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A Lewy-Stampacchia estimate for variational inequalities in the Heisenberg group

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Abstract. *We consider an obstacle problem in the Heisenberg group framework, and we prove that the operator on the obstacle bounds pointwise the operator on the solution. More explicitly, if $\bar{u}$ minimizes the functional*

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^2$$

among the functions with prescribed Dirichlet boundary condition that stay below a smooth obstacle ψ, then

$$0 \leq \Delta_{\mathbb{H}^n} \bar{u} \leq \Big(\Delta_{\mathbb{H}^n} \psi\Big)^+.$$

Moreover, we discuss how it could be possible to generalize the previous bound to a quasilinear setting once some regularity issues for the equation

$$\operatorname{div}_{\mathbb{H}^n} \Big(|\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u\Big) = f$$

are satisfied.

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1. Introduction

In this paper, we extend the so called Dual Estimate of [11] to the obstacle problem for the Kohn-Laplacian operator in the Heisenberg group.

The notation we use is the standard one: for $n \geq 1$, we consider $\mathbb{R}^{2n+1}$

endowed with the group law

$$\big(x^{(1)}, y^{(1)}, t^{(1)}\big) \circ \big(x^{(2)}, y^{(2)}, t^{(2)}\big) := \Big(x^{(1)} + x^{(2)}, y^{(1)} + y^{(2)}, t^{(1)} + t^{(2)} + 2(x^{(2)} \cdot y^{(1)} - x^{(1)} \cdot y^{(2)})\Big),$$

for any $(x^{(1)}, y^{(1)}, t^{(1)})$, $(x^{(2)}, y^{(2)}, t^{(2)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where the "$\cdot$" is the standard Euclidean scalar product.

Then, we denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, i.e., $\mathbb{R}^{2n+1}$ endowed with this group law.

The coordinates are usually written as $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and, as customary, we introduce the left invariant vector fields (X, Y) induced by the group law

$$X_j := \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \text{ and } Y_j := \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$, and the horizontal gradient $\nabla_{\mathbb{H}^n} := (X, Y)$. The main issue of the Heisenberg group is that X and Y do not commute, that is

$$[X, Y] = -4\frac{\partial}{\partial t} \neq 0.$$

We are interested in studying the obstacle problem in this framework. For this, we consider a smooth function $\psi : \mathbb{H}^n \to \mathbb{R}$, which will be our obstacle (more precisely, ψ is supposed to have continuous derivatives of second order in X and Y).

Fixed a bounded open set Ω with smooth boundary, and $p \in (1, +\infty)$, we consider the space $W^{1,p}_{\mathbb{H}^n}(\Omega)$ to be the set of all functions u in $L^p(\Omega)$ whose distributional horizontal derivatives $X_j u$ and $Y_j u$ belong to $L^p(\Omega)$, for $j = 1, \ldots, n$.

Such space is naturally endowed with the norm

$$\|u\|_{W^{1,p}_{\mathbb{H}^n}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{j=1}^{n} \Big(\|X_j u\|_{L^p(\Omega)} + \|Y_j u\|_{L^p(\Omega)}\Big).$$

We call $W^{1,p}_{\mathbb{H}^n,0}(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to this norm.

We fix a smooth domain $\Omega_\star \Supset \Omega$, $u_\star \in W^{1,2}_{\mathbb{H}^n}(\Omega_\star) \cap L^\infty(\Omega_\star)$ and we introduce the space

$$\mathcal{K} := \big\{u \in W^{1,2}_{\mathbb{H}^n}(\Omega) \text{ s.t. } u \leq \psi, \text{ and } u - u_\star \in W^{1,2}_{\mathbb{H}^n,0}(\Omega)\big\}. \tag{1}$$

Loosely speaking, $\mathcal{K}$ is the space of all the functions having prescribed Dirichlet boundary datum equal to $u_\star$ along $\partial\Omega$ and that stay below the obstacle ψ.

We deal with the variational problem

$$\inf_{u \in \mathcal{K}} \mathcal{F}(u; \Omega), \text{ where } \mathcal{F}(u; \Omega) := \int_\Omega |\nabla_{\mathbb{H}^n} u|^2. \tag{2}$$

By direct methods, it is seen that such infimum is attained (see, e.g., the compactness result in [18, 5] or references therein) and so we consider a minimizer $\bar{u}$. It is worth pointing out that such minimizer may be written in terms of a variational inequality, namely

$$\int_\Omega \nabla_{\mathbb{H}^n} \bar{u} \cdot \nabla_{\mathbb{H}^n}(v - \bar{u}) \,\geq\, 0, \tag{3}$$

for any $v \in W^{1,2}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v - \bar{u} \in W^{1,2}_{\mathbb{H}^n,0}(\Omega)$. These kind of variational inequalities[1] are now receiving a considerable attention (see, e.g., [6] and references therein).

Our main result is:

THEOREM 1.1. *Let $\bar{u}$ and ψ as above then*

$$0 \leq \Delta_{\mathbb{H}^n} \bar{u} \leq \Big(\Delta_{\mathbb{H}^n} \psi\Big)^+ \tag{4}$$

in the sense of distributions. As usual, the superscript "+" denotes the positive part of a function, i.e. $f^+(x) := \max\{f(x), 0\}$.

The result in Theorem 1.1 is quite intuitive: when $\bar{u}$ does not touch the obstacle, it is free to make the operator vanish. When it touches and sticks to it, the operator computed in $\bar{u}$ is driven by the positive part of the same operator computed in the obstacle – and on these touching points the obstacle has to bend in a somewhat convex fashion, which justifies the first inequality in (4) and superscript "+" in the right hand side of (4).

Figure 1, in which the thick curve represents the touching between $\bar{u}$ and the obstacle, tries to describe this phenomena. On the other hand, the set in which $\bar{u}$ touches the obstacle may be very wild, so the actual proof of Theorem 1.1 needs to be more technical than this.

In fact, the first inequality of (4) is quite obvious since it follows, for instance, by taking $v := \bar{u} - \varphi$ in (3), with an arbitrary $\varphi \in C_0^\infty(\Omega, [0, +\infty)))$, so the core of (4) lies on the second inequality: nevertheless, we think it is useful to write (4) in this way to emphasize a control from both the sides of the operator applied to the solution.

