NON-HERMITIAN SPECTRA and ANDERSON LOCALIZATION

Luca G Molinari
Physics Department
Universita' degli Studi di Milano

Abstract: a determinantal identity (spectral duality) and Jensen's theorem imply a formula for the exponents of a single generic transfer matrix in terms of the spectrum of the corresponding Hamiltonian matrix, with non Hermitian boundary conditions. Applications to Anderson model and BRM are presented.

Brunel, 20 dec 08

summary

- Determinants of block tridiagonal matrices and spectral duality
- Fifty years of Anderson Localization
- Jensen's theorem and the spectrum of (Lyapunov) exponents
- Energy spectra of non Hermitian Anderson matrices and BRM
- the smallest exponent

(LGM, arXiv:0808.1241 [math-ph])

determinants
of tridiagonal matrices

$$\det \begin{bmatrix} a_1 & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix} = \left[\begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_2 & -b_1 c_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 1 & 0 \end{pmatrix} \right]_{11}$$

$$\det \begin{bmatrix} a_1 & b_1 & & c_0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ b_n & & c_{n-1} & a_n \end{bmatrix} = (-1)^{n+1}(b_n \cdots b_1 + c_{n-1} \cdots c_0)$$

$$+ \operatorname{tr} \left[\begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_2 & -b_1 c_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & -b_n c_0 \\ 1 & 0 \end{pmatrix} \right]$$

determinants of block tridiagonal matrices

$$\mathrm{M}(z) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z}C_0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ zB_n & & C_{n-1} & A_n \end{bmatrix}$$

$$\det \mathrm{M}(\mathrm{z}) = \frac{(-1)^{\mathrm{nm}}}{(-\mathrm{z})^{\mathrm{m}}} \det[\mathrm{T} - \mathrm{z}\mathrm{I}_{2\mathrm{m}}] \det[\mathrm{B}_1 \ldots \mathrm{B}_\mathrm{n}]$$

$$\det \mathrm{M}^{(0)} = (-1)^{nm} \det[\mathrm{T}_{11}^{(0)}] \det[B_1 \cdots B_{n-1}]$$

$$\mathrm{T} = \begin{bmatrix} -B_n^{-1}A_n & -B_n^{-1}C_{n-1} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1}C_0 \\ I_m & 0 \end{bmatrix}$$

Theorem 1 (The Duality Relation)

$$\det[\lambda I_{nm} - \mathrm{M}(z)] = (-z)^{-m} \det[\mathrm{T}(\lambda) - zI_{2m}] \det[B_1 \cdots B_n]$$

L.G.M, Linear Algebra and its Applications 429 (2008) 2221

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

I. INTRODUCTION

A NUMBER of physical phenomena seem to involve quantum-mechanical motion, without any particular thermal activation, among sites at which the mobile entities (spins or electrons, for example) may be localized. The clearest case is that of spin diffusion[1,2]; another might be the so-called impurity band conduction at low concentrations of impurities. In such situations we suspect that transport occurs not by motion of free carriers (or spin waves), scattered as they move through a medium, but in some sense by quantum-mechanical jumps of the mobile entities from site to site. A second common feature of these phenomena is randomness: random spacings of impurities, random interactions with the "atmosphere" of other impurities, random arrangements of electronic or nuclear spins, etc.

Our eventual purpose in this work will be to lay the foundation for a quantum-mechanical theory of transport problems of this type. Therefore, we must start with simple theoretical models rather than with the complicated experimental situations on spin diffusion or impurity conduction. In this paper, in fact, we attempt only to construct, for such a system, the simplest model we can think of which still has some expectation of representing a real physical situation reasonably well, and to prove a theorem about the model. The theorem is that at sufficiently low densities, transport does not take place; the exact wave functions are localized in a small region of space. We also obtain a fairly good estimate of the critical density at which the theorem fails. An additional criterion is that the forces be of sufficiently short range—actually, falling off as $r\rightarrow\infty$ faster than $1/r^3$—and we derive a rough estimate of the rate of transport in the $V\propto 1/r^3$ case.

Such a theorem is of interest for a number of reasons: first, because it may apply directly to spin diffusion among donor electrons in Si, a situation in which Feher[3] has shown experimentally that spin diffusion is negligible; second, and probably more important, as an example of a real physical system with an infinite number of degrees of freedom, having no obvious oversimplification, in which the approach to equilibrium is simply impossible; and third, as the irreducible minimum from which a theory of this kind of transport, if it exists, must start. In particular, it re-emphasizes the caution with which we must treat ideas such as "the thermodynamic system of spin interactions" when there is no obvious contact with a real external heat bath.

The simplified theoretical model we use is meant to represent reasonably well one kind of experimental situation: namely, spin diffusion under conditions of

[1] N. Bloembergen, Physica 15, 386 (1949).
[2] A. M. Portis, Phys. Rev. 104, 584 (1956).
[3] G. Feher (private communication).

