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Rumin’s Complex and Intrinsic Graphs in Carnot Groups

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ABSTRACT. This thesis is concerned with some aspects of geometric analysis on Carnot groups. In the first chapter, we study differential forms and Rumin's complex on Carnot groups. In particular, we undertake the analysis of Rumin's Laplacian Δ_R on the Heisenberg group. We obtain a decomposition of the space of Rumin's forms with L^2 coefficients into invariant subspaces and describe the action of Δ_R restricted to these subspaces up to unitary equivalence. We also obtain that this decomposition provide a L^p decomposition of the space of Rumin's forms.

In the second chapter, we study intrinsic Lipschitz graphs and intrinsic differentiable graphs within Carnot groups. Both seem to be the natural analogues inside Carnot groups of the corresponding Euclidean notions. In particular, we prove that one codimensional intrinsic Lipschitz graphs are sets with locally finite $\mathbb{G}$-perimeter. From this a Rademacher's type theorem for one codimensional graphs in a general class of groups is proved.

Contents

Preface

This thesis is concerned with some aspects of geometric analysis on Carnot groups.

A Carnot group is a connected, simply connected, stratified nilpotent Lie group. These groups arise naturally in a variety of settings, they have drawn a great deal of interest in the recent years and are still a fertile field of research. As basic references for Carnot groups we refer to [**15**], [**21**], [**65**].

The interest in Carnot groups is multi-fold. On one hand they represent the simples setting of sub-Riemannian geometry. On another hand, stratified nilpotent Lie algebras are the simplest example of non-abelian Lie algebras. In this setting, representation theory and more specific Lie algebras (such as Heisenberg-type, and of Métivier, see [**38, 58**], [**46, 16**]) are of current great interest.

Moreover, the first step in the stratification, the subspace $\mathfrak{g}_1$ plays a fundamental role in analysis of distinguished partial differential operators. The naturally arising differential operator given by the sum of squares of a (left invariant) basis of $\mathfrak{g}_1$ is the basic example of a hypoelliptic, sub-elliptic differential operator, by the celebrated theorem by L. Hörmander, see [**37**].

The most famous and studied Carnot group is certainly the Heisenberg group $\mathbb{H}_n$. A detailed analysis of sub-elliptic partial differential operators was undertaken by E. M. Stein and G. B. Folland in the seminal paper [**22**]. The main object of the their study was the so-called *tangential Cauchy–Riemann* complex on $\mathbb{H}_n$ and the naturally arising Laplacian, the so-called *Kohn Laplacian.* This operator is defined on forms on $\mathbb{H}_n$, but it acts diagonally (with respect to a natural basis), so that its analysis immediately reduces to the analysis of scalar differential operators.

The tangential Cauchy–Riemann complex on $\mathbb{H}_n$ arises as the "trace" of a complex structure, since the group $\mathbb{H}_n$ can be identified with the boundary of a domain in $\mathbb{C}^{n+1}$. On a more general Carnot group such a structure is not present, while one is interested on the objects defined intrisically in terms of the first level of the stratification.

The differential operator d_H obtained as the exterior differential defined only by differentiation along directions of $\mathfrak{g}_1$ does not give rise to a complex, that is, $d_H^2 \neq 0$. In order to obtain a complex, M. Rumin introduced a subspace of the exterior algebra of the cotangent bundle and the associated exterior differential, that we denote by d_R, that turns out to form a complex, [**59**], [**61**], [**62**]. Rumin proceeded to study the main properties of the associated Laplacian Δ_R. In particular he proved that, on $\mathbb{H}_n$, Δ_R is hypoelliptic and maximal hypoelliptic in the sense of [**35**]. Rumin's Laplacian has been intesively studied by several authors. In particular here we mention [**9**], [**10**], [**11**], [**31**].

An important area of research involves the spectral analysis of Laplacians on differential manifolds, as in the spirit of the Hodge Laplacian on a Riemannian manifold. In this setting there exists a vast literature and we simply refer to [**8**] and the references therein.

Such an analysis is intimately connected with the so-called Riesz transforms, their L^p-boundedness, the strong L^p-decomposion of the space of forms. In this spirit see the recent work [**49**] and [**50**] for the case of the Hodge Laplacian on the Heisenberg group.

In the first part of this thesis, following the scheme in [**50**], we study the spectral properties of Δ_R on $\mathbb{H}_n$. In particular, we decompose the space of the smooth intrinsic forms as direct sum of subspaces that are invariant under the action of Δ_R and on which the action of Δ_R can be expressed, up to unitary equivalence, by well-understood scalar operators. These invariant subspaces are orthogonal with respect to the natural L^2-inner product, thus providing an orthogonal decomposition of the space of intrisinc forms with L^2 coefficients.

At this point, we are able to prove a Mihlin–Hörmander multiplier theorem for Δ_R, with minimal smoothness assumption on the multiplier given by half of the topologica dimension of underlying manifold $\mathbb{H}_n$. This result allows us in particular to show that the decomposition for L^2 extend to the case of the space of intrisinc forms with L^p coefficients, $1 < p < \infty$.[1]

In the second chapter of the thesis we recall the notion of *intrinsic graphs* within Carnot groups and specifically of *intrinsic Lipschitz graphs*, that has been introduced with different degrees of generality in [**30**], [**7**], [**29**]. In [**24**] B. Franchi and R. Serapioni provide a comprehensive presentation of this theory.

Then we introduce the notions of *intrinsic differentiable functions* within a general Carnot group $\mathbb{G}$ and we prove the almost everywhere differentiability of one dimensional intrinsic Lipschitz functions inside a wide class of Carnot groups.

Our interest in intrinsic Lipschitz functions originates from the problem of defining appropriately rectifiable sets inside Carnot groups.

Several notions of rectifiability have been proposed in the last few years: in this regard the reader is referred to [**2**], [**3**], [**25**], [**28**], [**55**], [**42**].

In Euclidean spaces, rectifiable sets are obtained, up to a negligible subset, by "gluing up" countable families of C^1 or of Lipschitz submanifolds. Hence, understanding C^1 and Lipschitz submanifolds within Carnot groups, is preliminary in order to develop a satisfactory theory of *intrinsic* rectifiable sets. We refer the reader to [**24**] for a complete discussion of the problem that lies behind the notion of Lipschitz graph. Let us sketch here the main points of the discussion of [**24**].

First of all, we stress that considering Euclidean regular submanifolds may be both too general and too restrictive (see e.g. [**39**] for a striking example related to the second instance).

On the other hand, in Euclidean spaces C^1 submanifolds can be locally viewed, equivalently, as (i) C^1 injective images of an open subset of a linear space; (ii) non-critical level sets of C^1-functions; (iii) graphs of C^1 maps between complementary linear subspaces.

Notion (i) has a natural counterpart in general metric spaces that goes back at least to Federer's book (see [**20**] and [**2**]). According to this definition, Lipschitz submanifolds of a metric spaces are Lipschitz images of open subsets of Euclidean spaces. When working with a Carnot group $\mathbb{G}$, open subsets of homogeneous subgroups of $\mathbb{G}$ might be more natural parameter spaces (see [**55**] and [**42**]). Rectifiable curves are instances of this class of sets. On the other hand, notion (ii) has been largely studied in the recent literature, thanks to the implicit function theorem in Carnot groups proved in [**25**] and [**26**].

However, neither point of view (i) nor (ii) seem to describe the complexity of the geometry of Carnot groups, even of the Heisenberg groups (see [**2**],[**40**],[**12**],[**6**]).

The notion of graph appears at the first glance as ill-suited to be generalized outside of the Euclidean setting, since Carnot groups in general fail to be Cartesian products of subgroups. However, this obstacle can be overcome when the group $\mathbb{G}$ can be decomposed as a product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ of two homogeneous *complementary subgroups* $\mathbb{M}$, $\mathbb{H}$ (see [**24**]). Indeed, if such a composition is given, we can recall the following definition introduced in [**24**]: let $\mathbb{M}$, $\mathbb{H}$ be complementary homogeneous subgroups of a group $\mathbb{G}$, then the intrinsic (left) graph of $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ is the set

$$\mathrm{graph}\,(f) = \{g \cdot f(g) : g \in \mathcal{A}\}.$$

[1] We expect the Riesz transform $d_R\Delta_R^{-1/2}$ to be bounded on L^p, for $1 < p < \infty$. However, at this time we have not completed the proof of this fact and we leave it as a future project.

Intrinsic graphs appeared naturally in [**30**], [**7**], [**29**] in relation with non critical level sets of differentiable functions from $\mathbb{G}$ to $\mathbb{R}^k$. Indeed, implicit function theorems for groups ([**25**], [**28**], [**26**], [**17**], [**18**]) can be rephrased stating precisely that these level sets are always, locally, intrinsic graphs.

Intrinsic graphs are 'intrinsic' since they keep being intrinsic graphs after left translations or dilations δ_λ: see [**24**]. Moreover, the notion of intrinsic graph is more general and flexible than parametrizations or level sets. For instance, already in Heisenberg groups, both non critical level sets and images of regular maps are locally intrinsic differentiable graphs (see [**7**]).

A further step consists now in the characterization of Lipschitz graphs when the group $\mathbb{G}$ admits a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. Since the Carnot group is endowed with its Carnot-Carathéodory distance, it might appear natural to consider graphs of maps from $\mathbb{M}$ to $\mathbb{H}$ that are Lipschitz w.r.t. such distance. Unfortunately, this notion is not invariant under group translations (except in trivial cases), and must be abandoned. A notion of intrinsic Lipschitz graphs, which is invariant under translations, is introduced in [**24**].

Going back to our original motivation related to the notion of rectifiability, we can wonder whether the notion of Lipschitz graph yields suitable differentiability properties as in the Euclidean setting.

With this aim, we give two equivalent definitions of *intrinsically differentiable* functions $f : \mathbb{M} \to \mathbb{H}$ in Definition 2.3.7 and Theorem 2.3.14.

Another more algebraic definition of differentiability using *intrinsic linear functions*, i.e. functions whose graphs are homogeneous subgroups, is proved to be equivalent to the previous one in Theorem 2.3.14. We recall also that an extensive study of intrinsic differentiability following an alternative but (likely) equivalent approach has been carried on in [**5**], [**13**], [**14**] [**53**].

In the last part of the chapter we consider the case $\mathbb{G} = \mathbb{M} \cdot \mathbb{V}$ with $\mathbb{V}$ one dimensional and horizontal. In this case we prove the general fact that the graphs of intrinsic Lipschitz functions $f : \mathbb{M} \to \mathbb{V}$ are boundaries of sets with locally finite $\mathbb{G}$-perimeter.

This fact is not only interesting in itself but also yields a Rademacher's type theorem (see Theorem 2.4.15 below).

Indeed the rectifiability of the reduced boundary of a set of finite $\mathbb{G}$-perimeter (the so-called De Giorgi's theorem in Carnot groups) yields, in particular, the almost everywhere intrinsic differentiability of the boundary, when the boundary is an intrinsic graph. Now, De Giorgi's theorem in Carnot groups, proved for Heisenberg groups in [**25**] and for step 2 groups in [**27**] (see also [**4**] for a different approach) has been recently generalized by the author to the much larger class of type $\star$ Carnot groups (see Definition 2.1.1) in [**43**].

Clearly it is natural to ask, and it is an open problem to the best of our knowledge, whether a Rademacher's type theorem holds for one dimensional valued intrinsic Lipschitz functions inside a general Carnot groups. One should also ask the equivalently natural question if Rademacher's theorem holds for intrinsic Lipschitz functions valued in higher dimensional horizontal homogeneous subgroups.

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Introduction

In this chapter we recall some basic definition, well-known facts and set up our notation. We will not provide any proofs since they are easily available in the literature. We will begin by recalling some basic facts about Carnot groups. For a general account, see for instance [**15, 21, 34**].

I.1. Carnot groups

Definition I.1.1 (Carnot group). A Carnot group $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group with stratified Lie algebra $\mathfrak{g}$. This means that the Lie algebra $\mathfrak{g}$ of the left-invariant vector fields on $\mathbb{G}$ has finite dimension N and there exist linear subspaces (so-called layers) $\mathfrak{g}_1, \dots, \mathfrak{g}_\kappa$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_\kappa \neq \{0\}, \tag{I.1.1}$$

where $\mathfrak{g}_i = \{0\}$ if $i > \kappa$, and $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$. Here κ is the step of the stratification and is also called step of the group.

Let $X_1, \dots, X_N$ be a base for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a base for $\mathfrak{g}_1$ and, for $1 < j \leq \kappa$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for $\mathfrak{g}_j$. Here we have $m_0 = 0$ and $m_j - m_{j-1} = n_j$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_{m_1}$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span}\,\{X_1(x), \dots, X_{m_1}(x)\}, \qquad x \in \mathbb{G}.$$

The sections of $H\mathbb{G}$ are called *horizontal sections*, a vector of $H\mathbb{G}_x$ is a *horizontal vector* while any vector in $T\mathbb{G}_x$ that is not horizontal is a vertical vector.

Definition I.1.2. An absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is a sub-unit curve with respect to $X_1, \dots, X_{m_1}$ if there exist measurable real functions $c_1(s), \dots, c_{m_1}(s)$, $s \in [0, T]$ such that $\sum_j c_j^2 \leq 1$ and

$$\dot{\gamma}(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \quad \text{for a.e. } s \in [0, T].$$

The Carnot-Carathéodory distance d_c of $x, y \in \mathbb{G}$ is defined as

$$d_c(x, y) := \inf \{T > 0 : \text{ there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = x,\, \gamma(T) = y\}.$$

By Chow's Theorem, the set of sub-unit curves joining x and y is not empty, furthermore d_c is a distance on $\mathbb{G}$ that induces the Euclidean topology (see [**15**] or Theorem 1.6.2 in [**48**]).

Since the exponential map is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, any $x \in \mathbb{G}$ can be written, in a unique way, as $x = \exp(x_1 X_1 + \cdots + x_N X_N)$ and we identify x with the N-tuple $(x_1, \dots, x_N) \in \mathbb{R}^N$ and $\mathbb{G}$ with $(\mathbb{R}^N, \cdot)$, i.e. $\mathbb{R}^N$ endowed with the product $\cdot$. The identity of $\mathbb{G}$ is denoted as $e = (0, \dots, 0)$.

If $\mathbb{G}$ is a Carnot group, for all $\lambda > 0$, the *(non isotropic) dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorphisms of $\mathbb{G}$ defined as

$$\delta_\lambda(x_1, ..., x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, ..., \lambda^{\alpha_N} x_N),$$

where $\alpha_i = j$, if $m_{j-1} < i \leq m_j$. Dilations are defined also for $\lambda \in \mathbb{R}$ setting

$$\delta_\lambda x := \delta_{|\lambda|} x^{-1} = \left(\delta_{|\lambda|} x\right)^{-1}, \text{ when } \lambda < 0.$$

We denote the product of x and $y \in \mathbb{G}$ as $x \cdot y$ or as xy. Moreover, for every $x \in \mathbb{G}$ we define the left translation by x as

$$\begin{aligned} &\tau_x : \mathbb{G} \to \mathbb{G} \\ &\tau_x y = xy \quad \text{for all } y \in \mathbb{G}. \end{aligned}$$

The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula (see [**21**]). It has the form

$$x \cdot y = x + y + \mathcal{Q}(x, y), \qquad \text{for all } x, y \in \mathbb{R}^N, \tag{I.1.2}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_N) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$. More explicitly,

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_i} \mathcal{Q}_i(x, y), \quad \text{for all } x, y \in \mathbb{G} \text{ and } \lambda > 0.$$

It is useful to think $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$, where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{n_i}$ is the i^{th} layer of $\mathbb{G}$ and to write $x \in \mathbb{G}$ as $(x^1, \dots, x^\kappa)$, with $x^i \in \mathbb{G}^i$. According to this notation [2]

$$x \cdot y = \left(x^1 + y^1, x^2 + y^2 + \mathcal{Q}^2(x, y), \dots, x^\kappa + y^\kappa + \mathcal{Q}^\kappa(x, y)\right), \quad \text{for all } x, y \in \mathbb{G},$$

where each $\mathcal{Q}^j$ is a vector valued polynomial homogeneous of degree j with respect to the non isotropic dilations δ_λ.

It is always possible to define a *homogeneous norm* on $\mathbb{G}$. A homogeneous norm is a function $\mathbb{G} \to \mathbb{R}^+$, vanishing only at the origin, such that $\|p\| = \left\|p^{-1}\right\|$ and

$$\|p \cdot q\| \leq \|p\| + \|q\|, \qquad \|\delta_\lambda p\| = \lambda \|p\|,$$

for all $p, q \in \mathbb{G}$ and $\lambda > 0$. Given any homogeneous norm $\|\cdot\|$ on $\mathbb{G}$, it is possible to define a distance $d(\cdot, \cdot)$ in $\mathbb{G}$ as

$$d(p, q) = d(q^{-1} \cdot p, 0) = \left\|q^{-1} \cdot p\right\|, \qquad \text{for all } p, q \in \mathbb{G}. \tag{I.1.3}$$

Any such distance d in (I.1.3) is comparable with the Carnot-Carathéodory distance d_c; in particular, they all induce the Euclidean topology on $\mathbb{G}$. Both d and d_c are left translation invariant and 1-homogeneous, i.e.

$$d(g \cdot p, g \cdot q) = d(p, q), \qquad d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q) \tag{I.1.4}$$

for all $p, q, g \in \mathbb{G}$ and all $\lambda > 0$, and similarly for d_c. For $r > 0$ and $p \in \mathbb{G}$, $U_c(p, r)$, $B_c(p, r)$ will be the open and closed balls associated with the distance d_c and $U(p, r)$, $B(p, r)$ the ones associated with d.

From now on we will assume that a norm $\|\cdot\|$ and the associated distance d are chosen in $\mathbb{G}$.

Using either the distance d_c or the distance d, Hausdorff measures and spherical Hausdorff measures of dimension $m \geq 0$ are obtained following Carathéodory's construction (see [**20**, Section 2.10.2.]). They are denoted respectively $\mathcal{H}_c^m$, $\mathcal{S}_c^m$, $\mathcal{H}_d^m$, $\mathcal{S}_d^m$. Translation invariance and dilation homogeneity of Hausdorff measures follow from (I.1.4) and, for $\mathcal{A} \subseteq \mathbb{G}$, $p \in \mathbb{G}$ and $r \in [0, \infty)$,

$$\mathcal{S}_d^m(p \cdot \mathcal{A}) = \mathcal{S}_d^m(\mathcal{A}) \quad \text{and} \quad \mathcal{S}_d^m(\delta_r \mathcal{A}) = r^m \mathcal{S}_d^m(\mathcal{A}).$$

[2] When using this notation, it will be clear that the superscript does not indicate an exponent.

DEFINITION I.1.3. The integer $Q = \sum_{j=1}^{N} \alpha_j = \sum_{i=1}^{\kappa} i \dim V_i$ is the *homogeneous dimension* of $\mathbb{G}$.

We stress that Q is also the Hausdorff dimension of $\mathbb{R}^N$ with respect to d_c (see [**47**]).

Finally we recall (see e.g. [**65**]) that the N-dimensional Lebesgue measure $m_{\mathbb{G}}$ is the Haar measure of the group $\mathbb{G}$. Therefore if $\mathcal{E} \subset \mathbb{R}^N$ is measurable, then $m_{\mathbb{G}}(g \cdot \mathcal{E}) = m_{\mathbb{G}}(\mathcal{E})$ for every $g \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $m_{\mathbb{G}}(\delta_\lambda(\mathcal{E})) = \lambda^Q m_{\mathbb{G}}(\mathcal{E})$. We note that

$$m_{\mathbb{G}}(U_c(p,r)) = r^Q m_{\mathbb{G}}(U_c(p,1)) = r^Q m_{\mathbb{G}}(U_c(0,1)).$$

I.2. Differential forms on Carnot groups

Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$, with basis given by $\{X_1, \dots, X_N\}$. Then we can consider the dual space of $\mathfrak{g}$, denoted by $\Lambda^1\mathfrak{g}$, that is, the real vector space of all linear functionals on $\mathfrak{g}$. In particular we can consider a particular basis of $\Lambda^1\mathfrak{g}$, the so-called dual basis, which will be denoted by $\{\theta_1, \dots, \theta_N\}$. Denoting by $\langle \cdot | \cdot \rangle$ the pairing of duality, we have that $\langle \theta_i | X_j \rangle = \delta_{ij}$. Moreover, we can introduce an inner product $\langle \cdot, \cdot \rangle$ on $\Lambda^1\mathfrak{g}$ such that the dual basis above will be an orthonormal basis. In general, we can define the exterior algebras of $\mathfrak{g}$ and $\Lambda^1\mathfrak{g}$ as follows:

$$\Lambda_*\mathfrak{g} := \bigoplus_{k=0}^{N} \Lambda_k\mathfrak{g}\,, \quad \Lambda^*\mathfrak{g} := \bigoplus_{k=0}^{N} \Lambda^k\mathfrak{g},$$

where $\Lambda_0\mathfrak{g} = \mathbb{R} = \Lambda^0\mathfrak{g}$, whereas for $1 \le k \le N$:

$$\Lambda_k\mathfrak{g} = \operatorname{span}\{X_{i_1} \wedge \ldots \wedge X_{i_k} | 1 \le i_1 < \ldots < i_k \le N\},$$
$$\Lambda^k\mathfrak{g} = \operatorname{span}\{\theta_{i_1} \wedge \ldots \wedge \theta_{i_k} | 1 \le i_1 < \ldots < i_k \le N\},$$

respectively called k-vectors and k-covectors.

REMARK I.2.1. Obsviously $\mathfrak{g} = \Lambda_1\mathfrak{g}$. By definition $\Lambda^1\mathfrak{g} = (\Lambda_1\mathfrak{g})^*$, where $(\Lambda_1\mathfrak{g})^*$ is the dual space of $\Lambda_1\mathfrak{g}$. Since $\mathfrak{g}$ is finite-dimensional, $\Lambda^k\mathfrak{g} \cong (\Lambda_k\mathfrak{g})^*$, that is, the space of k-covectors is naturally isomorphic to the dual space of k-vectors.

We will denote by $\Theta^k = \{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} | 1 \le i_1 < \cdots i_k \le N\}$ the basis of $\Lambda^k\mathfrak{g}$ obtained from the basis $(\theta_1, \dots, \theta_N)$ of $\Lambda^1\mathfrak{g}$. Moreover, the inner products $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{g}$ and $\Lambda^1\mathfrak{g}$ extend canonically to $\Lambda_k\mathfrak{g}$ and $\Lambda^k\mathfrak{g}$ making their basis orthonormal too.

We would like to define the Hodge $\star$-operator[3] wich provides a linear isomorphism and a duality between $\Lambda_k\mathfrak{g}$ and $\Lambda_{N-k}\mathfrak{g}$ and between $\Lambda^k\mathfrak{g}$ and $\Lambda^{N-k}\mathfrak{g}$ for all $k \in \{1, \dots, N\}$.

Defining the $\star$-operator requires choosing an orientation of the space V. In this way any other basis which is obtained from one with positive orientation through an automorphism with positive determinant will be positive as well. The remaining ones will be called negative.

Now, the action of the $\star$-operator on the elements of the orthonormal bases is extremely simplified:

$$\star : \Lambda_k\mathfrak{g} \to \Lambda_{N-k}\mathfrak{g}, \quad \star(X_{i_1} \wedge \ldots \wedge X_{i_k}) = X_{j_1} \wedge \ldots \wedge X_{j_{N-k}},$$
$$\star : \Lambda^k\mathfrak{g} \to \Lambda^{N-k}\mathfrak{g}, \quad \star(\theta_{i_1} \wedge \ldots \wedge \theta_{i_k}) = \theta_{j_1} \wedge \ldots \wedge \theta_{j_{N-k}},$$

where the multi indices $(j_1, \dots, j_{N-k})$ are chosen so that $(X_{i_1}, \dots, X_{i_k}, X_{j_1}, \dots, X_{j_{N-k}})$ is a positive basis for $\mathfrak{g}$ and similarly $(\theta_{i_1}, \dots, \theta_{i_k}, \theta_{j_1}, \dots, \theta_{j_{N-k}})$ is a positive basis for $\Lambda^1\mathfrak{g}$.

[3] Since the symbol $*$ already appears in several occurrences, we choose to denote the Hodge operator by $\star$.

Definition I.2.2. If $v \in \Lambda_k\mathfrak{g}$, then we can uniquely define $v^\natural \in \Lambda^k\mathfrak{g}$ as the element of $\Lambda^k\mathfrak{g}$ such that

$$\langle v^\natural, w\rangle = w(v) = \langle v|w\rangle, \quad \text{for all } w \in \Lambda^k\mathfrak{g}.$$

If $w \in \Lambda^k\mathfrak{g}$, then we can uniquely define $w^\natural \in \Lambda_k\mathfrak{g}$ as the element of $\Lambda_k\mathfrak{g}$ such that

$$\langle w^\natural, v\rangle = w(v) = \langle v|w\rangle, \quad \text{for all } v \in \Lambda_k\mathfrak{g}.$$

Definition I.2.3. Let $\Phi : V \to W$ be a linear map between two finite dimensional vector spaces V and W, we can extend it to k-vectors and k-covectors by:

$$\Lambda_k\Phi : \Lambda_k V \to \Lambda_k W, \quad (\Lambda_k\Phi)(v_1 \wedge \ldots \wedge v_k) := \Phi(v_1) \wedge \ldots \wedge \Phi(v_k);$$

and for $\alpha \in \Lambda^k W$ and $v_1 \wedge \ldots \wedge v_k \in \Lambda_k V$

$$\Lambda^k\Phi : \Lambda^k W \to \Lambda^k V, \quad \langle (\Lambda^k\Phi)(\alpha)|(v_1 \wedge \ldots \wedge v_k)\rangle := \langle \alpha|(\Lambda_k\Phi)(v_1 \wedge \ldots \wedge v_k)\rangle.$$

We will then consider $d\tau_p$ instead of Φ as a linear map from $\mathfrak{g}$ to itself. This implies that we are allowed to move the fibre from the origin $e \in \mathbb{G}$ by using the differential of left translations. More precisely, $\forall\ g \in \mathbb{G}$ we can extend the linear map $d\tau_g : \mathfrak{g} \to \mathfrak{g}$ to k-vectors and k-covectors as done above, so we can define the spaces:

$$\Lambda_{k,g}\mathfrak{g} := (\Lambda_k d\tau_g)(\Lambda_k\mathfrak{g}),$$
$$\Lambda^k_g\mathfrak{g} := (\Lambda^k d\tau_g)(\Lambda^k\mathfrak{g}),$$

where we have $\Lambda_{k,e}\mathfrak{g} = \Lambda_k\mathfrak{g}$ and $\Lambda^k_e\mathfrak{g} = \Lambda^k\mathfrak{g}$. One can also define a basis of any of these fibers using a basis Θ^k of $\Lambda^k g$:

$$\Theta^k_g := (\Lambda^k d\tau_{g^{-1}})(\Theta^k).$$

We refer to the section $g \mapsto \Theta^k_g$ of $\Lambda^k\mathfrak{g}$ as the left-invariant frame associated with Θ^k. The inner product on $\Lambda_k\mathfrak{g} = \Lambda_{k,e}\mathfrak{g}$ and $\Lambda^k\mathfrak{g} = \Lambda^k_e\mathfrak{g}$ induces an inner product on $\Lambda_{k,g}\mathfrak{g}$ and $\Lambda^k_g\mathfrak{g}$ respectively.

$$\langle \Lambda_k d\tau_g(v), \Lambda_k d\tau_g(w)\rangle := \langle v, w\rangle, \quad \forall\ v, w \in \Lambda_k\mathfrak{g};$$
$$\langle \Lambda^k d\tau_g(\varphi), \Lambda^k d\tau_g(\psi)\rangle := \langle \varphi, \psi\rangle, \quad \forall\ \varphi, \psi \in \Lambda_k\mathfrak{g}.$$

We will denote by Ω^k or $C^\infty\Lambda^k\mathfrak{g}$ the set of smooth sections $\mathbb{G} \ni g \mapsto \Lambda^k_g\mathfrak{g}$, i.e. the (differential) k-forms on $\mathbb{G}$. Let $f : \mathbb{G}_1 \to \mathbb{G}_2$ be a smooth map and ω be a k-form on $\mathbb{G}_2$, then the pull-back of ω by f is given by:

$$f^\sharp\omega(g) := (\Lambda^k(df_g))\omega(f(g)), \quad \forall\ g \in \mathbb{G}_1.$$

Definition I.2.4 (Left-invariant forms). A k-form on $\mathbb{G}$ is said to be left-invariant if $\tau_p^\sharp\alpha = \alpha$, for any $p \in \mathbb{G}$.

Proposition I.2.5. *Let $\xi \in \Lambda^k\mathfrak{g}$ and $g \in \mathbb{G}$. Setting $I_\xi(g) := \left(\Lambda^k d\tau_{g^{-1}}\right)(\xi)$, we obtain:*

(i) *the map $g \mapsto I_\xi(g)$ belongs to Ω^k and is left-invariant;*
(ii) *any left-invariant form $\alpha \in \Omega^k$ takes the form $\alpha = I_{\alpha(e)}$.*

I.3. Rumin's complex on Carnot groups

We now introduce one of the main objects of our studies.

Definition I.3.1. Let $\alpha \in \Lambda^1\mathfrak{g}$, $\alpha \neq 0$. We say that α has a pure weight p if $\alpha^\natural \in \mathfrak{g}_p$. In this case we write $w(\alpha) = p$. In general, given a k-covector $\beta \in \Lambda^k\mathfrak{g}$, we say that β has pure weight p if β can be expressed as a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ such that $w(\theta_{i_1}) + \cdots + w(\theta_{i_k}) = p$.

PROPOSITION I.3.2. *Let $\alpha, \beta \in \Lambda^k \mathfrak{g}$ be two k-covectors of different weights, then they are orthogonal.*

COROLLARY I.3.3. *We can express the space of k-covectors as a direct sum of subspaces which depend on the weight. The simplest case is obtained for $\Lambda^1\mathfrak{g}$:*

$$\Lambda^1\mathfrak{g} = (\Lambda^1\mathfrak{g})_1 \oplus (\Lambda^1\mathfrak{g})_2 \oplus \cdots \oplus (\Lambda^1\mathfrak{g})_\kappa \,,$$

where $(\Lambda^1\mathfrak{g})_\ell$ is the span of 1-covectors of weight ℓ. An analogous decomposition holds for arbitrary k-covectors. Precisely,

$$\Lambda^k\mathfrak{g} = \bigoplus_{p=P^{(k)}_{\min}}^{P^{(k)}_{\max}} (\Lambda^k\mathfrak{g})_p \,,$$

where $(\Lambda^k\mathfrak{g})_p$ is the span of k-covectors of weight p, $P^{(k)}_{\min}$, $P^{(k)}_{\max}$ are respectively the smallest and the largest weight that can be attained by k-covectors.

REMARK I.3.4. This particular decomposition of $\Lambda^k\mathfrak{g}$ is well described also at the level of the basis Θ^k. Indeed, all of its elements have pure weight, hence for any possible $p \in \{P^{(k)}_{\min}, \dots, P^{(k)}_{\max}\}$ we can define a basis of $(\Lambda^k\mathfrak{g})_p$ simply by:

$$(\Theta^k)_p := \Theta^k \cap (\Lambda^k\mathfrak{g})_p.$$

All the considerations we have gone through for k-covectors can be extended to k-forms: the space of all smooth sections of $(\Lambda^k\mathfrak{g})_p$ will be denoted by $(\Omega^k)_p$ or $C^\infty(\Lambda^k)_p$, which is the vector space of all smooth k-forms of pure weight p on $\mathbb{G}$. More generally, if Λ is any fiber bundle over $\mathbb{G}$ we denote by $L^2\Lambda$, $C^\infty\Lambda$, etc., the space of global sections of Λ with L^2, C^∞, etc. coefficients.

The decomposition we have seen for $\Lambda^k\mathfrak{g}$ will induce an analogous decomposition on k-forms:

$$\Omega^k = \bigoplus_{p=P^{(k)}_{\min}}^{P^{(k)}_{\max}} (\Omega^k)_p.$$

PROPOSITION I.3.5. [**62**] *Let $\alpha \in (\Omega^k)_p$ be a left-invariant k-form of pure weight p, such that $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$. In other words:*

$$d\left((\Omega^k)_p\right) \subset (\Omega^{k+1})_p.$$

We would like to use this result in order to find a more sophisticated way to express the exterior differential d in terms of weights. In general, an k-form of pure weight p, $\alpha \in (\Omega^k)_p$, will not be a left-invariant form. However it can be expressed as a linear combination of the elements of the orthogonal basis $(\Theta^k)_p$ of left-invariant forms as follows:

$$\alpha = \sum_i f_i\theta^k_i, \quad \text{where } \{\theta^k_i\}_i = (\Theta^k)_p.$$

Therefore the expression for the differential in local coordinated will be:

$$d\alpha = \sum_i d(f_i\theta^k_i) = \sum_i df_i \wedge \theta^k_i + f_i d\theta^k_i = \sum_i \left(\sum_{j=1}^N X_j(f_i)\theta_j \wedge \theta^k_i\right) + f_i d\theta^k_i.$$

This provides us with a well-given decomposition of the differential operator d by weights.

DEFINITION I.3.6. Let $\alpha = \sum_i f_i\theta^k_i$ be an arbitrary k-form of pure weight p as above, then we can write:

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha \,,$$

where d_i denotes the part of d which increases the weight of the form α by i. Then we will have:

$$d_0\alpha = \sum_i f_i d\theta_i^k \in (\Omega^{k+1})_p \,,$$
$$d_\ell\alpha = \sum_i \sum_{X_j\in\mathfrak{g}_\ell} X_j(f_i)\theta_j \wedge \theta_i^k \in (\Omega^{k+1})_{p+\ell} \quad \text{for } \ell \in \{1,\dots,\kappa\}.$$

In particular, d_0 is an algebraic operator, in the sense that its action can be identified at any point with the action of an operator from $\Lambda^k\mathfrak{g} \to \Lambda^{k+1}\mathfrak{g}$ (that we denote again by d_0). In other words, d_0 is C^∞-linear.