We remark that the right hand side of (4) is always finite (due to the regularity of the obstacle). Hence, (4) is an L^∞-bound and may be seen as a regularity result for the solution of the obstacle problem.

In the Euclidean setting, the analogue of (4) was first obtained in [11] for the Laplacian case, and it is therefore often referred to with the name of Lewy-Stampacchia Estimate. It is also called Dual Estimate, for it is, in a sense, obtained by the duality expressed by the variational inequality (3).

[1] The proof of (3) is standard. See however the footnote on page 27 for a general argument.

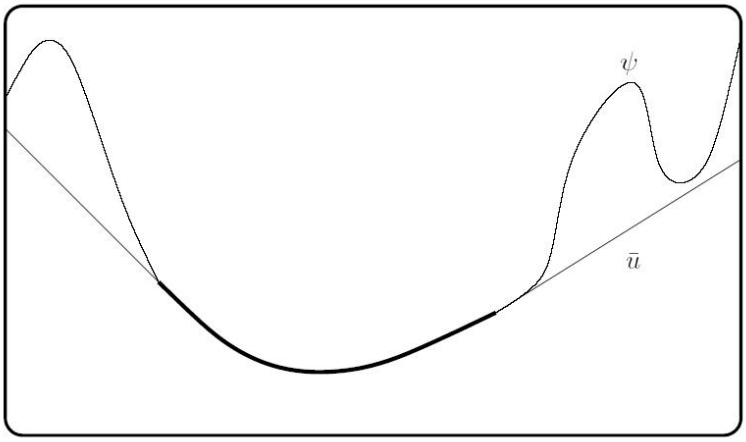

Figure 1: *Touching the obstacle*

After [11], estimates of these type became very popular and underwent many important extensions and strengthenings: see, among the others, [15, 9, 8, 1, 14].

The paper is organized as follows. First, in § 2, we discuss some possible extensions of Theorem 1.1 to the quasilinear case, once a more comprehensive regularity theory will become available. This will lead to a somewhat more general form of Theorem 1.1, namely Theorem 2.2 below (which will introduce an auxiliary parameter $\varepsilon \geq 0$). Then, in § 3, we prove Theorem 2.2 when $\varepsilon > 0$. The proof when $\varepsilon = 0$ is contained in § 4–5 and it is based on a limit argument, i.e., we consider the problem with $\varepsilon > 0$, we use Theorem 2.2 in such a case, and then we pass $\varepsilon \searrow 0$. The paper ends with an Appendix that collects some ancillary results needed in § 4.

2. Possible extension to the quasilinear case (waiting for a more exhaustive regularity theory)

Now we try to give some ideas of how Theorem 1.1 could be generalized to the quasilinear setting. In particular, we prove that for a suitable set of exponents $\mathcal{P}(\psi, \Omega)$ (see Definition 2.1 and Theorem 2.2) an analogue of Theorem 1.1 holds for the Heisenberg group version of the p-Laplace operator[2].

[2]We inform the reader that our result in Theorem 2.2 is far from being exhaustive in the quasilinear case, since, in principle, we are only able to prove explicitly that $2 \in \mathcal{P}(\psi, \Omega)$. The primary source of difficulties to decide whether $p \in \mathcal{P}(\psi, \Omega)$ is the absence of a satisfactory Hölder regularity theory for the horizontal gradient for solutions of quasilinear equations in

The notation we use is the following. Given $p \in (1, +\infty)$, a smooth domain $\Omega_\star \Supset \Omega$, $u_\star \in W^{1,p}_{\mathbb{H}^n}(\Omega_\star) \cap L^\infty(\Omega_\star)$ and $\varepsilon \geq 0$, we consider the minimization problem

$$\inf_{u \in \mathcal{K}_p} \mathcal{F}_\varepsilon(u;\Omega), \text{ where } \mathcal{F}_\varepsilon(u;\Omega) := \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}, \tag{5}$$

and

$$\mathcal{K}_p := \big\{ u \in W^{1,p}_{\mathbb{H}^n}(\Omega) \text{ s.t. } u \leq \psi, \text{ and } u - u_\star \in W^{1,p}_{\mathbb{H}^n,0}(\Omega) \big\}. \tag{6}$$

By comparing (1) and (6), we observe that $\mathcal{K}_p$ reduces to $\mathcal{K}$ when $p = 2$. Hence, the minimization problem in (5) reduces to the one in (2) when $p = 2$ and $\varepsilon = 0$.

We notice that $\bar{u}_\varepsilon$ is a solution of the variational inequality[3]

$$\int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n} (v - \bar{u}_\varepsilon) \ \geq\ 0, \tag{7}$$

for any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$. Now, we introduce the set of p's for which our main result holds. The definition we give is slightly technical, but, roughly speaking, consists in taking the set of all the p's for

the Heisenberg group. Namely, if one knew that for a given p bounded solutions of $\mathrm{div}_{\mathbb{H}^n}\,((\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u) = f$, with f bounded, have Hölder continuous horizontal gradient, with interior estimates (this would be the Heisenberg counterpart of classical regularity results for the Euclidean case, see, e.g., Theorem 1 in [17]) then $p \in \mathcal{P}(\psi, \Omega)$. As far as we know, such a theory has not been developed yet, not even for minimal solutions (see, however, [3, 12, 13, 19] where good $C^{1,\alpha}$ estimates are proved for the case of homogeneous equations). On the other hand, we think it is worth pointing out how Theorem 1.1 could be generalized in the generality allowed by the set $\mathcal{P}(\psi, \Omega)$, since once the regularity theory becomes available, our result would be valid in general – and also because the setting we use is somewhat more general and weaker than the regularity theory itself.

We stress that the quasilinear case in the Heisenberg group is more problematic than expected at a first glance, and many basic fundamental questions are still open (see, e.g., [7], [12], [13] and [19]).