The Nobel Prize in Physics 1977

"for their fundamental theoretical investigations of the electronic structure of magnetic and disordered systems"


Philip Warren Anderson


Sir Nevill Francis Mott


John Hasbrouck van Vleck

Isaac Newton Institute for Mathematical Sciences

Mathematics and Physics of Anderson localization: 50 Years After

14 July - 19 December 2008

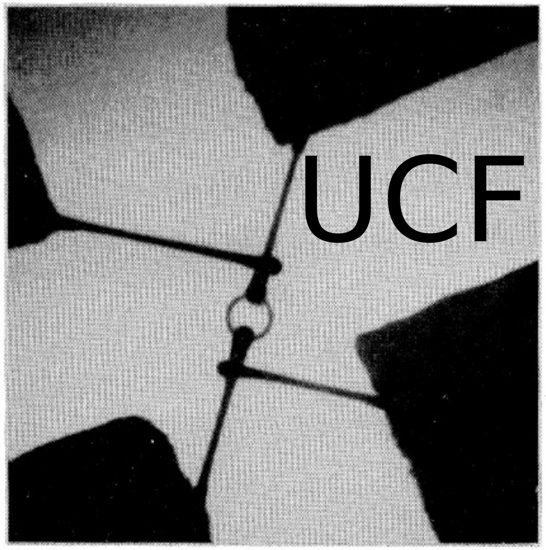


FIG. 1. Transmission electron micrograph of one of the Au rings measured in this experiment. The inside diameter of the ring was 280 nm and the width of the lines forming the ring was roughly 45 nm.

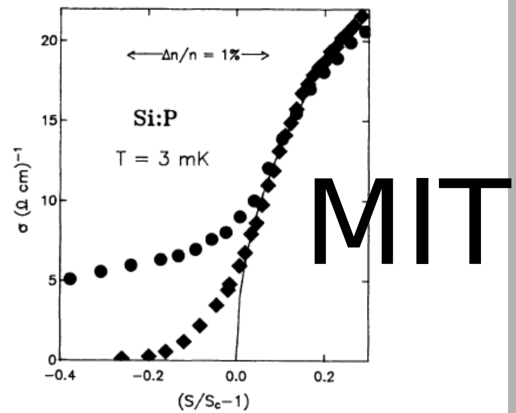
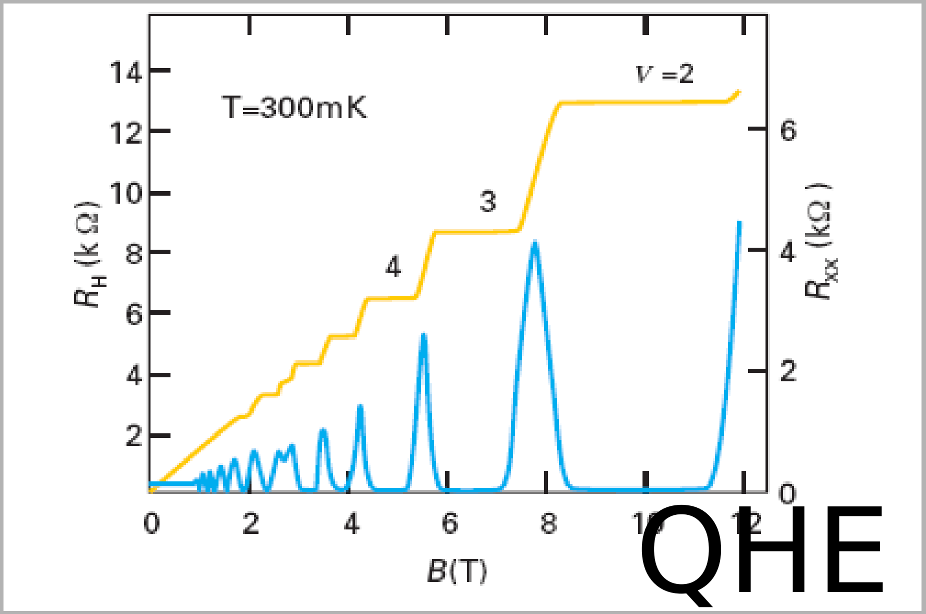
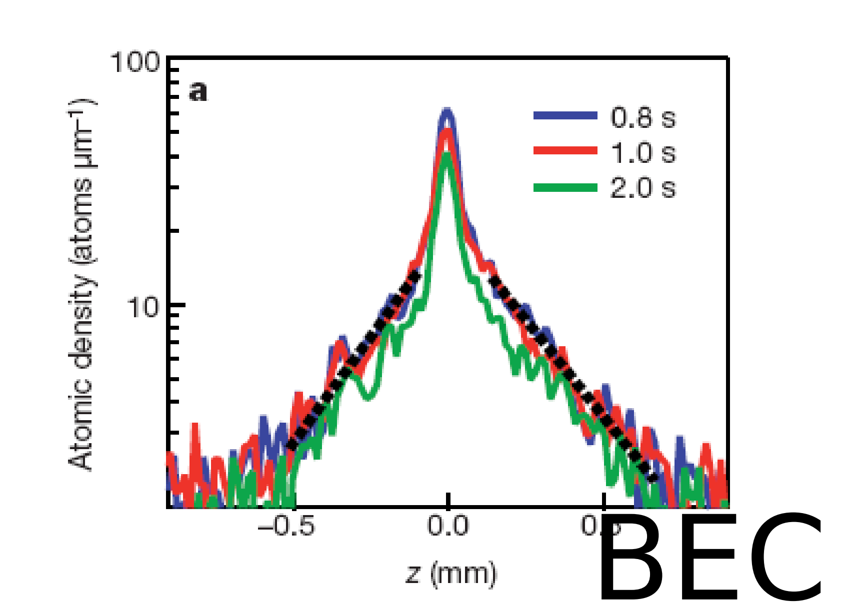


