G-zips and Ekedahl-Oort strata for Hodge type Shimura varieties

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0 Introduction

This thesis mainly deals with geometric structures of reductions of certain Shimura varieties. We will explain what are Shimura varieties and their good reductions in "Introduction for general readers", including our motivation of studying Ekedahl-Oort strata. Then we will list known results in the subsection "Known results". There we DO assume that the reader is familiar with algebraic geometry. Then we explain the structure of the thesis.

0.1 Introduction for general readers

0.1.1 Elliptic curves and modular curves

As Shimura varieties are higher-dimensional analogues of modular curves which classify elliptic curves, we will recall what they are before moving on.

An elliptic curve E over a field K is a smooth connected projective curve of genus 1 with a given rational point. An elliptic curve E can be realized as a plane curve defined by a single equation (called a Weierstrass equation) of the form

$$(W) \qquad Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

s.t. the distinguished rational point corresponds to $(0, 1, 0)$. Note that it is not true in general that a Weierstrass equation (W) gives an elliptic curve, as the curve defined by (W) might be singular. A Weierstrass equation of an elliptic curve is uniquely determined up to a change of variables of the form

$$X = u^2X' + v, \quad Y = u^3Y' + \alpha u^2X' + \beta, \quad Z = Z'$$

with $u, v, \alpha, \beta \in K$ and $u \neq 0$. It will be convenient sometimes to work with $E - \{(0, 1, 0)\}$ which is an open subvariety in E. Because it determines E, and the equation (W) becomes

$$(W') \qquad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

which is simpler than (W).

We will write

$$\begin{aligned}
&b_2 = a_1^2 + 4a_2,\ b_4 = 2a_4 + a_1a_3,\ b_6 = a_3^2 + 4a_6 \\
&b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
&\Delta = 9b_2b_4b_6 - b_2^2b_8 - 8b_4^3 - 27b_6^2.
\end{aligned}$$

One can check that (W) defines an elliptic curve if and only $\Delta \neq 0$ in K, and that E has a unique structure of a commutative group variety s.t. the distinguished rational point is the identity. In particular, for an elliptic curve with Weierstrass equation (W), $j := (b_2^2 - 24b_4)^3/\Delta$ is a well defined element in K. We remark that j actually only depends on the elliptic curve, not the Weierstrass equation, and hence is called the j-invariant of E. If K is algebraically closed, two elliptic curves are isomorphic if and only if they have the same j-invariant.

If $\mathrm{char}(K) \neq 2, 3$, then for an elliptic curve E, we can take its Weierstrass equation to be of the form

$$(W') \qquad y^2 = x^3 + ax + b, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

And (W') is unique up to change of variables

$$x = u^2x', \qquad y = u^3y', \quad \text{for some } u \in K^\times.$$

In this case, we see easily that

$$j(E) := 1728\frac{4a^3}{\Delta}$$

is an invariant of of E which does not depend on the choice of the Weierstrass equation.

When $K = \mathbb{C}$, $E(\mathbb{C})$ is isomorphic to $\mathbb{C}/\Lambda$ as complex analytic manifolds for a lattice $\Lambda \subseteq \mathbb{C}$. The group law corresponds to the usual addition on $\mathbb{C}/\Lambda$. More precisely, let $y^2 = x^3 + ax + b$ be a Weierstrass equation of E, then for point $P \in E(\mathbb{C}) \subseteq \mathbb{P}^2_{\mathbb{C}}(\mathbb{C})$, there is a point

$$f(P) = \int_0^P dx/y (\text{mod } \Lambda) \in \mathbb{C}/\Lambda.$$

The association $P \mapsto f(P)$ induces an isomorphism of complex Lie groups $E(\mathbb{C}) \to \mathbb{C}/\Lambda$. The inverse of f is described as follows. Let $\wp(z)$ be the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda, \lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

Then the association $z \mapsto (\wp(z), \wp'(z), 1)$ induces an embedding $\mathbb{C}/\Lambda \to \mathbb{P}^2_{\mathbb{C}}$. And the image of this embedding is an elliptic curve defined by the equation $Y^2 = 4x^3 + g_2 X + g_3$, where

$$g_2 = -60 \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{z^4}, \quad g_3 = -140 \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{z^6}.$$

Thus the study of isomorphism classes of complex elliptic curves is the same as that of complex tori, and hence that of lattices.

One can show that every complex elliptic curve E is isomorphic to some $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with τ in the upper half plane $\mathcal{H}$ which is uniquely determined up to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ given by Möbius transformations. Thus, the j-invariant induces a map $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \to \mathbb{C}$ which is continuous, bijective and holomorphic.

Let N be a positive integer, $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup defined to be

$$\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), b \equiv c \equiv 0 \text{ mod } N, a \equiv d \equiv 1 \text{ mod } N\}.$$

Then the complex manifold $Y(N)^0 := \Gamma\backslash\mathcal{H}$ has an unique structure of smooth complex algebraic curve. Moreover, $Y(N)^0$ is defined over $\mathbb{Q}(\zeta_N)$, i.e. there is a positive integer n, there are finitely many homogenous polynomials f_i (resp. g_j) with $n+1$ variables and coefficients in $\mathbb{Q}(\zeta_N)$, such that the complex curve $Y(N)^0$ is defined by $f_i = 0$ for all i and $g_j \neq 0$ for some j in $\mathbb{P}^n_{\mathbb{C}}$. We remark that $Y(N)^0$ is actually defined over $\mathbb{Z}[\zeta_N]$, and that the statement that $Y_\Gamma = \Gamma\backslash\mathcal{H}$ is algebraic and defined over a number field holds for a large class of discrete subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. The $\mathbb{Q}(\zeta_N)$-curve $Y(N)^0$ is called a modular curve. It has a nice moduli interpretation. Let $E[N]$ be the N-kernel of E. For an algebraically closed field K, K-points of $Y(N)^0$ are isomorphism classes of elliptic curves E with an isomorphism $\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z} \to E[N]$ compatible with the Weil pairing. If $N > 2$, one can drop the hypothesis that K is algebraically closed.

Another interesting object is obtained by disjoint union of $Y(N)^0$. Let $\mathbb{A}_f$ be the ring of finite adeles, i.e.

$$\mathbb{A}_f := \{(\cdots, a_p, \cdots) \in \prod_p \mathbb{Q}_p \mid a_p \in \mathbb{Q}_p, \text{ and almost all } a_p \text{ lie in } \mathbb{Z}_p\}.$$

Consider the double quotient

$$Y(N) := \mathrm{GL}_2(\mathbb{Q})\backslash(\mathcal{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K_N)$$

where $K_N \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ is the compact open subgroup given by

$$\{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid g \equiv \mathrm{id} \bmod N\}.$$

Then the complex manifold $Y(N)$ has a unique structure of an algebraic variety over $\mathbb{C}$, and $Y(N)$ is an union of finitely many copies of $Y(N)^0$. Moreover, $Y(N)$ is defined over $\mathbb{Q}$. One advantage of studying $Y(N)$ is that, unlike $Y(N)^0$, all the $Y(N)$ are defined over the same field, namely $\mathbb{Q}$.

The $\mathbb{Q}$-curve $Y(N)$ also has a nice moduli interpretation. For an algebraically closed field K, K-points of $Y(N)$ are isomorphism classes of elliptic curves E with an isomorphism $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \cong E[N]$. If $N > 2$, one can drop the hypothesis that K is algebraically closed.

Let p be a prime such that $(p, N) = 1$. Let $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ be the set

$$\{\frac{a}{b} \mid a, b \in \mathbb{Z}, (p, b) = 1\}.$$

It is a subring. Then the algebraic curve $Y(N)$ extends to a smooth scheme $\mathscr{Y}(N)$ over $\mathbb{Z}_{(p)}$, and hence $\mathscr{Y}_0(N) = \mathscr{Y}(N) \otimes \mathbb{F}_p$ is a smooth algebraic curve over $\mathbb{F}_p$. Moreover, $\overline{\mathbb{F}_p}$-points of $\mathscr{Y}_0(N)$ are isomorphism classes of elliptic curves E over $\overline{\mathbb{F}_p}$ with an isomorphism $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E[N]$. If $N > 2$, the same statement holds for any finite extension of $\mathbb{F}_p$. The variety $\mathscr{Y}_0(N)$ is called reduction of $Y(N)$. As points of $\mathscr{Y}_0(N)$ are elliptic curves, we should use knowledge of elliptic curves to study $\mathscr{Y}_0(N)$.

Let E be an elliptic curve over $\mathbb{F}_q$ which is a finite extension of $\mathbb{F}_p$. It is well known that the group of $\overline{\mathbb{F}_p}$-points of its p-kernel $E[p]$ has precisely two different possibilities, namely, $E[p](\overline{\mathbb{F}_p}) \cong \mathbb{Z}/p\mathbb{Z}$ or $E[p](\overline{\mathbb{F}_p}) = \{0\}$ (This is very different from those over $\mathbb{C}$ where $E[p](\mathbb{C})$ has to be $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$). The elliptic curve E is said to be ordinary if $E[p](\overline{\mathbb{F}_p}) \cong \mathbb{Z}/p\mathbb{Z}$, and supersingular if $E[p](\overline{\mathbb{F}_p}) = \{0\}$. Points on $\mathscr{Y}_0(N)$ whose corresponding elliptic curve is supersingular form a finite subset of $\mathscr{Y}_0(N)$.

There are many equivalent descriptions for supersingular elliptic curves (and hence for ordinary elliptic curves). For an elliptic curve E over $\mathbb{F}_q$, the following are equivalent.

1) E is supersingular.

2) $E[p^r](\overline{\mathbb{F}_p}) = \{0\}$ for all $r \geqslant 1$.

3) The multiplication by p map $p : E \to E$ is purely inseparable (i.e. $p : E \to E$ induces an endomorphism of field of rational functions on E, and this injection is a purely inseparable field extension).

4) The endomorphism ring $\mathrm{End}(E \times \overline{\mathbb{F}_p})$ is an order of a quaternion algebra.

To end this subsection, we mention the following result of Deuring.

Theorem 0.1.2. *(Deuring's mass formula) Let p be a prime. Let I be the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_p}$. For each $i \in I$, take any representative E_i. Then*

$$\sum_{i \in I} \frac{1}{\#(\mathrm{Aut}(E_i))} = \frac{p-1}{24}.$$

0.1.3 Abelian varieties

Abelian varieties and Siegel modular varieties are higher dimensional generalizations of elliptic curves and modular curves respectively. Let K be a field. By an abelian variety A over K, we mean a smooth connected projective group variety over K. In a more elementary language, an abelian variety A is defined as follows.