Lemma I.3.7. *Let $\alpha \in \Omega^k$ be a left invariant differential form, then: $d\alpha = d_0\alpha$ is still a left invariant form.*

We want to define an inverse of the operator d_0. We can exploit the following isomorphism:

$$d_0 : \Lambda^k\mathfrak{g}/\ker d_0 \xrightarrow{\cong} \Lambda^{k+1}\mathfrak{g} \cap \operatorname{ran} d_0 \,,$$

so that taking any $\beta \in \Lambda^{k+1}\mathfrak{g}$, there exists a unique $\alpha \perp \ker d_0$ such that $d_0\alpha = \beta+\xi$, with $\xi \in (\text{ran } d_0)^\perp$.

Definition I.3.8. We define

$$\begin{aligned} d_0^{-1} : \Lambda^{k+1}\mathfrak{g} &\to \{\alpha \in \Lambda^k\mathfrak{g} \mid \alpha \perp \ker d_0\} \\ \beta &\mapsto d_0^{-1}\beta = \alpha \,. \end{aligned}$$

Notice that if $\beta \in (\operatorname{ran} d_0)^\perp$, then $d_0^{-1}\beta = 0$.

Now we are ready to construct Rumin's complex. Let us consider the following operator:

$$d_0^{-1}d : \operatorname{ran} d_0^{-1} \to \operatorname{ran} d_0^{-1}.$$

We can split this new operator depending on the filtration of $\mathfrak{g}$:

$$\begin{aligned} d_0^{-1}d &= d_0^{-1}(d_0 + d_1 + \cdots + d_\kappa) \\ &= d_0^{-1}d_0 + d_0^{-1}d_1 + d_0^{-1}d_2 + \cdots + d_0^{-1}d_\kappa. \end{aligned}$$

Proposition I.3.9 ([**62**]). *The map $d_0^{-1}d\big|_{\operatorname{ran} d_0^{-1}} = \mathrm{Id}+D$, where $D := d_0^{-1}(d-d_0)$, is an isomorphism from $\operatorname{ran} d_0^{-1}$ to itself. Moreover the differential operator*

$$P := \sum_{k=0}^{r-1}(-D)^k, \quad \textit{with } r \in \mathbb{N} \textit{ such that } D^r \equiv 0,$$

is the inverse of $d_0^{-1}d$ on $\operatorname{ran} d_0^{-1}$.

Remark I.3.10. This means that, when restricted to this subspace, de Rham differential itself has a left inverse which we will denote by $Q := Pd_0^{-1}$, i.e. $Qd = \mathrm{Id}$ on $(\ker d_0)^\perp$.

Theorem I.3.11 ([**62**]). *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) \quad \textit{and} \quad F := \operatorname{ran} d_0^{-1} + \operatorname{ran}(dd_0^{-1}),$$

such that the projection Π_E on E along F is given by $\Pi_E = \mathrm{I} - Qd - dQ$.

Remark I.3.12. Π_E is not orthogonal.

Definition I.3.13 ([**62**]). For $0 \le k \le N$ we set

$$E_0^k := \ker d_0|_{\Lambda^k\mathfrak{g}} \cap (\operatorname{ran} d_0|_{\Lambda^{k-1}\mathfrak{g}})^\perp \subset \Lambda^k\mathfrak{g}.$$

The elements of $C^\infty E_0^k$ will be called Rumin's k-forms (or intrinsic k-forms) on $\mathbb{G}$.

Remark I.3.14. E_0^k inherits from $\Lambda^k\mathfrak{g}$ the scalar product. Moreover, there exists a orthogonal basis $\mathcal{E}_0^k = \{\xi_j\}$ of E_0^k that is adapted to the filtration of $\mathfrak{g}$. It is straightforward to see that $E_0^1 = \operatorname{span}\{\theta_1, \dots, \theta_{m_1}\}$, the space of horizontal 1-covectors, therefore we can assume that $\xi_j = \theta_j$ without loss of generality. We will denote by $P^{(k)}_{\min}$ and $P^{(k)}_{\max}$ respectively the lowest and highest weight of covectors in E_0^k and setting $(E_0^k)_p := E_0^k \cap (\Lambda^k\mathfrak{g})_p$ we obtain the following decomposition:

$$E_0^k = \bigoplus_{p=P^{(k)}_{\min}}^{P^{(k)}_{\max}} (E_0^k)_p.$$

Let us stress the fact that $(E_0^k)_p$ has an orthonormal basis given by $(\mathcal{E}_0^k)_p := \mathcal{E}_0^k \cap (\Lambda^k\mathfrak{g})_p$, so that all the elements in $(\mathcal{E}_0^k)_p$ have pure weight p.

Theorem I.3.15 ([**62**]). *Using the same notations and definition above we have:*

(i) *if we denote by Π_{E_0} the orthogonal projection from Ω^* to $C^\infty E_0^*$, we have:*

$$\Pi_{E_0} = \mathrm{Id} - d_0^{-1}d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0 d_0^{-1};$$

(ii) *if $d_R := \Pi_{E_0} d \Pi_E$, then $(C^\infty E_0^*, d_R)$ is an exact complex.*

Definition I.3.16. The exact complex $(C^\infty E_0^*, d_R)$ is called Rumin's complex, and d_R is called Rumin's differential.

Remark I.3.17. If we denote Rumin's k-forms with L^r-coefficients by $L^r E_0^k$, then the projection from $L^r\Lambda^k\mathfrak{g}$ onto $L^r E_0^k$ is bounded, since E_0^k is a subspace of $\Lambda^k\mathfrak{g}$.

Proposition I.3.18 ([**62**]). *Let $\alpha \in C^\infty(E_0^k)_p$, we denote by $(\Pi_E\alpha)_j$ the component of $(\Pi_E\alpha)$ of weight j (that is necessarily greater than or equal to p). Then*

$$(\Pi_E\alpha)_p = \alpha\,,$$

$$(\Pi_E\alpha)_{p+r+1} = -d_0^{-1}\left(\sum_{\ell=1}^{r+1} d_\ell (\Pi_E\alpha)_{p+r+1-\ell}\right), \quad \textit{for } r \in \mathbb{N}.$$

The Hodge $\star$-duality holds on Rumin's complex. In order to understand this fact, we need some technical results concerning the formal adjoint in $L^2\Lambda^*\mathfrak{g}$ of the differential operators d_i's for $i = 0, \dots, \kappa$.

Definition I.3.19 (L^2-adjoint of d). We begin by recalling the definition of the formal adjoint d^* of the de Rham exterior differential d in $L^2\Lambda^*\mathfrak{g}$:

$$\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle,$$

for all compactly supported smooth forms $\alpha \in \Omega^{k-1}$, $\beta \in \Omega^k$. The direct consequence of the definition is that $d^* : \Omega^k \to \Omega^{k-1}$ satisfies the equality

$$d^* = (-1)^{N(k+1)+1} \star d \star,$$

where $\star$ denotes the Hodge-$\star$ operator.

Proposition I.3.20. *Recalling the formal decomposition in Remark I.3.4, if $k = 0, 1, \dots, N$, $P^{(k)}_{\min} \le p \le P^{(k)}_{\max}$ and $i = 0, 1, \dots, \kappa$, we have:*

$$d_i^*((\Omega^k)_p) \subset (\Omega^{k-1})_{p-i} \text{ and } d_i^* = (-1)^{N(k+1)+1} \star d_i \star .$$

Proposition I.3.21 ([**62**],[**31**]). *Let us take $0 \le k \le N$ and let $\star$ denote the Hodge-$\star$ operator, then*

$$\star E_0^k = E_0^{N-k}$$

and

$$d_R^* = (-1)^{N(k+1)+1} \star d_R \star .$$

Remark I.3.22. Let $\eta \in \Omega^k$, then

$$\star\star\eta = (-1)^{k(N-k)}\eta.$$

In the Heisenberg group the previous relations are simpler.

Remark I.3.23. Let $\mathbb{G} = \mathbb{H}_n$ and $\omega \in \Omega^k$, then

$$d_R^*\omega = (-1)^k \star d_R \star \omega, \quad \star\star\omega = \omega. \tag{I.3.1}$$

Proposition I.3.24. $\star\Delta_R = \Delta_R\star$

Proof. It follows from Remark I.3.23. □

Remark I.3.25. First of all, the "naif Hodge Laplacian" associated with d_R, i.e.

$$d_R^* d_R + d_R d_R^* \tag{I.3.2}$$

generally is not homogeneous (and therefore, to the best of our knowledge, we lack Rockland- type hypoellipticity results (see, e.g., [**35**]) and sharp a priori estimates in a natural scale of Sobolev spaces). This because d_R itself may not be homogeneous, but mainly because the two terms in (I.3.2) may have different orders.

However, if $\mathbb{G} = \mathbb{H}_n$, then d_R is always homogeneous and it is of the second order when it acts on n-forms, while it is of the first order in the other cases. Hence, it is possibile to construct a homogeneous "Laplacian" operator as defined by Rumin in [**59**].

Definition I.3.26 ([**59, 9**]). If $\Delta_{R,k}$ is Rumin's laplacian acting on k-forms, then

$$\Delta_{R,k} := \begin{cases} d_R^* d_R + d_R d_R^* & \text{if } k \neq n, n+1 \\ d_R^* d_R + (d_R d_R^*)^2 & \text{if } k = n \\ (d_R^* d_R)^2 + d_R d_R^* & \text{if } k = n+1\,. \end{cases}$$

If $k = n, n+1$, $\Delta_{R,k}$ is a homogeneous fourth order operator, while it is a homogeneous second order operator in the other cases.

Moreover we recall an important theorem.

Theorem I.3.27 ([**59**]). *If $\mathbb{G} = \mathbb{H}_n$, Δ_R is hypoelliptic and maximal hypoelliptic in the sense of* [**35**].

Then, after a definition, we state an easy consequence of Proposition 6.18 in [**31**].

Definition I.3.28 (Folland-Stein Sobolev spaces). Let $I = (i_1, \dots, i_s)$ be a multi-index, we set $X^I = X_1^{i_1} \cdots X_s^{i_s}$. Furthermore we set $|I| := i_1 + \cdots + i_s$ the order of the differential operator X^I and $d(I) := d_1 i_1 + \cdots + d_s i_s$ its degree of homogeneity with respect to group dilations. Let k be a positive integer, $1 \le p < \infty$, and let Ω be an open set in $\mathbb{G}$. The Folland-Stein Sobolev space $W^{k,p}_{\mathbb{G}}(\Omega)$ associated with the vector fields $X_1, ..., X_{m_1}$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for any X^I with $d(I) \le k$.

Now we consider the operator $\Delta_{R,k}$ in $L^2 E_0^k$.

Proposition I.3.29. *For $k \neq n, n+1$, $\mathrm{dom}(\Delta_{R,k}) = W^{2,2}_{\mathbb{G}} E_0^k$, while $\mathrm{dom}(\Delta_{R,k}) = W^{4,2}_{\mathbb{G}} E_0^k$ for $k = n, n+1$. In particular Δ_R is self-adjoint on its domain.*

Proof. The proof follows the same arguments as Proposition 6.18 of [**31**]. □

I.4. Complementary subgroups and graphs

Here we recall some facts about subgroups and graphs of Carnot groups.

From now on $\mathbb{G}$ will be a Carnot group identified with $\mathbb{R}^N$ through exponential coordinates. A *homogeneous subgroup* $\mathbb{H}$ of $\mathbb{G}$ (see [**64**, 5.2.4]) is a Lie subgroup such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda > 0$.

DEFINITION I.4.1. Let $\mathbb{M}$, $\mathbb{H}$ be homogeneous subgroups of $\mathbb{G}$. We say that $\mathbb{M}, \mathbb{H}$ are *complementary subgroups* in $\mathbb{G}$, if $\mathbb{M} \cap \mathbb{H} = \{e\}$ and if

$$\mathbb{G} = \mathbb{M} \cdot \mathbb{H},$$

that is for each $g \in \mathbb{G}$, there are $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = m \cdot h$.

If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G}$ and one of them is a normal subgroup then $\mathbb{G}$ is said to be the *semi-direct product* of $\mathbb{M}$ and $\mathbb{H}$. If both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $\mathbb{G}$ is said to be the *direct product* of $\mathbb{M}$ and $\mathbb{H}$.

The elements $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = mh$ are unique because of $\mathbb{M} \cap \mathbb{H} = \{e\}$ and are denoted as *components* of g along $\mathbb{M}$ and $\mathbb{H}$ or as *projections* of g on $\mathbb{M}$ and $\mathbb{H}$.

PROPOSITION I.4.2 (see [**7, 24**]). *If* $\mathbb{M}, \mathbb{H}$ *are complementary subgroups in* $\mathbb{G}$ *there is* $c_0 = c_0(\mathbb{M}, \mathbb{H})$, $0 < c_0 \le 1$, *such that for all* $g = mh$

$$c_0 \left(\|m\| + \|h\|\right) \le \|g\| \le \|m\| + \|h\| . \tag{I.4.1}$$

From now on, we will keep the following convention: when $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $\mathbb{M}$ will always be the first 'factor' and $\mathbb{H}$ the second one and $g_\mathbb{M} \in \mathbb{M}$ and $g_\mathbb{H} \in \mathbb{H}$ are the unique elements such that

$$g = g_\mathbb{M} g_\mathbb{H}.$$

Observe that, because in general the group is not abelian,

$$(g_\mathbb{M})^{-1} \ne (g^{-1})_\mathbb{M}, \qquad (g_\mathbb{H})^{-1} \ne (g^{-1})_\mathbb{H}.$$

Moreover we stress that the notation $g = g_\mathbb{M} g_\mathbb{H}$ is ambiguous because each component $g_\mathbb{M}$ or $g_\mathbb{H}$ depends on both the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$ and also on the order under which they are taken.

The projection maps $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ are defined as

$$\mathbf{P}_\mathbb{M}(g) := g_\mathbb{M}, \qquad \mathbf{P}_\mathbb{H}(g) := g_\mathbb{H} \tag{I.4.2}$$

Notations $g_\mathbb{M}$ or $\mathbf{P}_\mathbb{M}(g)$ will be used indifferently, the choice being suggested by typographical reasons only.

Notice that, differently from Euclidean spaces, $\mathbf{P}_\mathbb{M}$ and $\mathbf{P}_\mathbb{H}$, in general, are not Lipschitz maps, from $\mathbb{G}$ to $\mathbb{M}$ or to $\mathbb{H}$, when $\mathbb{G}$, $\mathbb{M}$ and $\mathbb{H}$ are endowed with the restriction of the left invariant distance d of $\mathbb{G}$, (see the example in [**24**]). Nevertheless $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ are C^∞ (indeed polynomial) maps from $\mathbb{G} = \mathbb{R}^N \to \mathbb{G} = \mathbb{R}^N$. This is the content of the following proposition.

PROPOSITION I.4.3 (See Proposition 2.2.16 in [**24**]). *Let* $\mathbb{M}$, $\mathbb{H}$ *be complementary subgroups of* $\mathbb{G}$, *then the projection maps* $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ *and* $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ *defined in* (I.4.2) *are polynomial maps. More precisely, if* κ *is the step of* $\mathbb{G}$, *there are* 2κ *matrices* $A^1, \dots, A^\kappa, B^1, \dots, B^\kappa$, *depending on* $\mathbb{M}$ *and* $\mathbb{H}$, *such that*

$$A^j \textit{ and } B^j \textit{are } (n_j, n_j)\textit{-matrices, for all } 1 \le j \le \kappa,$$

and, with the notations of (I.1.2),

$$\mathbf{P}_\mathbb{M} g = \left(A^1 g^1, A^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, A^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\right);$$

$$\mathbf{P}_\mathbb{H} g = \left(B^1 g^1, B^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, B^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\right);$$

$$A^j \textit{ is the identity on } \mathbb{M}^j, \textit{ and } B^j \textit{ is the identity on } \mathbb{H}^j, \textit{ for } 1 \le j \le \kappa.$$

Now we come to the main definition.

DEFINITION I.4.4. Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. We say that a set $S \subset \mathbb{G}$ is a *(left)* $\mathbb{H}$*-graph* (or a *left graph in direction* $\mathbb{H}$) if S intersects each left coset of $\mathbb{H}$ in one point at most.

One has an important special case when $\mathbb{H}$ admits a complementary subgroup $\mathbb{M}$. Indeed, in this case, there is a one to one correspondence between left cosets of $\mathbb{H}$ and points of $\mathbb{M}$ and we get that S is a left $\mathbb{H}$-graph if and only if there is $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ such that

$$S = \mathrm{graph}\,(\varphi) := \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}.$$

By uniqueness of the components along $\mathbb{M}$ and $\mathbb{H}$, if $S = \mathrm{graph}\,(\varphi)$ then φ is uniquely determined among all functions from $\mathbb{M}$ to $\mathbb{H}$. From now on we will consider mainly graphs of functions acting between complementary subgroups.

If a set $S \subset \mathbb{G}$ is an intrinsic (left) graph in direction $\mathbb{H}$ then it keeps being an intrinsic (left) graph in direction $\mathbb{H}$ after left translations or group dilations.

PROPOSITION I.4.5 (See [**7, 24**]). *Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. If S is a $\mathbb{H}$-graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $\delta_\lambda S$ and $q \cdot S$ are $\mathbb{H}$-graphs. If, in particular, $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, if $S = \mathrm{graph}\,(\varphi)$ with $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, then*

$$\begin{aligned} &\textit{For all } \lambda > 0,\ \delta_\lambda S = \mathrm{graph}\,(\varphi_\lambda), \textit{with}\\ &\varphi_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{M} \to \mathbb{H} \textit{ and}\\ &\varphi_\lambda(m) = \delta_\lambda \varphi(\delta_{1/\lambda} m), \ \textit{for } m \in \delta_\lambda \mathcal{E}. \end{aligned}$$

$$\begin{aligned} &\textit{For any } q \in \mathbb{G}, \quad q \cdot S = \mathrm{graph}\,(\varphi_q), \ \textit{where}\\ &\varphi_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}, \quad \mathcal{E}_q = \{m : \mathbf{P}_{\mathbb{M}}(q^{-1}m) \in \mathcal{E}\} \textit{ and}\\ &\varphi_q(m) = (\mathbf{P}_{\mathbb{H}}(q^{-1}m))^{-1} \cdot \varphi\big(\mathbf{P}_{\mathbb{M}}(q^{-1}m)\big), \ \textit{for all } m \in \mathcal{E}_q. \end{aligned}$$

The algebraic expression of φ_q in Proposition I.4.5 can be made more explicit when $\mathbb{G}$ is a semi-direct product of $\mathbb{M}, \mathbb{H}$ (see e.g. Remark 2.2.23 in [**24**]).

REMARK I.4.6. From Proposition I.4.5 and the continuity of the projections $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ it follows that the continuity of a function is preserved by translations. Precisely, given q and $f : \mathbb{M} \to \mathbb{H}$, then the translated function f_q is continuous in $m \in \mathbb{M}$ if and only if the function f is continuous in the corresponding point $\mathbf{P}_{\mathbb{M}}(q^{-1}m)$.

CHAPTER 1

Analysis of Rumin's Laplacian on the Heisenberg group

In this chapter we undertake the analysis of Rumin's Laplacian Δ_R on the Heisenberg group. Our goal is multi-fold. On one hand we wish to obtain a decomposition of the space of Rumin's forms with L^2 coefficients into invariant subspaces and describe the action of Δ_R restricted to these subspaces up to unitary equivalence. If we prove that the projections of these subspaces are L^p-bounded, we also obtain that this decomposition provide a L^p decomposition of the space of Rumin's forms. Finally we wish to prove a multiplier theorem for Δ_R.

It turns out that the CR structure on $\mathbb{H}_n$ plays a fundamental role in our analysis. Therefore, it is convenient to identify $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$, thus stressing the complex structure on the horizontal part.

1.1. The Heisenberg group $\mathbb{H}_n$

Let $\mathbb{H}_n$ be the $(2n+1)$-dimensional Heisenberg group with coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ and product given by

$$(z,t)(z',t') = (z+z', t+t' - \frac{1}{2}\mathrm{Im}(z \cdot \bar{z}')).$$

A basis of left-invariant vector fields is formed by

$$Z_j = \sqrt{2}\left(\partial_{z_j} - \frac{i}{4}\bar{z}_j\partial_t\right), \quad \bar{Z}_j = \sqrt{2}\left(\partial_{\bar{z}_j} + \frac{i}{4}z_j\partial_t\right), \quad T = \partial_t,$$

for $1 \le j \le n$, where

$$[Z_j, \bar{Z}_j] = iT.$$

Moreover we set

$$L := -\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)\,.$$

Definition 1.1.1. The anisotropic dilations are defined as

$$\delta_\lambda(z,t) := (\lambda z, \lambda^2 t), \quad \text{for all } \lambda > 0.$$

Let D be a differential operator on $\mathbb{H}_n$. It is homogeneous of degree $\alpha \in \mathbb{N}$ if

$$(D(f\,\delta_\lambda))(x) = \lambda^\alpha (Df)(\delta_\lambda(x))$$

for all $\lambda > 0$, $f \in C^\infty(\mathbb{H}_n)$.

Remark 1.1.2. Let $\alpha \in \mathbb{C}$, we recall that the operator $L + i\alpha T$ is locally solvable, hypoelliptic and sub-elliptic if and only if $\alpha \neq \pm(n+2k)$ for every $k \in \mathbb{N}$. These operators play a central role in the analysis on the Heisenberg group. They are often called Folland-Stein operators, see [**22**].

As in any Carnot group, in $\mathbb{H}_n$ the ordinary Lebesgue measure is the Haar measure.

DEFINITION 1.1.3. Given two functions $f, g \in L^1(\mathbb{H}_n)$ we define the convolution $f * g$ as

$$(f * g)(x, y, t) = \int_{\mathbb{H}_n} f((x, y, t)(x', y', t')^{-1}) g(x', y', t')\, dx' dy' dt'.$$

The dual basis of complex 1-covectors is

$$\zeta_j = \frac{1}{\sqrt{2}} dz_j, \quad \bar{\zeta}_j = \frac{1}{\sqrt{2}} d\bar{z}_j, \quad \theta = dt + \frac{i}{4}\sum_{j=1}^{n} (\bar{z}_j dz_j - z_j d\bar{z}_j).$$

DEFINITION 1.1.4. For simplicity of notation, we simply write Λ^k instead of $\Lambda^k \mathfrak{g}$. We denote by $\Lambda^k_H := (\Lambda^k)_k$ the space of horizontal k-covectors, i.e. the space of k-covectors of weight k. For p, q non-negative integers with $p + q = k$, we call $\Lambda^{p,q}$ the space of k-covectors of bidegree (p, q) in Λ^k_H:

$$\Lambda^{p,q} = \operatorname{span}\left\{ \sum_{|I|=p, |I'|=q} \zeta^I \wedge \bar{\zeta}^{I'} \right\}.$$

We denote by $L^p\Lambda^k$, $\mathcal{S}\Lambda^{p,q}$, etc. the space of L^p-sections, $\mathcal{S}$-sections, etc. of the corresponding bundle over $\mathbb{H}_n$, i.e. the space of k-forms with L^2-coefficients, (p, q)-forms with $\mathcal{S}$-coefficients.

Therefore the differential of a function f can be expressed as

$$df = \sum_{j=1}^{n} (Z_j f \zeta_j + \bar{Z}_j f \bar{\zeta}_j) + T f \theta.$$

Knowing that

$$d\zeta_j = d\bar{\zeta}_j = 0, \quad d\theta = -i \sum_{j=1}^{n} \zeta_j \wedge \bar{\zeta}_j,$$

the exterior derivative of differential forms can be easily computed.

1.1.1. Bargmann representations. In our spectral analysis of Δ_R we are going to need a particular space of test forms that was introduced in [**50**]. In order to describe such space we need to briefly describe a family of unitary irreducible representation for $\mathbb{H}_n$. We follow the presentation in [**50**]. For proofs see the cited reference.

The L^2-Fourier analysis on the Heisenberg group involves the family of infinite dimensional irreducible unitary representations $\{\pi_\lambda\}_{\lambda \neq 0}$ such that $\pi_\lambda(0, t) = e^{i\lambda t} I$. These representations are most conveniently realized for our purposes in the Bargmann form. Let $\mathcal{F} = \mathcal{F}(\mathbb{C}^n)$ be the space of entire functions F on $\mathbb{C}^n$ such that

$$\|F\|^2_{\mathcal{F}} = \int_{\mathbb{C}^n} |F(w)|^2 e^{-\frac{1}{2}|w|^2} dw < \infty.$$

Let $\lambda \neq 0$, the family of Bargmann representations π_λ on $\mathcal{F}$ is defined as follows:

- for $\lambda = 1$,
$$(\pi_1(z, t)F)(w) = e^{it} e^{-\frac{1}{2}\langle w, z\rangle - \frac{1}{4}|z|^2} F(w + z);$$
- for $\lambda > 0$,
$$\pi_\lambda(z, t) = \pi_1(\lambda^{\frac{1}{2}} z, \lambda t);$$
- for $\lambda < 0$,
$$\pi_\lambda(z, t) = \pi_{-\lambda}(\bar{z}, -t).$$

The unitary group $U(n)$ acts on $\mathbb{H}_n$ through the automorphisms

$$(z,t) \mapsto (z,t)^g = (gz,t), \quad \text{where } g \in U(n),$$

and on $L^2(\mathbb{H}_n)$ through the representation

$$(\alpha(g)f)(z,t) = f((z,t)^{g^{-1}}).$$

We also consider the pair of contragradient representations U, $\bar{U}$ of $U(n)$ on $\mathcal{F}$, given by

$$U_g F = F \circ g^{-1}, \bar{U}_g = U_{\bar{g}}. \tag{1.1.1}$$

Then, for $\lambda > 0$,

$$\pi_\lambda(gz,t) = U_g \pi_\lambda(z,t) U_{g^{-1}},$$
$$\pi_{-\lambda}(gz,t) = \bar{U}_g \pi_{-\lambda}(z,t) \bar{U}_{g^{-1}}.$$

The representation U in (1.1.1) splits into irreducibles according to the decomposition of $\mathcal{F}$

$$\mathcal{F} = \sum_{j\geq 0} P_j, \tag{1.1.2}$$

where P_j denotes the space of homogeneous polynomials of degree j.

We denote by P_j the orthogonal projection of $\mathcal{F}$ on $\mathcal{P}_j$, and by $\mathcal{F}^\infty$ the space of functions $F \in \mathcal{F}$ such that

$$\|P_j F\|_{\mathcal{F}} = o(j^{-N}), \quad \text{for all } N \in \mathbb{N}.$$

Then $\mathcal{F}^\infty$ is the space of C^∞-vectors for all representations π_λ. The differential of π_λ is given by [1] $\pi_\lambda(T) = i\lambda$ and

$$\pi_\lambda(Z_\ell) = \begin{cases} \sqrt{2\lambda}\partial_{w_\ell} & \text{if } \lambda > 0 \\ -\sqrt{\frac{|\lambda|}{2}} w_\ell & \text{if } \lambda < 0\,, \end{cases}$$

$$\pi_\lambda(\bar{Z}_\ell) = \begin{cases} -\sqrt{\frac{\lambda}{2}} w_\ell & \text{if } \lambda > 0 \\ \sqrt{2|\lambda|}\partial_{w_\ell} & \text{if } \lambda < 0\,. \end{cases}$$

We adopt the following definition of $\pi_\lambda(f)$:

$$\pi_\lambda(f) = \int_{\mathbb{H}_n} f(x)\pi_\lambda(x)^{-1}dx \in \mathcal{L}(\mathcal{F},\mathcal{F}). \tag{1.1.3}$$

Notice that $\pi_\lambda(f * g) = \pi_\lambda(g)\pi_\lambda(f)$, but this disadvantage is compensated by a simpler formalism when dealing with forms.

One way to write the Plancherel formula for $f \in L^2$ is

$$\|f\|_2^2 = c_n \int_{-\infty}^{+\infty} \|\pi_\lambda(f)\|_{HS}^2 |\lambda|^n d\lambda = c_n \int_{-\infty}^{+\infty} \sum_{j,j'} \|P_j \pi_\lambda(f) P_{j'}\|_{HS}^2 |\lambda|^n d\lambda\,.$$

Let V be a finite dimensional Hilbert space. Defining $\pi_\lambda(f)$ for V-valued functions f by (1.1.3), we have

$$\pi_\lambda(f) \in \mathcal{L}(\mathcal{F},\mathcal{F}) \otimes V \cong \mathcal{L}(\mathcal{F},\mathcal{F}\otimes V).$$

Suppose now that V is the representation space of a unitary representation ρ of $U(n)$, and consider the two representations $U \times \rho$, $\bar{U} \otimes \rho$ of $U(n)$ on $\mathcal{F} \otimes V$. Denote by $\Sigma^+ = \Sigma^{\rho,+}$ (resp. $\Sigma^- = \Sigma^{\rho,-}$) the set of irreducible representations $\sigma \in \widehat{U(n)}$ contained in $U \otimes \rho$ (resp. in $\bar{U} \otimes \rho$), and let

$$\mathcal{F} \otimes V = \bigoplus_{\sigma \in \Sigma^\pm} E_\sigma^\pm \tag{1.1.4}$$

[1] Here we denote $d\pi_\lambda$ by π_λ in order to simplify the notation.

be the corresponding orthogonal decompositions into $U(n)$-types. When $V = \mathbb{C}$, the decomposition (1.1.4) reduces to (1.1.2). To indicate the dependence on ρ, we shall sometime also write $\mathcal{E}_\sigma^\pm = \mathcal{E}_\sigma^{\rho,\pm}$.

LEMMA 1.1.5. *Each $\mathcal{E}_\sigma^\pm$ is finite dimensional and decomposes into $U(n)$-invariant subspaces*

$$\mathcal{E}_\sigma^\pm = \bigoplus_j \mathcal{E}_\sigma^\pm \cap (\mathcal{P}_j \otimes V).$$

In particular $\mathcal{E}_\sigma^\pm \subset \mathcal{F} \otimes V$. More precisely, $\mathcal{E}_\sigma^\pm \subset \mathcal{F}^\infty \otimes V$.

The decomposition of $\mathcal{F} \otimes V$ given above leads to the following form of the Plancherel formula for L^2V , with $P_\sigma^\pm$ denoting the orthogonal projection of $\mathcal{F} \otimes V$ onto $E_\sigma^\pm$:

$$(1.1.5) \qquad \begin{aligned} \|f\|_2^2 &= c_n \int_{-\infty}^{+\infty} \sum_{j\in\mathbb{N},\sigma\in\Sigma^{\operatorname{sgn}\lambda}} \left\| P_\sigma^{\operatorname{sgn}\lambda} \pi_\lambda(f) P_j \right\|_{HS}^2 |\lambda|^n d\lambda \\ &= c_n \int_{-\infty}^{+\infty} \sum_{\sigma\in\Sigma^{\operatorname{sgn}\lambda}} \left\| P_\sigma^{\operatorname{sgn}\lambda} \pi_\lambda(f) \right\|_{HS}^2 |\lambda|^n d\lambda. \end{aligned}$$

Let ρ' be another unitary representation of $U(n)$ on a finite dimensional Hilbert space V'. The convolution

$$f * K(x) = \int_{\mathbb{H}_n} K(y^{-1}x) f(y) dy$$

of integrable functions f with values in V and K with values in $\mathcal{L}(V, V')$ produces a function taking values in V'. In the representations π_λ, $\lambda \neq 0$,

$$\pi_\lambda(K) \in \mathcal{L}(\mathcal{F}, \mathcal{F} \otimes \mathcal{L}(V, V') \cong \mathcal{L}(\mathcal{F} \otimes V, \mathcal{F} \otimes V'),$$

and

$$\pi_\lambda(f * K) = \pi_\lambda(K)\pi_\lambda(f) \in \mathcal{L}(\mathcal{F}, \mathcal{F} \otimes V').$$

Let $\tilde{\rho}$ (resp. $\tilde{\rho}'$) be the representation $\alpha \otimes \rho$ on L^2V (resp. $\alpha \otimes \rho'$ on L^2V') of $U(n)$ and suppose that convolution by K is an equivariant operator, i.e.

$$(1.1.6) \qquad \tilde{\rho}'(g)(f * K) = (\tilde{\rho}(g)f) * K$$

for $g \in U(n)$ and $f \in \mathcal{S}V$. Since for $f \in \mathcal{S}V$ and $\xi \in \mathcal{F}$, if $\lambda > 0$,

$$\pi_\lambda(\tilde{\rho}'(g)(f * K))\xi = \iint \rho'(g) K(y^{-1}x) f(y) U_g \pi_\lambda(x^{-1}) U_{g^{-1}} \xi \, dy dx,$$

$$\pi_\lambda((\tilde{\rho}(g)f) * K)\xi = \iint K(y^{-1}x) \rho(g) f(y^{g^{-1}}) \pi_\lambda(x^{-1}) \xi \, dy dx,$$

by letting f tend weakly to $\delta_0 \otimes v$, with $v \in V$, we see that (1.1.6) implies

$$\int \rho'(g) K(x) v U_g \pi_\lambda(x^{-1}) U_{g^{-1}} \xi dx = \int K(x) \rho(g) v \pi_\lambda(x^{-1}) \xi dx.$$

Replacing ξ by $U_g\xi$, we obtain

$$U_g \otimes \rho'(g)(\pi_\lambda(K)(\xi \otimes v)) = \pi_\lambda(K)(U_g\xi \otimes \rho(g)v)$$

for every $\xi \in \mathcal{F}$, $v \in V$. A similar formula holds for $\lambda < 0$ with $\bar{U}$ in place of U. Thus (1.1.6) implies the following identities, for K defining an equivariant convolution operator:

$$(1.1.7) \qquad \begin{aligned} (U \otimes \rho')(g)\pi_\lambda(K) &= \pi_\lambda(K)(U \otimes \rho)(g), \quad \lambda > 0 \\ (\bar{U} \otimes \rho')(g)\pi_\lambda(K) &= \pi_\lambda(K)(\bar{U} \otimes \rho)(g), \quad \lambda < 0 \end{aligned}$$

for $g \in U(n)$, i.e. $\pi_\lambda(K)$ intertwines $U \otimes \rho$ and $U \otimes \rho'$, or $\bar{U} \otimes \rho$ and $\bar{U} \otimes \rho'$ depending on the sign of λ. The following is an immediate consequence.

