}} \frac{(w_2\overline{z_2})^j}{8\pi} I_j(w_1-\overline{z_1}) =$$
$$= \frac{\nu_\beta e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))\frac{(w_1-\overline{z_1})\nu_\beta}{2}}}{2\pi}\Bigg[\frac{e^{\frac{i\pi\nu_\beta}{2}}}{(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}-1} + \frac{e^{-\frac{i\pi\nu_\beta}{2}}(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}}{1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}} +$$
$$+ E(w_1-\overline{z_1}, w_2\overline{z_2})\Bigg] + \frac{e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))(w_1-\overline{z_1})h}}{2\pi}\Bigg[\frac{e^{2\beta ih}}{i(w_1-\overline{z_1})+2\beta} + \frac{-e^{-2\beta ih}}{i(w_1-\overline{z_1})-2\beta} +$$
$$+ \frac{-e^{2\beta ih}}{(i(w_1-\overline{z_1})+2\beta)(1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}})} + \frac{e^{-2\beta ih}}{(i(w_1-\overline{z_1})-2\beta)(1-(w_2\overline{z_2})e^{\beta-\frac{\pi}{2}})} +$$
$$+ \frac{\psi_1(w_2\overline{z_2})}{(i(w_1-\overline{z_1})+2\beta)(1-(w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_2(w_1-\overline{z_1}, w_2\overline{z_2})}{(i(w_1-\overline{z_1})-2\beta)(1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}})}$$
$$+ \frac{\psi_3(w_1-\overline{z_1}, w_2\overline{z_2})}{(1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}})(1-(w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})})} + \frac{\psi_4(w_1-\overline{z_1}, w_2\overline{z_2})}{(1-(w_2\overline{z_2})e^{\beta-\frac{\pi}{2}})(1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}})} +$$
$$+ \frac{\mathbf{\Psi}_1^{(1)}(w_1-\overline{z_1}, w_2\overline{z_2})}{e^{\beta-\frac{\pi}{2}}-(w_2\overline{z_2})} + \frac{\mathbf{\Psi}_1^{(2)}(w_1-\overline{z_1}, w_2\overline{z_2})}{e^{-(\beta-\frac{\pi}{2})}-(w_2\overline{z_2})} + \mathbf{\Psi}_1^{(3)}(w_1-\overline{z_1}, w_2\overline{z_2}) +$$
$$+ \frac{\mathbf{\Phi}_{\mathbf{2}}^{(\mathbf{1})}(w_1-\overline{z_1}, w_2\overline{z_2})}{1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}} + \frac{\mathbf{\Phi}_2^{(2)}(w_1-\overline{z_1}, w_2\overline{z_2})}{(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}-1} + \mathbf{\Phi}_2^{(3)}(w_1-\overline{z_1}, w_2\overline{z_2}) + \mathbf{\Theta}(w_1-\overline{z_1}, w_2\overline{z_2})\Bigg].$$

Simplifying a little bit, using the notation of Theorem 2.17, we obtain

$$K_{D'_\beta}(w,z) = e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))\frac{(w_1-\overline{z_1})\nu_\beta}{2}} K(w,z) + e^{-\operatorname{sgn}(\operatorname{Re}(w_1-\overline{z_1}))(w_1-\overline{z_1})h}\widetilde{K}(w,z),$$

where

$$
\begin{aligned}
K(w,z) &= \frac{F_1(w,z)}{1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}} + \frac{F_2(w,z)}{1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}} + E(w,z) \\
&= K_1(w,z) + K_2(w,z) + E(w,z)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{K}(w,z) &= \frac{G_1(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}]} + \frac{G_2(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}]} \\
&+ \frac{G_3(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}][1-(w_2\overline{z_2})e^{\beta-\frac{\pi}{2}}]} \\
&+ \frac{G_4(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})+\pi}{2}}][i(w_1-\overline{z_1})+2\beta]} \\
&+ \frac{G_5(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}][i(w_1-\overline{z_1})-2\beta]} \\
&+ \frac{G_6(w,z)}{[1-(w_2\overline{z_2})e^{\frac{i(w_1-\overline{z_1})-\pi}{2}}][1-(w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})}]} \\
&+ \frac{G_7(w,z)}{[i(w_1-\overline{z_1})+2\beta][1-(w_2\overline{z_2})e^{-(\beta-\frac{\pi}{2})}]} \\
&+ \frac{G_8(w,z)}{[i(w_1-\overline{z_1})-2\beta][1-(w_2\overline{z_2})e^{\beta-\frac{\pi}{2}}]} + \widetilde{E}(w,z) \\
&= \widetilde{K}_1(w,z) + \ldots + \widetilde{K}_8(w,z) + \widetilde{E}(w,z)
\end{aligned}
$$

as we wished.

# Part II

# Holomorphic extension on product Lipschitz surfaces in two complex variables

# Chapter 4

# Biparameter Littlewood-Paley-Stein Theory

In this chapter we develop some biparameter Littlewood-Paley-Stein theory that we will use to prove our biparameter $Tb$ theorem.

We work in arbitrary dimension $\mathbb{R}^n$ where $n = n_1 + n_2$. We will use the subscripts of $x_j$ to distinguish between functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. In the first part of the chapter we fix notation and define biparameter Littlewood-Paley-Stein operators and square functions. Thus, we prove a reproducing formula in our setting which is the analogous of a result by Han (Theorem 4.7) and we conclude the chapter proving Theorem 0.2.

## 4.1 Background results

In this short section we fix notation and recall some known results. We do not prove these results, but we provide the references.

**Definition 4.1.** For $0 < \delta \leq 1$, define $C_0^{0,\delta}(\mathbb{R}^n)$ to be the collection of all $\delta$-Hölder continuous, compactly supported functions $f : \mathbb{R}^n \to \mathbb{C}$ with norm

$$\|f\|_\delta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\delta} < \infty.$$

Since $C_0^{0,\delta}(\mathbb{R}^n)$ is made up of compactly supported functions, it follows that $\| \cdot \|_\delta$ is a norm, and we endow $C_0^{0,\delta}(\mathbb{R}^n)$ the topology generated by the norm $\| \cdot \|_\delta$. Given a function $b \in L^\infty(\mathbb{R}^n)$ such that $b^{-1} \in L^\infty(\mathbb{R}^n)$, let $bC_0^{0,\delta}(\mathbb{R}^n)$ be the collection of functions $bf$ such that $f \in C_0^{0,\delta}(\mathbb{R}^n)$. We define $\|bf\|_{b,\delta} = \|f\|_\delta$ for $bf \in bC_0^{0,\delta}(\mathbb{R}^n)$, and endow $bC_0^{0,\delta}(\mathbb{R}^n)$ the topology generated by the norm $\|\cdot\|_{b,\delta}$. Finally, given a function space $X$, we define $X'$ to be the continuous dual of $X$ with the weak* topology. In our situation, we will primarily use this definition for $X = bC_0^{0,\delta}(\mathbb{R}^n)$.

For $k \in \mathbb{Z}, N > 0$ and $x \in \mathbb{R}^n$, define

$$\Phi_k^N(x) = \frac{2^{nk}}{(1 + 2^k|x|)^N}.$$

The following proposition will be used in later sections.

**Proposition 4.2.** *If $M, N > n$, then, for all $j, k \in \mathbb{Z}$,*

$$\int_{\mathbb{R}^n} \Phi_j^M(x-u)\Phi_k^N(u-y)\, du \lesssim \Phi_j^M(x-y) + \Phi_k^N(x-y).$$

*Proof.* Fix $x, y \in \mathbb{R}$ and $j, k \in \mathbb{Z}$. Then, $|x-y| \leq |x-u| + |u-y|$ for all $u \in \mathbb{R}^n$. Thus, either $|x-u| \geq |x-y|/2$ or $|u-y| \geq |x-y|/2$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_j^M(x-u)\Phi_k^N(u-y)\, du \leq \\
&\quad\leq \int_{|x-u|\leq|x-y|/2} \Phi_j^M(x-u)\Phi_k^N(u-y)\, du + \int_{|u-y|\geq|x-y|/2} \Phi_j^M(x-u)\Phi_k^N(u-y)\, du \\
&\quad= A + B.
\end{aligned}$$

Then,

$$\begin{aligned}
A &\leq \int_{|x-u|\geq|x-y|/2} \frac{2^{jn}}{(1+2^j|x-u|)^M}\frac{2^{kn}}{(1+2^k|u-y|)^N}\, du \\
&\leq \frac{2^{jn}}{(1+2^j|x-y|/2)^M}\int_{\mathbb{R}^n} \frac{2^{kn}}{(1+2^k|u-y|)^N}\, du \\
&\lesssim \Phi_j^M(x-y).
\end{aligned}$$

The estimate for the term $B$ is analogous. The proof is complete. □

**Definition 4.3.** For a measurable function $f : \mathbb{R}^{n_1+n_2} \to \mathbb{C}$, the biparameter strong maximal function is

$$\mathcal{M}_{\mathcal{S}} f(x) = \sup_{Q_i \ni x_i} \frac{1}{|Q_1||Q_2|}\int_{Q_1\times Q_2} |f(y_1, y_2)|\, dy_1 dy_2,$$

where the supremum is taken over cubes $Q_1 \subseteq \mathbb{R}^{n_1}$ and $Q_2 \subseteq \mathbb{R}^{n_2}$.

**Proposition 4.4.** *Let* $\Phi_{k_i}^{N_i} : \mathbb{R}^{n_i} \to \mathbb{C}$ *for* $i = 1, 2$ *and* $N_i > n_1$. *Then, for all* $f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$,

$$\sup_{k_1,k_2\in\mathbb{Z}} (\Phi_{k_1}^{N_1} \otimes \Phi_{k_2}^{N_2}) * |f|(x) \lesssim \mathcal{M}_{\mathcal{S}} f(x).$$

*Proof.* We have

$$\begin{aligned}
|(\Phi_{k_1}^{N_1} \otimes \Phi_{k_2}^{N_2}) * |f|(x)| &= \int_{\mathbb{R}^{n_1+n_2}} |f(y_1, y_2)|\Phi_{k_1}^{N_1}(x_1 - y_1)\Phi_{k_2}^{N_2}(x_2 - y_2)\ dy_1dy_2 \\
&= \int_{\substack{|x_1-y_1|\leq 2^{-k_1}\\ |x_2-y_2|\leq 2^{-k_2}}} + \int_{\substack{|x_1-y_1|\leq 2^{-k_1}\\ |x_2-y_2|\geq 2^{-k_2}}} + \int_{\substack{|x_1-y_1|\geq 2^{-k_1}\\ |x_2-y_2|\leq 2^{-k_2}}} + \int_{\substack{|x_1-y_1|\geq 2^{-k_1}\\ |x_2-y_2|\geq 2^{-k_2}}} \\
&= I + II + III + IV.
\end{aligned}$$

We estimate explicitly the term $II$. The other estimates follow similarly. Thus,

$$\begin{aligned}
II &= \int_{\substack{|x_1-y_1|\leq 2^{-k_1}\\ |x_2-y_2|\geq 2^{-k_2}}} |f(y_1, y_2)|\Phi_{k_1}^{N_1}(x_1 - y_1)\Phi_{k_2}^{N_2}(x_2 - y_2)\ dy_1dy_2 \\
&\leq \sum_{j=0}^{+\infty} \int_{\substack{|x_1-y_1|\leq 2^{-k_1}\\ 2^{j-k_2}<|x_2-y_2|\leq 2^{j+1-k_2}}} \frac{2^{k_1n_1}2^{k_2n_2}|f(y_1, y_2)|}{(2^{k_2}|x_2 - y_2|)^{N_2}}\ dy_1dy_2 \\
&\leq \sum_{j=0}^{+\infty} \int_{\substack{|x_1-y_1|\leq 2^{-k_1}\\ |x_2-y_2|\leq 2^{j+1-k_2}}} \frac{2^{k_1n_1}2^{k_2n_2}|f(y_1, y_2)|}{2^{jN_2}}\ dy_1dy_2 \\
&\lesssim \mathcal{M}_{\mathcal{S}} f(x_1, x_2) \sum_{j=0}^{\infty} 2^{-j(N_2-n_2)} \\
&\lesssim \mathcal{M}_{\mathcal{S}}(x_1, x_2)
\end{aligned}$$

as we wished. The proof is complete. □

Now, we recall the definition of para-accretive functions firstly introduced in [DJS85].

**Definition 4.5.** A function $b$ in $L^\infty(\mathbb{R}^n)$ is para-accretive if $b^{-1}$ is in $L^\infty(\mathbb{R}^n)$ and there exists a constant $c_0 > 0$ such that for all cubes $Q \subseteq \mathbb{R}^n$, there exists a cube $R \subseteq Q$ such that

$$\frac{1}{|Q|}\left|\int_R b(x)dx\right| \geq c_0.$$

Let $\varphi$ be a function in $C_0^\infty$ such that $\varphi$ is non-negative, it has integral 1 and $\text{supp}(\varphi) \subseteq B(0, 1/8)$. For every $k \in \mathbb{Z}$, let us denote $\varphi_k$ the function $\varphi_k(x) = 2^{kn}\varphi(2^k x)$. Define the operator

$$S_k^b f(x) = P_k M_{(P_k b)^{-1}} P_k f(x) \qquad \text{and} \qquad D_k^b f(x) = S_{k+1}^b f(x) - S_k^b f(x), \tag{4.1}$$

where

$$M_b f(x) := b(x)f(x) \qquad \text{and} \qquad P_k f(x) = \varphi_k * f(x).$$

These operators were introduced in [DJS85], where it is proved that $|P_k b(x)| \geq Cc_0$ with the constant $C > 0$ depending only on the dimension $n$. This assures that the operator $M_{(P_k b)^{-1}}$ is well-defined. Moreover, they proved the following results. We refer to [DJS85] or [Han94] for the proofs.

**Theorem 4.6. [DJS85]** *For every function $f$ in $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, where $1 < q \leq p < \infty$, it holds*

$$\lim_{k\to\infty} S_k^b M_b f = f \qquad \textit{and} \qquad \lim_{k\to\infty} S_{-k}^b M_b f = 0 \tag{4.2}$$

*in $L^p(\mathbb{R}^n)$. Moreover,*

$$S_k^b f(x) = \int_{\mathbb{R}^n} s_k^b(x,y)f(y)\ dy \qquad \textit{and} \qquad D_k^b f(x) = \int_{\mathbb{R}^n} d_k^b(x,y)f(y)\ dy, \tag{4.3}$$

*where the kernels $s_k^b$ and $d_k^b$ satisfy*

$$\begin{aligned}
&s_k^b(x,y) = d_k^b(x,y) = 0 \quad \textit{for } 2^k|x-y| > 1,\\
&|s_k^b(x,y)| + |d_k^b(x,y)| \lesssim 2^{kn},\\
&|s_k^b(x,y) - s_k^b(x',y)| + |d_k^b(x,y) - d_k^b(x',y)| \lesssim 2^{kn}(2^k|x-x'|)^\gamma,\\
&|s_k^b(x,y) - s_k^b(x,y')| + |d_k^b(x,y) - d_k^b(x,y')| \lesssim 2^{kn}(2^k|y-y'|)^\gamma.
\end{aligned}$$

We have the following important reproducing formula.

**Theorem 4.7. [Han94]** *Let $b \in L^\infty(\mathbb{R}^n)$ a para-accretive function. There exist operators $\widetilde{D}_k^b$ for $k \in \mathbb{Z}$ such that*

$$\sum_{k\in\mathbb{Z}} \widetilde{D}_k^b M_b D_k^b M_b f = f \tag{4.4}$$

*in $L^p(\mathbb{R}^n)$ for any function $f : \mathbb{R}^n \to \mathbb{C}$ such that $|f(x)| \lesssim \Phi_0^N(x)$ for some $N > n$, $|f(x) - f(y)| \lesssim |x-y|^\gamma$ for some $\gamma > 0$, and $bf$ has mean zero. Furthermore, $\widetilde{D}_k^d$ is given by integration against its kernel $\widetilde{d}_k^b : \mathbb{R}^{2n} \to \mathbb{C}$,*

$$\widetilde{D}_k^b f(x) = \int_{\mathbb{R}^n} \widetilde{d}_k^b(x,y)f(y)dy,$$

*and* $\widetilde{d}_k^b$ *satisfies*

$$
\begin{aligned}
&|\widetilde{d}_k^b(x,y)| \lesssim \Phi_k^{N+\gamma}(x-y),\\
&|\widetilde{d}_k^b(x,y) - \widetilde{d}_k^b(x',y)| \lesssim (2^k|x-x'|)^\gamma \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y)\right),\\
&\int_{\mathbb{R}^n} \widetilde{d}_k^b(x,y)b(y)dy = \int_{\mathbb{R}^n} \widetilde{d}_k^b(x,y)b(x)dx = 0
\end{aligned}
$$

*for some* $N > n$ *and* $0 < \gamma \leq 1$*.*

## 4.2 Biparameter Littlewood-Paley-Stein operators and square functions

**Definition 4.8.** A collection of functions $\theta_{\vec{k}} : \mathbb{R}^{2n} \to \mathbb{C}$ for $\vec{k} \in \mathbb{Z}^2$ is a collection of biparameter Littlewood-Paley-Stein kernels if for all $x_1, y_1, x_1', y_1' \in \mathbb{R}^{n_1}$ and $x_2, y_2, x_2', y_2' \in \mathbb{R}^{n_2}$

$$|\theta_{\vec{k}}(x,y)| \lesssim \Phi_{k_1}^{N_1+\gamma}(x_1-y_1)\Phi_{k_2}^{N_2+\gamma}(x_2-y_2) \tag{4.5}$$

$$|\theta_{\vec{k}}(x,y) - \theta_{\vec{k}}(x_1',x_2,y)| \lesssim (2^{k_1}|x_1-x_1'|)^\gamma \times \left(\Phi_{k_1}^{N_1+\gamma}(x_1-y_1) + \Phi_{k_1}^{N_1+\gamma}(x_1'-y_1)\right)\Phi_{k_2}^{N_2}(x_2-y_2) \tag{4.6}$$

$$|\theta_{\vec{k}}(x,y) - \theta_{\vec{k}}(x_1,x_2',y)| \lesssim (2^{k_2}|x_2-x_2'|)^\gamma \times \Phi_{k_1}^{N_1}(x_1-y_1)\left(\Phi_{k_2}^{N_2+\gamma}(x_2-y_2) + \Phi_{k_2}^{N_2+\gamma}(x_2'-y_2)\right) \tag{4.7}$$

$$|\theta_{\vec{k}}(x,y) - \theta_{\vec{k}}(x,y_1',y_2)| \lesssim (2^{k_1}|y_1-y_1'|)^\gamma \times \left(\Phi_{k_1}^{N_1+\gamma}(x_1-y_1) + \Phi_{k_1}^{N_1+\gamma}(x_1-y_1')\right)\Phi_{k_2}^{N_2}(x_2-y_2) \tag{4.8}$$

$$|\theta_{\vec{k}}(x,y) - \theta_{\vec{k}}(x,y_1,y_2')| \lesssim (2^{k_2}|y_2-y_2'|)^\gamma \times \Phi_{k_1}^{N_1}(x_1-y_1)\left(\Phi_{k_2}^{N_2+\gamma}(x_2-y_2) + \Phi_{k_2}^{N_2+\gamma}(x_2-y_2')\right) \tag{4.9}$$

for some $N_1 > n_1$, $N_2 > n_2$, and $0 < \gamma \leq 1$.

**Definition 4.9.** We say that a collection of operators $\Theta_{\vec{k}}$ for $\vec{k} \in \mathbb{Z}^2$ is a collection of biparameter Littlewood-Paley-Stein operators if

$$\Theta_{\vec{k}} f(x) = \int_{\mathbb{R}^n} \theta_{\vec{k}}(x,y)f(y)dy. \tag{4.10}$$

for some collection of biparameter Littlewood-Paley-Stein kernels $\theta_{\vec{k}}$ satisfying (4.5)-(4.9).

**Remark 4.10.** Properties (4.5)-(4.9) hold if and only if $\theta_{\vec{k}}$ satisfies the alternate condition set:

$$\begin{aligned}
&|\theta_{\vec{k}}(x,y)| \lesssim \Phi_{k_1}^{N_1'}(x_1-y_1)\Phi_{k_2}^{N_2'}(x_2-y_2),\\
&|\theta_{\vec{k}}(x,y)-\theta_{\vec{k}}(x_1',x_2,y)| \lesssim 2^{n_1k_1}\,2^{n_2k_2}(2^{k_1}|x_1-x_1'|)^{\gamma'},\\
&|\theta_{\vec{k}}(x,y)-\theta_{\vec{k}}(x_1,x_2',y)| \lesssim 2^{n_1k_1}\,2^{n_2k_2}(2^{k_2}|x_2-x_2'|)^{\gamma'},\\
&|\theta_{\vec{k}}(x,y)-\theta_{\vec{k}}(x,y_1',y_2)| \lesssim 2^{n_1k_1}\,2^{n_2k_2}(2^{k_1}|y_1-y_1'|)^{\gamma'},\\
&|\theta_{\vec{k}}(x,y)-\theta_{\vec{k}}(x,y_1,y_2')| \lesssim 2^{n_1k_1}\,2^{n_2k_2}(2^{k_2}|y_2-y_2'|)^{\gamma'}
\end{aligned}$$

for some $N_1' > n_1$, $N_2' > n_2$, and $0 < \gamma' \le 1$.

*Proof.* It is obvious that (4.5)-(4.9) imply the above condition set since $\Phi_{k_j}^{N_j}(x_j) \le 2^{k_jn_j}$. Assume there exist $N_1' > n_1$, $N_2' > n_2$, and $0 < \gamma' \le 1$ such that the alternate condition set holds and choose $\eta \in (0,1)$ small enough so that $N_1 = (1-\eta)N_1' - \eta\gamma' > n_1$ and $N_2 = (1-\eta)N_2' - \eta\gamma' > n_2$, which is possible since $N_1' > n_1$ and $N_2' > n_2$. Also define $\gamma = \eta\gamma'$, and it follows that

$$\begin{aligned}
|\theta_{\vec{k}}(x,y)-\theta_{\vec{k}}(x_1',x_2,y)| &\lesssim \left(2^{k_1n_1}\,2^{k_2n_2}(2^{k_1}|x_1-x_1'|)^{\gamma'}\right)^{\eta}\\
&\qquad\times\left(\Phi_{k_1}^{N_1'}(x_1-y_1)+\Phi_{k_1}^{N_1'}(x_1'-y_1)\right)^{1-\eta}\Phi_{k_2}^{N_2'}(x_2-y_2)^{1-\eta}\\
&\lesssim (2^{k_1}|x_1-x_1'|)^{\gamma}\left(\Phi_{k_1}^{N_1+\gamma}(x_1-y_1)+\Phi_{k_1}^{N_1+\gamma}(x_1'-y_1)\right)\Phi_{k_2}^{N_2+\gamma}(x_2-y_2).
\end{aligned}$$

The other conditions follow by symmetry, and hence the condition sets are equivalent. □

We now prove an almost orthogonality lemma.