[3]Formula (7) may be easily obtained this way. Fixed $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, for any $t \geq 0$, let $u^{(t)} := \bar{u}_\varepsilon + t(v - \bar{u}_\varepsilon)$. Notice that

$$u^{(t)} := (1-t)\bar{u}_\varepsilon + tv \leq (1-t)\psi + t\psi \leq \psi,$$

hence $u^{(t)} \in \mathcal{K}_p$. So, by the minimality of $\bar{u}_\varepsilon$, we have $\mathcal{F}_\varepsilon(u^{(0)};\Omega) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon;\Omega) \leq \mathcal{F}_\varepsilon(u^{(t)};\Omega)$ for any $t \geq 0$. Consequently,

$$\begin{aligned} 0 \ &\leq\ \lim_{t \searrow 0} \frac{\mathcal{F}_\varepsilon(u^{(t)};\Omega) - \mathcal{F}_\varepsilon(u^{(0)};\Omega)}{t} \\ &=\ \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n} (v - \bar{u}_\varepsilon), \end{aligned}$$

that is (7). Once again, (7) reduces to (3) when $p = 2$ (and in this case ε does not play any role).

which a pointwise bound for the operator of a sequence of minimal solutions is stable under uniform limits.

DEFINITION 2.1. *Let $p \in (1, +\infty)$. We say that $p \in \mathcal{P}(\psi, \Omega)$ if the following property holds true: For any $\varepsilon > 0$, any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, any $M > 0$, any sequence $F_k = F_k(r, \xi) \in C([-M, M] \times \Omega)$, with $F_k(\cdot, \xi) \in C^1([-M, M])$ and*

$$0 \leq \partial_r F_k \leq \Big(\mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi\Big)\Big)^+, \tag{8}$$

if $u_k : \Omega \to [-M, M]$ is a sequence of minimizers of the functional

$$\int_\Omega \frac{1}{p} (\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_k(u(\xi), \xi) \, d\xi \tag{9}$$

over the functions $u \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u - v \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, with the property that u_k converges to some u_∞ uniformly in Ω, we have that

$$\begin{aligned} 0 &\leq \mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\infty|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\infty\Big) \\ &\leq \Big(\mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi\Big)\Big)^+ \end{aligned} \tag{10}$$

in the sense of distributions.

As remarked in Lemma 5.7 at the end of this paper, we always have that

$$2 \in \mathcal{P}(\psi, \Omega). \tag{11}$$

We think that it is an interesting open problem to decide whether or not other values of p belong to $\mathcal{P}(\psi, \Omega)$, in general, or at least when the right hand side of (10) is close to zero (e.g., when the obstacle is almost flat).

With this notation, the following result holds true:

THEOREM 2.2. *If $p \in \mathcal{P}(\psi, \Omega)$ then*

$$\begin{aligned} 0 &\leq \mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon\Big) \\ &\leq \Big(\mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi\Big)\Big)^+ \end{aligned} \tag{12}$$

in the sense of distributions.

Notice that Theorem 1.1 is a particular case of Theorem 2.2 when $p = 2$, thanks to (11). Therefore, in the sequel, we will prove Theorem 2.2 and so Theorem 1.1 will follow as a consequence.

We point out that the arguments that we present are step-free, i.e. they do not directly depend on the stratification step of $\mathbb{H}^n$ apart from the definition of the homogeneous dimension. More precisely, since our arguments are based only on the intrinsic gradient concept $\nabla_{\mathbb{H}^n}$ and the homogeneous dimension of $\mathbb{H}^n$, we can restate Theorems 1.1 and 2.2 in any nilpotent stratified Lie groups of any step $\mathbb{G}$ simply changing $\mathrm{div}_{\mathbb{H}^n}$ and $\nabla_{\mathbb{H}^n}$ with $\mathrm{div}_{\mathbb{G}}$ and $\nabla_{\mathbb{G}}$. Here $\mathrm{div}_{\mathbb{G}}$ and $\nabla_{\mathbb{G}}$ are respectively the intrinsic divergence and the intrinsic gradient in $\mathbb{G}$ (see [18]). So here we work in $\mathbb{H}^n$ only for the sake of notational simplicity.

3. Proof of Theorem 2.2 when $\varepsilon > 0$

We prove (12) in the simpler case $\varepsilon > 0$ (the case $\varepsilon = 0$ will be dealt with in § 5). The technique used in this proof is a variation of a classical penalized test function method (see, e.g., [15, 9, 8, 1, 14] and references therein), and several steps of this proof are inspired by some estimates obtained by [4] in the Euclidean case.

First of all, we set

$$\mu := -1 + \min\left\{ \inf_{\overline{\Omega}} \psi,\ \inf_{\overline{\Omega}} u_\star \right\} \in \mathbb{R}$$

and we observe that

$$\bar{u}_\varepsilon \geq \mu \tag{13}$$

a.e. in Ω. Indeed, let $w := \max\{\bar{u}_\varepsilon, \mu\}$. Since ψ and $u_\star$ are below μ in Ω, we have that $w \in \mathcal{K}$, thus

$$0 \leq \mathcal{F}_\varepsilon(w; \Omega) - \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \Omega) = -\int_{\Omega \cap \{\bar{u}_\varepsilon < \mu\}} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \leq 0,$$

and, from this, (13) plainly follows.

Now, let $\eta \in (0, 1)$, to be taken arbitrarily small in the sequel. Let also

$$h := \Big(\mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}\psi\Big)\Big)^+ . \tag{14}$$

Notice that

$$\|h\|_{L^\infty(\Omega)} < +\infty, \tag{15}$$

because $\varepsilon > 0$. For any $t \in \mathbb{R}$, we consider the truncation function

$$H_\eta(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t/\eta & \text{if } 0 < t < \eta, \\ 1 & \text{if } t \geq \eta. \end{cases}$$

Now, we take u_η to be a weak solution of

$$\begin{cases} \mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta\Big) = h \cdot \big(1 - H_\eta(\psi - u_\eta)\big) & \text{in } \Omega, \\ u_\eta = \bar{u}_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{16}$$

where, as usual, the boundary datum is attained in the trace sense: such a function u_η may be obtained by the direct method in the calculus of variations, by minimizing the functional

$$\int_\Omega \frac{1}{p}(\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_\eta(u(\xi),\xi)\,d\xi$$

over $u \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u - \bar u_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, where

$$F_\eta(r,\xi) := \int_0^r h(\xi)\cdot\big(1 - H_\eta(\psi(\xi)-\sigma)\big)\,d\sigma.$$

