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Spacelike hypersurfaces in Generalized Robertson-Walker spacetimes

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Introduction

Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds are of great interest both in physical and in mathematical research. The most relevant aspect is, probably, their role in General Relativity. For instance, they are involved in the initial value formulation of the field equations. This latter consists in specifying an initial state of the universe and, then, describing the evolution of this data. One obtains a spacetime foliated by spacelike hypersurfaces and, assuming the mean curvature to be constant, the constraint equations are enormously simplified.

In Chapter 3 we study spacelike hypersurfaces in generalilzed Robertson-Walker spacetimes. Our results are contained in the paper [ARS]. Consider an open interval $I \subseteq \mathbb{R}$, a smooth real function $\varrho \in \mathcal{C}^\infty(I)$ and a Riemannian manifold $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$. Following the terminology introduced in [ARoS], we define *generalized Robertson-Walker spacetime* (GRW spacetime) $-I \times_\varrho \mathbb{P}^n$ the product manifold $I \times \mathbb{P}^n$ endowed with the metric

$$\langle , \rangle := -\pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_{\mathbb{P}}^*(\langle , \rangle_{\mathbb{P}}),$$

where π_I and $\pi_{\mathbb{P}}$ denote the projections onto I and $\mathbb{P}^n$ respectively.

When the fiber $\mathbb{P}^n$ has constant sectional curvature and $\dim(\mathbb{P}^n) = 3$ we have the classical Robertson-Walker spacetimes. In this perspective, these latter are the spatially homogeneous and spatially isotropic model of the universe. Homogeneity and isotropy (known as the *cosmological principle*) seem to be reasonable assumptions if one wants to describe the universe on large scale. These notions have both an intellectual credibility and several empirical confirmations. On smaller scales, however, they are no more appropriate, so that GRW spacetimes represent a motivated generalization.

The mathematical interest on constant mean curvature spacelike hypersurfaces is, for instance, due to the fact that they exhibit nice Bernstein properties. Calabi in [C] showed that the only complete maximal spacelike hypersurfaces in the Minkowski space $\mathbb{R}^{n+1}_1$, with $n \le 4$, are spacelike hyperplanes. Later on, Cheng and Yau in [CY3] extended this theorem to any dimension. This result shows the fact that, as in the Riemannian asset, constant mean curvarture spacelike hypersurfaces in Lorentzian manifolds are rigid in some sense. One way to generalize the Bernstein property mentioned above is to consider a spacetime foliated with many constant mean curvature hypersurfaces and to investigate under which geometric conditions an immersed constant mean curvature hypersurface is one of the slices of the foliation (see [M1]).

Extending an idea of Salavessa [S], we first obtain a curvature estimate for spacelike graphs and derive some geometric consequences. Namely, we prove

Theorem 1. *Let $(\mathbb{P}^n, \langle,\rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and let $u \in \mathcal{C}^\infty(\mathbb{P}^n)$, with $u : \mathbb{P}^n \to I = (a,b)$, be such that its graph $\Sigma(u)$ is a spacelike hypersurface of $-\mathbb{R}\times_\rho \mathbb{P}^n$ with bounded hyperbolic angle θ. We have*

$$\inf_{\mathbb{P}^n} H \le \frac{\sinh(\theta^*)}{n\rho(u^*)} h(\mathbb{P}^n) \qquad \text{if } \rho' \le 0,$$

$$\sup_{\mathbb{P}^n} H \ge -\frac{\sinh(\theta^*)}{n\rho(u_*)} h(\mathbb{P}^n) \qquad \text{if } \rho' \ge 0,$$

where H is the mean curvature of $\Sigma(u)$, $\theta^* = \sup_{\mathbb{P}} \theta < \infty$, $u^* := \sup_{\mathbb{P}} u \leq b$, $u_* := \inf_{\mathbb{P}} u \geq a$ and $h(\mathbb{P}^n)$ denotes the Cheeger constant of $\mathbb{P}^n$.
We allow $\rho(u^*)$ and $\rho(u_*)$ to be zero and in these cases we have the trivial inequalities.

Then, we furnish a height estimate for spacelike graphs in a GRW-spacetime

Theorem 2. *Consider a generalized Robertson-Walker spacetime $-I \times_\varrho \mathbb{P}^n$, and assume on $\mathbb{P}^n$ the validity of the weak maximum principle for the Lorentzian mean curvature operator. Let $\Sigma(u)$ be an entire spacelike maximal graph in $-I \times_\varrho \mathbb{P}^n$, with $I = (a,b)$, $-\infty \leq a < b \leq +\infty$, which is not a slice. Then*

$$\text{either } u^* = b \text{ or } u^* \leq \inf\{\lambda \in I : \varrho'(t) < 0 \text{ on } [\lambda, b)\}. \tag{1}$$

Similarly,

$$\text{either } u_* = a \text{ or } u_* \geq \sup\{\mu \in I : \varrho'(t) > 0 \text{ on } (a, \mu]\}. \tag{2}$$

As a consequence we obtain a nice rigidity result.

Corollary 3. *Consider a generalized Robertson-Walker spacetime $-I \times_\varrho \mathbb{P}^n$, and assume on $\mathbb{P}^n$ the validity of the weak maximum principle for the Lorentzian mean curvature operator. For $a, b \in I$, $a < b$, let*

$$(a,b) \times \mathbb{P}^n = \{(t,x) : a < t < b, x \in \mathbb{P}^n\}$$

be an open slab in $-I \times_\varrho \mathbb{P}^n$, and assume that there exists $t_0 \in (a,b)$ with the property that $\varrho'(t) > 0$ on $[a, t_0)$ and $\varrho'(t) < 0$ on $(t_0, b]$. Then the only entire maximal graph contained in $(a,b) \times \mathbb{P}^n$ is the slice $u \equiv t_0$.

The proof of Theorem 2 is based on a local form of the weak maximum principle introduced in [AMR] with the name *open weak maximum*

principle. Chapter 2 of the present dissertation is devoted to a discussion on this useful analytic tool. We state the open weak maximum principle (open-WMP) for a class of differential operators including the Lorentzian mean curvature operator and we prove the equivalence between the open-WMP and its classical version (see [ARS]). Moreover, we give a geometric sufficient condition which guarantees the validity of the WMP for Lorentzian mean curvature operator.
In the second part of Chapter 3 we give some height estimates for constant kth-mean curvature spacelike hypersurfaces immersed in GRW spacetimes and in the special case of Lorentzian products. These estimates relate the height of the hypersurface (or a portion of it) with its kth mean curvature. Again, the argument is based on the open-WMP for differential operators arising naturally in this context, the *Newton operators*. To give an example, in the case of products, we prove

Theorem 4. *Let $F : \Sigma^n \to -\mathbb{R}\times\mathbb{P}^n$ be a stochastically complete spacelike hypersurface with constant mean curvature $H > 0$. Suppose that for some $\alpha > 0$*

$$\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha. \tag{3}$$

Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \varnothing$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$. Assume

$$\beta^2 = \sup_{\Omega} \Theta^2 < \frac{\alpha + H^2}{\alpha}. \tag{4}$$

Then

$$F(\Omega) \subset \left[\frac{(1-\beta)H}{H^2 - \alpha(\beta^2 - 1)}, 0 \right] \times \mathbb{P}^n. \tag{5}$$

Our height estimates can be put in the context of many results in the same spirit, both in the Riemannian and in the Lorentzian asset. The first step in this direction is a theorem of Serrin [Se].

Theorem 5. *Let H be a positive constant and $u(x, y)$ a $\mathcal{C}^2$ solution of the equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 2H \tag{6}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. If u is continuous up to the boundary of Ω, then we have

$$m - \frac{1}{H} \leq u < M \qquad \text{in } \Omega, \tag{7}$$

where m and M are respectively the minimum and the maximum boundary values of u.

Later on, many generalizations of this result have been proposed by geometers. For instance, one can consider higher order mean curvatures, hypersurfaces which are not necessarily graphs or an ambient space that is not the Euclidean space and even not Riemannian. In Chapter 1 we give a brief survey on this topic in order to put our results contained in Chapter 3 in the right context.
At last, in Chapter 4 we consider the problem of height estimates for compact hypersurfaces in the Affine space. The problem, here, is that we do not have in general a distance function between points in an affine set. What we do is introducing a distance between points and *special* hyperplanes that is invariant under affine transformations. Using this notion we achieve a sharp estimate that looks formally the same as in the Riemannian case. This result is contained in the paper [Sc].

Chapter 1

Rigidity results for constant mean curvature hypersurfaces: a brief survey.

Constant mean curvature (briefly, CMC) surfaces in $\mathbb{R}^3$ are the mathematical model for soap films and bubbles. This is due to the fact that mean curvature is strictly linked to a variational problem regarding the area functional. Specifically, given a simple, closed curve (i.e. a *Jordan curve*) $\Gamma \subset \mathbb{R}^3$, surfaces with boundary Γ minimizing the area must have zero mean curvature. Analogously, non-zero constant mean curvature hypersurfaces immersed in an ambient manifold are critical points of the area functional for variations preserving volume. The surface tension of a soap film works to minimize the surface area, so that the investigation of these materials from a physical viewpoint motivate a geometric study of CMC hypersurfaces.
One of the most relevant facts in this topic is that CMC hypersurfaces are rigid in some sense, since their shape can not be any. In 1962 Alexandrov showed that the only compact CMC hypersurface embedded in $\mathbb{R}^{n+1}$ is the round sphere [Al], while H. Hopf in [Ho] obtained the

same conclusion for immersed closed surfaces of genus $g = 0$. The sharpness of these results is proved by the fact that there exist examples of genus greater than zero, non-embedded hypersurfaces of constant mean curvature (see [We], [K]).
On the other hand, if $u \in \mathcal{C}^2(\mathbb{R}^n)$ and its graph is a minimal hypersurface of $\mathbb{R}^{n+1}$ then u must be linear, provided $n < 7$. This problem is known as *Bernstein problem*, since he introduced it and proved it in the case $n = 2$. Fleming in [F] gave a new proof of the theorem deducing it from the fact that the falsity of Berstein theorem would imply the existence of minimal cones in $\mathbb{R}^3$. De Giorgi in [DeG] improved Fleming's idea showing that, if in $\mathbb{R}^n$ Bernstein theorem does not hold true, then there exist minimal cones in $\mathbb{R}^{n-1}$. Hence he extended Bernstein result to $\mathbb{R}^4$. Almgren proved the non-existence of minimal cones in $\mathbb{R}^4$ and Simons generalized this result up to dimension 7 (see [Alm], [Si]). Therefore we have the validity of Bernstein theorem through dimension 8. In [Si] Simons gave also examples of locally stable cones in $\mathbb{R}^{2m}$ for $m \leq 4$. Finally, in [BDeGG] the authors proved that Simons' cones are indeed global minimizing and furnished examples of minimal graph that are not hyperplane in $\mathbb{R}^n$ with $n \geq 9$.
The case of prescribed boundary compact CMC hypersurfaces required a different approach. Consider a Jordan curve $\Gamma \subset \mathbb{R}^n$ and a real parameter H. The Plateau problem for constant mean curvature consists in looking for a hypersurface S of constant mean curvature H whose boundary is Γ. In the literature there are several existence results with some assumptions on Γ and H. For instance, Heinz in [H54] proved that if Γ is contained in the unit ball about the origin of $\mathbb{R}^3$ and $H < \frac{1}{8}(\sqrt{17} - 1)$ there exists a solution to Plateau problem for H-constant mean curvature surfaces with boundary Γ. Later on, Werner [Wer] improved Heinz' assumption to $H < \frac{1}{2}$. The sharpest result in this direction is due to Hildebrandt [Hi] who assumed $H \leq 1$. We remark

that the authors above showed the existence of a *small* solution to the Plateau problem, that is a surface contained in the unit disc of $\mathbb{R}^3$.
On the other hand, for a geometric reasoning one can imagine that the hypersurface S could cease to exist if H is larger than some constant related to the prescribed boundary Γ. In [H69] Heinz proved that the Plateau problem mentioned above has no solution $x \in \mathcal{C}^2(B, \mathbb{R}^3) \cap \mathcal{C}^0(\overline{B}, \mathbb{R}^3)$ with

$$H > \frac{l(\Gamma)}{k(\Gamma)}, \tag{1.1}$$

where $l(\Gamma)$ is the length of the boundary, $k(\Gamma) := |\int_\Gamma x \times dx|$ and B denotes the unit disc in $\mathbb{R}^2$. The boundary Γ is assumed to be rectifiable. This result is related to the fact that there can not exist a CMC graph over a circle of radius larger than $\frac{1}{H}$ (see [H55]). Our Theorem 39, proved in Chapter 3, is linked to Heinz result (1.1).
In [Se] Serrin gives a sufficient condition for the surface solution to the Plateau problem to be unique. Specifically, he proved that if $0 < H \leq 1$ there are exactly two solutions of constant mean curvature H contained in the unit ball and with no self intersections. In the same paper, the author gives a limitation on the diameter of the region where a CMC surface spanning a Jordan curve Γ contained in the unit disc must lie, accordingly with its mean curvature H. Namely, he gave the following theorem

Theorem 6. *Let Γ be a Jordan curve contained in the closed unit ball B about the origin of $\mathbb{R}^3$. Suppose that S is a solution of the Plateau problem for constant mean curvature $H > 0$ with no self intersections. Then it must be contained in the open ball of radius $1 + 2/H$ about the origin.*

Note that the result is sharp in the sense that constant $1 + 2/H$ can not be improved. Indeed, consider a sphere Σ of radius $1/H$ intersecting the unit ball along a circle. Choosing this latter to be as little as we want, moving the center of Σ away from the origin, the part of the sphere lying outside the unit ball furnish an example of solution for the H-Plateau problem which is not contained in a ball with radius smaller than $1 + 2/H$.

In order to prove Theorem 1, Serrin obtained a height estimate for a constant mean curvature graph over a planar domain Ω. He showed the following maximum principle

Theorem 7. *Let H be a positive constant and $u(x, y)$ a $\mathcal{C}^2$ solution of the equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 2H \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. If u is continuous up to the boundary of Ω, then we have

$$m - \frac{1}{H} \leq u < M \qquad \text{in } \Omega, \tag{1.3}$$

where m and M are respectively the minimum and the maximum boundary values of u.

Proof. We will not give the original proof by Serrin. Instead, we use an argument inspired to an idea of Korevaar, Kusner, Meeks and Solomon, who obtained an analogous height estimate in the hyperbolic space (see [KKMS]).

First of all, note that there can not be an interior maximum of the graph of u since it has positive mean curvature. This proves the right-hand side of (1.3). Suppose, now, without loss of generality, that $m = 0$. Just observe that the mean curvature equation (1.2) is invariant under vertical traslations. Let $\{e_1, e_2, e_3\}$ be a Darboux frame for the graph, $\{\omega^1, \omega^2, \omega^3\}$ the corresponding co-frame and $\{\omega^i_j\}^3_{i,j=1}$ the connection forms. Denote by x the position vector of the graph and $f := \langle x, a\rangle$, where $a \in \mathbb{R}^3$ and $\langle , \rangle$ is the canonical scalar product in $\mathbb{R}^3$. We have

$$
\begin{aligned}
df &= \langle e_i, a\rangle \omega^i =: f_i \omega^i \\
f_{ij}\omega^j &= d\langle e_i, a\rangle - \langle e_k, a\rangle \omega_i^k \\
&= \langle e_s, a\rangle \omega_i^s + \langle e_3, a\rangle \omega_i^3 - \langle e_k, a\rangle \omega_i^k \\
&= \langle e_3, a\rangle h_{ij}\omega^j
\end{aligned}
$$

where $i = 1,2$. Hence $\Delta\langle x, a\rangle = f_{ii} = 2H\langle e_3, a\rangle$, denoting with Δ the Laplacian of the metric on the graph induced by the isometric immersion in $\mathbb{R}^3$. Analogously, denoting with $z := \langle e_3, a\rangle$ we have

$$
\begin{aligned}
dz &= \langle e_k, a\rangle \omega_3^k = -h_{ks}\langle e_k, a\rangle \omega^s := z_s\omega^s \\
z_{sj}\omega^j &= -d(h_{ks}\langle e_k, a\rangle) + h_{kt}\langle e_k, a\rangle \omega_s^t \\
&= -dh_{ks}\langle e_k, a\rangle - h_{ks}\langle e_t, a\rangle \omega_k^t - h_{ks}\langle e_3, a\rangle \omega_k^3 + h_{kt}\langle e_k, a\rangle \omega_s^t \\
&= -h_{ksj}\langle e_k, a\rangle \omega^j - h_{js}h_{jk}\langle e_3, a\rangle \omega^k.
\end{aligned}
$$

Hence $\Delta\langle e_3, a\rangle = -|II|^2\langle e_3, a\rangle - 2H_{,k}\langle e_k, a\rangle$.
Therefore, we introduce the well known function (see, for instance, [R])

$$
\psi := H\langle x, a\rangle + \langle e_3, a\rangle,
$$

where we set $a := (0,0,1)$.
We choose

$$
e_3 := (-u_x, -u_y, 1)\frac{1}{\sqrt{1+|\nabla u|^2}}.
$$

We have

$$
\begin{cases}
\Delta\psi = (2H^2 - |II|^2)\langle e_3, a\rangle \le 0 \text{ on } \Omega \\
\psi = \frac{1}{\sqrt{1+|\nabla u|^2}} \ge 0 \text{ on } \partial\Omega,
\end{cases}
$$

recalling that the mean curvature H is constant. Applying the classical maximum principle, we deduce $\psi \geq 0$ on Ω, that is

$$Hu \geq -\frac{1}{\sqrt{1+|\nabla u|^2}} \geq -1.$$

We conclude $u \geq -1/H$. □

The proof we gave for Serrin's theorem works equal in several dimensions, so we may extend the result for CMC graphs in $\mathbb{R}^n$.
In [Ro] Rosenberg generalized this height estimate for constant k-th mean curvature hypersurfaces Σ embedded in space forms with boundary $\partial\Sigma$ contained in a hyperplane. For $2 \leq k \leq n$, k-th mean curvatures are defined via the symmetric functions of the principal curvatures and represent the natural generalization of the mean curvature H. The bound found by Rosenberg for embedded constant k-mean curvare hypersurfaces is $h \leq 2(1/H_k)^{\frac{1}{k}}$, where h denotes the height over the hyperplane containing the boundary. The author first assumes the hypersurface to be a graph and achieved the estimate $h \leq 1/H_k{}^{\frac{1}{k}}$. Then, using a moving plane argument, he observes that one can consider an embedded hypersurface, obtaining the height estimate $h \leq 2(1/H_k)^{\frac{1}{k}}$.
Another possible generalization of Serrin's height estimate constists in substituting $\mathbb{R}^3$ with $M \times \mathbb{R}$, where (M, g) is a Riemannian manifold. This is the natural ambient for the graph of a real function defined on a differential manifold. Consider, first, the case where M is a Riemannian surface. The first height estimate in this asset is due to Hoffman, de Lira and Rosenberg, see [HLR].

Theorem 8. *Let Σ be a constant mean curvature graph over a compact region of M. Assume the boundary $\partial\Sigma$ to be contained in a slice, that is an hypersurface of the form $M \times \{a\}$, where a is a real number (assume WLOG $a = 0$). If the Gaussian curvature of M satisfies $K_M \geq 2\tau$, with*

$\tau < 0$, and the mean curvature of Σ satisfies $|H|^2 \geq |\tau|$ then

$$h \leq \frac{H}{H^2 - |\tau|}, \tag{1.4}$$

denoting with h the projection onto $\mathbb{R}$ restricted to Σ.

This result has been improved by Aledo, Espinár, Gálvez in [AEG]. We remark here that a-priori height estimates have a great importance in the study of topological and geometric properties of submanifolds and, in terms of a classification of CMC hypersurfaces, they reveal a useful tool in order to achieve rigidity and uniqueness results. For instance, in [HLR] the authors used their height estimate in order to show a non-existence result. Specifically

Theorem 9. *Let Σ be a non-compact surface embedded in $M^2 \times \mathbb{R}$ with constant mean curvature H. Assume M^2 to be closed and to have Gaussian curvature bounded from below by 2τ. In case $\tau < 0$, assume also $H^2 > |\tau|$. Then Σ can not lie in a halfspace.*

We call *halfspace* a subset of $M^2 \times I \subset M^2 \times \mathbb{R}$ where I is an interval of the type $(-\infty, a]$ or $[a, +\infty)$, with $a \in \mathbb{R}$. The fact that Σ can not lie in a halfspace gives informations on its topology at infinity. Indeed, it means that Σ must have *top* and *bottom* ends. We can think of the number of *ends* of a manifold, roughly speaking, as a way of counting the connected components of a manifold at infinity. We say that a topological space X has at least k ends if there exists an open, relatively compact set $A \subset X$ such that $X - A$ has k connected non-compact components. Several authors introduced independently the notion of ends, mostly because they are linked with compactification of topological spaces. We give here Freudenthal's definition (see [Fr])

Definition 10. Let X be a connected, locally connected, locally compact Hausdorff space. We say that an *end* of X is an equivalence class of descending sequences $\{G_j\}$ of connected, open sets with compact boundaries such that $\bigcap_{j=1}^{\infty} G_j = \emptyset$. We say that two such sequences are equivalent if each set of one sequence is contained in some set of the other one and vice versa.

In [CR] Cheng and Rosenberg generalized the height estimates in $M^2 \times \mathbb{R}$ mentioned above to the case of n dimensions and of higher order mean curvatures. Specifically, they considered compact vertical graphs in $M^n \times \mathbb{R}$ with positive constant r-mean curvature H_r, for $1 \leq r \leq n$, and with boundary contained in $M^n \times \{0\}$. In order to obtain their result they assume positive sectional curvature for generic r and sectional curvature bounded from below for $r = 2$.
In [AD] Alías and Dajczer generalized the estimate contained in [HLR] in the following sense. Consider $f : \Sigma^n \to M^n \times \mathbb{R}$ a compact hypersurface of constant mean curvature H and boundary $\partial\Sigma^n$ contained in the slice M_0. Assume $\mathrm{Ric}_M \geq n/(n-1)\alpha$ for some $\alpha \leq 0$, $H^2 \geq |\alpha|$ and $\Theta := \bar{g}(N, \partial_t) \leq 0$, where $\bar{g}$ is the product metric, N is the outward directed unit normal and ∂_t the coordinate field in $\mathbb{R}$. Then we have

$$f(\Sigma^n) \subset M \times \left[0, \frac{H}{H^2 - |\alpha|}\right].$$

Moreover, they introduced a further generalization. Let $I \subseteq \mathbb{R}$ be an open interval, $\varrho : I \to \mathbb{R}^+$ be a smooth function and $(\mathbb{P}^n, g)$ a complete n-dimensional Riemannian manifold. The warped product $\bar{M} := I \times_\varrho \mathbb{P}^n$ is the product manifold $I \times \mathbb{P}^n$ endowed with the Riemannian metric

$$g_{\bar{M}} := \pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_M^*(g),$$

where π_I and π_M denote the projections onto I and $\mathbb{P}^n$ respectively. Unlike the product case, in general warped products the slices $M_t = \{t\} \times M$ have not necessarily zero mean curvature, specifically M_t has mean curvature $H_t := \varrho'(t)/\varrho(t)$. This class of manifolds, as Montiel pointed out in [M], represents a natural choice in order to extend Alexandrov's rigidity result to the case of non-constant sectional curvature ambient space. The slices $\mathbb{P}^n_t$ form a foliation of the manifold $\bar{M}$ with constant mean curvature complete hypersurfaces. An interesting problem can consist, now, in finding some geometric conditions on an immersed hypersurface forcing it to be a slice.

