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On the relativistic Lagrange-Laplace secular dynamics for extrasolar systems

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Abstract. We study the secular dynamics of extrasolar planetary systems by extending the Lagrange-Laplace theory to high order and by including the relativistic effects. We investigate the long-term evolution of the planetary eccentricities via normal form and we find an excellent agreement with direct numerical integrations. Finally we set up a simple analytic criterion that allows to evaluate the impact of the relativistic effects in the long-time evolution.

Keywords. celestial mechanics, methods: analytical, secular dynamics, relativistic effects, extrasolar planetary systems, normal forms

1. Introduction

The study of the secular evolution of planetary systems is a long standing and challenging problem. The discoveries of hundreds of extrasolar planetary systems raised many interesting problems concerning their long-term evolution. In the present paper, we study the secular dynamics of two non-resonant coplanar planets in an extrasolar system. We extend the Lagrange-Laplace theory to high order and include the main relativistic effects.

The study of extrasolar system raised two particularly relevant problems, namely: (i) most exoplanets have highly eccentric orbits, in contrast with the almost circular orbits of the Solar System; (ii) there are many giant planets orbiting at a low distance from the central star, with periods of a few months or even a few days. In the latter case relativistic effects could have a significant impact and should be taken into account. The General Theory of Relativity, despite having been widely used in astrophysics, is not commonly adopted in the study of planetary system dynamics.

The generalization of the Lagrange-Laplace secular theory to high order in the eccentricities has been exploited so as to obtain an analytic model that gives an accurate description of the behavior of planetary systems, up to surprisingly high eccentricities (see, Libert & Henrard (2005, 2006)). The results appear to be quite good for systems which are not close to a mean-motion resonance. In Libert & Sansottera (2013) the secular theory has also been extended to order two in the masses, by using a first-order approximation of an elliptic lower dimensional torus in place of the usual circular approximation. In particular this allows to deal with systems close to a mean-motion resonance. The relevance of the relativistic corrections and tidal effects on the long-term evolution of extrasolar planetary systems has been studied, e.g., in Adams & Laughlin (2006) and Migaszewski & Goździewski (2008).

On the other hand, the application of Kolmogorov and Nekhoroshev theorems, allowed to make substantial progress for the problem of stability of the Solar System. Indeed, in recent years, the estimates for the applicability of both theorems to realistic models of some part of the Solar System have been improved by some authors (e.g., Robutel (1995), Celletti & Chierchia (2005), Locatelli & Giorgilli (2007), Giorgilli *et al.* (2009, 2014) and Sansottera *et al.* (2011, 2013)).

In the present paper we exploit the idea of extending the Lagrange-Laplace theory, already used in the above-cited papers, to the case of high eccentricities. The technical tool is the construction of a suitable normal form which allows us to investigate the long-time evolution of the planetary eccentricities. In this contribution we neglect the tidal effects, although we know that for many system they can be relevant. We decided to just consider the relativistic correction in order to keep the discussion at a simple level and to show that the extension of the Lagrange-Laplace theory, including relativistic effects, produces accurate results. We plan to further investigate the problem in a forthcoming work.

2. Classical expansion of the Hamiltonian

We consider a system of three coplanar point bodies, mutually interacting according to Newton's gravitational law: a central star P_0 of mass m_0 and two planets P_1 and P_2 of mass m_1 and m_2 and semi-major axis a_1 and a_2, respectively.

We refer to Libert & Sansottera (2013) for a detailed exposition concerning the expansion of the Hamiltonian in the Poincaré canonical variables, that reads

$$H(\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) = H_0(\boldsymbol{\Lambda}) + \varepsilon H_1(\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}), \tag{2.1}$$

where H_0 is the Keplerian part and εH_1 the perturbation due to the mutual attraction between the planets. Using the standard notation, we will refer to $(\boldsymbol{\Lambda},\boldsymbol{\lambda})$ as the *fast variables* and to $(\boldsymbol{\xi},\boldsymbol{\eta})$ as the *secular variables*.