**Lemma 4.11.** *Assume that $\Theta_{\vec{k}}$ and $\Psi_{\vec{k}}$ are operators defined by* (4.10) *with kernels respectively $\theta_{\vec{k}}$ and $\psi_{\vec{k}}$. Also assume that $\theta_{\vec{k}}$ satisfies* (4.5), (4.8), *and* (4.9) *and that $\psi_{\vec{k}}$ satisfies* (4.5), (4.6), *and* (4.7). *If there exist para-accretive functions $b_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2 \in L^\infty(\mathbb{R}^{n_2})$ such that*

$$\int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x,y)b_j(y_j)dy_j = \int_{\mathbb{R}^{n_j}} \psi_{\vec{k}}(x,y)b_j(x_j)dx_j = 0$$

*for $j=1,2$ all $x \in \mathbb{R}^n$ and $k_1,k_2 \in \mathbb{Z}$, then for all $\vec{k}=(k_1,k_2), \vec{j}=(j_1,j_2) \in \mathbb{Z}^2$*

$$|\Theta_{\vec{k}}M_b\Psi_{\vec{j}}f(x)| \lesssim 2^{-\epsilon|j_1-k_1|}2^{-\epsilon|j_2-k_2|}\mathcal{M}_Sf(x)$$

*for some $\epsilon > 0$, where $b(x)=b_1(x_1)b_2(x_2)$ for $x=(x_1,x_2)\in\mathbb{R}^n$.*

*Proof.* Using the cancellation of $\psi_{\vec{j}}$ and conditions (4.5) and (4.8), it follows that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \theta_{\vec{k}}(x,u)b(u)\psi_{\vec{j}}(u,y)du\right| &\lesssim \int_{\mathbb{R}^n} |\theta_{\vec{k}}(x,u)-\theta_{\vec{k}}(x,y_1,u_2)|\,|\psi_{\vec{j}}(u,y)|du \\
&\lesssim \int_{\mathbb{R}^n} (2^{k_1}|u_1-y_1|)^\gamma \left(\Phi_{k_1}^{N_1+\gamma}(x_1-u_1)+\Phi_{k_1}^{N_1+\gamma}(x_1-y_1)\right) \\
&\qquad \times \Phi_{k_2}^{N_2+\gamma}(x_2-u_2)\Phi_{j_1}^{N_1+\gamma}(u_1-y_1)\Phi_{j_2}^{N_2+\gamma}(u_2-y_2)du \\
&= 2^{\gamma(k_1-j_1)}\int_{\mathbb{R}^n} (2^{j_1}|u_1-y_1|)^\gamma \Phi_{j_1}^{N_1+\gamma}(u_1-y_1)\left(\Phi_{k_1}^{N_1+\gamma}(x_1-u_1)+\Phi_{k_1}^{N_1+\gamma}(x_1-y_1)\right) \\
&\qquad \times \Phi_{k_2}^{N_2+\gamma}(x_2-u_2)\Phi_{j_2}^{N_2+\gamma}(u_2-y_2)du \\
&\leq 2^{\gamma(k_1-j_1)}\int_{\mathbb{R}^n} \Phi_{j_1}^{N_1}(u_1-y_1)\left(\Phi_{k_1}^{N_1+\gamma}(x_1-u_1)+\Phi_{k_1}^{N_1+\gamma}(x_1-y_1)\right)du_1 \\
&\qquad \times \int_{\mathbb{R}^n} \Phi_{k_2}^{N_2+\gamma}(x_2-u_2)\Phi_{j_2}^{N_2+\gamma}(u_2-y_2)du_2 \\
&\lesssim 2^{\gamma(k_1-j_1)}\left(\Phi_{k_1}^{N_1}(x_1-y_1)+\Phi_{j_1}^{N_1}(x_1-y_1)\right)\left(\Phi_{k_2}^{N_2}(x_2-y_2)+\Phi_{j_2}^{N_2}(x_2-y_2)\right).
\end{aligned}$$

By similar computations using the cancellation of $\theta_{\vec{k}}$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \theta_{\vec{k}}(x,u)b(u)\psi_{\vec{j}}(u,y)du\right| \\
&\qquad \lesssim 2^{-\gamma(j_1-k_1)}\left(\Phi_{k_1}^{N_1}(x_1-y_1)+\Phi_{j_1}^{N_1}(x_1-y_1)\right)\left(\Phi_{k_2}^{N_2}(x_2-y_2)+\Phi_{j_2}^{N_2}(x_2-y_2)\right).
\end{aligned}$$

Then it follows that

$$|\Theta_{\vec{k}}M_b\Psi_{\vec{j}}f(x)| \lesssim 2^{-\gamma|j_1-k_1|}\mathcal{M}_S f(x).$$

Our assumptions are symmetric in $k_1, j_1$ and $k_2, j_2$, so it follows that

$$|\Theta_{\vec{k}}M_b\Psi_{\vec{j}}f(x)| \lesssim 2^{-\gamma|j_2-k_2|}\mathcal{M}_S f(x).$$

Then taking the geometric mean of these two estimates, we have

$$|\Theta_{\vec{k}}M_b\Psi_{\vec{j}}f(x)| \lesssim 2^{-\gamma|j_1-k_1|/2}2^{-\gamma|j_2-k_2|/2}\mathcal{M}_S f(x).$$

This completes the proof. □

**Lemma 4.12.** *Let $b_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions and $D_{k_1}^{b_1}$ and $D_{k_2}^{b_2}$ be the operators in (4.1). Also define $D_{\vec{k}} = D_{k_1}^{b_1}D_{k_2}^{b_2}$ for $\vec{k} \in \mathbb{Z}^2$. Then*

$$\left\|\left(\sum_{\vec{k}\in\mathbb{Z}^2} |D_{\vec{k}}f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

*for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$.*

This proof is essentially the same as the one due to R. Fefferman and Stein in Theorem 2 of [FS82]. We reproduce the argument to demonstrate that there are no problems that arise by introducing para-accretive perturbations.

*Proof.* We start by viewing the operator $\{D_{k_1}^{b_1}\}$ defined initially from $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ into $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$ in the following way: for $\{F_{k_2}\} \in L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$, define

$$\{D_{k_1}^{b_1}\}(\{F_{k_2}\})(x_1) = \{D_{k_1}^{b_1} F_{k_2}(x_1)\}_{k_1,k_2\in\mathbb{Z}}; \quad \text{for } x_1 \in \mathbb{R}^{n_1}.$$

Let $\{F_{k_2}\} \in L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$. For each $k_2 \in \mathbb{Z}$, we use the square function bound for $D_{k_1}^{b_1}$ from [DJS85], and it follows that

$$\int_{\mathbb{R}^{n_1}} \sum_{k_1\in\mathbb{Z}} |D_{k_1}^{b_1} F_{k_2}(x_1)|^2 dx_1 \lesssim \int_{\mathbb{R}^{n_1}} |F_{k_2}(x_1)|^2 dx_1.$$

Then it follows that

$$\begin{aligned} \|\{D_{k_1}^{b_1}\}(\{F_{k_2}\})\|_{L^2(\mathbb{R}^{n_1},\ell^2(\mathbb{Z}^2))}^2 &= \sum_{k_2\in\mathbb{Z}} \left( \int_{\mathbb{R}^{n_1}} \sum_{k_1\in\mathbb{Z}} |D_{k_1}^{b_1} F_{k_2}(x_1)|^2 dx_1 \right) \\ &\lesssim \sum_{k_2\in\mathbb{Z}} \left( \int_{\mathbb{R}^{n_1}} |F_{k_2}(x_1)|^2 dx_1 \right) = \|\{F_{k_2}\}\|_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}))}. \end{aligned}$$

That is, $\{D_{k_1}^{b_1}\}$ is bounded from $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ into $L^2(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$. Now the kernel of $\{D_{k_1}^{b_1}\}$ is given by $\{d_{k_1}^{b_1}(x_1, y_1)\} \in \mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))$ for all $x_1, y_1 \in \mathbb{R}^{n_1}$, where $\mathcal{L}(X, Y)$ for Banach spaces $X$ and $Y$ denotes the collection of all linear operators from $X$ into $Y$. For fixed $x_1, y_1 \in \mathbb{R}^{n_1}$, the kernel $\{d_{k_1}^{b_1}(x_1, y_1)\}$ is realized as a linear operator by the scalar multiplication: $\{a_{k_2}\} \mapsto \{d_{k_1}^{b_1}(x_1, y_1)a_{k_2}\}_{(k_1,k_2)\in\mathbb{Z}^2}$. Furthermore for $x_1 \neq y_1$

$$\begin{aligned} \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}),\ell^2(\mathbb{Z}^2))} &= \sup_{\|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})}=1} \|\{d_{k_1}^{b_1}(x_1, y_1)a_{k_2}\}\|_{\ell^2(\mathbb{Z}^2)} \\ &= \sup_{\|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})}=1} \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\ell^2(\mathbb{Z})} \|\{a_{k_2}\}\|_{\ell^2(\mathbb{Z})} \\ &= \|\{d_{k_1}^{b_1}(x_1, y_1)\}\|_{\ell^2(\mathbb{Z})} \lesssim \frac{1}{|x_1 - y_1|^{n_1}}. \end{aligned}$$

The last inequality is a well-known vector-valued Calderón-Zygmund kernel result, see e.g. Coifman–Meyer [CM78]. It also follows that

$$\|\{d_{k_1}^{b_1}(x_1, y_1)\} - \{d_{k_1}^{b_1}(x_1', y_1)\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}),\ell^2(\mathbb{Z}^2))} \lesssim \frac{|x_1 - x_1'|^\gamma}{|x_1 - y_1|^{n_1+\gamma}}; \text{ for } |x_1 - x_1'| < |x_1 - y_1|/2,$$

$$\|\{d_{k_1}^{b_1}(x_1, y_1)\} - \{d_{k_1}^{b_1}(x_1, y_1')\}\|_{\mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))} \lesssim \frac{|y_1 - y_1'|^\gamma}{|x_1 - y_1|^{n_1+\gamma}}; \text{ for } |y_1 - y_1'| < |x_1 - y_1|/2.$$

Then $\{D_{k_1}^{b_1}\}$ is bounded from $L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ into $L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))$ for $1 < p < \infty$ by the vector-valued Calderón-Zygmund theory developed by Benedek - Calderón - Panzone in [BCP62] and by Rubio de Francia - Ruiz - Torrea in [RdFRT83]. Alternatively, see Theorem 4.6.1 in Grafakos [Gra08] for a statement of the result applied here. Now we fix $f \in L^p(\mathbb{R}^n)$ and define for $x_2 \in \mathbb{R}^{n_2}$ and $k_2 \in \mathbb{Z}$,

$$F_{k_2}^{x_2}(x_1) = D_{k_2}^{b_2} f(x) = \int_{\mathbb{R}^{n_2}} d_{k_2}^{b_2}(x_2, y_2) f(x_1, y_2) dy_2.$$

For almost every $x_2 \in \mathbb{R}^{n_2}$, we have $\{F_{k_2}^{x_2}\} \in L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))$ and hence

$$\begin{aligned}
\int_{\mathbb{R}^{n_1}} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{\vec{k}} f(x)|^2 \right)^{\frac{p}{2}} dx_1 &= \int_{\mathbb{R}^{n_1}} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{k_1}^{b_1} F_{k_2}^{x_2}(x_1)|^2 \right)^{\frac{p}{2}} dx_1 \\
&= \|\{D_{k_1}^{b_1}\}(\{F_{k_2}^{x_2}\})\|_{L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}^2))} \\
&\lesssim \|\{F_{k_2}^{x_2}\}\|_{L^p(\mathbb{R}^{n_1}, \ell^2(\mathbb{Z}))} = \int_{\mathbb{R}^{n_1}} \left( \sum_{k_2 \in \mathbb{Z}} |D_{k_2}^{b_2} f(x)|^2 \right)^{\frac{p}{2}} dx_1. \quad (4.11)
\end{aligned}$$

Now integrate both sides of (4.11) in $x_2$, and using the square function bound for $D_{k_2}^{b_2}$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left( \sum_{\vec{k} \in \mathbb{Z}^2} |D_{\vec{k}} f(x)|^2 \right)^{\frac{p}{2}} dx &\lesssim \int_{\mathbb{R}^{n_1}} \left[ \int_{\mathbb{R}^{n_2}} \left( \sum_{k_2 \in \mathbb{Z}} |D_{k_2}^{b_2} f(x)|^2 \right)^{\frac{p}{2}} dx_2 \right] dx_1 \\
&\lesssim \int_{\mathbb{R}^{n_1}} \left[ \int_{\mathbb{R}^{n_2}} |f(x)|^p dx_2 \right] dx_1 = \|f\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned}$$

This completes the proof. □

We prove a lemma analogous to Theorem [Han94].

**Lemma 4.13.** *Let $b_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions and $b(x) = b_1(x_1) b_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^n$. For $j = 1, 2$ let $D_{k_j}^{b_i}$ be as in (4.1) and $\widetilde{D}_{k_j}^{b_i}$ be as in (4.4). Define $E_{k_j}^{b_j} = \widetilde{D}_{k_j} M_{b_j} D_{k_j}^{b_j}$ for $k_j \in \mathbb{Z}$ and $j = 1, 2$. For any differentiable compactly supported function $f : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0$$

*for* $x = (x_1, x_2) \in \mathbb{R}^n$, *we have the following convergence*

$$\lim_{T\to\infty} \left\| \sum_{|j_1|<T, |j_2|<N_T} E_{\vec{j}} M_b f - f \right\|_{L^p(\mathbb{R}^n)} = 0$$

*for some sequence* $N_T \geq T$.

*Proof.* Let $f : \mathbb{R}^n \to \mathbb{C}$ be differentiable and compactly supported such that

$$\int_{\mathbb{R}^{n_1}} f(x)b(x)dx_1 = \int_{\mathbb{R}^{n_2}} f(x)b(x)dx_2 = 0.$$

For each $x_2 \in \mathbb{R}^{n_2}$, $f(\cdot, x_2)$ is differentiable, compactly supported, and $b_1 \cdot f(\cdot, x_2)$ has mean zero. Then by Theorem 4.7, for every $x_2 \in \mathbb{R}^{n_2}$

$$\lim_{T\to\infty} \left\| \sum_{|j_1|<T} E_{j_1} M_{b_1} f(\cdot, x_2) - f(\cdot, x_2) \right\|_{L^p(\mathbb{R}^{n_1})} = 0$$

Since $f$ is compactly supported and the above quantity is bounded uniformly in $T$, it follows by dominated convergence that

$$\lim_{T\to\infty} \left\| \sum_{|j_1|<T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^{n_2}} \lim_{T\to\infty} \left\| \sum_{|j_1|<T} E_{j_1} M_{b_1} f(\cdot, x_2) - f(\cdot, x_2) \right\|_{L^p(\mathbb{R}^{n_1})}^p dx_2 = 0. \tag{4.12}$$

We also know that for each $T > 0$, define

$$F_T^{x_1}(x_2) = \sum_{|j_1|<T} E_{j_1} M_{b_1} f(x_1, x_2).$$

It follows that

$$|F_T^{x_1}(x_2)| \leq \sum_{|j_1|<T} |E_{j_1} M_{b_1} f(x_1, x_2)| \leq 2T\mathcal{M}_1 f(x) \leq 2T \sup_{x_1\in\mathbb{R}^{n_1}} |f(x_1, x_2)|.$$

Therefore $F_T^{x_1} : \mathbb{R}^{n_2} \to \mathbb{C}$ is bounded (depending on $T$) and compactly supported. Furthermore

$$|F_T^{x_1}(x_2) - F_T^{x_1}(y_2)| \leq \sum_{|j_1|<T} |E_{j_1} M_{b_1} f(x_1, x_2) - f(x_1, y_2)|$$

$$\leq \sum_{|j_1|<T} \int_{\mathbb{R}^{n_2}} |\widetilde{d}_{j_1}^{b_1}(x_2, u_2) - \widetilde{d}_{j_1}^{b_1}(y_2, u_2)| |M_{b_1} D_{j_1}^{b_1} M_{b_1} f(x_1, u_2)| du_2$$

$$\begin{aligned}
&\lesssim \sum_{|j_1|<T} \int_{\mathbb{R}^{n_2}} (2^{j_2}|x_2 - y_2|)^\gamma |D_{j_1}^{b_1} M_{b_1} f(x_1, u_2)| du_2 \\
&\lesssim 2^T |x_2 - y_2|^\gamma \sum_{|j_1|<T} \|D_{j_1}^{b_1} M_{b_1} f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} \\
&\leq 2^T |x_2 - y_2|^\gamma \sum_{|j_1|<T} \|f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} \leq T 2^{T+1} \|f(x_1, \cdot)\|_{L^1(\mathbb{R}^{n_2})} |x_2 - y_2|^\gamma.
\end{aligned}$$

Finally, we have that

$$\int_{\mathbb{R}^{n_2}} F_T^{x_1} b_2(x_2) dx_2 = \sum_{|j_1|<T} E_{j_1} M_{b_1} \int_{\mathbb{R}^{n_2}} f(x_1, x_2) b_2(x_2) dx_2 = 0.$$

Then by Theorem 4.7, it follows that

$$\lim_{N\to\infty} \left\| \sum_{|j_2|<N} E_{j_2} M_{b_2} F_T^{x_1} - F_T^{x_1} \right\|_{L^p(\mathbb{R}^{n_2})} = 0.$$

Then by dominated convergence

$$\begin{aligned}
&\lim_{N\to\infty} \left\| \sum_{|j_1|<T, |j_2|<N} E_{\vec{j}} M_b f - \sum_{|j_1|<T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^{n_2})}^p \\
&\qquad = \int_{\mathbb{R}^{n_1}} \lim_{N\to\infty} \left\| \sum_{|j_2|<N} E_{j_2} M_{b_2} F_T^{x_1} - F_T^{x_1} \right\|_{L^p(\mathbb{R}^{n_2})}^p dx_1 = 0. \qquad (4.13)
\end{aligned}$$

For each $T > 0$, using (4.13) there exists $N_T > T$ such that

$$\left\| \sum_{|j_1|<T, |j_2|<N_T} E_{\vec{j}} M_b f - \sum_{|j_1|<T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^{n_2})} < \frac{1}{T}.$$

This defines the sequence $N_T$, and so now we verify the conclusion of Lemma 4.13. Let $\epsilon > 0$. Fix $M > \frac{2}{\epsilon}$ large enough so that for $T > M$

$$\left\| \sum_{|j_1|<T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

Then

$$
\begin{aligned}
&\left\| \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b f - f \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b f - \sum_{|j_1|<T} E_{j_1} M_{b_1} f \right\|_{L^p(\mathbb{R}^n)} + \left\| \sum_{|j_1|<T} E_{j_1} M_{b_1} f - f \right\|_{L^p(\mathbb{R}^n)} \\
&< \frac{1}{T} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}
$$

This completes the proof. □

Finally, we prove the $L^p$ bounds for the square function associated to a collection of biparameter Littlewood-Paley-Stein operators.