Alías and Dajczer in [AD] obtained, then, height estimates for constant mean curvature hypersurfaces in some special cases of the above introduced warped products, namely where $\varrho(t) := e^t$ and $\varrho(t) := \cosh(t)$. These spaces are known as *pseudo-hyperbolic spaces* (see [T]).

Theorem 11. *Let* $f : \Sigma^n \to I \times_{e^t} \mathbb{P}^n$ *be a compact hypersurface of constant mean curvature* $H \notin [0,1)$ *and nonempty boundary* $f(\partial\Sigma^n) \subset \mathbb{P}^n_t$*. Assume that* $\mathrm{Ric}_{\mathbb{P}} \geq 0$ *and that the angle function* Θ *does not change sign. Set* $C := \log(H/(H-1))$*. Then we have*

1. *if* $H < 0$ *then* $f(\Sigma^n) \subset [\tau + C, \tau] \times \mathbb{P}^n$*;*
2. *if* $H > 1$ *then* $f(\Sigma^n) \subset [\tau, \tau + C] \times \mathbb{P}^n$*;*
3. *if* $H = 1$ *then* $f(\Sigma^n) \subset \mathbb{P}^n_t$*.*

Theorem 12. *Let* $f : \Sigma^n \to I \times_{\cosh(t)} \mathbb{P}^n$ *be a compact hypersurface of constant mean curvature and nonempty boundary* $f(\partial\Sigma^n) \subset \mathbb{P}^n_0$*. Assume that* $\mathrm{Ric}_{\mathbb{P}} \geq -1$ *and that the angle function* Θ *does not change sign. Set* $\tanh(C) := 1/H$*. Then we have*

1. *if* $H < -1$ *then* $f(\Sigma^n) \subset [C, 0] \times \mathbb{P}^n$*;*
2. *if* $H > -1$ *then* $f(\Sigma^n) \subset [0, C] \times \mathbb{P}^n$*;*
3. *if* $H = 0$ *then* $f(\Sigma^n) \subset \mathbb{P}^n_0$*.*

In [AMR] the authors gave a generalization of [AD, Theorem 3.5] considering open subsets with nonempty boundary in complete noncompact hypersurfaces. This generalization is due to the main result of the paper, the equivalence of the weak maximum principle to a local version of it, the *open weak maximum principle* (for further details see Chapter 2). They proved the following

Theorem 13. *Let $f : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a complete hypersurface with constant mean curvature $H > 0$. Assume that $\beta := \sup_\Sigma \Theta < 0$ and suppose that $\mathbb{K}_\mathbb{P} \geq -\alpha$ and $H^2 > \alpha$, for some $\alpha > 0$. Furthermore, assume that for the Weingarten operator A of Σ*

$$|A(x)| \leq G(r(x))$$

for some $G \in \mathcal{C}^1([0, +\infty))$ satisfying

$$(i)\ G(0) > 0 \qquad (ii)\ G'(t) \geq 0 \qquad 1/G(t) \notin L^1(+\infty),$$

where $r(x)$ denotes the distance in Σ from some fixed origin o. If $\Omega \subset \Sigma$ is an open set with $\partial\Omega \neq \varnothing$ for which $f(\Omega)$ is contained in a slab and $f(\partial\Omega) \subset \mathbb{P}_0$ then

$$f(\Omega) \subset \left[0, \frac{(1+\beta)H}{H^2 - \alpha}\right] \times \mathbb{P}^n.$$

In the same paper the authors generalized Theorem 13 for constant higher order mean curvatures.
In Chapter 3 we find some results, in the spirit of the above mentioned ones, where the ambient space is a Lorentzian manifold. These results are contained in the paper [ARS]. Our focus will be constant mean curvature *spacelike* hypersurfaces, that are hypersurfaces where the restriction of the Lorentzian ambient metric is Riemannian. The study of these objects is motivated by physical and mathematical interests. Indeed, Lorentzian geometry is the mathematical base for general rel-

ativity and CMC spacelike hypersurfaces play a central role in this physical area. For instance, they are involved in the so called initial value formulation of the field equations. Just to give a flavour, consider as initial data a triple (Σ, g, K) where (Σ, g) is a Riemannian manifold and K a symmetric tensor field on it. One looks for a spacetime $(M, \langle , \rangle)$ satisfying Einstein equations (see Chapter 3) and possessing a spacelike hypersurface isometric to (Σ, g), with second fundamental form K. This spacetime is foliated by hypersurfaces Σ_t representing the time evolution of Σ (see [W] and [HE] for further details).
From a geometric point of view, spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{R}^n_1$ have a nice Bernstein-type property. In [C] Calabi proved that the only complete maximal spacelike hypersurfaces in $\mathbb{R}^n_1$, with $n \le 5$, are spacelike hyperplanes. Later on, Cheng and Yau in [CY3] extended Calabi's result to any dimension. In another direction, this result can be generalized to prove that spacelike hyperplanes are the only complete constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}_1$ with image of the Gauss map contained in a geodesic ball of the hyperbolic space (see [Ai], [P], [X]).
Now, as we commented above in the Riemannian case, we can look for a generalization of Bernstein theorem in a larger class of Lorentzian manifolds. Namely, we consider the natural Lorentzian analogue to Riemannian warped products. Following the terminology introduced in [ARoS], we call *generalized Robertson-Walker spacetime* $\bar{M} := -I \times_\varrho \mathbb{P}^n$ the product manifold $I \times \mathbb{P}^n$ endowed with the metric

$$\langle , \rangle := -\pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_{\mathbb{P}}^*(\langle , \rangle_{\mathbb{P}}),$$

where I is an open interval, $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$ is a Riemannian manifold and ϱ is a positive smooth function on I. As in the Riemannian case, $\bar{M}$ is foliated by constant kth mean curvature spacelike slices $\{t\} \times \mathbb{P}^n$, for

$1 \leq k \leq n$, and Bernstein problem consists in investigating under which circumstances a complete k-CMC spacelike hypersurface has to be a leave of the foliation.

In [ARoS] Alías, Romero and Sanchez showed that a CMC compact spacelike hypersurface in a spatially closed generalized Robertson Walker spacetime obeying the *timelike convergence condition* must be a slice, except very exceptional cases. We say that generalized Robertson-Walker spacetime is spatially closed if the Riemannian factor is compact. Montiel in [M1] studied the same problem obtaining that the only CMC compact hypersurfaces in a generalized spatially closed Robertson Walker spacetime satisfying the *null convergence condition* are the spacelike slices, unless the case of round umbilical spheres in De Sitter space. We recall that a spacetime satisfies the timelike convergence condition if its Ricci curvature is non-negative on timelike vectors while it obeys the null convergence condition if its Ricci curvature is non-negative on lightlike (null) directions. Observe that the first requirement implies the second one because of continuity. For the case of constant higher order mean curvature see [AC1].

In [AIR] Alías, Impera and Rigoli considered the case of complete noncompact hypersurfaces. The assumptions they made on the ambient space is that the sectional curvature of the Riemannian factor is bounded from below and the warping function satisfies $\log(\varrho)'' \leq 0$. Regarding hypersurface Σ in exam, apart the constancy of a higher order mean curvature, the authors assumed $\sup_{\Sigma}|H_1| < \infty$ and that the hypersurface Σ is contained in a slab, conditions automatically satisfied in the compact case. The conclusion is that, in the above assumptions, Σ is forced to be a slice.

As in the Riemannian case, a-priori height estimates for hypersurfaces immersed in spacetimes settle in the context of uniqueness results as they furnish a quantitative measure of the deviation of the hypersurface from being a slice. For instance, in [deL] de Lima found a sharp height estimate for compact spacelike hypersurfaces with constant rth mean curvature immersed in Lorentz-Minkowski space and used it to have informations on the topology of such hypersurfaces at infinity. Namely, he obtained

Theorem 14. *Let $\psi : \Sigma^n \to \mathbb{R}^{n+1}_1$ be a compact spacelike hypersurface whose boundary is contained in $\{0\} \times \mathbb{R}^n$. Suppose that H_r is a positive constant and that the hyperbolic image of Σ is contained in a geodesic ball of radius $\varrho > 0$ and center e_{n+1} in the hyperbolic space $\mathbb{H}^n$. Then the height h of Σ satisfies the inequality*

$$|h| \le \frac{\cosh \varrho - 1}{H_r^{\frac{1}{r}}}. \tag{1.5}$$

For the case of spacelike surfaces in $\mathbb{R}^3_1$ we cite the work of López [Lo].
As an application of estimate (1.5), the author considered a complete spacelike non-zero r-CMC hypersurface with one end immersed in $\mathbb{R}^{n+1}_1$ with hyberbolic image contained into a geodesic ball of $\mathbb{H}^n$ and he showed that its end can not be divergent.

Remark 15. *Consider the hyperbolic caps*

$$\Sigma^n_\lambda := \{x \in \mathbb{R}^{n+1}_1 | \langle x, x\rangle = -\lambda^2, \lambda < x_{n+1} < \sqrt{1+\lambda^2}\}, \quad \lambda > 0,$$

where $\langle , \rangle$ denotes the Lorentzian metric

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

They are spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$ with $H_r = 1/\lambda^r > 0$ and with hyperbolic image contained in a ball of $\mathbb{H}^n$ centered in e_{n+1} and with *radius* $\varrho := \cosh^{-1}\sqrt{1-\frac{1}{\lambda^2}}$. *Hence, their height satisfies*

$$h = \sqrt{1+\lambda^2} - \lambda = \frac{\cosh\varrho - 1}{H_r^{\frac{1}{r}}}$$

and we deduce that estimate (1.5) *is sharp.*

In [CL] Colares and Lima generalized Theorem 14 considering as ambient space a Lorentzian product $-\mathbb{R}\times\mathbb{P}^n$. They gave

Theorem 16. *Let* Σ^n *be a compact spacelike hypersurface immersed in a Lorentzian product* $-\mathbb{R}\times\mathbb{P}^n$*, where* $(\mathbb{P}^n, g)$ *is a Riemannian manifold with non-negative constant sectional curvature* $k_{\mathbb{P}}$*. Suppose that* Σ^n *has positive constant* r*th mean curvature* H_r*, for some* $1 \le r \le n$*, and that its boundary* $\partial\Sigma^n$ *is contained in the slice* $\{0\}\times\mathbb{P}^n$*. Then the vertical height of* Σ *satisfies the inequality*

$$|h| \le \frac{C-1}{H_r^{\frac{1}{r}}},$$

where $C := \sup_{\partial\Sigma}|\Theta|$.

As an application of Theorem 16 the authors found the following result regarding the topology at infinity of the hypersurface in exam

Corollary 17. *Let* Σ^n *be a complete spacelike hypersurface immersed in a spatially closed Lorentzian product* $-\mathbb{R}\times\mathbb{P}^n$*. Suppose that one of the following conditions is satisfied*

1. *the Riemannian fiber* $\mathbb{P}^n$ *has non-negative Ricci curvature and* Σ^n *has positive constant mean curvature*

2. *the Riemannian fiber* $\mathbb{P}^n$ *has non-negative constant sectional curvature and* Σ^n *has positive constant* r*th mean curvature* H_r*, for some* $1 \le r \le n$.

If the hyperbolic angle Θ is bounded then the number of ends of Σ^n is not one.

We conclude here our brief survey on rigidity of constant mean curvature hypersurfaces in Riemannian and Lorentzian ambient spaces. In Chapter 3 we report some results obtained in the same spirit of the ones commented in the present Chapter and which settle in the geometric context outlined here. They are contained in the paper [ARS] and the main tool in the proof on many of them is the *open weak maximum principle*, a useful analytic tool that we are going to introduce and discuss in the following Chapter.

Chapter 2

The open weak maximum principle

2.1 The maximum principle at infinity in a nutshell

The weak maximum principle turns out to be a powerful analytic tool in the study of several geometric problems. In [AMR] the authors introduce a local form of the principle for a wide class of operators and they prove that it is equivalent to the classical version. Although the reasoning in the proof of this result seems to be straightforward, the new form of the principle, called by the authors the *open weak maximum principle*, allows them to yeald some interesting geometric applications.

Clearly, a smooth function f on a compact Riemannian manifold (M, g)

admits a point $p \in M$ such that

$i)$ $f(q) \leq f(p)$ for any $q \in M$;

$ii)$ $\nabla f(p) = 0$;

$iii)$ $\mathrm{Hess} f(p)$ is negative semi-definite.

We may look for an analogous property replacing compactness with completeness and assuming f to be bounded above. For instance, we may question if, fixed any $\varepsilon > 0$, there exists a point $p \in M$ such that

$i)$ $f(p) > \sup(f) - \varepsilon$;

$ii)$ $|\nabla f(p)| < \varepsilon$;

$iii)$ $Hess(f)(p) < \varepsilon g,$ in the sense of the quadratic forms.

This problem has been introduced by Omori in [Om]. He observes that, while in $\mathbb{R}$ each smooth function bounded from above satisfies the property discussed, in general complete Riemannian manifolds this may fail to be true. Indeed, the author exhibits a complete metric in $\mathbb{R}^2$ and a function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ bounded from above such that there exists $a > 0$ for which

$$m(p) = \max\{Hess_p(f)(X_p, X_p) \mid \|X_p\| = 1\} > a,$$

for every $p \in \mathbb{R}^2$.

A central point in this topic is that the property investigated, that has a pure analytic formulation, looks strongly linked with the geometry of the underlying manifold. Omori in [Om] proved that if the sectional curvature of a complete Riemannian manifold (M, g) has a lower bound, then for any $f \in \mathcal{C}^\infty(M)$ bounded from above ($f^* := \sup_M f < \infty$) and for any $\varepsilon > 0$ there exists a point $p \in M$ such that $f(p) > f^* - \varepsilon$, $|\nabla f(p)| < \varepsilon$

and $m(p) = \max\{Hess_p(f)(X_p, X_p) \mid \|X_p\| = 1\} < \varepsilon$. Therefore, in some cases the geometry of (M, g) makes available an analytic tool in the form of a maximum principle and we come full circle using it in order to investigate further geometric properties of the manifold. In fact, Omori introduced this maximum principle for geometric purposes: he proved that a complete manifold isometrically immersed in a cone of $\mathbb{R}^n$, with sectional curvature bouded from below admits some point where the second fundamental form is positive definite, for a choice of the unit normal. Later on, Yau refined the principle replacing the Hessian with its trace, the Laplacian, and he used it to study several geometric problems (see [Y], [CY2]). Again, the geometry of (M, g) is involved in Yau's result, since he assumed a lower bound on the Ricci tensor.
We can now introduce the following

Definition 18. We say that the *Omori-Yau maximum principle* for the Laplacian holds on the Riemannian manifold (M, g) if, for any $u \in \mathcal{C}^2(M)$ bounded from above, there is a sequence $\{x_n\} \subset M$ such that

$$\text{i)}u(x_n) > u^* - 1/n \quad \text{ii)}|\nabla u(x_n)| < 1/n \quad \text{iii)}\Delta u(x_n) < 1/n. \tag{2.1}$$

As Pigola, Rigoli and Setti pointed out in [PRS], it turns out that, in several geometric applications, the gradient condition plays no role. So they relaxed (2.1) introducing the following

Definition 19. We say that on (M, g) the *weak maximum principle* for the Laplacian holds if, in the same assumptions as in Definition 18, we have the validity of i) and iii) in (2.1).

The adjustment in Definition 19 may seem to be technical but it is indeed deep. In fact, in [PRS] the authors proved that the weak maximum principle is equivalent to the stochastic completeness of (M, g).

This is a striking result if one is interested in studying a manifold from the stochastic analysis viewpoint. Since we do not adopt this perspective in our work, we only observe that one could think of Brownian completeness as the property that the time-life of each random path is almost surely infinite.

Despite what appears in the works of Omori and Yau, the validity of the maximum principle in Definition 18 does not depend so strictly on curvature bounds. In fact, in [PRS1, Theorem 1.9] the authors give a sufficient condition for the validity of the maximum principle which is of functional theoretic type. The innovative viewpoint contained in this result is based on the existence of an auxiliary proper function γ, whose gradient and Laplacian satisfy some estimates. This new approach paved the way for several new applications, since it does not require any curvature bound. The argument consists in transforming the function u in exam (see Definition 18) by means of γ, in order to obtain a new function with a *finite* maximum. The idea of passing from bounded functions to functions admitting a maximum was introduced by Cheng and Yau in [Y], [CY2] and is inspired to an old work of Ahlfors [A].

We give, below, a recent improvement of [PRS1, Theorem 1.9], see [AAR, Theorem B], which deals with a large class of linear differential operators we are going to define. Indeed, we stress the fact that the differential operator arising naturally in several geometric applications is not necessarly the Laplacian. Hence, it turns out to be desirable to have a global maximum principle as in Definition 18 or 19 where condition *iii*) is replaced by an analogous control on the appropriate operator.

Let T be a symmetric positive semi-definite $(0,2)$-tensor field on a Riemannian manifold (M,g) and X a vector field. We denote by $L = L_{T,X}$

the differential operator

$$Lu := \mathrm{Tr}(T \circ \mathrm{Hess}(u)) + \mathrm{div}(T\nabla u) - g(X, \nabla u), \tag{2.2}$$

where $u \in \mathcal{C}^2(M)$. If $X = (\operatorname{div} T)^\sharp$, then

$$Lu = \mathrm{Tr}(T \circ \mathrm{Hess}(u))$$

is a typical trace operator, where we denoted with $\sharp : T^*M \to TM$ the musical isomorphism. Moreover, if T coincides with the metric g, we have

$$Lu = \Delta u - g(\nabla u, X),$$

which is the so called X-Laplacian, denoted by Δ_X, an operator arising in the study of general soliton structures.
We can now state

Theorem 20. [AAR, Theorem B] *Let (M, g) be a Riemannian manifold and L as in (2.2). Let $q \in \mathcal{C}^0(M)$, $q \geq 0$ and suppose*

$$q > 0 \qquad \textit{outside a compact set.}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$\begin{cases} \gamma(x) \to +\infty & \textit{as } x \to +\infty \\ q(x)L\gamma(x) \leq B & \textit{outside a compact set} \\ |\nabla\gamma| \leq A & \textit{outside a compact set,} \end{cases} \tag{2.3}$$

for some constants $A, B > 0$. If $u \in \mathcal{C}^2(M)$ and $u^* < \infty$ then there exists a sequence $\{x_k\} \subset M$ such that

$$\textit{i) } u(x_k) > u^* - 1/k \quad \textit{ii) } |\nabla u(x_k)| < 1/k \quad \textit{iii) } q(x_k)Lu(x_k) < 1/k \tag{2.4}$$

for each $k \in \mathbb{N}$.

In the case $q \equiv 1$, conclusion (2.4) is the Omori-Yau maximum principle. On the other hand, if $q \not\equiv 1$, we refer to (2.4) as the *q-maximum principle.*

Now, we move to the case of nonlinear differential operators of geometric interest. In [PRS1] the authors introduced the Omori-Yau maximum principle for the so-called ϕ-Laplacian. Consider a function $\phi \in \mathcal{C}^1((0,+\infty)) \cap \mathcal{C}^0([0,+\infty))$ such that

$$i)\ \phi(0) = 0 \quad ii)\ \phi(t) > 0 \text{ for } t > 0 \quad iii)\ \phi(t) \le At^{\delta} \text{ for } t \in [0,\varepsilon), \tag{2.5}$$

for some constants $A, \delta, \varepsilon > 0$. The differential operator

$$L_{\phi}(u) = \operatorname{div}\left(|\nabla u|^{-1}\phi(|\nabla u|)\nabla u\right) \quad u \in \mathcal{C}^1(M) \tag{2.6}$$

is called ϕ-Laplacian. Even for $u \in \mathcal{C}^2(M)$, the vector field in brackets may fail to be $\mathcal{C}^1$ where $\nabla u = 0$. In these cases the divergence has to be interpreted in distributional sense. If $\phi(t) = t$ we have the Laplacian; more generally, if $\phi(t) = t^{p-1}$, $p > 1$, we have the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In case $\phi(t) = \frac{t}{(1+t^2)^{\alpha}}$, $\alpha > 0$, we have the generalized mean curvature operator

$$L_{\phi} u = \operatorname{div}\left(\frac{\nabla u}{(1 + |\nabla u|^2)^{\alpha}}\right).$$

Theorem (1.9) in [PRS1] has been proved with a technique that strictly uses the linearity of the Laplacian. Hence, in order to achieve an Omori-Yau maximum principle for non-linear operators, such as the ϕ-Laplacian, one must change perspective. We give, below, a result which guarantees the validity of the q-maximum principle for the ϕ-Laplacian. This theorem is stated in [AAR] for a class of nonlinear differential operators but we will restrict, for the moment, to L_ϕ.