Now, we claim that

$$u_\eta \le \psi \text{ a.e. in } \Omega. \tag{17}$$

To establish this, we use the test function $(u_\eta - \psi)^+$ in (16). Since, on $\partial\Omega$, we have $(u_\eta - \psi)^+ = (\bar u_\varepsilon - \psi)^+ = 0$, we obtain that

$$\begin{aligned} & -\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u_\eta\Big)\cdot \nabla_{\mathbb{H}^n}(u_\eta-\psi)^+ \\ = & \int_\Omega h\cdot\big(1 - H_\eta(\psi - u_\eta)\big)(u_\eta-\psi)^+ = \int_\Omega h\cdot(u_\eta-\psi)^+. \end{aligned}$$

Consequently, by (14),

$$\begin{aligned} & \int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u_\eta\Big) \\ & \qquad -\Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \psi\Big)\Big]\cdot \nabla_{\mathbb{H}^n}(u_\eta-\psi)^+ \\ = & \int_\Omega \Big[\mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \psi\Big) - h\Big]\cdot(u_\eta-\psi)^+ \\ \le & \ 0. \end{aligned}$$

By the strict monotonicity of the operator (i.e., by the strict convexity of the function $\mathbb{R}^{2n} \ni \zeta \mapsto (\varepsilon + |\zeta|^2)^{p/2}$), it follows that $(u_\eta - \psi)^+$ vanishes almost everywhere in Ω, proving (17).

Now, we claim that

$$\bar u_\varepsilon \ge u_\eta \text{ a.e. in } \Omega. \tag{18}$$

To verify this, we consider the test function

$$\tau := \bar u_\varepsilon + (u_\eta - \bar u_\varepsilon)^+.$$

We notice that

$$\tau = \begin{cases} \bar u_\varepsilon & \text{in } \{u_\eta \le \bar u_\varepsilon\}, \\ u_\eta & \text{in } \{u_\eta > \bar u_\varepsilon\}, \end{cases}$$

hence $\tau \le \psi$, due to (17). Furthermore, on $\partial\Omega$, we have that $\tau = \bar{u}_\varepsilon$, due to the boundary datum in (16). Therefore the obstacle problem variational inequality (3) gives that

$$
\begin{aligned}
0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\tau - \bar{u}_\varepsilon) \\
&= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+ .
\end{aligned}
\tag{19}
$$

On the other hand, from (16),

$$
\begin{aligned}
&\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+ \\
&\qquad = - \int_\Omega h \cdot \big(1 - H_\eta(\psi - u_\eta)\big) \cdot (u_\eta - \bar{u}_\varepsilon)^+ \le 0.
\end{aligned}
\tag{20}
$$

By (19) and (20), we obtain that

$$
\begin{aligned}
&\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \\
&\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \Big] \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+ \le 0.
\end{aligned}
$$

This and the strict monotonicity of the operator implies that $(u_\eta - \bar{u}_\varepsilon)^+$ vanishes almost everywhere in Ω, hence proving (18).

Now, we claim that

$$
\bar{u}_\varepsilon \le u_\eta + \eta \text{ in } \Omega. \tag{21}
$$

To do this, we set

$$
\theta := \bar{u}_\varepsilon - (\bar{u}_\varepsilon - u_\eta - \eta)^+ .
$$

Notice that $\theta \le \bar{u}_\varepsilon \le \psi$, and also that, on $\partial\Omega$, $\theta = \bar{u}_\varepsilon$. As a consequence, (3) gives that

$$
\begin{aligned}
0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\theta - \bar{u}_\varepsilon) \\
&= - \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+ .
\end{aligned}
\tag{22}
$$

On the other hand, $(\bar{u}_\varepsilon - u_\eta - \eta)^+ = 0$ on $\partial\Omega$, and

$$
\begin{aligned}
\{\bar{u}_\varepsilon - u_\eta - \eta > 0\} &\subseteq \{\psi - u_\eta > \eta\} \\
&\subseteq \big\{1 - H_\eta(\psi - u_\eta) = 0\big\},
\end{aligned}
$$

and therefore, by (16),

$$\begin{aligned}
&\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta+\eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}(u_\eta+\eta) \Big) \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \\
&= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \\
&= -\int_\Omega h \cdot \big(1 - H_\eta(\psi - u_\eta)\big) \cdot (\bar u_\varepsilon - u_\eta - \eta)^+ = 0.
\end{aligned} \tag{23}$$

Then, (22) and (23) yield that

$$\begin{aligned}
&\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \\
&\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta+\eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}(u_\eta+\eta) \Big) \Big] \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \\
&\qquad \le 0.
\end{aligned}$$

Thus, in this case, the strict monotonicity of the operator implies that $(\bar u_\varepsilon - u_\eta - \eta)^+$ vanishes almost everywhere in Ω, and so (21) is established.

In particular, by (17), (21) and (13),

$$\|u_\eta\|_{L^\infty(\Omega)} \le 2 + \|\psi\|_{L^\infty(\Omega)} + \|u_\star\|_{L^\infty(\Omega)}. \tag{24}$$

Moreover, by (18) and (21), we have that

$$u_\eta \text{ converges uniformly in } \Omega \text{ to } \bar u_\varepsilon \tag{25}$$

as $\eta \searrow 0$.

Furthermore

$$0 \le \partial_r F_\eta(r,\xi) \le h(\xi) = \Big(\mathrm{div}_{\mathbb{H}^n} \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}\psi \Big) \Big)^+$$

hence (12) follows[4] from (25) and the fact that $p \in \mathcal{P}(\psi,\Omega)$ (recall (10) in Definition 2.1). $\square$

4. Estimating the L^p-distance between $\nabla_{\mathbb{H}^n}\bar u_0$ and $\nabla_{\mathbb{H}^n}\bar u_\varepsilon$

The purpose of this section is to consider the solution $\bar u_\varepsilon$ of the ε-problem and the solution $\bar u_0$ of the problem with $\varepsilon = 0$, and to bound the L^p-norm of $|\nabla_{\mathbb{H}^n}\bar u_0 - \nabla_{\mathbb{H}^n}\bar u_\varepsilon|$. Such estimate is quite technical and it is different according to the cases $p \in (1,2]$ and $p \in [2,+\infty)$: see the forthcoming Propositions 4.1 and 4.2.

---

[4] It is worth pointing out that this is the only place in the paper where we use the condition that $p \in \mathcal{P}(\psi,\Omega)$. In particular, all the estimates in § 4 are valid for all $p \in (1,+\infty)$.