**Theorem 4.14.** *Let $b_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions, and define $b(x) = b_1(x_1)b_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$. Also let $\Theta_{\vec{k}}$ for $\vec{k} \in \mathbb{Z}^2$ be a collection of biparameter Littlewood-Paley-Stein operators with kernels $\theta_{\vec{k}}$. If*

$$
\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}(x,y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}(x,y) b_2(y_2) dy_2 = 0
$$

*for all $\vec{k} \in \mathbb{Z}^2$ and $x, y \in \mathbb{R}^n$, then*

$$
\left\| \Big[ \sum_{\vec{j} \in \mathbb{Z}} |\Theta_{\vec{k}} f|^2 \Big]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}
$$

*for all $f \in L^p(\mathbb{R}^n)$ when $1 < p < \infty$.*

*Proof.* Let $b(x) = b_1(x_1)b_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^n$, and $f, g_{\vec{k}}$ be differentiable, compactly supported such that

$$
\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0
$$

and

$$
\left\| \left( \sum_{\vec{k} \in \mathbb{Z}^2} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \leq 1.
$$

Let $R > 1$, and define

$$\Lambda_R(f) = \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b f(x) g_{\vec{k}}(x) dx \right|,$$

which satisfies

$$0 \leq \Lambda_R(f) \lesssim \int_{\mathbb{R}^n} \mathcal{M}_S f(x) \sum_{|k_1|,|k_2|<R} |g_{\vec{k}}(x)| dx \lesssim R\|f\|_{L^p}. \tag{4.14}$$

Let $S_{k_j}^{b_j}$, $D_{k_j}^{b_j} = S_{k_j+1}^{b_j} - S_{k_j}^{b_j}$, $\widetilde{D}_{k_j}^{b_j}$, and $D_{\vec{k}} = D_{k_1}^{b_1} D_{k_2}^{b_2}$ be the operators defined in (4.1). Also define $E_{k_j}^{b_j} = \widetilde{D}_{k_j}^{b_j} M_{b_j} D_{k_j}^{b_j}$ and $E_{\vec{k}} = E_{k_1}^{b_1} E_{k_2}^{b_2}$, where $\widetilde{D}_{k_j}^{b_j}$ are the operators from (4.4) that were constructed in Theorem 4.7. Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous, compactly supported such that

$$\int_{\mathbb{R}^{n_1}} f(x) b_1(x_1) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b_2(x_2) dx_2 = 0$$

for all $x = (x_1, x_2) \in \mathbb{R}^n$. For $T > 1$ it follows that

$$\begin{aligned}\Lambda_R(f) \leq \sum_{|k_1|,|k_2|<R} &\left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b - \Theta_{\vec{k}} M_b \left( \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b \right) \right] f(x) g_{\vec{k}}(x) dx \right| \\ &+ \sum_{|k_1|,|k_2|<R} \left| \sum_{|j_1|<T,|j_2|<N_T} \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x) dx \right| = I_T + II_T.\end{aligned}$$

where $N_T$ are chosen as in Lemma 4.13. We first estimate $I_T$ using (4.14):

$$\begin{aligned} I_T &= \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( f(x) - \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b f(x) \right) \right] g_{\vec{k}}(x) dx \right| \\ &\leq \Lambda_R \left( f - \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b f \right) \lesssim R \left\| f - \sum_{|j_1|<T,|j_2|<N_T} E_{\vec{j}} M_b f \right\|_{L^p}, \end{aligned}$$

which tends to 0 as $T \to \infty$ by Lemma 4.13. Now we estimate $II_T$ by putting the absolute value inside and summing more terms,

$$II_T \leq \sum_{\vec{k},\vec{j} \in \mathbb{Z}^2} \int_{\mathbb{R}^n} |\Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x)| dx,$$

So we now estimate $II_T$. By Lemma 4.11, there exists $\epsilon > 0$ such that

$$|\Theta_{\vec{k}} M_b E_{\vec{j}} f(x)| \lesssim 2^{-\epsilon|k_1-j_1|} 2^{-\epsilon|k_2-j_2|} \mathcal{M}_S D_{\vec{j}} M_b f(x).$$

Then it follows that

$$\begin{aligned}
\Lambda_R(f) &\leq \int_{\mathbb{R}^n} \sum_{\vec{j},\vec{k}\in\mathbb{Z}^2} |\Theta_{\vec{k}} M_b E_{\vec{j}} M_b f(x) g_{\vec{k}}(x)| dx \\
&\lesssim \int_{\mathbb{R}^n} \sum_{\vec{j},\vec{k}\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|k_1-j_1|+|k_2-j_2|)} \mathcal{M}_S\left(D_{\vec{j}} M_b f\right)(x) |g_{\vec{k}}(x)| dx \\
&\leq \left\| \left( \sum_{\vec{j},\vec{k}\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|k_1-j_1|+|k_2-j_2|)} \left[\mathcal{M}_S\left(D_{\vec{j}} M_b f\right)\right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \\
&\qquad \times \left\| \left( \sum_{\vec{j},\vec{k}\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|k_1-j_1|+|k_2-j_2|)} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\lesssim \left\| \left( \sum_{\vec{j}\in\mathbb{Z}^2} \left[\mathcal{M}_S\left(D_{\vec{j}} M_b f\right)\right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \left\| \left( \sum_{\vec{k}\in\mathbb{Z}^2} |g_{\vec{k}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\lesssim \left\| \left( \sum_{\vec{j}\in\mathbb{Z}^2} |D_{\vec{j}} M_b f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

In the last two lines we use the Fefferman-Stein strong maximal function bound from [FS82] twice and the multiparameter Littlewood-Paley bound from Lemma 4.12. The estimate for general functions $f \in L^p(\mathbb{R}^n)$ follows by density. □

**Remark 4.15.** To prove Theorem 4.14, one does not need to assume that $\Theta_{\vec{k}}$ for $\vec{k} \in \mathbb{Z}^2$ is a collection of biparameter Littlewood-Paley-Stein operators as initially stated in Theorem 4.14. Instead, we only need to assume that $\theta_{\vec{k}}$ satisfies (4.5), (4.8), and (4.9). In short, we can remove the assumption that $\theta_{\vec{k}}$ satisfies conditions (4.6) and (4.7) from Theorem 4.14. In particular, this means that the square function associated to $\widetilde{D}^*_{\vec{k}}$ is bounded as well: let $\widetilde{D}^{b_1}_{k_1}$ and $\widetilde{D}^{b_2}_{k_2}$ be the operators constructed in Theorem 4.7. Define $\widetilde{D}_{\vec{k}} = \widetilde{D}^{b_1}_{k_1} \widetilde{D}^{b_2}_{k_2}$ for $\vec{k} \in \mathbb{Z}^2$, and it follows that

$$\left\| \left( \sum_{\vec{k}\in\mathbb{Z}^2} |\widetilde{D}^*_{\vec{k}} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$ when $1 < p < \infty$.

Next we prove a sort of dual pairing bound for biparameter Littlewood-Paley-Stein operators. This is the estimate that we use to bound the truncations of singular integral operators in the next chapter.

**Proposition 4.16.** *Let $\Theta_{\vec{k}}$ be a collection of biparameter Littlewood-Paley-Stein operators with kernels $\theta_{\vec{k}}$ for $\vec{k} \in \mathbb{Z}^2$ and $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions. If*

$$\int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x,y) b_j(y_j) dy_j = \int_{\mathbb{R}^{n_j}} \theta_{\vec{k}}(x,y) \tilde{b}_j(x_j) dx_j = 0$$

*for $j = 1, 2$, then for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$*

$$\sum_{k_1,k_2 \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b f(x) \tilde{b}(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)},$$

*where $b(x) = b_1(x_1) b_2(x_2)$ and $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^n$.*

*Proof.* Let $f, g$ be differentiable, compactly supported functions such that

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = \int_{\mathbb{R}^{n_1}} g(x) \tilde{b}(x) dx_1 = \int_{\mathbb{R}^{n_2}} g(x) \tilde{b}(x) dx_2 = 0.$$

Define for $R > 1$

$$\Lambda_R(f,g) = \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \Theta_{\vec{k}} M_b f(x) \tilde{b}(x) g(x) dx \right|,$$

which satisfies

$$0 \le \Lambda_R(f,g) \lesssim \sum_{|k_1|,|k_2|<R} \|\mathcal{M}_S f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \lesssim R^2 \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{4.15}$$

Let $S_{k_j}^{b_j}$, $D_{k_j}^{b_j} = S_{k_j+1}^{b_j} - S_{k_j}^{b_j}$, $\widetilde{D}_{k_j}^{b_j}$, $D_{\vec{k}}^{b} = D_{k_1}^{b_1} D_{k_2}^{b_2}$, and $\widetilde{D}_{\vec{k}}^{b} = \widetilde{D}_{k_1}^{b_1} \widetilde{D}_{k_2}^{b_2}$ be the operators defined in (4.1). Also define $E_{k_j}^{b_j} = \widetilde{D}_{k_j}^{b_j} M_{b_j} D_{k_j}^{b_j}$ and $E_{\vec{k}}^{b} = E_{k_1}^{b_1} E_{k_2}^{b_2}$, where $\widetilde{D}_{k_j}^{b_j}$ are the operators constructed in Theorem 2.3 in [Han94]. We also construct the corresponding operators with $b_j$ replaced by $\tilde{b}_j$. Then for $f, g \in C_0^\delta(\mathbb{R}^n)$ for some $0 < \delta \le 1$ where $bf$ and $\tilde{b}g$ have mean zero and $T > 1$, it follows that

$$\Lambda_R(f,g) \le I_T + II_T + III_T,$$

where

$$I_T = \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b - \Theta_{\vec{k}} M_b \left( \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b \right) \right] f(x) M_{\tilde{b}} g(x) dx \right|,$$

$$II_T = \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b \right) \right.\right.$$
$$\left.\left. - \left( \sum_{|m_1|<T,|m_2|<M_T} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} \right) \Theta_{\vec{k}} M_b \left( \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b \right) \right] f(x) M_{\tilde{b}} g(x) dx \right|,$$

$$III_T = \sum_{|k_1|,|k_2|<R} \left| \sum_{|j_1|<T,|j_2|<N_T,|m_1|<T,|m_2|<M_T} \int_{\mathbb{R}^n} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} \Theta_{\vec{k}} M_b E^b_{\vec{j}} M_b f(x) M_{\tilde{b}} g(x) dx \right|,$$

where $N_T$ and $M_T$ are chosen as in Lemma 4.13 for $f$ and $g$ respectively. We first estimate $I_T$ using (4.15) and Lemma 4.13:

$$\begin{aligned} I_T &= \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \left[ \Theta_{\vec{k}} M_b \left( f(x) - \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b f(x) \right) \right] M_{\tilde{b}} g(x) dx \right| \\ &\leq \Lambda_R \left( f - \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b f, g \right) \lesssim R \left\| f - \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b f \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

which tends to 0 as $T \to \infty$. Now we estimate $II_T$ again using (4.15) and Lemma 4.13,

$$\begin{aligned} II_T &= \sum_{|k_1|,|k_2|<R} \left| \int_{\mathbb{R}^n} \left[ \mathbf{I} - \sum_{|m_1|<T,|m_2|<M_T} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} \right] \Theta_{\vec{k}} M_b \left( \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{k}} M_b \right) f(x) M_{\tilde{b}} g(x) dx \right| \\ &= \Lambda_R \left( \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b f, g - \sum_{|m_1|<T,|m_2|<M_T} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} g \right) \\ &\lesssim R \left\| \sum_{|j_1|<T,|j_2|<N_T} E^b_{\vec{j}} M_b f \right\|_{L^p(\mathbb{R}^n)} \left\| g - \sum_{|m_1|<T,|m_2|<M_T} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} g \right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim R \|f\|_{L^p(\mathbb{R}^n)} \left\| g - \sum_{|m_1|<T,|m_2|<M_T} E^{\tilde{b}}_{\vec{m}} M_{\tilde{b}} g \right\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

where $\mathbf{I}$ is the identity operator. This term also tends to 0 as $T \to \infty$ by Lemma 4.13. So we are left with the third term, to estimate $\Lambda_R$

$$\begin{aligned}\Lambda_R(f,g) &\le \lim_{T\to\infty}\sum_{|k_1|,|k_2|<R}\left|\sum_{|j_1|<T,|j_2|<N_T,|m_1|<T,|m_2|<M_T}\int_{\mathbb{R}^n} E^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} M_b f(x) M_{\tilde b} g(x)dx\right| \\ &\le \sum_{\vec k,\vec j,\vec m\in\mathbb{Z}^2}\left|\int_{\mathbb{R}^n} M_b D^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} M_b f(x)(\widetilde{D}^{\tilde b}_{\vec m})^* M_{\tilde b} g(x)dx\right|. \end{aligned} \tag{4.16}$$

So we now estimate (4.16). By Lemma 4.11, there exists $\epsilon > 0$ such that

$$\begin{aligned} &|D^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} f(x)| \lesssim 2^{-\epsilon|m_1-k_1|}2^{-\epsilon|m_2-k_2|}\mathcal{M}^2_S D^b_{\vec j} f(x), \quad \text{and} \\ &|D^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} f(x)| \lesssim \mathcal{M}_S(\Theta_{\vec k} M_b E^b_{\vec j} f)(x) \lesssim 2^{-\epsilon|k_1-j_1|}2^{-\epsilon|k_2-j_2|}\mathcal{M}^2_S D^b_{\vec j} f(x). \end{aligned}$$

Therefore we also have

$$|D^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} f(x)| \lesssim 2^{-\frac{\epsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|}\mathcal{M}^2_S D^b_{\vec j} f(x). \tag{4.17}$$

Using (4.17) we have

$$\begin{aligned} &\int_{\mathbb{R}^n}\sum_{\vec j,\vec k,\vec m\in\mathbb{Z}^2}|M_{\tilde b} D^{\tilde b}_{\vec m} M_{\tilde b}\Theta_{\vec k} M_b E^b_{\vec j} M_b f(x)(\widetilde{D}^{\tilde b}_{\vec m})^* M_{\tilde b} g(x)|dx \\ &\lesssim \int_{\mathbb{R}^n}\sum_{\vec j,\vec k,\vec m\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)}\mathcal{M}^2_S\left(D^b_{\vec j} M_b f\right)(x)(\widetilde{D}^{\tilde b}_{\vec m})^* M_{\tilde b} g(x)|dx \\ &\le \left\|\left(\sum_{\vec j,\vec k,\vec m\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)}\left[\mathcal{M}^2_S\left(D^b_{\vec j} M_b f\right)\right]^2\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)} \\ &\qquad \times\left\|\left(\sum_{\vec j,\vec k,\vec m\in\mathbb{Z}^2} 2^{-\frac{\epsilon}{2}(|m_1-k_1|+|m_2-k_2|+|k_1-j_1|+|k_2-j_2|)}|(\widetilde{D}^{\tilde b}_{\vec m})^* M_{\tilde b} g|^2\right)^{\frac12}\right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim \left\|\left(\sum_{\vec j\in\mathbb{Z}^2}\left[\mathcal{M}^2_S\left(D^b_{\vec j} M_b f\right)\right]^2\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\left\|\left(\sum_{\vec m\in\mathbb{Z}^2}|(\widetilde{D}^{\tilde b}_{\vec m})^* M_{\tilde b} g|^2\right)^{\frac12}\right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim \left\|\left(\sum_{\vec j\in\mathbb{Z}^2}|D^b_{\vec j} M_b f|^2\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} \end{aligned}$$

In the last two lines we use the Fefferman-Stein maximal function bound from [FS82] twice and the biparameter Littlewood-Paley-Stein bound proved in Theorem 4.14. Recall that the square function associated to $(\widetilde{D}_{\vec{m}}^{\tilde{b}})^*$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ from Remark 4.15. The estimate for general functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ follows by density. $\square$

# Chapter 5

# A reduced biparameter $Tb$ theorem

In this chapter we use the theory developed so far to prove a reduced $Tb$ theorem. In the first section we define biparameter singular integral operators of Calderòn-Zygmund type associated to para-accretive functions and we define what we called the biparameter weak boundedness and the mixed biparameter weak boundedness properties. The second section is devoted to the proof of our reduced $Tb$ theorem.

## 5.1 Biparameter singular integral operators

We start defining standard kernels.

**Definition 5.1.** We say that $K$ a standard biparameter kernel on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ if it satisfies the following conditions:

$$|K(x,y)| \lesssim \frac{1}{|x_1 - y_1|^{n_1}\,|x_2 - y_2|^{n_2}} \quad \text{for } |x_1 - y_1|, |x_2 - y_2| \neq 0 \tag{5.1}$$

$$|K(x,y) - K(x_1', x_2, y) - K(x_1, x_2', y) + K(x_1', x_2', y)| \lesssim \frac{|x_1 - x_1'|^\gamma |x_2 - x_2'|^\gamma}{|x_1 - y_1|^{n_1+\gamma}|x_2 - y_2|^{n_2+\gamma}} \tag{5.2}$$

$$\text{whenever } |x_1 - x_1'| < |x_1 - y_1|/2 \text{ and } |x_2 - x_2'| < |x_2 - y_2|/2,$$

$$|K(x,y) - K(x, y_1', y_2) - K(x, y_1, y_2') + K(x, y_1', y_2')| \lesssim \frac{|y_1 - y_1'|^\gamma |y_2 - y_2'|^\gamma}{|x_1 - y_1|^{n_1+\gamma}|x_2 - y_2|^{n_2+\gamma}} \tag{5.3}$$

$$\text{whenever } |y_1 - y_1'| < |x_1 - y_1|/2 \text{ and } |y_2 - y_2'| < |x_2 - y_2|/2.,$$

$$|K(x,y) - K(x, y_1', y_2) - K(x_1, x_2', y) + K(x_1, x_2', y_1', y_2)| \lesssim \frac{|y_1 - y_1'|^\gamma |x_2 - x_2'|^\gamma}{|x_1 - y_1|^{n_1+\gamma}|x_2 - y_2|^{n_2+\gamma}} \tag{5.4}$$

$$\text{whenever } |y_1 - y_1'| < |x_1 - y_1|/2 \text{ and } |x_2 - x_2'| < |x_2 - y_2|/2.,$$

$$|K(x,y) - K(x,y_1,y_2') - K(x_1',x_2,y) + K(x_1',x_2,y_1,y_2')| \lesssim \frac{|x_1-x_1'|^\gamma |y_2-y_2'|^\gamma}{|x_1-y_1|^{n_1+\gamma}|x_2-y_2|^{n_2+\gamma}} \quad (5.5)$$

$$\text{whenever } |x_1-x_1'| < |x_1-y_1|/2 \text{ and } |y_2-y_2'| < |x_2-y_2|/2.$$

Now we give the definition of biparameter singular integral operator of Caldeón-Zygmund type associated to para-accretive functions. We recall that $bC_0^{0,\delta}(\mathbb{R}^n)$ is defined in Definition 4.1.

**Definition 5.2.** Let $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions and define $b(x) = b_1(x_1)b_2(x_2)$ and $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x = (x_1,x_2) \in \mathbb{R}^n$. A linear operator $T$ that is continuous from $bC_0^{0,\delta}(\mathbb{R}^n)$ into $(\tilde{b}C_0^{0,\delta}(\mathbb{R}^n))'$ for some $0 < \delta \le 1$ is a biparameter singular integral operator of Calderón-Zygmund type associated to $b, \tilde{b}$ if

$$\langle M_{\tilde{b}} T M_b f, g\rangle = \int_{\mathbb{R}^n} K(x,t) f(t) g(x) \tilde{b}(x) b(y) dx\, dy$$

is an absolutely convergent integral whenever $f, g \in C_0^{0,\delta}(\mathbb{R}^n)$ and

$$\bigcup_{x_1,y_1\in\mathbb{R}^{n_1}} \operatorname{supp}(f(y_1,\cdot)) \cap \operatorname{supp}(g(x_1,\cdot)) = \bigcup_{x_2,y_2\in\mathbb{R}^{n_2}} \operatorname{supp}(f(\cdot,y_2)) \cap \operatorname{supp}(g(\cdot,x_2)) = \emptyset.$$

We end this section stating the boundedness properties that we will need to assume for our operator $T$ in order to prove our $Tb$ theorem. Before, we recall what a normalized bump is.

**Definition 5.3.** A function $\phi \in C_0^\infty(\mathbb{R}^n)$ is a normalized bump of order $m \in \mathbb{N}$ if $\operatorname{supp}(\phi) \subset B(0,1) \subset \mathbb{R}^n$ and for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$

$$\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \le 1.$$

Then,

**Definition 5.4.** Let $T$ be a biparameter singular integral operator of Calderón-Zygmund type associated to $b(x) = b_1(x_1)b_2(x_2)$ and $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x = (x_1,x_2) \in \mathbb{R}^n$, where $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$ are para-accretive functions. We say $T$ satisfies the biparameter weak boundedness property if there exists $m \in \mathbb{N}$ such that the following holds: let $\varphi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$ be normalized bumps of order $m$. Let $x = (x_1,x_2) \in \mathbb{R}^n$ and $R_1, R_2 > 0$. Assume that either $b_1\varphi_1^{x_1,R_1}$ or $\tilde{b}_1\psi_1^{x_1,R_1}$ has mean zero and that either $b_2\varphi_2^{x_2,R_2}$ or $\tilde{b}_2\psi_2^{x_2,R_2}$ has mean zero. Then

$$\left|\left\langle M_{\tilde{b}} T M_b(\varphi_1^{x_1,R_1} \otimes \varphi_2^{x_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2}\right\rangle\right| \lesssim R_1^{n_1} R_2^{n_2}, \quad (5.6)$$

where $\varphi^{x_j,R_j}(u_j) = \varphi\left(\frac{u_j-x_j}{R_j}\right)$.

**Definition 5.5.** Let $T$ be a biparameter singular integral operator of Calderón-Zygmund type associated to $b(x) = b_1(x_1)b_2(x_2)$ and $\tilde{b}(x) = \tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^n$, where $b_1, \tilde{b}_1 \in L^\infty(\mathbb{R}^{n_1})$ and $b_2, \tilde{b}_2 \in L^\infty(\mathbb{R}^{n_2})$ are para-accretive functions. We say $T$ satisfies the mixed biparameter weak boundedness property if there exists $m \in \mathbb{N}$ and $0 < \gamma \leq 1$ such that the following two conditions hold: (1) Let be $R_1, R_2 > 0$, $x_1, y_1 \in \mathbb{R}^{n_1}$ with $|x_1 - y_1| > 4R_1$, and $x_2 \in \mathbb{R}^{n_2}$ and let $\varphi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$ be normalized bumps of order $m$. Then

$$\left|\left\langle M_{\tilde{b}}TM_b(\varphi_1^{y_1,R_1} \otimes \varphi_2^{x_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2}\right\rangle\right| \lesssim \frac{R_1^{n_1}R_2^{n_2}}{(R_1^{-1}|x_1-y_1|)^{n_1}}. \tag{5.7}$$

Further assume that either $b_1\varphi_1^{y_1,R_1}$ or $\tilde{b}_1\psi_1^{x_1,R_1}$ has mean zero and that either $b_2\varphi_2^{x_2,R_2}$ or $\tilde{b}_2\psi_2^{x_2,R_2}$ has mean zero. Then

$$\left|\left\langle M_{\tilde{b}}TM_b(\varphi_1^{y_1,R_1} \otimes \varphi_2^{x_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2}\right\rangle\right| \lesssim \frac{R_1^{n_1}R_2^{n_2}}{(R_1^{-1}|x_1-y_1|)^{n_1+\gamma}}. \tag{5.8}$$

(2) Let be $R_1, R_2 > 0$, $x_2, y_2 \in \mathbb{R}^{n_1}$ with $|x_2 - y_2| > 4R_2$, and $x_2 \in \mathbb{R}^{n_2}$ and let $\varphi_j, \psi_j \in C_0^\infty(\mathbb{R}^{n_j})$ be normalized bumps of order $m$. Then

$$\left|\left\langle M_{\tilde{b}}TM_b(\varphi_1^{x_1,R_1} \otimes \varphi_2^{y_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2}\right\rangle\right| \lesssim \frac{R_1^{n_1}R_2^{n_2}}{(R_2^{-1}|x_2-y_2|)^{n_2}}. \tag{5.9}$$

Further assume that either $b_1\varphi_1^{x_1,R_1}$ or $\tilde{b}_1\psi_1^{x_1,R_1}$ has mean zero and that either $b_2\varphi_2^{y_2,R_2}$ or $\tilde{b}_2\psi_2^{x_2,R_2}$ has mean zero. Then,

$$\left|\left\langle M_{\tilde{b}}TM_b(\varphi_1^{x_1,R_1} \otimes \varphi_2^{y_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2}\right\rangle\right| \lesssim \frac{R_1^{n_1}R_2^{n_2}}{(R_2^{-1}|x_2-y_2|)^{n_2+\gamma}}. \tag{5.10}$$

**Definition 5.6.** A biparameter singular integral operator satisfies the biparameter $Tb = T^*\tilde{b} = 0$ condition if the following two conditions hold: (1) Let $\psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$, $\psi_2, \varphi_2 \in C_0^\infty(\mathbb{R}^{n_2})$, and $\eta_R \in C_0^\infty(\mathbb{R}^{n_1})$ such that $\eta_R = 1$ on $B_1(0,R) \subset \mathbb{R}^{n_1}$ and $\mathrm{supp}(\eta_R) \subset B_1(0,2R) \subset \mathbb{R}^{n_1}$. If $b_1\psi_1$ has mean zero and either $b_2\varphi_2$ or $b_2\psi_2$ has mean zero, then

$$\left\langle T(b_1 \otimes b_2\psi_2), \tilde{b}_1\psi_1 \otimes \tilde{b}_2\varphi_2\right\rangle := \lim_{R\to\infty} \langle M_{\tilde{b}}TM_b(\eta_R \otimes \psi_2), \psi_1 \otimes \varphi_2\rangle = 0, \tag{5.11}$$

$$\left\langle T(b_1\psi_1 \otimes b_2\psi_2), \tilde{b}_1 \otimes \tilde{b}_2\varphi_2\right\rangle := \lim_{R\to\infty} \langle M_{\tilde{b}}TM_b(\psi_1 \otimes \psi_2), \eta_R \otimes \varphi_2\rangle = 0, \tag{5.12}$$

and (2) let $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$, $\psi_1, \varphi_1 \in C_0^\infty(\mathbb{R}^{n_1})$, and $\eta_R \in C_0^\infty(\mathbb{R}^{n_2})$ such that $\eta_R = 1$ on $B_2(0,R) \subset \mathbb{R}^{n_1}$ and $\mathrm{supp}(\eta_R) \subset B_2(0,2R) \subset \mathbb{R}^{n_2}$. If $b_2\psi_2$ has mean zero and either $b_1\varphi_1$ or $b_1\psi_1$ has mean zero, then

$$\left\langle T(b_1\psi_1 \otimes b_2), \tilde{b}_1\varphi_1 \otimes \tilde{b}_2\psi_2\right\rangle := \lim_{R\to\infty} \langle M_{\tilde{b}}TM_b(\psi_1 \otimes \eta_R), \varphi_1 \otimes \psi_2\rangle = 0,$$

$$\left\langle T(b_1\psi_1\otimes b_2\psi_2),\tilde{b}_1\varphi_1\otimes\tilde{b}_2\right\rangle := \lim_{R\to\infty}\langle M_{\tilde{b}}TM_b(\psi_1\otimes\psi_2),\varphi_1\otimes\eta_R\rangle = 0.$$

## 5.2 The proof of the theorem

In this section we finally prove our $Tb$ theorem.