Theorem 21. [AAR, Theorem B"]
Let (M,g) a Riemannian manifold and ϕ as above. Let $q \in \mathcal{C}^0(M)$, $q \geq 0$, $q > 0$ outside some compact set $K \subset M$ and $1/q \in L^1_{loc}(M)$. Assume that there exists a telescoping exhaustion of relatively compact open sets $\{\Sigma_j\}_{j\in\mathbb{N}}$ such that $K \subset \Sigma_1$, $\bar{\Sigma}_j \subset \Sigma_{j+1}$ for every $j \in \mathbb{N}$ and, for any pair $\Omega_1 = \Sigma_{j_1}$, $\Omega_2 = \Sigma_{j_2}$, $j_1 < j_2$, and for each $\varepsilon > 0$, there exists $\gamma \in \mathcal{C}^0(M - \Omega_1) \cap \mathcal{C}^1(M - \overline{\Omega}_1)$ with the following properties:

i) $\gamma \equiv 0$ on $\partial\Omega_1$;
ii) $\gamma > 0$ on $M - \Omega_1$;
iii) $\gamma \leq \varepsilon$ on $\Omega_2 - \Omega_1$;
iv) $\gamma(x) \to \infty$ when $x \to \infty$;
v) $q(x)L\gamma \leq \varepsilon$ on $M - \bar{\Omega}_1$ in the weak sense;
vi) $|\nabla u| < \varepsilon$ on $M - \bar{\Omega}_1$.

Then if $u \in \mathcal{C}^1(M)$, $u^* = \sup u < \infty$, for each $\eta > 0$ we have

$$\inf_{B_\eta}\{q(x)L_\phi u(x)\} \leq 0 \qquad \textit{in the weak sense,}$$

where $B_\eta := \{x \in M | u(x) > u^* - \eta$ and $|\nabla u(x)| < \eta\}$.

Note that condition $v)$ means that

$$L_\phi\gamma \leq \frac{\varepsilon}{q} \text{ weakly on } M - \overline{\Omega}_1,$$

that is, $\forall\psi \in \mathcal{C}^\infty_c(M - \overline{\Omega}_1)$, $\psi \geq 0$,

$$\int_{M-\overline{\Omega}_1} |\nabla\gamma|^{-1}\phi(|\nabla\gamma|)g(\nabla\gamma, \nabla\psi) + \frac{\varepsilon}{q}\psi \geq 0.$$

In the next section we are going to introduce a *local* version of the WMP for a differential operator we will use in Chapter 3, the *Lorentzian mean curvature operator*. We will also give a geometric sufficient condition which guarantees the validity of the WMP for the Lorentzian mean curvature operator. This condition is about volume growth in our manifold. In [PRS1] it is shown that, if the volume growth of geodesic balls of a Riemannian manifold (M, g) is sub-exponential, the weak maximum principle holds on (M, g) for L_ϕ. Moreover, function u is not assumed to be bounded from above but, more generally, we require to have a control on its growth. Namely, we have the following

Theorem 22. [PRS1, Theorem 4.1] *Let (M, g) be a complete manifold. Given $\sigma, \mu \in \mathbb{R}$ let*

$$\eta = \mu - (1+\delta)(1-\sigma), \tag{2.7}$$

and suppose that $\sigma \geq 0$, $\sigma - \eta > 0$. Assume that

$$\liminf_{r \to +\infty} \frac{\log \mathrm{Vol}(B_r)}{r^{\sigma-\eta}} = d_0 < \infty. \tag{2.8}$$

Let $u \in \mathcal{C}^2(M)$ be such that $|\nabla u|^{-1}\phi(|\nabla u|)\nabla u$ is a vector field of class at least $\mathcal{C}^1$ and suppose that

$$\hat{u} := \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^\sigma} < +\infty. \tag{2.9}$$

Then, given $\gamma > 0$ such that

$$\Omega_\gamma := \{x \in M \mid u(x) > \gamma\} \neq \varnothing,$$

we have

$$\inf_{\Omega_\gamma} (1 + r(x))^\mu \,\mathrm{div}(|\nabla u|^{-1}\phi(|\nabla u|)\nabla u) \leq A d_0 \max\{\hat{u}, 0\}^\delta C(\sigma, \eta, \delta),$$

with

$$C(\sigma,\eta,\delta) = \begin{cases} 0 & \textit{if } \sigma = 0 \\ (\sigma-\eta)^{1+\delta} & \textit{if } \ \sigma > 0, \eta < 0 \\ \sigma^{\delta}(\sigma-\eta) & \textit{if } \ \sigma \geq 0, \eta \geq 0. \end{cases}$$

2.2 The open weak maximum principle for the Lorentzian mean curvature operator

As we will see in Chapter 3, a natural non-linear differential operator arising in the study of spacelike graphs in a Lorentzian manifold is the *Lorentzian mean curvature operator*. Let (M,g) be a Riemannian manifold. For $0 < \omega, A, \delta < +\infty$ let $\varphi \in \mathcal{C}^0([0,\omega)) \cap \mathcal{C}^1((0,\omega))$ satisfying

$$i)\ \varphi(0) = 0 \tag{2.10}$$

$$ii)\ \varphi(t) > 0 \qquad \text{for } 0 < t < \omega \tag{2.11}$$

$$iii)\ \varphi(t) \leq At^{\delta} \qquad \text{for } 0 < t < \omega. \tag{2.12}$$

We introduce a class of operators which generalize the ϕ-Laplacian, namely we define

$$L_{\varphi}(u) := \operatorname{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u) \quad \text{in the weak sense,} \tag{2.13}$$

where

$$u \in \mathcal{A}_{\omega}(M) = \{u \in \mathrm{Lip}_{\mathrm{loc}}(M) : |\nabla u| < \omega \text{ and } |\nabla u|^{-1}\varphi(|\nabla u|) \in L^1_{\mathrm{loc}}(M)\}.$$

Operator (2.13) differs from the ϕ-Laplacian because here function φ is not defined on the whole real line.

If $\varphi(t) = t/\sqrt{1-t^2}$ we obtain the Lorentzian mean curvature operator

$$L_\varphi u = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right). \tag{2.14}$$

Following [AAR], for $q(x) \in \mathcal{C}^0(M)$, $q(x) > 0$, we say that the q-weak maximum principle (shortly q-WMP) holds on M for the operator L_φ if for each $u \in \mathcal{A}_\omega(M)$ bounded from above and for each $\gamma < u^* = \sup_M u$ we have

$$\inf_{\Omega_\gamma}\{q(x)L_\varphi u\} \le 0 \tag{2.15}$$

in the weak sense, where $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$. In case $q(x)$ is a positive constant we will simply say that L_φ satisfies the WMP (the weak maximum principle). Recall that the validity of (2.15) in the weak sense means that for any $\gamma < u^*$ and for each $\varepsilon > 0$ there exists $\psi \in \mathcal{C}_c^\infty(\Omega_\gamma)$, $\psi \ge 0$ and $\psi \not\equiv 0$ such that

$$-\int_{\Omega_\gamma} |\nabla u|^{-1}\varphi(|\nabla u|)g(\nabla u, \nabla\psi)dv < \varepsilon \int_{\Omega_\gamma} \frac{\psi}{q}dv.$$

Now, following [AMR] we introduce a local version of the weak maximum principle for operators L_φ.

Definition 23. Let (M, g) be a Riemannian manifold, $q \in \mathcal{C}^0(M)$, $q > 0$ and $\varphi \in \mathcal{C}^0([0,\omega)) \cap \mathcal{C}^1((0,\omega))$, for $\omega > 0$, satisfying structural conditions (2.10), (2.11) and (2.12).
We say that the *open q-weak maximum principle* holds on M for the operator L_φ if for each $f \in \mathcal{C}^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \varnothing$ and for each $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{A}_\omega(\Omega)$ satisfying

$$\begin{cases} \text{i) } q(x)L_\varphi v \ge f(v) \text{ on } \Omega; \\ \\ \text{ii) } \sup_\Omega v < +\infty, \end{cases} \tag{2.16}$$

we have that either

$$\sup_{\Omega} v = \sup_{\partial\Omega} v \tag{2.17}$$

or

$$f(\sup_{\Omega} v) \leq 0. \tag{2.18}$$

Note that $i)$ in (2.16) has to be understood in the weak sense, that is, for each $\psi \in \mathcal{C}_c^\infty(\Omega)$, $\psi \geq 0$,

$$-\int_{\Omega_\gamma} |\nabla u|^{-1}\varphi(|\nabla u|)g(\nabla u, \nabla\psi) \geq \int_{\Omega_\gamma} \frac{f(v)}{q(x)}\psi.$$

In our geometric results of Chapter 3 we will make use of the local form of the principle contained in Definition 23. Therefore, it is important showing that this latter is actually equivalent to the classical version.

Theorem 24. *In the above assumptions, the validity of the q-WMP for the operator L_φ is equivalent to that of the open q-WMP.*

Proof. Assume that the q-WMP holds for the operator L_φ on M and let Ω, f, v be as above, with v satisfying (2.16). We suppose $\sup_{\partial\Omega} v < \sup_\Omega v$ and we claim $f(v^*) \leq 0$. Fix $\sup_{\partial\Omega} v < \gamma < v^*$ and define $\Omega_\gamma := \{x \in \Omega \ : \ v(x) > \gamma\}$. In our setting $\Omega_\gamma \subset \Omega$. Consider the function

$$u := \begin{cases} v & \text{on } \Omega_\gamma \\ \gamma & \text{on } M - \Omega_\gamma \end{cases}$$

and observe that $u \in \mathcal{A}_\omega(M)$ and that $u^* = \sup_M u = v^* = \sup_\Omega v$.

Choose $\gamma < \sigma < u^* = v^*$. Since we are supposing the validity of the q-WMP, then for each $\varepsilon > 0$ there exists $\psi \in \mathcal{C}_c^\infty(\Omega_\sigma)$, $\psi \geq 0$ and $\psi \not\equiv 0$, such that

$$-\int_{\Omega_\sigma} |\nabla u|^{-1}\varphi(|\nabla u|)g(\nabla u, \nabla\psi) \leq \int_{\Omega_\sigma} \frac{\varepsilon}{q(x)}\psi. \tag{2.19}$$

On the other hand, since $\operatorname{supp}(\psi) \subset \Omega_\sigma \subset \Omega$, and since we are assuming $q(x)L_\phi v \ge f(v)$ on Ω in the weak sense, we also have that

$$\int_{\Omega_\sigma} \frac{f(v)}{q(x)}\psi \le -\int_{\Omega_\sigma} |\nabla v|^{-1}\varphi(|\nabla v|)g(\nabla v, \nabla\psi). \tag{2.20}$$

Note that $u = v$ on Ω_σ and therefore from (2.19) and (2.20) we deduce

$$\int_{\Omega_\sigma} \frac{f(v)}{q(x)}\psi \le \int_{\Omega_\sigma} \frac{\varepsilon}{q(x)}\psi.$$

Now, fix $\varepsilon > 0$ and, recalling that f in continuous, consider σ sufficiently close to v^* so that $f(v) > f(v^*) - \varepsilon$ on Ω_σ. Hence from the above we deduce

$$(f(v^*) - \varepsilon)\int_{\Omega_\sigma} \frac{\psi}{q(x)} \le \varepsilon\int_{\Omega_\sigma} \frac{\psi}{q(x)},$$

where ψ depends on the choice of ε and σ. Since $\int_{\Omega_\sigma} \frac{\psi}{q(x)} > 0$, we deduce $f(v^*) < 2\varepsilon$. But the choice of ε is arbitrary, so that $f(v^*) \le 0$.

Assume, now, the validity of the open q-WMP and consider $u \in \mathcal{A}_\omega(M)$ bounded above. Fix $\gamma < u^*$; we claim that $\inf_{\Omega_\gamma}\{q(x)L_\varphi u\} \le 0$ in the weak sense. By contradiction, suppose that there exists $\varepsilon > 0$ such that for each $\psi \in \mathcal{C}_c^\infty(\Omega_\gamma)$, $\psi \ge 0$

$$-\int_{\Omega_\gamma} |\nabla u|^{-1}\varphi(|\nabla u|)g(\nabla u, \nabla\psi) > \int_{\Omega_\gamma} \frac{\varepsilon}{q(x)}\psi.$$

This means $q(x)L_\varphi u \ge \varepsilon$ weakly on Ω_γ. Note that

$$\gamma = \sup_{\partial\Omega_\gamma} u < \sup_{\Omega_\gamma} u = u^* < \infty. \tag{2.21}$$

Applying the open q-WMP with $\Omega = \Omega_\gamma$, $v = u_{|\Omega_\gamma} \in \mathcal{A}_\omega$ and $f = \varepsilon$, inequality (2.21) yields the desired contradiction. $\square$

As we see in the first Section of the present Chapter, it is worth finding some geometric conditions on (M, g) assuring the validity of weak maximum principle for the operators we are analyzing. To this aim, we generalize Theorem [PRS1, Theorem 4.1] considering the new class of operators L_φ.

Proposition 25. *Let* (M, g) *be a complete Riemannian manifold. Assume that*

$$\liminf_{r\to\infty} \frac{\log \operatorname{Vol} B_r}{r^{1+\delta}} = d_0 < +\infty. \tag{2.22}$$

Let $u \in \mathcal{A}_\omega(M)$ *such that* $u^* := \sup_M u < \infty$. *Then for all* $\gamma < u^*$ *we have*

$$\inf_{\Omega_\gamma} L_\varphi u \leq 0 \tag{2.23}$$

in the weak sense, where $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$. *In other words, under assumption* (2.22) *the WMP holds on* M *for the operator* L_φ.

Proof. We follow the proof of Theorem 1.1 in [RSV] (see also the proof of Theorem 4.1 in [PRS1]). If $\nu \in \mathbb{R}$ and we set $u_\nu := u + \nu$, we have $L_\varphi(u_\nu) = L_\varphi(u)$ in the weak sense, so we can replace u with u_ν, where $\nu > 0$ is such that $u_\nu^* > 0$. With abuse of notation we are going to omit the subscript ν.

Fix $\gamma < u^*$ and note that $\Omega_t \subset \Omega_s$ if $t > s$, so that we may suppose without loss of generality that $\gamma \geq 0$. Next we let

$$\begin{aligned}
K &:= \inf_{\Omega_\gamma} L_\varphi u \\
&= \sup\{a \in \mathbb{R} : L_\varphi u \geq a\} \\
&= \sup\Big\{a \in \mathbb{R} : \forall \psi \in \mathcal{C}_c^\infty(\Omega_\gamma),\ \psi \geq 0, -\int_{\Omega_\gamma} |\nabla u|^{-1}\varphi(|\nabla u|) g(\nabla u, \nabla\psi) \\
&\qquad\qquad \geq a \int_{\Omega_\gamma} \psi\Big\},
\end{aligned}$$

and assume by contradiction that $K > 0$. Observe that, in this case, u is nonconstant on any connected component of Ω_γ.

We fix $\theta \in (1/2, 1)$ and choose $R_0 > 0$ such that $B_{R_0} \cap \Omega_\gamma \neq \emptyset$. Given $R > R_0$, let $\psi \in \mathcal{C}^\infty(M)$ be a cut-off function such that

i) $0 \le \psi \le 1$;

ii) $\psi \equiv 1$ on $B_{\theta R}$;

iii) $\psi \equiv 0$ on $M \backslash B_R$;

iv)$|\nabla \psi| \le \frac{2}{R(1-\theta)}$.

Let also $\xi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \le \xi \le 1$, $\mathrm{supp}(\xi) = [\gamma, +\infty)$ and $\xi' \ge 0$. Consider the test function

$$f = \psi^{1+\delta} \xi(u) \exp(z) \ge 0$$

where $z := -q(\alpha - u)r^{1+\delta}$ for some $q > 0$ and $\alpha > u^*$ to be chosen later. Denoting with $\Omega := \Omega_\gamma \cap B_R$, observe that $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\mathrm{supp}(f) \subset \Omega$, and therefore it is an admissible test function for (2.24). Then we have

$$\begin{aligned} -\int_\Omega |\nabla u|^{-1} \varphi(|\nabla u|) g(\nabla u, \nabla f) &= -\int_{\Omega_\gamma} |\nabla u|^{-1} \varphi(|\nabla u|) g(\nabla u, \nabla f) \\ &\ge K \int_{\Omega_\gamma} f = K \int_\Omega f. \end{aligned} \tag{2.24}$$

Computing

$$\begin{aligned}\nabla f &= (1+\delta)\psi^{\delta}\xi(u)\exp(z)\nabla\psi + \psi^{1+\delta}\xi'(u)\exp(z)\nabla u \\ &\quad -(1+\delta)q(1+r)^{\delta}\psi^{1+\delta}\xi(u)\exp(z)(\alpha-u)\nabla r \\ &\quad +q\psi^{1+\delta}\xi(u)\exp(z)(1+r)^{1+\delta}\nabla u\end{aligned}$$

and substituting it into (2.24), using $\xi' \geq 0$ and Cauchy-Schwarz inequality, we obtain the estimate

$$\int_{\Omega}\big(-\varphi(|\nabla u|)(1+\delta)\psi^{\delta}\xi(u)e^{z}|\nabla\psi| + \psi^{1+\delta}\xi(u)e^{z}q(1+r)^{1+\delta}B(|\nabla u|,r)\big) \leq 0 \tag{2.25}$$

where

$$\begin{aligned}B(|\nabla u|,r) &= \frac{K}{q(1+r)^{1+\delta}} + |\nabla u|\varphi(|\nabla u|) - (1+\delta)(\alpha-u)(1+r)^{-1}\varphi(|\nabla u|) \\ &\geq \frac{K}{q(1+r)^{1+\delta}} + \frac{1}{A^{1/\delta}}\varphi(|\nabla u|)^{1+1/\delta} - (1+\delta)\alpha(1+r)^{-1}\varphi(|\nabla u|)\end{aligned} \tag{2.26}$$

on Ω, since $\gamma \geq 0$. Observe that in the last inequality we have used that

$$t\varphi(t) \geq A^{-1/\delta}\varphi(t)^{1+1/\delta},$$

which follows from the structural condition $\varphi(t) \leq At^{\delta}$. At this time we need to estimate the right hand side of (2.26) so as to have

$$B(|\nabla u|,r) \geq \Lambda\varphi(|\nabla u|)^{1+1/\delta} \tag{2.27}$$

for some positive constant Λ independent of $|\nabla u|$ and r. Towards this end, we apply Lemma 4.2 in [PRS1] with the choices (following the notation of the Lemma 4.2): $\omega = 1/A^{1/\delta}$, $\rho = (K/q)(1+r)^{-(1+\delta)}$, and $\beta = \alpha(1+\delta)(1+r)^{-1}$. Applying the Lemma 4.2 with $s = \varphi(|\nabla u|)$ and

$r \geq 0$ fixed, it is easy to verify the validity of (2.27) provided

$$\Lambda \leq \frac{1}{A^{1/\delta}} - \frac{\delta \alpha^{1+1/\delta} q^{1/\delta}}{K^{1/\delta}}. \tag{2.28}$$

Since the right hand side of the above inequality is independent of r, for every such Λ (2.27) holds. In particular, if $\tau \in (0,1)$ and we choose

$$q = \frac{\tau^\delta K}{A \delta^\delta \alpha^{1+\delta}} \tag{2.29}$$

then

$$\Lambda = \frac{1-\tau}{A^{1/\delta}} > 0$$

and it satisfies (2.28).

We insert now (2.27) into (2.25) to obtain

$$\begin{aligned} \frac{q\Lambda}{1+\delta} \int_\Omega \psi^{1+\delta} \xi(u) e^z (1+r)^{1+\delta} \varphi(|\nabla u|)^{1+1/\delta} \\ \leq \int_\Omega \psi^\delta \xi(u) e^z |\nabla \psi| \varphi(|\nabla u|). \end{aligned}$$

Applying Hölder inequality with conjugate exponents $1+\delta$ and $1+1/\delta$ to the integral on the right hand side and simplifying, we obtain

$$\begin{aligned} \left(\frac{q\Lambda}{1+\delta}\right)^{1+\delta} \int_\Omega \psi^{1+\delta} \xi(u) e^z (1+r)^{1+\delta} \varphi(|\nabla u|)^{1+1/\delta} \\ \leq \int_\Omega \xi(u) e^z (1+r)^{-\delta(1+\delta)} |\nabla \psi|^{1+\delta}. \end{aligned} \tag{2.30}$$

Recall that $\Omega = \Omega_\gamma \cap B_R$. By the volume growth assumption (2.22), for every $d > d_0$ there exists a diverging sequence $R_k \uparrow +\infty$ with $R_1 > 2R_0$ such that

$$\log \operatorname{Vol} B_{R_k} \leq d R_k^{1+\delta}.$$

Noting that $\theta R_k > R_k/2 > R_0$, we apply (2.30) with $R = R_k$, and use the bound for $|\nabla\psi|$ and the fact that $\xi \le 1$ to get

$$E = \left(\frac{q\Lambda}{1+\delta}\right)^{1+\delta} \int_{\Omega_\gamma \cap B_{R_0}} \xi(u) e^z \varphi(|\nabla u|)^{1+1/\delta} \le$$

$$\left(\frac{q\Lambda}{1+\delta}\right)^{1+\delta} \int_{\Omega_\gamma \cap B_{R_0}} \psi^{1+\delta} \xi(u) e^z (1+r)^{\delta(1+\delta)} \varphi(|\nabla u|)^{1+1/\delta} \le$$

$$\int_{\Omega_\gamma \cap B_{R_k}} \xi(u) e^z (1+r)^{-\delta(1+\delta)} |\nabla\psi|^{1+\delta} \le$$

$$\int_{\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})} \xi(u) e^z (1+r)^{-\delta(1+\delta)} |\nabla\psi|^{1+\delta} \le$$

$$\frac{2^{1+\delta}}{(1+\theta R_k)^{\delta(1+\delta)}(1-\theta)^{1+\delta} R_k^{1+\delta}} \int_{\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})} e^z.$$

It follows from here that

$$E \le C R_k^{-(1+\delta)^2} \int_{\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})} e^z, \tag{2.31}$$

where $C > 0$ is a constant independent of k. Now we observe that since $|\nabla u| \not\equiv 0$ on $\Omega_\gamma \cap B_{R_0}$, then $E > 0$. On the other hand, taking into account that

$$e^z \le e^{-q(\alpha - u^*)(1+\theta R_k)^{1+\delta}}$$

on $\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})$, inserting this into (2.31) we obtain the inequality

$$0 < E \le C R_k^{-(1+\delta)^2} \exp\left(d R_k^{1+\delta} - q(\alpha - u^*)(1+\theta R_k)^{1+\delta}\right).$$

In order for this inequality to hold for every k, we must have

$$d \ge q(\alpha - u^*)\theta^{1+\delta},$$

and letting $\theta \to 1$,

$$d \ge q(\alpha - u^*).$$

Set $\alpha = t u^*$, $t > 1$, and insert the definition of q (2.29) in the above inequality, solve with respect to K and let $\tau \to 1$ to obtain

$$K \leq Ad(u^*)^\delta \delta^\delta \frac{t^{1+\delta}}{t-1}.$$

Taking into account that

$$\min_{t>1} \frac{t^{1+\delta}}{t-1} = \frac{(1+\delta)^{1+\delta}}{\delta^\delta}$$

and letting $d \to d_0$ we obtain

$$K \leq Ad_0(u^*)^\delta (1+\delta)^{1+\delta}.$$

Now fix $\varepsilon > 0$. As we observed at the beginning of the proof K does not depend on adding a constant to u, and therefore we can suppose that $u^* = \varepsilon$. Since ε is arbitrary, this yields $K \leq 0$, contradiction. □

Chapter 3

Height estimates in generalized Robertson-Walker spacetimes

3.1 Basic tools for Lorentzian geometry

In 1851 Riemann introduced the concept of *manifolds*, in his doctoral thesis. He understood what was the *right* way to generalize Gauss study on the intrinsic geometry of surfaces in the Euclidean space $\mathbb{R}^3$. Specifically, it is desirable to have a measure of infinitesimal distance between points. To this aim, each tangent space should be provided with an inner product.