FIG. 1. Electrical conductivity σ as a function of uniaxial stress S at $T=3$ mK for two crystals of Si:P (circles and diamonds), indicating the reproducibility that leads us to identify the region of intrinsic behavior. The solid line is a fit to the data in this region, leading to a critical exponent $\mu=\frac{1}{2}$. Extrapolation to $T=0$ of the reproducible data does not change the exponent (Ref. [3]). The critical stress values defined by the fits, S_c, differ by 10% for the two samples because the donor densities, n (the zero-stress points), of the samples differ. We have obtained the range of relative densities, as shown by the arrows, by a separate determination of the critical density, n_c, vs S.

- **QUANTUM CHAOS:** dynamical localization
- sound-light -matter waves

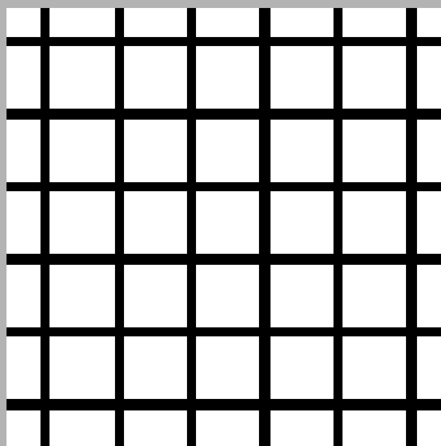




theory:

- **Theorems** (Spencer, Ishii, Martinelli,...)
- **Kubo formula** in weak disorder (Stone, Altshuler, ...)
- **Energy levels and b.c.** (Thouless, Hatano & Nelson mod., curvatures, ...)
- **Transfer matrix and Lyap spectrum** scaling (Kramer&MacKinnon), DMPK eq., conductance &scattering (Buttiker-Landauer),...
- **Supersymmetry, BRM** (Efetov, Fyodorov & Mirlin)

ANDERSON MODEL:



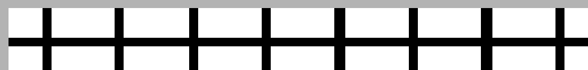
$$\sum_{\mathbf{i}} u_{\mathbf{n}+\mathbf{i}} + v_{\mathbf{n}} u_{\mathbf{n}} = E\, u_{\mathbf{n}}$$

$$\frac{w}{2} \leq v \leq \frac{w}{2}$$

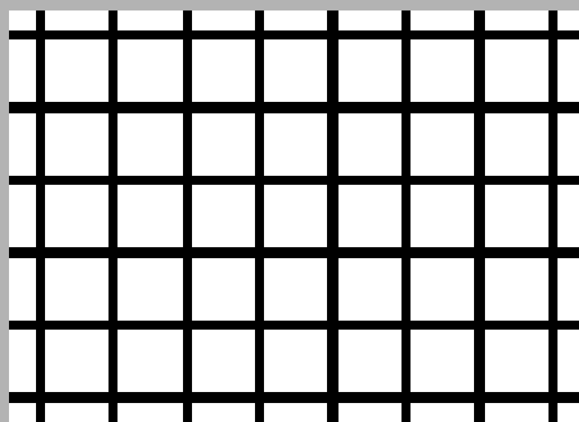
$$|u_{\mathbf{n}}| \approx e^{-\lambda|\mathbf{n}-\mathbf{n_0}|}$$

- d=1,2: exponential localization
- d=3 : metal-insulator transition

HAMILTONIAN and TRANSFER matrices



$$A_i = \begin{bmatrix} v_{i,1} & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & v_{i,m} \end{bmatrix}$$



$$H = \begin{bmatrix} A_1 & I_m & & I_m \\ I_m & \ddots & \ddots & \\ & \ddots & \ddots & I_m \\ I_m & & I_m & A_n \end{bmatrix}$$

$$T(E) = \begin{bmatrix} EI_m - A_n & -I_m \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} EI_m - A_1 & -I_m \\ I_m & 0 \end{bmatrix}$$

SPECTRAL DUALITY

$$u_{i+1} + A_i u_i + u_{i-1} = E u_i \qquad u_i \in \mathbb{C}^m$$

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

$$u_{n+1} = z^n u_1, \quad u_0 = \frac{1}{z^n} u_n$$

$$H(z^n) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z^n} I_m \\ I_m & \ddots & \ddots & \\ & \ddots & \ddots & I_m \\ z^n I_m & & I_m & A_n \end{bmatrix}$$

z^n is an eigenvalue of T(E)
iff
E is eigenvalue of H(z^n)