LEMMA 1.1.6. *Assume that convolution by $K \in L^1 \otimes \mathcal{L}(V, V')$ is an equivariant operator. Then, setting $\Sigma^{\rho,\rho',\mathrm{sgn}\,\lambda} = \Sigma^{\rho,\mathrm{sgn}\,\lambda} \cap \Sigma^{\rho',\mathrm{sgn}\,\lambda}$,*

$$\pi_\lambda(K) = \bigoplus_{\sigma \in \Sigma^{\rho,\rho',\mathrm{sgn}\,\lambda}} \pi_{\lambda,\sigma}(K),$$

with $\pi_{\lambda,\sigma}(K) : \mathcal{E}_\sigma^{\rho,\mathrm{sgn}\,\lambda} \to \mathcal{E}_\sigma^{\rho',\mathrm{sgn}\,\lambda}$.

By a variant of Schwartz's Kernel Theorem, the convolution operators D with kernels $K \in \mathcal{S}'(\mathbb{H}_n) \otimes \mathcal{L}(V, V')$ are characterized as the continuous operators from $\mathcal{S}(\mathbb{H}_n) \otimes V$ to $\mathcal{S}'(\mathbb{H}_n) \otimes V'$ that commute with left translations on $\mathbb{H}_n$. Lemma 1.1.6 applies to operators of this kind, provided the Fourier transform $\pi_\lambda(K)$ is well defined for $\lambda \neq 0$. This is surely the case if K has compact support, and in particular for a left-invariant differential operator $Df = f * (D\delta_0)$. We then have

$$\pi_\lambda(Df) = \pi_\lambda(D\delta_0)\pi_\lambda(f) = \pi_\lambda(D)\pi_\lambda(f)\,.$$

With ρ_k denoting the representation of $U(n)$ on Λ^k induced from its action on $\mathbb{H}_n$ by automorphisms, and, as before let $\tilde\rho_k = \alpha \otimes \rho_k$ be the tensor product acting on $L^2\Lambda^k$. Then d, d^*, Δ_k are equivariant operators. The same applies to $\partial, \bar\partial, d_H$ etc. on the appropriate L^2-subbundles. Notice that $\Box, \overline{\Box}$ and Δ_H have the special property of acting scalarly on (p,q)-forms, by Remark 1.2.2. Since the sub-Laplacian L has the property that $\pi_\lambda(L)$ acts as a scalar multiple of the identity (namely, as $|\lambda|(2m+n)I$) on $\mathcal{P}_m \subset \mathcal{F}$, the same is true for the image of $\Box, \overline{\Box}, \Delta_H$ under π_λ, see [**22**].

For $0 < \delta < R$ and $N \in \mathbb{N}$, denote by $\mathcal{S}_{\delta,R,N}(\mathbb{H}_n)$ the space of functions f satisfying the following properties:

(i) $f \in \mathcal{S}(\mathbb{H}_n)$;
(ii) $\pi_\lambda(f) = 0$ for $|\lambda| \le \delta$ and $|\lambda| \ge R$;
(iii) for $\delta < |\lambda| < R, P_j\pi_\lambda(f) = 0$ for $j > N$.

We set $\mathcal{S}_0 = \bigcup_{\delta,R,N} \mathcal{S}_{\delta,R,N}$.

LEMMA 1.1.7. *$\mathcal{S}_0$ is invariant under left translations, and dense in L^2 .*

PROOF. See Lemma 3.1 of [**50**]. □

1.2. Differential forms on $\mathbb{H}_n$

We are interested in horizontal k-forms, which can be written as

$$\omega = \sum_{|I|+|I'|=k} f_{I,I'}\zeta^I \wedge \bar\zeta^{I'},$$

with

$$\zeta^I = \zeta_{i_1} \wedge \zeta_{i_2} \wedge \cdots \wedge \zeta_{i_p}.$$

DEFINITION 1.2.1. We define the following differential operators.

$$\partial\omega = \sum_{j=1}^n \zeta_j \wedge Z_j\omega, \quad \bar\partial\omega = \sum_{j=1}^n \bar\zeta_j \wedge \bar Z_j\omega, \quad d_H\omega = \partial\omega + \bar\partial\omega,$$

$$\Box = \partial\partial^* + \partial^*\partial, \quad \overline{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial, \quad \Delta_H = \Box + \overline{\Box},$$

REMARK 1.2.2. The following identities hold (see also [**49**]):

$$\partial^2 = \bar\partial^2 = \partial\bar\partial^* + \bar\partial^*\partial = \bar\partial\partial^* + \partial^*\bar\partial = 0. \tag{1.2.1}$$

The operators $\Box$, $\overline{\Box}$, Δ_H act on (p,q)-forms as scalar operators:

$$\begin{aligned}\Box &= \frac{1}{2}L + i\left(\frac{n}{2} - p\right)T,\\ \overline{\Box} &= \frac{1}{2}L - i\left(\frac{n}{2} - q\right)T,\\ \Delta_H &= L + i(q-p)T.\end{aligned}$$

As we will see in Proposition 1.3.1, an important role is played by $e(d\theta)$ (the operator of exterior multiplication by $d\theta$) and $e^*(d\theta)$ (its adjoint w.r.t. the inner product in Λ^*). Moreover, the Lefschetz decomposition plays a special role in our analysis. In order to describe it we need the following.

Proposition 1.2.3 ([**49**]). *Consider the following subspaces of $\Lambda^{p,q}$,*

$$\begin{aligned}V_j^{p,q} &= e(d\theta)^j \ker e^*(d\theta)|_{\Lambda^{p-j,q-j}}\\ W_l^{p,q} &= e^*(d\theta)^l \ker e(d\theta)|_{\Lambda^{p+l,q+l}}\end{aligned}$$

Then $V_j^{p,q}$ is non-trivial if and only if $\max\{0, k-n\} \le j \le \min\{p,q\}$, $W_l^{p,q}$ is non-trivial if and only if $\max\{0, n-k\} \le l \le \min\{n-p, n-q\}$, and we have the equality

$$V_j^{p,q} = W_l^{p,q}, \quad \text{for } l = j + n - k = l(j).$$

Moreover, $\Lambda^{p,q}$ is the orthogonal sum of the non-trivial $V_j^{p,q}$, and

$$e(d\theta)e^*(d\theta) = j(j+1+n-k) = (l(j)+1)(l(j)+k-n), \tag{1.2.2}$$

$$e^*(d\theta)e(d\theta) = (j+1)(j+n-k) = l(j)(l(j)+1+k-n) \tag{1.2.3}$$

on $V_j^{p,q}$.

Corollary 1.2.4. *$e(d\theta) : \Lambda_H^k \to \Lambda_H^{k+2}$ is injective for $0 \le k \le n-1$ and surjective for $n-1 \le k \le 2n-1$, while $e^*(d\theta) : \Lambda_H^k \to \Lambda_H^{k-2}$ is surjective for $2 \le k \le n+1$ and injective for $n+1 \le k \le 2n+1$.*

Proof. From (1.2.2) we get that $e^*(d\theta)$ is injective if $j(j+1+n-k) \neq 0$ for all j such that $\max\{0, k-n\} \le j \le \min\{p,q\}$. In particular $j \ge 1$ and $(j+1+n-k) \ge 1$ if $j \ge k-n \ge 1$, that is, $k \ge n+1$.

From (1.2.3) we get that $e(d\theta)$ is injective if $(j+1)(j+n-k) \neq 0$ for all j such that $\max\{0, k-n\} \le j \le \min\{p,q\}$. In particular $j+1 \ge 1$ and $(j+n-k) \ge 1$ if $j \ge 0$ and $n-k \ge 1$, that is, $k \le n-1$. $\Box$

Remark 1.2.5 ([**49**]). $[e^*(d\theta), e(d\theta)] = (n-k)I$ on horizontal k-forms.

Remark 1.2.6 ([**49**]). Some relations that we will need are:

- $\Box\bar{\partial} = \bar{\partial}(\Box - iT)$, $\quad \overline{\Box}\partial = \partial(\overline{\Box} + iT)$;
- $\bar{\partial}^*\Box = (\Box - iT)\bar{\partial}^*$, $\quad \partial^*\overline{\Box} = (\overline{\Box} + iT)\partial^*$;
- $[i(d\theta), \partial] = -i\bar{\partial}^*$, $\quad [i(d\theta), \bar{\partial}] = i\partial^*$;
- $[\partial^*, e(d\theta)] = i\bar{\partial}$, $\quad [\bar{\partial}^*, e(d\theta)] = -i\partial$;
- $[i(d\theta), d_H] = i\partial^* - i\bar{\partial}^*$, $\quad [d_H^*, e(d\theta)] = i\bar{\partial} - i\partial$.

1.3. Rumin's Laplacian on $\mathbb{H}_n$

We begin our analysis of Rumin's Laplacian Δ_R on $\mathbb{H}_n$ by characterising the space E_0^k in this case.

Proposition 1.3.1 (see [**9**]). *The subspaces of Rumin's forms E_0^k are*

$$E_0^k = \begin{cases} \Lambda_H^k \cap \ker e^*(d\theta) & \text{if } 0 \le k \le n\\ \left\{\theta \wedge \omega : \omega \in \Lambda_H^{k-1} \cap \ker e(d\theta)\right\} & \text{if } n+1 \le k \le 2n+1.\end{cases}$$

PROOF. By definition $E_0^k = \ker d_0 \cap (\operatorname{Im} d_0)^\perp$.

If $k=0$, we have $E_0^0 = \Lambda^0 = \Lambda_H^0 \cap \ker e^*(d\theta)$.

If $k=1$, $\ker d_0 = \Lambda_H^1$ and $\operatorname{ran} d_0 = \{0\}$. Hence $E_0^1 = \Lambda_H^1 \cap \ker e^*(d\theta)$.

Let $k>1$ and $\alpha + \theta\wedge\beta \in \Lambda^k$. Note that

$$\alpha + \theta\wedge\beta \in \ker d_0 \text{ if and only if } d\theta\wedge\beta = 0.$$

Hence $\ker d_0 = \{\alpha+\theta\wedge\beta \,|\, \beta \in \ker e(d\theta)\}$. Moreover

$$\alpha+\theta\wedge\beta \in \operatorname{ran} d_0 \iff \beta = 0 \text{ and } \alpha = d\theta\wedge\alpha' \text{ for some } \alpha' \in \Lambda^{k-2}.$$

Hence $\operatorname{ran} d_0 = \Lambda_H^k \cap \operatorname{ran} e(d\theta)$ and $(\operatorname{ran} d_0)^\perp = \{\alpha+\theta\wedge\beta \,|\, \alpha \in \ker e^*(d\theta)\}$. Finally $E_0^k = \{\alpha+\theta\wedge\beta \,|\, \alpha\in\ker e^*(d\theta) \text{ and } \beta \in \ker e(d\theta)\}$.

If $k\le n$, then $\ker e(d\theta)\cap\Lambda_H^{k-1} = \{0\}$. If $k \ge n+1$, then $\ker e^*(d\theta)\cap\Lambda_H^k = \{0\}$. □

We stress the fact that Rumin's differential d_R is in general a non-homogeneous differential operator. However in Heisenberg groups $\mathbb{H}_n$, d_R is homogeneous with respect to the dilations. Precisely $d_R : C^\infty E_0^k \to C^\infty E_0^{k+1}$ can be identified with a matrix-valued operator $L^k := (L_{i,j}^k)_{1\le i\le r, 1\le j\le s}$ with $s = \dim E_0^k$, $r = \dim E_0^{k+1}$ and $L_{i,j}^k$ are homogeneous left-invariant differential operators of the same degree: there exists $a\in\mathbb{N}$ such that

$$(L_{i,j}^k(f\,\delta_\lambda))(p) = \lambda^\alpha (L_{i,j}^k f)(\delta_\lambda(p))$$

for all $\lambda>0$, $f\in C^\infty(\mathbb{H}_n)$, $1\le i\le r, 1\le j\le s$.

LEMMA 1.3.2. *If $\alpha \in C^\infty E_0^k(\mathbb{H}_n)$, $\Pi_E\alpha = \alpha - d_0^{-1}d_1\alpha$.*

PROOF. Thanks to Proposition I.3.18 it suffices to show that if

$$(\Pi_E\alpha)_j = 0$$

for $j>k+1$. Since $\Pi_E\alpha$ is a k-form, its weight is $k+1$ at most. □

In particular the following proposition holds.

PROPOSITION 1.3.3 (see [**9**]). *If $0\le k\le n-1$, Rumin's differential is*

$$\begin{aligned} d_R : C^\infty E_0^k &\to C^\infty E_0^{k+1} \\ \omega &\mapsto \Pi_{\ker e^*(d\theta)} d_H\omega \end{aligned}$$

and it is a first order operator, where $\Pi_{\ker e^*(d\theta)}$ is the orthogonal projection on $\ker e^*(d\theta)$.

If $k=n$, Rumin's differential is

$$\begin{aligned} d_R : C^\infty E_0^n &\to C^\infty E_0^{n+1} \\ \omega &\mapsto \theta\wedge(T\omega + d_H e(d\theta)^{-1} d_H\omega) \end{aligned}$$

and it is a second order operator.

If $k\ge n+1$, we may reduce to the case $k\le n$; indeed, if we set $\omega' := \star\omega \in C^\infty E_0^{2n+1-k}$, we can apply Remark I.3.23 and obtain $d_R^*\omega' = (-1)^{2n+1-k}\star d_R\star\omega' = (-1)^{k+1}\star d_R\omega$.

PROOF. We recall that $d_R = \Pi_{E_0} d\Pi_E$, where $\Pi_E = I - d_0^{-1}d_1$. If $\omega\in C^\infty E_0^k$ with $0\le k\le n-1$, then

$$\begin{aligned} d_R\omega &= \Pi_{E_0}d_0\omega + \Pi_{E_0}(d_1 - d_0d_0^{-1}d_1)\omega + \Pi_{E_0}(d_2 - d_1d_0^{-1}d_1) + \Pi_{E_0}(-d_2d_0^{-1}d_1)\omega \\ &= \Pi_{E_0}d_1\omega - \Pi_{E_0}\Pi_{\mathcal{R}(d_0)}d_1\omega \\ &= \Pi_{E_0}d_1\omega \\ &= \Pi_{\ker e^*(d\theta)}d_H\omega. \end{aligned}$$

If $\omega \in C^\infty E_0^n$, then

$$\begin{aligned} d_R\omega &= \Pi_{E_0} d_0\omega + \Pi_{E_0}(d_1 - d_0 d_0^{-1} d_1)\omega + \Pi_{E_0}(d_2 - d_1 d_0^{-1} d_1)\omega + \Pi_{E_0}(-d_2 d_0^{-1} d_1)\omega \\ &= \Pi_{E_0}(d_2 - d_1 d_0^{-1} d_1)\omega. \end{aligned}$$

Now $(d_2 - d_1 d_0^{-1} d_1)\omega = \theta \wedge (T\omega + d_H e(d\theta)^{-1} d_H\omega)$ since

$$e(d\theta) : \Lambda_H^{n-1} \to \Lambda_H^{n+1}$$

is invertible. Moreover $T\omega + d_H e(d\theta)^{-1} d_H\omega \in \ker e(d\theta)$, since

$$e(d\theta)(T\omega + d_H e(d\theta)^{-1} d_H\omega) = Te(d\theta)\omega + d_H^2\omega = 0.$$

Then

$$d_R\omega = \Pi_{E_0}\left(\theta \wedge (T\omega + d_H e(d\theta)^{-1} d_H\omega)\right) = \theta \wedge (T\omega + d_H e(d\theta)^{-1} d_H\omega).$$

$\square$

1.4. First properties of Δ_R

Now we consider Rumin's forms with coefficients in $L^2(\mathbb{H}_n)$.

$$L^2 E_0^k = \begin{cases} L^2\Lambda_H^k \cap \ker e^*(d\theta) & \text{if } 0 \le k \le n \\ \left\{\theta \wedge \omega : \omega \in L^2\Lambda_H^{k-1} \cap \ker e(d\theta)\right\} & \text{if } n+1 \le k \le 2n+1 \end{cases}.$$

PROPOSITION 1.4.1. *The operators* $\square$, $\overline{\square}$, T *map* $\mathcal{S}_0 E_0^k$ *into itself. Moreover* T *is invertibile in* $\mathcal{S}_0 E_0^k$.

PROPOSITION 1.4.2. *If* $k < n$, $\Delta_{R,k}$ *is injective on* $\mathrm{dom}\,(\Delta_{R,k}) = W_{\mathbb{G}}^{2,2} E_0^k$. $\Delta_{R,n}$ *is injective on* $\mathrm{dom}\,(\Delta_{R,n}) = W_{\mathbb{G}}^{4,2} E_0^n$.

PROOF. Note that $\Delta_{R,k}$ is hypoelliptic as proved in [**59**]. Since Proposition 3.2 in [**10**], if $\omega \in L^2 E_0^k$ and $\Delta_R\omega = 0$, then ω has polynomial coefficients in L^2, hence $\omega = 0$. $\square$

In the sections that follow, we will need to restrict ourselves where $\square$ and $\overline{\square}$ are injective, therefore we state here some facts related to the problem of their injectivity.

REMARK 1.4.3. We recall Remark 1.1.2 and observe that: $\square$ is injective and hypoelliptic on (p,q)-forms with $p \neq 0, n$, whereas $\overline{\square}$ is injective and hypoelliptic on (p,q)-forms with $q \neq 0, n$. If $p = 0$ $\ker \square_0 = \ker \partial$, while if $p = n$ $\ker \square_n = \ker \partial^*$. If $q = 0$ $\ker \overline{\square}_0 = \ker \bar\partial$, while if $q = n$ $\ker \overline{\square}_n = \ker \bar\partial^*$.

Hence we define the following injective operators.

DEFINITION 1.4.4.

$$\square'\omega := \begin{cases} \square\omega & \text{if } 1 \le p \le n-1 \\ \square|_{\ker\partial}\omega & \text{if } p = 0 \\ \square|_{\ker\partial^*}\omega & \text{if } p = n\,, \end{cases}$$

$$\overline{\square}'\omega := \begin{cases} \overline{\square}\omega & \text{if } 1 \le q \le n-1 \\ \overline{\square}|_{\ker\bar\partial}\omega & \text{if } q = 0 \\ \overline{\square}|_{\ker\bar\partial^*}\omega & \text{if } q = n\,. \end{cases}$$

REMARK 1.4.5. Observe that

$$\square_0 = \overline{\square}_n = \frac{1}{2}(L + inT), \quad \square_n = \overline{\square}_0 = \frac{1}{2}(L - inT).$$

We denote by $\mathcal{C}$ the orthogonal projection from $L^2(\mathbb{H}_n)$ to $\ker(L + inT)$, while $\bar{\mathcal{C}}$ is the orthogonal projection from $L^2(\mathbb{H}_n)$ to $\ker(L - inT)$.

DEFINITION 1.4.6. Consider ∂ as a closed operator from $L^2\Lambda^{p,q}$ to $L^2\Lambda^{p+1,q}$. We denote by $\mathcal{R}_p$ the holomorphic Riesz transforms defined on $\mathcal{S}_0\Lambda^{p,q}$ (with values in $\mathcal{S}_0\Lambda^{p+1,q}$).

$$\mathcal{R}_p = \begin{cases} \partial\Box_p^{-\frac12} = \Box_{p+1}^{-\frac12}\partial & \text{for } 1 \le p \le n-2 \\ \partial{\Box'}_0^{-\frac12}(I-\mathcal{C}) = \Box_1^{-\frac12}\partial & \text{for } p = 0 \\ \partial\Box_{n-1}^{-\frac12} = {\Box'}_n^{-\frac12}\partial & \text{for } p = n-1. \end{cases} \tag{1.4.1}$$

The adjoint operators $\mathcal{R}_p^*$ from $\mathcal{S}_0\Lambda^{p+1,q}$ to $\mathcal{S}_0\Lambda^{p,q}$ are:

$$\mathcal{R}_p^* = \begin{cases} \Box_p^{-\frac12}\partial^* = \partial^*\Box_{p+1}^{-\frac12} & \text{for } 1 \le p \le n-2 \\ {\Box'}_0^{-\frac12}\partial^* = \partial^*\Box_1^{-\frac12} & \text{for } p = 0 \\ \Box_{n-1}^{-\frac12}\partial^* = \partial^*{\Box'}_n^{-\frac12}(I-\bar{\mathcal{C}}) & \text{for } p = n-1. \end{cases}$$

When omitting the subscript will not cause confusion, we will write $\mathcal{R}$ instead of $\mathcal{R}_p$.

REMARK 1.4.7. The following identities hold:

$$\begin{aligned} &\mathcal{R}_p\Box_p^{\frac12} = \Box_{p+1}^{\frac12}\mathcal{R}_p = \partial\,; \\ &\mathcal{R}_{p+1}\mathcal{R}_p = \mathcal{R}_p^*\mathcal{R}_{p+1}^* = 0\,; \\ &\mathcal{R}_p^*\mathcal{R}_p + \mathcal{R}_{p-1}\mathcal{R}_{p-1}^* = I, \quad (1 \le p \le n-1); \\ &\mathcal{R}_0^*\mathcal{R}_0 = I - \mathcal{C}\,; \\ &\mathcal{R}_{n-1}\mathcal{R}_{n-1}^* = I - \bar{\mathcal{C}}\,. \end{aligned}$$

PROPOSITION 1.4.8 ([**50**]). *For* $0 \le p \le n-1$,

$$(L^2\Lambda^{p,q})_{\partial\text{-cl}} = \ker\mathcal{R}_p = \ker\partial.$$

For $1 \le p \le n$,

$$(L^2\Lambda^{p,q})_{\partial\text{-ex}} = \operatorname{ran}\mathcal{R}_{p-1}\mathcal{R}_{p-1}^* = \operatorname{ran}\mathcal{R}_{p-1} = \overline{\partial(\mathcal{S}_0\Lambda^{p-1,q})}.$$

For $1 \le p \le n-1$,

$$(L^2\Lambda^{p,q})_{\partial\text{-cl}} = (L^2\Lambda^{p,q})_{\partial\text{-ex}}.$$

For $1 \le p \le n$,

$$(L^2\Lambda^{p,q})_{\partial^*\text{-cl}} = \ker\mathcal{R}_{p-1}^* = \ker\partial^*.$$

For $0 \le p \le n-1$,

$$(L^2\Lambda^{p,q})_{\partial^*\text{-ex}} = \operatorname{ran}\mathcal{R}_p^*\mathcal{R}_p = \operatorname{ran}\mathcal{R}_p^* = \overline{\partial^*(\mathcal{S}_0\Lambda^{p+1,q})}.$$

For $1 \le p \le n-1$,

$$(L^2\Lambda^{p,q})_{\partial^*\text{-cl}} = (L^2\Lambda^{p,q})_{\partial^*\text{-ex}}.$$

The antiholomorphic Riesz transforms $\overline{\mathcal{R}}_q$ and their adjoints $\overline{\mathcal{R}}_q^*$ are easily defined and the following formula holds for all q

$$\overline{\mathcal{R}}_q\overline{\Box}_q^{\frac12} = \overline{\Box}_{q+1}^{\frac12}\overline{R}_q = \bar\partial.$$

DEFINITION 1.4.9. On (p,q)-forms we define

$$C_p = I - \mathcal{R}_p^*\mathcal{R}_p, \quad \overline{C}_q = I - \overline{\mathcal{R}}_q^*\overline{\mathcal{R}}_q, \text{ for } 0 \le p,q \le n-1, C_n = \overline{C}_n = I.$$

PROPOSITION 1.4.10 ([**50**]). *C_p is the orthogonal projection of $L^2\Lambda^{p,q}$ onto the kernel of ∂, and $\overline{C}_q$ is the orthogonal projection of $L^2\Lambda^{p,q}$ onto the kernel of $\bar\partial$, Moreover, if $\omega \in \mathcal{S}_0\Lambda^{p,q}$, with $1 \le p \le n-1$, then*

$$C_p\omega = 0 \iff \omega \in \overline{\partial^*(\mathcal{S}_0\Lambda^{p+1,q})} \iff \partial^*\omega = 0,$$

whereas for $p = 0$,

$$C_0\omega = 0 \iff \omega \in \overline{\partial^*(\mathcal{S}_0\Lambda^{1,q})},$$

and for $p = n$,

$$C_n\omega = 0 \iff \omega = 0.$$

Similar statements hold for $\overline{C}_q$*, if we replace* p *with* q *and conjugate all terms. In particular,* $C_0 = \mathcal{C}$ *and* $\overline{C}_0 = \overline{\mathcal{C}}$.

1.5. A decomposition of $L^2E_0^k$

In this part, we assume that $0 \leq k \leq n$. The case where $k > n$ can be reduced to the case $k \leq n$ by means of Hodge duality. Clearly, Δ_R maps $\mathcal{S}_0E_0^k$ into itself. We begin by decomposing $\mathcal{S}_0E_0^k$ into orthogonal subspaces which are invariant under Δ_R and on which Δ_R takes a simple form.

We recall that $\mathcal{S}_0E_0^k = \mathcal{S}_0\Lambda_H^k \cap \ker e^*(d\theta)$ for $0 \leq k \leq n$. Therefore we start by decomposing $\mathcal{S}_0\Lambda_H^k$.

1.5.1. The subspaces. The decomposition is based on the following lemma.

LEMMA 1.5.1. *Every* $\omega \in \mathcal{S}_0\Lambda_H^k$ *decomposes as*

$$\omega = \omega' + \partial\xi + \bar\partial\eta,$$

where $\xi, \eta \in \mathcal{S}_0\Lambda_H^{k-1}$*, and* $\omega' \in \mathcal{S}_0\Lambda_H^k$ *satisfies the condition*

$$\partial^*\omega' = \bar\partial^*\omega' = 0. \tag{1.5.1}$$

The term ω' *is uniquely determined, and we can assume, in addition, that*

$$\xi \in (\ker\partial)^\perp, \eta \in (\ker\bar\partial)^\perp. \tag{1.5.2}$$

Notice that, even with the extra assumption, ξ and η are not uniquely determined.

PROOF. See Lemma 5.1 [**50**]. □

REMARK 1.5.2. Observe that the decomposition, without the extra assumptions (1.5.2) on ξ and η, can be iterated, so to obtain in a next step that

$$\begin{aligned}\omega &= \omega' + \partial(\xi' + \partial\alpha_1 + \bar\partial\beta_1) + \bar\partial(\eta' + \partial\alpha_2 + \bar\partial\beta_2)\\ &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial\beta_1 + \bar\partial\partial\alpha_2,\end{aligned}$$

where now each of the primed symbols represents a form satisfying (1.5.1). Now we iterate it a second time and get

$$\begin{aligned}\omega &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial(\alpha' + \partial\sigma_1 + \bar\partial\sigma_2) + \bar\partial\partial(\beta' + \partial\tau_1 + \bar\partial\tau_2)\\ &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial\alpha' + \partial\bar\partial\partial\sigma_1 + \bar\partial\partial\beta' + \bar\partial\partial\bar\partial\tau_2\\ &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial\alpha' + \bar\partial\partial\beta' + d_H^2(\partial\sigma_1 + \bar\partial\tau_2)\\ &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial\alpha' + \bar\partial\partial\beta' - Te(d\theta)(\partial\sigma_1 + \bar\partial\tau_2).\end{aligned} \tag{1.5.3}$$

DEFINITION 1.5.3. We set

$$\begin{aligned}W_0^{p,q} &= \left\{\omega \in \mathcal{S}_0\Lambda^{p,q} : \partial^*\omega = \bar\partial^* = 0\right\},\\ W_1^{p,q} &= \left\{\omega = \partial\xi + \bar\partial\eta : \xi, \eta \in W_0^{p,q}\right\},\\ W_2^{p,q} &= \left\{\omega = \bar\partial\partial\xi + \partial\bar\partial\eta : \xi, \eta \in W_0^{p,q}\right\}.\end{aligned}$$

It is clear that these subspaces are mutually orthogonal. We also set

$$W_0^k = \sum_{p+q=k} W_0^{p,q} = \left\{\omega \in \mathcal{S}_0\Lambda_H^k : \partial^*\omega = \bar\partial^*\omega = 0\right\}.$$

LEMMA 1.5.4.

$$\mathcal{S}_0E_0^k \subset \sum_{p+q=k} W_0^{p,q} \oplus \sum_{p+q=k-1} W_1^{p,q} \oplus \sum_{p+q=k-2} W_2^{p,q}\,.$$

PROOF. It follows from Remark 1.5.2. Since $\mathcal{S}_0E_0^k = \Lambda_H^k \cap \ker e^*(d\theta)$, we can stop after the second iteration in formula (1.5.3). $\square$

LEMMA 1.5.5. *$W_0^{p,q}$ and $W_1^{p,q}$ are in the kernel of $e^*(d\theta)$.*
Moreover

$$\begin{aligned}(W_2^{p,q})' &:= W_2^{p,q} \cap \ker e^*(d\theta)|_{\mathcal{S}_0\Lambda_H^{p+q+2}} \\ &= \{\omega = \bar\partial\partial\xi + \partial\bar\partial\eta : \xi,\eta \in W_0^{p,q} \text{ and } \Box\xi = \overline{\Box}\eta\}.\end{aligned}$$

PROOF. Let $\omega \in W_0^{p,q}$, then $e^*(d\theta)\omega = T^{-1}{d_H^*}^2\omega = 0$. Let $\omega = \partial\xi+\bar\partial\eta \in W_1^{p,q}$, then $e^*(d\theta)(\partial\xi + \bar\partial\eta) = -i\bar\partial^*\xi + \partial e^*(d\theta)\xi + i\partial^*\eta + \bar\partial e^*(d\theta)\eta = 0$.

We claim that $W_2^{p,q}$ decomposes as an orthogonal sum

$$W_2^{p,q} = (W_2^{p,q})' \oplus e(d\theta)W_0^{p,q}. \tag{1.5.4}$$

It is obvious by (1.12) that $e(d\theta)W_0^{p,q} \subset W_2^{p,q}$, and clearly the two subspaces on the right-hand side are orthogonal. Assume that $\omega = \bar\partial\partial\xi + \partial\bar\partial\eta \in W_2^{p,q}$, with $\xi, \eta \in W_0^{p,q}$. Then

$$e^*(d\theta)\omega = i(\Box\xi - \overline{\Box}\eta) \in W_0^{p,q}.$$

Indeed, by (1.13) and (1.20) we have

$$\begin{aligned}e^*(d\theta)(\partial\bar\partial\eta + \bar\partial\partial\xi) &= T^{-1}{d_H^*}^2(\partial\bar\partial\eta + \bar\partial\partial\xi) \\ &= T^{-1}(-\partial^*\partial\overline{\Box}\eta + \bar\partial^*\Box\bar\partial\eta + \partial^*\overline{\Box}\partial\xi - \bar\partial^*\bar\partial\Box\xi) \\ &= T^{-1}(-\Box\overline{\Box}\eta + (\Box - iT)\overline{\Box}\eta + (\overline{\Box} + iT)\Box\xi - \overline{\Box}\Box\xi) \\ &= i(\Box\xi - \overline{\Box}\eta).\end{aligned}$$

Hence $(W_2^{p,q})' = \{\omega = \bar\partial\partial\xi + \partial\bar\partial\eta : \xi,\eta \in W_0^{p,q} \text{ and } \Box\xi = \overline{\Box}\eta\}$

We have seen that $e^*(d\theta)W_2^{p,q} \subset W_0^{p,q}$, and therefore $\omega \in W_2^{p,q} \cap (e(d\theta)W_0^{p,q})^\perp$ if and only if $\omega \in (W_2^{p,q})'$. $\square$

COROLLARY 1.5.6.

$$\mathcal{S}_0E_0^k = \sum_{p+q=k} W_0^{p,q} \oplus \sum_{p+q=k+1} W_1^{p,q} \oplus \sum_{p+q=k+2} (W_2^{p,q})'\,.$$

If $k = n$ there are some cases where $W_0^{p,q}$ is trivial. We will need the following proposition in Subsection 1.6.4.

PROPOSITION 1.5.7 ([**50**]). *Let $0 \le k \le n$ and $p + q = k$. Then $W_0^{p,q}$ is trivial if and only if $k = n$ and $1 \le p, q \le n - 1$.*

PROOF. The complete proof can be found in [**50**], Proposition 5.3. However, there is an another way to see that $W_0^{p,q}$ is trivial if $k = n$ and $1 \le p, q \le n-1$. Let $\omega \in W_0^{p,q}$, then $e^*(d\theta)d_H\omega = 0$. But $e^*(d\theta)$ is injective on $n + 1$ forms, therefore $d_H\omega = 0$. Also $d_H^*\omega = 0$, since $\omega \in W_0^{p,q}$. Therefore $\Delta_H\omega = (L + i(q - p)T)\omega = 0$ with $|q - p| < n$, which implies $\omega = 0$. $\square$

PROPOSITION 1.5.8. *$L^2E_0^k$ decomposes as the orthogonal sum*

$$L^2E_0^k = \sum_{p+q=k} \overline{W_0^{p,q}} \oplus \sum_{p+q=k-1} \overline{W_1^{p,q}} \oplus \sum_{p+q=k-2} \overline{(W_2^{p,q})'}\,.$$

PROOF. It follows from Corollary 1.5.6. $\square$

LEMMA 1.5.9. *Given $\xi \in W_0^{p,q}$, there exists a unique $\xi' \in W_0^{p,q}$ such that $\partial\xi = \partial\xi'$ and $C_p\xi' = 0$. An analogous statement holds for $\bar\partial$ in place of ∂.*

PROOF. See Lemma 5.8 [**50**]. □

REMARK 1.5.10. Set $X^{p,q} = \{\xi \in W_0^{p,q} : C_p\xi = 0\}$, $Y^{p,q} = \{\eta \in W_0^{p,q} : \bar{C}_q\eta = 0\}$, $Z^{p,q} = X^{p,q} \times Y^{p,q}$. The previous lemma implies that the spaces $Z^{p,q}$ provide parametrizations for the spaces $W_1^{p,q}$, $W_2^{p,q}$. Moreover, if we set $\Xi^{p,q} := X^{p,q} \cap Y^{p,q}$, then $(W_2^{p,q})' = \{\bar\partial\partial\overline{\Box}\sigma + \partial\bar\partial\Box\sigma : \sigma \in \Xi^{p,q}\}$. By Lemma 12.3 in [**50**], $\Box\xi \in X^{p,q}$ and $\overline{\Box}\eta \in Y^{p,q}$. Hence $\Box\xi = \overline{\Box}\eta = \alpha \in \Xi^{p,q}$. We set $\sigma := (\Box\overline{\Box})^{-1}\alpha$, hence $\xi = \overline{\Box}\sigma$ and $\eta = \Box\sigma$.