Einstein's theory of special relativity of 1905 brought out the need for a further generalization. In prerelativity physics the time interval Δt between two events and the spatial interval $|\Delta x|$ between two simultaneous events have absolute significance. This is no longer valid in special relativity. In this theory, space and time can be considered as a continuum (spacetime) composed of *events*, which can be labeled by four numbers $(t = x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, giving the points of the space at an instant of time. We assume that there exist preferred families of motion in spacetime, referred to as *inertial* motions, but among these latter there are no preferred (inertial) observers. In this setting, the time interval Δt and the space interval $|\Delta x|$ do not have intrinsic meaning. However, the spacetime interval between the events $(t = x^0, x^1, x^2, x^3)$ and $(\bar{t} = \bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ defined by

$$-(x^0 - \bar{x}^0)^2 + (x^1 - \bar{x}^1)^2 + (x^2 - \bar{x}^2)^2 + (x^3 - \bar{x}^3)^2 \tag{3.1}$$

has the same value for all inertial observers, so it represents an intrinsic property of spacetime. Equation (3.1) suggests that we can consider on spacetime manifold $\mathbb{R}^4$ tensor

$$\eta := -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

This symmetric tensor is not a Riemannian metric on $\mathbb{R}^4$ since it is not positive definite. Therefore, we need to extend the definition of inner product, substituting the positivity with a weaker request, the nondegeneracy.

We are going to begin this chapter with a brief survay on the basic tools for Lorentzian geometry, with the aim of fixing notation and touching upon the geometric background of our results. Note that the difference in metric signature makes no great differences in the geometric treatment of manifolds.

3.1.1 Lorentzian vector spaces

Let V be a real vector space of dimension n.

Definition 26. A *scalar product* η is a symmetric nondegenerate bilinear form on V. A *Lorentzian scalar product* is a scalar product of index 1.

The *nondegeneracy* means that for all $v \in V$ fixed, $\eta(v,w) = 0$ for all $w \in V$ implies $v = 0$. The *index* of a bilinear form is the largest integer that is the dimension of a subspace $W < V$ on which $\eta_{|W}$ is negative definite.
For every scalar product η, we can find a basis $e_1, \dots e_n$ such that

$$\begin{aligned}
&\eta(e_i, e_i) = -1 \quad \text{for } i = 1, \dots, k \\
&\eta(e_i, e_i) = 1 \quad \text{for } i = k+1, \dots, n \\
&\eta(e_i, e_j) = 0 \quad i \neq j.
\end{aligned}$$

On $\mathbb{R}^n$ there is a natural Lorentzian scalar product, defined by

$$\langle\langle x, \bar{x} \rangle\rangle = x^1\bar{x}^1 + \dots + x^{n-1}\bar{x}^{n+1} - x^n\bar{x}^n.$$

We refer to $(\mathbb{R}^n, \langle\langle ., . \rangle\rangle)$ as the Minkowski space.
Consider now a Lorentzian space (V, η). Note that the product $\eta(v,v)$ is not forced to be positive, it can also be negative or even null.

Definition 27. We define $v \in V - \{0\}$ to be *timelike* if $\eta(v,v) < 0$, *lightlike* (or *null*) if $\eta(v,v) = 0$ and *spacelike* if $\eta(v,v) > 0$. Moreover, we will refer to lightlike and timelike vectors as *causal* vectors.

For $n \geq 2$ the set of timelike vectors consists of two cones. Recall that a *cone* is a subset of a vector space that is closed under multiplication by positive scalars. Choosing a *time orientation* on V means picking up one of these two cones. The timelike vectors in this latter are said to be *future pointing*, the other ones *past pointing*. We say that two timelike vectors v and w have the same time orientation if they stay in the same cone and this is equivalent to requiring $\eta(v,w) < 0$.
Now, consider a subspace W of (V, η). We say that W is

- *spacelike* if $\eta_{|W}$ is positive definite;

- *timelike* if $\eta_{|W}$ is nondegenerate of index 1 (hence Lorentzian if $\dim(W) \geq 2$);
- *lightlike* if $\eta_{|W}$ is degenerate.

Two vectors $v, w \in V$ are said to be *orthogonal* if $\eta(v, w) = 0$. For a subspace $W < V$ we can introduce the subspace

$$W^{\perp} = \{v \in V \mid \eta(v, w) = 0 \ \forall w \in V\}.$$

Nevertheless, we do not refer to $W^{\perp}$ as the orthogonal complement of W since, unlike the Euclidean case, we do not have in general $V = W + W^{\perp}$. Specifically, it can be proved that $V = W \oplus W^{\perp}$ if and only if W is nondegenerate, that is $\eta_{|W}$ is nondegenerate.

Lemma 28. *If $w \in V$ is a timelike vector in a Lorentzian vector space then $\langle w \rangle^{\perp}$ is spacelike.*

Proof. Define $|w| := \sqrt{-\eta(w, w)}$ and $e_1 := w/|w|$. Using a Gram-Schmidt argument we complete e_1 to an orthonormal basis of V, $\{e_1, e_2, \ldots, e_n\}$. Since the index does not depend on the orthonormal basis we have that vectors $e_2, \ldots, e_n$ are spacelike. Since they span $\langle w \rangle^{\perp}$ we conclude that this latter is spacelike. □

As for the orthogonal complement, it is worthwhile stressing the fact that some of the basic results valid for Euclidean scalar products must be slightly modified in the Lorentzian case. For example, for timelike vectors we have a reverse Cauchy-Schwarz inequality and a reverse triangular inequality.

Proposition 29 (Reverse Cauchy-Schwarz inequality). *Let η be a Lorentzian scalar product on a vector space V. For v, $w \in V$ timelike vectors we have*

$$|\eta(v, w)| \geq |v||w|, \tag{3.2}$$

where $|v| \coloneqq \sqrt{-\eta(v,v)}$ *and the equality holds if and only if* v *and* w *are colinear.*

Proof. If we decompose w as $w = \lambda v + \bar{w}$, where $\lambda \in \mathbb{R}$ and $\eta(\bar{w}, v) = 0$ we have

$$\begin{aligned}\eta(w,w) &= \lambda^2 \eta(v,v) + \eta(\bar{w},\bar{w}) \\ \eta(v,w) &= \lambda \eta(v,v).\end{aligned}$$

Hence

$$\eta(v,w)^2 = \lambda^2 \eta(v,v)^2 = \eta(v,v)(\eta(w,w) - \eta(\bar{w},\bar{w})) \geq \eta(v,v)\eta(w,w),$$

where, in the last inequality, we used the fact that since v is timelike then $\bar{w}$ is spacelike. So, we conclude $|\eta(v,w)| \geq |v||w|$.
Note that the equality holds if and only if $\eta(\bar{w},\bar{w}) = 0$, that is $\bar{w} = 0$, and this is true if and only if $w = \lambda v$. □

Inequality (3.2) allows us to introduce the *hyperbolic angle* between two timelike vectors lying in the same timecone.

Definition 30. Let $v, w \in (V, \eta)$ be two timelike vectors such that $\eta(v,w) < 0$. The *hyperbolic angle* between v and w is the unique number $\theta \geq 0$ such that

$$\eta(v,w) = -|v||w| \operatorname{Ch}(\theta). \tag{3.3}$$

Proposition 31 (Reverse triangular inequality). *Let* η *be a Lorentzian scalar product on a vector space* V*. If* $v,\ w \in V$ *are timelike vectors in the same cone, we have*

$$|v + x| \geq |v| + |w| \tag{3.4}$$

and equality holds if and only if v *and* w *are colinear.*

Proof. The assumption on v and w to stay in the same cone gives $\eta(v, w) < 0$. We have

$$|v + w|^2 = -\eta(v + w, v + w) = |v|^2 + |w|^2 + 2|\eta(v, w)|.$$

Using the reverse Cauchy-Schwarz inequality we deduce

$$|v + w|^2 \geq |v|^2 + |w|^2 + 2|v||w| = (|v| + |w|)^2,$$

and equality holds if and only if $|\eta(v, w)| = |v||w|$ and this happens when v and w are colinear. $\square$

In a vector space endowed with a scalar product (V, g), the group of all distance-preserving endomorphisms (i.e. *isometries*) is called *orthogonal group.* If M is the matrix associated to g with respect to some basis of V, the elements of the orthogonal group are transformations represented by matrix A satisfying $A^T M A = M$. In the Euclidean space $\mathbb{R}^n$, the orthogonal group $O(n)$ is composed by matrix A such that $A^T A = I_n$. In the Minkowski space, the orthogonal group is called *Lorentz group* and is denoted by $O_1(n)$. A matrix $A \in \mathrm{Mat}(n, \mathbb{R})$ represents a Lorentz transformation if and only if $A^T \eta A = \eta$, where

$$\eta = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & -1 \end{array}\right).$$

Lorentz transformations play a central role in special relativity. A material body, if subjected to no external forces, undergoes a nonaccelerating motion, named *inertial motion.* The first postulate of special relativity (*principle of relativity*) asserts that *the laws of phisics are the same in all intertial frames of reference.* An inertial observer can label each event of spacetime with four numbers. If observer O labels an event p with coordinates (t, x, y, z) and O', for instance, moves with velocity v

in the x-direction, then the labeling by O' are the transformed of that of O by the Lorentz transformation

$$\begin{cases} t' = \frac{t-vx}{(1-v^2)^{\frac{1}{2}}} \\ x' = \frac{x-vt}{(1-v^2)^{\frac{1}{2}}} \\ y' = y \\ z' = z, \end{cases}$$

where we assume the speed of light to be $c = 1$. This assumption is due to the second postulate of special relativity (*invariance of* c), according to which *the speed of light c has the same value in all inertial frames of references*

3.1.2 Robertson-Walker spacetimes

Special relativity neglects the effects of gravitational fields. Newton's theory of gravitation is not consistent with special relativity since it assumes that certain *signals* trasmit instantaneously.
Einstein's idea, in order to include gravitation in a relativity theory, was to describe physical quantities by geometrical objects and to express physical laws as geometric relationships between these objects. In inertial frames all test particles move along straight lines. These trajectories constitute a preferred family of curves, which are geodesics for the flat metric in Minkowski space. Analogously, paths of bodies *freely falling* in a gravitational field can be thought as geodesics for a non-flat metric. This innovative approach was supported by the *equivalence principle*, which asserts that an observer can not distinguish wheather he is in presence of a uniform gravitational field or he is in an accelerating reference frame in absence of gravitational field. Hence, the fact that acceleration imparted to a body is independent of the nature of

the body suggests to ascribe properties of the gravitational field to the geometric structure of the spacetime.

Definition 32. A *pseudo-Riemannian* (or *semi-Riemannian*) manifold is a couple (M, g), where M is a smooth manifold and g is a symmetric, non-degenerate $(0,2)$ tensor field of constant index, called *metric tensor*.

Hence in a pseudo-Riemannian manifold each tangent space is furnished with a scalar product η_p and the index of η_p, ν, does not depend on the point $p \in M$. If $\nu = 0$ M is a Riemannian manifold, if $\nu = 1$ and $\dim(M) \geq 2$ M is called *Lorentzian manifold.*
For tangent vectors to M we use the terminology we introduced in the previous section. Hence, for example, a tangent vector $v \in T_pM$ is said spacelike (resp. timelike, lightlike) if it is spacelike (resp. timelike, lightlike) in the Lorentzian vector space T_pM. Analogously, we already know what is a time orientation on a Lorentzian vector space, so we can time-orient each tangent space (T_pM, η_p); the problem is how to do this in a *continuous* or even *smooth* way, on our manifold M. Choosing one of the two timecones on a Lorentzian vector space V is equivalent to saying that a specific timelike vector $v \in V$ is future-pointing. So, it is natural to interpret the smoothness of the choice of a time orientation on tangent spaces as the smoothness of a *representative* future-pointing timelike vector field. We are, then, led to the following

Definition 33. A Lorentzian manifold M is *time-orientable* if and only if there exists a smooth timelike vector field globally defined on M.

Observe that Definition 33 is equivalent to its explicitly local version. More precisely, consider a function Or such that, for any $p \in M$, $Or(p)$ is one of the two timecones in T_pM. We can say that our choice Or is smooth if, for any $p \in M$, there exists an open neighborhood $\mathcal{U}_p$

of p and a vector field $V^p \in \mathfrak{X}(\mathcal{U}_p)$ such that $V^p_q \in Or(q)$ for any $q \in \mathcal{U}_p$. Obviously, Definition 33 implies the existence of a smooth choice Or. Using partition of unity, one can prove that also the converse is true. Indeed, consider the partition of unity $\{f_p\}$ subordinate to the neighborhoods of points of M with the property discussed above. Vector field $X_q := \sum_p f_p V^p_q$ is smooth and globally defined on M. Since, for any $p, q \in M$, $V^p_q \in Or(q)$ and $f_p \geq 0$ it follows that X is timelike.
If (M, g) is a Riemannian manifold and $N \subset M$ is a submanifold of M, the pullback $j^*(g)$ is a Riemannian metric on N, where $j : N \hookrightarrow M$ denotes the inclusion map. However, if (M, η) is more generally a pseudo-Riemannian manifold and η is indefinite then $j^*(\eta)$ might be degenerate. So, we need the following

Definition 34. Let (M, η) be a pseudo-Riemannian manifold and $N \subset M$ a submanifold. If the pullback $j^*(\eta)$ is a metric tensor on N, we call N pseudo-Riemannian submanifold of M.

Let, now, N be a submanifold of a pseudo Riemannian manifold (M, η). If, for every $p \in N$, T_pN has the same causal character as subspace of T_pM, then we say that N itself has that causal character. Clearly, pseudo-Riemannian submanifolds can only be spacelike or timelike and, in general, a submanifold of (M, η) need not have a causal character. For instance, tangent vectors to a circle in the Minkowski plane $\mathbb{R}^2_1$ are spacelike in two arcs, timelike in two arcs and null in four points. This is not the case for geodesics. In a pseudo Riemannian manifold (M, η) (as in Riemannian geometry) a curve $\gamma : I \to M$ is a geodesic - by definition - if its velocity γ' is parallel, that is $\nabla_{\gamma'}\gamma' \equiv 0$. We denote by ∇ the Levi-Civita connection of η, defined in the same way as in the Riemannian case. Parallel translation is a linear isometry, so it preserves causal character of vectors. Therefore a geodesic always has a causal character.

Example 35 (**The Schwarzschild Half-plane**)**.** Let

$$\mathfrak{h}(r) = 1 - \frac{2M}{r},$$

where M is a positive constant. The *Schwarzschild Half-plane* P_I is the half-plane $\{(r,t) \in \mathbb{R}^2 | r > 2M\}$ endowed with Lorentzian metric

$$ds^2 = \mathfrak{h}^{-1}dr^2 - \mathfrak{h}dt^2.$$

In the Schwarszchild half-plane null geodesics are given by the transformed of

$$t = s + 2M\log s \quad r = s + 2M \quad \text{for} \quad s > 0$$

by isometries $(r,t) \to (r, \pm t + b)$.
The Schwarzschild halfplane is the essential element used to build up the simplest relativistic model of space around a star. Its null geodesics model light rays approaching (or departing) radius $2M$.

Our aim, now, is to describe special examples of Lorentzian manifolds, the generalized Robertson-Walker spacetimes, that will be the geometric framework of our results in this chapter. We start dealing with classical Robertson-Walker spacetimes.
The purpose of cosmology is to find appropriate models of the universe. More precisely, by *cosmological model* we mean a four-dimensional, time-orientable Lorentzian manifold (M, g), called *spacetime*, where g obeys a fundamental tensorial equation (*Einstein field equation*)

$$G_{ij} := R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij}. \tag{3.5}$$

We denote by R_{ij} the Ricci curvature of g and by R the trace of R_{ij}, the scalar curvature. The *stress-energy tensor* T_{ij} describes continuous matter distribution so that equation (3.5) gives a relationship between the geometry of (M, g) (namely, its curvature) and matter. In the large scale viewpoint one can consider the universe as a perfect fluid whose *particles* are galaxies, ignoring the internal structure of these latter. In this asset, one characterizes the gas by a 4-velocity u (the velocity of an observer for whom galaxies in his neighborhood have no mean motion), by a density of mass-energy ρ and by a pressure p. The stress-energy momentum has the form

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}. \tag{3.6}$$

Cosmologists want to find solutions to the Einstein field equations consistent with matter distribution of the universe. The main point is how to choose a good model for the universe. The right approach to the problem consists of a mixture of coherence with observational data and theoretical assumptions on the nature of the universe, with a philosophical rather than scientifical taste. This may seem to be a uroboric attitude, but it is in some sense unescapable since we cannot discuss experimental data without a model and, at the same time, we cannot construct a significative model without empirical evidences.

Hence we may start observing that there is no theoretical evidence that we occupy a privileged position in the universe, so that we can think that the portion of the universe we can observe is a fair sample. This means that it is natural to assume that the universe is spatially *homogeneous*, when viewed on a suitable (large) scale. Analogously, we suppose the universe to be *isotropic*, that is we suppose that there are no preferred directions, so that observations on large scale should yeald to the same conclusions independently from the direction we choose.

However, despite the philosophical nature of our reasoning above, the assumptions of the homogeneity and the isotropy of the universe (the so called *cosmological principle*) have strong empirical confirmations. We mention the very highly isotropy of x-ray, γ-ray background radiation and of the cosmic microwave background, a thermal radiation that is the remnant heat left over by the Big Bang. Regarding the homogeneity, it is quite difficult to test it directly but, as we will observe later, isotropy around any point implies homogeneity, so the isotropies of the extragalactic observations represent a confirmation of homogeneity too. Now, we want to give a precise mathematical definition of homogeneity and isotropy. Concerning spatial homogeneity, we want to formalize the idea that, at any instant of time, the universe looks the same everywhere. In special relativity, the expression *at a given instant of time* is ambiguous since there is no universal meaning of simultaneity, so we should specify which inertial frame we consider. In general relativity, moreover, there are no global inertial frames so we have to refine further our definition. We replace the concept of *a given moment of time* with the more general concept of spacelike hypersurface. At any point of a spacelike hypersurface there is a local Lorentz frame whose surface of simultaneity locally coincides with the hypersurface.

Definition 36. A spacetime (M, g) is said to be spatially homogeneous if it is foliated by a one-parameter family of spacelike surfaces Σ_t such that for each $t \in \mathbb{R}$ and for any points p, q in Σ_t there exists an isometry of the metric g which takes p into q.

Hence, on each Σ_t, called *homogeneity surface*, there exists a group of isometries acting transitively on it. This definition implies that at each time of a hypersurface Σ_t pressure p, density ρ and the curvature of the spacetime must be the same.
Regarding the concept of isotropy, observe first that the universe cannot look isotropic to all observers: it can be isotropic only for observers moving with the cosmological fluid.

Definition 37. A spacetime (M, g) is said to be spatially isotropic at each point if there exists a congruence of timelike curves, with tangents denoted by u, filling M and such that, for any $p \in M$ and $s_1, s_2 \in T_pM$ unit vectors with $g(s_1, u_p) = g(s_2, u_p) = 0$ (so s_1 and s_2 are spacelike vectors) there exists an isometry of (M, g) which leaves p and u_p fixed and rotates s_1 in s_2.

Recall that a *congurence* of curves in a spacetime (M, g) is the set of the integral curves of a nonvanishing vector field on M. A timelike curve in a M represents the world line of an observer, that is the trajectory of an observer moving in a spacetime. We will refer to the timelike curves in Definition 37 as the *isotropic observers*.

We remark that Definition 36 and Definition 37 are not completely independent: a spacetime isotropic around any point must be also homogeneous, while there exist homogeneous anisotropic cosmological models. Indeed, assume that we can foliate the spacetime M with complete spacelike hypersurfaces Σ_t orthogonal to the isotropic observers. Then, for each $t \in \mathbb{R}$ and for each $p, q \in \Sigma_t$ we can consider the geodesic $\gamma : [0, 1] \to \Sigma$ connecting p to q. Now, consider the isometry $F : M \to M$ that fixes $\gamma(1/2)$ and send $\gamma'(1/2)$ to $-\gamma'(1/2)$. Since $F \circ \gamma$ must be a geodesic with the same tangent vector as $\gamma(1 - s)$ in $\gamma(1/2)$, due to a unicity result for geodesics it must be the same curve and so it must be $F(q) = p$.