As a matter of fact, we think that the estimates proved in Propositions 4.1 and 4.2 are of independent interest, since they also allow to get around the more difficult (and in general not available in the Heisenberg group) Hölder-type estimates.

We recall the standard notation of balls in the Heisenberg group (in fact, we deal with the so called Folland-Korány balls, but the Carnot-Carathéodory balls would be good for our purposes too). For all $\xi := (z,t) \in \mathbb{R}^{2n} \times \mathbb{R}$, we define

$$\|\xi\|_{\mathbb{H}^n} := \sqrt[4]{|z|^4 + t^2}.$$

Then, for any $r > 0$, we set

$$\mathcal{B}_r := \big\{\xi \in \mathbb{R}^{2n+1} \text{ s.t. } \|\xi\|_{\mathbb{H}^n} < r\big\}.$$

We denote by $\mathcal{L}$ the $(2n+1)$-dimensional Lebesgue measure, and we observe that $\mathcal{L}(\mathcal{B}_r)$ equals, up to a multiplicative constant r^Q, where $Q := 2(n+1)$ is the homogeneous dimension of $\mathbb{H}^n$. Also, for all $g \in L^1(\mathcal{B}_r)$, we define the average of g in $\mathcal{B}_r$ as

$$(g)_r := \frac{1}{\mathcal{L}(\mathcal{B}_r)} \int_{\mathcal{B}_r} g.$$

In what follows, we focus on L^p-estimates around a fixed point, say $\xi_\star$, of Ω. Without loss of generality, we take $\xi_\star$ to be the origin, and we fix $R \in (0,1)$ so small that $\mathcal{B}_R \Subset \Omega$.

Then, we denote by $\bar{u}_0 : \Omega \to \mathbb{R}$ the minimizer of problem (2) with $\varepsilon = 0$. Then, for a fixed $\varepsilon > 0$, we take $\bar{u}_\varepsilon : \mathcal{B}_R \to \mathbb{R}$ to be the minimizer of $\mathcal{F}_\varepsilon(u; \mathcal{B}_R)$ among all the functions $u \in W^{1,p}_{\mathbb{H}^n}(\mathcal{B}_R)$, $u \le \psi$, and $u - \bar{u}_0 \in W^{1,p}_{\mathbb{H}^n,0}(\mathcal{B}_R)$. We can then extend $\bar{u}_\varepsilon$ also on $\Omega \setminus \mathcal{B}_R$ by setting it equal to $\bar{u}_0$ in such a set. By construction

$$\begin{aligned} \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p &= \mathcal{F}_0(\bar{u}_0; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p \\ &\le \mathcal{F}_0(\bar{u}_\varepsilon; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p = \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \end{aligned} \tag{26}$$

and

$$\begin{aligned} \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} &= \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \mathcal{B}_R) \\ &\le \mathcal{F}_\varepsilon(\bar{u}_0; \mathcal{B}_R) = \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \tag{27}$$

Proposition 4.1. *Assume that*

$$p \in (1,2]. \tag{28}$$

Then, there exists $C > 0$, only depending on n and p, such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \le C\big(1 + (|\nabla_{\mathbb{H}^n} \bar{u}_0|^p)_R\big)^{1-(p/2)} \varepsilon^{(p/2)^2} R^Q. \tag{29}$$

Proof. The technique for this proof is inspired by the one of Lemma 2.3 of [16], where a similar result was obtained in the quasilinear Euclidean case (however, our proof is self-contained). We have

$$\begin{aligned} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 &\le \big(|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| + |\nabla_{\mathbb{H}^n} \bar{u}_0|\big)^2 \\ &\le C\big(|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2\big). \end{aligned} \tag{30}$$

Here, C is a positive constant, which is free to be different from line to line. By (28), (27) and (30), we obtain

$$\begin{aligned} &\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 \\ &\le C \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \\ &= C \Bigg(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \\ &\qquad + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \Bigg) \\ &\le C \Bigg(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{1-(p/2)}} \Bigg) \\ &\le C \Bigg(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} + \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2} \Bigg) \\ &\le C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \tag{31}$$

Thus, (31) and Lemma 5.4, applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $b := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$, yield that

$$\begin{aligned} &\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \\ &\quad \le C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 + \\ &\qquad + C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} \\ &\quad \le C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)}. \end{aligned} \tag{32}$$

Now, from (26),

$$\begin{aligned}
&\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)} \\
&\qquad \le \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\qquad \le \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^p.
\end{aligned} \tag{33}$$

Moreover, using (28) and some elementary calculus, we see that

$$|(1+\tau)^{p/2} - \tau^{p/2}| \le C$$

for any $\tau \ge 0$. Therefore, taking $\tau := \theta/\varepsilon$, we obtain that

$$|(\varepsilon+\theta)^{p/2} - \theta^{p/2}| \le C\varepsilon^{p/2} \tag{34}$$

for any $\theta \ge 0$. Thus, using (33) and (34) with $\theta := |\nabla_{\mathbb{H}^n}\bar{u}_0|^2$, we conclude that

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)} \le C\varepsilon^{p/2}R^Q. \tag{35}$$

Now, we estimate the left hand side of (35) from below. For this scope, we define

$$\begin{aligned}
&h := t\nabla_{\mathbb{H}^n}\bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon, \\
&J := p\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \\
\text{and}\quad &\tilde{J} := p\int_{\mathcal{B}_R} \Big[\int_0^1 (1-t)\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1} h\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\Big)\,dt\Big].
\end{aligned}$$

We observe that the variational inequality in (3) for $\bar{u}_\varepsilon$ gives that

$$J \ge 0. \tag{36}$$

Also, using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)} \\
&\quad = \int_{\mathcal{B}_R} \Big[\int_0^1 \frac{d}{dt}(\varepsilon + |t\nabla_{\mathbb{H}^n}\bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)}\,dt\Big] \\
&\quad = p\int_{\mathcal{B}_R} \Big[\int_0^1 (\varepsilon + |t\nabla_{\mathbb{H}^n}\bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1} \\
&\qquad\quad \times (t\nabla_{\mathbb{H}^n}\bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\,dt\Big] \\
&\quad = p\int_{\mathcal{B}_R} \Big[\int_0^1 (\varepsilon + |h|^2)^{(p/2)-1} h\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\,dt\Big].
\end{aligned}$$