1) There is a positive integer n, such that A is defined by finitely many K-coefficient homogenous polynomials in $\mathbb{P}^n_K$. We will denote the embedding $A \hookrightarrow \mathbb{P}^n_K$ by i.

2) There is a K-rational point $e \in A$, i.e. $e \in A \subseteq \mathbb{P}^n_K$ has all its coordinates in K.

3) The variety A is smooth and connected.

4) There are maps $m : A \times A \to A$ and $\iota : A \to A$ making A a group with identity e. Moreover, m and ι are are Zariski locally given by fractions of homogenous polynomials.

One dimensional abelian varieties are elliptic curves. It is a direct consequence of the so called rigidity lemma that the group structure of an abelian variety is automatically commutative.

Now we will describe abelian varieties over $\mathbb{C}$. Let A be an abelian variety of dimension n over $\mathbb{C}$, then $A(\mathbb{C})$ is a commutative compact connected complex Lie group (i.e. a complex torus). The universal covering of $A(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$, and $A(\mathbb{C}) \simeq \mathbb{C}^n/\Lambda$, where $\Lambda = \pi_1(A(\mathbb{C}), 0)$ which is a lattice of $\mathbb{C}^n$. Therefore, $\mathrm{H}_1(A(\mathbb{C}), \mathbb{Z}) \simeq \Lambda$, and $\mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}) \simeq \mathrm{Hom}(\Lambda, \mathbb{Z})$.

For a complex torus, we know when it is an abelian variety. To explain this, we need to introduce the following definition.

Definition 0.1.4. Let R be a subring of $\mathbb{R}$ (for our purpose, it suffices to know that R can be taken to be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$). Let M be a free R-module of finite rank. Then by a Hodge structure of type $(-1, 0) + (0, -1)$ on M, we mean an action

$$h : \mathbb{C}^\times \to \mathrm{GL}(\mathrm{M}_\mathbb{R})$$

given a complex structure on $M_{\mathbb{R}}$.

A polarization of the Hodge structure on M is a perfect alternating form $\psi : M \times M \to R$ such that $\psi_{\mathbb{R}}(h(i)u, h(i)v) = \psi_{\mathbb{R}}(u, v)$ for all $u, v \in M \otimes \mathbb{C}$ and that $\psi_{\mathbb{R}}(-, h(i)-)$ is positive definite.

A Hodge structure on M is said to be polarizable if it admits a polarization.

To an abelian variety A over $\mathbb{C}$, we get a polarized $\mathbb{Z}$-Hodge structure of type $(-1, 0) + (0, -1)$ as follows. Let M be $\mathrm{H}^1(A(\mathbb{C}), \mathbb{Z})$. There exists a line bundle on A whose Chern class gives a polarization on M.

Now we can describe when a complex torus is a complex abelian variety. We have the following result due to Riemann.

Theorem 0.1.5. *A complex torus $\mathbb{C}^n/\Lambda$ is an abelian variety if and only if the Hodge structure Λ is polarizable. The functor $A \mapsto \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ is an equivalence from the category of abelian varieties over $\mathbb{C}$ to the category polarizable $\mathbb{Z}$-Hodge structures of type $(-1, 0) + (0, -1)$.*

Remark 0.1.6. In the statement of the previous theorem, we use the language of categories. In the category of abelian varieties, morphisms are morphism of algebraic varieties compatible with group structures. Again by the rigidity lemma, they are the same as morphisms of algebraic varieties taking identity to identity. For the definition of a morphism between two polarizable Hodge structures, we refer to [34].

Note that a polarization of a $\mathbb{Z}$-Hodge structure of type $(-1, 0) + (0, -1)$ on Λ gives a morphism of Hodge structures $\lambda : \Lambda \to \Lambda^\vee(1)$. Here $\Lambda^\vee(1)$ is $\Lambda^\vee \otimes \mathbb{Z}(1)$, with $\mathbb{Z}(1)$ the free $\mathbb{Z}$ module of rank 1 equipped with the $\mathbb{C}^\times$-action on $\mathbb{Z}(1) \otimes \mathbb{R}$ given by

$$c \cdot r = c\bar{c}r, \quad \forall\, c \in \mathbb{C}^\times \text{ and } r \in \mathbb{Z}(1) \otimes \mathbb{R}.$$

The point to consider $\Lambda^\vee(1)$ is that $\Lambda^\vee$ is of type $(1, 0) + (0, 1)$, while $\Lambda^\vee(1)$ is of type $(-1, 0) + (0, -1)$. Theorem 0.1.5 implies that Λ (resp. $\Lambda^\vee(1)$) gives an abelian variety A (resp. $A^\vee$), and that λ induces a morphism of abelian varieties $\lambda : A \to A^\vee$. We will call $A^\vee$ the dual of A and $\lambda : A \to A^\vee$ a polarization. A polarization is called principal, if it is an isomorphism. By a principally polarized abelian variety, we mean an abelian variety A together with a principal polarization λ.

We remark that we can define the dual for any abelian variety over any field. Moreover, we can define polarizations for abelian varieties (and hence principally polarized abelian varieties) over general fields. For a morphism of abelian varieties $f : A' \to A$, it is called an isogeny if f is surjective and A' has the same dimension as A. In this case, the inverse image of a point in A is finite and non-empty. Principally polarized abelian varieties are important because any abelian variety is isogenous to an abelian variety which affords a principal polarization.

0.1.7 Siegel modular varieties

Now we will introduce Siegel modular varieties, and their interpretation as moduli spaces of complex abelian varieties.

Let V be a $2g$ dimensional $\mathbb{Q}$-vector space with basis $e_1, e_2, \cdots, e_{2g}$. Let ψ be a perfect alternating form on V and assume that under the basis $e_1, e_2, \cdots, e_{2g}$, ψ has matrix

$$\begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix}.$$

Here 1_g is the $g \times g$ identity matrix. If we denote by $V_{\mathbb{Z}}$ the free $\mathbb{Z}$-module in V generated by $e_1, e_2, \cdots, e_{2g}$, then ψ induces a perfect alternating form on $V_{\mathbb{Z}}$. Let $\mathrm{Sp}(V, \psi)$ be the algebraic group over $\mathbb{Q}$ such that, for every $\mathbb{Q}$-algebra R, the group of R-points of $\mathrm{Sp}(V, \psi)$ is

$$\mathrm{Sp}(V, \psi)(R) = \{g \in \mathrm{GL}(V)(R) \mid g^t J g = J\}.$$

Similarly, we write $\mathrm{GSp}(V, \psi)$ for the algebraic group over $\mathbb{Q}$ such that, for every $\mathbb{Q}$-algebra R, the group of R-points of $\mathrm{GSp}(V, \psi)$ is

$$\mathrm{GSp}(V, \psi)(R) = \{g \in \mathrm{GL}(V)(R) \mid g^t J g = c(g) J, \text{ for some } c(g) \in R^\times\}.$$

For any $n \in \mathbb{Z}_{>0}$, let $\Gamma(n)$ be the subgroup of $\mathrm{Sp}(V_{\mathbb{Z}}, \psi)(\mathbb{Z})$ defined by

$$\Gamma(n) = \{g \in \mathrm{Sp}(V_{\mathbb{Z}}, \psi)(\mathbb{Z}) \mid g \equiv \mathrm{id} \pmod{n}\}.$$

Note that $\Gamma(1)$ is $\mathrm{Sp}(V_{\mathbb{Z}}, \psi)(\mathbb{Z})$.

Let

$$\mathbb{H} = \{\Omega \in \mathrm{M}_g(\mathbb{C}) \mid \Omega = \Omega^t, \text{ and } \mathrm{Im}\,\Omega \text{ is positive definite}\},$$

where $\mathrm{M}_g(\mathbb{C})$ means the set of complex $g \times g$ matrixes. Then $\mathrm{Sp}(V, \psi)(\mathbb{R})$ acts on $\mathbb{H}$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B) \cdot (C\Omega + D)^{-1},$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(V, \psi)(\mathbb{R})$, $\Omega \in \mathbb{H}$. The complex manifold $\Gamma(n)\backslash\mathbb{H}$ is algebraic and defined over $\mathbb{Q}(\zeta_n)$ for any $n \geq 3$. Moreover, $\Gamma(n)\backslash\mathbb{H}$ classifies abelian varieties.

More precisely, we define a level n structure on a g-dimensional abelian variety A to be an isomorphism $(\mathbb{Z}/n\mathbb{Z})^{2g} \cong A[n]$. Let $\mathscr{A}^0_{g,n}(\mathbb{C})$ be the set of isomorphism classes of principally polarized complex abelian varieties with level n structure compatible with the Weil pairing. We remark that when $n = 1$, this is precisely the set of isomorphism classes of principally polarized complex abelian varieties. For an element A in $\mathscr{A}^0_{g,n}(\mathbb{C})$, one gets a point in $\mathbb{H}$ using Section 6 of [46]. This association identifies $\mathscr{A}^0_{g,1}(\mathbb{C})$ with $\Gamma(1)\backslash\mathbb{H}$, and $\mathscr{A}^0_{g,n}(\mathbb{C})$ with $\Gamma(n)\backslash\mathbb{H}$ for $n > 1$.

As in the case of GL_2, we can also consider the double quotient

$$\mathscr{A}_{g,n} := \mathrm{GSp}(V, \psi)(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GSp}(V, \psi)(\mathbb{A}_f)/K_n),$$

where $K_n \subseteq \mathrm{GSp}(V, \psi)(\mathbb{A}_f)$ is the compact open subgroup

$$\{g \in \mathrm{GSp}(V_{\mathbb{Z}}, \psi)(\widehat{\mathbb{Z}}) \mid g \equiv \mathrm{id} \bmod n\}.$$

Precisely as what happened before, the complex manifold $\mathscr{A}_{g,n}$ has a unique structure of an algebraic variety over $\mathbb{C}$, and $\mathscr{A}_{g,n}$ is finitely many copies of $\mathscr{A}^0_{g,n}$. Moreover, $\mathscr{A}_{g,n}$ is defined over $\mathbb{Q}$.