**Theorem 5.7.** *Let $b_1,\tilde{b}_1\in L^\infty(\mathbb{R}^{n_1})$ and $b_2,\tilde{b}_2\in L^\infty(\mathbb{R}^{n_2})$ be para-accretive functions, and define $b(x)=b_1(x_1)b_2(x_2)$ and $\tilde{b}(x)=\tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x=(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$. Also let $T$ be a biparameter operator of Calderón-Zygmund type associated to $b$ and $\tilde{b}$. If $T$ satisfies the weak boundedness property, mixed weak boundedness properties, and the $Tb=T^*\tilde{b}=0$ conditions, then $T$ can be continuously extended to a bounded linear operator on $L^p(\mathbb{R}^n)$ for $1<p<\infty$.*

To prove our theorem, we need the following fundamental lemma.

**Lemma 5.8.** *Suppose $b_1,\tilde{b}_1\in L^\infty(\mathbb{R}^{n_1})$ and $b_2,\tilde{b}_2\in L^\infty(\mathbb{R}^{n_2})$ are para-accretive functions, and define $b(x)=b_1(x_1)b_2(x_2)$ and $\tilde{b}(x)=\tilde{b}_1(x_1)\tilde{b}_2(x_2)$ for $x=(x_1,x_2)\in\mathbb{R}^n$. Let $T$ be a biparameter singular integral operator of Calderón-Zygmund type associated to $b$ and $\tilde{b}$ with standard biparameter kernel $K$. Also assume that $M_{\tilde{b}}TM_b$ satisfies the biparameter weak boundedness and the mixed weak boundedness properties. Define $\Theta_{\vec{k}}$ for $\vec{k}\in\mathbb{Z}^2$ by integration against its kernel $\theta_{\vec{k}}$, as in (4.10), where*

$$\theta_{\vec{k}}(x,y)=\left\langle M_{\tilde{b}}TM_b(s_{k_1}^{b_1}(\cdot,y_1)\otimes s_{k_2}^{b_2}(\cdot,y_2)),d_{k_1}^{\tilde{b}_1}(x_1,\cdot)\otimes d_{k_2}^{\tilde{b}_2}(x_2,\cdot)\right\rangle. \tag{5.13}$$

*Then $\Theta_{\vec{k}}$ for $\vec{k}\in\mathbb{Z}^2$ is a collection of Littlewood-Paley-Stein operators and*

$$\int_{\mathbb{R}^{n_1}}\theta_{\vec{k}}(x,y)\tilde{b}_1(x_1)dx_1=\int_{\mathbb{R}^{n_2}}\theta_{\vec{k}}(x,y)\tilde{b}_2(x_2)dx_2=0.$$

*Proof.* Fix $x,y\in\mathbb{R}^n$ such that $|x_1-y_1|\le 2^{-k_1+2}$ and $|x_2-y_2|\le 2^{-k_2+2}$. Then using (5.6)

$$\begin{aligned}&|\theta_{\vec{k}}(x,y)|\\&=2^{2k_1n_1}2^{2k_2n_2}\left|\left\langle M_{\tilde{b}}TM_b\left(\phi_1^{\frac{x_1+y_1}{2},2^{-k_1+2}}\otimes\phi_2^{\frac{x_2+y_2}{2},2^{-k_2+2}}\right),\phi_3^{\frac{x_1+y_1}{2},2^{-k_1+2}}\otimes\phi_4^{\frac{x_2+y_2}{2},2^{-k_2+2}}\right\rangle\right|\\&\lesssim 2^{k_1n_1}2^{k_2n_2}\lesssim\Phi_{k_1}^{n_1+\gamma}(x_1-y_1)\Phi_{k_2}^{n_2+\gamma}(x_2-y_2).\end{aligned}$$

where $\phi_1,\phi_2,\phi_3,\phi_4$ are normalized bumps of order $m$ (up to a constant multiple independent of $x$, $y$, and $\vec{k}$) of the form

$$\phi_1(u_1)=2^{-k_1n_1}s_{k_1}^{b_1}\left(2^{-k_1+2}u_1+\frac{x_1+y_1}{2},y_1\right)\quad\phi_2(u_2)=2^{-k_2n_2}s_{k_2}^{b_2}\left(2^{-k_2+2}u_1+\frac{x_2+y_2}{2},y_2\right),$$

$$\phi_3(v_1) = 2^{-k_1n_1} d_{k_1}^{\tilde{b}_1}\left(x_1, 2^{-k_1+2}v_1 + \frac{x_1+y_1}{2}\right) \quad \phi_4(v_2) = 2^{-k_2n_2} d_{k_2}^{\tilde{b}_2}\left(x_2, 2^{-k_2+2}v_2 + \frac{x_2+y_2}{2}\right).$$

It is not hard to verify that $2^{k_1n_1}\phi_1^{\frac{x_1+y_1}{2}, 2^{-k_1+2}}(u_1) = s_{k_1}(u_1, y_1)$ for $u_1 \in \mathbb{R}^{n_1}$ and likewise for the other three terms. This completes the proof of (4.5) when both $x_1, y_1$ and $x_2, y_2$ are close. Now fix $x, y \in \mathbb{R}^n$ such that $|x_1 - y_1| > 2^{-k_1+2}$ and $|x_2 - y_2| > 2^{-k_2+2}$. It follows that

$$\operatorname{supp}(s_{k_1}^{b_1}(\cdot, y_1)) \cap \operatorname{supp}(d_{k_1}^{\tilde{b}_1}(x_1, \cdot)) = \operatorname{supp}(s_{k_2}^{b_2}(\cdot, y_2)) \cap \operatorname{supp}(d_{k_2}^{\tilde{b}_2}(x_2, \cdot)) = \emptyset.$$

Then we can use the kernel representation of $T$ to write

$$\begin{aligned}
|\theta_{\vec{s}}(x,y)| &= \left|\iint_{\mathbb{R}^{2n}} K(u,v) s_{k_1}^{b_1}(v_1,y_1) d_{k_1}^{\tilde{b}_1}(x_1,u_1) s_{k_2}^{b_2}(v_2,y_2) d_{k_2}^{\tilde{b}_2}(x_2,u_2)\tilde{b}(u)b(v)du\,dv\right| \\
&\lesssim \int_{\mathbb{R}^{2n}} |K(u,v) - K(x_1,u_2,v_1,v_2) - K(u_1,x_2,v_1,v_2) + K(x_1,x_2,v_1,v_2)| \\
&\qquad\qquad \times |s_{k_1}^{b_1}(v_1,y_1) d_{k_1}^{\tilde{b}_1}(x_1,u_1) s_{k_2}^{b_2}(v_2,y_2) d_{k_2}^{\tilde{b}_2}(x_2,u_2)|du\,dv \\
&\le \int_{|y_i-v_i|<2^{-k_i}} \int_{|x_i-u_i|<2^{-k_i}} \frac{|x_1-u_1|^\gamma |x_2-u_2|^\gamma}{|x_1-v_1|^{n_1+\gamma}|x_2-v_2|^{n_2+\gamma}} 2^{2k_1n_1}2^{2k_2n_2} du\,dv \\
&\le \int_{|y_i-v_i|<2^{-k_i}} \int_{|x_i-u_i|<2^{-k_i}} \frac{2^{k_1(2n_1-\gamma)}2^{k_2(2n_2-\gamma)}}{(|x_1-y_1|/2 + 2^{-k_1})^{n_1+\gamma}(|x_2-y_2|/2+2^{-k_2})^{n_2+\gamma}} du\,dv \\
&\lesssim \frac{2^{-\gamma k_1}2^{-\gamma k_2}}{(|x_1-y_1| + 2^{-k_1})^{n_1+\gamma}(|x_2-y_2|+2^{-k_2})^{n_2+\gamma}} = \Phi_{k_1}^{n_1+\gamma}(x_1-y_1)\Phi_{k_2}^{n_2+\gamma}(x_2-y_2).
\end{aligned}$$

Fix $x, y \in \mathbb{R}^n$ such that $|x_1 - y_1| \le 2^{-k_1+2}$ and $|x_2 - y_2| > 2^{-k_2+2}$. Then we can write

$$\begin{aligned}
|\theta_{\vec{s}}(x,y)| &= \left|\left\langle M_{\tilde{b}} T M_b \left(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)\right), d_{k_1}^{\tilde{b}_1}(x_1, \cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2,\cdot)\right\rangle\right| \\
&= 2^{2k_1n_1}2^{2k_2n_2}\left|\left\langle M_{\tilde{b}} T M_b \left(\widetilde{\phi}_1^{y_1, 2^{-k_1}} \otimes \phi_2^{\frac{x_2+y_2}{2}, 2^{-k_2+2}}\right), \widetilde{\phi}_3^{x_1, 2^{-k_1}} \otimes \phi_4^{\frac{x_2+y_2}{2}, 2^{-k_2+2}}\right\rangle\right|,
\end{aligned}$$

where

$$\widetilde{\phi}_1(u_1) = 2^{-k_1n_1} s_{k_1}^{b_1}(2^{-k}u_1 + y_1, y_1) \qquad \text{and} \qquad \widetilde{\phi}_3(v_1) = 2^{-k_1n_1} d_{k_1}^{\tilde{b}_1}(x_1, 2^{-k}v_1 + x_1)$$

again are normalized bumps of order $m$ (up to a constant multiple independent of $x$, $y$, and $\vec{k}$). Since $|x_2 - y_2| > 4 \cdot 2^{-k_2}$, we can apply (5.10) to obtain the following estimate.

$$\begin{aligned}
|\theta_{\vec{k}}(x,y)| &\lesssim 2^{2k_1n_1}2^{2k_2n_2}\left(\frac{2^{-k_1n_1}2^{-k_2n_2}}{(2^{k_2}|x_2-y_2|)^{n_2+\gamma}}\right) \\
&\lesssim \frac{2^{k_1n_1}2^{k_2n_2}}{(1+2^{k_2}|x_2-y_2|)^{n_2+\gamma}} \lesssim \Phi_{k_1}^{n_1+\gamma}(x_1-y_1)\Phi_{k_2}^{n_2+\gamma}(x_2-y_2).
\end{aligned}$$

A similar argument using (5.8) proves that (4.5) holds when $|x_1-y_1| > 2^{-k_1+2}$ and $|x_2-y_2| \leq 2^{-k_2+2}$. This verifies that $\theta_{\vec{k}}$ satisfies condition (4.5) for all $x, y \in \mathbb{R}^n$. Now to verify (4.6), recall that for $W \in (C_0^\infty(\mathbb{R}^n))'$, $f \in C_0^\infty(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, $F(x) = \langle W, f^x\rangle$ is a differentiable function where $\partial_{x_i} F(x) = \langle W, (\partial_{x_i} f)^x\rangle$. Then $\theta_{\vec{k}}$ is differentiable, and we can estimate

$$\begin{aligned}|\nabla_{x_1}\theta_{\vec{k}}(x,y)|^2 &= \sum_{j=1}^{n_1}\left|\left\langle M_{\tilde{b}}TM_b(s_{k_1}^{b_1}(\cdot,y_1)\otimes s_{k_2}^{b_2}(\cdot,y_2)), \partial_{x_{1,j}}(d_{k_1}^{\tilde{b}_1}(x_1,\cdot))\otimes d_{k_2}^{\tilde{b}_2}(x_2,\cdot)\right\rangle\right|^2\\ &\lesssim 2^{2k_1(n_1+1)}2^{2k_2n_2},\end{aligned}$$

since $2^{-k_1(n_1+1)}\partial_{x_{1,j}}(d_{k_1}^{\tilde{b}_1}(x_1,\cdot))$ is again a normalized bump for $x_1 = (x_{1,1}, ..., x_{1,n_1}) \in \mathbb{R}^{n_1}$ (up to a constant multiple independent of $x$, $y$, and $\vec{k}$). Therefore

$$|\theta_{\vec{k}}(x,y) - \theta_{\vec{k}}(x_1', x_2, y)| \leq ||\nabla_{x_1}\theta_{\vec{k}}(x,y)||_{L^\infty}\,|x_1 - x_1'| \lesssim 2^{k_1n_1}2^{k_2n_2}(2^{k_1}|x_1 - x_1'|).$$

This proves that $\theta_{\vec{k}}$ verifies (4.6) via the equivalence in Remark 4.10. By the same argument, it follows that $\theta_{\vec{k}}$ verifies (4.7)-(4.9). Now by the continuity of $T$ from $bC_0^\delta(\mathbb{R}^n)$ into $(\tilde{b}C_0^\delta(\mathbb{R}^n))'$, we have that

$$\int_{\mathbb{R}^{n_1}}\theta_{\vec{k}}(x,y)\tilde{b}_1(x_1)dx_1 = \lim_{R\to\infty}\left\langle M_{\tilde{b}}TM_b(s_{k_1}^{b_1}(\cdot,y_1)\otimes s_{k_2}^{b_2}(\cdot,y_2)), \lambda_{R,k_1}\otimes d_{k_2}^{\tilde{b}_2}(x_2,\cdot)\right\rangle$$

where

$$\lambda_{R,k_1}(u_1) = \int_{|x_1|\leq R} d_{k_1}^{\tilde{b}_1}(x_1,u_1)\tilde{b}_1(x_1)dx_1.$$

Note that for $|u_1| > R+2^{-k_1}$, we have $|u_1 - x_1| \geq |u_1| - |x_1| > 2^{-k_1}$ and hence $\lambda_{R,s_1}(u_1) = 0$ for such $u_1$. Also for $|u_1| < R - 2^{-k_1}$ and $x \in \text{supp}(d_{k_1}^{\tilde{b}_1}(\cdot,u_1))$, it follows that $|x_1| \leq |u_1| + |u_1 - x_1| < R$. Since $D_{k_1}^{\tilde{b}_1}\tilde{b}_1 = 0$, $\lambda_{R,s_1}(u_1) = 0$ for $|u_1| < R - 2^{-k_1}$. That is $\text{supp}(\lambda_{R,s_1}) \subset B(0, R+2^{-k_1})\backslash B(0, R-2^{-k_1})$. Now take $R > |y_1| + 2^{-k_1+1}$ so that $\lambda_{R,k_1}$ and $s_{k_1}^{b_1}(\cdot,y_1)$ have disjoint support. Now we split into two cases: (1) where $|x_2 - y_2| \leq 2^{-k_1+2}$ and (2) where $|x_2 - y_2| > 2^{-k_2+2}$.

Case 1: $(|x_2 - y_2| \leq 2^{-k_1+2})$ Here we take $R > 2^{-k_1+6} + 2|y_1|$. Consider

$$\mathcal{B} = \{B(u_1, 2^{-k_1}) : u_1 \in \text{supp}(\lambda_{R,k_1})\},$$

which is an open cover of $\text{supp}(\lambda_{R,k_1})$. Then by Vitali's covering lemma, there exists finite collection $\{B_1, ..., B_J\} \subset \mathcal{B}$ of disjoint balls such that $\{3B_1, ..., 3B_J\}$ forms an open cover of $\text{supp}(\lambda_{R,k_1})$. Let $c_j \in \mathbb{R}^{n_1}$ be the center of $B_j$ for each $j = 1,..,J$. Fix $\chi \in C_0^\infty(\mathbb{R}^{n_1})$ such that $\chi = 1$ on $B(0,1)$ and $\text{supp}(\chi) \subset B(0,2)$. Let $\widetilde{\chi}_j(u_1) = \chi\left(\frac{u_1-c_j}{3\cdot 2^{-k_1}}\right)$, and it follows that $\widetilde{\chi}_j = 1$ on $3B_j$ and $\widetilde{\chi}_j$ is supported inside $6B_j$. Finally define the partition of unity for $3B_1 \cup \cdots \cup 3B_J$,

$$\chi_j(u_1) = \frac{\widetilde{\chi}_j(u_1)}{\sum_{k=1}^J \widetilde{\chi}_k(u_1)} \quad \text{for } j = 1,...,J.$$

Let $m \in \mathbb{N}_0$ be the integer specified by the weak boundedness and mixed weak boundedness properties for $M_bTM_b$. It follows that

$$\eta_j(u_1) = \frac{1}{\max_{|\alpha|\leq m} ||\partial^\alpha(\lambda_{R,k_1}\chi_j)||_{L^\infty}} \chi_j(2^{-k_1+3}u_1 + c_j)\lambda_{R,k_1}(2^{-k_1+3}u_1 + c_j)$$

is a normalized bump of order $m$ for each $j = 1,...,J$. Note that for each $\beta \in \mathbb{N}_0^{n_1}$ with $|\beta| \leq |\alpha| \leq m$

$$\begin{aligned}|\partial^\beta \lambda_{R,k_1}(u_1)| &\leq \int_{|x_1|\leq R} |\partial_{u_1}^\beta d_{k_1}^{\tilde{b}_1}(x_1,u_1)\tilde{b}_1(x_1)|dx_1 \\ &\leq 2^{k_1|\beta|}\int_{\mathbb{R}^{n_1}} |\partial_{u_1}^\beta d_{k_1}^{\tilde{b}_1}(x_1,u_1)\tilde{b}_1(x_1)|dx_1 \lesssim 2^{k_1|\beta|}.\end{aligned}$$

The importance here is that this estimate does not depend on $R$; it does depend on $k_1$ and $\beta$, but since we are taking a limit in $R$ for a fixed $k_1$ and $|\beta| \leq m$, this is not of consequence. Likewise for $|\beta| \leq |\alpha| \leq m$ and $u \in \text{supp}(\lambda_{R,k_1}) \cap 3B_j$

$$|\partial^\beta \chi_j(u)| = \left|\partial^\beta\left[\frac{\widetilde{\chi}\left(3\frac{u_1-c_j}{2^{-k_1}}\right)}{\sum_{k=1}^J \widetilde{\chi}_k\left(3\frac{u_1-c_j}{2^{-k_1}}\right)}\right]\right| = 3^{|\beta|}2^{|\beta|k_1}\left|\left|\partial^\beta\left[\frac{\widetilde{\chi}}{\sum_{k=1}^J \widetilde{\chi}_k}\right]\right|\right|_{L^\infty(B(0,1))} \leq A_\beta 2^{|\beta|k_1},$$

for some constant $A_\beta > 0$ depending only on $\beta \in \mathbb{N}_0^{n_1}$. Note that we use $\widetilde{\chi}_j \in C_0^\infty(\mathbb{R}^{n_1})$ and $\sum_{k=1}^J \widetilde{\chi}_k \geq 1$ on $\text{supp}(\lambda_{R,k_1}) \cap 3B_j$. Again the importance here is that this estimate does not depend on $R$; it does depend on $k_1$, $\beta$, and derivatives of $\chi$, but that is not a problem. Also define $\phi(u_1) = 2^{-k_1n_1}s_{k_1}^{b_1}(2^{-k_1+3}u_1 + y_1, y_1)$, and it follows that $\phi$ is a normalized bump up to a constant multiple. We now use that

$$\sum_{j=1}^J \max_{|\alpha|\leq m} ||\partial^\alpha(\lambda_{R,k_1}\chi_j)||_{L^\infty}\eta_j^{c_j,2^{-k_1+3}}(u_1) = \sum_{j=1}^J \chi_j(u_1)\lambda_{R,k_1}(u_1) = \lambda_{R,k_1}(u_1),$$

$$\phi^{y_1,2^{-k_1+3}}(u_1) = 2^{-k_1n_1}s_{k_1}^{b_1}\left(2^{-k_1+3}\frac{u_1-c_j}{2^{-k_1+3}} + y_1, y_1\right) = 2^{-k_1n_1}s_{k_1}^{b_1}(u_1,y_1),$$

and since $R > 2^{-k_1+6} + 2|y_1|$, it follows that

$$|c_j - y_1| \geq |c_j| - |y_1| \geq R - 2^{-k_1} - |y_1| > 2^{-k_1+6} - 2^{-k_1} \geq 4 \cdot 2^{-k_1+3}.$$

Then we can apply (5.7) in the following way

$$\begin{aligned}
&\left|\left\langle M_{\tilde{b}} T M_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot)\right\rangle\right| \\
&\qquad \leq \sum_{j=1}^{J} \max_{|\alpha|\leq m} ||\partial^\alpha(\lambda_{R,k_1}\chi_j)||_{L^\infty} \left|\left\langle T(\phi^{y_1, 2^{-k_1+3}} \otimes s_{k_2}(\cdot, y_2)), \eta_j^{c_j, 2^{-k_1+3}} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot)\right\rangle\right| \\
&\qquad \leq \sum_{j=1}^{J} A_{k_1,m} \frac{2^{k_2 n_2} 2^{-k_1 n_1}}{(2^{k_1}|y_1 - c_j|)^{n_1}} \lesssim \sum_{j=1}^{J} A_{k_1,m} \frac{2^{k_2 n_2} 2^{-2k_1 n_1}}{R^{n_1}} = A_{k_1,m} \frac{2^{k_2 n_2} 2^{-2k_1 n_1}}{R^{n_1}} J, \\
&\qquad\quad \text{where} \quad A_{k_1,m} = \max_{|\beta|+|\gamma|\leq m} 2^{k_1(|\beta|+|\gamma|)} A_\gamma.
\end{aligned}$$

Now we use that $B_1, ..., B_J$ is a disjoint collection of open sets to estimate $J$:

$$\begin{aligned}
J \lesssim 2^{-k_1 n_1} \sum_{j=1}^{J} |B_j| = 2^{-k_1 n_1} \left|\bigcup_{j=1}^{J} B_j\right| &\leq 2^{-k_1 n_1} |B(0, R + 2^{-k_1+3}) \backslash B(0, R - 2^{-k_1+3})| \\
&\lesssim 2^{-k_1(n_1+1)} R^{n_1 - 1}.
\end{aligned}$$

Note that each $B_j \subset B(0, R + 2^{-k_1+3}) \backslash B(0, R - 2^{-k_1+3})$ since $c_j \in \operatorname{supp}(\lambda_{R,k_1}) \subset B(0, R + 2^{-k_1+3}) \backslash B(0, R - 2^{-k_1+3})$ and each $B_j$ has radius $2^{-k_1}$. Therefore

$$\begin{aligned}
&\left|\left\langle M_{\tilde{b}} T M_b(s_{k_1}(\cdot, y_1) \otimes s_{k_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_2, \cdot)\right\rangle\right| \\
&\qquad \lesssim A_{k_1,m} \frac{2^{-k_1(2n_1+\gamma)} 2^{k_2 n_2}}{R^{n_1}} 2^{-k_1(n_1+1)} R^{n_1-1} = A_{k_1,m} \frac{2^{-k_1(n_1-1)} 2^{k_2 n_2}}{R},
\end{aligned}$$

which tends to zero as $R \to \infty$. This completes the proof for the first case.