Integrating by parts the latter integral in t (by writing $dt = \frac{d}{dt}(t-1)\,dt$), and exploiting (36), we obtain

$$\begin{aligned}\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)} \\ = J + \tilde{J} \geq \tilde{J}.\end{aligned} \tag{37}$$

Making use of Lemma 5.3 – applied here with $a := \nabla_{\mathbb{H}^n}\bar{u}_0$ and $b := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$ – we have that

$$\tilde{J} \geq \frac{1}{C}\int_{\mathcal{B}_R}\Big[\int_0^1 (1-t)(\varepsilon+|t\nabla_{\mathbb{H}^n}\bar{u}_0+(1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2\,dt\Big].$$

From this and Lemma 5.5 – applied here with $\kappa := 1$ and $\Psi(x) := x^{1-(p/2)}$, which is nondecreasing, thanks to (28) – we deduce that

$$\tilde{J} \geq \frac{1}{C}\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2 + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2. \tag{38}$$

In view of (35), (37) and (38), we conclude that

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2 + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2 \leq C\varepsilon^{p/2}R^Q. \tag{39}$$

Then, (29) follows from (32), (39) and Lemma 5.6, applied here with $f := \nabla_{\mathbb{H}^n}\bar{u}_0$ and $g := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$. □

In the degenerate case $p \in [2, +\infty)$ the estimate obtained in Proposition 4.1 for the singular case $p \in (1, 2]$ needs to be modified according to the following result:

Proposition 4.2. *Suppose that*

$$p \in [2, +\infty). \tag{40}$$

Then, there exists $C > 0$, only depending on n and p, such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \leq C\big(1 + (|\nabla_{\mathbb{H}^n}\bar{u}_0|^p)_R\big)^{1-(1/p)}\varepsilon R^Q.$$

Proof. The variational inequalities (3) for $\bar{u}_0$ and $\bar{u}_\varepsilon$ imply that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 \cdot (\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - \nabla_{\mathbb{H}^n}\bar{u}_0) \geq 0$$

$$\text{and}\qquad \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \geq 0.$$

Consequently,

$$\int_{\mathcal{B}_R} \Big(|\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 - (\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \le 0.$$

Using this and (46) of Lemma 5.1, applied here with $A := \nabla_{\mathbb{H}^n}\bar{u}_0$ and $B := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\le C\int_{\mathcal{B}_R} \Big(|\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \\
&\le C\int_{\mathcal{B}_R} \Big((\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon).
\end{aligned}$$

This and Corollary 5.2, applied here with $a := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$, give

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\le C\int_{\mathcal{B}_R} \Big((\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1} - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\Big)\,|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|\,|\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon| \\
&\le C\varepsilon\int_{\mathcal{B}_R} (\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p-2)/2}\big(|\nabla_{\mathbb{H}^n}\bar{u}_0|+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|\big).
\end{aligned}$$

Therefore, recalling (40), noticing that

$$\frac{p-2}{p}+\frac{1}{p}+\frac{1}{p}=1$$

and using the Generalized Hölder Inequality with the three exponents $p/(p-2)$, p and p, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\le C\varepsilon\left(\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-2)/p}\left(\int_{\mathcal{B}_R}\big(|\nabla_{\mathbb{H}^n}\bar{u}_0|^p+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\big)\right)^{1/p} R^{Q/p}.
\end{aligned}$$

Then, by the minimal property of $\bar{u}_0$ in (26),

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
\le{}& C\varepsilon\left(\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-2)/p}\left(\int_{\mathcal{B}_R}|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\right)^{1/p} R^{Q/p} \\
\le{}& C\varepsilon\left(\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-1)/p} R^{Q/p} \\
\le{}& C\varepsilon\left(R^Q+\int_{\mathcal{B}_R}|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\right)^{(p-1)/p} R^{Q/p} \\
\le{}& C\varepsilon\left(R^Q+\int_{\mathcal{B}_R}|\nabla_{\mathbb{H}^n}\bar{u}_0|^p\right)^{(p-1)/p} R^{Q/p},
\end{aligned}$$

from which the desired result follows. □

Corollary 4.3. *For all $p\in(1,+\infty)$, we have that*

$$\lim_{\varepsilon\searrow 0}\|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - \nabla_{\mathbb{H}^n}\bar{u}_0\|_{L^p(\mathcal{B}_R)} = 0. \tag{41}$$

Also, there exist a subsequence of ε's and a function $G\in L^p(\mathcal{B}_R)$ such that

$$|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon(x)| \le G(x) \tag{42}$$

for almost every $x\in\mathcal{B}_R$.

Furthermore, if we set

$$\Gamma_\varepsilon := (\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon, \tag{43}$$

then there exist a subsequence of ε's and a function $G_\star\in L^1(\mathcal{B}_R)$ such that

$$|\Gamma_\varepsilon(x)| \le G_\star(x) \tag{44}$$

for almost every $x\in\mathcal{B}_R$.

Proof. We obtain (41) from Propositions 4.1 and 4.2, according to whether $p\in(1,2)$ or $p\in[2,+\infty)$.

From (41), one deduces (42) (see, e.g., Theorem 4.9(b) in [2]).

Now, we define $G_\star := 2^{(p/2)}(G+G^{p-1})$. We observe that $G_\star\in L^1(\mathcal{B}_R)$, since $G\in L^p(\mathcal{B}_R)\subseteq L^1(\mathcal{B}_R)$ and $G^{p-1}\in L^{p/(p-1)}(\mathcal{B}_R)\subseteq L^1(\mathcal{B}_R)$. So, in order to obtain the desired result, we have only to show that the inequality in (44) holds true.

For this, we notice that, if $p \in (1,2)$,

$$
\begin{aligned}
|\Gamma_\varepsilon| &= \frac{|\nabla_{\mathbb{H}^n} \bar u_\varepsilon|}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{1-(p/2)}} = \frac{|\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^{p-1} |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^{2-p}}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{1-(p/2)}} \\
&\le \frac{|\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^{p-1} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(2-p)/2}}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{1-(p/2)}} = |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^{p-1} \le G^{p-1},
\end{aligned}
$$

which implies (44) in this case.