We will from now on assume that $n \geq 3$. The variety $\mathscr{A}_{g,n}$ is defined over $\mathbb{Q}$. Moreover, let p be a prime number such that $(n,p) = 1$, then $\mathscr{A}_{g,n}$ can be defined by polynomials with coefficients in $\mathbb{Z}_{(p)}$, i.e. it has a model (far from being unique) over $\mathbb{Z}_{(p)}$. If one puts a condition on integral points of the model as in Section 2, then there is a unique model satisfying the condition. We can then reduce the model modulo p. Then we get a smooth variety over $\mathbb{F}_p$ denoted by $\mathscr{A}_0$. The varieties $\mathscr{A}_0$ is important and interesting in the sense that for any field extension k of $\mathbb{F}_p$, the set of k-points of $\mathscr{A}_0$ is the set of isomorphism classes of principally polarized abelian varieties over k with level n structure.

We remark that the statements in the previous paragraph do NOT quite reflect the history. As it sounds like we knew what kind of condition should we put first, and then we constructed the model. The right order is as follows. The fine moduli scheme $\mathscr{A}_{g,n}$ of principally polarized abelian schemes with level n structure is constructed by Mumford [41] around 1960's. It is defined over $\mathbb{Z}$, and smooth at $\mathbb{Z}_{(p)}$ for all p prime to n. Then Vasiu [52] shows that $\varprojlim_n \mathscr{A}_{g,n}$ satisfies a certain extension property using works by Faltings [13] and Faltings-Chai [12]. This implies that $\mathscr{A}_{g,n}$ is uniquely determined by the so called Shimura datum (and of course together with $\Gamma(n)$). In particular, $\mathscr{A}_0$ is smooth, uniquely determined by the Shimura datum and classifies abelian varieties.

As for modular curves, one has a stratification on $\mathscr{A}_0$ using p-kernel of the associated abelian varieties of points. Two $\overline{\mathbb{F}_p}$-points are in the same stratum if and only if the p-kernel of their attached abelian varieties have the same p-rank. Let us take $g = 2$ as an example. In this case, $\mathscr{A}_0$ classifies principally polarized abelian surfaces with level structures. Let x be a point of $\mathscr{A}_0$, and A_x be the corresponding abelian variety. Let $k(x)$ be the residue field of x and $\overline{k(x)}$ be the algebraic closure of $k(x)$. Then $A_x[p](\overline{k(x)})$, as an abelian group up to isomorphism, can only be 0, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Then the set of points x in $\mathscr{A}_0$ whose associated abelian variety A_x is such that $A_x[p](\overline{k(x)}) = 0$

is of dimension 1. And the set of points x in $\mathscr{A}_0$ whose associated abelian variety A_x is such that $A_x[p](\overline{k(x)}) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is open dense, and in particular, of dimension 3. The set of point x such that $A_x[p](\overline{k(x)}) \simeq \mathbb{Z}/p\mathbb{Z}$ is of dimension 2.

The above stratification is called the p-rank stratification. There are $g+1$ p-rank strata for $\mathscr{A}_0$, where g is the dimension of abelian varieties that $\mathscr{A}_0$ classifies. One can define a even finer stratification, namely the Ekedahl-Oort stratification, using the structure of $A_x[p]$ as a group scheme. We will not give the precise definition here, as then we have to assume the reader is familiar with finite group schemes which contradicts the very introductive feature of this general introduction. We remark that there are 2^g Ekedahl-Oort strata for $\mathscr{A}_0$, and they are all smooth. But the p-rank strata are not necessarily smooth. The p-rank stratification coincides with the Ekedahl-Oort stratification for modular curves, but they do not coincide for $g \geq 2$. Still take the case $g = 2$ as an example. There are 3 p-rank strata in this case, but 4 Ekedahl-Oort strata. The set of points x in $\mathscr{A}_0$ whose associated abelian variety A_x is such that $A_x[p](\overline{k(x)}) \simeq 0$ decomposes into two Ekedahl-Oort strata. Actually, for each integer n such that $0 \leq n \leq 3$, there is a unique Ekedahl-Oort stratum in $\mathscr{A}_0$ which is of dimension n.

0.1.8 Motivations

This thesis mainly deals with the following questions.

1) How to get a good theory of Ekedahl-Oort stratification for reductions of general Shimura varieties of Hodge type?

2) What are the basic geometric properties of the stratification, and how to find a list for all the strata.

In this part, I will explain why these questions are interesting and important. Before doing this, I will remark the following.

1) Shimura varieties are generalizations of Siegel modular varieties. I will not explain what is a Shimura variety here, and the reader can just think

about a modular curve or a Siegel modular variety when I talk about a Shimura variety. The only fact I want to remark is that a Shimura variety is determined by a Shimura datum.

2) We are interested in reductions of Shimura varieties of Hodge type in this paper, because we DO NOT know how to work with general Shimura varieties. For a Shimura variety of Hodge type, there is a closed embedding into some Siegel modular variety.

3) Let S be a Shimura variety of Hodge type. It is defined over a number field E. For a "good" prime p (I will not specify what does good mean in general, but just refer to the body of the paper and mention that almost all primes are good), if S is equipped with hyperspecial level structure, then the variety S extends to a smooth scheme $\mathscr{S}$ over $O_{E,(p)}$. As examples of $\mathscr{S}$, one can think about $\mathscr{Y}(N)$ or $\mathscr{A}_{g,n}$ with $(N,p)=1$ or $(n,p)=1$.

4) The scheme $\mathscr{S}$ is constructed using the embedding to a Siegel modular variety, but $\mathscr{S}$ is uniquely determined by a certain extension property. In particular, the special fiber $\mathscr{S}_0$ is unique and smooth over $O_{E,(p)}/pO_{E,(p)}$. As examples of $\mathscr{S}_0$, one can think about $\mathscr{Y}_0(N)$ or $\mathscr{A}_0$ with $(N,p)=1$ or $(n,p)=1$.

5) Let $\mathfrak{p}$ be a place of E over p, and let $\kappa = O_E/\mathfrak{p}O_E$. Then $\mathrm{Spec}(\kappa)$ gives a connected component of $\mathrm{Spec}(O_{E,(p)}/pO_{E,(p)})$. From now on, the fiber of $\mathscr{S}_0/\mathrm{Spec}(O_{E,(p)}/pO_{E,(p)})$ at κ will be denote by $\mathscr{S}_0$.

Clearly, 1), 4) and 5) of the above remarks simply imply that all invariants of the variety $\mathscr{S}_0$ are determined by the Shimura datum. Hence all invariants and relations between different invariants of $\mathscr{S}_0$ should be expressed in terms of the Shimura datum. For example, The Langlands-Rapoport conjecture predicts an explicit expression of the set of $\mathbb{F}_q$-points of $\mathscr{S}_0$ in terms of its Shimura datum. Kisin has recently claimed a proof of this deep conjecture for Shimura varieties of abelian type. Of course one can always ask how to describe points in each stratum when a stratification is given. This is definitely important and interesting. But we want to mention another invariant which is also very important but very difficult to study.

For a connected smooth quasi-projective variety X over an algebraically closed field k (a connected component of our $\mathscr{S}_0 \otimes \overline{\mathbb{F}_p}$ is always smooth and quasi-projective), there is a deep invariant–the Chow ring which is constructed as follows. The main reference here is [14] Chapter 1, 6 and 7.

1) An algebraic cycle is a finite formal linear sum $\sum_i n_i V_i$ where n_i is an integer and V_i is an irreducible closed subvariety. The abelian group of algebraic cycles is denoted by Z. The subgroup generated by irreducible subvarieties of codimension r will be denoted by Z^r.

2) An element $\alpha \in Z^r$ is said to be rationally equivalent to zero if there exist finitely many irreducible closed subvarieties $W_i \subseteq X$ of codimension $r-1$ and a non-zero rational function f_i on W_i, s.t. $\alpha = \sum_i \mathrm{div}(f_i)$. Here $\mathrm{div}(f_i)$ is the divisor associated to f_i which is given by

$$\mathrm{div}(f_i) = \sum_{V \subseteq W_i \text{ of codimension } 1} \big(\mathrm{length}(O_{W_i,V}/(g_{i,V})) - \mathrm{length}(O_{W_i,V}/(h_{i,V}))\big) V,$$

where $g_{i,V}, h_{i,V} \in O_{W_i,V}$ are such that $f_{i,V} = g_{i,V}/h_{i,V}$. Note that $O_{W_i,V}$ is a 1-dimensional local ring, and that there are only finitely many V s.t. $\mathrm{length}(O_{W_i,V}/(f_i))$ is non-zero.

Cycles rationally equivalent to zero form a subgroup R^r of Z^r, and we write $\mathrm{CH}^r(X) = Z^r/R^r$.

3) There is a commutative ring structure on $\mathrm{CH}^*(X) = \bigoplus_i \mathrm{CH}^i(X)$ s.t. the multiplication takes $\mathrm{CH}^i(X) \otimes \mathrm{CH}^j(X)$ to $\mathrm{CH}^{i+j}(X)$. The ring structure is constructed as follows. For any $\alpha \in Z^i(X)$ and $\beta \in Z^j(X)$, there is a $\beta' \in Z^j(X)$ which is rationally equivalent to β and that $\alpha \cap \beta'$ is of codimension $i+j$. Moreover, the choice of β' does not affect the class of $\alpha \cap \beta'$ in $\mathrm{CH}^{i+j}(X)$. We need to remark that there is a intersection multiplicity involved in $\alpha \cap \beta'$ which is not easy to describe.

This intersection product gives a commutative ring structure on $\mathrm{CH}^*(X)$. The commutative ring $\mathrm{CH}^*(X)$ is called the Chow ring of X.

4) For a prime number l which is different from the characteristic of k,

there is a cycle map

$$\mathrm{CH}^i(X) \to \mathrm{H}^{2i}_{\text{ét}}(X, \mathbb{Q}_l)(i),$$

where the right hand side is the i-th twist of the i-th l-adic cohomology.

The basic observation is as follows. The i-th Chow group, the Chow ring and the cycle map are invariants of $\mathscr{S}_{0,\overline{\mathbb{F}_p}}$, and hence are determined by the Shimura datum and the compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. So one should ask for an expression of these invariants in terms of (G, X, K^p). We remark that the Chow ring contains much more information than the Chow group, as it gives the whole intersection theory.

The Ekedahl-Oort stratification provides some locally closed subvarieties (whose closures will be closed subvarieties, or cycles). We also know their dimensions by the dimension formula. So it is natural to ask what is the subring generated by closures of Ekedahl-Oort strata in the Chow ring. For a Shimura variety of Hodge type, one can even ask what is the subring of the Chow ring generated by it Ekedahl-Oort strata and all those of its special subvarieties.