REMARK 1.5.11. For convenience, we define the constant

$$c_{pq} := \frac{1}{n-p-q}\,. \tag{1.5.5}$$

COROLLARY 1.5.12. *The maps*

$$\begin{pmatrix}\partial & \bar\partial\end{pmatrix} : Z^{p,q} \to W_1^{p,q}, \quad (\xi,\eta) \mapsto \partial\xi + \bar\partial\eta, \tag{1.5.6}$$

$$\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix} : Z^{p,q} \to W_2^{p,q}, \quad (\xi,\eta) \mapsto \bar\partial\partial\xi + \partial\bar\partial\eta, \tag{1.5.7}$$

$$\begin{pmatrix}\bar\partial\partial\overline{\Box} + \partial\bar\partial\Box\end{pmatrix} : \Xi^{p,q} \to (W_2^{p,q})', \quad \sigma \mapsto \bar\partial\partial\overline{\Box}\sigma + \partial\bar\partial\Box\sigma, \tag{1.5.8}$$

are bijections.

PROOF. This corollary is already stated in [**50**]. Here we give a more detailed proof. The maps are surjective by construction. Moreover the injectivity of map (1.5.6) follows easily from Lemma 1.5.9 and Remark 1.5.10.

Now we prove the injectivity of maps (1.5.7) and (1.5.8). Since (1.5.4), if $\omega = \bar\partial\partial\xi + \partial\bar\partial\eta \in W_2^{p,q}$, then

$$\omega = \alpha + e(d\theta)\beta,$$

with $\alpha \in (W_2^{p,q})'$ and $\beta \in W_0^{p,q}$. Recalling the notation of Proposition 1.2.3, $\beta \in V_0^{p,q}$, hence

$$e^*(d\theta)e(d\theta)\beta = (n-p-q)\beta,$$

with $p+q = k-2 \le n-2$. We conclude that $\beta = c_{pq}e^*(d\theta)\omega$ and $\alpha = \Pi_{\ker e^*(d\theta)}w = \omega - (n-p-q)^{-1}e(d\theta)e^*(d\theta)\omega$. A simple computation shows that

$$\begin{aligned} &\alpha = \bar\partial\partial(ic_{p,q}T^{-1}\Box\xi + \xi - ic_{p,q}T^{-1}\overline{\Box}\eta) + \partial\bar\partial(ic_{p,q}T^{-1}\Box\xi + \eta - ic_{p,q}T^{-1}\overline{\Box}\eta), \\ &\beta = ic_{p,q}(\Box\xi - \overline{\Box}\eta). \end{aligned} \tag{1.5.9}$$

Since on (p,q)-forms $\Box - \overline{\Box} = ic_{p,q}^{-1}T$, we can rewrite (1.5.9) as

$$\begin{aligned} \alpha &= \bar\partial\partial(ic_{p,q}T^{-1}\overline{\Box}(\xi-\eta)) + \partial\bar\partial(ic_{p,q}T^{-1}\Box(\xi-\eta)) \\ &= \bar\partial\partial(\overline{\Box}\sigma) + \partial\bar\partial(\Box\sigma)\,, \end{aligned}$$

with $\sigma = ic_{p,q}T^{-1}(\xi-\eta)$. Notice that if $\Box\xi = \overline{\Box}\eta$, then $\overline{\Box}\sigma = \xi$ and $\Box\sigma = \eta$ in accordance with Remark 1.5.10. Suppose $\omega = 0$, then $\alpha = 0$ and $e(d\theta)\beta = 0$. Moreover $e(d\theta)$ is injective, because β is a $(k-2)$-form with $k \le n$. Hence $\beta = 0$, that is, $\Box\xi = \overline{\Box}\eta$. In order to prove the injectivity of map (1.5.7), we need to show that $\xi = \eta = 0$. Now we compute $\partial^*\alpha$ and obtain

$$\begin{aligned} \partial^*\alpha &= (\partial^*\bar\partial\partial\overline{\Box} + \partial^*\partial\bar\partial\Box)\sigma \\ &= (-\bar\partial\Box\overline{\Box} + \Box\bar\partial\Box)\sigma \\ &= (-\bar\partial\overline{\Box} + \Box\bar\partial)\Box\sigma \\ &= (\Box - \overline{\Box})\bar\partial\Box\sigma \\ &= ic_{p,q}^{-1}T\bar\partial\Box\sigma\,. \end{aligned}$$

Since $\partial^*\alpha = 0$ and $\bar\partial$ is injective on $Y^{p,q}$, $\sigma = 0$, that is, $\xi = \eta$. This proves the injectivity of map (1.5.8). We already know that $\Box\xi = \overline{\Box}\eta$, hence $(\Box - \overline{\Box})\xi = 0$. Finally we conclude that $\xi = \eta = 0$.

$\Box$

REMARK 1.5.13. Recall that, by Proposition 4.12,

$$X^{p,q} = \begin{cases} W_0^{p,q} & \text{if } 1 \le p \le n-1 \\ \{0\} & \text{if } p = n \\ \{\xi \in \mathcal{S}_0\Lambda^{0,q} : \mathcal{C}\xi = 0, \bar\partial^*\xi = 0\} & \text{if } p = 0\,. \end{cases}$$

By the proof of Lemma 5.9, the latter space is indeed nothing but $(I - \mathcal{C})W_0^{0,q}$. Analogous statements hold true for $Y^{p,q}$. Finally, notice that the spaces $Z^{p,q}$ are non-trivial if $p + q \le n-1$.

1.5.2. The action of Δ_R. Repeating the arguments formulated in [**50**] it is possibile to prove the following lemmas.

LEMMA 1.5.14. *The following propositions hold:*
(i) $d_H^*(W_0^{p,q}) = 0\,;$
(ii) $d_H^*(W_1^{p,q}) \subset W_0^{p,q}\,;$
(iii) $d_H^*(W_2^{p,q}) \subset W_1^{p,q}\,;$
(iv) $d_H^*(e(d\theta)W_1^{p,q}) \subset W_2^{p,q}\,;$
(v) $e^*(d\theta)(W_2^{p,q}) \subset W_0^{p,q}\,.$

PROOF.
(i) If $\omega \in W_0^{p,q}$ then $d_H^*(\omega) = 0$.
(ii) If $\partial\omega_1 + \bar\partial\omega_2 \in W_1^{p,q}$ then

$$d_H^*(\partial\omega_1 + \bar\partial\omega_2) = \underbrace{\bar\partial^*\partial\omega_1}_{=0} + \partial^*\partial\omega_1 + \underbrace{\partial^*\bar\partial\omega_2}_{=0} + \bar\partial^*\bar\partial\omega_2 = \Box\omega_1 + \overline{\Box}\omega_2.$$

Now $\Box\omega_1 + \overline{\Box}\omega_2 \in W_0^{p,q}$ since

$$\partial^*(\Box\omega_1 + \overline{\Box}\omega_2) = 0 \quad \text{and} \quad \bar\partial^*(\Box\omega_1 + \overline{\Box}\omega_2) = 0.$$

Let us see the first identity:

$$\partial^*(\Box\omega_1 + \overline{\Box}\omega_2) = \partial^*\Box\omega_1 + \partial^*\overline{\Box}\omega_2 = \partial^*\partial^*\partial\omega_1 + \overline{\Box}\partial^*\omega_2 + iT\partial^*\omega_2 = 0.$$

The second one is proved similarly. Therefore,

$$\begin{gathered} d_H^* : W_1^{p,q} \to W_0^{p,q} \\ \begin{pmatrix} \partial & \bar\partial \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \Box & \overline{\Box} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \end{gathered}$$

(iii) If $\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2 \in W_2^{p,q}$ then, using formula (1.2.1) and Remark 1.2.6,

$$\begin{aligned} d_H^*(\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2) &= \partial^*(\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2) + \bar\partial^*(\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2) \\ &= -\bar\partial\partial^*\partial\omega_1 - \partial\bar\partial^*\bar\partial\omega_2 + \underbrace{\partial^*\partial\bar\partial\omega_2}_{=\Box\bar\partial\omega_2} + \underbrace{\bar\partial^*\bar\partial\partial\omega_1}_{=\overline{\Box}\partial\omega_1} \\ &= -\bar\partial\partial^*\partial\omega_1 - \partial\bar\partial^*\bar\partial\omega_2 + (\bar\partial\Box - iT\bar\partial)\omega_2 + (\partial\overline{\Box} + iT\partial)\omega_1 \\ &= \partial((\overline{\Box} + iT)\omega_1 - \overline{\Box}\omega_2) + \bar\partial((\Box - iT)\omega_2 - \Box\omega_1). \end{aligned}$$

This belongs to $W_1^{p,q}$ since

$$\begin{aligned} \partial^*((\overline{\Box} + iT)\omega_1 - \overline{\Box}\omega_2) &= 0, \qquad & \partial^*((\Box - iT)\omega_2 - \Box\omega_1) &= 0, \\ \bar\partial^*((\overline{\Box} + iT)\omega_1 - \overline{\Box}\omega_2) &= 0, \qquad & \bar\partial^*((\Box - iT)\omega_2 - \Box\omega_1) &= 0. \end{aligned}$$

Finally,

$$d_H^* : W_2^{p,q} \to W_1^{p,q}$$
$$\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\Box + iT & -\overline{\Box}\\ -\Box & \overline{\Box} - iT\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

(iv) Now,

$$\begin{aligned} d_H^* e(d\theta)(\partial\xi + \bar\partial\eta) &= e(d\theta)d_H^*(\partial\xi + \bar\partial\eta) + (i\bar\partial - i\partial)(\partial\xi + \bar\partial\eta)\\ &= e(d\theta)(\Box\xi + \overline{\Box}\eta) + i(\bar\partial\partial\xi - \partial\bar\partial\eta)\\ &= (\bar\partial\partial + \partial\bar\partial)(-T^{-1})(\Box\xi + \overline{\Box}\eta) + i(\bar\partial\partial\xi - \partial\bar\partial\eta). \end{aligned}$$

Hence,

$$d_H^* : e(d\theta)W_1^{p,q} \to W_2^{p,q}$$
$$e(d\theta)\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}-T^{-1}\Box + i & -T^{-1}\overline{\Box}\\ -T^{-1}\Box & -T^{-1}\overline{\Box} - i\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

(v) It follows from $(d_H^*)^2 = Te^*(d\theta)$. $e^*(d\theta)(\bar\partial\partial\xi + \partial\bar\partial\eta) = i(\Box\xi - \overline{\Box}\eta)$.

$$e^*(d\theta) : W_2^{p,q} \to W_0^{p,q}$$
$$\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}i\Box & -i\overline{\Box}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

□

LEMMA 1.5.15. *The following propositions hold:*

(i) $d_H(W_0^{p,q}) \subset W_1^{p,q}$;
(ii) $d_H(W_1^{p,q}) \subset W_2^{p,q}$;
(iii) $d_H(W_2^{p,q}) \subset e(d\theta)W_1^{p,q}$;
(iv) $e(d\theta)W_0^{p,q} \subset W_2^{p,q}$.

PROOF.
(i) If $\omega \in W_0^{p,q}$ then $d_H(\omega) = \partial\omega + \bar\partial\omega \in W_1^{p,q}$. The matrix form of d_H is quite simple, but it will turn out to be useful later.

$$d_H : W_0^{p,q} \to W_1^{p,q}$$
$$w_0 \mapsto \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}(w_0).$$

(ii) If $\partial\omega_1 + \bar\partial\omega_2 \in W_1^{p,q}$ then $d_H(\partial\omega_1 + \bar\partial\omega_2) = \bar\partial\partial\omega_1 + \partial\bar\partial\omega_2 \in W_2^{p,q}$

$$d_H : W_1^{p,q} \to W_2^{p,q}$$
$$\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

(iii) If $\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2 \in W_2^{p,q}$ then $d_H(\bar\partial\partial\omega_1 + \partial\bar\partial\omega_2) = \partial\bar\partial\partial\omega_1 + \bar\partial\partial\bar\partial\omega_2 = -Te(d\theta)(\partial\omega_1 + \bar\partial\omega_2) \in e(d\theta)W_1^{p,q}$.

$$d_H : W_2^{p,q} \to e(d\theta)W_1^{p,q}$$
$$\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto e(d\theta)\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}-T & 0\\ 0 & -T\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

(iv) It follows from $(d_H)^2 = -Te(d\theta)$. In particular $e(d\theta)\omega_0 = -T^{-1}(d_H)^2\omega_0$.

$$e(d\theta) : W_0^{p,q} \to W_2^{p,q}$$
$$\omega_0 \mapsto \begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}-T^{-1}\\ -T^{-1}\end{pmatrix}(\omega_0).$$

$\square$

REMARK 1.5.16. Let $\Pi := \Pi_{\ker e^*(d\theta)}$ be the orthogonal projection on $\ker e^*(d\theta)$. Obviously, Π is the identity on $W_0^{p,q}$ and on $W_1^{p,q}$.

If $\omega \in W_2^{p,q}$ with $p+q = k-2$, using (1.5.9) we get

$$\Pi : W_2^{p,q} \to (W_2^{p,q})'$$
$$(\bar\partial\partial \quad \partial\bar\partial)\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto (\bar\partial\partial \quad \partial\bar\partial)\begin{pmatrix} ic_{pq}T^{-1}\square + 1 & -ic_{pq}T^{-1}\overline{\square}\\ ic_{pq}T^{-1}\square & 1 - ic_{pq}T^{-1}\overline{\square}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

Thanks to the previous lemmas, the following theorem holds.

THEOREM 1.5.17. *If $k \le n$, $W_0^{p,q}$, $W_1^{p,q}$ and $(W_2^{p,q})'$ are Δ_R-invariant, that is,*
(i) $\Delta_R(W_0^{p,q}) \subset W_0^{p,q}$;
(ii) $\Delta_R(W_1^{p,q}) \subset W_1^{p,q}$;
(iii) $\Delta_R((W_2^{p,q})') \subset (W_2^{p,q})'$.

1.6. Intertwining operators and different scalar forms for Δ_R

First we study the subcritical case of Rumin's Laplacian: $0 \le k \le n-1$.

1.6.1. Subcritical case: $W_0^{p,q}$. In this case Δ_R is a scalar operator.

PROPOSITION 1.6.1. *Let $p+q \le n-1$. Then on $W_0^{p,q}$*

$$\Delta_R = \Delta_H. \tag{1.6.1}$$

PROOF. Let $\omega_0 \in W_0^{p,q}$. Since $d_H^*(\omega_0) = 0$,

$$\Delta_R(\omega_0) = d_H^*\Pi d_H(\omega_0) = (\square \quad \overline{\square})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}(\omega_0) = \Delta_H\omega_0.$$

$\square$

1.6.2. Subcritical case: $W_1^{p,q}$. According to Corollary 1.5.12, we can write

$$W_1^{p,q} = \{w = \partial\xi + \bar\partial\eta : (\xi,\eta) \in Z^{p,q}\}.$$

Let $\omega_1 \in W_1^{p,q}$, then $\omega_1 = \partial\xi + \bar\partial\eta$ for some $\xi, \eta \in W_0^{p,q}$. Recall that $\Delta_R = \Pi d_H d_H^* + d_H^*\Pi d_H$.

REMARK 1.6.2. We recall the constant c_{pq} defined in (1.5.5). Since $p+q = k-1 < n-1$, $n-p-q > 1$. Hence $0 < c_{pq} < 1$.

$$\begin{aligned}
\Delta_R\omega_1 &= (\partial \;\; \bar\partial)\left[\begin{pmatrix}1\\ 1\end{pmatrix}(\square \quad \overline{\square}) + \begin{pmatrix}\overline{\square} + iT & -\overline{\square}\\ -\square & \square - iT\end{pmatrix}\begin{pmatrix} ic_{p,q}T^{-1}\square + 1 & -ic_{p,q}T^{-1}\overline{\square}\\ ic_{p,q}T^{-1}\square & 1 - ic_{p,q}T^{-1}\overline{\square}\end{pmatrix}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&= (\partial \;\; \bar\partial)\left[\begin{pmatrix}\square & \overline{\square}\\ \square & \overline{\square}\end{pmatrix} + \begin{pmatrix}-c_{p,q}\square + \overline{\square} + iT & \overline{\square}\,(c_{p,q}-1)\\ \square\,(c_{p,q}-1) & -c_{p,q}\overline{\square} + \square - iT\end{pmatrix}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&= (\partial \;\; \bar\partial)\begin{pmatrix}-c_{p,q}\square + \Delta_H + iT & c_{p,q}\overline{\square}\\ c_{p,q}\square & \Delta_H - iT - c_{p,q}\overline{\square}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.
\end{aligned}$$

Now we would like to diagonalize

$$M = \begin{pmatrix}-c_{p,q}\square + \Delta_H + iT & c_{p,q}\overline{\square}\\ c_{p,q}\square & \Delta_H - iT - c_{p,q}\overline{\square}\end{pmatrix}.$$

Proceeding formally, using the fact that all scalar entries commute, and recalling that $\square - \overline{\square} = ic_{p,q}^{-1}T$ on (p,q)-forms we have

$$\begin{aligned}
\det M &= (1-c_{p,q})(\Delta_H)^2 + ic_{p,q}T(\square - \overline{\square}) + T^2\\
&= (1-c_{p,q})(\Delta_H)^2,
\end{aligned}$$

and moreover,

$$\operatorname{tr} M = (2 - c_{p,q})\,\Delta_H.$$

The characteristic polynomial is

$$p(x) = x^2 - (2 - c_{p,q})\,\Delta_H x + (\Delta_H)^2(1 - c_{p,q}),$$

hence the eigenvalues are

$$\lambda_1 = (1 - c_{p,q})\,\Delta_H \text{ and } \lambda_2 = \Delta_H.$$

We observe that $1 - c_{p,q} > 0$. In particular we have

$$\begin{aligned} M &= \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix} \tilde{M} \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix} \tilde{M} \begin{pmatrix} \Delta_H^{-1} & -\Delta_H^{-1} \\ \Delta_H^{-1}\Box & \Delta_H^{-1}\overline{\Box} \end{pmatrix}, \end{aligned}$$

with $\tilde{M} = \begin{pmatrix} (1 - c_{p,q})\,\Delta_H & 0 \\ 0 & \Delta_H \end{pmatrix}$, since

$$\begin{aligned} \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix} \tilde{M} \begin{pmatrix} \Delta_H^{-1} & -\Delta_H^{-1} \\ \Delta_H^{-1}\Box & \Delta_H^{-1}\overline{\Box} \end{pmatrix} &= \begin{pmatrix} (1 - c_{p,q})\,\overline{\Box}\Delta_H & \Delta_H \\ -(1 - c_{p,q})\,\Box\Delta_H & \Delta_H \end{pmatrix} \begin{pmatrix} \Delta_H^{-1} & -\Delta_H^{-1} \\ \Delta_H^{-1}\Box & \Delta_H^{-1}\overline{\Box} \end{pmatrix} \\ &= \begin{pmatrix} \Delta_H - c_{p,q}\overline{\Box} & c_{p,q}\overline{\Box} \\ c_{p,q}\Box & \Delta_H - c_{p,q}\Box \end{pmatrix} = M. \end{aligned}$$

We set $Q := \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix}$,

$$\tilde{Z}^{p,q} := \Xi^{p,q} \times W_0^{p,q},$$

where $\Xi^{p,q} = X^{p,q} \cap Y^{p,q}$ and we define $A_1 : \tilde{Z}^{p,q} \to W_1^{p,q}$ as

$$A_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix} = (\partial \quad \bar{\partial}) \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Moreover $A_1^+\xi := A_1 \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $A_1^-\eta := A_1 \begin{pmatrix} 0 \\ \eta \end{pmatrix}$.

Lemma 1.6.3. *If $p+q \le n-1$, then the operator matrix $Q : \mathcal{S}_0\Lambda^{p,q} \times \mathcal{S}_0\Lambda^{p,q} \to \mathcal{S}_0\Lambda^{p,q} \times \mathcal{S}_0\Lambda^{p,q}$ is invertible with inverse*

$$Q^{-1} = \begin{pmatrix} \Delta_H^{-1} & -\Delta_H^{-1} \\ \Delta_H^{-1}\Box & \Delta_H^{-1}\overline{\Box} \end{pmatrix}. \tag{1.6.2}$$

Moreover Q maps the subspace $W_0^{p,q} \times W_0^{p,q}$ bijectively onto itself.

Proof. Formally $\det Q = \Delta_H$, which is invertible since $p+q \le n-1$. Therefore (1.6.2) is obvious. The operators $\Box$ and $\overline{\Box}$ leave $W_0^{p,q}$ invariant, therefore the lemma is proved. □

Lemma 1.6.4. *It holds that*

$$\begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} Q(\tilde{Z}^{p,q}) = Z^{p,q}.$$

Proof. Since

$$Z^{p,q} = \begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} (W_0^{p,q} \times W_0^{p,q}) = \begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} Q(W_0^{p,q} \times W_0^{p,q}),$$

we need to show that

$$\begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} Q(\tilde{Z}^{p,q}) = \begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} Q(W_0^{p,q} \times W_0^{p,q}).$$

Since $\tilde{Z}^{p,q} = \Xi^{p,q} \times W_0^{p,q} = (I - C_p - \bar{C}_q)W_0^{p,q} \times W_0^{p,q}$, it suffices to show that

$$\begin{pmatrix} I - C_p & 0 \\ 0 & I - \bar{C}_q \end{pmatrix} Q \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} (I - C_p)\overline{\Box}\xi \\ -(I - \bar{C}_q)\Box\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.6.3}$$

for every $\xi = (C_p + \bar{C}_q)\xi'$. Note that

$$(I - C_p)(C_p + \bar{C}_q)\overline{\Box}\xi' = \bar{C}_q\overline{\Box}\xi' = 0$$

and

$$(I - \bar{C}_q)(C_p + \bar{C}_q)\Box\xi' = C_p\Box\xi' = 0\,;$$

hence (1.6.3) is verified. $\square$

The following proposition follows immediately from the previous lemmas and computations.

PROPOSITION 1.6.5. *The space $W_1^{p,q}$ decomposes as the direct sum*

$$W_1^{p,q} = A_1^+(\Xi^{p,q}) + A_1^-(W_0^{p,q}).$$

Moreover, A_1^+ and A_1^- are injective on $\Xi^{p,q}, W_0^{p,q}$ respectively. The following identities hold, on $\Xi^{p,q}, W_0^{p,q}$ respectively:

$$\begin{aligned} &(A_1^+)^{-1}\Delta_R(A_1^+) = (1 - c_{pq})\,\Delta_H \\ &(A_1^-)^{-1}\Delta_R(A_1^-) = \Delta_H\,. \end{aligned} \tag{1.6.4}$$

Finally we set

$$(W_1^{p,q})^+ := A_1^+(\Xi^{p,q}), \quad (W_1^{p,q})^- := A_1^-(W_0^{p,q}). \tag{1.6.5}$$

We will show that $(W_1^{p,q})^+$, $(W_1^{p,q})^-$ are orthogonal by analysing the intertwining operators in the next section.

1.6.3. Subcritical case: $(W_2^{p,q})'$.

We will need the following remark.

REMARK 1.6.6. If $\Box\xi = \overline{\Box}\eta$, then

$$\begin{pmatrix} \overline{\Box} + iT & -\overline{\Box} \\ -\Box & \Box - iT \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -i(n-p-q-1)T & 0 \\ 0 & i(n-p-q-1)T \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

and

$$\begin{pmatrix} a\Box & -a\overline{\Box} \\ -b\Box & b\overline{\Box} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \tag{1.6.6}$$

for every $a, b \in \mathbb{C}$.

Also note that $n - p - q = c_{pq}^{-1}$, since (1.5.5).

PROPOSITION 1.6.7. *Let $A_2 = \left(\bar\partial\partial\overline{\Box} + \partial\bar\partial\Box\right) : X^{p,q} \cap Y^{p,q} \to (W_2^{p,n-1-p})'$. Then A_2 is injective on $X^{p,q} \cap Y^{p,q}$. The operator Δ_R restricted to $(W_2^{p,q})'$ is given by*

$$\Delta_R|_{(W_2^{p,q})'} = A_2\,(1 - c_{pq})\,\Delta_H A_2^{-1}\,. \tag{1.6.7}$$

PROOF. Recall that $\Delta_R = \Pi d_H d_H^* + d_H^*\Pi d_H$, where Π is the orthogonal projection onto $\ker e^*(d\theta)$, and observe that $\Pi d_H\omega_2 = 0$ if $\omega_2 \in W_2^{p,q}$. Hence $\Delta_R\omega_2 = \Pi d_H d_H^*\omega_2$.

Then,

$$
\begin{aligned}
&\Delta_R(\bar\partial\partial\xi+\partial\bar\partial\eta)\\
&=\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix} ic_{pq}T^{-1}\Box+1 & -ic_{pq}T^{-1}\overline{\Box}\\ ic_{pq}T^{-1}\Box & 1-ic_{pq}T^{-1}\overline{\Box}\end{pmatrix}\begin{pmatrix}\overline{\Box}+iT & -\overline{\Box}\\ -\Box & \Box-iT\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&=\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix} ic_{pq}T^{-1}\Box+1 & -ic_{pq}T^{-1}\overline{\Box}\\ ic_{pq}T^{-1}\Box & 1-ic_{pq}T^{-1}\overline{\Box}\end{pmatrix}\begin{pmatrix}-i(c_{pq}^{-1}-1)T & 0\\ 0 & i(c_{pq}^{-1}-1)T\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&=\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}(c_{pq}^{-1}-1)\,(c_{pq}\Box-iT) & \overline{\Box}\,(1-c_{pq})\\ \Box\,(1-c_{pq}) & (c_{pq}^{-1}-1)\,\big(iT+c_{pq}\overline{\Box}\big)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.
\end{aligned}
$$

We can add (1.6.6) to the previous formula with $a=b=1-c_{pq}$ and we get

$$
\begin{aligned}
&\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}(1-c_{pq})(2\Box-ic_{pq}^{-1}T) & 0\\ 0 & (1-c_{pq})(2\overline{\Box}+ic_{pq}^{-1}T)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&\quad=\begin{pmatrix}\bar\partial\partial & \partial\bar\partial\end{pmatrix}\begin{pmatrix}(1-c_{pq})\,\Delta_H & 0\\ 0 & (1-c_{pq})\,\Delta_H\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&\quad=\big(\bar\partial\partial\overline{\Box}+\partial\bar\partial\Box\big)\,(1-c_{pq})\,\Delta_H\sigma\,.
\end{aligned}
$$

Here we used the identity $\Box-\overline{\Box}=ic_{pq}^{-1}T$.

$\Box$

Now we consider the critical case: $k=n$.

Lemma 1.6.8. *The following identity holds:*

$$\ker e(d\theta)\cap\Lambda_H^n=\ker e^*(d\theta)\cap\Lambda_H^n.$$

As a consequence, $E_0^{n+1}=\{\theta\wedge\omega\,|\,\omega\in\Lambda_H^n\cap\ker e^*(d\theta)\}=\theta\wedge E_0^n$.

Proof. Let us prove that $\ker e(d\theta)\cap\Lambda_H^n\subset\ker e^*(d\theta)\cap\Lambda_H^n$. The other inclusion is analogous. Now $[e^*(d\theta),e(d\theta)]=(n-k)I$, therefore $[e^*(d\theta),e(d\theta)]=0$ on horizontal n-form. Let $\omega\in\ker e(d\theta)\cap\Lambda_H^n$, then $e^*(d\theta)e(d\theta)\omega=0$; hence $e(d\theta)e^*(d\theta)\omega=0$. But $e(d\theta)$ is injective on $(n-2)$-forms, therefore $e^*(d\theta)\omega=0$. $\Box$

Lemma 1.6.9. *If* $\omega\in\ker e^*(d\theta)\cap\Lambda_H^n=\ker e(d\theta)\cap\Lambda_H^n$, *then* $e(d\theta)^{-1}d_H\omega=e^*(d\theta)d_H\omega$.

Proof. $[e^*(d\theta),e(d\theta)]d_H\omega=-d_H\omega$, but $e(d\theta)d_H\omega=0$ since $\omega\in\ker e(d\theta)$. Therefore

$$e(d\theta)e^*(d\theta)d_H\omega=d_H\omega.$$

$\Box$

We will need the following remark.

Remark 1.6.10. We recall that if $\omega\in\ker e^*(d\theta)\cap\mathcal{S}_0\Lambda_H^n$, then

$$
\begin{aligned}
d_R\omega&=\theta\wedge(T\omega+d_He(d\theta)^{-1}d_H\omega),\\
d_R^*(\theta\wedge\omega)&=-T\omega+d_H^*e^*(d\theta)^{-1}d_H^*\omega,\\
\Delta_R\omega&=d_R^*d_R+\Pi d_Hd_H^*\Pi d_Hd_H^*.
\end{aligned}
$$

Since Lemma 1.6.9, $e(d\theta)^{-1}d_H\omega=e^*(d\theta)d_H\omega$; hence we need to study the action of $e^*(d\theta)$ on $W_2^{p,n-1-p}$ (see Lemma 1.5.14, (v)) and on $e(d\theta)W_1^{p,n-2-p}$. It is easy to see that

$$
\begin{aligned}
e^*(d\theta):e(d\theta)W_1^{p,n-2-p}&\to W_1^{p,n-2-p}\\
e(d\theta)\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}&\mapsto\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.
\end{aligned}
$$

In fact, if $\omega \in W_1^{p,q}$ with $p+q=n-2$, $e^*(d\theta)e(d\theta)\omega = \omega$, because of Proposition 1.2.3.

Moreover, in order to compute d_R^*, we need the action of $e(d\theta)$ on $W_0^{p,n-1-p}$ (see Lemma 1.5.15, (iv)) and on $W_1^{p,n-2-p}$.

$$e(d\theta) : W_1^{p,n-2-p} \to e(d\theta)W_1^{p,n-2-p}$$
$$\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto e(d\theta)\begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

1.6.4. Critical case: $W_0^{p,q}$ $(p+q=n)$.

REMARK 1.6.11. If $\omega_0 \in W_0^{n,0}$ or $\omega_0 \in W_0^{0,n}$, then $i(d\theta)d_H\omega_0 = 0$ and $\Box\omega_0 = \overline{\Box}\omega_0 = 0$. $\Delta_H\omega_0 = (L \pm inT)\omega_0 = 0$. Hence $-T^2 = L^2/n^2$.