Anderson D=1

Hatano and Nelson (1996)

$$e^{-\xi}u_{i-1} + v_i u_i + e^{\xi}u_{i+1} = E u_i$$

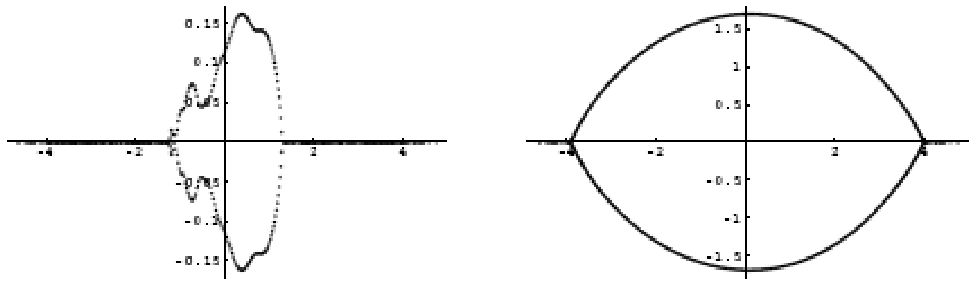


FIG. 2: Spectra of two matrices for disorder w=7, n=600, $\xi = 0.5$ (left) and $\xi = 1$ (right).

$$\xi(E) = \int dE' \rho(E') \log |E - E'|$$

(Herbert-Jones-Thouless formula)

Anderson model: **duality**

$$\det[T(E) - z^n] = (-1)^m z^{nm} \det[E - H(z^n)]$$

$$\xi_a(n, m, E, \{v\}) = \frac{1}{n} \log |z_a^n|$$

Exponents describe **decay lenghts** of Anderson model. They are obtained from non-Herm. energy spectrum via **Jensen's identity**

Let f be an analytic function in the open disk of radius R, where it has zeros $z_1, \ldots, z_k$ that are ordered according to increasing modulus. Then, if $0 < |z_1|$ and for r such that $|z_\ell| \leq r \leq |z_{\ell+1}|$ we have:

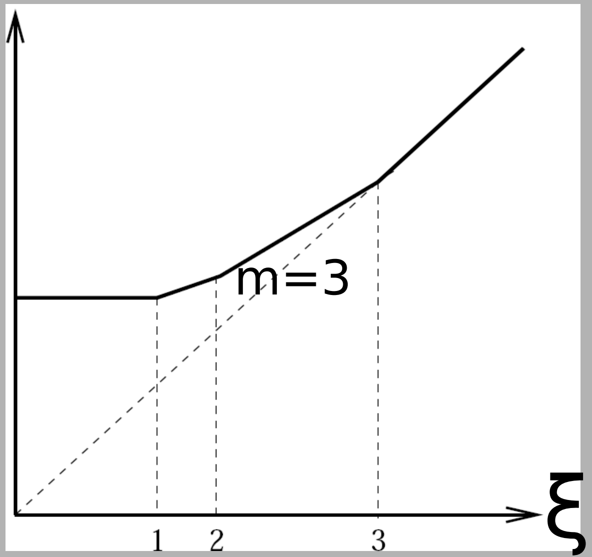
$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \log |f(re^{i\varphi})| = \log \frac{r^\ell |f(0)|}{|z_1 \cdots z_\ell|}$$

A formula for the exponents

(a deterministic variant of Thouless formula)

$$\frac{1}{m}\sum_{a=1}^{2m}(\xi-\xi_a)\theta(\xi-\xi_a)-\xi \;=$$

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{1}{nm}\log|\det[E-H(e^{n\xi+i\varphi})]|$$