Recently, a good compactification $\overline{\mathscr{S}_0}$ for $\mathscr{S}_0$ has been constructed by Keerthi Madapusi Pera [32]. The smooth projective variety $\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}$ is determined by (G, X, K) and a toroidal decomposition. So $\mathrm{CH}^*(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}})$, $\mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$ and the cycle map $\mathrm{CH}^i(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}) \to \mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$ should be expressed in terms of (G, X, K) and the toroidal decomposition. This is of course a difficult question. But one can also consider the subring generated by Ekedahl-Oort strata and cycle classes of them as a first step approximation to the solutions of these questions.

Finally, I will make one remark toward the Tate's conjecture. The variety $\overline{\mathscr{S}_0}$ is also defined over κ, the field of definition of $\mathscr{S}_0$. One can also define cycles, the group $Z^i(\overline{\mathscr{S}_0})$ and the cycle map $Z^i(\overline{\mathscr{S}_0}) \to \mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$. There is an action of $Gal(\overline{\mathbb{F}_p}/\kappa)$ on $\mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$, and the image of a cycle is always $Gal(\overline{\mathbb{F}_p}/\kappa)$-invariant. The Tate conjecture predicts that the cycle map

$Z^i(\overline{\mathscr{S}_0}) \to \mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$ induces a surjection

$$Z^i(\overline{\mathscr{S}_0}) \otimes \mathbb{Q}_l \to \mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)^{Gal(\overline{\mathbb{F}_p}/\kappa)}.$$

Of course the Tate's conjecture is raised for general varieties, but here we only restrict to Shimura varieties. One remarkable result about Tate's conjecture is a theorem of King-Fai Lai [28]. It asserts that the Tate conjecture is true for any projective Shimura surface of PEL type. The proof is based on the study of Shimura curves on that surface. As an analog, one should consider the contribution of Galois orbits of Shimura subvarieties and their Ekedahl-Oort strata to the Galois invariant subspace of $\mathrm{H}^{2i}_{\text{ét}}(\overline{\mathscr{S}_0}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)(i)$.

0.2 Known results

Now the reader is assumed to know more about algebraic geometry.

The Ekedahl-Oort strata were first defined and studied by Ekedahl and Oort for Siegel modular varieties in late 1990's, see [42]. Let g, n be integers s.t. $g > 0$ and $n > 2$, and $\mathscr{A}_{g,n}$ be the moduli scheme of principally polarized abelian schemes over locally Noetherian $\mathbb{F}_p$-schemes with full level n structure. Then $\mathscr{A}_{g,n}$ is smooth over $\mathbb{F}_p$. Let $\mathcal{A}$ be the universal abelian scheme over $\mathscr{A}_{g,n}$, then for any field k of characteristic $p > 0$, a k-point s of $\mathscr{A}_{g,n}$ gives a polarized abelian variety $(\mathcal{A}_s, \psi)$ over k. The polarization $\psi : \mathcal{A}_s \to \mathcal{A}_s^\vee$ induces an isomorphism $\mathcal{A}_s[p] \simeq \mathcal{A}_s^\vee[p]$ which will still be denoted by ψ.

Definition 0.2.1. We define an equivalence relation on the underlying topological space of $\mathscr{A}_{g,n}$ as follows. Two points $s, s' \in \mathscr{A}_{g,n}$ are equivalent if and only if there exists an algebraically closed field $\overline{k}$ and embeddings of $k(s)$ and $k(s')$, s.t. the pairs $(\mathcal{A}_s[p], \psi) \otimes \overline{k}$ and $(\mathcal{A}_{s'}[p], \psi') \otimes \overline{k}$ are isomorphic, i.e. there is an isomorphism of group schemes $f : \mathcal{A}_s[p] \otimes \overline{k} \to \mathcal{A}_{s'}[p] \otimes \overline{k}$, s.t. the diagram

$$\begin{tikzcd}
\mathcal{A}_s[p] \otimes \overline{k} \arrow[r, "\psi\otimes\overline{k}"] \arrow[d, "f"] & \mathcal{A}_s^\vee[p] \otimes \overline{k} \\
\mathcal{A}_{s'}[p] \otimes \overline{k} \arrow[r, "\psi\otimes\overline{k}"] & \mathcal{A}_s^\vee[p] \otimes \overline{k} \arrow[u, "f^\vee"']
\end{tikzcd}$$

commutes. Let C be the set of equivalence classes of points in $\mathscr{A}_{g,n}$, and for each $c \in C$, let

$$\mathscr{A}_{g,n}^c := \{s \in \mathscr{A}_{g,n} \mid s \text{ lies in the class } c\}.$$

Then each $\mathscr{A}_{g,n}^c$ is called an Ekedahl-Oort stratum.

Here is an incomplete collection of results in [42].

1) Each $\mathscr{A}_{g,n}^c$ is locally closed in $\mathscr{A}_{g,n}$ (see [42] Proposition 3.2).

2) The set C is finite of cardinality 2^g ([42] Lemma 5.7). And each stratum $\mathscr{A}_{g,n}^c$ is quasi-affine, see [42] Theorem 1.2 with also a dimension formula.

3) There is a unique stratum of dimension zero which is called the superspecial stratum and denoted by Σ. It is closed. If $g > 1$, then a point s lies in Σ if and only if its attached abelian variety $\mathcal{A}_s$ is isomorphic to E^g (as abelian varieties without any extra structures) over an algebraically closed field containing $k(s)$. Here E is a supersingular elliptic curve. See also Section 1 of [42].

4) There is a unique one dimensional stratum $L^0 \subseteq \mathscr{A}_{g,n}$. Moreover, if $g > 1$ and if we denote by $\overline{L^0}$ the Zariski closure of L^0 in $\mathscr{A}_{g,n}$, then $L^0 = \overline{L^0} - \Sigma$. And $\overline{L^0}$ is a connected curve contained in the supersingular locus. See also [42] Theorem 1.1 and Section 7.

The works of van der Geer [16] and Ekedahl-van der Geer [11] are also closely related to [42]. They computed the cycle classes of Ekedahl-Oort strata, and compared them with tautological cycles.

The theory of Ekedahl-Oort strata has been generalized to PEL-type Shimura varieties by works of Moonen [38], [39], Wedhorn [55], Moonen-Wedhorn [40] and Viehmann-Wedhorn [53].

Let $\mathscr{D} = (B, {}^*, V, \psi, O_B, \Lambda, h)$ be a PEL-type Shimura datum as in [53] 1.1. Here B is a finite-dimensional semi-simple $\mathbb{Q}$-algebra, * is a $\mathbb{Q}$-linear positive involution on B, V is a finitely generated faithful left B-module, $\psi : V \times V \to \mathbb{Q}$ is a symplectic form, O_B is a $\mathbb{Z}_{(p)}$-order of B, Λ is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$. There are conditions on these data for which we refer to [53] 1.1. Note that our notation here is precisely the same as in [53] 1.1, except that the symplectic form $\langle , \rangle : V \times V \to \mathbb{Q}$ is denoted by ψ here. Let G be the group scheme over $\mathbb{Z}_p$ s.t. for any $\mathbb{Z}_p$-algebra R,

$$\begin{aligned} G(R) = & \{g \in \mathrm{GL}_{O_B}(\Lambda \otimes R) \mid \text{there exists } c(g) \in R^\times, \text{s.t.} \\ & \forall\ v, w \in \Lambda \otimes R, \psi(gv, gw) = c(g)\psi(v, w)\}. \end{aligned}$$

Then G is a reductive group over $\mathbb{Z}_p$ (see [53] page 10). Let E be the reflex field of the PEL-Shimura datum $\mathscr{D}$, note that the conditions on $\mathscr{D}$ implie that $E/\mathbb{Q}$ is unramified at p (see [19] page 306, the second paragraph from the bottom, or our proof of Proposition 2.2.4). Take a place v of E over (p), and let $O_{E,v}$ be the completion of O_E with respect to the maximal ideal v. Then for each compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$, there is a moduli functor $\mathscr{A}_{\mathscr{D},K^p}$ associated to $(\mathscr{D}, K^p)$ defined to be

$$\begin{aligned} &((\text{locally Noetherian } O_{E,v}\text{-schemes})) \to ((\text{Sets})) \\ &\qquad S \mapsto \{\text{isomorphism classes of } (A, \iota, \lambda, \eta), \text{where } (A, \iota, \lambda) \\ &\qquad\qquad \text{is a } \mathbb{Z}_{(p)}\text{-polarized abelian scheme with } O_B\text{-action} \\ &\qquad\qquad \text{and } \eta \text{ is a level } K^p \text{ structure. Moreover, Lie}(A), \\ &\qquad\qquad \text{as an} O_B\text{-module, satisfies Kottwitz's} \\ &\qquad\qquad \text{determinant condition}\}. \end{aligned}$$

We refer to [5] Definition 2.11 for the definition of a $\mathbb{Z}_{(p)}$-polarized abelian scheme with an O_B-action, [5] 2.3 for the definition of a level K^p structure, and [5] 2.2 for the definition of Kottwitz's determinant condition. We remark that the moduli interpretation of $\mathscr{A}_{\mathscr{D},K^p}$ works for general non-Noetherian schemes, but we won't need that.

By [5] Theorem 3.2, $\mathscr{A}_{\mathscr{D},K^p}$ is represented by a smooth quasi-projective scheme over $O_{E,v}$ when K^p is small enough. We will simply write $\mathscr{A}_{\mathscr{D}}$ for $\mathscr{A}_{\mathscr{D},K^p}$ and $\mathscr{A}_{\mathscr{D},0}$ for the special fiber of $\mathscr{A}_{\mathscr{D}}$ when there is no risk for confusion. And we will always assume that K^p is small enough.

For a morphism from an $\mathbb{F}_p$-scheme S to $\mathscr{A}_{\mathscr{D},0}$, there is a tuple $(A, \iota, \lambda, \eta)$ on S which is the pull back of the universal one. Let $A[p]$ be the p-kernel of A, it is a finite flat group scheme over S of rank p^{2g}. The O_B-action ι induces a ring homomorphism $O_B/pO_B \to \mathrm{End}(A[p])$ which will still be denoted by ι. Also, the $\mathbb{Z}_{(p)}$-polarization λ induces an isomorphism $A[p] \to A^\vee[p]$ which is still denoted by λ. Isomorphism classes of triples $(A[p], \iota, \lambda)$ are studied in [38] and [55].