PROPOSITION 1.6.12. *Let $pq=0$ and $p+q=n$. Then on $W_0^{p,q}$*

$$\Delta_R = -T^2 = L^2/n^2. \tag{1.6.8}$$

PROOF. Note that $d_H^*\omega = 0$ and $d_H\omega \in \ker e^*(d\theta)$ so that $\Delta_R\omega = d_R^* d_R\omega = -T^2\omega = L^2/n^2$. $\square$

1.6.5. Critical case: $W_1^{p,q}$ $(p+q=n-1)$.

PROPOSITION 1.6.13. *Let $A_1 = \begin{pmatrix}\partial & \bar\partial\end{pmatrix} : Z^{p,q} \to W_1^{p,n-1-p}$. Then A_1 is injective. The operator Δ_R restricted to $W_1^{p,n-1-p}$ is given by*

$$\Delta_R|_{W_1^{p,n-1-p}} = A_1(\Delta_H)^2 A_1^{-1} \tag{1.6.9}$$

PROOF. Observing that $\Box - \overline{\Box} = iT$, we have

$$\Delta_R(\partial\xi + \bar\partial\eta) = \begin{pmatrix}\partial & \bar\partial\end{pmatrix} M_1M_2\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \begin{pmatrix}\partial & \bar\partial\end{pmatrix}(M_3)^2\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

where

$$\begin{aligned}
M_1 &= \begin{pmatrix}-T & 0\\ 0 & -T\end{pmatrix} + \begin{pmatrix}\overline{\Box}+iT & -\overline{\Box}\\ -\Box & \Box - iT\end{pmatrix}\begin{pmatrix}-T^{-1}\\ -T^{-1}\end{pmatrix}\begin{pmatrix}\Box & \overline{\Box}\end{pmatrix},\\
M_2 &= \begin{pmatrix}T & 0\\ 0 & T\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}i\Box & -i\overline{\Box}\end{pmatrix},\\
M_3 &= \begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}\Box & \overline{\Box}\end{pmatrix}.
\end{aligned}$$

Now,

$$\begin{aligned}
M_1M_2 &= \begin{pmatrix}-i\Box - T & -i\overline{\Box}\\ i\Box & i\overline{\Box} - T\end{pmatrix}\begin{pmatrix}i\Box + T & -i\overline{\Box}\\ i\Box & -i\overline{\Box} + T\end{pmatrix}\\
&= \begin{pmatrix}\Box(\Delta_H - 2iT) - T^2 & -\overline{\Box}\Delta_H\\ -\Box\Delta_H & \overline{\Box}(\Delta_H + 2iT) - T^2\end{pmatrix},
\end{aligned}$$

and

$$(M_3)^2 = \begin{pmatrix}\Box & \overline{\Box}\\ \Box & \overline{\Box}\end{pmatrix}^2 = \begin{pmatrix}\Box\Delta_H & \overline{\Box}\Delta_H\\ \Box\Delta_H & \overline{\Box}\Delta_H\end{pmatrix}.$$

Hence

$$\begin{aligned}
\Delta_R(\partial\xi + \bar\partial\eta) &= \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}2\Box(\Delta_H - iT) - T^2 & 0\\ 0 & 2\overline{\Box}(\Delta_H + iT) - T^2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\
&= \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}4\Box\overline{\Box} - T^2 & 0\\ 0 & 4\Box\overline{\Box} - T^2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.
\end{aligned}$$

Since $2\Box(\Delta_H - iT) - T^2 = 2\overline{\Box}(\Delta_H + iT) - T^2 = 4\Box\overline{\Box} - T^2$, we have

$$4\Box\overline{\Box} - T^2 = (2\Box(\Delta_H - iT) - T^2)/2 + (2\overline{\Box}(\Delta_H + iT) - T^2)/2 = (\Delta_H)^2.$$

Therefore,

$$\Delta_R(\partial\xi + \bar{\partial}\eta) = \begin{pmatrix}\partial & \bar{\partial}\end{pmatrix}\begin{pmatrix}(\Delta_H)^2 & 0 \\ 0 & (\Delta_H)^2\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}.$$

□

1.6.6. Critical case: $(W_2^{p,q})'$ $(p+q=n-2)$.

Proposition 1.6.14. *Let $A_2 = \left(\bar{\partial}\partial\overline{\Box} + \partial\bar{\partial}\Box\right) : X^{p,q}\cap Y^{p,q} \to (W_2^{p,n-2-p})'$. Then A_2 is injective. The operator Δ_R restricted to $W_2^{p,n-2-p}$ is given by*

$$\Delta_R|_{W_2^{p,n-2-p}} = A_2\frac{(\Delta_H)^2}{4}A_2^{-1} \tag{1.6.10}$$

Proof. Observing that $\Box - \overline{\Box} = 2iT$, we have

$$\Delta_R(\bar{\partial}\partial\xi + \partial\bar{\partial}\eta) = \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix} M_1M_2\begin{pmatrix}\xi \\ \eta\end{pmatrix} + \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix}(M_3)^2\begin{pmatrix}\xi \\ \eta\end{pmatrix},$$

where

$$\begin{aligned}
M_1 &= \begin{pmatrix}-T & 0 \\ 0 & -T\end{pmatrix} + \begin{pmatrix}-T^{-1}\Box + i & -T^{-1}\overline{\Box} \\ -T^{-1}\Box & -T^{-1}\overline{\Box} - i\end{pmatrix}\begin{pmatrix}\overline{\Box} + iT & -\overline{\Box} \\ -\Box & \Box - iT\end{pmatrix}, \\
M_2 &= \begin{pmatrix}T & 0 \\ 0 & T\end{pmatrix} + \begin{pmatrix}-T & 0 \\ 0 & -T\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}, \\
M_3 &= \begin{pmatrix}\frac{1}{2}iT^{-1}\Box + 1 & -\frac{1}{2}iT^{-1}\overline{\Box} \\ \frac{1}{2}iT^{-1}\Box & 1 - \frac{1}{2}iT^{-1}\overline{\Box}\end{pmatrix}\begin{pmatrix}\overline{\Box} + iT & -\overline{\Box} \\ -\Box & \Box - iT\end{pmatrix}.
\end{aligned}$$

Using the identity $\begin{pmatrix}\overline{\Box} + iT & -\overline{\Box} \\ -\Box & \Box - iT\end{pmatrix} = \begin{pmatrix}-iT & 0 \\ 0 & +iT\end{pmatrix}$ on $(W_2^{p,q})'$, we have

$$\Delta_R(\bar{\partial}\partial\xi + \partial\bar{\partial}\eta) = \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix}\begin{pmatrix}\left(-\frac{1}{2}\Box + iT\right)^2 + \frac{1}{4}\Box\overline{\Box} & \frac{1}{4}\overline{\Box}\Delta_H \\ \frac{1}{4}\Box\Delta_H & \left(-\frac{1}{2}\overline{\Box} - iT\right)^2 + \frac{1}{4}\Box\overline{\Box}\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}.$$

We recall that $\begin{pmatrix}a\Box & -a\overline{\Box} \\ -b\Box & b\overline{\Box}\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = 0$ for all $a, b \in \mathbb{C}$, since $\Box\xi = \overline{\Box}\eta$. Then,

$$\begin{aligned}
&\Delta_R(\bar{\partial}\partial\xi + \partial\bar{\partial}\eta) \\
&= \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix}\begin{pmatrix}\left(-\frac{1}{2}\Box + iT\right)^2 + \frac{1}{4}\Box\left(\overline{\Box} + \Delta_H\right) & 0 \\ 0 & \left(-\frac{1}{2}\overline{\Box} - iT\right)^2 + \frac{1}{4}\overline{\Box}\left(\Box + \Delta_H\right)\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\
&= \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix}\begin{pmatrix}\frac{1}{4}\left[(\Box - 2iT)^2 + \Box\left(\overline{\Box} + \Delta_H\right)\right] & 0 \\ 0 & \frac{1}{4}\left[(\overline{\Box} + 2iT)^2 + \overline{\Box}\left(\Box + \Delta_H\right)\right]\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\
&= \begin{pmatrix}\bar{\partial}\partial & \partial\bar{\partial}\end{pmatrix}\begin{pmatrix}\frac{1}{4}(\Delta_H)^2 & 0 \\ 0 & \frac{1}{4}(\Delta_H)^2\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\
&= \left(\bar{\partial}\partial\overline{\Box} + \partial\bar{\partial}\Box\right)\frac{1}{4}(\Delta_H)^2\sigma.
\end{aligned}$$

□

1.7. Unitary intertwining operators and projections

The intertwining operators for Δ_R that we have defined in the previous section were non-unitary and unbounded. In order to verify that the forms to which Δ_R, when restricted to the subspaces $W_0^{p,q}$,$W_1^{p,q}$ and $(W_2^{p,q})'$ had been reduced on the corresponding parameter spaces by means of the formulas are indeed describing the spectral theory of Δ_R on these subspaces, we need to replace the previous intertwining operators by unitary ones. Our next tasks will therefore be the following ones:

(i) replace these intertwining operators with unitary ones;
(ii) determine the orthogonal projections from $L^2E_0^k$ onto $\overline{W_0^{p,q}}$, $\overline{W_1^{p,q}}$ and $\overline{(W_2^{p,q})'}$, the L^2-closures of the invariant subspaces $W_0^{p,q}$, $W_1^{p,q}$ and $(W_2^{p,q})'$.

These two tasks can be accomplished simultaneously by making use of the polar decomposition of the intertwining operators.

1.7.1. Known facts from operator theory.

We shall use the following fact from spectral theory. Compare [**57**] for the case $H = K$.

PROPOSITION 1.7.1. *Let H,K be Hilbert spaces and $A : \operatorname{dom} A \subset H \to K$ be a densely defined, closed operator. Then there exist a positive self-adjoint operator $|A| : \operatorname{dom} A \subset H \to H$, with $\operatorname{dom}|A| = \operatorname{dom} A$, and a partial isometry $U : H \to K$ with $\ker U = \ker A$ and $\operatorname{ran} U = \overline{\operatorname{ran} A}$, so that $A = U|A|$. $|A|$ and U are uniquely determined by these properties together with the additional condition $\ker|A| = \ker A$. Moreover, $|A| = \sqrt{A^*A}$, U^*U is the orthogonal projection from H onto $(\ker A)^\perp = \overline{\operatorname{ran} A^*}$, and UU^* is the orthogonal projection from K onto $\overline{\operatorname{ran} A} = (\ker A^*)^\perp$.*

We also need the following principle.

PROPOSITION 1.7.2 ([**50**]). *Let H_1,H_2 be Hilbert spaces and let $\mathcal{D}_1 \subset H_1$, $\mathcal{D}_2 \subset H_2$ be dense subspaces. Assume that for $j = 1, 2$, $S_j : \operatorname{dom} S_1 \subset H_j \to H_j$ is a self-adjoint operator on H_j for which $\mathcal{D}_j$ is a core such that $S_j(D_j) \subset D_j$. Moreover, let $A : \operatorname{dom} A \subset H_1 \to H_2$ be a closed operator such that the following properties hold true:*

(i) *$\mathcal{D}_1 \subset \operatorname{dom} A$ and $A(\mathcal{D}_1) \subset \mathcal{D}_2$;*
(ii) *A intertwines S_1 and S_2 on the core $\mathcal{D}_1$, i.e.,*

$$AS_1\xi = S_2A\xi \quad \text{for all } \xi \in \mathcal{D}_1. \tag{1.7.1}$$

Consider the polar decomposition $A = U|A|$ from Proposition 7.1, where $|A| = \sqrt{A^*A}$, and where $U : H_1 \to H_2$ is a partial isometry, and assume furthermore that $\mathcal{D}_1 \subset \operatorname{dom}|A|$, and that

(iii) *$|A|(\mathcal{D}_1) = \mathcal{D}_1$;*
(iv) *the commutation relation*

$$S_1|A|\xi = |A|S_1\xi \quad \text{for all } \xi \in \mathcal{D}_1 \tag{1.7.2}$$

holds true on the core $\mathcal{D}_1$.

Then, also U intertwines S_1 and S_2 on the core $\mathcal{D}_1$, i.e., $U(\mathcal{D}_1) = A(\mathcal{D}_1) \subset \mathcal{D}_2$, and

$$US_1\xi = S_2U\xi \quad \text{for all } \xi \in \mathcal{D}_1. \tag{1.7.3}$$

Moreover, we have $\overline{\operatorname{ran} A} = \overline{A(\mathcal{D}_1)} = U(H_1)$, $\ker A = \ker|A| = \ker U$, and $P := UU^*$ is the orthogonal projection from H_2 onto $\overline{A(\mathcal{D}_1)}$. Let us finally denote by $S_2^r = S_2|_{\overline{A(\mathcal{D}_1)}}$ the restriction of S_2 to $\overline{A(\mathcal{D}_1)}$, with domain $\operatorname{dom} S_2^r := \operatorname{dom} S_2 \cap \overline{A(\mathcal{D}_1)}$. If we assume in addition that

(v) $\ker|A| = \{0\}$;
(vi) $(I - iS_1)^{-1}(\mathcal{D}_1) \subset \mathcal{D}_1$;
(vii) $P(\mathcal{D}_2) = A(\mathcal{D}_1)$,

then U is injective, and we even that $U(\operatorname{dom} S_1) = \operatorname{dom} S_2^r$, and

$$S_2^r = US_1U^{-1} \text{ on } \operatorname{dom} S_2^r.$$

The next lemma will often facilitate the computation of the corresponding operators A^*A.

LEMMA 1.7.3 ([**50**]). *Let H, K be Hilbert spaces and $H_1 \subset H$ and $K_1 \subset K$ be closed subspaces. Let $A : \operatorname{dom} A \subset H \to K$ be a densely defined, closed operator, and assume that $\mathcal{D} \subset \operatorname{dom} A$ is a core for A. Assume furthermore that $\mathcal{D}_1 := \mathcal{D} \cap H_1$ is dense in H_1 and that $\operatorname{dom} A_1 := \operatorname{dom} A \cap H_1$ is mapped under A into K_1, so that the operator $A_1 : \operatorname{dom} A_1 \subset H_1 \to K_1$, given by restricting A to $\operatorname{dom} A_1 := \operatorname{dom} A \cap H_1$, is densely defined and closed.*

Under these conditions, also A^* is densely defined, and $\operatorname{dom} A^* \cap K_1 \subset \operatorname{dom} A_1^*$. We shall further assume that $\mathcal{E} \subset K$ is a subspace of $\operatorname{dom} A^*$ such that $A(\mathcal{D}) \subset \mathcal{E}$ and $A^*(\mathcal{E}) \subset \mathcal{D}$ (so that, in particular, $\mathcal{E}_1 := \mathcal{E} \cap K_1$ is contained in $\operatorname{dom} A_1^*$). Then we have

$$A_1^*A_1\xi = P_{H_1}A^*A\xi \quad \text{for all } \xi \in \mathcal{D}_1,$$

where $P_{H_1} : H \to H_1$ denotes the orthogonal projection from the Hilbert space H onto its closed subspace H_1. In particular, if we know that A^*A maps $\mathcal{D}_1$ into H_1, then $A_1^*A_1\xi = A^*A\xi$ for every $\xi \in \mathcal{D}_1$.

In the case of $W_0^{p,q}$, with $p+q \le n$, the unitary intertwining operator is trivially the identity. Hence, we consider the other cases.

1.7.2. Unitary interwining operators for $W_1^{p,q}$ (subcritical case). We would like to compute $U_1^{p,q} = U_1 = A_1(A_1^*A_1)^{-\frac{1}{2}}$. We have

$$A_1 = \begin{pmatrix} \partial & \bar\partial \end{pmatrix} \begin{pmatrix} \overline{\Box} & 1 \\ -\Box & 1 \end{pmatrix} = \begin{pmatrix} \partial\overline{\Box} - \bar\partial\Box & \partial + \bar\partial \end{pmatrix},$$

so that

$$A_1^*A_1 = \begin{pmatrix} \overline{\Box}\partial^* - \Box\bar\partial^* \\ \partial^* + \bar\partial^* \end{pmatrix} \begin{pmatrix} \partial\overline{\Box} - \bar\partial\Box & \partial + \bar\partial \end{pmatrix} = \begin{pmatrix} \Delta_H\Box\overline{\Box} & 0 \\ 0 & \Delta_H \end{pmatrix}$$

and

$$(A_1^*A_1)^{\frac{1}{2}} = \begin{pmatrix} (\Delta_H\Box\overline{\Box})^{\frac{1}{2}} & 0 \\ 0 & \Delta_H^{\frac{1}{2}} \end{pmatrix}.$$

LEMMA 1.7.4. *We have that*

$$A_1^*A_1 = \begin{pmatrix} \Delta_H\Box\overline{\Box} & 0 \\ 0 & \Delta_H \end{pmatrix},$$

$A_1^*A_1 : \tilde{Z}^{p,q} \to \tilde{Z}^{p,q}$ is a bijection.

Moreover the subspaces $(W_1^{p,q})^+$ and $(W_1^{p,q})^-$ defined in (1.6.5) *are orthogonal.*

PROOF. It is clear that

$$\begin{pmatrix} \Delta_H\Box\overline{\Box} & 0 \\ 0 & \Delta_H \end{pmatrix}$$

maps $\tilde{Z}^{p,q}$ into itself, because $\Box\overline{\Box}$ is a bijection on $\Xi^{p,q}$. Since the matrix is diagonal $(W_1^{p,q})^+$ and $(W_1^{p,q})^-$ are orthogonal. $\Box$

PROPOSITION 1.7.5. *We have that*

$$U_1 = A_1(A_1^*A_1)^{-\frac{1}{2}} = \begin{pmatrix} U_1^+ & U_1^- \end{pmatrix},$$

with

$$U_1^+ = \mathcal{R}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}} - \bar{\mathcal{R}}\Delta_H^{-\frac{1}{2}}\Box^{\frac{1}{2}} \quad \textit{and} \quad U_1^- = \mathcal{R}\Delta_H^{-\frac{1}{2}}\Box^{\frac{1}{2}} + \bar{\mathcal{R}}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}}.$$

PROOF. It suffices to notice that

$$\begin{aligned} A_1(A_1^*A_1)^{-\frac{1}{2}} &= \left(\partial(\Delta_H\Box)^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}} - \bar\partial(\Delta_H\overline{\Box})^{-\frac{1}{2}}\Box^{\frac{1}{2}} \quad (\partial+\bar\partial)\Delta_H^{-\frac{1}{2}}\right) = \\ &= \left(\mathcal{R}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}} - \bar{\mathcal{R}}(\Delta_H)^{-\frac{1}{2}}\Box^{\frac{1}{2}} \quad \mathcal{R}\Delta_H^{-\frac{1}{2}}\Box^{\frac{1}{2}} + \bar{\mathcal{R}}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}}\right). \end{aligned}$$

□

PROPOSITION 1.7.6. *The operators $U_1^\pm$ map $\Xi^{p,q}$, respectively $W_0^{p,q}$, onto $(W_1^{p,q})^\pm$ and intertwine $D^+ := (1-c_{pq})\Delta_H$ and $D^- := \Delta_H$ respectively, with Δ_R on the core $\mathcal{S}_0$. Here c_{pq} is the constant defined in* (1.5.5)*. Moreover $U_1^+ : \overline{\Xi^{p,q}} \to L^2E_0^k$ and $U_1^- : \overline{W_0^{p,q}} \to L^2E_0^k$ are linear isometries onto their ranges $\overline{(W_1^{p,q})^+}$ and $\overline{(W_1^{p,q})^-}$, respectively, which intertwine D^+ respectively D^- with the restriction of Δ_R to $\overline{(W_1^{p,q})^\pm}$.*

$$\Delta_R|_{\overline{(W_1^{p,q})^\pm}} = U_1^\pm D^\pm (U_1^\pm)^{-1} \ \textit{on}\ \mathrm{dom}\,\Delta_R|_{\overline{(W_1^{p,q})^\pm}}.$$

$(U_1^\pm)^{-1}$ denotes the inverse of $U_1^\pm$ when viewed as an operator into its range $(W_1^{p,q})^\pm$. Finally, if we consider $U_1^\pm$ as an operator mapping into $L^2E_0^k$, then $P_1^{p,q,\pm} := P_1^\pm := U_1^\pm(U_1^\pm)^*$ is the orthogonal projection from $L^2E_0^k$ onto $\overline{(W_1^{p,q})^\pm}$. Hence $P_1^{p,q} := P_1^{p,q,+} + P_1^{p,q,-}$ is the orthogonal projection from $L^2E_0^k$ onto $\overline{(W_1^{p,q})}$.

PROOF. It suffices to see that all the hypothesis of Proposition 1.7.2 are satisfied by $U_1^\pm$. We consider U_1^+. We set $\mathcal{D}_1 = \Xi^{p,q}$, $H_1 = \overline{\Xi^{p,q}}$, $\mathcal{D}_2 = \mathcal{S}_0E_0^k$, $H_2 = L^2E_0^k$, $S_1 = D^+$, $S_2 = \Delta_R$ and we denote by A the closure of A_1^+. Condition (i) of Proposition 1.7.2 is trivial. Condition (ii) is satisfied because of (1.6.4). Conditions (iii) and (v) follows from Lemma 1.7.4.

According to Lemma 1.7.4, A^*A and S_1 are positive scalar operators. Also $|A| = \sqrt{A^*A}$ is a scalar operator, therefore it commutes with S_1. Thus condition (iv) is satisfied. Condition (vi) is trivial. The explicit formula for $U = U_1^+$ in Proposition 1.7.5 show that U maps the space $\Xi^{p,q}$ into $\mathcal{S}_0E_0^k$ and U^* maps $\mathcal{S}_0E_0^k$ into $\Xi^{p,q}$. Finally, if $P = UU^*$ then $P(\mathcal{D}_2) = P(\mathcal{S}_0E_0^k) = U(\Xi^{p,q}) = A(|A|^{-1}(\Xi^{p,q})) = A(\Xi^{p,q}) = A(\mathcal{D}_1)$. This proves condition (vii).

The proof is analogous for the case $U = U_1^-$.

□

1.7.3. A unitary intertwining operator for $(W_2^{p,q})'$. The arguments and computations in this subsection are the same for both the subcritical and the critical case. In particular we have $p+q \le n-2$.

We wish to replace the intertwining operator A_2 by a unitary one, denoted by U_2, which should be given by $A_2(A_2^*A_2)^{-\frac{1}{2}}$.

Now we compute $U_2^{p,q} = U_2 = A_2(A_2^*A_2)^{-\frac{1}{2}}$.

$$\begin{aligned} A_2^*A_2 &= \left(\overline{\Box}\partial^*\bar\partial^* + \Box\bar\partial^*\partial^*\right)\left(\bar\partial\partial\overline{\Box} + \partial\bar\partial\Box\right) = \\ &= \left(\overline{\Box}\partial^*\bar\partial^*\bar\partial\partial\overline{\Box} + \overline{\Box}\partial^*\bar\partial^*\partial\bar\partial\Box + \Box\bar\partial^*\partial^*\bar\partial\partial\overline{\Box} + \Box\bar\partial^*\partial^*\partial\bar\partial\Box\right) = \\ &= \left(\overline{\Box}^2\Box\left(\overline{\Box}+iT\right) - (\Box\overline{\Box})^2 - (\Box\overline{\Box})^2 + \Box^2\overline{\Box}\left(\Box - iT\right)\right) = \\ &= \Box\overline{\Box}((\Box-\overline{\Box})^2 - iT(\Box-\overline{\Box})) = \\ &= -T^2c_{pq}^{-1}(c_{pq}^{-1}-1)\Box\overline{\Box}, \end{aligned}$$

where c_{pq} is defined in (1.5.5).

LEMMA 1.7.7. *It holds that*

$$A_2^*A_2 = -T^2 C_k(C_k-1)\Box\overline{\Box}\,,$$
$$(A_2^*A_2)^{\frac12} = |T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)\Box\overline{\Box}}\,.$$

Moreover, $(A_2^*A_2)^{\frac12}$ *maps* $\Xi^{p,q}$ *bijectively onto itself and on* $\Xi^{p,q}$ *we have*

$$(A_2^*A_2)^{-\frac12} = \frac{\Box^{-\frac12}\overline{\Box}^{-\frac12}}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}\,.$$

PROOF.
$$A_2^*A_2 = -T^2 c_{pq}^{-1}(c_{pq}^{-1}-1)\Box\overline{\Box} > 0$$
since $c_{pq}^{-1} = (n-p-q) \geq 2$. It is clear that $A_2^*A_2$ maps $\Xi^{p,q}$ onto itself. $\Box$

REMARK 1.7.8. We want to ensure the invertibility of $\Box$ and $\overline{\Box}$. Hence we recall that if $p=0$, then $X^{p,q} = (I-\mathcal{C})X^{p,q}$, and if $q=0$, then $Y^{p,q} = (I-\bar{\mathcal{C}})Y^{p,q}$. Therefore we set

$$\Box_r = \begin{cases} \Box & \text{if } p\geq 1 \\ \Box' & \text{if } p = 0 \end{cases}\,, \quad \overline{\Box}_r = \begin{cases} \overline{\Box} & \text{if } q\geq 1 \\ \overline{\Box}' & \text{if } q = 0 \end{cases}\,.$$

PROPOSITION 1.7.9. *The operator* U_2 *which acts on* $\Xi^{p,q}$ *is given by*

$$M\left(\frac12 e(d\theta)\Delta_H\Box^{-\frac12}\overline{\Box}^{-\frac12} + ic_{pq}^{-1}\bar{\mathcal{R}}\mathcal{R}(\overline{\Box}+iT)^{\frac12}\overline{\Box}^{-\frac12} + \frac12 c_{pq}^{-1}e(d\theta)iT\Box^{-\frac12}\overline{\Box}^{-\frac12}\right),$$

where

$$M := -\frac{T}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}\,.$$

PROOF. We have that

$$\begin{aligned} A_2(A_2^*A_2)^{-\frac12} &= \frac{(\bar\partial\partial\overline{\Box}+\partial\bar\partial\Box)(\Box\overline{\Box})^{-\frac12}}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}} \\ &= \frac{\left[(\bar\partial\partial+\partial\bar\partial)\Delta_H + (\bar\partial\partial-\partial\bar\partial)(\overline{\Box}-\Box)\right](\Box\overline{\Box})^{-\frac12}}{2|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}} \\ &= \frac{\left[-Te(d\theta)\Delta_H + (2\bar\partial\partial + Te(d\theta))(-ic_{pq}^{-1}T)\right](\Box\overline{\Box})^{-\frac12}}{2|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}} \\ &= H_1 + H_2 + H_3\,, \end{aligned}$$

with

$$H_1 = -\frac{Te(d\theta)}{2|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}\Delta_H\Box^{-\frac12}\overline{\Box}^{-\frac12},$$
$$H_2 = -\frac{ic_{pq}^{-1}T}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}\bar\partial\partial\Box^{-\frac12}\overline{\Box}^{-\frac12},$$
$$H_3 = -\frac{c_{pq}^{-1}Te(d\theta)}{2|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}iT\Box^{-\frac12}\overline{\Box}^{-\frac12}.$$

Observe that $\bar\partial\partial = \bar{\mathcal{R}}\mathcal{R}(\overline{\Box}+iT)^{\frac12}\Box^{\frac12}$ on $W_0^{p,q}$, with $p+q\leq n-2$.

Now we can write

$$H_2 = -\frac{ic_{pq}^{-1}T}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}}\bar{\mathcal{R}}\mathcal{R}(\overline{\Box}+iT)^{\frac{1}{2}}\overline{\Box}^{-\frac{1}{2}}.$$

□

PROPOSITION 1.7.10. *The operator U_2 maps the space $\Xi^{p,q}$ onto $(W_2^{p,q})'$ and intertwines D with Δ_R on the core $\mathcal{S}_0$, where*

$$D := \begin{cases} (1-c_{pq})\,\Delta_H & \text{if } p+q+2<n \\ \frac{1}{4}(\Delta_H)^2 & \text{if } p+q+2=n. \end{cases}$$

Here c_{pq} is the constant defined in (1.5.5). *Moreover $U_2 : \overline{\Xi^{p,q}} \to L^2E_0^k$ is a linear isometry onto $\overline{(W_2^{p,q})'}$ which intertwines D with the restriction of Δ_R to $\overline{(W_2^{p,q})'}$.*

$$\Delta_R|_{\overline{(W_2^{p,q})'}} = U_2D(U_2)^{-1} \text{ on } \operatorname{dom}\Delta_R|_{\overline{(W_2^{p,q})'}}.$$

$(U_2)^{-1}$ denotes the inverse of U_2 when viewed as an operator into its range $\overline{(W_2^{p,q})'}$. Finally, if we consider U_2 as an operator mapping into $L^2E_0^k$, then $P_2^{p,q} := P_2 := U_2(U_2)^*$ is the orthogonal projection from $L^2E_0^k$ onto $\overline{(W_2^{p,q})'}$.

PROOF. We would like to apply Proposition 1.7.2 to A_2. We set $\mathcal{D}_1 = \Xi^{p,q}$, $H_1 = \overline{\Xi^{p,q}}$, $\mathcal{D}_2 = \mathcal{S}_0E_0^k$, $H_2 = L^2E_0^k$, $S_1 = D$, $S_2 = \Delta_R$ and we denote by A the closure of A_2. Condition (i) Proposition 1.7.2 of is trivial. Condition (ii) is satisfied because of (1.6.7). According to Lemma 1.7.7, A^*A is a positive matrix with scalar operator entries and S_1 is a scalar operator. Also $|A|$ is a matrix with scalar operator entries, therefore commutes with S_1. Thus condition (iv) is satisfied. Conditions (iii) and (v) follows from Lemma 1.7.7 . It is clear that condition (vi) is satisfied. Finally, the explicit formula for $U = U_2$ in Proposition 1.7.9 shows that U maps the space $\Xi^{p,q}$ into $\mathcal{S}_0E_0^k$, so that U^* maps $\mathcal{S}_0E_0^k$ into $\Xi^{p,q}$. If $P = UU^*$ then $P(\mathcal{D}_2) = P(\mathcal{S}_0E_0^k) = U(\Xi^{p,q}) = A(|A|^{-1}(\Xi^{p,q})) = A(\Xi^{p,q}) = A(\mathcal{D}_1)$. Thus (vii) is satisfied.

□

1.7.4. A unitary interwining operator for $W_1^{p,q}$ (critical case). We would like to compute $U_1^{p,q} = U_1 = A_1(A_1^*A_1)^{-\frac{1}{2}}$. We have

$$A_1^*A_1 = \begin{pmatrix} \partial^* \\ \bar{\partial}^* \end{pmatrix}\begin{pmatrix} \partial & \bar{\partial} \end{pmatrix} = \begin{pmatrix} \partial^*\partial & \partial^*\bar{\partial} \\ \bar{\partial}^*\partial & \bar{\partial}^*\bar{\partial} \end{pmatrix} = \begin{pmatrix} \Box & 0 \\ 0 & \overline{\Box} \end{pmatrix}$$

and

$$(A_1^*A_1)^{\frac{1}{2}} = \begin{pmatrix} \Box^{\frac{1}{2}} & 0 \\ 0 & \overline{\Box}^{\frac{1}{2}} \end{pmatrix}.$$

Therefore

$$A_1(A_1^*A_1)^{-\frac{1}{2}} = \begin{pmatrix} \partial\Box^{-\frac{1}{2}} & \bar{\partial}\overline{\Box}^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \mathcal{R} & \bar{\mathcal{R}} \end{pmatrix}.$$

PROPOSITION 1.7.11. *The operator U_1 which acts on $Z^{p,q}$ is given by*

$$\begin{pmatrix} \mathcal{R} & \bar{\mathcal{R}} \end{pmatrix}.$$

PROPOSITION 1.7.12. *The operator U_1 maps the space $Z^{p,q}$ onto $W_1^{p,q}$ and intertwines $D := (\Delta_H)^2$ with Δ_R on the core $\mathcal{S}_0$. Moreover $U_1 : \overline{Z^{p,q}} \to L^2E_0^k$ is a linear isometry onto $\overline{W_1^{p,q}}$ which intertwines D with the restriction of Δ_R to $\overline{W_1^{p,q}}$.*

$$\Delta_R|_{\overline{W_1^{p,q}}} = U_1D(U_1)^{-1} \text{ on } \operatorname{dom}\Delta_R|_{\overline{W_1^{p,q}}}.$$

$(U_1)^{-1}$ denotes the inverse of U_1 when viewed as an operator into its range $\overline{W_1^{p,q}}$. Finally, if we consider U_1 as an operator mapping into $L^2E_0^k$, then $P_1^{p,q} := P_1 := U_1(U_1)^*$ is the orthogonal projection from $L^2E_0^k$ onto $\overline{W_1^{p,q}}$.

PROOF. We would like to apply Proposition 1.7.2 to A_1. We set $\mathcal{D}_1 = Z^{p,q}$, $H_1 = \overline{Z^{p,q}}$, $\mathcal{D}_2 = \mathcal{S}_0 E_0^k$, $H_2 = L^2E_0^k$, $S_1 = D$, $S_2 = \Delta_R$ and we denote by A the closure of A_1. Condition (i) Proposition 1.7.2 of is trivial. Condition (ii) is satisfied because of (1.6.7). According to Lemma 1.7.7, A^*A is a positive matrix with scalar operator entries and S_1 is a scalar operator. Also $|A|$ is a matrix with scalar operator entries, therefore commutes with S_1. Thus condition (iv) is satisfied. Conditions (iii) and (v) follows from Lemma 1.7.7 . It is clear that condition (vi) is satisfied. Finally, the explicit formula for $U = U_1$ in Proposition 1.7.9 show that U maps the space $Z^{p,q}$ into $\mathcal{S}_0E_0^k$, so that U^* maps $\mathcal{S}_0E_0^k$ into $Z^{p,q}$. If $P = UU^*$ then $P(\mathcal{D}_2) = P(\mathcal{S}_0E_0^k) = U(Z^{p,q}) = A(|A|^{-1}(Z^{p,q})) = A(Z^{p,q}) = A(\mathcal{D}_1)$. Thus (vii) is satisfied.

□

1.8. Decomposition of $L^2E_0^k$ for $k > n$

We recall some facts about the Hodge operator. For more details see for example [**56**].

PROPOSITION 1.8.1. *The Hodge $\star$-operator satisfies the following properties:*

(i) *for $\omega_1, \omega_2 \in L^2E_0^k(\mathbb{H}_n)$,*

$$\int_{\mathbb{H}_n} \omega_1 \wedge \star\overline{\omega}_2 = \langle \omega_1, \omega_2 \rangle_{L^2\Lambda^k} \,;$$

(ii) *the operator $\star : L^2E_0^k \to L^2E_0^{2n+1-k}$ is unitary;*
(iii) $d_R^* = (-1)^k \star d_R \star$ *(Remark I.3.23)*;
(iv) $\star\Delta_{R,k} = \Delta_{R,2n+1-k}\star$ *(Proposition I.3.24).*

We set $\overset{*}{W_0}{}^{r,s} = \{\omega' \in \mathcal{S}_0\Lambda^{r,s} : \partial\omega' = \bar\partial\omega' = 0\}$ and define

$$\begin{aligned} Z_0^{r,s} &= \{\omega = \theta \wedge \omega' : \omega' \in \overset{*}{W_0}{}^{r,s}\},\\ Z_1^{r,s} &= \{\omega = \theta \wedge \omega' : \omega' = \partial^*\sigma + \bar\partial^*\tau, \text{ with } \sigma, \tau \in \overset{*}{W_0}{}^{r,s}\},\\ (Z_2^{r,s})' &= \{\omega = \theta \wedge \omega' : \omega' = \bar\partial^*\partial^*\sigma + \partial^*\bar\partial^*\tau, \text{ with } \sigma, \tau \in \overset{*}{W_0}{}^{r,s} \text{ and } \Box\sigma = \overline{\Box}\tau\}. \end{aligned}$$

LEMMA 1.8.2. *Given p, q, we put $r = n - p$ and $s = n - q$. Then*

(i) $\star(W_0^{p,q}) = Z_0^{r,s}$;
(ii) $\star(W_1^{p,q}) = Z_1^{r,s}$;
(iii) $\star((W_2^{p,q})') = (Z_2^{r,s})'$.

THEOREM 1.8.3. *Let $n < k \le 2n+1$. Then $L^2E_0^k$ admits the orthogonal decomposition*

$$L^2E_0^k = \sum_{r+s=k-1} \overline{Z_0^{r,s}} \oplus \sum_{r+s=k} \overline{Z_1^{r,s}} \oplus \sum_{r+s=k+1} \overline{(Z_2^{r,s})'}.$$

PROOF. It follows from Proposition 1.8.1 and Lemma 1.8.2.

□

1.9. L^p-multipliers

The decomposition of $L^2\Lambda^k$ presented in the previous sections, together with the description of the action of Δ_R on the various subspaces, can be used for the L^p-functional calculus of Δ_R. For this purpose, we are going to show that $L^p\Lambda^k$ admits the same decomposition when $1 < p < \infty$. Concretely this means proving that the orthogonal projections on the various invariant subspaces and the intertwining operators that reduce Δ_R to scalar forms are L^p-bounded.