Case 2: ($|x_2 - t_2| > 2^{-k_2+2}$) Since $\lambda_{R,k_1}$ and $s_{k_1}(\cdot, y_1)$ have disjoint support, we can use the full kernel representation for $T$ to compute

$$\begin{aligned}
&\left|\left\langle M_{\tilde{b}} T M_b(s_{k_1}^{b_1}(\cdot, y_1) \otimes s_{k_2}^{b_2}(\cdot, y_2)), \lambda_{R,k_1} \otimes d_{k_2}^{\tilde{b}_2}(x_1, \cdot)\right\rangle\right| \\
&\quad = \left| \iint_{\mathbb{R}^{2n}} K(u, v) s_{k_1}^{b_1}(v_1, y_1) s_{k_2}^{b_2}(v_2, y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2, u_2) \tilde{b}(u) b(v) du\, dv \right| \\
&\quad \lesssim \iint_{\mathbb{R}^{2n}} \frac{1}{|u_1 - v_1|^{n_1} |u_2 - v_2|^{n_2}} |s_{k_1}^{b_1}(v_1, y_1) s_{k_2}^{b_2}(v_2, y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2, u_2)| du\, dv
\end{aligned}$$

$$\lesssim \iint_{\mathbb{R}^{2n}} \frac{2^{k_2 n_2}}{(|u_1|-|t_1|-|t_1-v_1|)^{n_1}} |s_{k_1}^{b_1}(v_1,y_1) s_{k_2}^{b_2}(v_2,y_2) \lambda_{R,k_1}(u_1) d_{k_2}^{\tilde{b}_2}(x_2,u_2)| du\, dv$$
$$\lesssim 2^{k_2 n_2} R^{-n_1} \int_{\mathbb{R}^{n_1}} |\lambda_{R,s_1}(u_1)| du_1 \lesssim 2^{k_2 n_2} 2^{-k_1} R^{-1},$$

which again tends to zero as $R \to \infty$. Therefore $\theta_{\vec{k}}$ has integral zero in $x_1$, and a similar argument proves that it has integral zero in $x_2$ as well. □

By symmetry, it follows that each of the following define collections of biparameter Littlewood-Paley-Stein operators:

$$\theta_{\vec{k}}^2(x,y) = \left\langle M_{\tilde{b}} T M_b (s_{k_1}^{b_1}(\cdot,y_1) \otimes d_{k_2}^{b_2}(\cdot,y_2)), d_{k_1}^{\tilde{b}_1}(x_1,\cdot) \otimes s_{k_2}^{\tilde{b}_2}(x_2,\cdot) \right\rangle,$$
$$\theta_{\vec{k}}^3(x,y) = \left\langle M_{\tilde{b}} T M_b (d_{k_1}^{b_1}(\cdot,y_1) \otimes s_{k_2}^{b_2}(\cdot,y_2)), s_{k_1}^{\tilde{b}_1}(x_1,\cdot) \otimes d_{k_2}^{\tilde{b}_2}(x_2,\cdot) \right\rangle, \quad \text{and}$$
$$\theta_{\vec{k}}^4(x,y) = \left\langle M_{\tilde{b}} T M_b (d_{k_1}^{b_1}(\cdot,y_1) \otimes d_{k_2}^{b_2}(\cdot,y_2)), s_{k_1}^{\tilde{b}_1}(x_1,\cdot) \otimes s_{k_2}^{\tilde{b}_2}(x_2,\cdot) \right\rangle.$$

Furthermore, these kernels satisfy

$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}^2(x,y) \tilde{b}_1(x_1) dx_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}^2(x,y) b_2(y_2) dy_2 = 0,$$
$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}^2(x,y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}^2(x,y) \tilde{b}_2(x_2) dx_2 = 0, \quad \text{and}$$
$$\int_{\mathbb{R}^{n_1}} \theta_{\vec{k}}^2(x,y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta_{\vec{k}}^2(x,y) b_2(y_2) dy_2 = 0.$$

Finally, we are able to prove Theorem 5.7

*Proof.* Let $S_{\vec{k}}^b = S_{k_1}^{b_1} \otimes S_{k_2}^{b_2}$ and $S_{\vec{k}}^{\tilde{b}} = S_{k_1}^{\tilde{b}_1} S_{k_2}^{\tilde{b}_2}$, where $S_{k_1}^{b_1}$, $S_{k_2}^{b_2}$, $S_{k_1}^{\tilde{b}_1}$, and $S_{k_2}^{\tilde{b}_2}$ be the approximations to identity with respect to $b_1$ and $b_2$ respectively constructed in (4.1). Also define $D_{k_1}^{b_1} = S_{k_1+1}^{b_1} - S_{k_1}^{b_1}$, $D_{k_2}^{b_2} = S_{k_2+1}^{b_2} - S_{k_2}^{b_2}$, $D_{k_1}^{\tilde{b}_1} = S_{k_1+1}^{\tilde{b}_1} - S_{k_1}^{\tilde{b}_1}$, $D_{k_2}^{\tilde{b}_2} = S_{k_2+1}^{\tilde{b}_2} - S_{k_2}^{\tilde{b}_2}$, $D_{\vec{k}}^b = D_{k_1}^{b_1} D_{k_2}^{b_2}$, and $D_{\vec{k}}^{\tilde{b}} = D_{k_1}^{\tilde{b}_1} D_{k_2}^{\tilde{b}_2}$. It follows that $M_{b_j} S_{k_j}^{b_j} M_{b_j} f_j \to b_j f_j$ and $M_{b_j} S_{-k_j}^{b_j} M_{b_j} f_j \to 0$ in $b_j C_0^\delta(\mathbb{R}^{n_j})$ as $k_j \to \infty$ for $j = 1, 2$, whenever $f_j \in C_0^{0,1}(\mathbb{R}^{n_j})$ and

$$\int_{\mathbb{R}^{n_j}} f_j(x_j) b_j(x_j) dx_j = 0.$$

This was proved originally in [DJS85], and the proof is also available in [Har13a]. It follows that $M_{b_j} S_{k_j}^{b_j} M_b f \to bf$ and $M_{b_j} S_{-k_j}^{b_j} M_b f_j \to 0$ in $bC_0^\delta(\mathbb{R}^n)$ as $k_j \to \infty$ for $j = 1, 2$, whenever $f \in C_0^{0,1}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^{n_1}} f(x) b(x) dx_1 = \int_{\mathbb{R}^{n_2}} f(x) b(x) dx_2 = 0.$$

Let $f, g \in C_0^{0,1}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^{n_1}} f(x)b(x)dx_1 = \int_{\mathbb{R}^{n_2}} f(x)b(x)dx_2 = \int_{\mathbb{R}^{n_1}} g(x)\tilde{b}(x)dx_1 = \int_{\mathbb{R}^{n_2}} g(x)\tilde{b}(x)dx_2 = 0.$$

Then by the continuity of $T$ from $bC_0^\delta(\mathbb{R}^n)$ into $(\tilde{b}C_0^\delta(\mathbb{R}^n))'$,

$$\begin{aligned}
\langle M_{\tilde{b}}TM_b f, g\rangle &= \lim_{N_2\to\infty} \left\langle M_{\tilde{b}_2}TM_{b_2}S^{b_2}_{N_2}M_b f, S^{\tilde{b}}_{N_2}M_b g\right\rangle - \left\langle M_{\tilde{b}_2}TM_{b_2}S^{b_2}_{-N_2}M_b f, S^{\tilde{b}_2}_{-N_2}M_{\tilde{b}} g\right\rangle \\
&= \sum_{k_2\in\mathbb{Z}} \left\langle M_{\tilde{b}_2}TM_{b_2}S^{b_2}_{k_2+1}M_b f, D^{\tilde{b}_2}_{k_2}M_{\tilde{b}} g\right\rangle - \left\langle M_{\tilde{b}_2}TM_{b_2}D^{b_2}_{k_2}M_b f, S^{\tilde{b}_2}_{k_2}M_{\tilde{b}} g\right\rangle \\
&= \sum_{k_2\in\mathbb{Z}} \lim_{N_1\to\infty} \left\langle M_{\tilde{b}}TM_b S^{b_2}_{k_2+1}S^{b_1}_{N_1}M_b f, D^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{N_1}M_{\tilde{b}} g\right\rangle + \left\langle M_{\tilde{b}}TM_b D^{b_2}_{k_2}S^{b_1}_{N_1}M_b f, S^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{N_1}M_{\tilde{b}} g\right\rangle \\
&\quad - \left\langle M_{\tilde{b}}TM_b S^{b_2}_{k_2+1}S^{b_1}_{-N_1}M_b f, D^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{-N_1}M_{\tilde{b}} g\right\rangle - \left\langle M_{\tilde{b}}TM_b D^{b_2}_{k_2}S^{b_1}_{-N_1}M_b f, S^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{-N_1}M_{\tilde{b}} g\right\rangle \\
&= \sum_{k_1,k_2\in\mathbb{Z}} \left\langle M_{\tilde{b}}TM_b S^{b_2}_{k_2+1}S^{b_1}_{k_1+1}M_b f, D^{\tilde{b}_2}_{k_2}D^{\tilde{b}_1}_{k_1}M_{\tilde{b}} g\right\rangle + \left\langle M_{\tilde{b}}TM_b D^{b_2}_{k_2}S^{b_1}_{k_1+1}M_b f, S^{\tilde{b}_2}_{k_2}D^{\tilde{b}_1}_{k_1}M_{\tilde{b}} g\right\rangle \\
&\quad + \left\langle M_{\tilde{b}}TM_b S^{b_2}_{k_2+1}D^{\tilde{b}_1}_{k_1}M_b f, D^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{k_1}M_{\tilde{b}} g\right\rangle + \left\langle M_{\tilde{b}}TM_b D^{b_2}_{k_2}D^{b_1}_{k_1}M_b f, S^{\tilde{b}_2}_{k_2}S^{\tilde{b}_1}_{k_1}M_{\tilde{b}} g\right\rangle \\
&= \sum_{k_1,k_2\in\mathbb{Z}} \sum_{j=1}^{4} \left\langle \Theta^j_{\vec{k}}M_b f, M_{\tilde{b}} g\right\rangle
\end{aligned}$$

where $\Theta_j$ for $j = 1, 2, 3, 4$ are defined as follows with their respective kernels

$$\begin{aligned}
\Theta^1_{\vec{k}} &= D^{\tilde{b}}_{\vec{k}}M_{\tilde{b}}TM_b S_{\vec{k}+1}; \\
&\qquad \theta^1_{\vec{k}}(x,y) = \left\langle M_{\tilde{b}}TM_b(s^{b_1}_{k_1+1}(\cdot,y_1)\otimes s^{b_2}_{k_2+1}(\cdot,y_2)), d^{\tilde{b}_1}_{k_1}(x_1,\cdot)\otimes d^{\tilde{b}_2}_{k_2}(x_2,\cdot)\right\rangle, \\
\Theta^2_{\vec{k}} &= D^{\tilde{b}_1}_{k_1}S^{\tilde{b}_2}_{k_2}M_{\tilde{b}}TM_b S^{b_1}_{k_1+1}D^{b_2}_{k_2}; \\
&\qquad \theta^2_{\vec{k}}(x,y) = \left\langle M_{\tilde{b}}TM_b(s^{b_1}_{k_1+1}(\cdot,y_1)\otimes d^{b_2}_{k_2}(\cdot,y_2)), d^{\tilde{b}_1}_{k_1}(x_1,\cdot)\otimes s^{\tilde{b}_2}_{k_2}(x_2,\cdot)\right\rangle, \\
\Theta^3_{\vec{k}} &= S^{\tilde{b}_1}_{k_1}D^{\tilde{b}_2}_{k_2}M_{\tilde{b}}TM_b D^{b_1}_{k_1}S^{b_2}_{k_2+1}; \\
&\qquad \theta^3_{\vec{s}}(x,y) = \left\langle M_{\tilde{b}}TM_b(d^{b_1}_{k_1}(\cdot,y_1)\otimes s^{b_2}_{k_2+1}(\cdot,y_2)), s^{\tilde{b}_1}_{k_1}(x_1,\cdot)\otimes d^{\tilde{b}_2}_{k_2}(x_2,\cdot)\right\rangle, \\
\Theta^4_{\vec{k}} &= S^{\tilde{b}}_{\vec{k}}M_{\tilde{b}}TM_b D^b_{\vec{k}}; \\
&\qquad \theta^4_{\vec{s}}(x,y) = \left\langle M_{\tilde{b}}TM_b(d^{b_1}_{k_1}(\cdot,y_1)\otimes d^{b_2}_{k_2}(\cdot,y_2)), s^{\tilde{b}_1}_{k_1}(x_1,\cdot)\otimes s^{\tilde{b}_2}_{k_2}(x_2,\cdot)\right\rangle.
\end{aligned}$$

By Lemma 5.8, $\theta^1_{\vec{s}}$ satisfies (4.5)-(4.9) and

$$\int_{\mathbb{R}^{n_1}} \theta^1_{\vec{k}}(x,y)b_1(x_1)dx_1 = \int_{\mathbb{R}^{n_2}} \theta^1_{\vec{k}}(x,y)b_2(x_2)dx_2 = 0.$$

By the biparameter $Tb = T^*b = 0$ assumption on $T$, we also have

$$\int_{\mathbb{R}^{n_1}} \theta^1_{\vec{k}}(x,y) b_1(y_1) dy_1 = \int_{\mathbb{R}^{n_2}} \theta^1_{\vec{k}}(x,y) b_2(y_2) dy_2 = 0.$$

Then by Theorem (4.16),

$$\sum_{\vec{k} \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^n} \Theta^1_{\vec{k}} f(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

The same holds for $\Theta^j_{\vec{s}}$ when $j = 2, 3, 4$, and so it follows that

$$|\langle Tf, g\rangle| \leq \sum_{j=1}^{4} \sum_{\vec{k} \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^n} \Theta^j_{\vec{k}} f(x) g(x) dx \right| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Therefore, by density, $T$ can be extended to a bounded operator on $L^p$ for $1 < p < \infty$. $\square$

# Chapter 6

# An extension problem

In this chapter, we apply our reduced biparameter $Tb$ theorem to solve the holomorphic extension problem we presented in the Introduction.

The problem we are dealing with can be stated as follows. Given an appropriate Lipschitz boundary surface $\Gamma = \Gamma_1 \times \Gamma_2 \subset \mathbb{C}^2$ and a function $g : \Gamma \to \mathbb{C}$, there is a function $G$ that is, holomorphic on $(\mathbb{C}\backslash\Gamma_1) \times (\mathbb{C}\backslash\Gamma_2)$ satisfying

$$g(z) = g_{++}(z) - g_{+-}(z) - g_{-+}(z) + g_{--}(z), \tag{6.1}$$

for $z = (z_1, z_2) \in \Gamma$, where

$$g_{++}(z) = \lim_{t_1,t_2\to 0^+} G(z_1 + it_1, z_2 + it_2), \qquad g_{+-}(z) = \lim_{t_1,t_2\to 0^+} G(z_1 + it_1, z_2 - it_2), \tag{6.2}$$
$$g_{-+}(z) = \lim_{t_1,t_2\to 0^+} G(z_1 - it_1, z_2 + it_2), \qquad g_{--}(z) = \lim_{t_1,t_2\to 0^+} G(z_1 - it_1, z_2 - it_2).$$

For now we leave the sense in which (6.1) holds and the sense that the limits in (6.2) hold unspecified, but these things will be defined later in this section.

Now we define what is our Lipschitz boundary surface $\Gamma$. Let $L_1, L_2 : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions with Lipschitz constants $\lambda_1$ and $\lambda_2$ respectively. Define

$$\gamma_1(x_1) = x_1 + iL_1(x_1);$$
$$\gamma_2(x_2) = x_2 + iL_2(x_2);$$
$$\gamma(x) = (\gamma_1(x_1), \gamma_2(x_2)),$$

for $x = (x_1, x_2)$ in $\mathbb{R}^2$.

**Definition 6.1.** We say that

$$\Gamma := \Gamma_1 \times \Gamma_2 = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$$

is a product Lipschitz surface with small Lipschitz constants if the Lipshitz constants $\lambda_1$ and $\lambda_2$ of $L_1$ and $L_2$ are both smaller than 1.

**Definition 6.2.** The upper half space associated to $\Gamma$ is defined $\mathbb{H}_{\Gamma_1} \times \mathbb{H}_{\Gamma_2}$ where

$$\mathbb{H}_{\Gamma_j} = \{\gamma_j(x_j) + it_j : x_j \in \mathbb{Z}, t_j > 0\}.$$

We also define $L^p(\Gamma)$ for a product Lipschitz surface $\Gamma$ as follows: given a product Lipschitz surface $\Gamma = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$, let $L^p(\Gamma)$ be the collection of measurable functions $g : \Gamma \to \mathbb{C}$ such that

$$\|g\|_{L^p(\Gamma)}^p = \int_{\mathbb{R}^2} |g(\gamma(x))|^p |\gamma_1'(x_1)\gamma_2'(x_2)| dx_1\, dx_2 < \infty.$$

Our goal is to prove the following theorem.

**Theorem 6.3.** *Let $\Gamma$ be a product Lipschitz surface with small Lipschitz constants in $\mathbb{C}^2$ defined by $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^2 \to \mathbb{C}^2$. Assume that*

$$\lim_{|x_1|\to\infty} \frac{\gamma_1(x_1)}{x_1} = c_1 \qquad \text{and} \qquad \lim_{|x_2|\to\infty} \frac{\gamma_2(x_2)}{x_2} = c_2$$

*for some $c_1, c_2 \in \mathbb{C}$. If $g \in L^p(\Gamma)$ for some $1 < p < \infty$, then there exists a function $G : (\mathbb{C}\backslash\Gamma_1) \times (\mathbb{C}\backslash\Gamma_2) \to \mathbb{C}$ that is a holomorphic extension of $g$, where* (6.1) *and the limits in* (6.2) *hold in $L^p(\Gamma)$.*

To prove Theorem 6.3, we take an approach related to the ones in [MM77, Cha79, Fef79, GS79, Ste79, CF80], which uses the boundedness of biparameter and partial Hilbert transforms. In place of the Hilbert transforms, we define biparameter and partial Cauchy integral transforms for $z = (z_1, z_2) \in \Gamma$ and appropriate $g : \Gamma \to \mathbb{C}$,

$$\mathcal{C}_\Gamma g(z) = \lim_{t_1,t_2\to 0^+} \mathcal{C}_t g(z); \qquad \mathcal{C}_t g(z) = \frac{1}{(2\pi i)^2}\int_\Gamma \frac{z_1-\xi_1}{(z_1-\xi_1)^2+t_1^2}\frac{z_2-\xi_2}{(z_2-\xi_2)^2+t_2^2} g(\xi)d\xi,$$

$$\mathcal{C}_\Gamma^{p1} g(z) = \lim_{t_1,t_2\to 0^+} \mathcal{C}_t^{p1} g(z); \qquad \mathcal{C}_t^{p1} g(z) = \frac{1}{(2\pi i)^2}\int_\Gamma \frac{z_1-\xi_1}{(z_1-\xi_1)^2+t_1^2}\frac{t_2}{(z_2-\xi_2)^2+t_2^2} g(\xi)d\xi,$$

$$\mathcal{C}_\Gamma^{p2} g(z) = \lim_{t_1,t_2\to 0^+} \mathcal{C}_t^{p2} g(z); \qquad \mathcal{C}_t^{p2} g(z) = \frac{1}{(2\pi i)^2}\int_\Gamma \frac{t_1}{(z_1-\xi_1)^2+t_1^2}\frac{z_2-\xi_2}{(z_2-\xi_2)^2+t_2^2} g(\xi)d\xi.$$

**Remark 6.4.** The limits defining $\mathcal{C}_\Gamma$, $\mathcal{C}_\Gamma^{p1}$, and $\mathcal{C}_\Gamma^{p2}$ are taken in the following pointwise sense: given $c \in \mathbb{C}$ and $c_t \in \mathbb{C}$ for $t = (t_1, t_2) \in (0, \infty)^2$, we say $c_t \to c$ as $t_1, t_2 \to 0^+$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < t_1, t_2 < \delta$ implies $|c_t - c| < \epsilon$. We also define convergence in normed spaces as $t_1, t_2 \to 0^+$: given a normed function space $X$, $F \in X$, and $F_t \in X$ for $t = (t_1, t_2) \in (0, \infty)^2$, we say $F_t \to F$ as $t_1, t_2 \to 0^+$ if $\|F_t - F\|_X \to 0$ as $t_1, t_2 \to 0^+$.