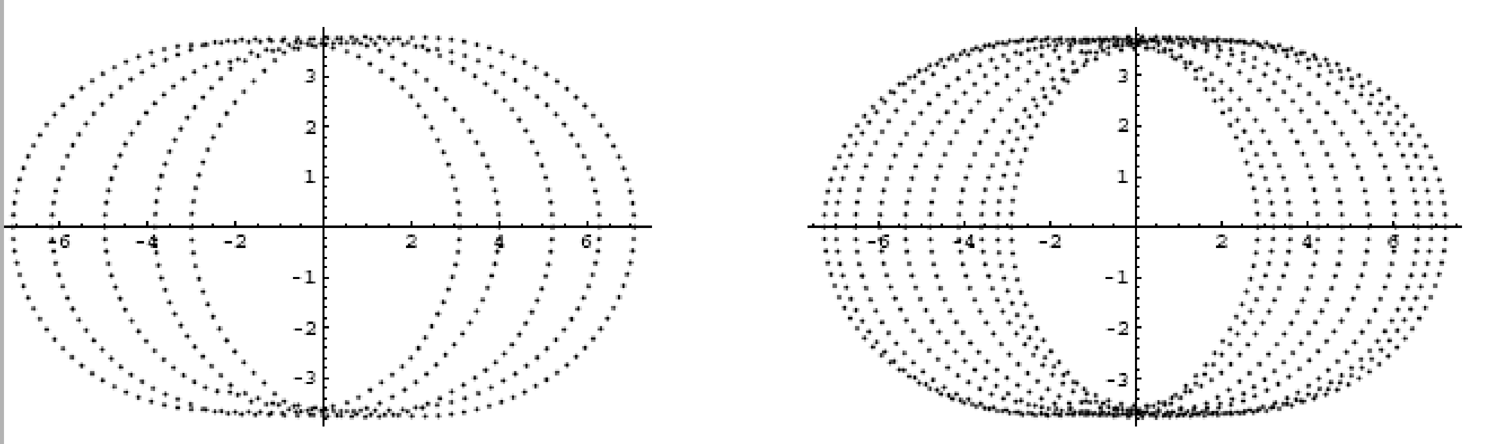


$$\begin{bmatrix} A_1 & I_m & & \frac{1}{z^n}I_m \\ I_m & \ddots & \ddots & \\ & \ddots & \ddots & I_m \\ z^n I_m & & I_m & A_n \end{bmatrix} \simeq \begin{bmatrix} A_1 & zI_m & & \frac{1}{z}I_m \\ \frac{1}{z}I_m & \ddots & \ddots & \\ & \ddots & \ddots & zI_m \\ zI_m & & \frac{1}{z}I_m & A_n \end{bmatrix}, \qquad 0 \le \arg z \le \frac{2\pi}{n}$$

no formula of Thouless type is known in D>1 (only for sum of exps, xi=0)

non-hermitian energy spectra

(Anderson 2D)



m=5 m=10

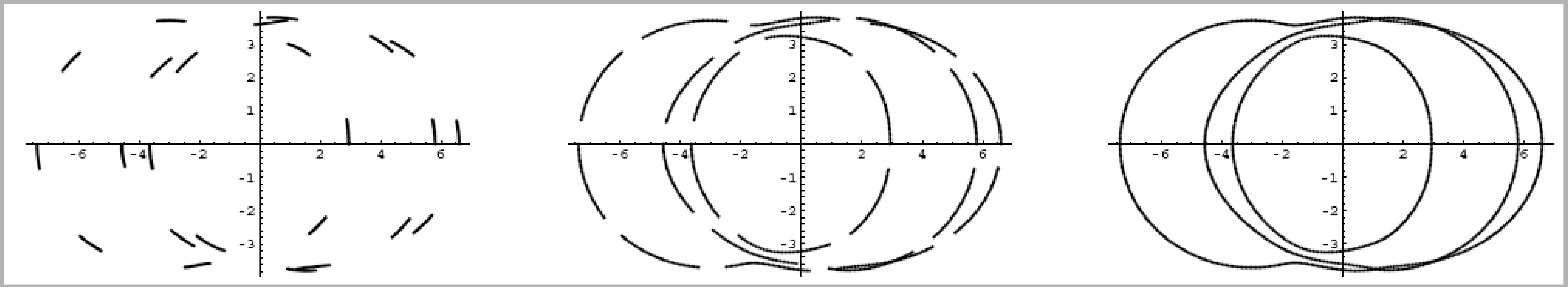
n=100, w=7, xi=1.5

$$\frac{(\mathrm{Re}E - E_0)^2}{4\cosh^2\xi} + \frac{(\mathrm{Im}E)^2}{4\sinh^2\xi} \leq 1$$

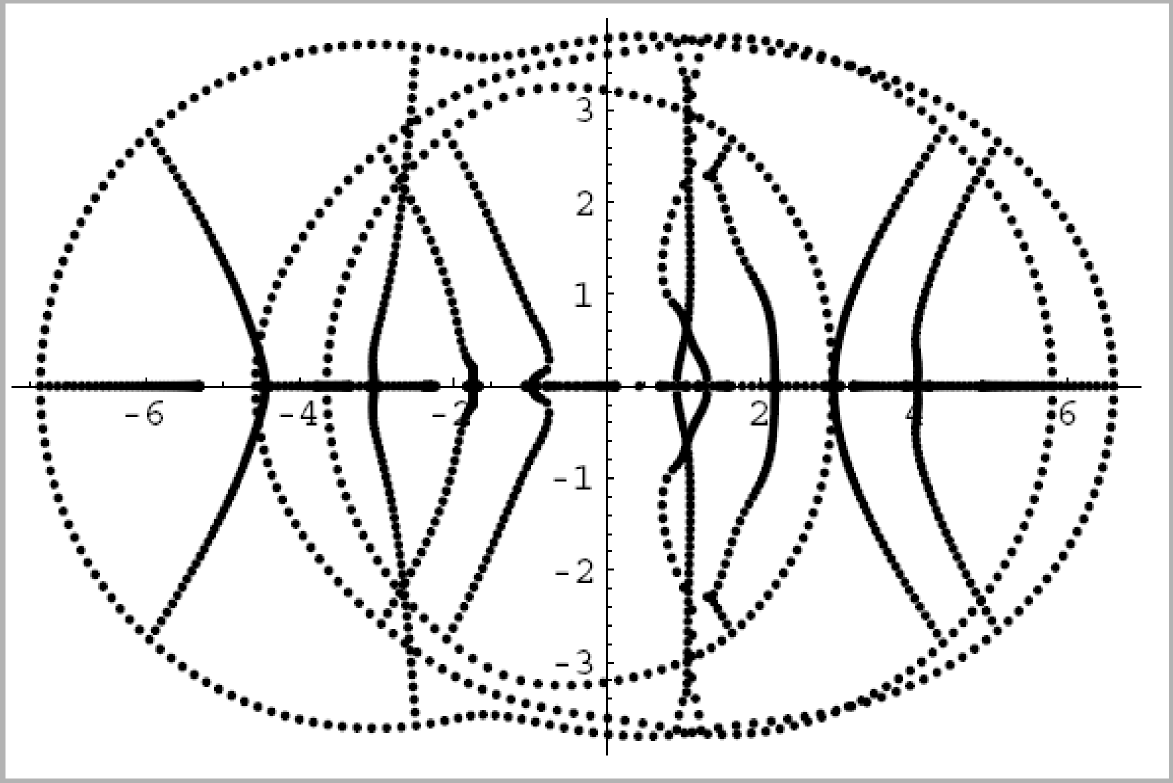
$$\xi_a(E) = \xi, \quad \varphi_a(E) = \varphi, \quad \mathrm{mod}\frac{2\pi}{n}, \qquad (a = 1, \ldots, m)$$