Now we are going to show that the geometry of a spacetime is strongly influenced by the assumption of the cosmological principle. Consider a homogeneous and isotropic spacetime (M, g) and denote by $\{\Sigma_t\}_{t\in\mathbb{R}}$ the foliation of M with homogeneity hypersurfaces. First we remark that each Σ_t is orthogonal to the world line of the isotropic observer (or, if the isotropic observers are not uniquely determined, one can construct a family of isotropic observers orthogonal to the homogeneity surfaces). Then, let h be the Riemannian metric on Σ_t defined by $h := j^*(g)$, where

$j : \Sigma_t \hookrightarrow M$ is the inclusion map, and let R_{ijkl} be the Riemann tensor of h. Contracting the last two indices with the metric

$$R_{ij}{}^{kl} := h^{rk}h^{tl}R_{ijrt},$$

where h^{ij} is defined by $h^{ij}h_{jk} = \delta^i_k$, we obtain, for each $p \in M$, a linear map $L : \Lambda^2(T^*_p\Sigma_t) \to \Lambda^2(T^*_p\Sigma_t)$ defined by

$$L(v_{ij}\omega^i \wedge \omega^j) := R_{ij}{}^{kl}v_{kl}\omega^i \wedge \omega^j,$$

where $\{\omega^i\}_{i=1}^n$ is a local orthonormal co-frame in p.
Recall that the metric tensor h induces a natural inner product on $\Lambda^2(T^*_p\Sigma_t)$

$$\langle v_{ij}\omega^i \wedge \omega^j, w_{kl}\omega^k \wedge \omega^l\rangle_p := h^{ki}h^{lj}v_{ij}w_{kl}. \tag{3.7}$$

Remark that operator L is a self-adjoint operator on $\Lambda^2(T^*_p\Sigma_t)$ furnished with inner product (3.7). Indeed, for any two 2-forms $v = v_{ij}\omega^i \wedge \omega^j$, $w = w_{kl}\omega^k \wedge \omega^l$ we have

$$\begin{aligned}\langle Lv, w\rangle &= h^{ti}h^{sj}R_{ij}{}^{kl}v_{kl}w_{ts} = h^{ti}h^{sj}h^{km}h^{ln}R_{ijmn}v_{kl}w_{ts}\\ &= h^{ti}h^{sj}h^{km}h^{ln}R_{mnij}v_{kl}w_{ts} = h^{km}h^{ln}R_{mn}{}^{ts}v_{kl}w_{ts}\\ &= \langle v, Lw\rangle.\end{aligned}$$

Therefore, there exists a basis of $\Lambda^2(T^*_p\Sigma_t)$ made up of eigenvectors of L, say e_1, e_2, e_3 associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively. Due to the assumption of isotropy, the eigenvalues of L can not be distinguished. Indeed, recall that $\dim(\Lambda^2_p(\Sigma_t)) = \dim(T_p\Sigma_t) = 3$ so we can consider a linear isometry between these two spaces. The hypothesis of isotropy, then, implies that if $i, j \in \{1, 2, 3\}$, with $i \neq j$, there exists an isometry F of M that fixes p and u_p and sends e_i in e_j. So we have

$$\lambda_i = h(Le_i, e_i) = h(F_* Le_i, F_* e_i) = h(LF_* e_i, F_* e_i) = h(Le_j, e_j) = \lambda_j,$$

where we used the fact that the Riemann operator L commutes with F_*.
Hence $\lambda_1 = \lambda_2 = \lambda_3 = K$ and

$$R_{ij}{}^{kl} = K(\delta_i{}^k \delta_j{}^l - \delta_i{}^l \delta_j{}^k). \tag{3.8}$$

We can repeat the same reasoning in any point $p \in \Sigma_t$ and we conclude that equation (3.8) holds on Σ_t, with K is independent of the point due to Schur theorem. Therefore Σ_t is a space form and, since two space forms with the same value of K are locally isometric, we can classify it. Namely, if $K > 0$ Σ_t is locally isometric to a 3-sphere, if $K = 0$ Σ_t to the flat space $\mathbb{R}^3$ and if $K < 0$ Σ_t to a three dimensional hyperboloid.
We stress the fact that the isotropy constraint alone implies that we have only three possibilities for spatial geometry. In special relativity the cosmological model is a flat universe. Since the 3-sphere is a compact hypersurface without boundary, we refer to the case $K > 0$ as the *closed* model, while if $K \leq 0$ we have the *open* one.
Observe that, until now, we have fixed one of the surfaces Σ_t, $t \in \mathbb{R}$, and we have obtained conclusions on its geometry. Now, we want to describe the geometry of the whole spacetime, that is, we want to relate to each other the behaviours of the homogeneity surfaces. To this purpose, we label each surface Σ_t with the proper time of any of the isotropic observers, $\tau \in \mathbb{R}$. Then we denote by ∂_τ the tangent vector field u to the isotropic observers and we introduce the scale function $a(\tau)$ that gives the rescaling of metrics of the homogeneity surfaces. So, we can represent M as the differential manifold $M = \mathbb{R} \times \Sigma$ endowed

with metric

$$g := -d\tau^2 + a^2(\tau)h, \tag{3.9}$$

where h is a metric on Σ with constant curvature. We denote the Lorentzian manifold M furnished with metric g by $-\mathbb{R} \times_a \Sigma$, where the subscipt a indicate that we are modifying the usual product metric with the scale factor a^2 and sign minus reminds that we are considering a Lorentzian metric. In differential geometry these kind of manifolds are called *Lorentzian warped products* and one refers to function a as the *warping function.*

Hence, we remark that the only assumption of homogeneity and isotropy yealds to the very specific cosmological model $-\mathbb{R}\times_a\Sigma$, called *Robertson–Walker* spacetime. These spaces are, then, a good aproximation for the large-scale geometry of our universe.

In order to study the dynamical evolution of the universe, we should substitute equation (3.9) in (3.5), where T_{ij} must have a suitable form in order to describe satisfactory the matter content of the universe. The general evolution equations for our homogeneous and isotropic model are

$$3\frac{\dot{a}^2}{a^2} = 8\pi\rho - \frac{3k}{a^2} \tag{3.10}$$

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3p), \tag{3.11}$$

where a is the scale factor introduced in (3.9) and $k = 1$ for the 3-sphere, $k = 0$ for the flat space and $k = -1$ for the hyperbolic space. These equations are known as *Friedmann-Lemaître equations.* From (3.11) we get $\ddot{a} < 0$, provided $\rho > 0$ and $p \geq 0$. This fact means that, in the homogeneous and isotropic model, universe must always be expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$). Einstein was not satisfied with this striking prediction, since he believed in a static universe. So, he modified equation (3.5) with the addition of a new term

$$G_{ij} + \Lambda g_{ij} = 8\pi T_{ij}, \tag{3.12}$$

where Λ is a constant called *cosmological constant*. This adjustment yeald static solutions but Hubble's observations in 1929 on the redshifts of distant galaxies confirmed the expansion of the universe. So the introduction of the cosmological constant turned out to be unjustified.
De Sitter and *anti-de Sitter* spaces are maximally symmetric (with constant sectional curvature), vacuum solutions ($T_{ij} = 0$) to Einstein field equations, with respectively positive and negative cosmological constant Λ.
Consider the hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

isometrically immersed in the Minkowski space $\mathbb{R}^5_1$ endowed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

We refer to this space as the *De Sitter space*, dS_4. It has the topology of $\mathbb{R} \times S^3$; indeed we can introduce the coordinate system (t, χ, θ, ϕ) such that

$$\begin{cases} x_0 = \mathrm{Sh}(t) \\ x_1 = \mathrm{Ch}(t)\cos(\chi) \\ x_2 = \mathrm{Ch}(t)\sin(\chi)\cos(\theta) \\ x_3 = \mathrm{Ch}(t)\sin(\chi)\sin(\theta)\cos(\phi) \\ x_4 = \mathrm{Ch}(t)\sin(\chi)\sin(\theta)\sin(\phi). \end{cases}$$

In these coordinates the metric has the form

$$g = -dt^2 + \mathrm{Ch}^2(t)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\},$$

so that $dS_4 = -\mathbb{R} \times_a S^3$, with $a(t) = \mathrm{Ch}(t)$. Hence, De Sitter space is a special case of Robertson-Walker spacetime.
The *anti-de Sitter space* can be represented as the hyperboloid

$$-u^2 - v^2 + x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^5$ with the metric induced by

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2.$$

This space has the topology of $S^1 \times \mathbb{R}^3$ and it is not simply connected. Its universal covering has the topology of $\mathbb{R}^4$ and usually one refers to this latter as the anti-de Sitter space. The metric induced on this space is

$$g = -dt^2 + \cos^2(t)(d\chi^2 + \mathrm{Sh}^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2))$$

Now, we want to generalize Robertson-Walker spacetimes considering a fiber with not necessarily constant sectional curvature and of generic dimension $n \geq 3$. Let $(\mathbb{P}^n, g)$ be a complete n-dimensional Riemannian manifold, $I \subseteq \mathbb{R}$ an open interval and $\varrho \in \mathcal{C}^\infty(I)$ a positive function (called *warping function*). We denote by $\bar{M}^{n+1} = -I \times_\varrho \mathbb{P}^n$ the differentiable manifold $I \times \mathbb{P}^n$ endowed with the metric

$$\bar{g} := -\pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_{\mathbb{P}}^*(g),$$

where π_I and $\pi_{\mathbb{P}}$ denote the projections onto I and $\mathbb{P}^n$ respectively.
We call $-I \times_\varrho \mathbb{P}^n$ *generalized Robertson-Walker (GRW) spacetime.* Since we allow $\mathbb{P}^n$ not to have constant sectional curvature, GRW spacetimes

need not be homogeneous. Observe that, as we commented before, spatial homogeneity seems to be a reasonable assumption in order describe universe at large scales but, obvioulsy, it is not realistic at smaller scales.

In any *GRW* spacetime $\bar{M}$, the coordinate vector field ∂_t is globally defined and timelike, so $\bar{M}$ is time-orientable. Consider a spacelike hypersurface $F : \Sigma^n \hookrightarrow \bar{M}$. There exists on Σ a unique timelike normal vector field N with the same orientation as ∂_t, that is such that $\bar{g}(N, \partial_t) \leq 0$. We will call N the *future pointing Gauss map* of the hypersurface. Due to (3.2) we have $\bar{g}(N, \partial_t) \leq -1$, so we can introduce smooth functions $\theta \geq 0$ and $\Theta(\leq 0)$, defined by

$$\bar{g}(N, \partial_t) = -\cosh(\theta) = \Theta. \tag{3.13}$$

By abuse of notation, we will refer both to θ and to Θ as *hyperbolic angle.*

Consider, now, the second fundamental form of the immersion $A : T\Sigma \to T\Sigma$ and let $k_1, \dots, k_n$ be its eigenvalues, the *principal curvatures* of Σ. The *mean curvature* of the immersion is defined by

$$H := -\frac{1}{n}\sum_{i=1}^{n} k_i. \tag{3.14}$$

3.2 Spacelike graphs immersed in Robertson-Walker spacetimes

The aim of this section is to present some bounds for the mean curvature of spacelike graphs in GRW spacetimes.

Let $(\mathbb{P}^n, \langle , \rangle_{\mathbb{P}})$ denote a Riemannian manifold of dimension n, $I \subset \mathbb{R}$ an open interval and $\rho \in \mathcal{C}^\infty(I)$. Given $u \in \mathcal{C}^\infty(\mathbb{P}^n)$, we denote by $\Sigma(u)$ the graph of u in the GRW spacetime $-I \times_\rho \mathbb{P}^n$

$$\Sigma(u) := \{(u(x), x) \in I \times \mathbb{P}^n | x \in \mathbb{P}^n\} \subset -I \times_\rho \mathbb{P}^n.$$

The metric induced on $\mathbb{P}^n$ by the Lorentzian metric of the ambient space via $\Sigma(u)$ is given by

$$\langle , \rangle = -du^2 + \rho(u)^2 \langle , \rangle_{\mathbb{P}}.$$

Assuming the graph to be spacelike means assuming the metric $\langle , \rangle$ to be Riemannian and this is equivalent to assuming $|Du|^2 < \varrho(u)^2$ everywhere on $\mathbb{P}^n$. We denote by Du the gradient of u in $\mathbb{P}^n$ and by $|Du|$ its norm, both with respect to the metric of $\mathbb{P}^n$.
Introduce the vector field

$$N := \frac{1}{\varrho(u)\sqrt{\varrho(u)^2 - |Du|^2}} (\varrho(u)^2 \partial_t + Du).$$

This latter is orthogonal to $\Sigma(u)$ and it is the future-pointing Gauss map of the graph. Indeed, N has the same orientation as ∂_t

$$\cosh(\theta) := -\Theta := -\langle N, \partial_t \rangle = \frac{\rho(u)}{\sqrt{\varrho(u)^2 - |Du|^2}}$$

In particular

$$\sinh(\theta) = \frac{|Du|}{\sqrt{\varrho(u)^2 - |Du|^2}} \tag{3.15}$$

The mean curvature of the graph, $H(u)$, corresponding to this choice of N is given by equation

$$\operatorname{div}_{\mathbb{P}} \left(\frac{Du}{\varrho(u)\sqrt{\varrho(u)^2 - |Du|^2}} \right) + \frac{\varrho'(u)}{\sqrt{\varrho(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{\varrho(u)^2} \right) = nH(u). \tag{3.16}$$

The above equation can be written in the following equivalent form

$$\frac{1}{\varrho(u)}\operatorname{div}_{\mathbb{P}}\left(\frac{Du}{\sqrt{\varrho(u)^2-|Du|^2}}\right)+\frac{n\varrho'(u)}{\sqrt{\varrho^2(u)-|Du|^2}}=nH(u). \tag{3.17}$$

To any complete manifold, as $\mathbb{P}^n$, we can associate the so called *Cheeger constant* defined by

$$h(\mathbb{P}^n):=\inf_{\Omega}\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(\Omega)},$$

where $\Omega \subset\subset \mathbb{P}^n$ ranges over all relatively compact subdomains of Ω. The geometric constant $h(\mathbb{P}^n)$ is related to the following basic spectral inequality, due to Cheeger

$$\lambda_1(\mathbb{P}^n)\geq\frac{1}{4}h(\mathbb{P}^n)^2,$$

where $\lambda_1(\mathbb{P}^n)$ is the spectral radius of $\mathbb{P}^n$ (see Appendix A). P. Buser in [Bu] found an upper bound for $\lambda_1(\mathbb{P}^n)$ in terms of the Cheeger constant: he showed that if the Ricci tensor of $\mathbb{P}^n$ satisfies $\mathrm{Ric}_{\mathbb{P}^n}\geq-(n-1)\delta^2$, for a $\delta\geq 0$, then

$$\lambda_1(\mathbb{P}^n)\leq 2\delta(n-1)h(\mathbb{P}^n)+10h(\mathbb{P}^n)^2.$$

Thus $\lambda_1(\mathbb{P}^n)$ and $h(\mathbb{P}^n)$ can be considered equivalent in some sense, that is $\lambda_1(\mathbb{P}^n)=0$ if and only if $h(\mathbb{P}^n)=0$, under opportune curvature bounds. Hence, one may ask under which geometric conditions $\lambda_1(\mathbb{P}^n)=0$; it turns out that if $\mathbb{P}^n$ is complete non-compact and it has subexponential volume growth then $\lambda_1(\mathbb{P}^n)=0$ (see [CY2], [Br]).
Now we want to give a result, that has been essentially proved by Salavessa in [S], concerning the maximality of a constant mean curvature spacelike graph $\Sigma(u)$ in a Lorentzian product $-\mathbb{R}\times\mathbb{P}^n$, provided the fiber $\mathbb{P}^n$ has vanishing Cheeger constant and $\Sigma(u)$ has bounded hyperbolic angle.

Theorem 38. *Let $(\mathbb{P}^n, \langle, \rangle)$ be an n-dimensional Riemannian manifold with vanishing Cheeger constant and let $u \in \mathcal{C}^\infty(\mathbb{P}^n)$ be such that $\Sigma(u)$ is a spacelike constant mean curvature graph in the Lorentzian product $-\mathbb{R} \times \mathbb{P}^n$. Assume the hyperbolic angle to be bounded. Then $\Sigma(u)$ is maximal.*

Proof. Consider $\Omega \subset\subset \mathbb{P}^n$ a relatively compact domain. The mean curvature equation in a Lorentzian product has the form

$$\operatorname{div}_{\mathbb{P}}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH(u), \tag{3.18}$$

where $\operatorname{div}_{\mathbb{P}}$ denotes the divergence operator on the fiber $\mathbb{P}^n$. Integrating (3.18) on Ω and taking the absolute values, we get

$$n|H(u)|\operatorname{Vol}(\Omega) = \left|\int_{\partial\Omega} \frac{\langle Du, \nu\rangle}{\sqrt{1-|Du|^2}}\right| \leq C_\theta \operatorname{Vol}_{n-1}(\partial\Omega),$$

where ν is the outward-pointing normal on $\partial\Omega$, $C_\theta := \sup_{\partial\Omega}|\operatorname{Sh}(\theta)|$ and this latter is finite because we are assuming the hyperbolic angle to be bounded. So we get

$$|H(u)| \leq \frac{C_\theta}{n} \inf_{\Omega\subset\subset\mathbb{P}^n} \frac{\operatorname{Vol}_{n-1}(\partial\Omega)}{\operatorname{Vol}(\Omega)} = 0$$

and we achieve the maximality of $\Sigma(u)$. □

Before going on, we should spend a few words on the assumption of the boundedness of Θ. First of all, observe that the boundedness of the hyperbolic angle is necessary for Theorem 3.2. Indeed, consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = \sqrt{1 + x^2 + y^2}$ and denote by $\Sigma(u)$ its graph, that is one sheet of the hyperboloid $z^2 - x^2 - y^2 = 1$ immersed in the Minkowski space $\mathbb{R}^3_1$. It is easy to see that for Euclidean spaces $h(\mathbb{R}^n) = 0$. Indeed, consider the sequence of balls of radius k, B_k. We have

$$\mathrm{Vol}(B_k) = \omega_n k^n \qquad \mathrm{Vol}_{n-1}(\partial B_k) = n\omega_n k^{n-1},$$

where ω_n denotes the volume of B_1. Hence

$$0 \le h(\mathbb{R}^n) \le \inf_{k\in\mathbb{N}} \frac{\mathrm{Vol}_{n-1}(\partial B_k)}{\mathrm{Vol}(B_k)} = 0.$$

The graph $\Sigma(u)$ is spacelike since

$$|Du|^2 = \frac{x^2 + y^2}{1 + x^2 + y^2} < 1.$$

The hyperbolic angle of $\Sigma(u)$ is

$$\Theta = \frac{1}{\sqrt{1 - |Du|^2}} = \sqrt{1 + x^2 + y^2}$$

and it is not bounded. Computing the mean curvature of the graph, we get $H(u) = 1 \neq 0$. We remark that the only assumption of Theorem 3.2 which is not fulfilled is the boundedness of Θ. Therefore this latter condition turns out to be necessary; it is desirable to know if it has also a physical interpretation. In a spacetime (M, g) the concept of *observer* is formalized as a future-pointing timelike curve. Often it suffices to consider an instantaneous observer, that is a couple (p, X), where $p \in M$ and $X \in T_pM$ is a future pointing timelike vector in T_pM. Along a spacelike hypersurface $\Sigma \hookrightarrow -\mathbb{R} \times_\rho \mathbb{P}^n$ we can consider two relevant observers: (p, N_p) and $(p, \partial_{t|_p})$. Computing the Newtonian velocity of N relative to ∂_t, say v, we get $|v| = \tanh(\theta)$ and so if Θ is bounded $|v|$ does not approach the speed of light in vacuum (see [SW]: pp. 41-45).

Now, we want to give an analogous of Theorem 38 for a spacelike graph in a GRW-spacetime. This result and the following ones in the rest of the present Chapter are contained in the paper [ARS].

Theorem 39. *Let $(\mathbb{P}^n, \langle,\rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and let $u \in \mathcal{C}^\infty(\mathbb{P}^n)$, with $u : \mathbb{P}^n \to I = (a,b)$, be such that $\Sigma(u)$ is a spacelike hypersurface of $-I \times_\rho \mathbb{P}^n$ with bounded hyperbolic angle. We have*

$$\inf_{\mathbb{P}^n} H \le \frac{\sinh(\theta^*)}{n\rho(u^*)} h(\mathbb{P}^n) \qquad \text{if } \rho' \le 0$$

$$\sup_{\mathbb{P}^n} H \ge -\frac{\sinh(\theta^*)}{n\rho(u_*)} h(\mathbb{P}^n) \qquad \text{if } \rho' \ge 0,$$

where $\theta^* = \sup_{\mathbb{P}} \theta < \infty$, $u^* := \sup u \le b$ and $u_* := \inf u \ge a$.
We allow $\rho(u^*)$ and $\rho(u_*)$ to be zero and in these cases we have the trivial inequalities.

Proof. Consider the case $\varrho' \le 0$ and let $\Omega \subset\subset \mathbb{P}^n$ be a relatively compact domain with smooth boundary. Integrating (3.16) over Ω we get

$$n \inf_{\Omega} H(u) \operatorname{Vol}\Omega \le \int_{\partial\Omega} \frac{\langle Du, \nu\rangle}{\varrho(u)\sqrt{\varrho(u)^2 - |Du|^2}} + \int_{\Omega} \frac{\varrho'(u)}{\sqrt{\varrho(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{\varrho(u)^2} \right)$$

Since $\varrho' \le 0$ we have $\varrho(u) \ge \varrho(u^*)$ and

$$n \inf_{\Omega} H(u) \operatorname{Vol}\Omega \le \int_{\partial\Omega} \frac{|Du|}{\varrho(u)\sqrt{\varrho(u)^2 - |Du|^2}} = \int_{\partial\Omega} \frac{\sinh\theta}{\varrho(u)} \le \frac{\sinh(\theta^*)}{\varrho(u^*)} \operatorname{Vol}_{n-1}(\partial\Omega).$$

Therefore, we get

$$\inf_{\mathbb{P}^n} H(u) \leq \frac{\sinh(\theta^*)}{n\varrho(u^*)} \inf_{\Omega\subset\subset\mathbb{P}^n} \frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(\Omega)} = \frac{\sinh(\theta^*)}{n\varrho(u^*)} h(\mathbb{P}^n).$$

The case where $\varrho' \geq 0$ follows from a similar argument. □

In the case of vanishing Cheeger constant we obtain, as a consequence of Theorem 39, a sign on $\inf_{\mathbb{P}^n} H$ and on $\sup_{\mathbb{P}^n} H$, related to the sign of ρ'.

Corollary 40. *Let $(\mathbb{P}^n, \langle,\rangle)$ be an n-dimensional, non-compact, complete Riemannian manifold with vanishing Cheeger constant and let $u \in \mathcal{C}^\infty(\mathbb{P}^n)$, with $u : \mathbb{P} \to I = (a,b)$, be such that $\Sigma(u)$ is a spacelike graph in $-I \times_\varrho \mathbb{P}^n$ with bounded hyperbolic angle. If $\varrho' \leq 0$ and $u^* < b$ then $\inf_{\mathbb{P}} H \leq 0$. Analogously if $\varrho' \geq 0$ and $u_* > a$ then $\sup_{\mathbb{P}} H \geq 0$.*

In the case of Lorentzian product we have $\rho \equiv 1$ and Theorem 39 becomes

Corollary 41. *Let $(\mathbb{P}^n, \langle,\rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and let $u \in \mathcal{C}^\infty(\mathbb{P}^n)$ such that $\Sigma(u) \to -\mathbb{R} \times \mathbb{P}^n$ is spacelike and has bounded hyperbolic angle. We have*

$$\inf_{\mathbb{P}^n} H \leq \frac{\sinh(\theta^*)}{n} h(\mathbb{P}^n) \qquad \textit{and}$$
$$\sup_{\mathbb{P}^n} H \geq -\frac{\sinh(\theta^*)}{n} h(\mathbb{P}^n),$$

where $H(u)$ is the mean curvature of the graph $\Sigma(u)$ and $\theta^* = \sup_{\mathbb{P}} \theta$.