Definition 0.2.2. We define an equivalence relation on the underlying topological space of $\mathscr{A}_{\mathscr{D},0}$ as follows. Two points $s, s' \in \mathscr{A}_{\mathscr{D},0}$ are equivalent if and only if there exists an algebraically closed field $\overline{k}$ with embeddings of $k(s)$ and $k(s')$, s.t. the pairs $(\mathcal{A}_s[p], \iota, \lambda) \otimes \overline{k}$ and $(\mathcal{A}_{s'}[p], \iota', \lambda') \otimes \overline{k}$ are isomorphic. We denote by C the set of equivalence classes of triples $(A[p], \iota, \lambda)$, and for each $c \in C$, let

$$\mathscr{A}^c_{\mathscr{D},0} := \{s \in \mathscr{A}_{\mathscr{D},0} \mid s \text{ lies in the class } c\}.$$

Then each $\mathscr{A}^c_{\mathscr{D},0}$ is called an Ekedahl-Oort stratum.

The main results in [38], [39] and [55] about Ekedahl-Oort strata are as follows.

1) Let $\mathrm{BT}_{\mathscr{D},1}$ be the stack in groupoids over (($\mathbb{F}_p$-schemes)) whose fibers are the categories of BT-1s with $\mathscr{D}$-structure (see [55] 1.3 and Definition 1.4 for the definition of a BT-1 with $\mathscr{D}$-structure, we only mention that the triple $(A[p], \iota, \lambda)$ is a BT-1 with $\mathscr{D}$-structure) with morphisms isomorphisms. Then $\mathrm{BT}_{\mathscr{D},1}$ is a smooth Artin stack over $\mathbb{F}_p$ (see [55] Proposition 1.8 and Corollary 3.3).

2) By associating to $(A, \iota, \lambda, \eta)_{/S} \in \mathscr{A}_{\mathscr{D},0}(S)$ the triple $(A[p], \iota, \lambda)$, we define a morphism of stacks $\Phi : \mathscr{A}_{\mathscr{D},0} \to \mathrm{BT}_{\mathscr{D},1}$. The morphism Φ is smooth when $p > 2$ ([55] Theorem 6.4). T. Wedhorn mentioned to us that the smoothness still holds when $p = 2$, but this has not appeared in literatures. Clearly, $\mathscr{A}^c_{\mathscr{D},0} = \Phi^{-1}(\xi)$ where ξ is the point of $\mathrm{BT}_{\mathscr{D},1}$ corresponding to c. In particular, $\mathscr{A}^c_{\mathscr{D},0}$ is a locally closed subset of $\mathscr{A}_{\mathscr{D},0}$ (see [55] 6.7 for more details).

3) The set of isomorphism classes of BT-1s with $\mathscr{D}$-structure over an algebraically closed field $\overline{k}$ is in bijection with a certain set of double cosets of the Weyl group of $G_{\overline{k}}$ (see [38] 5.4, 5.7 and 6.6 for the construction of the Weyl cosets, and Theorem 5.5, Theorem 6.7 for the bijection).

4) There is a dimension formula for each stratum as follows. Take a stratum $\mathscr{A}^c_{\mathscr{D},0}$ and any representative (G, ι, λ) in c, then $\mathscr{A}^c_{\mathscr{D},0}$ is equi-dimensional, and the codimension of $\mathscr{A}^c_{\mathscr{D},0}$ in $\mathscr{A}_{\mathscr{D},0}$ is $\dim(\mathbf{Aut}(G, \iota, \lambda))$ (see [55], page 465). If one takes the description in terms of Weyl cosets, then the dimension of $\mathscr{A}^c_{\mathscr{D},0}$ equals to the minimal length of elements in the Weyl coset corresponding to c ([39] Corollary 3.1.6).

5) There is a unique stratum which is open dense in $\mathscr{A}_{\mathscr{D},0}$. This stratum is called the ordinary stratum. And there is at most one zero dimensional stratum in $\mathscr{A}_{\mathscr{D},0}$ which is called the superspecial stratum([39] Corollary 3.2.1).

Remark 0.2.3. The advantage of using Φ to define Ekedahl-Oort strata is that it is a morphism of stacks. And one can really work in families. While the characterization using Weyl cosets has the advantage that one gets a list for all the possible strata and their dimensions in terms of Shimura datum.

Later, F-zips and algebraic zip data were introduced by Moonen-Wedhorn in [40] and Pink-Wedhorn-Ziegler in [43] respectively. Assume that the PEL Shimura data is of type A or C. Let X_J be the scheme as on page 1513 of [53]. It is a smooth scheme over $\kappa := O_{E,v}/pO_{E,v}$ (see the last sentence of [53] Remark 4.6) which is determined by the Shimura datum. Moreover, it is equipped with a G_κ-action (see the paragraph after [53] Remark 5.6). By Proposition 5.7 of [53], $\overline{\kappa}$-points of $[G_\kappa \backslash X_J]$ parameterizes isomorphism classes of BT-1s with $\mathscr{D}$-structure. There is also a morphism of stacks $\zeta : \mathscr{A}_{\mathscr{D},0} \to [G_\kappa \backslash X_J]$ (see 5.3 of [53]) s.t. two points are equivalent (w.r.t the relation defined in 0.2.2) if and only if their images in $[G_\kappa \backslash X_J]$ are equal. Using a group theoretic description of Dieudonné displays with $\mathscr{D}$-structure (see [53] 4.2, 5.1) and results about Newton strata (see [53] Proposition 9.17), the following is proved in [53].

The morphism ζ is faithfully flat (see [53] Theorem 6.1 for flatness, and

[53] Theorem 10.1 for non-emptiness of fibers).

Moreover, a partial order $\preceq$ on ${}^JW$ is defined s.t. the underlying topological space of $[G_\kappa\backslash X_J]\otimes\overline{\kappa}$ is homeomorphic to ${}^JW$ (see [53] Definition 5.8, Proposition 5.10). So the closure of an Ekedahl-Oort stratum $\mathscr{A}^c_{\mathscr{D},0}$ is the union of the strata $\mathscr{A}^{c'}_{\mathscr{D},0}$s, s.t. $c'\preceq c$ (Theorem 7.1 of [53]).

0.3 Structure of this paper

The existence of integral canonical models for abelian type Shimura varieties was proved in works of Vasiu and Kisin (see [23], [51]). So it is natural to ask whether we can extend the theory of Ekedahl-Oort strata to reductions of those integral canonical models. And get results similar to the PEL cases (e.g. the strata are listed using a certain Weyl cosets, there is a dimension formula,$\cdots$).

This paper mainly deals with reductions of Hodge type Shimura varieties. And we take the language of G-zips developed in [44] as our technical tool. One can work with $[G_\kappa\backslash X_J]$, but this will lead to some "unpleasant" definitions. Moreover, the language of G-zips is more "motivic", once a good theory of motives over finite fields is established and an intrinsic interpretation of mod p points on the integral model as "motives with G-structure" is given, one should be able to define Ekedahl-Oort strata intrinsically in terms of Shimura datum. Note that to define Ekedahl-Oort strata, we need to first fix a symplectic embedding, and we DO NOT know whether the strata would change or not if we choose another embedding. On the other hand, both the reduction of the Shimura variety and the stack of G-zips are determined by Shimura datum (see [23] Theorem 2.3.8 and [44] 1.4).

The structure of this paper is as follows.

First, we gather the necessary background definitions and results on F-zips and G-zips in Chapter 1. Then we describe Kisin's construction of the integral canonical model in the first part of Chapter 2, and construct a G-zip over it in part 4, using results in part 2 and part 3. The construction is actually the technical heart of this paper, Kisin's results and Faltings's deformation theory of p-divisible groups are used there. And the construction is inspired by the proof of [40] 4.3.

Ekedahl-Oort strata are defined and studied in Chapter 3. The construction in Chapter 2 induces a morphism $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}^{\mu}_{\kappa}$ from the reduction of the Shimura variety to the stack of G-zips of type μ (see Definition 1.2.1 for the definition). There are only finite many points in the topological space of $G\text{-}\mathtt{Zip}^{\mu}_{\kappa} \otimes \overline{\kappa}$. Let C be the set of points of $G\text{-}\mathtt{Zip}^{\mu}_{\kappa} \otimes \overline{\kappa}$, then for a $c \in C$, the Ekedahl-Oort stratum associated to c is defined to be $\zeta^{-1}(c)$ which is locally closed in $\mathscr{S}_0$. We prove that the morphism of stacks ζ is smooth, and hence the dimension formula and the description of the closure of a stratum also work in these cases.

The author believes that ζ is actually faithfully flat (or equivalently, the superspecial locus is always non-empty). Kisin recently announced a proof of the Langlands-Rapoport conjecture for reductions of abelian type Shimura varieties. His proof probably implies a variation (and generalization) of 8.1 of [53], especially Lemma 8.5, and hence a generalization of Proposition 8.17 in [53] which proves the faithfully flatness.

In the second part of Chapter 3, we will give some remarks and comments concerning extra structures on F-zips attached to reduction of a Hodge type Shimura variety.

The last chapter is devoted to one example, namely, the CSpin-varieties. For the construction of the integral canonical model of CSpin-varieties, one can also see [33]. But here still we follow an explicate approach and write down all the details. The purpose of doing this is to make this chapter more likely to be an example. Prof. Wedhorn informed us that the computation in 4.4 has been done in his recent preprint [57]. Here we also write down all the details for the same reason.

1 F-zips and G-zips

1.1 F-zips

In this section, we will follow [40] and [44] to introduce F-zips. Let S be a locally Noetherian scheme, M be a locally free O_S-module of finite rank. By a descending (resp. ascending) filtration $C^\bullet$ (resp. $D_\bullet$) on M, we always mean a separating and exhaustive filtration s.t. $C^{i+1}(M)$ is a locally direct summand of $C^i(M)$ (resp. $D_i(M)$ is a locally direct summand of $D_{i+1}(M)$).

Let $\mathtt{LF}(S)$ be the category of locally free O_S-modules of finite rank, $\mathtt{FilLF}^\bullet(S)$ be the category of locally free O_S-modules of finite rank with descending filtration. For two objects $(M, C^\bullet(M))$ and $(N, C^\bullet(N))$ in $\mathtt{FilLF}^\bullet(S)$, a morphism $f : (M, C^\bullet(M)) \to (N, C^\bullet(N))$ is a morphism f of O_S-modules s.t. $f(C^i(M)) \subseteq C^i(N)$. We also denote by $\mathtt{FilLF}_\bullet(S)$ the category of locally free O_S-modules of finite rank with ascending filtration. For two objects $(M, C^\bullet)$ and $(M', C'^\bullet)$ in $\mathtt{FilLF}^\bullet(S)$, their tensor product is defined to be $(M \otimes M', T^\bullet)$ with $T^i = \sum_j C^j \otimes C'^{i-j}$. Similarly for $\mathtt{FilLF}_\bullet(S)$. For an object $(M, C^\bullet)$ in $\mathtt{FilLF}^\bullet(S)$, one defines its dual to be

$$(M, C^\bullet)^\vee = ({}^\vee M := M^\vee, {}^\vee C^i := (M/C^{1-i})^\vee);$$

and for an object $(M, D_\bullet)$ in $\mathtt{FilLF}_\bullet(S)$, one defines its dual to be

$$(M, D_\bullet)^\vee = ({}^\vee M := M^\vee, {}^\vee D_i := (M/D_{-1-i})^\vee).$$

If S is over $\mathbb{F}_p$, we will denote by $\sigma : S \to S$ the morphism which is the identity on the topological space and p-th power on the sheaf of functions. For an S-scheme T, we will write $T^{(\sigma)}$ for the pull back of T via σ. For a quasi-coherent O_S-module M, $M^{(p)}$ means the pull back of M via σ. For a σ-linear map $\varphi : M \to M$, we will denote by $\varphi^{\mathrm{lin}} : M^{(p)} \to M$ its linearization.