3. Relativistic corrections

Starting from the Hamiltonian of the Newton model, we add the relativistic corrections due to the mutual interaction between the star and each of the two planets. That is, we consider the correction included in the relativistic Hamiltonian of the problem of two-body in heliocentric coordinates $(\mathbf{r},\mathbf{p})$. The relativistic Hamiltonian takes the form

$$H = H_0 + \varepsilon H_1 + \frac{1}{c^2} H_2, \tag{3.1}$$

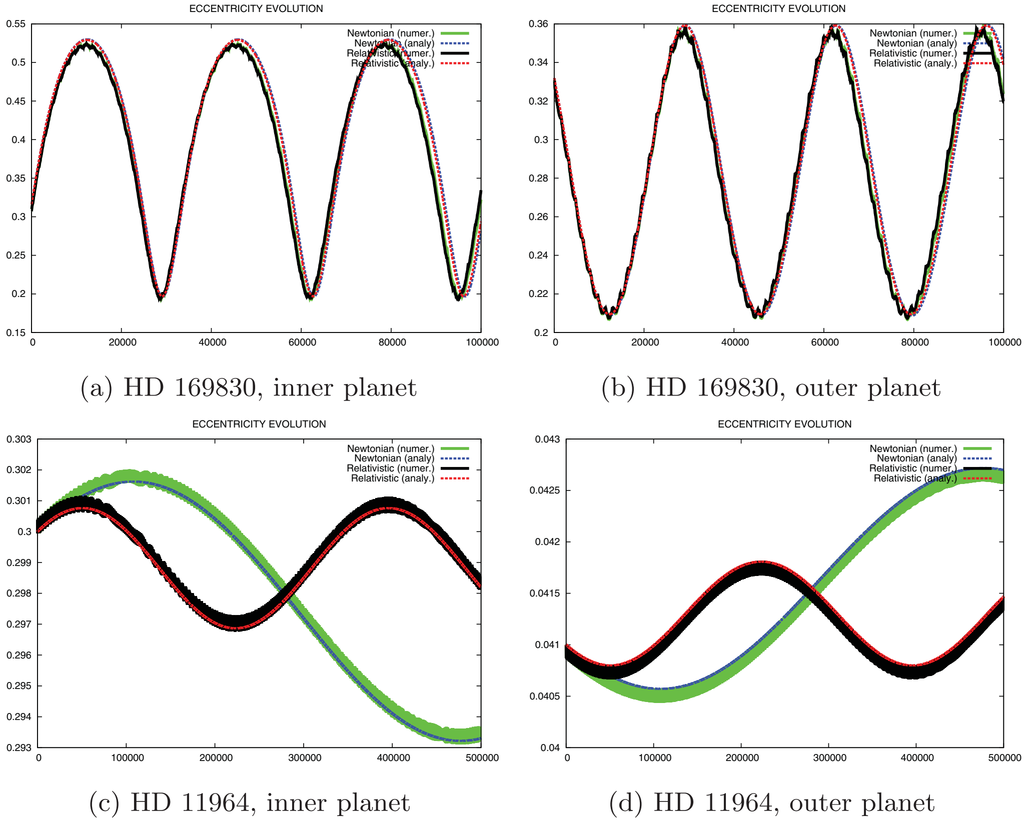
with H_0 and εH_1 as in the Newtonian model, while $\frac{1}{c^2}H_2$ is

$$\frac{1}{c^2}H_2 = \frac{1}{c^2}\sum_{i=1}^{2}\left[-\frac{\gamma_{1,i}}{\mu_i^3}(\mathbf{P}_i\cdot\mathbf{P}_i)^2 - \frac{\gamma_{2,i}}{\mu_i}\frac{\mathbf{P}_i\cdot\mathbf{P}_i}{\parallel \mathbf{r}_i \parallel} - \frac{\gamma_{3,i}}{\mu_i}\frac{(\mathbf{r}_i\cdot\mathbf{P}_i)^2}{\parallel \mathbf{r}_i \parallel^3} + \gamma_{4,i}\mu_i\frac{1}{\parallel \mathbf{r}_i \parallel^2}\right], \tag{3.2}$$

with

$$\begin{aligned} &\mu_i = \frac{m_0 m_i}{m_0+m_i}, \qquad \beta_i = \mathcal{G}(m_0+m_i), \qquad \upsilon_i = \frac{m_0 m_i}{(m_0+m_i)^2}, \\ &\gamma_{1,i} = \frac{1-3\upsilon_i}{8}, \qquad \gamma_{2,i} = \frac{\beta_i(3+\upsilon_i)}{2}, \qquad \gamma_{3,i} = \frac{\beta_i\upsilon_i}{2}, \qquad \gamma_{4,i} = \frac{\beta_i^2}{2}, \end{aligned} \tag{3.3}$$

and $\mathbf{P}_i = \mathbf{p}_i + \frac{\mu_i}{m_0}\mathbf{p}_{3-i} + \mathcal{O}(c^{-2})$ for $i = 1, 2$.