Observe that in the case of constant mean curvature we recover Salavessa's result Theorem 38.
Theorem 39 can be generalized to the case of spacelike hypersurfaces, not necessarily graphs, as follows.

Theorem 42. *Let $F : \Sigma^n \to -I \times_\varrho \mathbb{P}^n$ be a complete, non-compact spacelike hypersurface for which the hyperbolic angle is bounded, $\theta^* := \sup_\Sigma \theta < \infty$. If $\varrho' \le 0$ then*

$$\inf_\Sigma H \le \frac{\sinh\theta^*}{n} h_\Sigma,$$

where h_Σ denotes the Cheeger constant of Σ.
Analogously, if $\varrho' \ge 0$ then

$$\sup_\Sigma H \ge -\frac{\sinh(\theta^*)}{n} h_\Sigma.$$

Proof. Consider the height function $h = \pi_I \circ F : \Sigma \to I \subset \mathbb{R}$, whose gradient is given by

$$\nabla h = -\partial_t - \Theta N = -\partial_t + \cosh\theta N.$$

In particular,

$$|\nabla h|^2 = \Theta^2 - 1 = \sinh^2\theta.$$

Observe that

$$\nabla_X \nabla h = -\frac{\varrho'(h)}{\varrho(h)} \left(\langle X, \nabla h\rangle \nabla h + X\right) + \Theta A X$$

for every $X \in T\Sigma$. Therefore,

$$\Delta h = -\frac{\varrho'(h)}{\varrho(h)} \left(|\nabla h|^2 + n\right) - n\Theta H. \tag{3.19}$$

Consider the case where $\varrho' \le 0$. Then from the above we obtain

$$\Delta h \ge -n\Theta H = n\cosh\theta H.$$

If $\inf_\Sigma H < 0$ there is nothing to prove. So, without loss of generality we can assume that $\inf_\Sigma H \geq 0$. Fix $\Omega \subset\subset \Sigma$ a relatively compact domain with smooth boundary. Integrating over Ω and using divergence theorem we obtain

$$n \inf_\Omega H \operatorname{Vol}(\Omega) \leq \int_\Omega nH \cosh\theta \leq \int_\Omega \Delta h \leq \int_{\partial\Omega} |\nabla h| \leq \sinh\theta^* \operatorname{Vol}_{n-1}(\partial\Omega).$$

Therefore,

$$\inf_\Sigma H \leq \frac{\sinh\theta^*}{n} h_\Sigma.$$

The case where $\varrho' \geq 0$ is obtained in a a similar way. $\square$

As a consequence we get

Corollary 43. *Let $F : \Sigma^n \to -\mathbb{R} \times \mathbb{P}^n$ be a complete, non-compact, constant mean curvature spacelike hypersurface with bounded hyperbolic angle. If Σ^n has zero Cheeger constant then it is maximal, that is $H = 0$.*

Now, in the next result, we will use the open version of the weak maximum principle we introduced in Chapter 2, an analytic tool with interesting geometric applications. Namely, in the statement of the next result we will assume the validity on the Riemannian manifold $(\mathbb{P}^n, \langle , \rangle_\mathbb{P})$ of the weak maximum principle for the Lorentzian mean curvature operator

$$Lv := \operatorname{div}_\mathbb{P}\left(\frac{Dv}{\sqrt{1 - |Dv|^2}}\right),$$

with v in the class

$$\mathcal{A}_1(\mathbb{P}) = \{v \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{P}) : |Dv| < 1 \text{ and } (1 - |Dv|^2)^{-\frac{1}{2}} \in L^1_{\mathrm{loc}}(\mathbb{P})\}.$$

According to Proposition 25 of Chapter 2 this is guaranteed by the volume growth condition

$$\liminf_{r\to\infty} \frac{\log \operatorname{Vol} B_r}{r^2} < +\infty. \tag{3.20}$$

In terms of curvature, note that condition (3.20) is implied by

$$\mathrm{Ric}_{\mathbb{P}} \geq -c(1+r^2)\langle\, ,\, \rangle_{\mathbb{P}}$$

for some constant $c > 0$.
We achieve a height estimate for an entire maximal graph in a GRW spacetime.

Theorem 44. *Consider a generalized Robertson-Walker spacetime $-I \times_\varrho \mathbb{P}^n$, and assume on $\mathbb{P}^n$ the validity of the weak maximum principle for the Lorentzian mean curvature operator. Let $\Sigma(u)$ be an entire space-like maximal graph in $-I \times_\varrho \mathbb{P}^n$, with $I = (a,b)$, $-\infty \leq a < b \leq +\infty$, which is not a slice. Then*

$$\text{either } u^* = b \text{ or } u^* \leq \inf\{\lambda \in I : \varrho'(t) < 0 \text{ on } [\lambda, b)\}. \tag{3.21}$$

Similarly,

$$\text{either } u_* = a \text{ or } u_* \geq \sup\{\mu \in I : \varrho'(t) > 0 \text{ on } (a, \mu]\}. \tag{3.22}$$

Proof. Let us prove (3.21). The proof of (3.22) is analogous. Suppose that $u^* < b$ and, by contradiction, assume that

$$u^* > \inf\{\lambda \in I : \varrho'(t) < 0 \text{ on } [\lambda, b)\}.$$

Choose $\lambda < u^*$ such that $\varrho'(t) < 0$ on $[\lambda, b)$ and sufficiently near to u^* so that if $\Lambda_\lambda = \{x \in \Sigma : u(x) > \lambda\}$, then $\partial\Lambda_\lambda \neq \varnothing$. We fix an origin $o \in \mathbb{P}^n$ and for $u_0 := u(o)$ we consider the function

$$\psi(s) := \int_{u_0}^{s} \frac{dt}{\varrho(t)}, \quad \psi' = \frac{1}{\varrho} > 0.$$

Setting $v(x) = \psi(u(x))$, a calculation from (3.17) with $H = 0$ shows that

$$\operatorname{div}_{\mathbb{P}}\left(\frac{Dv}{\sqrt{1-|Dv|^2}}\right) = -n\frac{\varrho'(\psi^{-1}(v))}{\sqrt{1-|Dv|^2}}.$$

Let $\gamma = \psi(\lambda)$ and observe that

$$\Omega_\gamma = \{x \in \Sigma : v(x) > \gamma\} = \{x \in \Sigma : u(x) > \lambda\} = \Lambda_\lambda.$$

Since $\varrho'(\psi^{-1}(v)) < 0$ on Ω_γ, we have

$$\operatorname{div}_{\mathbb{P}}\left(\frac{Dv}{\sqrt{1-|Dv|^2}}\right) \geq -n\varrho'(\psi^{-1}(v)) \quad \text{ on } \Omega_\gamma.$$

Now observe that

$$\sup_{\Omega_\gamma} v = \psi(u^*) > \gamma = \sup_{\partial\Omega_\gamma} v.$$

Therefore, by the open form of the weak maximum principle we conclude that

$$-n\varrho'(u^*) \leq 0,$$

which is a contradiction. □

In the special case where the graph $\Sigma(u)$ is contained in a slab where ϱ has only one stationary point and this latter is a maximum, as an application of Theorem 44 we have a nice rigidity result.

Corollary 45. *Consider a generalized Robertson-Walker spacetime $-I \times_\varrho \mathbb{P}^n$, and assume on $\mathbb{P}^n$ the validity of the weak maximum principle for the Lorentzian mean curvature operator. For $a, b \in I$, $a < b$, let*

$$(a,b) \times \mathbb{P}^n = \{(t,x) : a < t < b, x \in \mathbb{P}^n\}$$

be an open slab in $-I \times_\varrho \mathbb{P}^n$, and assume that there exists $t_0 \in (a,b)$ with the property that $\varrho'(t) > 0$ on $[a, t_0)$ and $\varrho'(t) < 0$ on $(t_0, b]$. Then the *only entire maximal graph contained in $(a,b)\times\mathbb{P}^n$ is the slice $u\equiv t_0$.*

Proof. Choose $\varepsilon>0$ sufficiently small such that $\varrho'(t)>0$ on $(a-\varepsilon,t_0)$ and $\varrho'(t)<0$ on $(t_0,b+\varepsilon)$. Set $\alpha=a-\varepsilon$ and $\beta=b+\varepsilon$, and let $J=(\alpha,\beta)$. By contradicion, suppose that $\Sigma(u)$ is not a slice. By applying Theorem 44 to the generalized Robertson-Walker spacetime $-J\times_\varrho\mathbb{P}^n$ and taking into account that $u^*<b$ and $u_*>a$ we conclude that

$$u^*\leq t_0\leq u_*.$$

That is $u\equiv t_0$, contradiction. Therefore $\Sigma(u)$ must be a maximal slice in the slab $(a,b)\times\mathbb{P}^n$ and the only maximal slice contained in that slice is $u\equiv t_0$ □

In the next result we give a height estimate for spacelike graphs $\Sigma(u)$ which are not necessarily maximal but whose mean curvature satisfies $\inf H\leq 0$. This result is achieved using again the open form of the weak maximum principle for Lorentzian mean curvature operator.

Theorem 46. *Let $(\mathbb{P}^n,\langle,\rangle_{\mathbb{P}})$ be an n-dimensional Riemannian manifold and assume on it the validity of the WMP for the Lorentzian mean curvature operator. Consider $u\in\mathcal{C}^\infty(\mathbb{P}^n)$ bounded above. Assume the graph $\Sigma(u)$ to be a spacelike hypersurface of $-I\times_\varrho\mathbb{P}^n$ such that $H_*:=\inf_{\Sigma(u)}H\leq 0$.*
Then $\Sigma(u)$ is a slice $\mathbb{P}_{u_0}$ with $\mathcal{H}(u_0)=H_*\equiv H$ or $\mathcal{H}(u^*)\geq H_*$, with $u^*:=\sup u$.

Proof. If $\Sigma(u)$ is the slice $\mathbb{P}_{u_0}$ then, since $Du\equiv 0$, from (3.17) it follows directly $\mathcal{H}(u_0)=H_*\equiv H$. So, assume u non-constant. We reason by contradiction and we suppose $\mathcal{H}(u^*)<H_*$.
Define

$$\Omega_\gamma=\{x\in\mathbb{P}^n:u(x)>\gamma\}$$

having chosen $\gamma < u^*$ such that $\partial\Omega_\gamma \neq \varnothing$ and $\mathcal{H}(u) < H_*$ on Ω_γ. Note that this is always possible since $\mathcal{H}$ is continuous.
Reasoning as in Theorem 44, we consider the function

$$v(x) := \psi(u(x)) = \int_{u_0}^{u(x)} \frac{ds}{\varrho(s)}.$$

Then from (3.17), function v satisfies

$$\begin{aligned}\operatorname{div}\left(\frac{Dv}{\sqrt{1-|Dv|^2}}\right) &\geq n\varrho(u)\left(H_* - \frac{\mathcal{H}(u)}{\sqrt{1-|Dv|^2}}\right)\\ &\geq n\varrho(u)(H_* - \mathcal{H}(u)) \geq C(H_* - \mathcal{H}(\psi^{-1}(v))).\end{aligned}$$

on Ω_γ, where $C := n\min_{[\gamma,u^*]}\varrho$ and we have used the fact that $\mathcal{H}(u) < H_*$ on Ω_γ.
Since ψ is strictly increasing,

$$\sup_{\Omega_\gamma} v = v^* = \psi(u^*) > \psi(\gamma) = \sup_{\Omega_\gamma} v.$$

Then, because of the open form of the WMP we have

$$H_* - \mathcal{H}(\psi^{-1}(v^*)) = H_* - \mathcal{H}(u^*) \leq 0,$$

contradicting the fact that $H_* - \mathcal{H}(u^*) > 0$. $\square$

We comment the fact that in Theorem 46 conclusion $\mathcal{H}(u^*) \geq H_*$ gives interesting informations only if $\mathcal{H}(u^*) < 0$, that is $\varrho'(u^*) < 0$.

3.3 Height estimates for spacelike hypersurfaces

Let $F : \Sigma \to -I \times_{\varrho} \mathbb{P}^n$ be a spacelike hypersurface and denote with A the second fundamental form of the immersion with respect to the future-pointing Gauss map N. We are now going to define some differential operators which will be the key tool in the proof of our results in this section. Recall that the eigenvalues of A, $k_1, \dots, k_n$, are the principal curvatures of the hypersurface and their normalized elementary symmetric functions

$$H_k := \frac{(-1)^k}{\binom{n}{k}} \sum_{1 \le i_1 < \dots < i_k \le n} k_{i_1} \dots . k_{i_k}$$

define the future k-mean curvatures of the immersion. The *Newton tensors* $P_k : T\Sigma \to T\Sigma$, $k = 0, \dots, n$, associated to the immersion F are are inductively defined by

$$\begin{cases} P_0 := I \\ P_k := \binom{n}{k} H_k I + A \circ P_{k-1}. \end{cases}$$

The trace of P_k is $\operatorname{Tr} P_k = c_k H_k$, where $c_k := (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$. We can now define, via the Newton tensors, the second order linear differential operators

$$L_k := \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma),$$

associated to each P_k, by

$$L_k u = \operatorname{Tr}(P_k \circ \operatorname{hess}(u)).$$

Note that L_k is elliptic if and only if P_k is positive definite. We remark that for spacelike hypersurfaces a sufficient condition to guarantee the ellipticity of L_j for all $1 \le j \le k$ is the existence of an elliptic point jointly with the positivity of H_{k+1}, for some $1 \le k \le n-1$ (see [AC1, Section 3]). We introduce also, for $2 \le k \le n$, opportune linear combinations of operators L_i

$$\mathcal{L}_{k-1} = \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i} \mathcal{H}(h)^{k-1-i}(-\Theta)^i L_i = \mathrm{Tr}(\mathcal{P}_{k-1} \circ \mathrm{hess}),$$

where

$$\mathcal{P}_{k-1} = \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i} \mathcal{H}(h)^{k-1-i}(-\Theta)^i P_i.$$

Our aim is, now, to apply operators L_k and $\mathcal{L}_k$ on an appropriate function of h, the height of points of Σ, and to derive interesting geometric consequences of the validity of the open weak maximum principle for L_k. For the first point we refer to [AC1] and [AIR] and we recall

Proposition 47. *Let $F : \Sigma^n \to -I \times_\varrho \mathbb{P}^n$ be a spacelike hypersurface and let σ be defined on I by $\sigma(t) = \int_{t_0}^t \varrho(s)ds$, for some fixed $t_0 \in I$. Then*

$$\begin{cases} L_k h = -\mathcal{H}(h)(c_k H_k + \langle P_k \nabla h, \nabla h \rangle) - \Theta c_k H_{k+1} \\ \\ L_k \sigma(h) = -c_k(\varrho'(h) H_k + \Theta \varrho(h) H_{k+1}), \end{cases}$$

where $\Theta = \langle N, \partial_t \rangle$.

From Proposition 47 one obtains

$$\mathcal{L}_{k-1}\sigma(h) = -c_{k-1}\varrho(h)(\mathcal{H}(h)^k - (-\Theta)^k H_k),$$

for $2 \le k \le n$. Indeed, observe that, when $k = 2$,

$$\mathcal{L}_1 = (n-1)\mathcal{H}(h)\Delta - \Theta L_1.$$

Therefore, Proposition 47 implies directly

$$\mathcal{L}_1\sigma(h) = -c_1\varrho(h)(\mathcal{H}(h)^2 - (-\Theta)^2 H_2).$$

For the general case, proceeding by induction we get

$$\begin{aligned}
\mathcal{L}_{k-1}\sigma(h) =& \frac{c_{k-1}}{c_{k-2}}\mathcal{H}'(h)\sum_{i=0}^{k-2}\frac{c_{k-2}}{c_i}\mathcal{H}'(h)^{k-2-i}(-\Theta)^i L_i\sigma(h) \\
&+ (-\Theta)^{k-1}L_{k-1}\sigma(h) \\
=& \frac{c_{k-1}}{c_{k-2}}\mathcal{H}'(h)\mathcal{L}_{k-2}\sigma(h) + (-\Theta)^{k-1}L_{k-1}\sigma(h) \\
=& - c_{k-1}\varrho(h)\mathcal{H}'(h)^k + c_{k-1}\rho(h)\mathcal{H}'(h)(-\Theta)^{k-1}H_{k-1} \\
&- c_{k-1}\varrho(h)\left(\mathcal{H}'(h)(-\Theta)^{k-1}H_{k-1} - (-\Theta)^k H_k\right) \\
=& - c_{k-1}\varrho(h)\left(\mathcal{H}'(h)^k - (-\Theta)^k H_k\right).
\end{aligned}$$

Now, we start with an observation on the sign of H analogous to Corollary 40.

Proposition 48. *Consider $F : \Sigma^n \to -I \times_\varrho \mathbb{P}^n$ a spacelike hypersurface, where $I = (a, b)$ with $-\infty \le a < b \le +\infty$, and suppose the validity of the WMP on Σ for the Laplacian. If $\varrho' \le 0$ and $h^* < b$ then $H_* \le 0$; similarly, if $\varrho' \ge 0$ and $h_* > a$ then $H^* \ge 0$.*

Proof. We focus our attention on the first case, $\varrho' \le 0$. If h is constant then there is nothing to prove because, in this case, Σ is a slice $\{h^*\} \times \mathbb{P}$ with constant mean curvature $H = H_* = \mathcal{H}(h^*) \le 0$. If h is non-constant we reason by contradiction and assume that $H_* > 0$. Let $\gamma < h^*$ such that $\partial\Omega_\gamma \neq \varnothing$, where

$$\Omega_\gamma = \{x \in \Sigma : h(x) > \gamma\}.$$

Now, recall that h satisfies the equation

$$\Delta h = -\frac{\varrho'(h)}{\varrho(h)}(n + |\nabla h|^2) - n\Theta H,$$

so that, since $\varrho' \leq 0$, we get

$$\Delta h \geq -n\Theta H \geq nH \geq nH_* > 0.$$

Hence applying the open WMP on Ω_γ we get a contradiction, since $h^* = \sup_{\Omega_\gamma} h > \sup_{\partial\Omega_\gamma} h = \gamma$.

The case where $\varrho' \geq 0$ follows in a similar way. □

Now, we will give some height estimates for spacelike hypersurfaces with constant k mean curvature H_k in a generalized Robertson-Walker spacetime. These estimates are related to those discussed in Chapter 1. In that context, the authors assumed the timelike (TCC) or the null (NCC) convergence condition on the ambient space. We recall that a spacetime obeys the TCC if its Ricci curvature is nonnegative on timelike directions, while it satisfies the NCC if its Ricci curvature is nonnegative on lightlike directions. The first condition implies the second one, due to continuity. One can prove that a generalized Robertson-Walker spacetime $-I \times_\varrho \mathbb{P}^n$ obeys the TCC if and only if

$$\mathrm{Ric}_{\mathbb{P}} \geq (n-1)\sup_I (\log(\varrho)''\varrho^2)\langle,\rangle_{\mathbb{P}} \tag{3.23}$$

and

$$\varrho'' \leq 0, \tag{3.24}$$

where $\mathrm{Ric}_{\mathbb{P}}$ and $\langle , \rangle_{\mathbb{P}}$ are the Ricci and the metric tensors of the Riemannian factor. On the other hand, NCC is equivalent to (3.23). In what follows we will deal with the more general condition $\mathcal{H}' = (\log \varrho)'' \leq 0$. We start with an estimate that generalize Theorem 46 to the case of hypersurfaces (not necessarily graphs) and of higher order mean curvatures.

Theorem 49. *Let $F : \Sigma^n \to -I \times_\varrho \mathbb{P}^n$, $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$, be a spacelike hypersurface with non-zero constant k-mean curvature for some $2 \leq k \leq n$ and $h^* < b$. Assume the existence of an elliptic point with respect to the future pointing Gauss map and the validity on Σ of the WMP for the operator $\mathcal{L}_{k-1}$. If $\mathcal{H} \geq 0$ and $\mathcal{H}' \leq 0$, then*

$$\mathcal{H}(h^*)^k \geq H_k. \tag{3.25}$$

In the proof of Theorem 49 we will use the following particular case of Theorem 2.5 in [AMR] for trace operators, that are operators of the form

$$L_T(u) = \mathrm{Tr}(T \circ \mathrm{hess}(u)),$$

where T is a positive definite, symmetric endomorphism on TM.

Theorem 50. *The weak maximum principle holds on M for the operator L_T if and only if for each $f \in \mathcal{C}^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \varnothing$ and for each $v \in C^0(\bar{\Omega}) \cap C^1(\Omega)$ satisfying*

$$\begin{cases} i) \ L_T v \geq f(v) \ \text{on } \Omega \\ ii) \ \sup_\Omega v < +\infty, \end{cases} \tag{3.26}$$

we have that either

$$\sup_\Omega v = \sup_{\partial\Omega} \tag{3.27}$$

or

$$f\left(\sup_{\Omega} v\right) \leq 0. \tag{3.28}$$

We remark that the existence of an elliptic point on Σ and the fact that H_k is a non-zero constant imply that $H_k > 0$. Hence $\mathcal{P}_j$ is positive definite for all $1 \leq j < k-1$. Since $\mathcal{H}(h) > 0$ and $\Theta \leq -1 < 0$ we have that $\mathcal{P}_{k-1}$ itself is positive definite and, equivalently, operator L_k is elliptic.

Proof. If h is constant then there is nothing to prove because, in this case, Σ is a slice $\{h^*\} \times \mathbb{P}$ with constant k-mean curvature $H_k = \mathcal{H}(h^*)^k$. If h is non-constant we reason by contradiction and assume that $\mathcal{H}(h^*)^k < H_k$. Let $\gamma < h^*$ be such that $\partial\Omega_\gamma \neq \varnothing$ and $\mathcal{H}(\gamma)^k < H_k$, where

$$\Omega_\gamma = \{x \in \Sigma : h(x) > \gamma\}.$$

Define $v := \sigma(h)$, where $\sigma(t) := \int_{t_0}^{t} \varrho(s)ds$. Note that, since σ is an increasing function, $\sigma(h)^* = \sigma(h^*) < +\infty$.