The operators $\mathcal{C}_\Gamma g$, $\mathcal{C}_\Gamma^{p1} g$, and $\mathcal{C}_\Gamma^{p2} g$ are defined initially as pointwise limits, and we will prove later that these limits hold in $L^p(\Gamma)$ as well for $1 < p < \infty$ and appropriate $g$. A crucial part of the proof of these convergence results is the $L^p(\Gamma)$ boundedness of $\mathcal{C}_\Gamma$, $\mathcal{C}_\Gamma^{p1}$, and $\mathcal{C}_\Gamma^{p2}$, which we state now in Theorem 6.5.

**Theorem 6.5.** *Let $\Gamma$ be a product Lipschitz surface with small Lipschitz constant in $\mathbb{C}^2$ defined by $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^2 \to \mathbb{C}^2$. Assume that*

$$\lim_{|x_1| \to \infty} \frac{\gamma_1(x_1)}{x_1} = c_1 \qquad \text{and} \qquad \lim_{|x_2| \to \infty} \frac{\gamma_2(x_2)}{x_2} = c_2$$

*for some $c_1, c_2 \in \mathbb{C}$. Then operators $\mathcal{C}_\Gamma$, $\mathcal{C}_\Gamma^{p1}$, and $\mathcal{C}_\Gamma^{p2}$ are bounded on $L^p(\Gamma)$ and for $g \in L^p(\Gamma)$*

$$\lim_{t_1, t_2 \to 0^+} \mathcal{C}_t g = \mathcal{C}_\Gamma g, \qquad \lim_{t_1, t_2 \to 0^+} \mathcal{C}_t^{p1} g = \mathcal{C}_\Gamma^{p1} g, \qquad \text{and} \qquad \lim_{t_1, t_2 \to 0^+} \mathcal{C}_t^{p2} g = \mathcal{C}_\Gamma^{p2} g$$

*in $L^p(\Gamma)$ when $1 < p < \infty$.*

We will prove Theorem 6.5 in Section 6.3 using our biparameter reduced $Tb$ theorem in the same spirit David-Journé-Semmes used their $Tb$ theorem to prove $L^p$ bounds for Cauchy integral transform in [DJS85].

## 6.1 The holomorphic extension

In this section we prove Theorem 6.3 assuming the validity of Theorem 6.5. The proof of the latter theorem will be provided in later sections.

Let $\Gamma$ a product Lipschitz surface with small Lipschitz constants $\lambda_1$ and $\lambda_2$ as defined in Definition 6.1. It follows that

$$0 < 1 - \lambda_j^2 \leq \frac{(x_j - y_j)^2 - (L_j(x_j) - L_j(y_j))^2}{(x_j - y_j)^2} = \frac{|\operatorname{Re}[(\gamma_j(x_j) - \gamma_j(y_j))^2]|}{(x_j - y_j)^2} \leq 2,$$

therefore, $\operatorname{Re}[(\gamma_j(x_j) - \gamma_j(y_j))^2]$ and $(x_j - y_j)^2$ are comparable with constants only depending on the Lipschitz constants of $\gamma$, not on $x_j$ and $y_j$.

We also remark that the norms of $g$ and $g \circ \gamma$ are comparable in the following sense: for any $g \in L^p(\Gamma)$,

$$\begin{aligned} \|g \circ \gamma\|^p_{L^p(\mathbb{R}^2)} &\le \|(\gamma_1')^{-1}\|_{L^\infty(\mathbb{R})} \|(\gamma_2')^{-1}\|_{L^\infty(\mathbb{R})} \|g\|^p_{L^p(\Gamma)} \le \|g\|^p_{L^p(\Gamma)} \\ &\le \|\gamma_1'\|_{L^\infty(\mathbb{R})} \|\gamma_2'\|_{L^\infty(\mathbb{R})} \|g \circ \gamma\|^p_{L^p(\mathbb{R}^2)} \le 2\|g \circ \gamma\|^p_{L^p(\mathbb{R}^2)}. \end{aligned} \tag{6.3}$$

Note that since $\mathrm{Re}[\gamma_j'(x_j)] = 1$ for all $x_j \in \mathbb{R}$, we have $|\gamma_j'(x_j)| \ge \mathrm{Re}[\gamma_j'(x_j)] = 1$ for all $x_j \in \mathbb{R}$.

Now, given a function $g : \Gamma \to \mathbb{C}$, we define for $\omega = (\omega_{t_1}, \omega_{t_2}) = (z_1 + it_1, z_2 + it_2) \in (\mathbb{C}\backslash\Gamma_1) \times (\mathbb{C}\backslash\Gamma_2)$ where $(z_1, z_2) \in \Gamma$ and $t_1, t_2 \neq 0$,

$$G(\omega_{t_1}, \omega_{t_2}) = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{g(\xi)d\xi}{(\xi_1 - \omega_{t_1})(\xi_2 - \omega_{t_2})}. \tag{6.4}$$

It follows that

$$\begin{aligned} G(\omega_{t_1}, \omega_{t_2}) = \frac{1}{4} \int_\Gamma \Big( & p_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) - q_{t_1}(z_1 - \xi_1) q_{t_2}(z_2 - \xi_2) \\ & + i q_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) + i p_{t_1}(z_1 - \xi_1) q_{t_2}(z_2 - \xi_2) \Big) g(\xi) d\xi, \end{aligned}$$

where

$$p_{t_j}(\omega_j) = \frac{1}{\pi} \frac{t_j}{\omega_j^2 + t_j^2} \qquad \text{and} \qquad q_{t_j}(\omega_j) = \frac{1}{\pi} \frac{\omega_j}{\omega_j^2 + t_j^2} \qquad \text{for } \omega_j \in \mathbb{C}.$$

Finally, for $t = (t_1, t_2) \in (0, \infty)^2$, $g_1 : \Gamma_1 \to \mathbb{C}$, $g_2 : \Gamma_2 \to \mathbb{C}$, $g : \Gamma \to \mathbb{C}$, and $z = (z_1, z_2) \in \Gamma$, we define the operators

$$P_{t_1} g_1(z_1) = \int_{\Gamma_1} p_{t_1}(z_1 - \xi_1) g_1(\xi_1) d\xi_1, \qquad P_{t_2} g_2(z_2) = \int_{\Gamma_2} p_{t_2}(z_2 - \xi_2) g_2(\xi_2) d\xi_2,$$

$$\text{and} \quad P_t g(z) = \int_\Gamma p_{t_1}(z_1 - \xi_1) p_{t_2}(z_2 - \xi_2) g(\xi) d\xi.$$

We use the indices of $P_{t_1}$, $P_{t_2}$, and $P_t$ to identify the operators.

**Remark 6.6.** Note that $P_t g = P_{t_1} P_{t_2} g$ for $g : \Gamma \to \mathbb{C}$, where we use the notation

$$P_{t_1} g(z) = \int_{\Gamma_1} p_{t_1}(z_1 - \xi_1) g(\xi_1, z_2) d\xi_1 \quad \text{and} \quad P_{t_2} g(z) = \int_{\Gamma_2} p_{t_2}(z_2 - \xi_2) g(z_1, \xi_2) d\xi_2$$

This is an abuse of notation, but it is clear in context which operator is being used.

We start with a lemma about the convergence of the operators $P_{t_1}g$, $P_{t_2}g$, and $P_tg$ for $g \in L^p(\Gamma)$.

**Lemma 6.7.** *Let $\Gamma$ be a product Lipschitz surface with small Lipschitz constants in $\mathbb{C}^2$ and $g \in L^p(\Gamma)$ for some $1 < p < \infty$. Then*

$$\lim_{t_1 \to 0^+} P_{t_1} g = g, \qquad \lim_{t_2 \to 0^+} P_{t_2} g = g, \qquad \textit{and} \qquad \lim_{t_1, t_2 \to 0^+} P_t g = g,$$

*where each limit holds in the topology of $L^p(\Gamma)$ and pointwise almost everywhere on $\Gamma$.*

*Proof.* We first verify that $P_{t_j}1 = 1$ for each $j = 1, 2$. Let $R > 0$ and

$$E_R = \{z_j \in \Gamma_j : |z_j| \le R\} \cup \{z_j \in \mathbb{C} : |z_j| = R, \ \operatorname{Im} z_j > L_j(\operatorname{Re}(z_j))\}.$$

$E_R$ is a closed, and for $R$ sufficiently large, it defines the boundary of an open, simply connected region $U_R = \{z_j \in \mathbb{C} : |z_j| < R, \ \operatorname{Im}(z_j) > L_j(\operatorname{Re}(z_j))\}$. For $z_j \in \Gamma_j$,$t_j > 0$, and $R$ sufficiently large, it follows that $z_j + it_j \in U_R$ and $z_j - it_j \notin U_R$. Then

$$\frac{t_j}{\xi_j - (z_j - it_j)}$$

is holomorphic in $\xi_j$ on $U_R$ for such $z_j$, $t_j$, and $R$. Using the decay of $p_{t_j}$ and a residue theorem, it follows that

$$\begin{aligned} \int_{\Gamma_j} p_{t_j}(z_j - \xi_j) d\xi_j &= \lim_{R \to \infty} \frac{1}{\pi} \int_{E_R} \frac{t_j}{(\xi_j - (z_j + it_j))(\xi_j - (z_j - it_j))} d\xi_j \\ &= \lim_{R \to \infty} \frac{1}{\pi} \frac{2\pi i \, t_j}{(z_j + it_j) - (z_j - it_j)} = 1. \end{aligned}$$

Consider the following parameterized versions of $P_t$, $P_{t_1}$, and $P_{t_2}$: for $f : \mathbb{R}^2 \to \mathbb{C}$ and $x \in \mathbb{R}^2$

$$\begin{aligned} \widetilde{P}_{t_1} f(x) &= \int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) \gamma_1'(y_1) f(y_1, x_2) dy_1, \\ \widetilde{P}_{t_2} f(x) &= \int_{\mathbb{R}} p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) \gamma_2'(y_2) f(x_1, y_2) dy_2, \text{ and} \\ \widetilde{P}_t f(x) = \widetilde{P}_{t_1} \widetilde{P}_{t_2} f(x) &= \int_{\mathbb{R}^2} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) p_{t_2}(\gamma_2(x_2) - \gamma_1(y_2)) \gamma_1'(y_1) \gamma_2'(y_2) f(y) dy. \end{aligned}$$

The kernels of $\widetilde{P}_{t_1}$, $\widetilde{P}_{t_2}$, and $\widetilde{P}_t$ are

$$\widetilde{p}_{t_1}(x_1, y_1) = p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) \gamma_1'(y_1), \qquad \widetilde{p}_{t_2}(x_2, y_2) = p_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) \gamma_2'(y_2),$$

and $\quad \widetilde{p}_t(x,y)=\widetilde{p}_{t_1}(x_1,y_1)\widetilde{p}_{t_2}(x_2,y_2), \qquad$ respectively.

Note that $\widetilde{P}_{t_j}1(x_j)=P_{t_j}1(\gamma_j(x_j))=1$ for all $x_j\in\mathbb{R}$. Also, since the Lipschitz constant of $L_1$ and $L_2$ are small, it follows that

$$|\widetilde{p}_{t_j}(x_j,y_j)|=\frac{1}{\pi}\left|\frac{t_j|\gamma_j'(y_j)|}{t_j^2+(\gamma_j(x_j)-\gamma_j(y_j))^2}\right|\leq\frac{t_j}{t_j^2+(1-\lambda_j^2)(x_j-y_j)^2}\lesssim\frac{t_j^{-1}}{(1+t_j^{-1}|x_j-y_j|)^2}.$$

Then $\{\widetilde{p}_{t_j}:t_j>0\}$ forms an approximation to the identity on $\mathbb{R}$ for each $j=1,2$. Fix $g\in L^p(\Gamma)$ for some $1<p<\infty$. It follows that $g\circ\gamma\in L^p(\mathbb{R}^2)$, and hence that $g\circ\gamma(\cdot,x_2)\in L^p(\mathbb{R})$ for almost every $x_2\in\mathbb{R}$. Now fix $x_2\in\mathbb{R}$ outside of an appropriate exceptional set, so that $\|g\circ\gamma(\cdot,x_2)\|_{L^p(\mathbb{R})}<\infty$. It follows that $g\circ\gamma(\cdot,x_2)\in L^p(\mathbb{R})$ and hence that

$$\lim_{t_1\to0^+}\|\widetilde{P}_{t_1}(g\circ\gamma)(\cdot,x_2)-g\circ\gamma(\cdot,x_2)\|_{L^p(\mathbb{R})}=0.$$

By dominated convergence, it also follows that

$$\lim_{t_1\to0^+}\|\widetilde{P}_{t_1}(g\circ\gamma)-g\circ\gamma\|_{L^p(\mathbb{R}^2)}^p=\int_{\mathbb{R}}\lim_{t_1\to0^+}\|\widetilde{P}_{t_1}(g\circ\gamma)(\cdot,x_2)-g\circ\gamma(\cdot,x_2)\|_{L^p(\mathbb{R})}^p dx_2=0.$$

Therefore $\widetilde{P}_{t_1}(g\circ\gamma)\to g\circ\gamma$ in $L^p(\mathbb{R}^2)$, and in light of (6.3) it easily follows that $P_{t_1}g\to g$ in $L^p(\Gamma)$. By symmetry, it follows that $P_{t_2}g\to g$ in $L^p(\Gamma)$ as well. Now for $g\in L^p(\Gamma)$, we verify that $P_tg\to g$ in $L^p(\Gamma)$ as $t_1,t_2\to0^+$ for $1<p<\infty$, as defined in Remark 6.4. First, define $\mathcal{M}_1$ to be the Hardy-Littlewood maximal function acting on the first variable of a function $f:\mathbb{R}^2\to\mathbb{C}$, i.e.

$$\mathcal{M}_1f(x)=\sup_{I\ni x_1}\frac{1}{|I|}\int_I|f(y_1,x_2)|dy_1,$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$ that contain $x_1$. It is not hard to verify that $\mathcal{M}_1$ is bounded on $L^p(\mathbb{R}^2)$ for $1<p\leq\infty$ and that $|P_{t_1}h(\gamma(x))|\lesssim\mathcal{M}_1(h\circ\gamma)(x)$ uniformly in $t_1>0$ for any $h\in L^p(\Gamma)$. The $L^p(\Gamma)$ convergence of $P_tg$ follows:

$$\begin{aligned}\lim_{t_1,t_2\to0}\|P_tg-g\|_{L^p(\Gamma)}&\leq\lim_{t_1,t_2\to0}\|P_{t_1}(P_{t_2}g-g)\|_{L^p(\Gamma)}+\|P_{t_1}g-g\|_{L^p(\Gamma)}\\&\lesssim\lim_{t_1,t_2\to0}\|\mathcal{M}_1(\widetilde{P}_{t_2}(g\circ\gamma)-g\circ\gamma)\|_{L^p(\mathbb{R}^2)}+\|P_{t_1}g-g\|_{L^p(\Gamma)}\\&\lesssim\lim_{t_2\to0}\|\widetilde{P}_{t_2}(g\circ\gamma)-g\circ\gamma\|_{L^p(\mathbb{R}^2)}+\lim_{t_1\to0}\|P_{t_1}g-g\|_{L^p(\Gamma)}=0.\end{aligned}$$

In the last line, we use that $\widetilde{P}_{t_2}(g\circ\gamma)\to g\circ\gamma$ in $L^p(\mathbb{R}^2)$ and that $P_{t_1}(g\circ\gamma)\to g\circ\gamma$ in $L^p(\mathbb{R}^2)$. This completes the proof of the $L^p(\Gamma)$ convergence properties. Now we prove the pointwise convergence results. For $g \in L^p(\Gamma)$, it follows that $g \circ \gamma(\cdot, x_2) \in L^p(\mathbb{R})$ for almost every $x_2 \in \mathbb{R}$. For a fixed $x_2 \in \mathbb{R}$ outside of an appropriate measure zero set, by the Lebesgue differentiation theorem it follows that

$$\lim_{t_1 \to 0^+} \widetilde{P}_{t_1}(g \circ \gamma)(x_1, x_2) = g(\gamma(x_1, x_2))$$

for almost every $x_1 \in \mathbb{R}$. Hence $\widetilde{P}_{t_1}(g \circ \gamma) \to g \circ \gamma$ as $t_1 \to 0^+$ pointwise almost everywhere in $\mathbb{R}^2$ and hence that $P_{t_1} g \to g$ as $t_1 \to 0^+$ pointwise almost everywhere in $\Gamma$. By symmetry, $\widetilde{P}_{t_2}(g \circ \gamma) \to g \circ \gamma$ as $t_2 \to 0^+$ pointwise almost everywhere in $\mathbb{R}^2$ and hence that $P_{t_2} g \to g$ as $t_2 \to 0^+$ pointwise almost everywhere in $\Gamma$.

Now we verify the pointwise convergence for $P_t g$ on $\Gamma$. Fix $x \in \mathbb{R}^2$ such that $\widetilde{P}_{t_1}(g \circ \gamma)(x) \to g \circ \gamma(x)$ as $t_1 \to 0^+$ and $\|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})} < \infty$, which is true for almost every $x \in \mathbb{R}^2$. Now we bound

$$\begin{aligned} |\widetilde{P}_t(g \circ \gamma)(x) - g \circ \gamma(x)| &\le |\widetilde{P}_{t_1}(\widetilde{P}_{t_2}(g \circ \gamma) - (g \circ \gamma))(x)| + |\widetilde{P}_{t_1}(g \circ \gamma)(x) - (g \circ \gamma)(x)| \\ &\lesssim \int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) |\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - (g \circ \gamma)(y_1, x_2)| dy_1 \\ &\qquad + |\widetilde{P}_{t_1}(g \circ \gamma)(x) - (g \circ \gamma)(x)|. \end{aligned} \tag{6.5}$$

We verify that the first term of (6.5) tends to zero as $t_1, t_2 \to 0^+$: let $\epsilon > 0$. Since $\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) \to (g \circ \gamma)(y_1, x_2)$ pointwise as $t_2 \to 0^+$ for almost every $y_1 \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < t_2 < \delta$ implies $|\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - g \circ \gamma(y_1, x_2)| < \epsilon$ for almost every $y_1 \in \mathbb{R}$ such that $|x_1 - y_1| \le 1$ (recall we have fixed $x_1$ and $x_2$). The selection of $\delta$ does not depend on $y_1$ as long as it is within the compact set defined by $|x_1 - y_1| \le 1$. Now we take $0 < t_1, t_2 < \min(\delta, \epsilon)/(1 + \|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})})$, which is possible since $x \in \mathbb{R}^2$ was selected so that $\|g \circ \gamma(\cdot, x_2)\|_{L^p(\mathbb{R})}$ is finite. Then

$$\begin{aligned} &\int_{\mathbb{R}} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) |\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2) - g \circ \gamma(y_1, x_2)| dy_1 \\ &\quad \lesssim \epsilon \int_{|x_1 - y_1| \le 1} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) dy_1 \\ &\qquad\qquad + \int_{|x_1 - y_1| > 1} \frac{t_1(|\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2)| + |g \circ \gamma(y_1, x_2)|)}{(\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2} dy_1 \\ &\lesssim \epsilon + t_1 \int_{|x_1 - y_1| > 1} \frac{(|\widetilde{P}_{t_2}(g \circ \gamma)(y_1, x_2)| + |g \circ \gamma(y_1, x_2)|)}{(x_1 - y_1)^2} dy_1 \end{aligned}$$

$$\lesssim \epsilon + t_1 \left( \|\widetilde{P}_{t_2}(g\circ\gamma)(\cdot, x_2)\|_{L^p(\mathbb{R})} + \|g\circ\gamma(\cdot,x_2)\|_{L^p(\mathbb{R})} \right) \left( \int_{|x_1-y_1|>1} \frac{dy_1}{(x_1-y_1)^{2p'}} \right)^{\frac{1}{p'}}$$

$$\lesssim \epsilon + t_1 \|g\circ\gamma(\cdot,x_2)\|_{L^p(\mathbb{R})} \lesssim \epsilon.$$

It follows that the first term of (6.5) tends to zero as $t_1, t_2 \to 0^+$ for almost every $x \in \mathbb{R}^2$. The second term in (6.5) also tends to zero as $t_1, t_2 \to 0^+$ since $x$ was chosen so that $\widetilde{P}_{t_1} f(x) \to f(x)$ as $t_1 \to 0^+$. Again using (6.3), it easily follow that $P_t g \to g$ as $t_1, t_2 \to 0^+$ pointwise almost everywhere on $\Gamma$. □

Now we prove Theorem 6.3 assuming Theorem 6.5; we will prove Theorem 6.5 in the next chapter.