Definition 1.1.1. Let S be a locally Noetherian scheme over $\mathbb{F}_p$. An F-zip over S is a tuple $\underline{M} = (M,\ C^\bullet,\ D_\bullet,\ \varphi_\bullet)$ such that

a) M is an object in $\mathtt{LF}(S)$, i.e. M is a locally free sheaf of finite rank on S;

b) $(M, C^\bullet)$ is an object in $\mathtt{FilLF}^\bullet(S)$, i.e. $C^\bullet$ is a descending filtration on M;

c) $(M, D_\bullet)$ is an object in $\mathtt{FilLF}_\bullet(S)$, i.e. $D_\bullet$ is an ascending filtration on M;

d) $\varphi_i : C^i/C^{i+1} \to D_i/D_{i-1}$ is a σ-linear map whose linearization

$$\varphi_i^{\mathrm{lin}} : (C^i/C^{i+1})^{(p)} \to D_i/D_{i-1}$$

is an isomorphism.

By a morphism of F-zips

$$\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \to \underline{M'} = (M', C'^\bullet, D'_\bullet, \varphi'_\bullet),$$

we mean a morphism of O_S-modules $f : M \to N$, s.t. for all $i \in \mathbb{Z}$, $f(C^i) \subseteq C'^i$, $f(D_i) \subseteq D'_i$, and f induces a commutative diagram

$$\begin{array}{ccc} C^i/C^{i+1} & \xrightarrow{\varphi_i} & D_i/D_{i-1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ C'^i/C'^{i+1} & \xrightarrow{\varphi'_i} & D'_i/D'_{i-1}. \end{array}$$

Remark 1.1.2. Let S be a locally Noetherian $\mathbb{F}_p$-scheme, and X be an abelian scheme over S, $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ has a natural F-zip structure. More generally, let Y be a proper smooth scheme over S, then there are two spectral sequences constructed using filtrations in [56] 1.1. These two spectral sequences are the Hodge spectral sequence and the conjugate spectral sequence (see [56] 1.11). If Y/S satisfies

a) the Hodge spectral sequence degenerates at E_1,

b) the O_S-modules $\mathrm{R}^a f_*(\Omega^b_{Y/S})$ are locally free,

then $\mathrm{H}^i_{\mathrm{dR}}(Y/S)$ has a structure of an F-zip. The descending filtration is given by the Hodge spectral sequence, the ascending filtration is induced by the conjugate spectral sequence, and the φ_is are induced by the Cartier isomorphisms. See [56] 1.6, 1.7 and 1.11 for more details and examples when the conditions a) and b) are satisfied.

Example 1.1.3. Let M be a two dimensional vector space over $k = \overline{\mathbb{F}_p}$ with basis (e_1, e_2). Consider the F-zip structure $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ on M where $C^0 = M$, $C^1 = k \cdot e_1$, $C^2 = 0$, $D_{-1} = 0$, $D_0 = k \cdot e_1$, $D_1 = M$, φ_0 maps e_2 to e_1 and φ_1 maps e_1 to e_2. This is an F-zip, and it is isomorphic to the F-zip attached to the p-kernel of a supersingular elliptic curve over k. Let f be the endomorphism of M taking e_1 to 0 and e_2 to e_1. Then f induces an endomorphism of $(M, C^\bullet, D_\bullet, \varphi_\bullet)$. But f does not have a kernel in the category of F-zips.

Example 1.1.4. Here is another example of a 2-dimensional F-zip. Let M be as in Example 1.1.3. We consider the F-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ as follows. The filtrations are

$$C^\bullet : C^0 = M \supseteq C^1 = k \cdot e_1 \supseteq C^2 = 0$$
$$D_\bullet : D_{-1} = 0 \subseteq D_0 = k \cdot e_2 \subseteq D_1 = M,$$

and the σ-linear maps φ_0 and φ_1 are given by $\varphi_0(e_2) = e_2$ and $\varphi_1(e_1) = e_1$ respectively. Then $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ is isomorphic to the F-zip associated to an ordinary elliptic curve over $\overline{\mathbb{F}_p}$.

Definition 1.1.5. A morphism between two objects in $\mathtt{LF}(S)$ is said to be admissible if the image of the morphism is a locally direct summand. A morphism $f : (M, C^\bullet) \to (M', C'^\bullet)$ in $\mathtt{FilLF}^\bullet(S)$ (resp. $f : (M, D_\bullet) \to (M', D'_\bullet)$ in $\mathtt{FilLF}_\bullet(S)$) is called admissible if for all i, $f(C^i)$ (resp. $f(D_i)$) is equal to $f(M) \cap C'^i$ (resp. $f(M) \cap D'_i$) and is a locally direct summand of M'. A morphism between two F-zips $\underline{M} \to \underline{M'}$ in $F\text{-}\mathtt{Zip}(S)$ is called admissible if it is admissible with respect to the two filtrations.

Definition 1.1.6. ([44] Definition 6.4) Let $\underline{M}$, $\underline{N}$ be two F-zips over S, then their tensor product is the F-zip $\underline{M} \otimes \underline{N}$, consisting of the tensor product $M \otimes N$ with induced filtrations $C^\bullet$ and $D_\bullet$ on $M \otimes N$, and induced σ-linear maps

$$\begin{tikzcd}
\mathrm{gr}^i_C(M\otimes N) \arrow[d, "\cong"] & \mathrm{gr}^D_i(M\otimes N) \\
\bigoplus_j \mathrm{gr}^j_C(M)\otimes \mathrm{gr}^{i-j}_C(N) \arrow[r, "\bigoplus_j \varphi_j\otimes\varphi_{i-j}"] & \bigoplus_j \mathrm{gr}^D_j(M)\otimes \mathrm{gr}^D_{i-j}(N). \arrow[u, "\cong"']
\end{tikzcd}$$

whose linearization are isomorphisms.

Definition 1.1.7. ([44] Definition 6.5) The dual of an F-zip $\underline{M}$ is the F-zip $\underline{M}^\vee$ consisting of the dual sheaf of O_S-modules $M^\vee$ with the dual descending filtration of $C^\bullet$ and dual ascending filtration of $D_\bullet$, and σ-linear maps whose linearization are isomorphisms

$$(\mathrm{gr}^i_C(M^\vee))^{(p)} = ((\mathrm{gr}^{-i}_C M)^\vee)^{(p)} \xrightarrow{\left((\varphi^{\mathrm{lin}}_{-i})\right)^{-1\vee}} (\mathrm{gr}^D_{-i} M)^\vee \cong \mathrm{gr}^D_i(M^\vee).$$

Example 1.1.8. ([44] Example 6.6) The Tate F-zips of weight d is

$$\mathbb{1}(d) := (O_S, C^\bullet, D_\bullet, \varphi_\bullet),$$

where

$$C^i = \begin{cases} O_S & \text{for } i \le d; \\ 0 & \text{for } i > d; \end{cases} \qquad D_i = \begin{cases} 0 & \text{for } i < d; \\ O_S & \text{for } i \ge d; \end{cases}$$

and φ_d is the identity on $O_S^{(p)} = O_S$.

There are natural isomorphisms $\mathbb{1}(d)\otimes\mathbb{1}(d') \cong \mathbb{1}(d+d')$ and $\mathbb{1}(d)^\vee \cong \mathbb{1}(-d)$. The d-th Tate twist of an F-zip $\underline{M}$ is defined as $\underline{M}(d) := \underline{M}\otimes\mathbb{1}(d)$, and there is a natural isomorphism $\underline{M}(0) \cong \underline{M}$.

With admissible morphisms, tensor products and duals defined as above, the categories $\mathtt{LF}(S)$, $\mathtt{FilLF}^\bullet(S)$, $\mathtt{FilLF}_\bullet(S)$ become O_S-linear exact rigid tensor categories (see [44] 4.1, 4.3, 4.4, and see [45] for the definition of an exact category). The admissible morphisms, tensor products, duals and the Tate object $\mathbb{1}(0)$ makes F-$\mathtt{Zip}(S)$ an $\mathbb{F}_p$-linear exact rigid tensor category (page 27 of [44]). And the natural forgetful functors F-$\mathtt{Zip}(S) \to \mathtt{LF}(S)$, F-$\mathtt{Zip}(S) \to \mathtt{FilLF}^\bullet(S)$, F-$\mathtt{Zip}(S) \to \mathtt{FilLF}_\bullet(S)$ are exact functors.

Remark 1.1.9. For a morphism in $\mathtt{LF}(S)$, $\mathtt{FilLF}^{\bullet}(S)$, $\mathtt{FilLF}_{\bullet}(S)$ or $F\text{-}\mathtt{Zip}(S)$, the property of being admissible is local for the Zariski/étale/fppf/fpqc topology (see [44] Lemma 4.2, Lemma 6.8). The morphism f in Example 1.1.3 is not admissible, as $0 = f(C^1) \neq f(M) \cap C^1 = k \cdot e_1$.

1.2 G-zips

We will introduce G-zips following [44] Chapter 3. Note that the authors of [44] work with reductive groups over a general finite field $\mathbb{F}_q$ containing $\mathbb{F}_p$, and q-Frobenius. But we don't need the most general version of G-zips, as our reductive groups are connected and defined over $\mathbb{F}_p$.

Let G be a connected reductive group over $\mathbb{F}_p$, k be a finite extension of $\mathbb{F}_p$, and $\chi : \mathbb{G}_{m,k} \to G_k$ be a cocharacter over k. Let P_+ (resp. P_-) be the unique parabolic subgroup of G_k s.t. its Lie algebra is the sum of spaces with non-negative weights (resp. non-positive weights) in $\mathrm{Lie}(G_k)$ under $\mathrm{Ad} \circ \chi$. Let U_+ (resp. U_-) be the unipotent radical of P_+ (resp. P_-), and L be the common Levi subgroup of P_+ and P_-. Note that L is also the centralizer of χ.