Recalling that $\mathcal{H}$ is non-increasing and $\Theta \leq -1$ we have

$$\mathcal{H}(h)^k - (-\Theta)^k H_k \leq \mathcal{H}(\gamma)^k - H_k \quad \text{ on } \Omega_\gamma.$$

Therefore, since ϱ is non-decreasing and $(\mathcal{H}(\gamma)^k - H_k) < 0$, we get

$$\begin{aligned} \mathcal{L}_{k-1} v &= -c_{k-1}\varrho(h)(\mathcal{H}(h)^k - (-\Theta)^k H_k) \\ &\geq -c_{k-1}\varrho(h)(\mathcal{H}(\gamma)^k - H_k) \\ &\geq -c_{k-1}\varrho(\gamma)(\mathcal{H}(\gamma)^k - H_k) > 0 \end{aligned}$$

on Ω_γ, with $\sup_{\Omega_\gamma} v < +\infty$. Observe that

$$\sigma(h^*) = \sup_{\Omega_\gamma} v > \sup_{\partial\Omega_\gamma} v = \sigma(\gamma).$$

Then, applying Theorem 50 on Ω_γ to the elliptic trace operator $\mathcal{L}_{k-1}$, with $f \equiv -c_{k-1}\varrho(\gamma)(\mathcal{H}(\gamma)^k - H_k)$, we get

$$-c_{k-1}\varrho(\gamma)(\mathcal{H}(\gamma)^k - H_k) \leq 0$$

which is a contradiction. $\square$

Now, we will consider the case $\varrho \equiv 1$, that is the case of Riemannian products. We are going to give, first, a height estimate for open subsets of stochastically complete spacelike CMC hypersurfaces immersed in $-I \times \mathbb{P}^n$. We assume a curvature bound on the Riemannian factor, namely $\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha$, and we consider planar boundary open subsets Ω contained in a slab. Moreover, we will assume the square of the hyperbolic angle on Ω bounded from above by a constant depending on the mean curvature of Σ and the lower bound of $\mathrm{Ric}_{\mathbb{P}}$.

Theorem 51. *Let $F : \Sigma^n \to -\mathbb{R} \times \mathbb{P}^n$ be a stochastically complete spacelike hypersurface with constant mean curvature $H > 0$. Suppose that for some $\alpha > 0$*

$$\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha. \tag{3.29}$$

Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \varnothing$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$. Assume

$$\beta^2 = \sup_{\Omega} \Theta^2 < \frac{\alpha + H^2}{\alpha}. \tag{3.30}$$

Then

$$F(\Omega) \subset \left[\frac{(1-\beta)H}{H^2 - \alpha(\beta^2 - 1)}, 0 \right] \times \mathbb{P}^n. \tag{3.31}$$

Proof. If $\beta = 1$ then there is nothing to prove because, in this case, $\Theta \equiv -1$ is constant on Ω, or equivalently h is constant on Ω. Thus, $F(\Omega)$ is contained in the slice $\{0\} \times \mathbb{P}$.

Let $\beta > 1$. From (3.30), we can choose $\delta > 0$ sufficiently small such that

$$(\alpha+\delta)(\beta^2-1) < H^2.$$

We consider the function

$$\psi_\delta := \phi - \frac{\alpha+\delta}{H}(\beta^2-1)h = \Theta + \frac{H^2-(\alpha+\delta)(\beta^2-1)}{H}h,$$

where $\phi := \Theta + Hh$. From Proposition 47 (with $k=0$) we know that

$$\Delta h = -n\Theta H,$$

and by equation (8.10) in [AC1] we also have that

$$\Delta\Theta = \Theta|A|^2 + \Theta\,\mathrm{Ric}_{\mathbb{P}}(N^*,N^*),$$

where N^* denotes the projection of N onto the fiber $\mathbb{P}^n$. Therefore, using $|A|^2 = n^2H^2 - n(n-1)H_2$ we obtain

$$\Delta\psi_\delta = \Theta\{n(n-1)(H^2-H_2) + \mathrm{Ric}_{\mathbb{P}}(N^*,N^*) + n(\alpha+\delta)(\beta^2-1)\}. \quad (3.32)$$

From (3.29),

$$\mathrm{Ric}_{\mathbb{P}}(N^*,N^*) \geq -n\alpha|N^*|^2 = -n\alpha(\Theta^2-1) \geq -n\alpha(\beta^2-1) \quad \text{on } \Omega.$$

Thus using the basic inequality $H^2 \geq H_2$, (3.32) implies

$$\Delta\psi_\delta \leq n\Theta\delta(\beta^2-1) \leq -n\delta(\beta^2-1) < 0 \qquad \text{on } \Omega,$$

where the last inequality is due to $\beta > 1$. We define $w = \psi_{\delta|\overline{\Omega}}$. Since

$F(\Omega)$ is contained in a slab we have

$$\begin{cases} \Delta w \leq -n\delta(\beta^2 - 1) \text{ on } \Omega; \\ \\ \inf_\Omega w > -\infty, \end{cases}$$

Stochastic completeness of Σ and Theorem 24 give

$$\inf_{\Omega} w = \inf_{\partial\Omega} w.$$

By assumption $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$ and thus $h \equiv 0$ on $\partial\Omega$, so that $w = \psi_\delta = \Theta \geq -\beta$ on $\partial\Omega$. We then have

$$-\beta \leq \Theta + \frac{H^2 - (\alpha + \delta)(\beta^2 - 1)}{H} h \leq -1 + \frac{H^2 - (\alpha + \delta)(\beta^2 - 1)}{H} h.$$

That is, dividing by the positive quantity $H^2 - (\alpha + \delta)(\beta^2 - 1)$,

$$h \geq \frac{(1 - \beta)H}{H^2 - (\alpha + \delta)(\beta^2 - 1)}.$$

Taking the limit as $\delta \downarrow 0$ we deduce

$$h \geq \frac{(1 - \beta)H}{H^2 - \alpha(\beta^2 - 1)}.$$

On the other hand

$$\begin{cases} \Delta h = -nH\Theta \geq nH > 0; \\ \\ \sup_\Omega h < +\infty. \end{cases}$$

Using again Theorem 24 in Chapter 2 we deduce $\sup_\Omega h = \sup_{\partial\Omega} h = 0$, that is $h \leq 0$ on Ω. This completes the proof of the theorem. $\square$

We remark that in Theorem 51 what we need is that H has a sign, not necessarily $H > 0$. Indeed, in case $H < 0$ we may replace (3.31) with

$$F(\Omega) \subset \left[0, \frac{(1-\beta)H}{H^2 - \alpha(\beta^2 - 1)}\right] \times \mathbb{P}^n.$$

The proof is analogous, substituting ψ_δ with $-\psi_\delta$.
Moreover, in Theorem 51 we assume $\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha$ with $\alpha > 0$. In case $\mathrm{Ric}_{\mathbb{R}} \geq 0$, we can not use $\alpha = 0$ in the Theorem because (3.30) does not make sense. We observe, now, that a limit reasosing will allow us to substitute, in this asset, (3.30) with $\sup_\Omega |\Theta| < \infty$. Indeed, we can consider $\hat{\alpha}$ sufficiently small so that (3.30) holds with $\alpha := \hat{\alpha}$. Then, letting $\hat{\alpha} \downarrow 0$ we obtain

Corollary 52. *Let $F : \Sigma^n \to -\mathbb{R} \times \mathbb{P}^n$ be a stochastically complete spacelike hypersurface with constant mean curvature $H > 0$. Suppose that*

$$\mathrm{Ric}_{\mathbb{P}} \geq 0. \tag{3.33}$$

Let $\Omega \subset \Sigma$ be an open set with $\partial\Omega \neq \varnothing$ for which $F(\Omega)$ is contained in a slab and $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$, and assume

$$\beta = \sup_\Omega |\Theta| < +\infty. \tag{3.34}$$

Then

$$F(\Omega) \subset \left[\frac{1-\beta}{H}, 0\right] \times \mathbb{P}^n. \tag{3.35}$$

In the following Theorem we will give a geometric condition guaranteeing stochastic completeness of Σ

Theorem 53. *Let $F : \Sigma^n \to -\mathbb{R} \times \mathbb{P}^n$ be a complete spacelike hypersurface with constant mean curvature $H > 0$. Assume that the height function $h = \pi_{\mathbb{R}} \circ F : \Sigma \to \mathbb{R}$ satisfies*

$$\lim_{x \to \infty} h(x) = -\infty.$$

Suppose that for some $\alpha > 0$

$$\mathrm{Ric}_{\mathbb{P}} \geq -n\alpha. \tag{3.36}$$

Let $\Omega \subset \Sigma$ be a relatively compact open set with $\partial\Omega \neq \varnothing$ such that $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$. Assume

$$\beta^2 = \sup_{\Omega} \Theta^2 < \frac{\alpha + H^2}{\alpha}. \tag{3.37}$$

Then

$$F(\Omega) \subset \left[\frac{(1-\beta)H}{H^2 - \alpha(\beta^2 - 1)}, 0 \right] \times \mathbb{P}^n. \tag{3.38}$$

Proof. We only have to show that condition $\lim_{x \to \infty} h(x) = -\infty$ implies the validity of the WMP on Σ for the Laplacian and then apply Theorem 51. Towards this end we let $\gamma = -h$ so that it satisfies

$$\Delta\gamma = n\Theta H \leq -nH < 0$$

and

$$\gamma(x) \to +\infty \qquad \text{as } x \to \infty.$$

We then apply Theorem A of [AAR] to get the desired conclusion.

□

Now, we want to generalize Theorem 51 to the case of higher order mean curvatures. We will substitute the bound on Ricci tensor of $\mathbb{P}^n$ with an analogous bound on the sectional curvature and the stochastic completeness with the validity of the weak maximum principle for an opportune Newton operator.

Theorem 54. *Let $F : \Sigma \to -\mathbb{R} \times \mathbb{P}^n$ be an immersed hypersurface with constant, non-zero k-mean curvature H_k, for some $k = 2, \dots, n$ and with* *an elliptic point with respect to the future-pointing Gauss map. Suppose that the sectional curvature of* $\mathbb{P}^n$ *satisfies*

$$K_{\mathbb{P}^n} > -\alpha,$$

for some $\alpha > 0$ *and assume the validity of the WMP for the operator* L_{k-1} *on* Σ*. Let* $\Omega \subset \Sigma$ *be an open set with* $\partial\Omega \neq \varnothing$ *for which* $F(\Omega)$ *is contained in a slab and* $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$*. Assume*

$$\beta^2 = \sup_{\Omega} \Theta^2 < \frac{\alpha H^*_{k-1} + H_k^{(k+1)/k}}{\alpha H^*_{k-1}},$$

where $H^*_{k-1} := \sup_\Omega H_{k-1}$*. Then*

$$F(\Omega) \subseteq \left[\frac{(1-\beta) H_k}{H_k^{\frac{k+1}{k}} - \alpha(\beta^2 - 1) H^*_{k-1}}, 0 \right].$$

Observe that, under our assumptions, $H^*_{k-1} := \sup_\Omega H_{k-1} > 0$. In fact, the existence of an elliptic point on Σ and the fact that H_k is a non-zero constant imply that $H_k > 0$. Then, by Lemma 3.3 in [AC1], P_{k-1} is positive definite and, in particular, $H_{k-1}(x) > 0$ for every $x \in \Sigma$.

Proof. As in the proof of Theorem 51 we may assume that $\beta > 1$. Otherwise, $F(\Omega)$ is contained in the slice $\{0\} \times \mathbb{P}^n$ and there is nothing to prove.

Let us consider the function $\phi := H_k^{\frac{1}{k}} h + \Theta$. We know that

$$L_{k-1} h = -c_{k-1} \Theta H_k. \tag{3.39}$$

On the other hand H_k is constant, so by Corollary 8.5 in [AC1] we have

$$\begin{aligned} L_{k-1}\Theta &= \binom{n}{k}(nH_1H_k - (n-k)H_{k+1})\Theta \\ &\quad +\Theta\sum_{i=1}^{n}\mu_{k-1,i}K_{\mathbb{P}}(N^*\wedge E_i^*)|N^*\wedge E_i^*|^2, \end{aligned} \tag{3.40}$$

where $\{E_i\}_1^n$ is a local orthonormal frame that diagonalizes P_{k-1}, the $\mu_{k-1,i}$'s are the eigenvalues of this latter and star denotes projection onto $\mathbb{P}^n$. We then get

$$L_{k-1}\phi = \binom{n}{k}(nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}})\Theta \tag{3.41}$$

$$+\,\Theta\sum_{i=1}^{n}\mu_{k-1,i}K_{\mathbb{P}}(N^*\wedge E_i^*)|N^*\wedge E_i^*|^2. \tag{3.42}$$

Using Gårding inequalities we obtain

$$H_1H_k \geq H_k^{\frac{k+1}{k}},$$

and therefore

$$nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}} \geq (n-k)(H_k^{\frac{k+1}{k}} - H_{k+1}) \geq 0. \tag{3.43}$$

From the decompositions

$$N = N^* - \langle N,\partial_t\rangle\partial_t, \quad E_i = E_i^* - \langle E_i,\partial_t\rangle\partial_t, \quad \text{and } \partial_t = -\nabla h - \langle N,\partial_t\rangle N,$$

it follows easily that

$$|E_i^*\wedge N^*|^2 = |\nabla h|^2 - \langle E_i,\nabla h\rangle^2.$$

In particular,

$$|E_i^*\wedge N^*|^2 \leq |\nabla h|^2 = \Theta^2 - 1.$$

Now, recall that the existence of an elliptic point on Σ and the assumption that H_k is a non-zero constant imply that $H_k > 0$ and P_{k-1} is positive definite. So, using this latter fact and the assumption on $K_{\mathbb{P}}$, we have

$$\mu_{k-1,i} K_{\mathbb{P}}(N^* \wedge E_i^*)|N^* \wedge E_i^*|^2 \geq -\alpha \mu_{k-1,i}(\Theta^2 - 1). \tag{3.44}$$

Inserting (3.43) and (3.44) into (3.41), we estimate

$$L_{k-1}\phi \leq -\Theta\alpha(\Theta^2 - 1)\operatorname{Tr}(P_{k-1}) = -\Theta\alpha(\Theta^2 - 1)c_{k-1}H_{k-1}. \tag{3.45}$$

In particular

$$L_{k-1}\phi \leq -\Theta\alpha(\beta^2 - 1)c_{k-1}H_{k-1} \quad \text{ on } \Omega. \tag{3.46}$$

Now, choose $\delta > 0$ satisfying

$$(\alpha H_{k-1}^* + \delta)(\beta^2 - 1) < H_k^{\frac{k+1}{k}}$$

and define

$$\begin{aligned} \psi_\delta &= \phi - \frac{\alpha H_{k-1}^* + \delta}{H_k}(\beta^2 - 1)h \\ &= \Theta + \frac{H_k^{\frac{k+1}{k}} - (\alpha H_{k-1}^* + \delta)(\beta^2 - 1)}{H_k} h \end{aligned}$$

We let $w = \psi_\delta|_{\overline{\Omega}}$. Using (3.39) and (3.46) we obtain

$$\begin{aligned} L_{k-1}w &\leq c_{k-1}(\beta^2 - 1)\Theta\{\alpha(H_{k-1}^* - H_{k-1}) + \delta\} \\ &\leq c_{k-1}(\beta^2 - 1)\Theta\delta \leq -c_{k-1}(\beta^2 - 1)\delta < 0 \end{aligned}$$

on Ω, where the last inequality is due to $\beta > 1$. Since $F(\Omega)$ is contained in a slab we also have

$$\inf_{\Omega} w > -\infty.$$

Using Theorem 50 for the elliptic trace operator L_{k-1}, we deduce

$$\inf_{\Omega} w = \inf_{\partial\Omega} w.$$

Therefore,

$$\begin{aligned}
&-1 + \frac{H_k^{\frac{k+1}{k}} - (\alpha H_{k-1}^* + \delta)(\beta^2 - 1)}{H_k} h \\
&\geq \Theta + \frac{H_k^{\frac{k+1}{k}} - (\alpha H_{k-1}^* + \delta)(\beta^2 - 1)}{H_k} h \\
&\geq -\beta,
\end{aligned}$$

and letting $\delta \to 0$ we finally obtain

$$h \geq \frac{(1-\beta)H_k}{H_k^{\frac{k+1}{k}} - \alpha(\beta^2 - 1)H_{k-1}^*} \quad \text{on } \Omega, \tag{3.47}$$

since $H_k^{\frac{k+1}{k}} - \alpha(\beta^2 - 1)H_{k-1}^* > 0$. On the other hand, by (3.39) we also have

$$\begin{cases} L_{k-1}h = -c_{k-1}\Theta H_k \geq c_{k-1}H_k > 0 & \text{on } \Omega \\ \\ \sup_{\Omega} h < +\infty. \end{cases}$$

Reasoning as above we deduce

$$\sup_{\Omega} h = \sup_{\partial\Omega} h,$$

that implies $h \leq 0$ and, combining this inequality with (3.47), we get the desired conclusion. □

Finally, we conclude with a geometric condition that guarantees the validity of the weak maximum principle for the operator L_{k-1} on Σ.

Theorem 55. *Let $F : \Sigma \to -\mathbb{R} \times \mathbb{P}^n$ be an immersed hypersurface with constant, non-zero k-mean curvature H_k, for some $k = 2, \ldots, n$ and with an elliptic point with respect to the future-pointing Gauss map. Assume that the height function $h = \pi_{\mathbb{R}} \circ F : \Sigma \to \mathbb{R}$ satisfies*

$$\lim_{x \to \infty} h(x) = -\infty.$$

Suppose that the sectional curvature of $\mathbb{P}^n$ satisfies

$$K_{\mathbb{P}^n} > -\alpha,$$

for some $\alpha > 0$. Let $\Omega \subset \Sigma$ be a relatively compact open set with $\partial\Omega \neq \varnothing$ such that $F(\partial\Omega) \subset \{0\} \times \mathbb{P}^n$. Assume

$$\beta^2 = \sup_{\Omega} \Theta^2 < \frac{\alpha H^*_{k-1} + H_k^{(k+1)/k}}{\alpha H^*_{k-1}},$$

where $H^*_{k-1} := \sup_{\Omega} H_{k-1}$. Then

$$F(\Omega) \subseteq \left[\frac{(1-\beta) H_k}{H_k^{\frac{k+1}{k}} - \alpha(\beta^2 - 1) H^*_{k-1}}, 0 \right].$$

Proof. We only have to show the validity of the weak maximum principle on Σ for the operator L_{k-1} and then the result follows from Theorem 54. Towards this end we let $\gamma = -h$, so that it satisfies

$$L_{k-1}\gamma = c_{k-1} \Theta H_k \leq -c_{k-1} H_k < 0$$

and

$$\gamma(x) \to +\infty \qquad \text{as } x \to \infty.$$

We then apply Theorem A of [AAR] to get the desired conclusion. $\square$

Chapter 4

Height estimates in the Affine Space

4.1 Basic definitions in Affine Geometry

In his Erlangen Program, Felix Klein proposed to approach geometry as the study of invariants under some allowed transformations. Let S be a set of points and G a subgroup of the group of bijections of S. We say that $A \subset S$ and $B \subset S$ are equivalent in the geometry given by G if there exists a transformation $f \in G$ such that $f(A) = B$. So, we are intersted in those properties that are invariant under the action of G. A property P of a subset of S is a *geometric property* if, for any $g \in G$, P is true for $g(S)$. For instance, if $S = \mathbb{R}^n$ orthogonality is a geometric property for Euclidean geometry but not for Affine geometry.
Consider, again, $S = \mathbb{R}^n$. Affine geometry deals with properties invariant under *affine transformations*, that are transformations of the form

$$T(x) := Ax + b \qquad A \in GL(n, \mathbb{R}),\ b \in \mathbb{R}^n. \tag{4.1}$$

In a vector space V, we say that a subset $W \subset V$ is an *affine subspace* if there exists $b \in V$ such that $W + b$ is a linear subspace of V. An affine trasformation sends affine subspaces in affine subspaces and preserves parallelism. Indeed, consider two affine subspace W_1, W_2 in $\mathbb{R}^n$. They are said to be parallel if there exist $b \in \mathbb{R}^n$ such that $W_1 = W_2 + b$. Let $A \in \mathrm{GL}(n, \mathbb{R})$, we have $AW_1 = AW_2 + Ab$, so that AW_1 and AW_2 are parallel.
Before going on, we formalize the concepts we just treated.

Definition 56. Let X be a set of points and V a vector space. We say that $(X, V, +)$ is an *affine space* if $+ : X \times V \to X$ satisfies the following conditions

A1 $\forall P, Q \in X$ there exists a unique vector $v \in V$ such that $P + v = Q$;

A2 $\forall v, w \in V$ and $\forall P \in X$ we have $(P + v) + w = P + (v + w)$;

A3 $\forall P \in X$ we have $P + 0_V = P$.

For $P, Q \in X$, we denote by $P - Q$ the uniquely determined (due to **A1**) vector $v \in V$ such that $P = Q + v$.
Clearly if V is a vector space, $(V, V, +)$ is an affine space, where $+$ is the sum in V.
If $(X, V, +)$ is an affine space, $(Y, W, +)$ is an *affine subspace* if $Y \subset X$, $W < V$ is a subspace of the vector space V and for all $y \in Y$ and $w \in W$ we have $y + w \in Y$. This means that Y is closed under summation with vectors of W.
If we choose an origin $O \in X$, we may look at X as a vector space. Indeed, since for all $P \in X$ vector $P - O \in V$ is uniquely determined, we have a bijection between X and V. However, one should keep in mind that, since the choice of the origin is arbitrary, properties to be considered are only those origin-independent.

Now, we recall some facts on affine differential geometry, for more details see [LSZ],[CY1].

Let us consider a $n+1$-dimensional real affine space $\mathbb{A}^{n+1}$. We fix a coordinate system, that is an origin in $\mathbb{A}^{n+1}$ and a basis in the real vector space associated, so that we can think of $\mathbb{A}^{n+1}$ simply as $\mathbb{R}^{n+1}$. Considering the group of unimodular affine transformations acting over $\mathbb{A}^{n+1}$

$$x' = Ax + b \qquad A \in SL(n+1,\mathbb{R}),\ b \in \mathbb{R}^{n+1} \tag{4.2}$$

we call $\mathbb{A}^{n+1}$ *unimodular affine space.*

Under unimodular affine transformations (4.2), vectors transform as

$$v' = Av \quad A \in SL(n+1,\mathbb{R}),$$

so the determinant of $n+1$ vectors is an affine invariant. Indeed

$$\begin{aligned}\det(v'_1,\dots,v'_{n+1}) &= \det(Av_1,\dots,Av_{n+1})\\ &= \det(A(v_1,\dots,v_{n+1})) = \det(v_1,\dots,v_{n+1}).\end{aligned}$$

Let us consider, now, a hypersurface M^n immersed in $\mathbb{A}^{n+1}$ and let us call x its position vector. By *local adapted affine frame* we mean $n+1$ local vector fields $\{e_1,\dots,e_{n+1}\}$ in $\mathbb{A}^{n+1}$ such that $\det(e_1,\dots,e_{n+1}) = 1$ and $e_1,\dots,e_n$ are tangent to M.