*Proof.* Let $1 < p < \infty$, $g \in L^p(\Gamma)$, and define $G$ as in (6.4). Note that $p_{-t_j}(z_j - \xi_j) = -p_{t_j}(z_j - \xi_j)$ and $q_{-t_j}(z_j - \xi_j) = q_{t_j}(z_j - \xi_j)$ for $t_j \neq 0$, $z_j \in \Gamma_j$, and $j = 1, 2$. Then it follows that for $(z_1, z_2) \in \Gamma$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}
G(z_1 + it_1, z_2 + it_2) &= \frac{1}{4}\left(P_t g(z) - \mathcal{C}_t g(z) + i\mathcal{C}_t^{p1} g(z) + i\mathcal{C}_t^{p2} g(z)\right),\\
G(z_1 + it_1, z_2 - it_2) &= \frac{1}{4}\left(-P_t g(z) - \mathcal{C}_t g(z) - i\mathcal{C}_t^{p1} g(z) + i\mathcal{C}_t^{p2} g(z)\right),\\
G(z_1 - it_1, z_2 + it_2) &= \frac{1}{4}\left(-P_t g(z) - \mathcal{C}_t g(z) + i\mathcal{C}_t^{p1} g(z) - i\mathcal{C}_t^{p2} g(z)\right),\\
G(z_1 - it_1, z_2 - it_2) &= \frac{1}{4}\left(P_t g(z) - \mathcal{C}_t g(z) - i\mathcal{C}_t^{p1} g(z) - i\mathcal{C}_t^{p2} g(z)\right).
\end{aligned}$$

By Theorem 6.5, it follows that $\mathcal{C}_\Gamma g, \mathcal{C}_\Gamma^{p1} g, \mathcal{C}_\Gamma^{p2} g \in L^p(\Gamma)$ and $\mathcal{C}_t g \to \mathcal{C}_\Gamma g$, $\mathcal{C}_t^{p1} g \to \mathcal{C}_\Gamma^{p1} g$, and $\mathcal{C}_t^{p2} g \to \mathcal{C}_\Gamma^{p2} g$ as $t_1, t_2 \to 0^+$ in $L^p(\Gamma)$. Then for $z = (z_1, z_2) \in \Gamma$

$$\begin{aligned}
g_{++}(z) &= \frac{1}{4}\left(g(z) - \mathcal{C}_\Gamma g(z) + i\mathcal{C}_\Gamma^{p1} g(z) + i\mathcal{C}_\Gamma^{p2} g(z)\right),\\
g_{+-}(z) &= \frac{1}{4}\left(-g(z) - \mathcal{C}_\Gamma g(z) - i\mathcal{C}_\Gamma^{p1} g(z) + i\mathcal{C}_\Gamma^{p2} g(z)\right),\\
g_{-+}(z) &= \frac{1}{4}\left(-g(z) - \mathcal{C}_\Gamma g(z) + i\mathcal{C}_\Gamma^{p1} g(z) - i\mathcal{C}_\Gamma^{p2} g(z)\right), \text{ and}\\
g_{--}(z) &= \frac{1}{4}\left(g(z) - \mathcal{C}_\Gamma g(z) - i\mathcal{C}_\Gamma^{p1} g(z) - i\mathcal{C}_\Gamma^{p2} g(z)\right).
\end{aligned}$$

Then it also follows that (6.1) holds, i. e. $g = g_{++} - g_{+-} - g_{-+} + g_{--}$, as $L^p(\Gamma)$ functions. It is also not hard to verify that $G(\omega_1, \omega_2)$ is holomorphic for $(\omega_1, \omega_2) \in (\mathbb{C}\backslash\Gamma_1) \times (\mathbb{C}\backslash\Gamma_2)$: for $\zeta = (\zeta_1, \zeta_2) \in (\mathbb{C}\backslash\Gamma_1) \times (\mathbb{C}\backslash\Gamma_2)$, we have the following power series representation

$$G(\omega_1, \omega_2) = \frac{1}{(2\pi i)^2} \sum_{k_1, k_2 = 0}^{\infty} \left( \int_\Gamma \frac{g(\xi) d\xi}{(\xi_1 - \zeta_1)^{k_1+1} (\xi_2 - \zeta_2)^{k_2+1}} \right) (\omega_1 - \zeta_1)^{k_1} (\omega_2 - \zeta_2)^{k_2},$$

when $|\omega_1 - \zeta_1| < \text{dist}(\zeta_1, \Gamma_1)/2$ and $|\omega_2 - \zeta_2| < \text{dist}(\zeta_2, \Gamma_2)/2$. Therefore, $G$ is a holomorphic extension of $g$. □

## 6.2 Bounds for the biparameter Cauchy Integral Transform

In this section, we use Theorem 5.7 to prove bounds for $\mathcal{C}_\Gamma$ and its parameterized version $\widetilde{\mathcal{C}}_\Gamma$, which we define now. For appropriate $f : \mathbb{R}^n \to \mathbb{C}$, define

$$\widetilde{\mathcal{C}}_\Gamma M_b f(x) = \lim_{t_1,t_2 \to 0^+} \int_{\mathbb{R}^2} \frac{\gamma_1(x_1) - \gamma_1(y_1)}{(\gamma_1(x_1) - \gamma_1(y_1))^2 + t_1^2} \frac{\gamma_2(x_2) - \gamma_2(y_2)}{(\gamma_2(x_2) - \gamma_2(y_2))^2 + t_2^2} f(y)b(y)dy,$$

where $b(y) = \gamma_1'(y_1)\gamma_2'(y_2)$. We call this the parameterized version of $\mathcal{C}_\Gamma$ since

$$\widetilde{\mathcal{C}}_\Gamma M_b f(x) = \mathcal{C}_\Gamma(f \circ \gamma^{-1})(\gamma(x)),$$

and furthermore, the $L^p(\Gamma)$ bound for $\mathcal{C}_\Gamma$ can be reduced to $L^p(\mathbb{R}^2)$ bounds for $\widetilde{\mathcal{C}}_\Gamma$ via (6.3). It is not hard to see that the kernel of $\widetilde{\mathcal{C}}_\Gamma$ is

$$\frac{1}{(\gamma_1(x_1) - \gamma_1(y_1))(\gamma_1(x_2) - \gamma_1(y_2))},$$

which is a biparameter Calderón-Zygmund kernel. In the next proposition, we prove that $\widetilde{\mathcal{C}}_\Gamma f$ is well-defined for appropriate $f : \mathbb{R}^n \to \mathbb{C}$ and hence $\mathcal{C}_\Gamma g$ is also well defined for appropriate $g : \Gamma \to \mathbb{C}$.

Define the complex log function with the negative real branch cut, that is, for $z \in \mathbb{C}$ we define

$$\log(z) = \ln(|z|) + i\text{Arg}(z),$$

where $\ln : (0, \infty) \to \mathbb{R}$ logarithm base $e$ function with positive real domain and $\text{Arg}(z)$ is the principle argument of $z$ taking values in $(-\pi, \pi]$. Note that for $u \in (0, \infty)$, $\ln(u) = \log(u)$; we use this notation to emphasize when the input is real versus complex.

**Proposition 6.8.** *Assume that $\Gamma$ satisfies the hypotheses of Theorem 6.3. For all $f \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$,*

$$\widetilde{\mathcal{C}}_\Gamma(bf)(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \log\left((\gamma_1(x_1) - \gamma_1(y_1))^2\right) \log\left((\gamma_2(x_2) - \gamma_2(y_2))^2\right) \partial_{y_1}\partial_{y_2} f(y)dy.$$

*Also, for all $f, g \in C_0^\infty(\mathbb{R}^2)$, the pairing $\left\langle \widetilde{C}_\Gamma(bf), bg \right\rangle$ can be realized as any of the following absolutely convergent integrals:*

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\partial_{y_1}\partial_{y_2}f(y)g(x)b(x)dy\,dx,$$
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)f(y)\partial_{x_1}\partial_{x_2}g(x)b(y)dy\,dx,$$
$$-\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\partial_{y_1}f(y)\partial_{x_2}g(x)b(x_1,y_2)dy\,dx,$$
$$-\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\partial_{y_2}f(y)\partial_{x_1}g(x)b(y_1,x_2)dy\,dx.$$

*Proof.* We first note that for $x_j, y_j \in \mathbb{R}$

$$\begin{aligned} q_{t_j}(\gamma_j(x_j)-\gamma_j(y_j))\gamma_j'(y_j) &= \frac{1}{\pi}\frac{\gamma_j(x_j)-\gamma_j(y_j)}{(\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2}\gamma_j'(y_j) \\ &= -\frac{1}{2\pi}\partial_{y_j}\log\left((\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2\right). \end{aligned} \tag{6.6}$$

The derivative of log is well defined here since we defined it with the negative real branch cut, and for all $x_j, y_j \in \mathbb{R}$, we have $\mathrm{Re}\left((\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2\right) \geq t_j^2 > 0$. Now for $f \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, we compute the following pointwise limit

$$\begin{aligned} \widetilde{C}_\Gamma(bf)(x) &= \lim_{t_1,t_2\to 0^+}\int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1)-\gamma_1(y_1))q_{t_2}(\gamma_2(x_2)-\gamma_2(y_2))f(y)\gamma_1'(y_1)\gamma_2'(y_2)dy \\ &= \lim_{t_1,t_2\to 0^+}\int_{\mathbb{R}^2}\left[-\frac{1}{2\pi}\partial_{y_1}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\right] \\ &\qquad \times\left[-\frac{1}{2\pi}\partial_{y_2}\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2+t_2^2\right)\right]f(y)dy \\ &= \lim_{t_1,t_2\to 0^+}\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left[\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\right] \\ &\qquad \times\left[\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2+t_2^2\right)\right]\partial_{y_1}\partial_{y_2}f(y)dy \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\partial_{y_1}\partial_{y_2}f(y)dy. \end{aligned}$$

We integrate by parts in $y_1$ and $y_2$ above, and the boundary terms vanish since $f$ is compactly supported. Also to justify the last inequality, note the following holds for all $x_j \neq y_j$, so that we can apply dominated convergence: the following pointwise limit exists

$$\lim_{t_1,t_2\to 0^+}\log\left(\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2\right)\partial_{y_1}\partial_{y_2}f(y) = \log\left(\gamma_j(x_j)-\gamma_j(y_j))^2\right)\partial_{y_1}\partial_{y_2}f(y),$$

and the integrand is dominated by an integrable function function independent of $t_1, t_2 < 1/4$

$$|\log\left((\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2\right)| \leq |\ln\left(|(\gamma_j(x_j)-\gamma_j(y_j))^2+t_j^2|\right)|+\pi \lesssim |\ln\left((x_j-y_j)^2\right)|+1.$$

Since $\ln(|\cdot|)$ is locally integrable and $f \in C_0^\infty(\mathbb{R}^2)$, we may apply dominated convergence in the last line above. Now take $f, g \in C_0^\infty(\mathbb{R}^2)$, and it immediately follows that

$$\begin{aligned}\left\langle M_b\widetilde{C}_\Gamma M_b f, g\right\rangle = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} &\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\\ &\times \partial_{y_1}\partial_{y_2} f(y)g(x)\gamma_1'(x_1)\gamma_2'(x_2)dy\,dx.\end{aligned}$$

We also have that

$$\begin{aligned}&\left\langle M_b\widetilde{C}_\Gamma M_b f, g\right\rangle\\ &= \lim_{t_1,t_2\to 0^+}\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2+t_2^2\right)\\ &\qquad\times \partial_{y_1}\partial_{y_2} f(y)g(x)\gamma_1'(x_1)\gamma_2'(x_2)dy\,dx\\ &= \lim_{t_1,t_2\to 0^+}\int_{\mathbb{R}^4} q_{t_1}(\gamma_1(x_1)-\gamma_1(y_1))q_{t_2}(\gamma_2(x_2)-\gamma_2(y_2))\\ &\qquad\times f(y)g(x)\gamma_1'(y_1)\gamma_2'(y_2)\gamma_1'(x_1)\gamma_2'(x_2)dy\,dx\\ \\ &= \lim_{t_1,t_2\to 0^+}\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\left[\partial_{x_1}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\right]\\ &\qquad\times\left[-\partial_{y_2}\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2+t_2^2\right)\right] f(y)g(x)\gamma_1'(y_1)\gamma_2'(x_2)dy\,dx\\ &= \lim_{t_1,t_2\to 0^+} -\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\\ &\qquad\times\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2+t_2^2\right)\partial_{y_2} f(y)\partial_{x_1} g(x)\gamma_1'(y_1)\gamma_2'(x_2)dy\,dx\\ &= -\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\\ &\qquad\times\partial_{y_2} f(y)\partial_{x_1} g(x)\gamma_1'(y_1)\gamma_2'(x_2)dy\,dx.\end{aligned}$$

Here we integrate by parts in $x_1$ and $y_2$ and use dominated convergence in essentially the same way as above. A similar argument verifies the other formulas for $\left\langle \widetilde{C}_\Gamma(bf), bg\right\rangle$. □

Note that we cannot use properties of logs to replace the integrand above by

$$4\log\left(\gamma_1(x_1)-\gamma_1(y_1)\right)\log\left(\gamma_2(x_2)-\gamma_2(y_2)\right).$$

This is because $\mathrm{Re}\,[(\gamma_j(x_j)-\gamma(y_j))^2]>0$ for $x_j\neq y_j$, and furthermore recall that we showed that $\mathrm{Re}\,[(\gamma_j(x_j)-\gamma(y_j))^2]\geq(1-\lambda_j^2)(x_j-y_j)^2$. So this term avoids the branch cut of log, but $\mathrm{Re}\,[\gamma_j(x_j)-\gamma(y_j)]$ may change sign, which causes problems with the complex log function.

The next lemma prove an estimate we need later.

**Lemma 6.9.** *Suppose $L_j:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function with small Lipschitz constant $\lambda_j<1$ for $j=1,2$, and define $\gamma(x)=(\gamma_1(x_1),\gamma_2(x_2))=(x_1+iL_1(x_1),x_2+iL_2(x_2))$. If $\psi\in C_0^\infty(\mathbb{R})$ is a normalized bump of any order with mean zero, then*

$$\sup_{u_j\in\mathbb{R},R_j>0}\left|\int_{\mathbb{R}}\log\left((\gamma_j(x_j)-\gamma_j(y_j))^2\right)R_j^{-1}\psi\left(\frac{u_j-y_j}{R_j}\right)dy_j\right|\lesssim 1,$$

*where the suppressed constant does not depend on $\psi$, $x_j$, or $\gamma$.*

*Proof.* Let $\psi\in C_0^\infty(\mathbb{R})$ be a normalized bump with integral zero. For $|u_j-x_j|\leq 2R_j$

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\log\left((\gamma_j(x_j)-\gamma_j(y_j))^2\right)R_j^{-1}\psi\left(\frac{u_j-y_j}{R_j}\right)dy_j\right|\\
&\leq\frac{\|\psi\|_{L^\infty}}{R_j}\int_{u_j-x_j-R_j}^{u_j-x_j+R_j}\left|\log\left((\gamma_j(x_j)-\gamma_j(x_j+y_j))^2\right)-\log(R_j^2)\right|dy_j\\
&\leq\int_{-3}^{3}\left(\ln\left(\frac{|(\gamma_j(x_j)-\gamma_j(x_j+R_jy_j))^2|}{R_j^2}\right)+\pi\right)dy_j\\
&\lesssim\int_{-3}^{3}(1+|\ln(|y_j|)|)dy_j\lesssim 1.
\end{aligned}$$

Here we use that for $|y_j|\leq 3$

$$(1-\lambda_j^2)|y_j|^2\leq\frac{|(\gamma_j(x_j)-\gamma_j(x_j+R_jy_j))^2|}{R_j^2}\leq(1+\lambda_j)^2|y_j|^2\leq 4|y_j|^2\leq 36.$$

Now for $|u_j-x_j|>2R_j$, we estimate as follows

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\log\left((\gamma_j(x_j)-\gamma_j(y_j))^2\right)R_j^{-1}\psi\left(\frac{u_j-y_j}{R_j}\right)dy_j\right|\\
&\leq\frac{\|\psi\|_{L^\infty}}{R_j}\int_{u_j-x_j-R_j}^{u_j-x_j+R_j}\left|\log\left((\gamma_j(x_j)-\gamma_j(x_j+y_j))^2\right)-\log\left((\gamma_j(x_j)-\gamma_j(u_j))^2\right)\right|dy_j\\
&\lesssim 1+\frac{1}{R_j}\int_{u_j-x_j-R_j}^{u_j-x_j+R_j}\left|\ln\left(\frac{|\gamma_j(x_j)-\gamma_j(x_j+y_j)|^2}{|\gamma_j(x_j)-\gamma_j(u_j)|^2}\right)\right|dy_j\\
&\lesssim 1+\frac{1}{R_j}\int_{|y_j-(u_j-x_j)|<R_j}\left|\ln\left(\frac{|y_j|}{|u_j-x_j|}\right)\right|dy_j
\end{aligned}$$

$$\leq 1 + \frac{1}{R_j}\int_{|y_j-(u_j-x_j)|<R_j}\left|\ln\left(\frac{|u_j-x_j|+|y_j-(u_j-x_j)|}{|u_j-x_j|}\right)\right|dy_j$$
$$+\frac{1}{R_j}\int_{|y_j-(u_j-x_j)|<R_j}\left|\ln\left(\frac{|u_j-x_j|}{|u_j-x_j|-|y_j-(u_j-x_j)|}\right)\right|dy_j$$
$$\leq 1 + \frac{1}{R_j}\int_{|y_j-(u_j-x_j)|<R_j}(\ln(3/2)+\ln(2))dy_j \lesssim 1.$$

This completes the proof. □

Now we prove that $\widetilde{\mathcal{C}}_\Gamma$ satisfies the hypotheses of Theorem 5.7.

**Proposition 6.10.** *Assume that $\Gamma$ satisfies the hypotheses of Theorem 6.3. Then, the operator $M_b\widetilde{\mathcal{C}}_\Gamma M_b$ satisfies the weak boundedness and mixed weak boundedness properties, where $b(x)=\gamma_1'(x_1)\gamma_2'(x_2)$ for $x=(x_1,x_2)\in\mathbb{R}^2$.*

*Proof.* Let $\varphi_j,\psi_j\in C_0^\infty$ be normalized bumps, $x\in\mathbb{R}^2$, and $R_1,R_2>0$. Then

$$\left|\left\langle M_b\widetilde{\mathcal{C}}_\Gamma M_b(\varphi_1^{x_1,R_1}\otimes\varphi_2^{x_2,R_2}),\psi_1^{x_1,R_1}\otimes\psi_2^{x_2,R_2}\right\rangle\right|$$
$$=\frac{1}{4\pi^2}\left|\int_{\mathbb{R}^4}\log\left((\gamma_1(u_1)-\gamma_1(v_1))^2\right)\log\left((\gamma_2(u_2)-\gamma_2(v_2))^2\right)\right.$$
$$\left.\times(\varphi_1^{x_1,R_1})'(v_1)(\varphi_2^{x_2,R_2})'(v_2)\psi_1^{x_1,R_1}(u_1)\psi_2^{x_2,R_2}(u_2)du\,dv\right|$$
$$\leq\frac{1}{4\pi^2}\int_{x_1-R_1}^{x_1+R_1}\int_{x_2-R_2}^{x_2+R_2}\left|\int_{\mathbb{R}^2}\log\left((\gamma_1(u_1)-\gamma_1(v_1))^2\right)\log\left((\gamma_2(u_2)-\gamma_2(v_2))^2\right)\right.$$
$$\left.\times R_1^{-1}(\varphi_1')^{x_1,R_1}(v_1)R_2^{-1}(\varphi_2')^{x_2,R_2}(v_2)dv\right|du\lesssim R_1\,R_2.$$

The last inequality holds due to Lemma 6.9. Then $\widetilde{\mathcal{C}}_\Gamma$ satisfies the weak boundedness property. we first verify (5.7). Let $x_1\in\mathbb{R}$, $R_1>0$, and $\varphi_j,\psi_j\in C_0^\infty(\mathbb{R})$ be normalized bumps. Then for $x_1,x_2,y_2\in\mathbb{R}$ and $R_1,R_2>0$ such that $|x_1-y_1|>4R_1$

$$\left|\left\langle M_b\widetilde{\mathcal{C}}_\Gamma M_b(\varphi_1^{y_1,R_1}\otimes\varphi_2^{x_2,R_2}),\psi_1^{x_1,R_1}\otimes\psi_2^{x_2,R_2}\right\rangle\right|$$
$$=\lim_{t_1,t_2\to0^+}\left|\int_{\mathbb{R}^2}q_{t_1}(\gamma_1(u_1)-\gamma_1(v_1))\varphi_1^{y_1,R_1}(v_1)\psi_1^{x_1,R_1}(u_1)\gamma_1'(v_1)\gamma_1'(u_1)dv_1\,du_1\right|$$
$$\times\left|\int_{\mathbb{R}^2}q_{t_2}(\gamma_2(u_2)-\gamma_2(v_2))\varphi_2^{y_2,R_2}(v_2)\psi_2^{x_2,R_2}(u_2)\gamma_2'(v_2)\gamma_2'(u_2)dv_2\,du_2\right|$$
$$\leq\lim_{t_1,t_2\to0^+}\int_{\mathbb{R}^2}|q_{t_1}(\gamma_1(u_1)-\gamma_1(v_1))|\,|\varphi_1^{y_1,R_1}(v_1)\psi_1^{x_1,R_1}(u_1)\gamma_1'(v_1)\gamma_1'(u_1)|dv_1\,du_1$$

$$\times \left| \iint_{\mathbb{R}^2} \log \left( (\gamma_2(u_2) - \gamma_2(v_2))^2 \right) (\varphi_2^{y_2,R_2})'(v_2) \psi_2^{x_2,R_2}(u_2) \gamma_2'(u_2) dv_2\, du_2 \right|$$
$$= \lim_{t_1,t_2 \to 0^+} A_{t_1} \times B_{t_2}.$$

To estimate $A_{t_1}$, we use the kernel estimate for $q_{t_1}$ to conclude the following bound.

$$\int_{\mathbb{R}^2} |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1))|\, |\varphi_1^{y_1,R_1}(v_1) \psi_1^{x_1,R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1)| dv_1\, du_1$$
$$\lesssim \int_{\mathbb{R}^2} \frac{1}{|u_1 - v_1|} |\varphi_1^{y_1,R_1}(v_1) \psi_1^{x_1,R_1}(u_1)| dv_1\, du_1$$
$$\lesssim \frac{R_1^2}{|x_1 - y_1|} = \frac{R_1}{(R_1^{-1}|x_1 - y_1|)}.$$

For the second term, we argue exactly as in the full weak boundedness case using Lemma 6.9:

$$\begin{aligned} B_{t_2} &\lesssim \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \log \left( (\gamma_2(u_2) - \gamma_2(v_2))^2 \right) R_2^{-1} (\varphi_2')^{y_2,R_2}(v_2) dv_2 \right| |\psi_2^{x_2,R_2}(u_2)| du_2 \\ &\lesssim \int_{\mathbb{R}} |\psi_2^{x_2,R_2}(u_2)| du_2 \\ &\lesssim R_2. \end{aligned}$$