1.9.1. The multiplier theorem. The joint spectrum of L and $i^{-1}T$ is the Heisenberg fan $F \subset \mathbb{R}^2$ defined as follows. If

$$l_{k,\pm} = \{(\lambda, \xi) : \xi = \pm(n + 2k)\lambda, \lambda \in \mathbb{R}^*_+\},$$

then

$$F = \overline{\bigcup_{k\in\mathbb{N}} (l_{k,+} \cup l_{k,-})}.$$

The variable λ corresponds to $i^{-1}T$ and ξ to L, i.e., calling $dE(\lambda, \xi)$ the spectral measure on F, then

$$i^{-1}T = \int_F \lambda dE(\lambda, \xi), \quad L = \int_F \xi dE(\lambda, \xi).$$

If m is any bounded, continuous function on $\mathbb{R} \times \mathbb{R}^*_+$, we can then define the associated multiplier operator $m(i^{-1}T, L)$ by

$$m(i^{-1}T, L) := \int_F m(\lambda, \xi) dE(\lambda, \xi),$$

which is clearly bounded on $L^2(\mathbb{H}_n)$. It follows from Plancherel's formula that the spectral measure of the vertical half-line $\{(0, \xi) : \xi \geq 0\} \subset F$ is zero. A spectral multiplier is therefore a function $m(\lambda, \xi)$ on F whose restriction to each l_k is measurable with respect to $d\lambda$ for every k. Given $\rho, \sigma > 0$, we say that a measurable function $f(\lambda, \xi)$ is in the mixed Sobolev space $L^2_{\rho,\sigma} = L^2_{\rho,\sigma}(\mathbb{R}^2)$ if

$$\begin{aligned} \|f\|^2_{L^2_{\rho,\sigma}} :&= \int_{\mathbb{R}^2} (1 + |\xi'|)^{2\rho}(1 + |\lambda'| + |\xi'|)^{2\sigma} |\hat{f}(\lambda', \xi')|^2 d\lambda' d\xi' = \\ &= c\,\|(1 + |\partial_\xi|)^\rho (1 + |\partial_\lambda| + |\partial_\xi|)^\sigma f\|_2^2 < \infty. \end{aligned} \tag{1.9.1}$$

Let $\eta_0 \in C^\infty_0(\mathbb{R})$ be a non-trivial, non-negative, smooth bump function supported in $\mathbb{R}^*_+ := (0, \infty)$, put $\eta_1(x) := \eta_0(x) + \eta_0(-x)$ and set $\chi := \eta_1 \otimes \eta_0$. If $f(\lambda, \xi)$ is a continuous, bounded function on $\mathbb{R} \times \mathbb{R}^*_+$, then we put $f^r(\lambda, \xi) = f(r_1\lambda, r_2\xi)$, $r = (r_1, r_2) \in (\mathbb{R}^*_+)^2$, and we say that $f \in L^2_{\rho,\sigma,\mathrm{sloc}}(\mathbb{R} \times \mathbb{R}^*_+)$ if for every $r = (r_1, r_2) \in (\mathbb{R}^*_+)^2$, the function $f^r\chi \in L^2_{\rho,\sigma}$ and

$$\|f\|_{L^2_{\rho,\sigma,\mathrm{sloc}}} := \sup_r \|f^r\chi\|_{L^2_{\rho,\sigma}} < \infty. \tag{1.9.2}$$

Definition 1.9.1. A function m satisfying (9.2) is called a Marcinkiewicz multiplier of class (ρ,σ). A smooth Marcinkiewicz multiplier is a Marcinkiewicz multiplier of every class (ρ, σ), i.e., satisfying the pointwise estimates

$$|\partial^j_\lambda \partial^k_\xi m(\lambda, \xi)| \leq C_{jk}|\lambda|^{-j}|\xi|^{-k},$$

for every j, k.

Theorem 1.9.2 ([**52**]). *Let m be a Marcinkiewicz multiplier of class (ρ, σ) for some $\rho > n$ and $\sigma > \frac{1}{2}$. Then $m(i^{-1}T, L)$ is bounded on $L^p(\mathbb{H}_n)$ for $1 < p < \infty$, with norm controlled by $\|m\|_{L^2_{\rho,\sigma,\mathrm{sloc}}}$.*

1.9.2. Some classes of multipliers. We recall the definition of some classes $\Psi^{\rho,\sigma}_{\tau}$ of (possibly unbounded) smooth multipliers introduced in [**50**], in terms of which we will understand the behaviour of the projections and intertwining operators presented in the previous sections. These classes are defined by pointwise estimates on all derivatives, in analogy to (9.3), which must be satisfied on some open angle $\Lambda_{n-\epsilon} := \{(\lambda,\xi) \in \mathbb{R}^2 : \xi > (n-\epsilon)|\lambda|\}$ containing the Heisenberg fan F taken away the origin.

Definition 1.9.3. We say that $m \in \Psi^{\rho,\sigma}_{\tau}$ ($\rho,\sigma,\tau \in \mathbb{R}$) if

$$|\partial^j_\lambda \partial^k_\xi m(\lambda,\xi)| \lesssim \begin{cases} \xi^{\tau-j-k} & \text{for } \xi \le 1 \\ (\xi+\lambda^2)^{\rho-\frac{j}{2}}\xi^{\sigma-k} & \text{for } \xi > 1 \end{cases}$$

for every $j,k \in \mathbb{N}$. We also say that $m \in {}^*\Psi^{\rho,\sigma}_{\tau}$ if $m \in \Psi^{\rho,\sigma}_{\tau}$ and moreover

$$m(\lambda,\xi) \gtrsim \begin{cases} \xi^{\tau} & \text{for } \xi < 1 \\ (\xi+\lambda^2)^{\rho}\xi^{\sigma} & \text{for } \xi > 1 \end{cases}$$

Typical examples are given by the smooth functions m such that

$$m(\lambda,\xi) = \begin{cases} (\xi + p\lambda + a\lambda^2)^{\tau} & \text{for } \xi < 1 \\ (\xi+\lambda^2)^{\rho}(\xi+q\lambda)^{\sigma} & \text{for } \xi > 2 \end{cases}$$

with $|p|, |q| < n$.

We recall some properties proved in [**50**].

Lemma 1.9.4 ([**50**]). *The classes $\Psi^{\rho,\sigma}_{\tau}$ satisfy the following properties:*

(i) $\partial_\lambda \Psi^{\rho,\sigma}_{\tau} \subset \Psi^{\rho-\frac{1}{2},\sigma}_{\tau-1}$, $\partial_\xi \Psi^{\rho,\sigma}_{\tau} \subset \Psi^{\rho,\sigma-1}_{\tau-1}$

(ii) $\Psi^{\rho,\sigma}_{\tau}\Psi^{\rho',\sigma'}_{\tau'} \subset \Psi^{\rho+\rho',\sigma+\sigma'}_{\tau+\tau'}$

(iii) *if $m \in {}^*\Psi^{\rho,\sigma}_{\tau}$ and then $m^s \in \Psi^{s\rho,s\sigma}_{s\tau}$ for every $s \in \mathbb{R}$ (for $s \in \mathbb{N}$, $m \in \Psi^{\rho,\sigma}_{\tau}$ is sufficient)*

(iv) *if $\rho+\sigma \le \rho'+\sigma'$, $2\rho+\sigma \le 2\rho'+\sigma'$ and $\tau \ge \tau'$, then $\Psi^{\rho,\sigma}_{\tau} \subset \Psi^{\rho',\sigma'}_{\tau'}$*

(v) *In particular, if $\rho+\sigma \le 0$, $2\rho+\sigma \le 0$ and $\tau \ge 0$, then $\Psi^{\rho,\sigma}_{\tau} \subset \Psi^{0,0}_{0}$, and $\Psi^{\rho,\sigma}_{\tau}$ consists of Marcinkiewicz multipliers.*

Remark 1.9.5.

(i) Observe that if χ is a smooth cut-off function on $\mathbb{R}$, compactly supported in $\mathbb{R}\setminus\{0\}$ and with $0 \le \chi \le 1$, then $\eta = \chi(\xi/|\lambda|)$ and $1-\eta$ are in $\Psi^{0,0}_{0}$. By Lemma , multiplication by η or $1-\eta$ preserves the classes $\Psi^{\rho,\sigma}_{\tau}$.

(ii) If we are given a multiplier m, which satisfies the inequalities, but is only defined on an angle Γ leaving out a finite number of half-lines $\ell_{k,\pm}$ of F, we can easily extend m to a multiplier in $\Psi^{\rho,\sigma}_{\lambda}$ which vanishes identically on the missing lines.

(iii) Property also applies to the situation where $s > 0$, only holds on an angle omitting a finite number of half-lines in F, and m vanishes identically on these half-lines

We denote by the same symbol $\Psi^{\rho,\sigma}_{\tau}$ the class of operators defined by the multipliers in this class. For notational convenience we shall often use the same symbol to denote an operator $M \in \Psi^{\rho,\sigma}_{\tau}$ and its multiplier $M(\lambda,\xi)$.

1.10. Decomposition of $L^pE^k_0$

Theorem 1.10.1. *Let $1 < r < \infty$.*

(i) *For* $0 \le k \le n$, $L^r E_0^k$ *has a direct sum decomposition given by*

$$L^r E_0^k = \sum_{p+q=k} \overline{W_0^{p,q}}^{L^r} \oplus \sum_{p+q=k-1} \overline{W_1^{p,q}}^{L^r} \oplus \sum_{p+q=k-2} \overline{(W_2^{p,q})'}^{L^r}.$$

(ii) *For* $n+1 \le k \le 2n+1$, $L^r E_0^k$ *has a direct sum decomposition given by*

$$L^r E_0^k = \sum_{r+s=k-1} \overline{Z_0^{r,s}}^{L^r} \oplus \sum_{r+s=k} \overline{W_1^{r,s}}^{L^r} \oplus \sum_{r+s=k+1} \overline{(W_2^{r,s})'}^{L^r}.$$

PROOF. It follows from Lemma 1.10.5. $\square$

The following result is clear, since the holomorphic and antiholomorphic Riesz transforms of (1.4.1) are known to be Calderón-Zygmund type singular integral operators, and consequently are L^r-bounded for $1 < r < \infty$ (see, e.g., [**22**]).

LEMMA 1.10.2. $\mathcal{R}, \bar{\mathcal{R}}, e(d\theta)$ *are* L^r*-bounded for* $1 < r < \infty$.

In the following lemma, for the sake of simplicity, we use an abuse of notation.

LEMMA 1.10.3. $\square^{\frac12}, \overline{\square}^{\frac12}, (\overline{\square}+iT)^{\frac12} \in \Psi_{\frac12}^{0,\frac12}$ *for* $p+q+2 \le n$, $-iT \in \Psi_1^{\frac12,0} \subset \Psi_1^{0,1}$.

PROOF. The fact that $-iT \in \Psi_1^{\frac12,0}$ is obvious.

$(\overline{\square} + iT)^{\frac12}(\lambda,\xi) = \frac{1}{\sqrt2}(\xi + (n-2(q+1))\lambda)^{\frac12}$ where $p+q+2 \le n$. Hence $|n-2(q+1)| \le n-2$. Therefore $\xi + (n-2(q+1))\lambda \sim \xi$ on an angle containing the whole fan. This implies $(\overline{\square} + iT)^{\frac12} \in \Psi_{\frac12}^{0,\frac12}$.

Now we prove that $\square^{\frac12} \in \Psi_{\frac12}^{0,\frac12}$.

$$(\square)^{\frac12}(\lambda,\xi) = \frac{1}{\sqrt2}(\xi - (n-2p)\lambda)^{\frac12}.$$

We note that

$$\xi - (n-2p)\lambda \sim \xi$$

on an angle containing the whole fan if $p \ge 1$, and, if $p = 0$, on an angle avoiding just the half-line $\xi = n\lambda, \lambda > 0$. If we use Remark 9.5 (iii), we can deduce $\square^{\frac12} \in \Psi_{\frac12}^{0,\frac12}$. A similar argument applies to $\overline{\square}^{\frac12}$. $\square$

LEMMA 1.10.4. *Let* $p+q < n$, *then* $(\Delta_H) \in \Psi_1^{0,1}$ *and* $(\Delta_H)^{-\frac12} \in \Psi_{-\frac12}^{0,-\frac12}$.

PROOF. Δ_H has multiplier $\xi + (p-q)\lambda \sim \xi$, hence $\Delta_H \in {}^*\Psi_1^{0,1}$. Therefore $(\Delta_H)^{-\frac12} \in \Psi_{-\frac12}^{0,-\frac12}$. $\square$

LEMMA 1.10.5. *Let* U *denote any of the operators* $U_1^{p,q}$ *or* $U_2^{p,q}$. *Then each component of* U *consists of a multiplier operator in* $\Psi_0^{0,0}$, *possibly composed with* $e(d\theta)$ *and the holomorphic and antiholomorphic Riesz transforms* $\mathcal{R}$, $\bar{\mathcal{R}}$. *In particular, for* $1 < r < \infty$, *these operators are* L^r*-bounded.*

PROOF. It is sufficient to consider

$$U_2 = M\left(\frac12 e(d\theta)\Delta_H \square^{-\frac12}\overline{\square}^{-\frac12} + ic_{pq}^{-1}\bar{\mathcal{R}}\mathcal{R}(\overline{\square}+iT)^{\frac12}\overline{\square}^{-\frac12} + \frac12 c_{pq}^{-1} e(d\theta) iT \square^{-\frac12}\overline{\square}^{-\frac12}\right),$$

where

$$M = -\frac{T}{|T|\sqrt{c_{pq}^{-1}(c_{pq}^{-1}-1)}},$$

$c_{pq}^{-1} = n-k+2$, $0 \le k \le n$, and

$$U_1 = \left(\mathcal{R}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}} - \bar{\mathcal{R}}(\Delta_H)^{-\frac{1}{2}}\Box^{\frac{1}{2}} \quad \mathcal{R}\Delta_H^{-\frac{1}{2}}\Box^{\frac{1}{2}} + \bar{\mathcal{R}}\Delta_H^{-\frac{1}{2}}\overline{\Box}^{\frac{1}{2}} \right).$$

Observing that $\Box^{\frac{1}{2}}$, $\overline{\Box}^{\frac{1}{2}} \in {}^*\Psi^{0,\frac{1}{2}}_{\frac{1}{2}}$ the proof follows from Lemma 1.10.2, Lemma 1.10.3 and Lemma 1.10.4. □

1.10.1. Multipliers of Δ_R. A function μ defined on the positive half-line is a Mihlin-Hörmander multiplier of class $\rho > 0$ if, given a smooth function χ supported on $[\frac{1}{2}, 4]$ and equal to 1 on $[1,2]$,

$$\|\mu\|_{\rho,\mathrm{sloc}} := \sup_{t>0} \|\mu(t\cdot)\chi\|_{L^2_\rho} < \infty.$$

THEOREM 1.10.6. *Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded, continuous function in $L^2_{\rho,\mathrm{sloc}}(\mathbb{R})$ for some $\rho > (2n+1)/2$. Then for every $k = 0, \dots, 2n+1$, the operator $m(\Delta_{R,k})$ is bounded on $L^p(\mathbb{H}_n)E_0^k$ for $1 < p < \infty$, with norm controlled by $\|m\|_{\rho,sloc}$.*

PROOF. It is clear that

$$\Delta_{R,k} = \sum_{p+q=k} \Delta_{R,k} P_0^{p,q} + \sum_{p+q=k-1} \Delta_{R,k} P_1^{p,q} + \sum_{p+q=k-2} \Delta_{R,k} P_2^{p,q},$$

where $P_1^{p,q}$ are the orthogonal projections onto $\overline{W_1^{p,q}}$ defined in Proposition 1.7.6 for the subcritical case and in Proposition 1.7.12 for the critical case, $P_2^{p,q}$ are the orthogonal projections onto $\overline{W_2^{p,q}}$ defined in Proposition 1.7.10, and $P_0^{p,q}$ are the orthogonal projections onto $\overline{W_0^{p,q}}$. Obviously

$$\sum_{p+q=k} P_0^{p,q} = I - \sum_{p+q=k-1} P_1^{p,q} - \sum_{p+q=k-2} P_2^{p,q}.$$

Moreover, we recall that

$$\begin{aligned}
\Delta_{R,k} P_0^{p,q} &= D_0^{p,q} P_0^{p,q}, \\
\Delta_{R,k} P_1^{p,q} &= U_1^{p,q} D_1^{p,q} (U_1^{p,q})^* P_1^{p,q}, \\
\Delta_{R,k} P_2^{p,q} &= U_2^{p,q} D_2^{p,q} (U_2^{p,q})^* P_j^{p,q},
\end{aligned}$$

where $D_j^{p,q}$ is the scalar form of Δ_R restricted to $W_j^{p,q}$ as seen in formulas (1.6.1), (1.6.4) (1.6.7), (1.6.8), (1.6.9) and (1.6.10). Then

$$\begin{aligned}
&m(\Delta_R) \\
&\quad = \sum_{p,q} m(D_0^{p,q}) P_0^{p,q} + \sum_{p,q} U_1^{p,q} m(D_1^{p,q}) (U_1^{p,q})^* P_1^{p,q} + \sum_{p,q} U_2^{p,q} m(D_2^{p,q}) (U_2^{p,q})^* P_2^{p,q}.
\end{aligned}$$

Finally the proof follows from Theorem 1.9.2. □

CHAPTER 2

Differentiability for intrinsic Lipschitz functions

In this chapter we address a different problem, specifically the description of Lipschitz graphs on Carnot groups according to an intrinsic notion, see [**24**]. Again Rumin's complex play a crucial role here: in fact it is deeply related to the existence of complementary subgroups in Carnot groups.

As already stated in the introduction, our interest in intrinsic Lipschitz functions originates from the problem of defining appropriately rectifiable sets inside Carnot groups. To this end we begin by introducing a new class of Carnot groups, the so-called groups of type $\star$. These groups were introduced by the present author in [**43**], where he proved the rectifiability of the reduced boundary of sets of finite $\mathbb{G}$-perimeter (the so-called De Giorgi's theorem in Carnot groups).

Then we introduce the notions of *intrinsic differentiable functions* within a Carnot group $\mathbb{G}$ and we prove a Rademacher's type theorem for one dimensional intrinsic Lipschitz functions inside this class of Carnot groups.

We also point out that a good portion of this chapter is joint with B. Franchi and R. Serapioni, see [**23**].

2.1. Carnot groups of type $\star$

Definition 2.1.1 ([**43**]). We say that a stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is of type $\star$[1] if there exists a basis $\{X_1, \dots, X_{m_1}\}$ of $\mathfrak{g}_1$ such that

$$[X_j, [X_j, X_i]] = 0, \qquad \text{for all } i, j = 1, \dots, m_1. \tag{2.1.1}$$

A Carnot group $\mathbb{G}$ is said to be of type $\star$ if its Lie algebra $\mathfrak{g}$ is of type $\star$.

Remark 2.1.2. We do not require the validity of (2.1.1) for every basis of $\mathfrak{g}_1$, since that would be equivalent to require that the step is 2. The proof is quite straightforward. Suppose $[Y_1, [Y_1, Y_2]] = 0$ for every $Y_1, Y_2 \in \mathfrak{g}_1$. If $X, Y, Z \in \mathfrak{g}_1$ then

$$\begin{aligned} 0 &= [X+Y, [X+Y, Z]] = [X, [Y, Z]] + [Y, [X, Z]] \\ 0 &= [X+Z, [X+Z, Y]] = [X, [Z, Y]] + [Z, [X, Y]] = -2[X, [Y, Z]] + [Y, [X, Z]]. \end{aligned}$$

Therefore $[X, [Y, Z]] = 0$ for all $X, Y, Z \in \mathfrak{g}_1$.

Example 2.1.3. Obviously step 2 Carnot groups are of type $\star$.

The Lie groups of unit upper triangular $(m+1) \times (m+1)$ matrices are non-trivial examples of Carnot groups of type $\star$, for any $m \in \mathbb{N}$ $(m > 2)$, where m coincides with the step of the stratification. They are the nilpotent groups that come from the Iwasawa decomposition of $GL_{m+1}(\mathbb{R})$.

Now let $\mathbb{G}$ be one of these groups. We wish to prove that $\mathbb{G}$ is of type $\star$ for $m > 2$. The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is isomorphic to the one of strictly upper triangular $(m+1) \times (m+1)$ matrices (see [**33**], Part I, Chapter 2, Section 5.7, Example 1). If $E_{i,j}$ is the matrix with 1 in the (i, j)-th entry and 0 elsewhere, it is easy to see that a basis of $\mathfrak{g}$ is formed by the single-entry matrices $E_{k,k+\ell}$ for $\ell = 1, \dots, m$

[1] No confusion should arise with the notation of the Hodge $\star$-operator.

and $k = 1, \dots, m + 1 - \ell$, and $\dim \mathfrak{g} = \frac{m(m+1)}{2}$. The choice of using the particular parameters k and l will soon be explained.

The following formula, which can be proven by direct computation of the commutators of single-entry matrices, gives the expression of Lie brackets in $\mathfrak{g}$.

$$
[E_{k_1,k_1+\ell_1}, E_{k_2,k_2+\ell_2}] = \begin{cases} E_{k_1,k_1+(\ell_1+\ell_2)} & \text{if } k_1 < k_2 \text{ and } k_1 + \ell_1 = k_2 \\ -E_{k_2,k_2+(\ell_1+\ell_2)} & \text{if } k_1 > k_2 \text{ and } k_2 + \ell_2 = k_1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.2}
$$

From (2.1.2), it is easy to see that $E_{k,k+1}$ (for $k = 1, \dots, m$) are generators of $\mathfrak{g}$. Moreover, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ with

$$
\mathfrak{g}_\ell = \operatorname{span}\{E_{k,k+\ell} \mid k = 1, \dots, m + 1 - \ell\}
$$

for $\ell = 1, \dots, m$. This explains the use of the parameters k and ℓ. Moreover, we observe that m is the dimension of $\mathfrak{g}_1$ and the step of the stratification.

Now we can finally prove that $\mathbb{G}$ is of type $\star$. We set $X_k := E_{k,k+1}$ for $k = 1, \dots, m$. From (2.1.2) we obtain that $E_{k,k+2} = [X_k, X_{k+1}]$ for $k = 1, \dots, m-1$ and the other independent commutators of length 2 are zero, whereas $E_{k,k+3} = [[X_k, X_{k+1}], X_{k+2}] = [X_k, [X_{k+1}, X_{k+2}]]$ for $k = 1, \dots, m - 2$ and the other independent commutators of length 3 are zero. Hence (2.1.1) holds.

Finally we observe that, if $m = 2$, $\mathbb{G}$ is isomorphic to the Heisenberg group $\mathbb{H}_1$.

The rectifiability of the reduced boundary of finite perimeter sets, proved in [**25, 27**] inside Heisenberg groups and more generally inside step 2 groups, has been extended by the author to groups of type $\star$ in [**43**]. In particular we refer the reader to Theorem 2.4.14. This theorem plays a key role in the proof of Theorem 2.4.15.

2.2. Intrinsic functions and intrinsic Lipschitz functions

First we recall some relations between the Rumin's complex of intrinsic differential forms in a Carnot group $\mathbb{G}$ and the existence of complementary subgroups in $\mathbb{G}$ (see [**24**]). Then we describe some interesting facts about intrinsic Lipschitz functions.

2.2.1. Relations between Rumin's complex and complementary subgroups.

Remark 2.2.1 ([**24**]). A homogeneous subgroup $\mathbb{H}$ is stratified, that is, $\mathbb{H} = \mathbb{H}^1 \oplus \cdots \oplus \mathbb{H}^\kappa$, where $\mathbb{H}^i \subset \mathbb{G}^i$ and $\mathbb{H}^i$ is a linear subspace of $\mathbb{G}^i$. If we denote by $\mathfrak{h}$ the Lie algebra of $\mathbb{H}$, this follows once we prove that

$$
\mathfrak{h} = \oplus_{p=1}^{\kappa} \mathfrak{h}_p, \tag{2.2.1}
$$

where $\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{g}_p$. Indeed, if $v \in \mathfrak{h}$, we can write $v = \sum_p v_p$, with $v_p \in \mathfrak{g}_p$, $p = 1, \dots, \kappa$. Thus (2.2.1) follows if we show that

$$
v_p \in \mathfrak{h} \quad \text{for all } p = 1, \dots, \kappa. \tag{2.2.2}
$$

To this end, we remind that $\mathfrak{h}$ is a vector space and, in addition, it is homogeneous with respect to group dilations. Hence, for $\lambda > 0$,

$$
\frac{1}{\lambda}\delta_\lambda v := \frac{1}{\lambda}\sum_p \lambda^p v_p = v_1 + \sum_{p \geq 2} \lambda^{p-1} v_p \in \mathfrak{h}.
$$

But $\frac{1}{\lambda}\delta_\lambda v$ is bounded, and hence, if we choose $\lambda = \lambda_n$, with $\lambda_n \to 0$ as $n \to \infty$, we can assume $\left(\frac{1}{\lambda_n}\delta_{\lambda_n} v\right)_n$ has a limit in $\mathfrak{h}$. Thus, we can conclude that $v_1 \in \mathfrak{h}$. We can repeat now the argument replacing v by $v - v_1 \in \mathfrak{h}$, and we write

$$\frac{1}{\lambda^2}\delta_\lambda(v - v_1) = v_2 + \sum_{p\geq 3} \lambda^{p-2} v_p \in \mathfrak{h},$$

obtaining eventually that $v_2 \in \mathfrak{h}$. Iterating this argument, we get (2.2.2) and therefore (2.2.1).

The following result shows that a pair of non-parallel intrinsic simple covectors $\xi \in E_0^h$ and $\omega \in E_0^{N-h}$ naturally define a couple of complementary subgroups as in Definition I.4.1. Following the notations of [**36**], p.90, if X is a vector field, we denote by $i(X)$ the interior product and by $\theta(X)$ the Lie derivative along X.

Theorem 2.2.2 ([**24**]). *If $1 \leq h < N$, $\xi \in E_0^h$ and $\omega \in E_0^{N-h}$ are simple covectors such that*

$$\xi \wedge \omega \neq 0,$$

we set

$$\mathfrak{m} := \{X \in \mathfrak{g} \,:\, i(X)\xi = 0\}, \qquad \mathfrak{h} := \{X \in \mathfrak{g} \,:\, i(X)\omega = 0\}.$$

Then both $\mathfrak{m}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g}$. Moreover $\dim \mathfrak{m} = N - h$, $\dim \mathfrak{h} = h$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. If, in addition, $\xi = \xi_1 \wedge \cdots \wedge \xi_h$, $\omega = \omega_1 \wedge \cdots \wedge \omega_{N-h}$, where all the ξ_i's and the ω_i have pure weights p_i and q_i, respectively, then both $\mathfrak{m}$ and $\mathfrak{h}$ are homogeneous Lie subalgebras of $\mathfrak{g}$. Thus, if we set

$$\mathbb{M} := \exp(\mathfrak{m}) \quad \textit{and} \quad \mathbb{H} := \exp(\mathfrak{h}),$$

then $\mathbb{M}$ and $\mathbb{G}$ are complementary subgroups. In particular, since $*E_0^h = E_0^{N-h}$, if $\xi \in E_0^h$, we can choose $\omega := *\xi$. In this case, $\mathfrak{m}$ and $\mathfrak{h}$ are orthogonal.

Reciprocally, suppose $\mathfrak{m}$ and $\mathfrak{h}$ are homogeneous Lie subalgebras of $\mathfrak{g}$ such that $\dim \mathfrak{m} = N - h$, $\dim \mathfrak{h} = h$, and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Then there exist a scalar product $\langle \cdot, \cdot \rangle_0$ in $\mathfrak{g}$, $\xi \in E_0^h$ and $\omega \in E_0^{N-h}$ such that $\xi \wedge \omega \neq 0$ and

$$\mathfrak{m} := \{X \in \mathfrak{g} \,:\, i(X)\xi = 0\}, \qquad \mathfrak{h} := \{X \in \mathfrak{g} \,:\, i(X)\omega = 0\}.$$

2.2.2. Intrinsic Lipschitz functions. Intrinsic Lipschitz functions in $\mathbb{G}$ are functions, acting between complementary subgroups of $\mathbb{G}$, with graphs non intersecting naturally defined cones. Hence, the notion of *intrinsic Lipschitz graph* respects strictly the geometry of the ambient group $\mathbb{G}$.

We begin with two definitions of intrinsic cones.

Definition 2.2.3. Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$, $q \in \mathbb{G}$. The *cones* $X(q, \mathbb{H}, \alpha)$ with *axis* $\mathbb{H}$, *vertex* q, *opening* α, $0 \leq \alpha \leq 1$ are defined as

$$X(q, \mathbb{H}, \alpha) = q \cdot X(e, \mathbb{H}, \alpha), \text{ where } X(e, \mathbb{H}, \alpha) = \big\{p : \operatorname{dist}(p, \mathbb{H}) \leq \alpha \, \|p\| \,\big\},$$

where $\operatorname{dist}(p, \mathbb{H}) := \inf\{\|p^{-1}h\| : h \in \mathbb{H}\}$.

If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $q \in \mathbb{G}$ and $\beta \geq 0$, the cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$, with base $\mathbb{M}$, axis $\mathbb{H}$, vertex q, opening β are defined as

$$C_{\mathbb{M},\mathbb{H}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{H}}(e, \beta), \text{ where } C_{\mathbb{M},\mathbb{H}}(e, \beta) = \{p : \|p_{\mathbb{M}}\| \leq \beta \, \|p_{\mathbb{H}}\|\}.$$

The cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$ are 'equivalent' with the cones $X(q, \mathbb{H}, \alpha)$, indeed

Proposition 2.2.4 ([**24**]). *If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ then, for any $\alpha \in (0, 1)$ there is $\beta \geq 1$, depending on α, $\mathbb{M}$ and $\mathbb{H}$, such that*

$$C_{\mathbb{M},\mathbb{H}}(q, 1/\beta) \subset X(q, \mathbb{H}, \alpha) \subset C_{\mathbb{M},\mathbb{H}}(q, \beta),$$

Now we introduce the main definition of this subsection.

DEFINITION 2.2.5. Let $\mathbb{H}$ be a homogeneous subgroup, not necessarily complemented in $\mathbb{G}$.
(i) An $\mathbb{H}$-graph S is an *intrinsic Lipschitz* $\mathbb{H}$*-graph* if there is $\alpha \in (0,1)$ such that,

$$S \cap X(p, \mathbb{H}, \alpha) = \{p\}, \qquad \text{for all } p \in S.$$

(ii) If there is a subgroup $\mathbb{M}$ such that $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, we say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic Lipschitz* in $\mathcal{E}$ when graph$\,(f)$ is an *intrinsic Lipschitz* $\mathbb{H}$*-graph*.
(iii) We say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic L-Lipschitz* in $\mathcal{E}$ for some $L \geq 0$ if for all $\tilde{L} > L$

$$C_{\mathbb{M},\mathbb{H}}(p, 1/\tilde{L}) \cap \operatorname{graph}(f) = \{p\}, \quad \text{for all } p \in \operatorname{graph}(f). \tag{2.2.3}$$

The Lipschitz constant of f in $\mathcal{E}$ is the infimum of the $\tilde{L} > 0$ such that (2.2.3) holds.

It follows immediately from Proposition 2.2.4 that f is intrinsic Lipschitz in $\mathcal{E}$ if and only if it is intrinsic L-Lipschitz for an appropriate constant L, depending on α, f and $\mathbb{M}$.

Because of Proposition I.4.5 and Definition 2.2.3 left translations of intrinsic Lipschitz $\mathbb{H}$-graphs, or of intrinsic L-Lipschitz functions, keep being intrinsic Lipschitz $\mathbb{H}$-graphs, or intrinsic L-Lipschitz functions.

We collect without proofs a few results about intrinsic Lipschitz functions. All the proofs, here omitted, can be found in [**24**].

First we observe that the geometric definition of intrinsic Lipschitz graphs has equivalent algebraic forms (see also [**7**], [**30**], [**29**]).

PROPOSITION 2.2.6 ([**24**]). *Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $L > 0$. Then* (i) *to* (iii) *are equivalent.*

(i) $$f \text{ is intrinsic } L\text{-Lipschitz in } \mathcal{E}.$$

(ii) $$\left\|\mathbf{P}_{\mathbb{H}}(\bar{q}^{-1}q)\right\| \leq L \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f).$$

(iii) $$\left\|f_{\bar{q}^{-1}}(m)\right\| \leq L \left\|m\right\|, \quad \text{for all } \bar{q} \in \operatorname{graph}(f) \text{ and } m \in \mathcal{E}_{\bar{q}^{-1}}.$$

REMARK 2.2.7. f is intrinsic Lipschitz if and only if the distance of two points $q, \bar{q} \in \operatorname{graph}(f)$ is bounded by the norm of the projection on the domain $\mathbb{M}$ of $\bar{q}^{-1}q$. Precisely, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz if and only if there is a constant $C > 0$ such that

$$\left\|\bar{q}^{-1}q\right\| \leq C \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f).$$

The relations between the constant C and the Lipschitz constant L of f follow from (I.4.1):

- if f is intrinsic L-Lipschitz then
$$\left\|\bar{q}^{-1}q\right\| \leq (1+L) \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f);$$
- conversely, if $\left\|\bar{q}^{-1}q\right\| \leq c_0(1+L) \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|$ then
$$\left\|\mathbf{P}_{\mathbb{H}}(\bar{q}^{-1}q)\right\| \leq L \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f)$$
and f is intrinsic L-Lipschitz.

In general intrinsic Lipschitz functions are not metric Lipschitz functions. By this we mean that, if $f : \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz then this does not yields the existence of a constant C such that

$$\left\|f(\bar{m})^{-1}f(m)\right\| \leq C \left\|\bar{m}^{-1}m\right\| \qquad \text{for } m, \bar{m} \in \mathbb{M},$$

not even locally. For a more complete discussion about this see Remark 3.1.8 of [**24**]. Nevertheless, intrinsic Lipschitz functions are metric Hölder continuous and hence uniformly continuous.