Definition 1.2.1. Let S be a locally Noetherian scheme over k. A G-zip of type χ over S is a tuple $\underline{I} = (I, I_+, I_-, \iota)$ consisting of a right G_k-torsor I over S, a right P_+-torsor $I_+ \subseteq I$, a right $P_-^{(p)}$-torsor $I_- \subseteq I$, and an isomorphism of $L^{(p)}$-torsors $\iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$.

A morphism $(I, I_+, I_-, \iota) \to (I', I'_+, I'_-, \iota')$ of G-zips of type χ over S consists of equivariant morphisms $I \to I'$ and $I_\pm \to I'_\pm$ that are compatible with inclusions and the isomorphisms ι and ι'.

The category of G-zips of type χ over S will be denoted by $G\text{-}\mathtt{Zip}_k^\chi(S)$. Let $G\text{-}\mathtt{Zip}_k^\chi$ be the fibered category over the category of k-schemes as follows. For each k-scheme S, its fiber is the category whose objects are objects in $G\text{-}\mathtt{Zip}_k^\chi(S)$, and whose morphisms are isomorphisms in $G\text{-}\mathtt{Zip}_k^\chi(S)$. With the evident notion of pullback, $G\text{-}\mathtt{Zip}_k^\chi$ forms a fibered category over the category of locally Noetherian schemes over k.

Theorem 1.2.2. *The fibered category $G\text{-}\mathtt{Zip}_k^\chi$ is a smooth algebraic stack of dimension 0 over k.*

Proof. This is [44] Corollary 3.12. $\square$

1.2.3 Some technical constructions about G-zips

We need more information about the structure of $G\text{-}\mathtt{Zip}_k^\chi$. First, we need to introduce some standard G-zips as in [44].

Construction 1.2.4. ([44] Construction 3.4) Let S/k be a locally Noetherian scheme. For any section $g \in G(S)$, one associates a G-zip of type χ over S as follows. Let $I_g = S \times_k G_k$ and $I_{g,+} = S \times_k P_+ \subseteq I_g$ be the trivial torsors. Then $I_g^{(p)} \cong S \times_k G_k = I_g$ canonically, and we define $I_{g,-} \subseteq I_g$ as the image of $S \times_k P_-^{(p)} \subseteq S \times_k G_k$ under left multiplication by g. Then left multiplication by g induces an isomorphism of $L^{(p)}$-torsors

$$\iota_g : I_{g,+}^{(p)}/U_+^{(p)} = S\times_k P_+^{(p)}/U_+^{(p)} \cong S\times_k P_-^{(p)}/U_-^{(p)} \xrightarrow{\sim} g(S\times_k P_-^{(p)})/U_-^{(p)} = I_{g,-}/U_-^{(p)}.$$

We thus obtain a G-zip of type χ over S, denoted by $\underline{I}_g$.

Lemma 1.2.5. *Any G-zip of type χ over S is étale locally of the form $\underline{I}_g$.*

Proof. This is [44] Lemma 3.5. Unfortunately, their proof seems to be not correct. We first give our proof, and then point out the problem with theirs.

Let S be a k-scheme. Let (I, I_+, I_-, ι) be a G-zip of type χ on S. Étale locally, on $U \to S$, say, there are sections a of I_+ and b of I_- such that $\iota(a^{(p)}) = b$ in $I_-/U_-^{(p)}$ (first choose a, then b, and use that $I_- \to I_-/U_-^{(p)}$ is smooth). Let g be the element of $G(U)$ such that $b = a{\cdot}g$ in I.

We claim that $\underline{I}_g$ is isomorphic to (I, I_+, I_-, ι) on U. Let $f : I_g = G_U \to I_U$ be the isomorphism of G-torsors that sends 1 to a. Then f induces an isomorphism f_+ from $I_{g,+} = P_{+,U}$ to $I_{+,U} = a{\cdot}P_{+,U}$. We have

$$f(g) = f(1{\cdot}g) = f(1){\cdot}g = a{\cdot}g = b,$$

hence f sends $I_{g,-}$ to $I_{-,U}$, inducing an isomorphism f_-. It remains to check that f is compatible with the ιs. The isomorphism ι_g sends the image of 1 in $(P_+/U_+)_U^{(p)}$ to the image of g in $(g{\cdot}P_{-,U}^{(p)})/U_{-,U}^{(p)}$. The isomorphism f_+ induces an isomorphism $f_+^{(p)} : P_{+,U}^{(p)} \to I_{+,U}^{(p)}$ sending 1 to $a^{(p)}$. As f_- sends g to b, and ι sends $a^{(p)}$ to b, f is compatible with the ιs.

In the proof of [44] Lemma 3.5, a and b are called i_+ and i_-, but g is defined by the condition that $b = a^{(p)}{\cdot}g$ in $I^{(p)}$. Their f also sends 1 to a. But then the problem is that f sends $I_{g,-}$ to $a{\cdot}g{\cdot}P_-^{(p)}$, whereas $I_{-,U} = a^{(p)}g{\cdot}P_-^{(p)}$. $\square$

Now we will explain how to write $G\text{-}\mathtt{Zip}_k^\chi$ in terms of quotient of an algebraic variety by the action of a linear algebraic group following [44] Section 3.

Denote by $\mathrm{Frob}_p : L \to L^{(p)}$ the relative Frobenius of L, and by $E_{G,\chi}$ the fiber product

$$\begin{tikzcd} E_{G,\chi} \arrow[rr] \arrow[d] & & P_-^{(p)} \arrow[d] \\ P_+ \arrow[r] & L \arrow[r, "\mathrm{Frob}_p"] & L^{(p)}. \end{tikzcd}$$

Then $E_{G,\chi}$ acts on G_k from the left hand side as follows. The group
(1.2.6)
$$E_{G,\chi}(S) = \{(p_+ := lu_+,\ p_- := l^{(p)}u_-) : l \in L(S), u_+ \in U_+(S), u_- \in U_-^{(p)}(S)\}.$$

For $(p_+, p_-) \in E_{G,\chi}(S)$ and $g \in G_k(S)$, $(p_+, p_-) \cdot g := p_+ g p_-^{-1}$.

To relate $G\text{-}\mathtt{Zip}_k^\chi$ to the quotient stack $[E_{G,\chi}\backslash G_k]$, we need the following constructions in [44]. First, for any two sections $g, g' \in G_k(S)$, there is a natural bijection between the set

$$\mathrm{Transp}_{E_{G,\chi}(S)}(g, g') := \{(p_+, p_-) \in E_{G,\chi}(S) \mid p_+ g p_-^{-1} = g'\}$$

and the set of morphisms of G-zips $\underline{I}_g \to \underline{I}_{g'}$ (see [44] Lemma 3.10). So we define a category $\mathcal{X}$ fibered in groupoids over the category of locally Noetherian k-schemes $((\mathbf{Sch}/k))$ as follows. For any scheme S/k, let $\mathcal{X}(S)$ be the small category whose underly set is $G(S)$, and for any two elements $g, g' \in G(S)$, the set of morphisms is the set $\mathrm{Transp}_{E_{G,\chi}(S)}(g, g')$.

Theorem 1.2.7. *There is a fully faithful morphism $\mathcal{X} \to G\text{-}\mathtt{Zip}_k^{\chi}$ sending $g \in \mathcal{X}(S) = G(S)$ to $\underline{I}_g$ which induces an isomorphism $[E_{G,\chi} \backslash G_k] \to G\text{-}\mathtt{Zip}_k^{\chi}$.*

Proof. This is Proposition 3.11 in [44]. $\square$

2 Integral canonical models for Hodge type Shimura varieties and G-zips

2.1 Construction of integral canonical models

We will first follow [34] to introduce Shimura varieties, and then follow [23] to introduce integral canonical models.

Definition 2.1.1. Let G be a connected reductive group over $\mathbb{Q}$. We will write $\mathbb{S}$ for the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Let $h : \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism of algebraic groups and X be the $G(\mathbb{R})$-conjugacy class of h. Then the pair (G, X) is called a Shimura datum if the following conditions are satisfied

1) The Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ given by $\mathrm{Ad} \circ h$ is of type $(-1, 1) + (0, 0) + (1, -1)$.

2) The conjugation action of $h(i)$ on $G^{\mathrm{ad}}_{\mathbb{R}}$ gives a Cartan involution.

3) G^{ad} has no simple factor over $\mathbb{Q}$ onto which h has trivial projection.

Let (G, X) be a Shimura datum, K be a compact open subgroup of $G(\mathbb{A}_f)$. We write $G(\mathbb{R})^+$ for the identity component of $G(\mathbb{R})$ with respect to the real topology, and $G(\mathbb{R})_+$ for the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the adjoint map, $G(\mathbb{Q})_+ = G(\mathbb{Q}) \bigcap G(\mathbb{R})_+$. We set

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K),$$

where $G(\mathbb{Q})$ acts diagonally on $X \times G(\mathbb{A}_f)$. Let $X^+ \subseteq X$ be a connected component, let $g_1, \cdots, g_m$ be representatives in $G(\mathbb{A}_f)$ of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ (see [34] Lemma 5.12), then

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = \coprod_{i=1}^{m} \Gamma_i\backslash X^+,$$

where Γ_i is the image of $G(\mathbb{Q})_+ \cap g_i K g_i^{-1}$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$ which is an arithmetic subgroup in $\mathrm{Aut}(X^+)$ (see [34] Lemma 5.13 and the discussion after the proof).

For K small enough i.e. s.t. all the Γ_i's are torsion free, a theorem of Baily-Borel asserts that $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ has a natural structure of a quasi-projective smooth algebraic variety over $\mathbb{C}$. Results of Deligne, Milne, Borovoi, Shih and others show that $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ has a canonical model $\mathrm{Sh}_K(G,X)$ over the reflex field E. We refer to [34] Chapter 12 and [37] Chapter 2, especially 2.17 for more details.

Let $p \geq 3$ be a prime. Let $G_{\mathbb{Z}_p}$ be a reductive group over $\mathbb{Z}_p$ whose generic fiber is $G_{\mathbb{Q}_p}$. Let K_p be $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, and K^p be an open compact subgroup of $G(\mathbb{A}_f^p)$ and let $K = K_pK^p$. Let v be a prime of O_E, then v is unramified over p.