(a) HD 169830, inner planet

(b) HD 169830, outer planet

(c) HD 11964, inner planet

(d) HD 11964, outer planet

Figure 1. Long-term evolution of the eccentricities for the HD 169830 and HD 11964 planetary systems. Comparison of the results obtained via direct numerical integration (green-black) against normal form (blue-red), for the Newtonian approximation (green-blue) and the model including the relativistic effects (black-red).

4. Long-term evolution

As we are interested in the long-term dynamics, we remove the dependency of the Hamiltonian from the fast angles. The classical approach consists in replacing the Hamiltonian with its average, the so-called approximation at order one in the masses. We replace this procedure by a Kolmogorov-like step, that allows us to include in the secular model the effects of the main near-resonances effects (see Libert & Sansottera (2013) for a detailed exposition). This is the secular Hamiltonian at order two in the masses.

After averaging, the secular Hamiltonian has two degrees of freedom and its quadratic part differs from the one considered in the Lagrange-Laplace theory by relativistic corrections and contributions of order two in the masses, which however are small.

As in Libert & Sansottera (2013), we introduce the action-angle variables via normal form. In the normalized coordinates, the equations of motion take a simple form that can be analytically integrated. We validate the results by comparing the analytic integration with the direct numerical integration of the full three-body system.

In Figure 1a–1b we report the results for the HD 169830 system. In this case the relativistic effects are negligible and the Newtonian approximation allows to accurately describe the long-term evolution for a time interval of 10^5 years. Instead, for the HD 11964, the relativistic corrections play a major role, as it is clearly shown in Figure 1c–1d. In this case the calculations cover 5×10^5 years. In all cases, the evolutions via normal and via numerical integration are in excellent agreement. We emphasize that the use of normal form provides us with a natural criterion for deciding whether or not the relativistic corrections are relevant. Indeed the difference is seen in the precession frequencies: if the relativistic corrections are relevant then so is the difference, as the figures clearly show.

5. Relevance of the relativistic corrections

In order to evaluate the impact of the relativistic corrections, we look at the quadratic parts of the secular Hamiltonians, namely

$$H_q^{(\mathrm{New})}(\boldsymbol{\eta},\boldsymbol{\xi}) = \boldsymbol{\eta}\cdot A\boldsymbol{\eta} + \boldsymbol{\xi}\cdot A\boldsymbol{\xi} \quad \text{and} \quad H_q^{(\mathrm{Rel})}(\boldsymbol{\eta},\boldsymbol{\xi}) = \boldsymbol{\eta}\cdot B\boldsymbol{\eta} + \boldsymbol{\xi}\cdot B\boldsymbol{\xi},$$

where A and B are real symmetric 2×2 with

$$B = A - \frac{3}{2}\frac{\mathcal{G}^{3/2}}{c^2}\begin{bmatrix} \frac{(m_0+m_1)^{3/2}}{a_1^{5/2}} & 0 \\ 0 & \frac{(m_0+m_2)^{3/2}}{a_2^{5/2}} \end{bmatrix}.$$

Clearly, the relativistic effects are more important if

$$A_{ii} \sim -\frac{3}{2}\frac{\mathcal{G}^{3/2}}{c^2}\frac{(m_0+m_i)^{3/2}}{a_i^{5/2}}, \qquad \text{i.e. if} \quad \Pi_i \equiv \frac{4\mathcal{G}a_2^3 m_0(m_0+m_i)}{c^2 a_i^2 a_1^2 m_{3-i}} \sim 1. \tag{5.1}$$

In the following table we report the dimensionless quantities Π_i for the extrasolar systems considered above.

HD 169830	Π_1: 0.0021779	Π_2: 0.0001547
HD 11964	Π_1: 0.9651708	Π_2: 0.0399271

We observed that for the majority of the extrasolar systems taken into consideration, the relevance of relativistic corrections may be inferred from the difference between the matrices. This provides us with a rough criterion based on the first order approximation. Normal form provides a more refined criterion.

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