Therefore $\widetilde{C}_\Gamma$ satisfies (5.7). To prove (5.8), fix $x_1, x_2, y_2 \in \mathbb{R}$, $R_1, R_2 > 0$, and $\varphi_j, \psi_j$ for $j = 1, 2$ as above, but furthermore assume (without loss of generality) that $\gamma_1' \psi_1^{x_1,R_1}$ has mean zero. Since $|x_1 - y_1| > 4R_1$

$$\left| \left\langle M_b \widetilde{\mathcal{C}}_\Gamma M_b (\varphi_1^{y_1,R_1} \otimes \varphi_2^{x_2,R_2}), \psi_1^{x_1,R_1} \otimes \psi_2^{x_2,R_2} \right\rangle \right|$$
$$\leq \lim_{t_1,t_2 \to 0^+} \int_{\mathbb{R}^2} |q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1)) - q_{t_1}(\gamma_1(x_1) - \gamma_1(v_1))|\, |\varphi_1^{y_1,R_1}(v_1) \psi_1^{x_1,R_1}(u_1) \gamma_1'(v_1) \gamma_1'(u_1)| dv_1\, du_1$$
$$\times \left| \iint_{\mathbb{R}^2} \log \left( (\gamma_2(u_2) - \gamma_2(v_2))^2 \right) (\varphi_2^{y_2,R_2})'(v_2) \psi_2^{x_2,R_2}(u_2) \gamma_2'(u_2) dv_2 du_2 \right|$$
$$= \lim_{t_1,t_2 \to 0^+} \widetilde{A}_{t_1} \times B_{t_2}.$$

By the support properties of $\varphi_1$ and $\psi_1$, we may assume that $|y_1 - v_1| \leq R_1$ and $|x_1 - u_1| \leq R_1$ to estimate the following part of the integrand from $\widetilde{A}_{t_1}$:

$$|q_{t_1}(\gamma_1(u_1) - \gamma_1(v_1)) - q_{t_1}(\gamma_1(x_1) - \gamma_1(v_1))|$$
$$= \left| \frac{(\gamma_1(u_1) - \gamma_1(v_1))(\gamma_1(x_1) - \gamma_1(v_1))^2 - (\gamma_1(x_1) - \gamma_1(v_1))(\gamma_1(u_1) - \gamma_1(v_1))^2}{[(\gamma_1(x_1) - \gamma_1(v_1))^2 + t_1^2][(\gamma_1(u_1) - \gamma_1(v_1))^2 + t_1^2]} \right.$$

$$
\begin{aligned}
&\qquad + \frac{(\gamma_1(u_1)-\gamma_1(v_1))t_1^2 - (\gamma_1(x_1)-\gamma_1(v_1))t_1^2}{[(\gamma_1(x_1)-\gamma_1(v_1))^2+t_1^2][(\gamma_1(u_1)-\gamma_1(v_1))^2+t_1^2]}\Bigg| \\
&\le \frac{|\gamma_1(u_1)-\gamma_1(v_1)|\,|\gamma_1(x_1)-\gamma_1(v_1)|\,|\gamma_1(x_1)-\gamma_1(u_1)|}{[(\gamma_1(u_1)-\gamma_1(v_1))^2+t_1^2][(\gamma_1(x_1)-\gamma_1(v_1))^2+t_1^2]} \\
&\qquad + t_1^2 \frac{|\gamma_1(u_1)-\gamma_1(x_1)|}{|(\gamma_1(u_1)-\gamma_1(v_1))^2+t_1^2|\,|(\gamma_1(x_1)-\gamma_1(v_1))^2+t_1^2|} \\
&\lesssim \frac{|u_1-v_1|\,|x_1-v_1|\,|x_1-u_1|}{|u_1-v_1|^2|x_1-v_1|^2} + \frac{|x_1-u_1|}{|x_1-v_1|^2} \lesssim \frac{R_1}{|x_1-y_1|^2}.
\end{aligned}
$$

In the last line, we use that $|x_1-y_1| > R_1/4$, $|x_1-u_1| \le R_1$, $|y_1-v_1| \le R_1$,

$$|u_1-v_1| \ge |x_1-y_1|/2, \quad \text{and} \quad |x_1-v_1| \ge |x_1-y_1|/2.$$

It easily follows that

$$\widetilde{A}_{t_1} \lesssim \frac{R_1}{|x_1-y_1|^2} \int_{\mathbb{R}^2} |\varphi_1^{y_1,R_1}(v_1)\psi_1^{x_1,R_1}(u_1)|dv_1\,du_1 \lesssim \frac{R_1^3}{|x_1-y_1|^2} = \frac{R_1}{(R_1^{-1}|x_1-y_1|)^2},$$

as required in (5.8) with $n_1 = \gamma = 1$.

This verifies the first mixed weak boundedness properties (5.7) and (5.8) for $\mathcal{C}_\Gamma$, and the other two conditions follow by symmetry. $\square$

**Proposition 6.11.** *Assume $\Gamma$ satisfies the hypotheses of Theorem 6.3. The operator $\widetilde{\mathcal{C}}_\Gamma$ satisfies the $Tb = T^*\tilde{b} = 0$ conditions with $b(x) = \tilde{b}(x) = \gamma_1'(x_1)\gamma_2'(x_2)$ for $x = (x_1,x_2) \in \mathbb{R}^2$.*

*Proof.* Let $\eta_R \in C_0^\infty(\mathbb{R}^{n_1})$ be as above, $\varphi_1, \psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$, and $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$ such that $\gamma_1'\psi_1$ and $\gamma_2'\psi_2$ have mean zero. We use Proposition 6.8 to compute

$$
\begin{aligned}
\left\langle \widetilde{\mathcal{C}}_\Gamma(\gamma_1'\eta_R \otimes \gamma_2'\varphi_2), \gamma_1'\psi_1 \otimes \gamma_2'\psi_2 \right\rangle &= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left((\gamma_1(x_1)-\gamma_1(y_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right) \\
&\qquad \times (\eta_R)'(y_1)\varphi_2'(y_2)\psi_1(x_1)\psi_2(x_2)\gamma_1'(x_1)\gamma_2'(x_2)dy\,dx \\
= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} &\log\left((\gamma_1(x_1)-\gamma_1(Ry_1))^2\right)\log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right) \\
&\qquad \times \eta'(y_1)\varphi_2'(y_2)\psi_1(x_1)\psi_2(x_2)\gamma_1'(x_1)\gamma_2'(x_2)dy\,dx \\
= \int_{\mathbb{R}^2} F_R(x_1)&\left(\int_{\mathbb{R}} \log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\varphi_2'(y_2)dy_2\right)\psi_1(x_1)\psi_2(x_2)\gamma_1'(x_1)\gamma_2'(x_2)dx, \\
\text{where} \quad F_R(x_1) &= \int_{\mathbb{R}} \log\left((\gamma_1(x_1)-\gamma_1(Ry_1))^2\right)\eta'(y_1)dy_1.
\end{aligned}
$$

Since $\eta \in C_0^\infty(\mathbb{R})$, it follows that $\eta'$ has mean zero. Note also that $\mathrm{Re}(c_1) = 1$ since $\gamma_1(x_1) = x_1 + iL_1(x_1)$ and $L_1$ is real-valued, so $\log(y_1^2c_1^2)$ is well defined for $y_1 \ne 0$. Recall the definition of $c_1$ in the hypotheses of Theorem 6.3. Hence we can also write $F_R(x_1)$ in the following way.

$$\begin{aligned} F_R(x_1) &= \int_{\mathbb{R}} \left[\log\left((\gamma_1(x_1)-\gamma_1(Ry_1))^2\right) - \log\left(R^2\right)\right]\eta'(y_1)dy_1 \\ &= \int_{\mathbb{R}} \log\left(\frac{(\gamma_1(x_1)-\gamma_1(Ry_1))^2}{R^2}\right)\eta'(y_1)dy_1. \end{aligned}$$

Now we note that for all $x_1 \in \mathbb{R}$ and $y_1 \neq 0$

$$\lim_{R\to\infty} \log\left(\frac{(\gamma_1(x_1)-\gamma_1(Ry_1))^2}{R^2}\right) = \lim_{R\to\infty} \log\left(y_1^2\frac{(\gamma_1(x_1)-\gamma_1(Ry_1))^2}{y_1^2R^2}\right) = \log(y_1^2c_1^2).$$

Recall that we have assumed $\gamma_1(u_1)/u_1 \to c_1$ as $|u_1| \to \infty$. For $R$ large enough so that $\mathrm{supp}(\psi_1) \subset B(0,R/2)$, it follows that for $x_1 \in \mathrm{supp}(\psi_1)$ and $y_1 \in \mathrm{supp}(\eta') \subset B(0,2)\backslash B(0,1)$

$$\frac{|\gamma_1(x_1)-\gamma_1(Ry_1)|^2}{R^2} \geq (1-\lambda_1^2)\frac{|x_1-Ry_1|^2}{R^2} \geq (1-\lambda_1^2)\frac{R^2-|x_1|^2}{R^2} \geq 1-\lambda_1^2.$$

We also have

$$\frac{|\gamma_1(x_1)-\gamma_1(Ry_1)|^2}{R^2} \leq \frac{4|x_1-Ry_1|^2}{R^2} \leq \frac{4|x_1|^2}{R^2} + 4|y_1|^2 \leq 20$$

Therefore

$$\left|\log\left(\frac{(\gamma_1(x_1)-\gamma_1(Ry_1))^2}{R^2}\right)\eta'(y_1)\right| \lesssim \eta'(y_1).$$

Then by dominated convergence,

$$\lim_{R\to\infty} F_R(x_1) = \int_{\mathbb{R}} \log(y_1^2c_1^2)\eta'(y_1)dy_1 = c.$$

Now $F_R(x_1) \to c$ for some constant $c \in \mathbb{C}$, which does not depend on $x_1$. Since $F_R(x_1)$ is bounded independent of $x_1$, we apply dominated convergence again to conclude

$$\begin{aligned} \lim_{R\to\infty}\left\langle \widetilde{\mathcal{C}}_\Gamma(\gamma_1'\eta_R \otimes \gamma_2'\varphi_2), \gamma_1'\psi_1 \otimes \gamma_2'\psi_2\right\rangle &= \int_{\mathbb{R}^2} c\left(\int_{\mathbb{R}} \log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\varphi_2'(y_2)dy_2\right) \\ &\qquad \times \psi_1(x_1)\psi_2(x_2)\gamma_1'(x_1)\gamma_2'(x_2)dx \\ &= c\left(\int_{\mathbb{R}} \psi_1(x_1)\gamma_1'(x_1)dx_1\right)\left(\int_{\mathbb{R}^2} \log\left((\gamma_2(x_2)-\gamma_2(y_2))^2\right)\varphi_2'(y_2)\psi_2(x_2)\gamma_2'(x_2)dy_2\,dx_2\right) \\ &= 0. \end{aligned}$$

Here we use that $\gamma_1'\psi_1$ has mean zero. By symmetry, this holds when $\gamma_1'\varphi_1$ has mean zero in place of $\gamma_1'\psi_1$. Hence the $\widetilde{\mathcal{C}}_\Gamma(b) = 0$ condition is satisfied, and the adjoint condition follows by symmetry. $\square$

By Theorem 5.7, we conclude that $\widetilde{\mathcal{C}}_\Gamma$ can be extended to a bounded linear operator on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Hence $\mathcal{C}_\Gamma$ can be defined for $g \in L^p(\Gamma)$ for $1 < p < \infty$, and for $g \in L^p(\Gamma)$, it follows that

$$\begin{aligned}
\|\mathcal{C}_\Gamma g\|^p_{L^p(\Gamma)} &= \int_{\mathbb{R}^2} |\widetilde{\mathcal{C}}_\Gamma M_b(g\circ\gamma)(x)|^p |\gamma_1'(x_1)\gamma_2'(x_2)| dx \\
&\leq \|\gamma_1'\|_{L^\infty}\|\gamma_2'\|_{L^\infty}\|\widetilde{\mathcal{C}}_\Gamma\|^p_{L^p,L^p}\int_{\mathbb{R}^2}|(g\circ\gamma)(x)|^p dx \\
&\leq 4\|(\gamma_1')^{-1}\|_{L^\infty}\|(\gamma_2')^{-1}\|_{L^\infty}\|\widetilde{\mathcal{C}}_\Gamma\|^p_{L^p,L^p}\int_{\mathbb{R}^2}|g(x)|^p|\gamma_1'(x_1)\gamma_2'(x_2)|dx \leq 4\|\widetilde{\mathcal{C}}_\Gamma\|^p_{L^p,L^p}\|g\|^p_{L^p(\Gamma)}.
\end{aligned}$$

Furthermore for $f \in C_0^\infty(\mathbb{R}^2)$, there exists a constant $C_{f,p} > 0$ such that

$$|\widetilde{\mathcal{C}}_t M_b f(x)|^p \leq C_{f,p}\left(\chi_{|x_1|\leq 2R_0} + \frac{1}{|x_1|^p}\chi_{|x_1|>2R_0}\right)\left(\chi_{|x_2|\leq 2R_0} + \frac{1}{|x_2|^p}\chi_{|x_2|>2R_0}\right),$$

where $R_0$ is large enough so that $\mathrm{supp}(f) \subset B(0, R_0/2)$. Then by dominated convergence, it follows that

$$\lim_{t_1,t_2\to 0^+} \widetilde{\mathcal{C}}_t M_b f = \widetilde{\mathcal{C}}_\Gamma M_b f \quad \text{in} \quad L^p(\mathbb{R}^2).$$

One can argue by density to verify that $\widetilde{\mathcal{C}}_\Gamma$ extends to all of $L^p(\mathbb{R}^2)$ and that $\widetilde{\mathcal{C}}_t f \to \widetilde{\mathcal{C}}_\Gamma f$ in $L^p(\mathbb{R}^2)$ for $f \in L^p(\mathbb{R}^2)$ as $t_1, t_2 \to 0^+$ for all $1 < p < \infty$.

It easily follows that for $g \in L^p(\Gamma)$ where $1 < p < \infty$

$$\lim_{t_1,t_2\to 0^+} \mathcal{C}_t g = \mathcal{C}_\Gamma g$$

in $L^p(\Gamma)$. This completes the proof of the first part of Theorem 6.5, pertaining to $\mathcal{C}_\Gamma$.

## 6.3 Bounds for the partial Cauchy Integral Transform

Like in the last section, we define the parameterized versions of $\mathcal{C}^{p1}_\Gamma$ and $\mathcal{C}^{p2}_\Gamma$, for $f \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$

$$\begin{aligned}
\widetilde{\mathcal{C}}^{p1}_\Gamma M_b f(x) &= \lim_{t_1,t_2\to 0^+} \widetilde{\mathcal{C}}^{p1}_\Gamma M_b f(x), \quad \text{where} \\
&\qquad \widetilde{\mathcal{C}}^{p1}_t M_b f(x) = \int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1)-\gamma_1(y_1))p_{t_2}(\gamma_2(x_2)-\gamma_2(y_2))f(y)b(y)dy, \\
\widetilde{\mathcal{C}}^{p2}_\Gamma M_b f(x) &= \lim_{t_1,t_2\to 0^+} \widetilde{\mathcal{C}}^{p2}_t M_b f(x), \quad \text{where}
\end{aligned}$$

$$\widetilde{\mathcal{C}}_t^{p2} M_b f(x) = \int_{\mathbb{R}^2} p_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f(y) b(y) dy.$$

We prove these bounds by applying the single parameter $Tb$ theorem from [DJS85]. We outline the proof that $\widetilde{\mathcal{C}}_\Gamma^{p1}$ and $\widetilde{\mathcal{C}}_\Gamma^{p2}$ are bounded on $L^p(\Gamma)$. The details can be deciphered from the previous more complicated biparameter versions. Define for $f_1, f_2 : \mathbb{R} \to \mathbb{C}$ and $x_1, x_2 \in \mathbb{R}$

$$\widetilde{\mathcal{C}}_{\Gamma_1} M_{\gamma_1'} f_1(x_1) = \lim_{t_1 \to 0^+} \int_{\mathbb{R}} q_{t_1}(\gamma_1(x_1) - \gamma_1(y_1)) f_1(y_1) \gamma_1'(y_1) dy_1,$$
$$\widetilde{\mathcal{C}}_{\Gamma_2} M_{\gamma_2'} f_2(x_2) = \lim_{t_2 \to 0^+} \int_{\mathbb{R}} q_{t_2}(\gamma_2(x_2) - \gamma_2(y_2)) f_2(y_2) \gamma_2'(y_2) dy_2.$$

The following propositions are routine given the proofs of Propositions 6.8, 6.10, and 6.11.

**Proposition 6.12.** *Assume $\Gamma$ satisfies the hypotheses of Theorem 6.3. For all $f \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$,*

$$\widetilde{\mathcal{C}}_\Gamma^{p1}(bf)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \log\left((\gamma_1(x_1) - \gamma_1(y_1))^2\right) \partial_{y_1} f(y_1, x_2) dy_1,$$
$$\widetilde{\mathcal{C}}_\Gamma^{p2}(bf)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \log\left((\gamma_2(x_2) - \gamma_2(y_2))^2\right) \partial_{y_2} f(x_1, y_2) dy_2.$$

*Also, for all $f, g \in C_0^\infty(\mathbb{R}^2)$, the pairings $\left\langle \widetilde{\mathcal{C}}_\Gamma^{p1}(bf), bg \right\rangle$ and $\left\langle \widetilde{\mathcal{C}}_\Gamma^{p2}(bf), bg \right\rangle$ can be realized as any of the following absolutely convergent integrals:*

$$\left\langle \widetilde{\mathcal{C}}_\Gamma^{p1}(bf), bg \right\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^3} \log\left((\gamma_1(x_1) - \gamma_1(y_1))^2\right) \partial_{y_1} f(y_1, x_2) g(x) b(x) dy_1 \, dx,$$
$$\left\langle \widetilde{\mathcal{C}}_\Gamma^{p1}(bf), bg \right\rangle = -\frac{1}{2\pi} \int_{\mathbb{R}^3} \log\left((\gamma_1(x_1) - \gamma_1(y_1))^2\right) f(y_1, x_2) \partial_{x_1} g(x) b(y_1, x_2) dy_1 \, dx,$$
$$\left\langle \widetilde{\mathcal{C}}_\Gamma^{p2}(bf), bg \right\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^3} \log\left((\gamma_2(x_2) - \gamma_2(y_2))^2\right) \partial_{y_2} f(x_1, y_2) g(x) b(x) dy_2 \, dx,$$
$$\left\langle \widetilde{\mathcal{C}}_\Gamma^{p2}(bf), bg \right\rangle = -\frac{1}{2\pi} \int_{\mathbb{R}^3} \log\left((\gamma_2(x_2) - \gamma_2(y_2))^2\right) f(x_1, y_2) \partial_{x_2} g(x) b(x_1, y_2) dy_2 \, dx.$$

**Proposition 6.13.** *Assume $\Gamma$ satisfies the hypotheses of Theorem 6.3. The operator $\widetilde{\mathcal{C}}_{\Gamma_1}$ and $\widetilde{\mathcal{C}}_{\Gamma_2}$ satisfies the single parameter weak boundedness property.*

**Proposition 6.14.** *Assume $\Gamma$ satisfies the hypotheses of Theorem 6.3. The operator $\widetilde{\mathcal{C}}_{\Gamma_1}$ and $\widetilde{\mathcal{C}}_{\Gamma_2}$ satisfies the cancellation conditions $\widetilde{\mathcal{C}}_{\Gamma_1}(\gamma_1') = \widetilde{\mathcal{C}}_{\Gamma_1}^*(\gamma_1') = \widetilde{\mathcal{C}}_{\Gamma_2}(\gamma_2') = \widetilde{\mathcal{C}}_{\Gamma_2}^*(\gamma_2') = 0$.*

Then by the $Tb$ theorem of David-Journé-Semmes [DJS85], it follows that $\widetilde{\mathcal{C}}_{\Gamma_1}$ and $\widetilde{\mathcal{C}}_{\Gamma_2}$ are bounded on $L^p(\mathbb{R})$. It follows that for $f, g \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\left|\left\langle \widetilde{\mathcal{C}}_\Gamma^{p1}(bf), bg\right\rangle\right| &= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\lim_{t_1\to 0^+}\int_{\mathbb{R}^2}\log\left((\gamma_1(x_1)-\gamma_1(y_1))^2+t_1^2\right)\right.\\ &\qquad\left.\times\,\partial_{y_1} f(y_1,x_2)g(x)\gamma_1'(x_1)dy_1\,dx_1\right|\left|\gamma_2'(x_2)\right|dx_2\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\lim_{t_1\to 0^+}\int_{\mathbb{R}^2} q_{t_1}(\gamma_1(x_1)-\gamma_1(y_1))f(y_1,x_2)\gamma_1'(y_1)g(x)\gamma_1'(x_1)dy_1\,dx_1\right|\left|\gamma_2'(x_2)\right|dx_2\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\left\langle\widetilde{\mathcal{C}}_{\Gamma_1}(\gamma_1' f(\cdot,x_2)), \gamma_1' g(\cdot,x_2)\right\rangle\right|\left|\gamma_2'(x_2)\right|dx_2\\ &\lesssim \int_{\mathbb{R}}\|f(\cdot,x_2)\|_{L^p(\mathbb{R})}\|g(\cdot,x_2)\|_{L^{p'}(\mathbb{R})}dx_2 \le \|f\|_{L^p(\mathbb{R}^2)}\|g\|_{L^{p'}(\mathbb{R}^2)}.\end{aligned}$$

Therefore $\widetilde{\mathcal{C}}_\Gamma^{p1}$ is bounded on $L^p(\mathbb{R}^2)$ for $1<p<\infty$, and by symmetry $\widetilde{\mathcal{C}}_\Gamma^{p2}$ is as well. Again it follows that for $f \in L^p(\mathbb{R}^2)$

$$\lim_{t_1,t_2\to 0^+}\widetilde{\mathcal{C}}_t^{p1}M_b f = \widetilde{\mathcal{C}}_{\Gamma_1}M_b f \qquad\text{and}\qquad \lim_{t_1,t_2\to 0^+}\widetilde{\mathcal{C}}_t^{p2}M_b f = \widetilde{\mathcal{C}}_{\Gamma_2}M_b f \qquad \text{in } L^p(\mathbb{R}^2),$$

and for $g \in L^p(\Gamma)$

$$\lim_{t_1,t_2\to 0^+}\mathcal{C}_t^{p1}g = \mathcal{C}_\Gamma^{p1}g \qquad\text{and}\qquad \lim_{t_1,t_2\to 0^+}\mathcal{C}_t^{p2}g = \mathcal{C}_\Gamma^{p2}g \qquad \text{in } L^p(\Gamma).$$

This completes the proof.

**Remark 6.15.** The comment after Proposition 6.11 and Proposition 6.14 together prove Theorem 6.5.

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