Anderson 2D (m=3,n=8)

(xi fixed, change phase)



(change xi and phase)



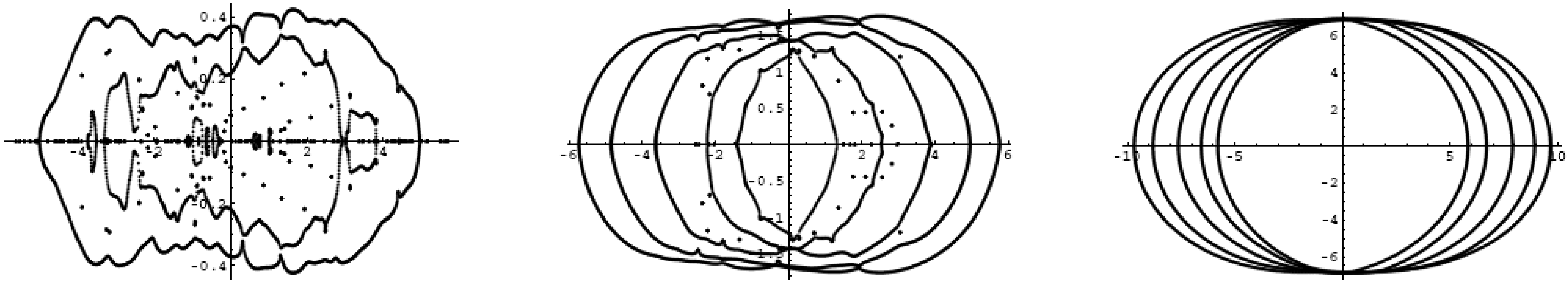


FIG. 6: w=7, m=5, n=100, $\xi = 0.5$ (left), $\xi = 1$ (center) and $\xi = 2$ (right), ten values of φ.