PROPOSITION 2.2.8 ([**24**]). *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in a step κ group $\mathbb{G}$. Let $L > 0$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ be an intrinsic L-Lipschitz function. Then* (i) *f is bounded on bounded subsets of $\mathcal{E}$. Precisely, for all $R > 0$, $p \in \mathcal{E}$ with $\|p\| \le R$, there is $C_1 = C_1(\mathbb{G}, \mathbb{M}, \mathbb{H}, L, R, f(p)) > 0$ such that*

$$\|f(m)\| \le C_1, \qquad \text{for all } m \in \mathcal{E} \text{ such that } \|m\| \le R.$$

(ii) *f is $\frac{1}{\kappa}$-Holder continuous on bounded subset of $\mathcal{E}$. Precisely, for all $R > 0$, there is $C_2 = C_2(\mathbb{G}, \mathbb{M}, \mathbb{H}, C_1, L, R) > 0$ such that*

$$\left\|f(\bar m)^{-1} f(m)\right\| \le C_2 \left\|\bar m^{-1} m\right\|^{1/\kappa}, \qquad \text{for all } m, \bar m \in \mathcal{E} \text{ with } \|m\|, \|\bar m\| \le R.$$

The graphs of intrinsic Lipschitz functions are Ahlfors regular sets. If $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz then the Hausdorff dimension of graph (f) is the same as the metric dimension of the domain $\mathcal{E}$. That is if s is this metric dimension of $\mathcal{E}$ then

$$\mathcal{S}_d^s\left(\mathrm{graph}(f) \cap \mathcal{U}\right) < \infty,$$

for any bounded $\mathcal{U} \subset \mathbb{G}$.

THEOREM 2.2.9. [**24**] *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Let d_m denote the metric dimension of $\mathbb{M}$. If $f : \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz in $\mathbb{M}$ then there is $c = c(\mathbb{M}, \mathbb{H}) > 0$ such that,*

$$\left(\frac{c_0}{1+L}\right)^{d_m} R^{d_m} \le \mathcal{S}_d^{d_m}\big(\mathrm{graph}\,(f) \cap B(p,R)\big) \le c(1+L)^{d_m} R^{d_m},$$

for all $p \in \mathrm{graph}\,(f)$ and $R > 0$, where c_0 is the structural constant in Proposition I.4.2. In particular, graph (f) *has metric dimension d_m.*

A non trivial corollary of Theorem 2.2.9 will be that 1-codimensional intrinsic Lipschitz graphs are boundaries of sets of locally finite $\mathbb{G}$-perimeter (see Theorem 2.4.10). The *extension property* for intrinsic Lipschitz functions taking values in one dimensional subgroups, as stated in Theorem 2.2.10, is one of the main results in [**24**] and is a key instrument in proving the equivalence of the two definitions of rectifiable sets given in Definition 2.4.18.

THEOREM 2.2.10. [**24**] *Let $\mathbb{M}$ and $\mathbb{V}$ be complementary subgroups with $\mathbb{V}$ one dimensional (and consequently horizontal). Let $\mathcal{B} \subset \mathbb{M}$ be a Borel subset of $\mathbb{M}$ and $f : \mathcal{B} \to \mathbb{V}$ be an intrinsic L-Lipschitz function. Then there are $\tilde f : \mathbb{M} \to \mathbb{V}$ and $\tilde L = \tilde L(L, \mathbb{G}, \mathbb{M}, \mathbb{V}) \ge L$ such that*

$$\tilde f \text{ is intrinsic } \tilde L\text{-Lipschitz in } \mathbb{M},$$
$$\tilde f(m) = f(m) \qquad \text{for all } m \in \mathcal{B}.$$

2.3. Intrinsic differentiable functions

2.3.1. Intrinsic linear Functions. A function $f : \mathbb{M} \to \mathbb{H}$, acting between complementary subgroups of $\mathbb{G}$, is *intrinsic differentiable* in a point $m \in \mathbb{M}$ if the graph of f has a tangent homogeneous subgroup in $mf(m) \in \mathrm{graph}\,(f)$ (see Definition 2.3.12). This notion can be stated also in terms of the existence of an approximating intrinsic linear function. Intrinsic linear functions, acting between complementary subgroups, are those functions whose graphs are homogeneous subgroups. We begin with this second approach.

DEFINITION 2.3.1. Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{M} \to \mathbb{H}$ is an *intrinsic linear function* if ℓ is defined on all of $\mathbb{M}$ and if graph $(\ell) = \{m\, \ell(m) : m \in \mathbb{M}\}$ is a homogeneous subgroup of $\mathbb{G}$.

Intrinsic linear functions are not necessarily group homomorphisms between their domains and codomains, as the following example shows.

EXAMPLE 2.3.2. We consider the Heisenberg group $\mathbb{H}_1$ identified with $(\mathbb{R}^3, \cdot)$. The group product is given by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) := (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - x_2y_1)/2)\,.$$

The linear subspaces $\mathbb{V}$, $\mathbb{W}$ of $\mathbb{R}^3$, defined as $\mathbb{V} := \{v = (v_1, 0, 0)\}$ and $\mathbb{W} := \{w = (0, w_2, w_3)\}$ are complementary homogeneous subgroups of $\mathbb{H}_1$. For any fixed $a \in \mathbb{R}$, the function $\ell : \mathbb{V} \to \mathbb{W}$ defined as

$$\ell(v) = (0, av_1, -av_1^2/2)$$

is intrinsic linear because graph $(\ell) = \{(t, at, 0) : t \in \mathbb{R}\}$ is a 1-dimensional homogeneous subgroup of $\mathbb{H}_1$. This ℓ is not a group homomorphism from $\mathbb{V}$ to $\mathbb{W}$.

Intrinsic linear functions can be algebraically characterized as follows.

PROPOSITION 2.3.3. *Let* $\mathbb{M}$ *and* $\mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$. *Then* $\ell : \mathbb{M} \to \mathbb{H}$ *is an* intrinsic linear function *if and only if*

$$\ell(\delta_\lambda m) = \delta_\lambda(\ell(m)), \qquad \textit{for all } m \in \mathbb{M} \textit{ and } \lambda \in \mathbb{R};$$

$$\ell(m_1m_2) = \left(\mathbf{P}_{\mathbb{H}}(\ell(m_1)^{-1}m_2)\right)^{-1} \ell\left(\mathbf{P}_{\mathbb{M}}(\ell(m_1)^{-1}m_2)\right), \qquad \textit{for all } m_1, m_2 \in \mathbb{M}.$$

PROOF. Because graph (ℓ) is a homogeneous subgroup, for each $m \in \mathbb{M}$ there is $\bar{m} \in \mathbb{M}$ such that $\delta_\lambda\left(m\ell(m)\right) = \bar{m}\ell(\bar{m})$; hence

$$\delta_\lambda m\, \delta_\lambda\left(\ell(m)\right) = \bar{m}\ell(\bar{m})$$

and by uniqueness of the components

$$\bar{m} = \delta_\lambda m \quad \text{and } \ell(\delta_\lambda m) = \ell(\bar{m}) = \delta_\lambda\left(\ell(m)\right).$$

Because graph (ℓ) is a subgroup, for all $m_1, m_2 \in \mathbb{M}$ there is $\bar{m}$ such that

$$m_1\ell(m_1)m_2\ell(m_2) = \bar{m}\ell(\bar{m}).$$

Hence

$$\begin{aligned}\bar{m} &= \left(m_1\ell(m_1)m_2\ell(m_2)\right)_{\mathbb{M}} = m_1\left(\ell(m_1)m_2\right)_{\mathbb{M}},\\ \ell(\bar{m}) &= \left(m_1\ell(m_1)m_2\ell(m_2)\right)_{\mathbb{H}} = \left(\ell(m_1)m_2\right)_{\mathbb{H}}\ell(m_2).\end{aligned}$$

This way we obtained

$$\ell\left(m_1\left(\ell(m_1)m_2\right)_{\mathbb{M}}\right) = \left(\ell(m_1)m_2\right)_{\mathbb{H}}\ell(m_2), \quad \text{for all } m_1, m_2 \in \mathbb{M}. \tag{2.3.1}$$

To get a more explicit expression we change variables. To do this, first we observe that for each couple $m \in \mathbb{M}$ and $h \in \mathbb{H}$ there is exactly one $\bar{m} \in \mathbb{M}$ such that

$$m = (h\bar{m})_{\mathbb{M}} \tag{2.3.2}$$

and $\bar{m}$ can be explicitly defined as

$$\bar{m} = \left(h^{-1}m\right)_{\mathbb{M}}. \tag{2.3.3}$$

Indeed

$$\left(h\left(h^{-1}m\right)_{\mathbb{M}}\right)_{\mathbb{M}} = \left(hh^{-1}m\left(\left(h^{-1}m\right)_{\mathbb{H}}\right)^{-1}\right)_{\mathbb{M}} = \left(m\left(\left(h^{-1}m\right)_{\mathbb{H}}\right)^{-1}\right)_{\mathbb{M}} = m.$$

To prove that the choice of $\bar{m}$ in (2.3.2) is unique observe that

$$(h\,m_1)_{\mathbb{M}} = (h\,m_2)_{\mathbb{M}} \implies m_1 = m_2$$

for all $m_1, m_2 \in \mathbb{M}$, $h \in \mathbb{H}$. Indeed

$$h\,m_1 = (h\,m_1)_{\mathbb{M}}\,(h\,m_1)_{\mathbb{H}} = (h\,m_2)_{\mathbb{M}}\,(h\,m_1)_{\mathbb{H}} = h\,m_2\,\left((h\,m_2)_{\mathbb{H}}\right)^{-1}(h\,m_1)_{\mathbb{H}}\,;$$

hence

$$m_1 = m_2\,\left((h\,m_2)_{\mathbb{H}}\right)^{-1}(h\,m_1)_{\mathbb{H}}\,,$$

that gives

$$m_2^{-1}\,m_1 = \left((h\,m_2)_{\mathbb{H}}\right)^{-1}(h\,m_1)_{\mathbb{H}} \in \mathbb{H},$$

and finally $m_2^{-1}\,m_1 = 0$ because $\mathbb{M}$ and $\mathbb{H}$ are complementary. This completes the proof of the existence and uniqueness of $\bar{m}$ as in (2.3.2) and (2.3.3).

Using (2.3.2) and (2.3.3), for all $m_1, m_3 \in \mathbb{M}$ we define

$$m_2 := \left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}}\,,$$

so that

$$m_3 = (\ell(m_1)m_2)_{\mathbb{M}}$$

and we substitute inside (2.3.1) to get

$$\begin{aligned}
\ell(m_1m_3) &= \left(\ell(m_1)\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}}\right)_{\mathbb{H}}\ell(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}})\\
&= \left(\ell(m_1)\ell(m_1)^{-1}m_3\left(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{H}}\right)^{-1}\right)_{\mathbb{H}}\ell(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}})\\
&= \left(m_3\left(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{H}}\right)^{-1}\right)_{\mathbb{H}}\ell(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}})\\
&= \left(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{H}}\right)^{-1}\ell(\left(\ell(m_1)^{-1}m_3\right)_{\mathbb{M}})
\end{aligned}$$

for all $m_1, m_3 \in \mathbb{M}$. □

Corollary 2.3.4. *Let $\mathbb{G}$ be the semidirect product of the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$. Let $\ell : \mathbb{M} \to \mathbb{H}$ be intrinsic linear. Then the second statement of Proposition 2.3.3 takes the following form:*

$$\begin{aligned}
\textit{if } \mathbb{M} \textit{ is normal in } \mathbb{G}\textit{:}\quad & \ell(m_1m_2) = \ell(m_1)\,\ell\left(\ell(m_1)^{-1}\,m_2\,\ell(m_1)\right),\\
\textit{if } \mathbb{H} \textit{ is normal in } \mathbb{G}\textit{:}\quad & \ell(m_1m_2) = m_2^{-1}\,\ell(m_1)\,m_2\,\ell(m_2),\\
\textit{if } \mathbb{M} \textit{ and } \mathbb{H} \textit{ are normal:}\quad & \ell(m_1m_2) = \ell(m_1)\cdot\ell(m_2),
\end{aligned}$$

for all $m_1, m_2 \in \mathbb{M}$. Hence, when $\mathbb{G}$ is the direct product of $\mathbb{M}$ and $\mathbb{H}$, an intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ is a homogeneous homomorphism from $\mathbb{M}$ to $\mathbb{H}$.

Proof. If $\mathbb{M}$ is normal then $(hm)_{\mathbb{M}} = hmh^{-1}$ and $(hm)_{\mathbb{H}} = h$. Hence the first one is proved. If $\mathbb{H}$ is normal then $(hm)_{\mathbb{M}} = m$ and $(hm)_{\mathbb{H}} = m^{-1}hm$ and we get the second one. Finally the last one follows from the first two. □

Proposition 2.3.5. *Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$.*
(i) *If $\ell : \mathbb{M} \to \mathbb{H}$ is* intrinsic linear *then the homogeneous subgroups* graph(ℓ) *and $\mathbb{H}$ are complementary subgroups and $\mathbb{G} =$ graph$(\ell)\cdot\mathbb{H}$.*
(ii) *If $\mathbb{V}$ is a homogeneous subgroup such that $\mathbb{V}$ and $\mathbb{H}$ are complementary in $\mathbb{G}$ then there is a unique intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ such that $\mathbb{V} =$ graph(ℓ).*

Proof. (i): Observe that for all $g \in \mathbb{G}$ we have $g = g_{\mathbb{M}}g_{\mathbb{H}} = g_{\mathbb{M}}\ell(g_{\mathbb{M}})\ell(g_{\mathbb{M}})^{-1}g_{\mathbb{H}}$, and $g_{\mathbb{M}}\ell(g_{\mathbb{M}}) \in$ graph(ℓ) while $\ell(g_{\mathbb{M}})^{-1}g_{\mathbb{H}} \in \mathbb{H}$. On the other side graph$(\ell) \cap \mathbb{H} = e$. Indeed, if $g = m\ell(m) \in \mathbb{H}$ then $m = 0$ and $g = \ell(0)$. By the first statement of Proposition 2.3.3 also $\ell(0) = 0$, hence $g = 0$.

(ii): We have to prove that $\mathbb{V}$ is a graph over $\mathbb{M}$ in direction $\mathbb{H}$. That is we have to prove that each coset of $\mathbb{H}$ intersects $\mathbb{V}$ in at most one point. Indeed, if there are $m \in \mathbb{M}$ and $h_1, h_2 \in \mathbb{H}$ such that $mh_1 \in \mathbb{V}$ and $mh_2 \in \mathbb{V}$, then $h_2^{-1}h_1 = h_2^{-1}m^{-1}mh_1 \in \mathbb{V} \cap \mathbb{H}$ hence $h_1 = h_2$ and $mh_1 = mh_2$. This shows that $\mathbb{V}$ is a graph over $\mathbb{M}$. To complete the proof we have to show that each coset of $\mathbb{H}$ intersects $\mathbb{V}$. Indeed, for each $m \in \mathbb{M}$, by the assumption that $\mathbb{V}$, $\mathbb{H}$ are complementary it follows that $m = m_{\mathbb{V}} m_{\mathbb{H}}$. Hence $mm_{\mathbb{H}}^{-1} = m_{\mathbb{V}} \in m \cdot \mathbb{H} \cap \mathbb{V}$. Finally the function having $\mathbb{V}$ as graph, is intrinsic linear by definition. □

Proposition 2.3.6. *Let* $\mathbb{M}$ *and* $\mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *and* $\ell : \mathbb{M} \to \mathbb{H}$ *be intrinsic linear, then*

$$\ell \text{ is a polynomial function;}$$

$$\ell \text{ is intrinsic } L\text{-Lipschitz with } L := \sup\{\|\ell(m)\| : \|m\| = 1\}.$$

Proof. Let $\mathbf{P}_1 : \mathbb{M} \to \operatorname{graph}(\ell)$ be the restriction to $\mathbb{M}$ of the projection on the first component, related with the decomposition $\mathbb{G} = \operatorname{graph}(\ell) \cdot \mathbb{H}$. Then

$$\mathbf{P}_1(m) = m\ell(m), \qquad \text{for all } m \in \mathbb{M}.$$

Let $\mathbf{P}_2 : \operatorname{graph}(\ell) \to \mathbb{H}$ be the restriction to $\operatorname{graph}(\ell)$ of the projection on the second component, related with the decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. Then

$$\mathbf{P}_2(m\ell(m)) = \ell(m), \qquad \text{for all } m\ell(m) \in \operatorname{graph}(\ell).$$

Then $\ell : \mathbb{M} \to \mathbb{H}$ is $\ell = \mathbf{P}_2 \circ \mathbf{P}_1$. By Proposition I.4.3 both $\mathbf{P}_1$ and $\mathbf{P}_2$ are polynomial maps, hence also ℓ is a polynomial map.

Observe that L is finite because ℓ is continuous. From the first statement of Proposition 2.3.3 we have also that $\|\ell(m)\| \le L \|m\|$, for all $m \in \mathbb{M}$. This inequality can be stated geometrically as

$$C_{\mathbb{M},\mathbb{H}}(0, \alpha) \cap \operatorname{graph}(\ell) = \{0\}, \quad \text{for all } \alpha \text{ s.t. } 0 < \alpha < 1/L. \tag{2.3.4}$$

By definition of intrinsic linear functions,

$$\operatorname{graph}(\ell) = p \cdot \operatorname{graph}(\ell), \quad \text{for all } p \in \operatorname{graph}(\ell),$$

moreover $C_{\mathbb{M},\mathbb{H}}(p, \alpha) = p \cdot C_{\mathbb{M},\mathbb{H}}(0, \alpha)$ for all $p \in \mathbb{G}$. Hence from (2.3.4) it follows

$$C_{\mathbb{M},\mathbb{H}}(p, \alpha) \cap \operatorname{graph}(\ell) = \{p\},$$

for all α with $0 < \alpha < 1/L$ and for all $p \in \operatorname{graph}(\ell)$. Hence by (iii) of Definition 2.2.5, ℓ is intrinsic L-Lipschitz. □

2.3.2. Intrinsic differentiable functions. We use intrinsic linear functions to define intrinsic differentiability in a way that is formally similar to the usual definition of differentiability.

Definition 2.3.7. Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$. For $\bar{m} \in \mathcal{A}$ let $\bar{p} := \bar{m} \cdot f(\bar{m}) \in \operatorname{graph}(f)$ and $f_{\bar{p}^{-1}} : \mathcal{A}_{\bar{p}^{-1}} \subset \mathbb{M} \to \mathbb{H}$. We say that f is *intrinsic differentiable in* $\bar{m} \in \mathcal{A}$ if $f_{\bar{p}^{-1}}$ is *intrinsic differentiable in* e that is if there is an intrinsic linear map $df = df_{\bar{m}} : \mathbb{M} \to \mathbb{H}$ such that, for all $m \in \mathcal{A}_{\bar{p}^{-1}}$,

$$\left\| df_{\bar{m}}(m)^{-1} \cdot f_{\bar{p}^{-1}}(m) \right\| = o(\|m\|), \quad \text{as } \|m\| \to 0.$$

The intrinsic linear map $df_{\bar{m}}$ is called the *intrinsic differential* of f.

Remark 2.3.8. If a function is intrinsic differentiable it keeps being intrinsic differentiable after a left translation of the graph. Precisely, let $q_1 = m_1 f(m_1)$ and $q_2 = m_2 f(m_2) \in \operatorname{graph}(f)$, then f is intrinsic differentiable in m_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv \left(f_{q_1^{-1}}\right)_{q_2}$ is intrinsic differentiable in m_2. In particular, f is intrinsic differentiable in m_1 if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e.

Proposition 2.3.9. *Let* $\mathbb{M}$, $\mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *and* $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ *with* $\mathcal{A}$ *relatively open in* $\mathbb{M}$. *If* f *is intrinsic differentiable in* $m \in \mathcal{A}$, *then* f *is continuous in* m.

PROOF. As observed in Remark I.4.6, it is enough to prove the continuity of $f_{p_1^{-1}}$ at the origin. This last fact is an immediate consequence of Definition 2.3.7 and of the continuity of intrinsic linear functions. $\square$

REMARK 2.3.10. Writing explicitly $f_{\bar p^{-1}}$ in Definition 2.3.7 when $\mathbb{G}$ is a semidirect product of $\mathbb{M}$ and $\mathbb{H}$ we obtain:

- if $\mathbb{M}$ is a normal subgroup then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar m \in \mathbb{M}$ if
$$\left\| df_{\bar m}(m)^{-1} \cdot f(\bar m)^{-1} \cdot f\left(\bar m f(\bar m) m f(\bar m)^{-1}\right)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0;$$
- if $\mathbb{H}$ is a normal subgroup then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar m \in \mathbb{M}$ if
$$\left\| df_{\bar m}(m)^{-1} \cdot m \cdot f(\bar m)^{-1} \cdot m^{-1} \cdot f\left(\bar m m\right)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0;$$
- if both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar m \in \mathbb{M}$ if
$$\left\| df_{\bar m}(m)^{-1} \cdot f(\bar m)^{-1} \cdot f\left(\bar m m\right)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0.$$

REMARK 2.3.11. P. Pansu introduced in [**54**] a notion of differentiability for maps between nilpotent groups, the differential being an approximating homogeneous homomorphisms. More precisely, a function f, acting between two nilpotent groups $\mathbb{G}_1$ and $\mathbb{G}_2$, is Pansu differentiable in $\bar g \in \mathbb{G}_1$ if there is a *homogeneous homomorphism* $h : \mathbb{G}_1 \to \mathbb{G}_2$ such that
$$\left\| h(\bar g^{-1}g)^{-1} \cdot f(\bar g)^{-1} \cdot f(g)\right\|_{\mathbb{G}_2} = o(\left\|\bar g^{-1}g\right\|_{\mathbb{G}_1}), \qquad \text{as } \left\|\bar g^{-1}g\right\|_{\mathbb{G}_1} \to 0.$$
We remark that Pansu differentiability and intrinsic differentiability, when both of them make sense, are in general *different notions*.

Indeed, let $\mathbb{V}$, $\mathbb{W}$ be the complementary subgroups of $\mathbb{H}_1$ defined as $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$. As observed before, an intrinsic linear function $\ell : \mathbb{V} \to \mathbb{W}$ is of the form
$$\ell(v) = (0, av_1, -av_1^2/2), \quad \text{for any fixed } a \in \mathbb{R}.$$
A homogeneous homomorphism $h : \mathbb{V} \to \mathbb{W}$ is of the form
$$h(v) = (0, av_1, 0), \quad \text{for any fixed } a \in \mathbb{R}.$$
Obviously, ℓ is intrinsic differentiable in $v = 0$ while h is Pansu differentiable in $v = 0$. On the other side, it is easy to check that neither ℓ is Pansu differentiable nor h is intrinsic differentiable in $v = 0$.

We remark also that Pansu differentiability is not preserved after graph translations. Indeed, consider once more the function $h : \mathbb{V} \to \mathbb{W}$, then $\operatorname{graph}(h) = \{(t, at, at^2/2) : \ t \in \mathbb{R}\}$. Let $p := (0, p_2, 0) \in \mathbb{H}_1$, $p \neq 0$. Then $p \cdot \operatorname{graph}(h) = \{(t, at + p_2, (at^2 - p_2 t)/2) : \ t \in \mathbb{R}\}$ is the graph of the function $h_p : \mathbb{V} \to \mathbb{W}$ defined as $h_p(v) = (0, av_1 + p_2, -p_2 v_1)$. It is easy to check that h_p is not Pansu differentiable in $v = 0$.

Finally, if $\mathbb{G}$ is the *direct product* of $\mathbb{M}$ and $\mathbb{H}$ it is easy to convince oneself that
$$f : \mathbb{M} \to \mathbb{H} \text{ is Pansu differentiable} \iff f \text{ is intrinsic differentiable.}$$

The algebraic definition of intrinsic differentiability of Definition 2.3.7 has an equivalent geometric formulation. Indeed intrinsic differentiability in one point is equivalent to the existence of a tangent subgroup to the graph. We begin with the definition of tangent subgroup.

DEFINITION 2.3.12. Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$ and let $\mathbb{T}$ be a homogeneous subgroup in $\mathbb{G}$. Let $m \in \mathcal{A}$ and $p = mf(m) \in \operatorname{graph}(f)$ we say that $p \cdot \mathbb{T}$ is a *tangent (affine) subgroup* or *tangent coset* to $\operatorname{graph}(f)$ in p if for all $\varepsilon > 0$ there is $\lambda = \lambda(\varepsilon) > 0$ such that
$$\operatorname{graph}(f) \cap \{q \in \mathbb{G} : \left\|\mathbf{P}_{\mathbb{M}}(p^{-1}q)\right\| < \lambda(\varepsilon)\} \subset X(p, \mathbb{T}, \varepsilon).$$

REMARK 2.3.13. The definition is translation invariant, that is $p \cdot \mathbb{T}$ is the tangent (affine) subgroup to graph (f) in p if and only if $\mathbb{T}$ is the tangent subgroup to graph $(f_{p^{-1}})$ in 0.

THEOREM 2.3.14. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$.*
(I) If f is intrinsic differentiable in $m \in \mathcal{A}$, set $\mathbb{T} :=$ graph (df_m). Then
- (i) *$\mathbb{T}$ is a homogeneous subgroup of $\mathbb{G}$;*
- (ii) *$\mathbb{T}$ and $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$;*
- (iii) *$p \cdot \mathbb{T}$ is the tangent coset to* graph (f) *in $p := mf(m)$.*

(II) Conversely, if $p := mf(m) \in$ graph (f) and if there is $\mathbb{T}$ such that (i), (ii), (iii) *hold, then f is intrinsic differentiable in m and the differential $df_m : \mathbb{M} \to \mathbb{H}$ is the unique intrinsic linear function such that $\mathbb{T} =$ graph (df_m).*

PROOF. By Remark 2.3.8 and Remark 2.3.13 we can assume without loss of generality that $m = 0$ and $f(0) = 0$.

Proof of (I). Because f is intrinsic differentiable in 0 there is an intrinsic linear function $df_0 : \mathbb{M} \to \mathbb{H}$ and $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\varepsilon > 0$

$$\left\| df_0(m)^{-1} \cdot f(m) \right\| < \varepsilon \left\| m \right\|, \quad \text{for all } m \in \mathbb{M} \text{ with } \|m\| < \delta(\varepsilon).$$

Define $\mathbb{T} :=$ graph (df_0); then, for all $m \in \mathbb{M}$,

$$\begin{aligned} \operatorname{dist}(mf(m), \mathbb{T}) &:= \inf\{\left\|x^{-1}mf(m)\right\| : x \in \mathbb{T}\} \\ &= \inf\{\left\|df_0(w)^{-1}w^{-1}mf(m)\right\| : w \in \mathbb{M}\} \\ &\le \left\|df_0(m)^{-1}f(m)\right\| \\ &\le \varepsilon\, c_0 \left\|m\right\|, \qquad \text{if } \|m\| < \delta(c_0\varepsilon) \\ &\le \varepsilon \left\|mf(m)\right\|, \qquad \text{if } \|m\| < \delta(c_0\varepsilon), \end{aligned}$$

where c_0 is the constant in (I.4.1). Hence we proved that for all $\varepsilon > 0$ there is $\lambda(\varepsilon) := \delta(c_0\, \varepsilon) > 0$ s.t.

$$\operatorname{dist}(mf(m), \mathbb{T}) \le \varepsilon \left\|mf(m)\right\|, \qquad \text{for all } m \text{ with } \|m\| < \lambda(\varepsilon),$$

that is

$$\operatorname{graph}(f) \cap \{q \in \mathbb{G} : \|\mathbf{P}_{\mathbb{M}}(q)\| < \lambda(\varepsilon)\} \subset X(0, \mathbb{T}, \varepsilon).$$

Proof of (II).
By assumption, the tangent subgroup $\mathbb{T}$ is complementary to $\mathbb{H}$; hence, from Lemma 2.3.5, there is an intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ such that $\mathbb{T} =$ graph (ℓ). We have to prove that $\ell = df_0$. Because $\mathbb{T}$ is the tangent subgroup to graph (f) in 0, for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\operatorname{dist}(mf(m), \mathbb{T}) \le \varepsilon \left\|mf(m)\right\|, \quad \text{for all } m \in \mathbb{M} \text{ with } \|m\| < \delta(\varepsilon). \tag{2.3.5}$$

Observe that, for all $x \in \mathbb{T}$,

$$x^{-1}mf(m) = x^{-1}m\ell(m)\, \ell(m)^{-1}f(m)$$

where $x^{-1}m\ell(m) \in \mathbb{T}$ and $\ell(m)^{-1}f(m) \in \mathbb{H}$. Hence, in the decomposition $\mathbb{G} = \mathbb{T}\cdot\mathbb{H}$,

$$\ell(m)^{-1}f(m) = \mathbf{P}_{\mathbb{H}}(x^{-1}mf(m)), \quad \text{for all } x \in \mathbb{T}.$$

Consequently, for all $x \in \mathbb{T}$,

$$\left\|\ell(m)^{-1}f(m)\right\| = \left\|\mathbf{P}_{\mathbb{H}}(x^{-1}mf(m))\right\| \le \frac{1}{\tilde{c}_0} \left\|x^{-1}mf(m)\right\|,$$

where $0 < \tilde{c}_0 < 1$ is the constant in Proposition I.4.2, but related here to the decomposition $\mathbb{G} = \mathbb{T} \cdot \mathbb{H}$. Eventually we have

$$\left\|\ell(m)^{-1}f(m)\right\| \le \frac{1}{\tilde{c}_0} \operatorname{dist}(mf(m), \mathbb{T}), \tag{2.3.6}$$

and from (2.3.5) and (2.3.6)

$$\left\|\ell(m)^{-1}f(m)\right\| \leq \frac{\varepsilon}{\tilde{c}_0}\left\|mf(m)\right\|, \quad \text{for all } m \in \mathbb{M} \text{ with } \|m\| < \delta(\varepsilon). \tag{2.3.7}$$

Denoting by L the Lipschitz constant of ℓ (remember Proposition 2.3.6), we have

$$\begin{aligned}\|f(m)\| &\leq \|\ell(m)\| + \left\|\ell(m)^{-1}f(m)\right\| \\ &\leq L\|m\| + \varepsilon\|mf(m)\|, \qquad \text{for all } \|m\| < \delta(\tilde{c}_0\varepsilon) \\ &\leq (L+\varepsilon)\|m\| + \varepsilon\|f(m)\|, \qquad \text{for all } \|m\| < \delta(\tilde{c}_0\varepsilon).\end{aligned}$$

Hence $\|f(m)\| \leq \frac{L+\varepsilon}{1-\varepsilon}\|m\|$ that gives $\|mf(m)\| \leq \frac{1+L}{1-\varepsilon}\|m\|$ and finally

$$\|mf(m)\| \leq 2(1+L)\|m\|, \qquad \text{for } \varepsilon < 1/2.$$

Eventually, from (2.3.7) we obtain that, for all $\varepsilon > 0$,

$$\left\|\ell(m)^{-1}f(m)\right\| \leq \frac{2(L+1)}{\tilde{c}_0}\varepsilon\|m\|,$$

for all $m \in \mathbb{M}$ with $\|m\| < \delta(\varepsilon)$ and $\varepsilon < 1/2$. Hence ℓ is the intrinsic differential of f in 0 and the proof is concluded. □

2.4. One codimensional intrinsic graphs

In this final section we assume that $\mathbb{G} = \mathbb{M} \cdot \mathbb{V}$, with $\mathbb{M}$ and $\mathbb{V}$ complementary homogeneous subgroups, $\mathbb{V}$ *one dimensional* and (therefore) *horizontal* and consequently $\mathbb{M}$ a normal subgroup. Precisely we fix $V \in \mathfrak{g}_1$ such that

$$\mathbb{V} = \{\exp(tV) : t \in \mathbb{R}, V \in \mathfrak{g}_1\}.$$

Since $\mathbb{V} = \{\exp(tV) : t \in \mathbb{R}\}$, it can be identified with $\mathbb{R}$, it carries an order and we can define the supremum and the infimum of families of $\mathbb{V}$-valued functions.

2.4.1. Approximate tangent subgroups and intrinsic differentiability. In this subsection we prove that intrinsic differentiability and the (weaker property of) existence of an *approximate tangent subgroup* (see Definition 2.4.1) are equivalent notions for 1-codimensional intrinsic Lipschitz graphs (see Theorem 2.4.4).