On the other hand, if $p \in [2, +\infty)$,

$$
\begin{aligned}
|\Gamma_\varepsilon| &\le 2^{(p/2)-1} \big(\varepsilon^{(p/2)-1} + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^{p-2}\big) |\nabla_{\mathbb{H}^n} \bar u_\varepsilon| \\
&\le 2^{(p/2)-1} (1 + G^{p-2}) G,
\end{aligned}
$$

which implies (44) in this case too. □

5. Proof of Theorem 2.2 when $\varepsilon = 0$

By Theorem 2.2 (for $\varepsilon > 0$, which has been proved in § 3), we know that, for a sequence $\varepsilon \searrow 0$,

$$
0 \le \int_{\mathcal{B}_R} \Gamma_\varepsilon \cdot \nabla\varphi \le \int_{\mathcal{B}_R} \Big(\mathrm{div}_{\mathbb{H}^n} \, \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \Big) \Big)^+ \varphi, \tag{45}
$$

for any $\varphi \in C_0^\infty(\mathcal{B}_R, [0, +\infty))$, as long as $\mathcal{B}_R \subset \Omega$, where Γ_ε is as in (43).

By possibly taking subsequences, in the light of (41) and (44), we have that

$$
\lim_{\varepsilon \searrow 0} \Gamma_\varepsilon = |\nabla_{\mathbb{H}^n} \bar u_0|^{p-2} \nabla_{\mathbb{H}^n} \bar u_0
$$

almost everywhere in $\mathcal{B}_R$, and that Γ_ε is equidominated in $L^1(\mathcal{B}_R)$. Consequently, we can pass to the limit in (45) via the Dominated Convergence Theorem and obtain (12) for $\bar u_0$. This completes the proof of Theorem 2.2 also when $\varepsilon = 0$. □

Appendix

In this appendix, we collect some technical and well known estimates of general interest that will be used in the proofs of the main results of this paper.

We start with some classical estimates (see, e.g. Lemma 3 in [10] and references therein), which turns out to be quite useful to deal with nonlinear operators of degenerate type:

Lemma 5.1. *Let $M \in \mathbb{N}$, $M \geq 1$, and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on M and p, such that, for any A, $B \in \mathbb{R}^M$,*

$$|A - B|^p \leq C\Big(|A|^{p-2}A - |B|^{p-2}B\Big) \cdot (A - B) \tag{46}$$

and

$$\Big||A|^{p-2}A - |B|^{p-2}B\Big| \leq C|A - B| \left(|A|^{p-2} + |B|^{p-2}\right). \tag{47}$$

Corollary 5.2. *Let $N \in \mathbb{N}$ and and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on N and p, such that for any $\varepsilon > 0$ and any $a \in \mathbb{R}^N$*

$$\left((\varepsilon + |a|^2)^{(p/2)-1} - |a|^{p-2}\right)|a| \leq C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2}.$$

Proof. We let $A := (a, \varepsilon)$ and $B := (a, 0) \in \mathbb{R}^{N+1}$ and we exploit (47). We obtain

$$\begin{aligned}
&2C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2} \\
&\quad \geq C\varepsilon\Big((\varepsilon + |a|^2)^{(p-2)/2} + |a|^{p-2}\Big) \\
&\quad = C|A - B| \left(|A|^{p-2} + |B|^{p-2}\right) \\
&\quad \geq \Big||A|^{p-2}A - |B|^{p-2}B\Big| \\
&\quad = \Big|(\varepsilon + |a|^2)^{(p-2)/2}(a, \varepsilon) - |a|^{p-2}(a, 0)\Big| \\
&\quad = \Big|\Big(\big((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2}\big)a,\ (\varepsilon + |a|^2)^{(p-2)/2}\varepsilon\Big)\Big| \\
&\quad \geq \left((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2}\right)|a|,
\end{aligned}$$

as desired. □

In the subsequent Lemmata 5.3 and 5.4, we collect some simple, but interesting, estimates that are used in Proposition 4.1:

Lemma 5.3. *Let $N \in \mathbb{N}$, $N \geq 1$, $t \in [0, 1]$, $\varepsilon > 0$, and a, $b \in \mathbb{R}^N$. Let $h(t) := ta + (1 - t)b$. Then, there exists $C > 1$, only depending on N and p, such that*

$$\frac{d}{dt}\Big((\varepsilon + |h|^2)^{(p/2)-1} h \cdot (a - b)\Big) \geq \frac{1}{C}(\varepsilon + |ta + (1 - t)b|^2)^{(p/2)-1}|a - b|^2.$$

Proof. We have

$$\begin{aligned}
&\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(a-b)\Big)=\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\Big)\cdot(a-b)\\
&\quad=(\varepsilon+|h|^2)^{(p/2)-2}\big(\varepsilon+(p-1)|h|^2\big)\frac{dh}{dt}\cdot(a-b)\\
&\quad\geq\frac{1}{C}(\varepsilon+|h|^2)^{(p/2)-1}|a-b|^2\\
&\quad=\frac{1}{C}(\varepsilon+|ta+(1-t)b|^2)^{(p/2)-1}|a-b|^2,
\end{aligned}$$

as desired. □

Lemma 5.4. *Let*

$$p\in(1,2]. \tag{48}$$

Let $N\in\mathbb{N}$, $N\geq 1$, $\varepsilon>0$, and a, $b\in\mathbb{R}^N$. Then, there exists $C>1$, only depending on N and p, such that

$$(\varepsilon+|a|^2+|b|^2)^{p/2}\leq C\Big[(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+(\varepsilon+|a|^2)^{(p/2)}\Big].$$

Proof. We have

$$|b|^2=|b-a+a|^2\leq\big(|b-a|+|a|\big)^2\leq C\big(|b-a|^2+|a|^2\big)$$

and so

$$\begin{aligned}
&\quad(\varepsilon+|a|^2+|b|^2)^{p/2}\\
&=\quad(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2+|b|^2)\\
&\leq\quad C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2+|b-a|^2)\\
&=\quad C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2).
\end{aligned}$$

Therefore, by (48),

$$\begin{aligned}
&(\varepsilon+|a|^2+|b|^2)^{p/2}\\
&\qquad\leq C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+C(\varepsilon+|a|^2)^{(p/2)},
\end{aligned}$$

that is the desired claim. □

The following result deals with some technical estimates on monotone integrands.