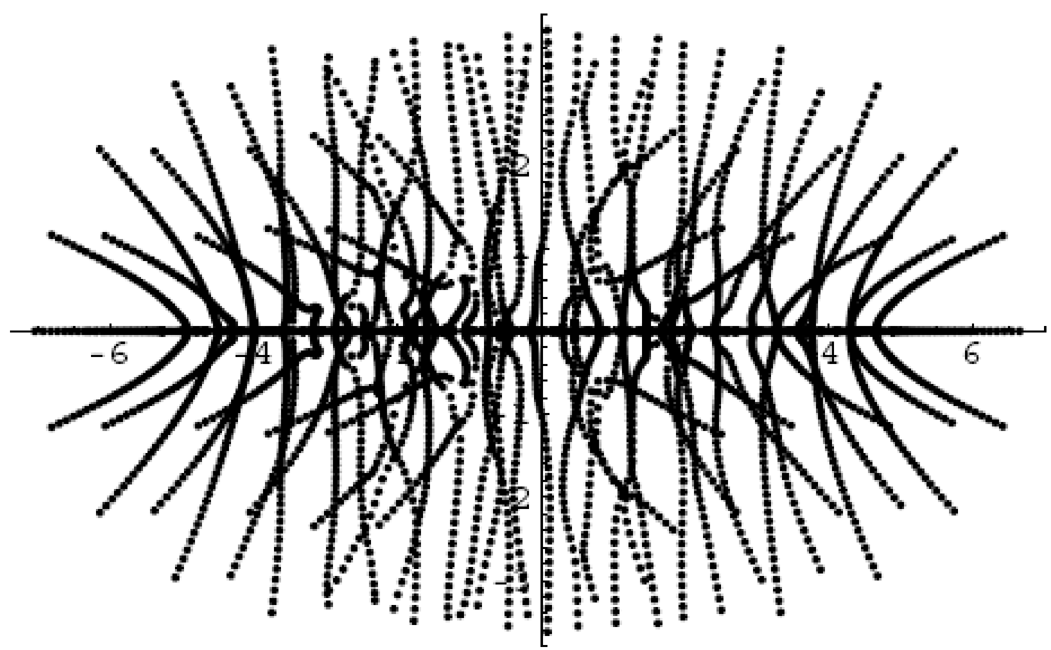
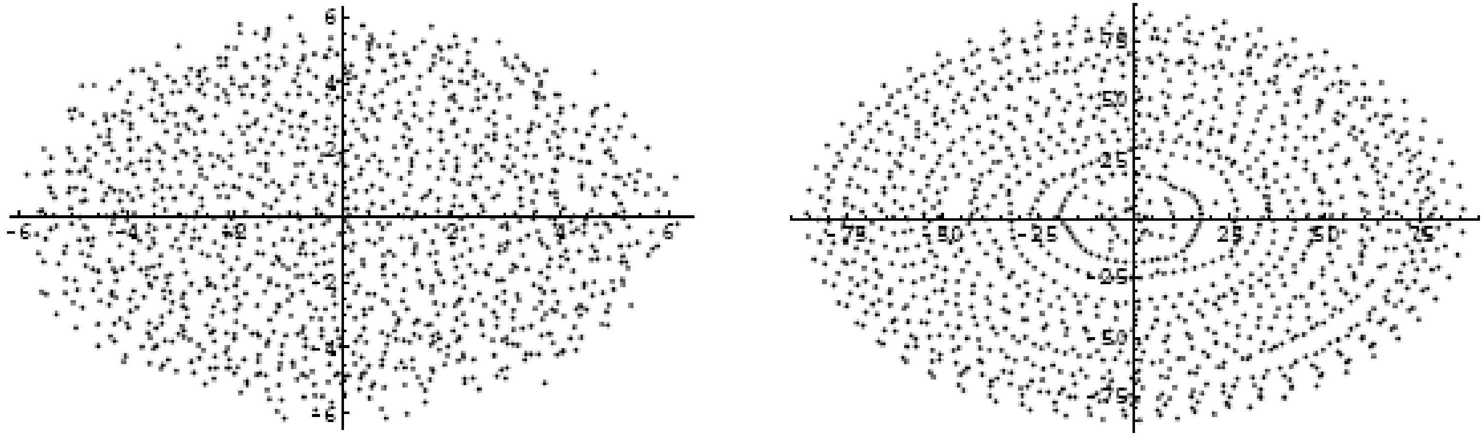
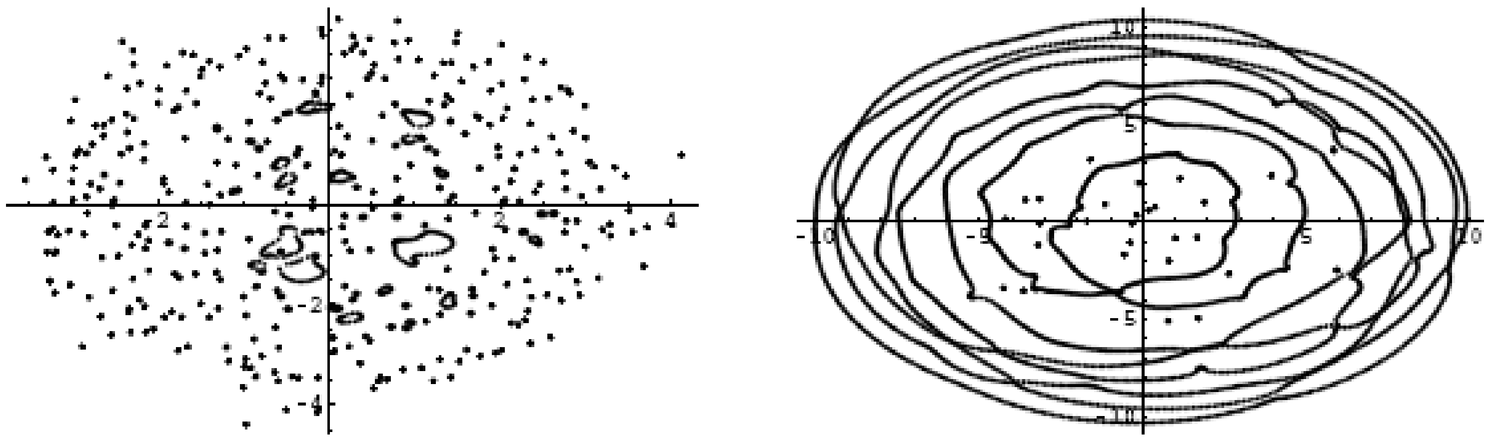


FIG. 3: ξ-evolution of eigenvalues, w=7, m5, n=20, $0 \leq \xi \leq 1.5$, $\varphi = 0$.

BAND RANDOM MATRICES complex, no symmetry



Eigenvalues in the complex plane of a non symmetric BRM, size 1250, $b = 25$, with matrix elements uniformly chosen in the unit square $[-1, 1] \times [-1, 1]$ of the complex plane. Left: $\xi = 0$ and $\varphi = 0$, Right: $\xi = 4$ and $\varphi = 0$.