We summarize the structure equations:

$$dx = \omega^\alpha e_\alpha \tag{4.3}$$

$$de_\alpha = \omega_\alpha^\beta e_\beta \tag{4.4}$$

$$d\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha \tag{4.5}$$

$$d\omega_\beta^\alpha = \omega_\beta^\gamma \wedge \omega_\gamma^\alpha, \tag{4.6}$$

where the forms ω^α and ω_β^α are defined by (4.3)-(4.4) and equations

(4.5)-(4.6) are obtained by exterior differentiation of the first two equations. We adopt this convention for the range of indices

$$1 \le \alpha, \beta, \gamma, \dots \le n+1$$

$$1 \le i, j, k, \dots \le n$$

and we adopt Einstein's summation convention.
The condition $\det(e_1, \dots, e_{n+1}) = 1$ implies that $\sum \omega_\alpha^\alpha = 0$ and the fact that the first n vector fields are tangent to M implies that $\omega^{n+1} = 0$ over M. So we have

$$\omega^\beta \wedge \omega_\beta^{n+1} = 0,$$

hence by Cartan's Lemma there exist functions h_{ij} such that

$$\omega_i^{n+1} = h_{ij}\omega^j \tag{4.7}$$

$$h_{ij} = h_{ji}. \tag{4.8}$$

We assume M to be locally strictly convex. This means that any point $p \in M$ admits a neighborhood U_p which is a convex graph under appropriate choice of coordinates. Due to this assumption, we can suppose (h_{ij}) to be a positive definite matrix. Then we introduce the tensor

$$II = |H|^{-\frac{1}{n+2}} h_{ij}\omega^i \otimes \omega^j,$$

where $H := \det(h_{ij})$. This tensor field can be proved to be invariant under unimodular affine transformations, so it defines an affinely invariant Riemannian metric, called *Blaschke metric.*
Defining the vector

$$Y := \frac{1}{n}\Delta x,$$

where Δ denotes the Laplacian with respect to the Blaschke metric, we obtain a trasversal vector field that is invariant under unimodular transformations. If we suppose e_{n+1} parallel to Y we have the so called *apolarity condition*

$$\omega_{n+1}^{n+1} + \frac{1}{n+2} d \log H = 0. \tag{4.9}$$

Suppose, now, $H = 1$, so that $\omega_{n+1}^{n+1} = 0$. Exterior differentiating (4.7) we obtain

$$(dh_{ij} - h_{ik}\omega_j^k - h_{kj}\omega_i^k) \wedge \omega^j = 0,$$

for all i, so by Cartan's lemma we can define coefficients h_{ijk} such that

$$\begin{aligned} &h_{ijk}\omega^k = dh_{ij} - h_{kj}\omega_i^k - h_{ik}\omega_j^k \\ &h_{ijk} = h_{ikj}. \end{aligned}$$

We remark now that we can consider on M two connections, the Levi-Civita connection of the Blaschke metric, $\tilde{\nabla}$, whose connection forms are denoted by $\tilde{\omega}_j^i$ and the so called *induced connection*, ∇, whose connection forms are ω_j^i. This is the restriction to M of the affine flat connection of $\mathbb{A}^{n+1}$, say $\overline{\nabla}$, defined by

$$\overline{\nabla}_{e_i} e_j = de_j(e_i).$$

One can prove that

$$\tilde{\omega}_j^i - \omega_j^i = \frac{1}{2} h^{ik} h_{kjs} \omega^s,$$

where (h^{ij}) denotes the inverse matrix of (h_{ij}).

Lemma 57. *If we assume $H = 1$ we have the apolarity condition*

$$h^{ij} h_{ijk} \omega^k = 0. \tag{4.10}$$

Proof. Using Jacobi relation $\mathrm{Tr}(A^{-1}dA) = d\log\det A$ we obtain

$$\begin{aligned} 0 = d\log H = h^{ij}dh_{ij} \\ = h^{ij}(h_{ijk}\omega^k + h_{kj}\omega_i^k + h_{ik}\omega_j^k) \\ = h^{ij}h_{ijk}\omega^k. \end{aligned}$$

□

Exterior differentiating (4.9) we obtain $d\omega_{n+1}^{n+1} = 0$ hence we have

$$\omega_{n+1}^i \wedge \omega_i^{n+1} = 0$$

and applying Cartan's lemma with respect to ω_i^{n+1}, that are linearly independent, we can define

$$\begin{aligned} &\omega_{n+1}^i =: -l^{ij}\omega_j^{n+1} = -l^{ij}h_{jk}\omega^k =: -l_k^i\omega^k \\ &l^{ij} = l^{ji} \\ &l_{ji} := l_j^k h_{ki} = l_i^k h_{kj} =: l_{ij} \end{aligned}$$

We introduce the symmetric quadratic form

$$B := l_{ij}\omega^i\omega^j = l_j^k h_{ki}\omega^i\omega^j$$

called *third fundamental form* and we call *affine shape operator* the self-adjoint operator implicitly defined by

$$B(v, w) = II(\sigma(v), w).$$

A direct calculation shows that $\sigma = -dY$. The eigenvalues of σ, say $k_1, \ldots, k_n$, are called *affine principal curvatures* and the normalized elementary symmetric functions of these eigenvalues are the *affine mean* *curvature functions*

$$\binom{n}{r} H_r := S_r := \sum_{1 \le i_1 < \cdots < i_r \le n} k_{i_1} \dots k_{i_r}.$$

H_1 is the *affine mean curvature* and H_n the *affine Gauss-Kronecker curvature.*

4.2 The Laplacian of the affine normal

In this section we compute the Laplacian with respect to the Blaschke metric of the affine normal Y of a locally strongly convex hypersurface M immersed in $\mathbb{A}^{n+1}$. See also [NO].
Let us consider a local unimodular affine frame along M, that is a local adapted frame $\{e_1, \dots, e_{n+1}\}$ such that $e_{n+1} = Y$. We are assuming $H = 1$. We have

$$dY = de_{n+1} = \omega^i_{n+1} e_i = -l^i_j \omega^j e_i = Y_j \omega^j$$

and

$$\begin{aligned} Y_{ij}\omega^j &= dY_i - Y_k \tilde{\omega}^k_i \\ &= d(-l^j_i e_j) + l^j_k e_j \tilde{\omega}^k_i \\ &= -dl^j_i e_j - l^j_i \omega^k_j e_k - l^j_i \omega^{n+1}_j e_{n+1} + l^j_k e_j \tilde{\omega}^k_i \\ &= -dl^j_i e_j - l^j_i \omega^k_j e_k + l^j_k e_j \tilde{\omega}^k_i - l_{ij}\omega^j e_{n+1}. \end{aligned}$$

Differentiating $\omega^i_{n+1} = -l^i_j \omega^j$ we obtain

$$(dl^i_j + l^k_j \omega^i_k - l^i_k \omega^k_j) \wedge \omega^j = 0,$$

so applying Cartan's lemma we can define coefficients $l^i_{jk} = l^i_{kj}$ by

$$l^i_{jk}\omega^k = dl^i_j + l^k_j\omega^i_k - l^k_i\omega^k_j.$$

Hence

$$\begin{aligned}
Y_{ij}\omega^j &= -l^j_{ik}\omega^k e_j + l^k_i\omega^j_k e_j - l^j_k\omega^k_i e_j - l^j_i\omega^k_j e_k + l^j_k e_j\tilde{\omega}^k_i - l_{ij}\omega^j e_{n+1} \\
&= -l^k_{ij}\omega^j e_k + l^k_j e_k(\tilde{\omega}^j_i - \omega^j_i) - l_{ij}\omega^j e_{n+1} \\
&= -l^k_{ij}\omega^j e_k + \frac{1}{2}l^k_m e_k h^{ms}h_{sij}\omega^j - l_{ij}\omega^j e_{n+1} \\
&= -l^k_{ij}\omega^j e_k + \frac{1}{2}l^k_s h_{sij}e_k\omega^j - l_{ij}\omega^j e_{n+1}.
\end{aligned}$$

Therefore we have

$$\Delta Y = h^{ij}Y_{ij} = -h^{ij}l^k_{ij}e_k + \frac{1}{2}l^k_s h^{ij}h_{sij}e_k - nH_1 e_{n+1}.$$

Using $h^{ij}l^k_{ij} = nH_{1,k}$ and (4.10) we have

$$\Delta Y = -nH_{1,k}e_k - nH_1 e_{n+1}.$$

Lemma 58. *If $M^n \hookrightarrow \mathbb{A}^{n+1}$ is a locally strongly convex immersed hypersurface, the Laplacian of the affine normal is parallel to the affine normal if and only if the affine mean curvature is constant and in this case we have*

$$\Delta Y = -nH_1 Y.$$

4.3 The main theorem

In this section we find a height estimate for compact, convex hypersurfaces in $\mathbb{A}^{n+1}$ with planar boundary and constant affine mean curvature. This result is contained in the paper [Sc].

Before stating our theorem we have to clarify what *affine height* means. In the affine space we do not have a notion of natural distance between points that is invariant under affine transformations but we can look for an affinely invariant distance between points and hyperplanes, using the volume form. Let us consider a hyperplane Π containing the origin and a point $P \in \mathbb{R}^{n+1}$. The main idea is to consider a basis for Π, say $e_1, \ldots, e_n$, to complete this basis with a vector e_{n+1} such that $\det(e_1, \ldots, e_{n+1}) = 1$ and then to consider the volume of the parallepiped defined by the vectors $e_1, \ldots, e_n$ and P

$$\det(e_1, \ldots, e_n, P).$$

This determinant gives the component of the position vector of P with respect to e_{n+1}. The problem is that in general we do not have a natural way to choose the vector e_{n+1} in a way that is affinely invariant. But if we consider a tangent hyperplane of a locally strongly convex hypersurface we can use the affine normal that is an affinely invariant transversal direction. Finally, we can give the following

Definition 59. (See also [NS], p. 62) Let $M^n \hookrightarrow \mathbb{A}^{n+1}$ be a locally strongly convex hypersurface. Let us consider $P \in \mathbb{A}^{n+1}$ and $Q \in M^n$. We define *affine distance* between P and the tangent hyperplane of M^n in Q, Π, as

$$d_a(P, \Pi) := \det(e_1, \ldots, e_n, P - Q),$$

where $\{e_1, \ldots, e_n, Y\}$ is a local unimodular affine frame in a neighborhood of Q and Y is the affine normal.

Observe, again, that $d_a(P, \Pi)$ can be viewed as the component of $P - Q$ along the affine normal Y_p.
We can now state and prove our main theorem.

Theorem 60. *Let $M^n \hookrightarrow \mathbb{A}^{n+1}$ be a strictly convex, compact, immersed hypersurface with boundary contained in a hyperplane Π. Let us suppose the affine mean curvature H_1 to be a positive constant. Then the maximal affine height that the points of the hypersurface can reach over Π is less or equal to $1/H_1$.*
If M^n is an ellipsoidal cap then we have the equality, so our estimate is sharp.

Remark 61. *We have defined, above, only the affine distance between points and hyperplanes tangent to locally strongly convex affine hypersurfaces. In Theorem 60 with* "maximal affine height that the points of the hypersurface can reach over Π" *we mean the maximal affine height over the hyperplane tangent to M at some point $P \in M$ and parallel to Π. This latter is unique due to the convexity assumption.*

Let us indicate with $\langle\ ,\ \rangle$ the inner product of $\mathbb{R}^{n+1}$. Assuming $O \in \Pi$, we can express Π as $\langle a, x\rangle = 0$, with $a \in S^n$ and $\langle a, z\rangle \leq 0$ for all $z \in M$. In what follows we will suppose $\langle a, Y\rangle \geq 0$ at each point of the boundary of the hypersurface (and so at each point of the hypersurface), where Y denote the affine normal. This is an affinely invariant property.
We can now prove the main theorem.

Proof. We suppose M^n to be the graph of a strictly convex function $f : D \subset \mathbb{A}^n \to \mathbb{R}$, where D is a domain in the affine space $\mathbb{A}^n$. We suppose now that the boundary of M^n is contained in the hyperplane $\{(x^1, \dots, x^{n+1}) \in \mathbb{A}^{n+1} | x^{n+1} = 0\}$. We call $P \in M^n$ a point where x^{n+1} is minimum and $\tilde{\Pi}$ the hyperplane tangent to M^n in P.
The maximal affine distance of the points of M^n to the hyperplane containing the boundary is the maximal affine distance of the points of M^n to $\tilde{\Pi}$. In order to obtain an estimate for this distance we compute the Laplacian with respect to the Blaschke metric of the (vectorial)

function $H_1x + Y$

$$\Delta(H_1x + Y) = nH_1Y - nH_1Y = 0,$$

where we used the fact that H_1 is constant and Lemma 58. On the boundary we have $nH_1x^{n+1} + Y^{n+1} = Y^{n+1} \geq 0$, for our assumption. Using the classical maximum principle we have $H_1x^{n+1} + Y^{n+1} \geq 0$ on M^n, so

$$x^{n+1} \geq -\frac{1}{H_1}Y^{n+1}. \tag{4.11}$$

Computing the affine normal for graphs (see [LSXJ], pag. 29) we have

$$Y = H^{\frac{1}{n+2}} f^{ij} \frac{\partial}{\partial x^i} \log \rho \ \tilde{e}_j + H^{\frac{1}{n+2}} \tilde{e}_{n+1},$$

where

$$\begin{aligned}
&f_{ij} := \frac{\partial^2}{\partial x^i \partial x^j} f \\
&f^{ik} f_{kj} = \delta^i_j \\
&\tilde{e}_i := \left(0, \ldots, 1, \ldots, \frac{\partial f}{\partial x^i}\right) \quad i = 1, \ldots, n \\
&\tilde{e}_{n+1} = (0, \ldots, 0, 1) \\
&\rho := \left(\det\left(\frac{\partial^2}{\partial x^i \partial x^j} f\right)\right)^{-\frac{1}{n+2}} =: H^{-\frac{1}{n+2}}.
\end{aligned}$$

Since at P the function f attains its minimum we have

$$Y^{n+1} = H^{\frac{1}{n+2}}.$$

The affine distance between a point Q of M^n and Π is

$$d_a(Q, \tilde{\Pi}) = \frac{Q^{n+1} - P^{n+1}}{H^{\frac{1}{n+2}}},$$

so it is maximal when the point Q is a boundary point and in this case we have

$$d_a(Q,\tilde{\Pi}) = -\frac{P^{n+1}}{H^{\frac{1}{n+2}}}.$$

Using (4.11) we have

$$d_a(Q,\Pi) \leq \frac{1}{H_1}\frac{H^{\frac{1}{n+2}}}{H^{\frac{1}{n+2}}} = \frac{1}{H_1}. \tag{4.12}$$

Let us suppose now M^n to be the ellipsoidal cap given by the intersection of the hyperquadric

$$(x^1)^2 + \cdots + (x^{n+1})^2 = r^2 \quad r > 0$$

with the halfspace where $x^{n+1} \leq 0$. For the ellipsoid we have $Y = -r^{-(2n+2)/(n+2)}x$ and $k_1 = \cdots = k_n = r^{-(2n+2)/(n+2)}$. So, we have that for a point $Q \in \partial M^n$ (with the same notation as above)

$$d_a(Q,\Pi) = -\frac{x_P{}^{n+1}}{H^{\frac{1}{n+2}}} = \frac{r^{\frac{2n+2}{n+2}}Y^{n+1}}{H^{\frac{1}{n+2}}} = \frac{1}{H_1}.$$

So, the inequality (4.12) is sharp.

$\square$

Appendix A

A.1 Cheeger inequality

Let (M^n, g) be an n–dimensional complete Rimannian manifold and $\Omega \subset\subset M$ a relatively compact domain with piecewise smooth boundary $\partial\Omega$. Consider the eigenvalue problem for the Laplacian, with boundary Dirichlet conditions

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where Δ denotes the Laplace-Beltrami operator on (M, g). We start requiring $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$. It can be shown that the set of λ such that A.1 has a solution is discrete and can be ordered in a sequence of the form

$$0 < \lambda_1 < \lambda_2 < \cdots \to +\infty,$$

and that each eigenspace associated to these eigenvalues is finite dimensional.

A deep question is to investigate the relationships between analytic properties of the eigenvalues λ_k and geometric features of the domain Ω. For instance, in dimension 2 the topology of the domain Ω imposes some bounds for the multiplicities of each λ_k (see [Cheng] for further details), while in dimensions greater than or equal to 3 it can be proved

that there are no restrictions on multiplicities (see [Co]). Specifically, if M is a closed, connected manifold with dimension $n \geq 3$ then any preassigned finite sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ is the sequence of the first $k+1$ eigenvalues for $-\Delta_g$, where g is a suitable Riemannian metric on M and λ_k's are repeated according with multiplicity.
A famous example of relation between analytic and geometric properties of Ω is Weyl's asymptotic formula (see [Ch], [Wey])

$$(\lambda_k)^{n/2} \sim \frac{(2\pi)^n}{\omega_n}\frac{k}{\mathrm{Vol}(\Omega)} \quad \text{as } k \to \infty, \tag{A.2}$$

where ω_n denotes the volume of the unit disc in $\mathbb{R}^n$. The important message of this formula is that one can infer the volume of Ω by studying the asymptotic behaviour of λ_k.
We focus our attention, now, on the first (non-zero, in the closed case) eigenvalue λ_1 and, in particular, we discuss some bounds that the geometry of Ω imposes on it. It is well known that Poincaré inequality is a milestone in analysis; since a lower bound on λ_1 gives an upper bound on Poincaré constant, it is very interesting looking for lower bounds for λ_1. The first step in this direction is due to Lichnerowicz (see [L])

Theorem 62. *Let (M^n, g) be a closed, connected Riemannian manifold of dimension $n \geq 2$ and let* Ric *be its Ricci tensor field. If*

$$\mathrm{Ric}(X, X) \geq a(n-1)|X|^2 \quad \forall X \in TM,$$

where a is a positive constant, then

$$\lambda_1 \geq na. \tag{A.3}$$

Obata in [O] showed that equality in (A.3) holds if and only if M is isometric to the n-sphere of constant sectional curvature a. Later on, Li and Yau extended Lichnerowicz' result to the case $a = 0$ (see [LY]). They proved that if (M, g) is compact, connected and has non-negative Ricci curvature, then

$$\lambda_1 \geq \frac{\pi^2}{4d^2}, \tag{A.4}$$

where d is the diameter of the manifold. Zhong and Yang in [ZY] improved Li and Yau's estimate (A.4) by showing that $\lambda_1 \geq \frac{\pi^2}{d^2}$.

Given a closed Riemannian manifold (M, g) we define the *Cheeger constant* $h(M)$ by

$$h(M) := \inf_S \frac{\mathrm{Vol}_{n-1}(S)}{\min\{\mathrm{Vol}(A), \mathrm{Vol}(B)\}},$$

where S runs over all the hypersurfaces dividing M into two parts, A and B. So, an equivalent definition of $h(M)$ is

$$h(M) := \inf_A \frac{\mathrm{Vol}_{n-1}(\partial A)}{\mathrm{Vol}(A)},$$

where A runs over all open subsets of M whose volume does not exceed half of the total volume of M.

In 1970 Cheeger ([Chee]) gave the following lower bound for λ_1

$$\lambda_1 \geq \frac{1}{4} h^2(M). \tag{A.5}$$

Later on, Buser in [Bu] found an upper bound for λ_1 in terms of $h(M)$, given a lower bound on the Ricci curvature. Specifically, let (M^n, g) be a closed Riemannian manifold whose Ricci tensor satisfies $\mathrm{Ric} \geq -(n-1)\delta^2$, for $\delta \geq 0$. Then

$$\lambda_1(M) \leq c_1(\delta h + h^2),$$

where c_1 is a constant depending only on the dimension of M.

Cheeger inequality (A.5) holds true also in the noncompact case, provided that we have clarified the correct definition of $h(M)$ and $\lambda_1(M)$. Let (M,g) be a noncompact Riemannian manifold. The Cheeger constant, in this setting, is defined by

$$h(M) = \inf_{\Omega} \frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(\Omega)},$$

where Ω ranges over all relatively compact open domains in M. Moreover, on M by $\lambda_1(M)$ we mean the bottom of the spectrum (also called first eigenvalue or spectral radius), defined by

$$\lambda_1(M) := \inf \frac{\int_M |\nabla u|^2 dv}{\int_M u^2 dv},$$

where u ranges over all non-zero smooth functions with compact support. Therefore, we have

Theorem 63 (Cheeger). *Let (M,g) be a complete Riemannian manifold. We have*

$$\lambda_1(M) \geq \frac{1}{4} h(M)^2$$

Proof. Consider a relatively compact open domain $\Omega \subset\subset M$ and let u be an eigenfunction relative to the first eigenvalue of $-\Delta$ in Ω, that is $\Delta u = -\lambda_1(\Omega) u$ on Ω and $u = 0$ on $\partial\Omega$. Using Cauchy-Schwarz inequality for $\nabla u^2 = 2u\nabla u$ we get

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \geq \frac{1}{4}\left(\frac{\int_\Omega |\nabla u^2|}{\int_\Omega u^2}\right)^2.$$

Now, using the co-area formula, we have

$$\begin{aligned}\int_\Omega |\nabla u^2| dv &= \int_0^\infty A(u^2 = t) dt \\ &\geq h(\Omega) \int_0^\infty V(u^2 \geq t) dt \\ &= h(\Omega) \int_\Omega u^2 dv.\end{aligned}$$

So, we get $\lambda_1(\Omega) \geq \frac{1}{4} h(\Omega)^2$. Then, since $\lambda_1(M) = \inf \lambda_1(\Omega)$, with $\Omega \subset M$ bounded domain, and $h(M) \leq h(\Omega)$ for every open $\Omega \subset M$, we achieve

$$\lambda_1(M) \geq \frac{1}{4} h^2(M).$$

$\square$

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