We will use the following notations. Let $f, g_n : \mathcal{U} \subset \mathbb{M} \to \mathbb{V}$, for $n \in \mathbb{N}$ be defined for $m \in \mathcal{U}$ as $f(m) := \exp(\varphi(m)V)$ and as $g_n(m) := \exp(\psi_n(m)V)$ with $\varphi, \psi_n : \mathcal{U} \to \mathbb{R}$. We will say that $g_n \to f$ uniformly in $\mathcal{U}$ or that $g_n \to f$ locally L^1 in $\mathcal{U}$ if $\psi_n \to \varphi$ uniformly or locally L^1 in $\mathcal{U}$ (as real valued functions).
We define the *supergraph* E_f^+ and the *subgraph* E_f^- of f as

$$E_f^- := \{m\exp(tV) : m \in \mathcal{U}, t < \varphi(m)\}, \quad E_f^+ := \{m\exp(tV) : m \in \mathcal{U}, t > \varphi(m)\}.$$

Notice that, if $f : \mathbb{M} \to \mathbb{V}$ is continuous,

$$\overline{E_f^-} = \{m\exp(tV) : m \in \mathbb{M}, t \leq \varphi(m)\}, \quad \overline{E_f^+} = \{m\exp(tV) : m \in \mathbb{M}, t \geq \varphi(m)\}.$$

For any subgroup $\mathbb{T}$ complementary to $\mathbb{V}$ we define the 'half-spaces' $S_{\mathbb{G}}^+(\mathbb{T},\mathbb{V})$ and $S_{\mathbb{G}}^-(\mathbb{T},\mathbb{V})$ as

$$\begin{aligned}S_{\mathbb{G}}^-(\mathbb{T},\mathbb{V}) &:= \{g = g_{\mathbb{T}}g_{\mathbb{V}} : g_{\mathbb{V}} = \exp(tV), t < 0\}, \\ S_{\mathbb{G}}^+(\mathbb{T},\mathbb{V}) &:= \{g = g_{\mathbb{T}}g_{\mathbb{V}} : g_{\mathbb{V}} = \exp(tV), t > 0\}.\end{aligned}$$

Definition 2.4.1. Let $\mathcal{O}$ be relatively open in $\mathbb{M}$ and $f : \mathcal{O} \to \mathbb{V}$. We say that f is *approximately intrinsic differentiable* in $m \in \mathcal{O}$ if graph$\,(f)$ has an *approximate tangent (affine) subgroup* in $p = mf(m)$, that is if there is a homogeneous subgroup $\mathbb{T}$ such that

$$\mathbb{T} \text{ and } \mathbb{V} \text{ are complementary subgroups in } \mathbb{G},$$

and the following convergence of characteristic functions holds

$$\lim_{r\to 0^+} \mathbf{1}_{(E_f^-)_{r,p}} = \mathbf{1}_{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{V})} \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{G},\mathcal{L}^n),$$

where, for $p\in\mathbb{G}$ and for $r>0$,

$$(E_f^-)_{r,p} := \{q\in\mathbb{G} : p\cdot\delta_r q\in E_f^-\} = \delta_{\frac{1}{r}}\left(p^{-1}\cdot E_f^-\right).$$

REMARK 2.4.2. The notion of approximate intrinsic tangent subgroup is invariant by left translations. Observe also that it is given in the spirit of De Giorgi's approach to tangents of finite perimeter sets (see [**19**], [**25**] for Heisenberg groups and Theorem 2.4.14 here).

REMARK 2.4.3. If $\mathbb{T}=\mathrm{graph}\,(\ell)$ then $\mathbb{T}$ is the approximate tangent subgroup to graph (f) in p if and only if the sequence $\left(f_{p^{-1}}\right)_\lambda \to \ell$ as $\lambda\to 0^+$ in $L^1_{\mathrm{loc}}(M)$. Moreover, it is clear that $\mathbb{T}=\mathrm{graph}\,(\ell)$ is the tangent group of f at p (see Definition 2.3.12 and Remark 2.3.13) if and only if $\left(f_{p^{-1}}\right)_\lambda\to\ell$ as $\lambda\to 0$ uniformly on each compact subset of $\mathbb{M}$.

THEOREM 2.4.4. *Let $\mathbb{M}$ and $\mathbb{V}$ be complementary subgroups with $\mathbb{V}$ one dimensional and horizontal. Let $f:\mathbb{M}\to\mathbb{V}$ be an intrinsic L-Lipschitz function. Then, for all $m\in\mathbb{M}$, f is approximately intrinsic differentiable in m if and only if f is intrinsic differentiable in m.*

PROOF. Keeping in mind Remark 2.4.3, it is clear that differentiability yields approximate differentiability. On the other side, simply observe that, being f intrinsic Lipschitz, the dilated functions $\left(f_{p^{-1}}\right)_\lambda$, $\lambda>0$ are a precompact family of functions, being equi Hölder-continuous and equibounded (see Proposition 2.2.8). □

2.4.2. Finite perimeter sets and intrinsic Lipschitz graphs. The local boundedness of the $(Q-1)$-dimensional Hausdorff measure of intrinsic Lipschitz graphs (Theorem 2.2.9) yields that 1-codimensional graphs of intrinsic Lipschitz functions are locally the boundary of sets with locally finite $\mathbb{G}$-perimeter (see Theorem 2.4.10). We recall a few notions related to the perimeter of sets in $\mathbb{G}$. For more details and proofs, see [**32**], [**25**] and [**66**].

Let $\Omega\subset\mathbb{G}$ be open, we say that $f\in C^1_{\mathbb{G}}(\Omega)$ if X_if are continuous for $i=1,\dots,m_1$.

Moreover, we denote as $C^1_{\mathbb{G}}(\Omega,H\mathbb{G})$ the set of all horizontal sections

$$\phi := \sum_{i=1}^{m_1}\phi_i X_i$$

of $H\mathbb{G}$ whose coordinates ϕ_i belong to $C^1_{\mathbb{G}}(\Omega)$, for $i=1,\dots,m_1$. Each horizontal section is identified by its coordinates with respect to this moving frame $X_1(x),\dots,X_{m_1}(x)$. This way, an horizontal section ϕ is identified with a function $\phi=(\phi_1,\dots,\phi_{m_1}):\mathbb{R}^N\to\mathbb{R}^{m_1}$.

The *horizontal gradient* of a regular function $f:\mathbb{G}\to\mathbb{R}$ is the horizontal section

$$\nabla_{\mathbb{G}}f := \sum_{i=1}^{m_1}(X_if)X_i,$$

whose coordinates are $(X_1f,...,X_{m_1}f)$.

If $\phi=\sum_{i=1}^{m_1}\phi_iX_i\in C^1_{\mathbb{G}}(\Omega,H\mathbb{G})$ the *horizontal divergence* of ϕ is the real valued function

$$\mathrm{div}_{\mathbb{G}}(\phi) := \sum_{j=1}^{m_1}X_j\phi_j.$$

A function $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ is said to be of *bounded variation* in Ω if $f \in L^1(\Omega)$ and if

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup \left\{ \int_\Omega f(p) \mathrm{div}_{\mathbb{G}} \phi(p)\, dp : \phi \in \mathbf{C}^1_{\mathbb{G},0}(\Omega, H\mathbb{G}),\ |\phi(p)|_p \le 1 \right\} < \infty,$$

where $\mathbf{C}^1_{\mathbb{G},0}(\Omega, H\mathbb{G})$ denote compactly supported smooth sections of $H\mathbb{G}$.

We denote by $BV_{\mathbb{G}}(\Omega)$ the normed space of bounded variation functions and by $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ the vector space of functions in $BV_{\mathbb{G}}(\mathcal{U})$ for every open set $\mathcal{U} \subset\subset \Omega$.

THEOREM 2.4.5. *If $f \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ induces a Radon measure on Ω, still denoted by $\|\nabla_{\mathbb{G}} f\|$. Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(p)|_p = 1$ for $\|\nabla_{\mathbb{G}} f\|$-a.e. $p \in \Omega$, and*

$$\int_\Omega f(p) \mathrm{div}_{\mathbb{G}} \phi(p)\, dp = \int_\Omega \langle \phi, \sigma_f \rangle\, d\|\nabla_{\mathbb{G}} f\|, \qquad \textit{for every } \phi \in \mathbf{C}^1_0(\Omega, H\mathbb{G}).$$

Thus, the notion of $\nabla_{\mathbb{G}} f$ can be extended to functions $f \in BV_{\mathbb{G}}$ defining $\nabla_{\mathbb{G}} f$ as the vector valued measure

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner \|\nabla_{\mathbb{G}} f\| = \left(-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{G}} f\|, \dots, -(\sigma_f)_{m_1} \llcorner \nabla_{\mathbb{G}} f\right),$$

where $(\sigma_f)_j$ are the components of σ_f with respect to the moving frame X_j.

DEFINITION 2.4.6. A measurable set $E \subset \mathbb{G}$ is a set with locally finite $\mathbb{G}$-perimeter in Ω if its characteristic function $\mathbf{1}_E \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$. In this case we call perimeter of E in Ω the measure

$$|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}} \mathbf{1}_E\|$$

and we call generalized horizontal inward $\mathbb{G}$-normal to ∂E in Ω the horizontal vector

$$\nu_E(p) := -\sigma_{\mathbf{1}_E}(p).$$

As in the Euclidean setting, given $E \subset \mathbb{G}$, we define the *essential boundary* or *measure theoretic boundary* $\partial_{*,\mathbb{G}} E$ and, if E is a set with locally finite $\mathbb{G}$-perimeter, the *reduced boundary* $\partial^*_{\mathbb{G}} E$.

DEFINITION 2.4.7.
(I) Let $E \subset \mathbb{G}$ be a measurable set, we say that $p \in \partial_{*,\mathbb{G}} E$ if

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^N(E \cap U_c(p,r))}{\mathcal{L}^N(U_c(p,r))} > 0 \qquad \text{and} \qquad \limsup_{r \to 0^+} \frac{\mathcal{L}^N(E^c \cap U_c(p,r))}{\mathcal{L}^N(U_c(p,r))} > 0.$$

(II) Let E be a a set with locally finite $\mathbb{G}$-perimeter. We say that $p \in \partial^*_{\mathbb{G}} E$ if

$$|\partial E|_{\mathbb{G}}(U_c(p,r)) > 0 \qquad \text{for any } r > 0;$$

$$\text{there exists} \quad \lim_{r \to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}};$$

$$\lim_{r \to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \left\| \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}} \right\|_{\mathbb{R}^{m_1}} = 1.$$

LEMMA 2.4.8 (Differentiation Lemma). *Assume E is a set with locally finite $\mathbb{G}$-perimeter, then*

$$\lim_{r \to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}} = \nu_E(p), \qquad \textit{for } |\partial E|_{\mathbb{G}}\textit{-a.e. } p,$$

hence $|\partial E|_{\mathbb{G}}$ is concentrated on the reduced boundary $\partial^*_{\mathbb{G}} E$. Moreover we can redefine ν_E in a $|\partial E|_{\mathbb{G}}$-negligible set, by assuming that $\nu_E(p)$ is equal to the limit of the averages at all point $p \in \partial^*_{\mathbb{G}} E$.

LEMMA 2.4.9. *There is $c = c(\mathbb{G}) > 0$ such that $|\partial U_c(p,R)|_{\mathbb{G}} = cR^{Q-1}$, for all $p \in \mathbb{G}$ and for a.a. $R > 0$.*

Proof. Because of the invariance of the $\mathbb{G}$-perimeter under group translations, we may assume $p = e$. Moreover, by its homogeneity with respect to group dilations, we have but to show that $|\partial U_c(e,1)|_{\mathbb{G}} < \infty$. We notice first that $\partial U_c(e,1) = \{q \in \mathbb{G}\,;\, d_c(e,q) = 1\}$, and that $|\partial U_c(e,1)| = 0$.

We put $u_k(p) := \psi_k(d_c(p,e))$, $\psi_k : [0,\infty[\to [0,1]$ is a smooth function such that $\psi_k \equiv 1$ on $[0,1]$, $\psi_k \equiv 0$ on $[1+1/k,\infty[$, $|\psi'(t)| \le 2/k$ for $t \ge 0$. Clearly, $\nabla_{\mathbb{G}} u_k$ is supported in the anulus $B_c(e,1+1/k)\setminus U_c(e,1)$. On the other hand, it is well known that $|\nabla_{\mathbb{G}} d_c(\cdot,e)| \le 1$, so that $|\nabla_{\mathbb{G}} u_k| \le 2/k$. Since $|B_c(e,1+1/k)\setminus U_c(e,1)| \sim k^{-1}$ as $k \to \infty$, it follows that its total $\mathbb{G}$-variation is bounded uniformly with respect to k. Thus, we can conclude the proof because of the lower L^1-semicontinuity of the $\mathbb{G}$-variation, since $(u_k)_{k\in\mathbb{V}}$ tends in L^1 to the characteristic function of $U_c(e,1)$. □

The following Theorem is the group version of a (special case of a) celebrated theorem of Federer (see [4.5.11] of [**20**] and also Proposition 3.6.2 of [**1**] and [**67**]).

Theorem 2.4.10. *Let $\mathcal{O}$ be an open subset of $\mathbb{G}$. If the measure $\mathcal{S}_d^{Q-1} \llcorner \partial\mathcal{O}$ is locally finite in $\mathbb{G}$ then also $|\partial\mathcal{O}|_{\mathbb{G}}$ is locally finite in $\mathbb{G}$ and there is a geometric constant $c = c(\mathbb{G}) > 0$ such that*

$$|\partial\mathcal{O}|_{\mathbb{G}} \le c\, \mathcal{S}_d^{Q-1} \llcorner \partial\mathcal{O}.$$

In particular, if $\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty$ then $|\partial\mathcal{O}|_{\mathbb{G}}(\mathbb{G}) < \infty$.

Proof. First we assume that $\mathcal{O}$ is bounded. Then, by hypothesis, $\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty$ and, for each $\varepsilon > 0$, we can cover $\partial\mathcal{O}$ with a finite number of open metric balls $U_{\varepsilon,j}$, $j = 1,2,\dots$, with radius $r_{\varepsilon,j} < \varepsilon$, such that

$$\sum_j r_{\varepsilon,j}^{Q-1} < (1+\varepsilon)\, \mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty.$$

Denote

$$S_\varepsilon := \bigcup_j U_{\varepsilon,j} \quad \text{and} \quad \mathcal{O}_\varepsilon := \mathcal{O} \cup S_\varepsilon.$$

It follows

$$\mathcal{O}_\varepsilon \to \mathcal{O} \qquad \text{in } L^1(\mathbb{G}, m_{\mathbb{G}}),\ \text{as } \varepsilon \to 0, \tag{2.4.1}$$

because $m_{\mathbb{G}}(\mathcal{O}_\varepsilon \triangle \mathcal{O}) = m_{\mathbb{G}}(\mathcal{O}_\varepsilon \setminus \mathcal{O}) \le m_{\mathbb{G}}(S_\varepsilon) \le \omega_{\mathbb{G}}^Q \sum_j r_{\varepsilon,j}^Q < (1+\varepsilon)\omega_{\mathbb{G}}^Q\, \varepsilon \mathcal{S}_d^{Q-1}(\partial\mathcal{O})$. Here $\omega_{\mathbb{G}}^Q$ is the geometric constant such that $m_{\mathbb{G}}(U) = \omega_{\mathbb{G}}^Q\, r^Q$ for any metric ball U with radius r.

Now observe that

$$\partial\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}} = \emptyset, \qquad \operatorname{dist}(\partial\mathcal{O}_\varepsilon, \bar{\mathcal{O}}) > 0$$

and that

$$\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c = S_\varepsilon \cap \bar{\mathcal{O}}^c.$$

From these and general properties of the perimeter we get

$$\begin{aligned} |\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\mathbb{G}) &= |\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\bar{\mathcal{O}}^c) = |\partial(\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{G}}(\bar{\mathcal{O}}^c) \\ &= |\partial(S_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{G}}(\bar{\mathcal{O}}^c) = |\partial S_\varepsilon|_{\mathbb{G}}(\bar{\mathcal{O}}^c) \\ &\le |\partial S_\varepsilon|_{\mathbb{G}}(\mathbb{G}), \end{aligned}$$

and

$$|\partial S_\varepsilon|_{\mathbb{G}}(\mathbb{G}) \le \sum_j |\partial U_{\varepsilon,j}|_{\mathbb{G}}(\mathbb{G}) \le c \sum_j r_{\varepsilon,j}^{Q-1} < (1+\varepsilon) c\, \mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty,$$

where $c = c(\mathbb{G}) > 0$ is the geometric constant such that $|\partial U_{\varepsilon,j}|_{\mathbb{G}}(\mathbb{G}) = c\, r_{\varepsilon,j}^{Q-1}$. Hence eventually we have

$$|\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\mathbb{G}) < (1+\varepsilon) c\, \mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty. \tag{2.4.2}$$

From (2.4.1), (2.4.2) and the L^1-lower semicontinuity of the perimeter it follows

$$|\partial\mathcal{O}|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty.$$

Now we drop the assumption of the boundedness of $\mathcal{O}$. Let U be any fixed open ball such that $U \cap \partial\mathcal{O} \neq \emptyset$. Then, by hypothesis, $\mathcal{S}_d^{Q-1}(U \cap \partial\mathcal{O}) < \infty$. Notice that $\partial(U \cap \mathcal{O}) \subset \partial U \cup (\partial\mathcal{O} \cap U)$. An elementary covering argument yields that

$$\mathcal{S}_d^{Q-1}(\partial U_c(p,r)) = r^{Q-1}\mathcal{S}_d^{Q-1}(\partial U_c(p,1)) < \infty, \quad \text{for all } p \in \mathbb{G}.$$

Then,

$$\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

Thus, applying the first part of the proof to the bounded set $U \cap \mathcal{O}$, we have

$$|\partial(U \cap \mathcal{O})|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

Once more by the locality of the $\mathbb{G}$-perimeter,

$$|\partial\mathcal{O}|_{\mathbb{G}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{G}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

This achieves the proof of the first part of the theorem.

Finally, if $\partial\mathcal{O} \cap U$ is an intrinsic Lipschitz graph, then its measure theoretic boundary in U coincides with $\partial\mathcal{O} \cap U$ and the assertion follows from [**25**], Theorem 7.1. □

THEOREM 2.4.11. *If $f : \mathbb{M} \to \mathbb{V}$ is intrinsic Lipschitz then the subgraph E_f^- is a set with locally finite $\mathbb{G}$-perimeter.*

PROOF. The proof is a consequence of Theorems 2.2.9 and 2.4.10. □

LEMMA 2.4.12. *Let $f : \mathbb{M} \to \mathbb{V}$ be an intrinsic Lipschitz function and let $\Phi_f : \mathbb{M} \to \mathbb{G}$ be the parametrization of* graph(f) *given by $\Phi_f(m) = m \cdot f(m)$. Then there exists $c(\mathbb{M},\mathbb{V}) > 0$ such that*

$$(\Phi_f)_\sharp(m_{\mathbb{M}}) = c(\mathbb{M},\mathbb{V})\langle \nu, V\rangle\, |\partial E_f^-|_{\mathbb{G}},$$

where $m_{\mathbb{M}}$ is the the Haar measure of $\mathbb{M}$, that is, the $(N-1)$-dimensional Lebesgue measure on M, $(\Phi_f)_\sharp(m_{\mathbb{M}})$ denotes the image of $m_{\mathbb{M}}$ under the map Φ_f and $\nu := \nu_{E_f^-}$ is the horizontal generalized inward normal to E_f^- defined in Definition 2.4.6.

PROOF. We have already fixed $V \in \mathfrak{g}^1$ such that $\mathbb{V} = \{\exp \lambda V : \lambda \in \mathbb{R}\}$. Now choose $W_2, \cdots, W_N \in \mathfrak{g}$ such that $\{V, W_2, \cdots, W_N\}$ is a base of $\mathfrak{g}$ and such that

$$\mathbb{M} = \big\{ \exp(\sum_{i=2}^{N} \lambda_i W_i) : \lambda_i \in \mathbb{R}\big\}.$$

Let $\Psi : \mathbb{R}^N \to \mathbb{G} = \mathbb{R}^N$ be the linear map identified by the conditions

$$\Psi(\xi_1, 0 \ldots, 0) = \exp \xi_1 V, \qquad \Psi(0, \xi_2 \ldots, \xi_N) = \exp\big(\sum_{i=2}^{N} \xi_i W_i\big),$$
$$\Psi(\xi_1, \ldots, \xi_N) = \Psi(0, \xi_2, \ldots, \xi_N) \cdot \Psi(\xi_1, 0, \ldots, 0).$$

Ψ is $1-1$ and we denote

$$c_1(\mathbb{M},\mathbb{V}) := \left|\det \frac{\partial \Psi}{\partial \xi}\right|. \tag{2.4.3}$$

Since E_f^- has locally finite $\mathbb{G}$-perimeter then,

$$\int_{E_f^-} \operatorname{div}_{\mathbb{G}}\phi\, dm_{\mathbb{G}} = \int_{\mathbb{G}} \langle \nu, \phi\rangle\, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}). \tag{2.4.4}$$

Choose $\phi = \psi V$, with $\psi \in C^1_c(\mathbb{G})$. Then $\mathrm{div}_{\mathbb{G}}\phi = V(\psi)$ and from (2.4.4) we get

$$\int_{E_f^-} V(\psi)\, dm_{\mathbb{G}} = \int_{\mathbb{G}} \langle \nu, V\rangle\, \psi\, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \psi \in C^1_c(\mathbb{G}). \tag{2.4.5}$$

Notice also that $\dfrac{\partial}{\partial \xi_1}(\psi \circ \Psi) = V(\psi)\circ \Psi$. Denoting $f(m) = \exp(\varphi(m)V)$ with $\varphi : \mathbb{M} \to \mathbb{R}$, we have

$$\hat{E}_f^- := \Psi^{-1}(E_f^-) = \left\{\xi \in \mathbb{R}^N : \xi_1 < \varphi\big(\Psi(0,\xi_2,\dots,\xi_n)\big)\right\}, \tag{2.4.6}$$

and also

$$\Psi\left(\varphi(\Psi(0,\cdot),\cdot)\right) = \Phi_f \circ \Psi(0,\cdot). \tag{2.4.7}$$

By (2.4.6) and (2.4.3), we get

$$\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(\psi\circ\Psi)\, dm_{\mathbb{G}} = \frac{1}{c_1(\mathbb{M},\mathbb{V})}\int_{E_f^-} V(\psi)\, dm_{\mathbb{M}}\,. \tag{2.4.8}$$

On the other hand, by Fubini theorem, (2.4.6) and (2.4.7), it follows

$$\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(\psi\circ\Psi)\, dm_{\mathbb{G}} = \int_{\{\xi_1=0\}} \psi\circ\Phi_f\circ\Psi\; d\xi_2,\dots,d\xi_N. \tag{2.4.9}$$

Then, by [**44**], Theorem 1.19, and the area formula for linear maps, there is $c_2(\mathbb{M}) > 0$ such that

$$\begin{aligned}\int_{\mathbb{G}} \psi\, d\big((\Phi_f\circ\Psi(0,\cdot))_\sharp(m_{\mathbb{M}})\big) &= \int_{\{\xi_1=0\}} \psi\circ\Phi_f\circ\Psi(0,\cdot)\, d\xi_2,\dots,d\xi_N\\ &= c_2(\mathbb{M})\int_{\mathbb{M}} \psi\circ\Phi_f\;\; d(m_{\mathbb{M}}), \qquad \text{for all } \psi\in C^1_c(\mathbb{G}).\end{aligned}$$

By (2.4.5), (2.4.8) and (2.4.9) we get eventually

$$\int_{\mathbb{G}} \psi\, d\big((\Phi_f)_\sharp(m_{\mathbb{M}})\big) = c(\mathbb{M},\mathbb{V})\int_{\mathbb{G}} \langle\nu, V\rangle\,\psi\, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \psi\in C^1_c(\mathbb{G}).$$

The proof is completed.

□

Corollary 2.4.13. *Under the same assumptions of Lemma 2.4.12, we have*

$$(m_{\mathbb{M}})\big(\mathbb{M}\setminus \mathbf{P}_{\mathbb{M}}(\partial^* E_f^-)\big) = 0.$$

Proof. By Lemma 2.4.12,

$$(m_{\mathbb{M}})\big(\mathbb{M}\setminus \mathbf{P}_{\mathbb{M}}(\partial^* E_f^-)\big) = c(\mathbb{M},\mathbb{V})\int_{\mathbb{G}\setminus\partial^* E_f^-} \langle \nu, V\rangle\, d|\partial E_f^-|_{\mathbb{G}} = 0,$$

since $|\partial E_f^-|_{\mathbb{G}}(\mathbb{G}\setminus\partial^* E_f^-) = 0$.

□

2.4.3. A Rademacher type theorem. Let $\varphi : \mathcal{U}\subset\mathbb{M}\to\mathbb{V}$ be an intrinsic Lipschitz function, where $\mathcal{U}$ is a (relatively) open subset; we want to prove here a Rademacher's type result, connecting that is, *if φ is intrinsic Lipschitz in $\mathcal{U}$ then φ is intrinsic differentiable almost everywhere in $\mathcal{U}$.* Such a result was known only inside Heisenberg groups. We will extend it here to the Carnot groups of type $\star$ (see Definition 2.1.1). A key role in our proof is played by the following theorem.

THEOREM 2.4.14 ([**43**]). *Let $\mathbb{G}$ be a Carnot group of type $\star$. If $E \subset \mathbb{G}$ is a set with locally finite $\mathbb{G}$-perimeter, then*

$$\partial^*_{\mathbb{G}} E \text{ is one-codimensional } \mathbb{G}\text{-rectifiable,}$$

that is $\partial^*_{\mathbb{G}} E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a $\mathbb{G}$-regular hypersurface S_h (see Definition 2.4.16);

$$\nu_E(p) \text{ is the horizontal } \mathbb{G}\text{-normal to } S_h \text{ in } p, \text{ for every } p \in K_h,$$

$$|\partial E|_{\mathbb{G}} = \theta_c \mathcal{S}_c^{Q-1} \llcorner \partial^*_{\mathbb{G}} E,$$

where $\theta_c = \theta_c(\mathbb{G}, E, x) > 0$.

The starting point in our proof of the Rademacher theorem is the fact, proved in Theorem 2.4.11, that the subgraph E^-_φ of an intrinsic Lipschitz function $\varphi : \mathbb{M} \to \mathbb{V}$ is a set with locally finite $\mathbb{G}$-perimeter. From this and Theorem 2.4.14, it follows that at $|\partial E^-_f|_{\mathbb{G}}$-almost every point of graph$\,(\varphi)$ there is an approximate tangent coset. This in turn, together with the intrinsic Lipschitz assumption, yields the intrinsic differentiability of φ.

THEOREM 2.4.15. *Let $\mathbb{M}$ and $\mathbb{V}$ be complementary subgroups of a Carnot group $\mathbb{G}$ of type $\star$, with $\mathbb{V}$ one-dimensional and horizontal. Let $\mathcal{U} \subset \mathbb{M}$ be relatively open in $\mathbb{M}$ and $\varphi : \mathcal{U} \to \mathbb{V}$ be intrinsic Lipschitz. Then φ is intrinsic differentiable $(m_{\mathbb{M}})$-almost everywhere in $\mathcal{U}$. Notice that $m_{\mathbb{M}}$ is the Haar measure of $\mathbb{M}$.*

PROOF. By Theorem 2.2.10, we may assume that φ is intrinsic Lipschitz and defined on all of $\mathbb{M}$. Hence, by Theorem 2.4.11, we know that E^-_φ has locally finite $\mathbb{G}$-perimeter. Then, by Theorem 2.4.14 we know that there is a subset

$$\partial^*_{\mathbb{G}} E^-_\varphi \subset \partial E^-_\varphi = \text{graph}\,(\varphi)$$

such that

$$|\partial E^-_f|_{\mathbb{G}}\big(\text{graph}\,(\varphi) \setminus \partial^*_{\mathbb{G}} E^-_\varphi\big) \equiv |\partial E^-_\varphi|_{\mathbb{G}}\big(\partial E^-_\varphi \setminus \partial^*_{\mathbb{G}} E^-_\varphi\big) = 0,$$

and for all $p = m\varphi(m) \in \partial^*_{\mathbb{G}} E^-_\varphi$, φ is approximately intrinsic differentiable in m (remember Definition 2.4.1). By Proposition 2.4.4, φ is differentiable at any point $m \in \mathbb{M}$ such that

$$m\varphi(m) \in \partial^*_{\mathbb{G}} E^-_\varphi.$$

Finally, from Corollary 2.4.13, we have

$$(m_{\mathbb{M}})\big(\mathbf{P}_{\mathbb{M}}\big(\text{graph}\,(\varphi) \setminus \partial^*_{\mathbb{G}} E^-_\varphi\big)\big) = (m_{\mathbb{M}})\big(\mathbb{M} \setminus \mathbf{P}_{\mathbb{M}} \partial^*_{\mathbb{G}} E^-_\varphi\big) = 0,$$

that completes the argument. □

2.4.4. One-codimensional rectifiable sets. The results of the previous sections can be applied to prove the equivalence of two intrinsic notions of one-codimensional rectifiable sets in $\mathbb{G}$. For a related and deeper analysis about equivalence of different notions of intrinsic rectifiable sets in $\mathbb{H}_n$ we refer to [**45**].

We begin by recalling the definitions of intrinsic regular hypersurfaces (or one codimensional surfaces), of their tangent groups or tangent cosets as well as the related implicit function theorem and the notion of intrinsically rectifiable set.

DEFINITION 2.4.16. $S \subset \mathbb{G}$ is a $\mathbb{G}$*-regular hypersurface* if for every $p \in S$ there exist a neighborhood $\mathcal{U}$ of p and a function $f \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

$$S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\};$$

$$df_q \neq 0 \quad \text{for all } q \in \mathcal{U}.$$

The *tangent affine group* or *tangent coset* to S at p is the coset of the kernel of df_p, i.e.

$$T_{\mathbb{G}} S(p) := \{q \in \mathbb{G} : \; df_p(p^{-1} \cdot q) = 0\}.$$

Theorem 2.35 of [**27**] can be restated as follows

THEOREM 2.4.17. *Let S be a $\mathbb{G}$-regular hypersurface in $\mathbb{G}$. Then, for all $p \in S$ there are an open $\mathcal{U} \ni p$, complementary subspaces $\mathbb{M}$ and $\mathbb{V}$, with $\mathbb{V}$ one-dimensional and horizontal, a relatively open $\mathcal{V} \subset \mathbb{M}$ and an intrinsic Lipschitz and intrinsic differentiable function $\varphi : \mathcal{V} \to \mathbb{V}$ such that*

$$S \cap \mathcal{U} = \{m\varphi(m) : m \in \mathcal{V}\}.$$

Moreover, for all $q = m\varphi(m) \in S \cap \mathcal{U}$, the tangent affine group $T_{\mathbb{G}}S(q)$ coincides with the tangent coset to graph (φ) *in q as introduced in Definition 2.3.12.*

It follows, in particular, that the definition of $T_{\mathbb{G}}S(x)$ does not depend on the particular function f defining the surface S.

Finally we recall two definitions of one codimensional intrinsic rectifiable sets. Each of them mimics a natural definition used in Euclidean context.

DEFINITION 2.4.18. $\Gamma \subset \mathbb{G}$ is said to be *one codimensional $\mathbb{G}$-rectifiable* if there are $\mathbb{G}$-regular hypersurfaces $(S_j)_{j\in\mathbb{V}}$ such that

$$\mathcal{H}_c^{Q-1}\big(\Gamma \setminus \bigcup_{j\in\mathbb{V}} S_j\big) = 0.$$

$\Gamma \subset \mathbb{G}$ is said to be *one codimensional $\mathbb{G}_L$-rectifiable* if there are one-codimensional intrinsic Lipshitz graphs G_i, $i = 1, 2, \dots$, such that,

$$\mathcal{H}_c^{Q-1}\big(\Gamma \setminus \bigcup_{j\in\mathbb{V}} G_j\big) = 0.$$

In both definitions, $\mathcal{H}_c^{Q-1}$ is the $(Q-1)$-Hausdorff measure related to the distance d_c.

PROPOSITION 2.4.19. *If $\mathbb{G}$ is a type $\star$ Carnot group then $\Gamma \subset \mathbb{G}$ is one codimensional $\mathbb{G}_L$-rectifiable if and only if it is one codimensional $\mathbb{G}$-rectifiable.*

PROOF. Since intrinsic regular surfaces are locally graphs of intrinsic Lipschitz functions it follows that the scope of the second definition is larger than the first one. On the other direction, by definition, each G_i is the graph of a one-dimensional valued, intrinsic Lipschitz function $\varphi_i : \mathcal{C}_i \subset \mathbb{M}_i \to \mathbb{V}_i$. By the extension theorem for one dimensional intrinsic Lipschitz functions (see [**24**]), we can assume that $\mathcal{C}_i = \mathbb{M}_i$ for all i. Hence, by Theorem 2.4.11, the subgraph of φ_i has locally finite $\mathbb{G}$-perimeter and, eventually, it is $\mathbb{G}$-rectifiable, by the structure theorem for sets of locally finite $\mathbb{G}$-perimeter proved. This proves that all of Γ is $\mathbb{G}$-rectifiable. $\square$

Afterword

The research project undertaken in thesis is far from being concluded. Here we mention a few points of directions for future work.

– Boundedness of the Riesz transform for Rumin's Laplacian and the strong L^p-decomposition of $L^pE_0^k$ on the Heisenberg group. A basic question in analysis on Riemannian manifold is the L^p-boundedness of the Riesz transform, defined as $d\Delta_k^{-\frac{1}{2}}$, where we denote by Δ_k the Hodge Laplacian. We define the Riesz transform for Rumin's Laplacian as

$$\mathcal{R}_R = \begin{cases} d_R\Delta_R^{-\frac{1}{2}} & \text{if } k \leq n \\ \Delta_R^{-\frac{1}{2}}d_R & \text{if } k > n\,. \end{cases}$$

Notice that for any $k = 0, \ldots, 2n$, $\mathcal{R}_R$ is an operator of order 0. We are confident that techniques and results of this current work will allow us to prove the following statements:

(1) $\mathcal{R}_R : L^pE_0^k \to L^pE_0^{k+1}$ is bounded for $1 < p < \infty$, $k = 0, \ldots, 2n$;

(2) the space $L^pE_0^k$ admits the strong L^p-decomposition

$$L^pE_0^k = (\mathcal{R}_R)_{k-1}L^pE_0^{k-1} \bigoplus (\mathcal{R}_R)_k^* L^pE_0^{k+1},$$

where $(\mathcal{R}_R)_k$ denotes the Riesz transform for Rumin's Laplacian acting on $L^pE_0^k$ and $(\mathcal{R}_R)_k^*$ denotes its adjoint.

– Analysis of Rumin's Laplacian on H-type groups. The extension of the detailed spectral analysis that we have carried out in the case of the Heisenberg group to more general groups is far from being trivial or straightforward. The role played by the CR structure on $\mathbb{H}_n$ is of key importance and other tools have to been developed. Also, the Lefschetz decomposition of horizontal forms $\mathbb{H}_n$ on is rather special and does not seem to have an obvious generalization on more general groups.

The most obvious setting on which try to extend the results of Chapter 1 are is the case of H-type groups. We began this work and we wish to pursue it in the near future.

– Problem of extension of Rademacher's type theorem for one dimensional valued intrinsic Lipschitz functions inside a general Carnot group and extension of Rademacher's theorem for intrinsic Lipschitz functions valued in higher dimensional horizontal homogeneous subgroups.

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