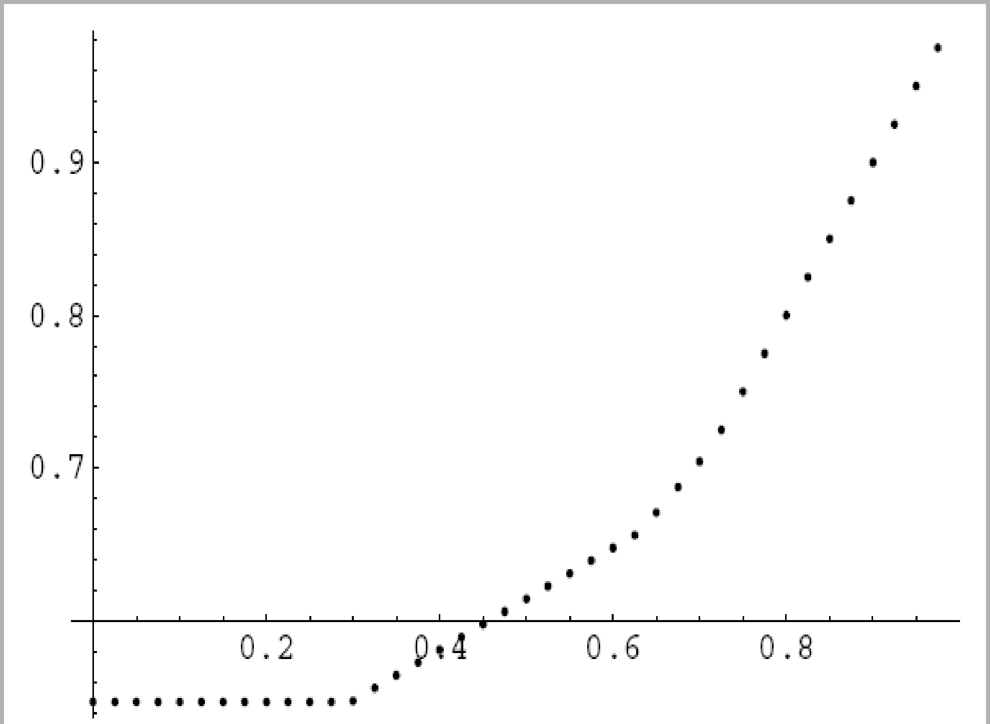


: Eigenvalues in the complex plane of 10 non symmetric BRMs, size 400, $b = 10$, with matrix elements uniformly chosen in the unit square $[-1, 1] \times [-1, 1]$ of the complex plane. Left: $\xi = 0$, Right: $\xi = 2$. φ is varied on 10 angles in 2π.

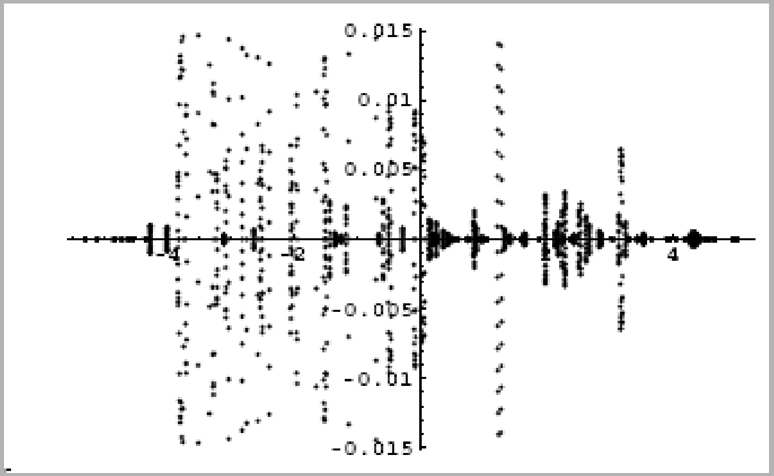
the exponents

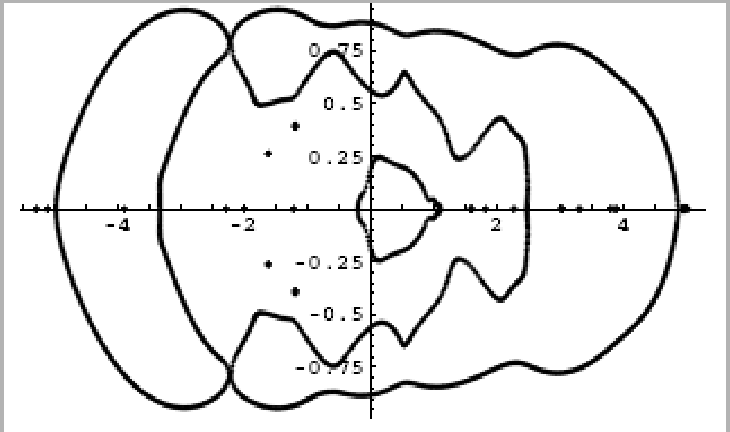
$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{nm} \log|\det[E - H(e^{n\xi + i\varphi})]|$$

ξ



m=3, n=50, w=7





conclusions

- Spectral duality and Jensen's identity yield the exponents of a single transfer matrix in terms of the eigenvalues of the Hamiltonian matrix with non-hermitian boundary conditions
- Theory can be extended to T*T (Lyapunov exponents)
- D=3?, band random matrices?