Lemma 5.5. *Let $N\in\mathbb{N}$, $N\geq 1$. Let $\kappa\in\{0,1\}$. Let ε, $\varepsilon'>0$. Let a, $b\in\mathbb{R}^N$. Let $\Psi:[\varepsilon,+\infty)\to[\varepsilon',+\infty)$ be a measurable and nondecreasing function. Then*

$$\int_0^1\frac{(1-t)^\kappa}{\Psi(\varepsilon+|ta+(1-t)b|^2)}\,dt\geq\frac{1}{2\Psi(\varepsilon+|a|^2+|b|^2)}. \tag{49}$$

Proof. If $|a| \le |b|$, for any $t \in [0,1]$,

$$\begin{aligned}|ta+(1-t)b|^2 &\le t^2|a|^2+(1-t)^2|b|^2+2t(1-t)|a||b|\\ &\le t^2|b|^2+(1+t^2-2t)|b|^2+2t(1-t)|b|^2=|b|^2.\end{aligned}$$

On the other hand, if $|a| \ge |b|$, for any $t \in [0,1]$,

$$\begin{aligned}|ta+(1-t)b|^2 &\le t^2|a|^2+(1-t)^2|b|^2+2t(1-t)|a||b|\\ &\le t^2|a|^2+(1+t^2-2t)|a|^2+2t(1-t)|a|^2=|a|^2.\end{aligned}$$

In any case,

$$\varepsilon+|ta+(1-t)b|^2 \le \varepsilon+|a|^2+|b|^2$$

and the claim follows from the monotonicity of Ψ. □

The next is a useful Hölder/L^p type estimate, that is exploited in Proposition 4.1.

LEMMA 5.6. *Let $N \in \mathbb{N}$, $N \ge 1$. Let f, $g \in L^p(\mathcal{B}_R, \mathbb{R}^N)$. Suppose that*

$$p \in (1,2]. \tag{50}$$

Then

$$\begin{aligned}&\int_{\mathcal{B}_R}|f-g|^p\\ &\quad\le\left(\int_{\mathcal{B}_R}(\varepsilon+|f|^2+|g|^2)^{(p/2)-1}|f-g|^2\right)^{p/2}\\ &\qquad\times\left(\int_{\mathcal{B}_R}(\varepsilon+|f|^2+|g|^2)^{p/2}\right)^{(2-p)/2}.\end{aligned}$$

Proof. We observe that

$$\begin{aligned}&|f-g|^p\\ =\ &\Big[(\varepsilon+|f|^2+|g|^2)^{(p/2)-1}|f-g|^2\Big]^{p/2}\Big[(\varepsilon+|f|^2+|g|^2)^{p/2}\Big]^{(2-p)/2},\end{aligned}$$

and so the desired result follows from the Hölder Inequality with exponents $2/p$ and $2/(2-p)$, which can be used here due to (50). □

To end this paper, we remark that Definition 2.1 is always nonvoid (independently of ψ and Ω), in the sense that

LEMMA 5.7. $2 \in \mathcal{P}(\psi, \Omega)$.

Proof. The functional in (9) when $p = 2$ boils down to

$$\int_\Omega \frac{1}{2}|\nabla_{\mathbb{H}^n} u(\xi)|^2 + F_k(u(\xi), \xi)\, d\xi, \tag{51}$$

up to an additive constant that does not play any role in the minimization. Hence, if u_k minimizes this functional, we have that

$$-\int_\Omega \nabla_{\mathbb{H}^n} u_k(\xi) \cdot \nabla_{\mathbb{H}^n} \varphi(\xi)\, d\xi = \int_\Omega \partial_r F_k(u_k(\xi), \xi)\, \varphi(\xi)\, d\xi$$

for any $\varphi \in C_0^\infty(\Omega)$.

Accordingly, if also u_k approaches some u_∞ uniformly in Ω, it follows that

$$\begin{aligned} \int_\Omega u_\infty \Delta_{\mathbb{H}^n} \varphi &= \lim_{k\to+\infty} \int_\Omega u_k \Delta_{\mathbb{H}^n} \varphi \\ &= \lim_{k\to+\infty} -\int_\Omega \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi = \lim_{k\to+\infty} \int_\Omega \partial_r F_k(u_k, \xi)\, \varphi \end{aligned} \tag{52}$$

for any $\varphi \in C_0^\infty(\Omega)$.

Also, from (8),

$$0 \le \partial_r F_k \le (\Delta_{\mathbb{H}^n} \psi)^+$$

and so (52) gives that

$$0 \le \int_\Omega u_\infty \Delta_{\mathbb{H}^n} \varphi \le \int_\Omega (\Delta_{\mathbb{H}^n} \psi)^+\, \varphi \tag{53}$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$.

On the other hand, since u_k is a minimizer for (51), we have that

$$\sup_{k\in\mathbb{N}} \|\nabla_{\mathbb{H}^n} u_k\|_{L^2(\Omega)} < +\infty$$

and so, up to a subsequence, we may suppose that $\nabla_{\mathbb{H}^n} u_k$ converges to some $\nu \in L^2(\Omega)$ weakly in $L^2(\Omega)$. It follows from the uniform convergence of u_k that

$$\begin{aligned} -\int_\Omega \nu \cdot \nabla_{\mathbb{H}^n} \varphi &= -\lim_{k\to+\infty} \int_\Omega \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi \\ &= \lim_{k\to+\infty} \int_\Omega u_k\, \Delta_{\mathbb{H}^n} \varphi = \int_\Omega u_\infty\, \Delta_{\mathbb{H}^n} \varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$. That is, $\nabla_{\mathbb{H}^n} u_\infty = \nu$ in the sense of distributions, and so as a function. In particular, $\nabla_{\mathbb{H}^n} u_\infty \in L^2(\Omega)$, and therefore (53) yields that

$$0 \le \int_\Omega \nabla_{\mathbb{H}^n} u_\infty \cdot \nabla_{\mathbb{H}^n} \varphi \le \int_\Omega (\Delta_{\mathbb{H}^n} \psi)^+\, \varphi,$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$. This shows that u_∞ satisfies (10) in the distributional sense. □

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