Assume that the Shimura datum (G,X) is of Hodge type, i.e. there is an embedding of Shimura data $(G,X) \hookrightarrow \big(\mathrm{GSp}(V,\psi), X'\big)$. Then by [23] Proposition 1.3.2, Lemma 2.3.1 and 2.3.2, for the chosen $G_{\mathbb{Z}_p}$, there exists a lattice $V_{\mathbb{Z}} \subseteq V$, s.t. ψ restricts to a pairing $V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ and $G_{\mathbb{Z}_p} \subseteq \mathrm{GL}(V_{\mathbb{Z}_p})$ is defined by a tensor $s \in V_{\mathbb{Z}_p}^{\otimes}$. Here $V_{\mathbb{Z}_p} := V_{\mathbb{Z}} \otimes \mathbb{Z}_p$, and $V_{\mathbb{Z}_p}^{\otimes}$ is a finite free $\mathbb{Z}_p$-module which is obtained from $V_{\mathbb{Z}_p}$ by using the operations of taking duals, tensor products, symmetric powers, exterior powers and direct sums finitely many times.

Let $K'_p \subseteq \mathrm{GSp}(V_{\mathbb{Q}_p}, \psi)$ be the stabilizer of $V_{\mathbb{Z}_p}$. Then by [23] Lemma 2.1.2, we can choose $K' = K'_pK'^p$ s.t. K'^p contains K^p and K' leaves $V_{\widehat{\mathbb{Z}}}$ stable, making the finite morphism

$$\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}(V,\psi), X')_E$$

a closed embedding.

Let $d = |V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}|$, and $g = \dim(V)/2$. Then $\mathrm{Sh}_{K'}(\mathrm{GSp}(V,\psi), X')$ is the generic fiber of $\mathscr{A}_{g,d,K'}$, the fine moduli scheme of g-dimensional abelian schemes over $\mathbb{Z}_{(p)}$-schemes equipped with degree d polarization and level K'-structure (see [41] Theorem 7.9). Let $\mathscr{S}_K(G,K)^-$ be the Zariski closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{A}_{g,d,K'} \otimes O_{E,(v)}$ with the reduced induced scheme structure, and $\mathscr{S}_K(G,X)$ be the normalization of $\mathscr{S}_K(G,K)^-$. Let $\mathcal{A}$ be the universal abelian scheme on $\mathscr{A}_{g,d,K'}$. Then

$$\mathcal{V}^{\circ} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A} \times_{\mathscr{A}_{g,d,K'}} \mathscr{S}_K(G,X)/\mathscr{S}_K(G,X))$$

is a vector bundle on $\mathscr{S}_K(G,X)$. By the construction of [23] Section 2.2, the tensor $s \in V_{\mathbb{Z}_p}^{\otimes}$ gives a section s_{dR} of $(\mathcal{V}^\circ \otimes \mathrm{Sh}_K(G,X))^{\otimes}$.

Here we collect some of the main results in [23].

Theorem 2.1.2.

1) The scheme $\mathscr{S}_K(G,X)$ is smooth over $O_{E,(v)}$. And

$$\mathscr{S}_{K_p}(G,X) := \varprojlim_{K^p} \mathscr{S}_{K_pK^p}(G,X)$$

is an inverse system with finite étale translation maps, whose generic fiber is $G(\mathbb{A}_f^p)$-equivariantly isomorphic to $\mathrm{Sh}_{K_p}(G,X) := \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(G,X)$.

2) The pro-scheme $\mathscr{S}_{K_p}(G,X)$ satisfies the following extension property. For any regular and formally smooth $O_{E,(v)}$-scheme X, any morphism

$$X \otimes E \to \mathscr{S}_{K_p}(G,X)$$

extends uniquely to a morphism $X \to \mathscr{S}_{K_p}(G,X)$. In particular, $\mathscr{S}_{K_p}(G,X)$ (and hence $\mathscr{S}_K(G,X)$) is independent of the choice of the symplectic embedding.

3) The section s_{dR} extends to a section of $\mathcal{V}^{\circ\otimes}$. For any closed point $x \in \mathscr{S}_K(G,X) \otimes \mathbb{F}_p$ and any lifting $\widetilde{x} \in \mathscr{S}_K(G,X)(W(k(x)))$, we have

3.a) the scheme $\mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^\circ_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\big)$ is a trivial right $G_{\mathbb{Z}_p} \otimes W(k(x))$-torsor.

3.b) for any $t \in \mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^\circ_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\big)(W(k(x)))$, $G_{\mathbb{Z}_p} \otimes W(k(x))$ acts faithfully on $\mathcal{V}^\circ_{\widetilde{x}}$ via $g(v) := tgt^{-1}(v)$, for all $v \in \mathcal{V}^\circ_{\widetilde{x}}$. The Hodge filtration on $\mathcal{V}^\circ_{\widetilde{x}}$ is induced by a cocharacter of $G_{\mathbb{Z}_p} \otimes W(k(x))$.

Proof. 1) and 2) are [23] Theorem 2.3.8 (1) and (2) respectively. The first sentence of 3) is Corollary 2.3.9 in [23].

The proof 3.a) is hidden inside [23]. We write F for $q.f.(W(k(x)))$ and $\widetilde{x}_F$ for the F-point of $\mathscr{S}_K(G,X)$ given by the composition

$$\mathrm{Spec}(F) \hookrightarrow \mathrm{Spec}(W(k(x))) \xrightarrow{\widetilde{x}} \mathscr{S}_K(G,X).$$

Let $\mathcal{A}$ be the universal abelian scheme defined six lines before the statement of this theorem. Then there is an isomorphism $V_{\mathbb{Z}_p} \to \mathrm{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x}_F}, \mathbb{Z}_p)$ taking s to $s_{\text{ét},\widetilde{x}_F}$. Here the right hand side is the p-adic étale cohomology of the abelian variety $\mathcal{A}_{\widetilde{x}_F}$ over F.

Then by 3) of Corollary 1.4.3 in [23], there is an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x}_F}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k(x)) \to \mathbb{D}(\mathcal{A}_x[p^\infty])(W(k(x)))$$

taking $s_{\text{ét},\widetilde{x}_F} \otimes 1$ to a Frobenius-invariant tensor s_0. Here we write $\mathbb{D}(-)$ for the Dieudonné functor as in [23]. We remark that this is just an isomorphism, which is highly non-canonical. But by the construction in [23] Corollary 2.3.9, s_0 gives $s_{\mathrm{dR},\widetilde{x}}$ by the canonical identification

$$\mathbb{D}(\mathcal{A}_x[p^\infty])(W(k(x))) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{x}}/W(k(x))).$$

This proves 3.a).

For 3.b), see the proof of [23] Corollary 1.4.3 4). Note that Kisin actually proves the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{x}})$ is a $G_{\mathbb{Z}_p}$-filtration, but in his statement, he only states it for the special fiber. $\square$

2.2 Construction of the G-zip at a point

In this and the following sections, we show how to get a G-zips over $\mathscr{S}_0$ using $\overline{\mathcal{V}^\circ}$. We will first say something about cocharacters inducing the Hodge filtrations. As they are crucial data in the definition of G-zips, and will also be used in 2.4 to get the torsor I over $\mathscr{S}_0$.

2.2.1 Basics about cocharacters

Let $V_{\mathbb{Z}_{(p)}}$ be the localization of $V_{\mathbb{Z}}$ as introduced in 2.1. Then the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is reductive and such that $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = G_{\mathbb{Z}_p}$. Let $Z := \mathbf{Hom}(\mathbb{G}_{m,\mathbb{Z}_{(p)}},\ G_{\mathbb{Z}_{(p)}})$ be the fppf-sheaf of cocharacters. We have the following

Proposition 2.2.2. *The scheme Z is represented by a smooth and separated scheme over $\mathbb{Z}_{(p)}$. The quotient of Z by the adjoint action of $G_{\mathbb{Z}_{(p)}}$ is a disjoint union of connected finite étale $\mathbb{Z}_{(p)}$-schemes.*

Proof. The first statement is a special case of Corollary 4.2 in [9] Chapter XI. To prove the second statement, first note that, by Corollary 3.20 in Chapter XIV of [9], maximal tori of $G_{\mathbb{Z}_{(p)}}$ exist Zariski locally. So there exists a maximal torus $T \subseteq G_{\mathbb{Z}_{(p)}}$, as $\mathbb{Z}_{(p)}$ is a local ring. The scheme of cocharacters $X_*(T) := \mathbf{Hom}(\mathbb{G}_{m,\mathbb{Z}_{(p)}}, T)$ is represented by an étale and locally finite scheme, and the Weyl group scheme $W_T := N_{G_{\mathbb{Z}_{(p)}}}(T)/T$ is represented by a finite étale group scheme (see 3.1 in Chapter XXII of [9]).

The inclusion $T \subseteq G_{\mathbb{Z}_{(p)}}$ induces an isomorphism of fppf-sheaves

$$W_T \backslash X_*(T) \cong G_{\mathbb{Z}_{(p)}} \backslash Z.$$

So to prove the second statement, it suffices to prove that $W_T \backslash X_*(T)$ is represented by an étale and locally finite scheme over $\mathbb{Z}_{(p)}$. To see that $W_T \backslash X_*(T)$ is representable, note that $W_T \times X_*(T)$ and $X_*(T) \times X_*(T)$ are both étale over $\mathrm{Spec}(\mathbb{Z}_{(p)})$, so the morphism

$$\alpha : W_T \times X_*(T) \to X_*(T) \times X_*(T), \quad (w, \nu) \mapsto (\nu, w \cdot \nu)$$

is also étale, and hence has open image. But it is also closed, since there is a finite étale cover S of $\mathrm{Spec}(\mathbb{Z}_{(p)})$, s.t. W_T and $X_*(T)$ become constant, and then the image of α is just copies of S, which is closed in $(X_*(T) \times X_*(T))_S$. So by [24] Chapter 2 Definition 1.6 and Corollary 6.16, $W_T \times X_*(T)$ is represented by an étale separated scheme over $\mathbb{Z}_{(p)}$ of relative dimension 0.

To see that the quotient is étale and locally finite, one still works over S. For an S-point of $X_*(T)_S$, its orbit under $W_T(S)$ is just copies of S. Let $X' \subseteq X_*(T)_S$ be an open and closed subscheme s.t. it contains precise one copy of S in each W_T-orbit. Then $X' \cong (W_T \backslash X_*(T))_S$. And hence $W_T \backslash X_*(T)$ is locally finite and étale. □

Remark 2.2.3. In this remark, we propose another point of view to understand the previous proposition. We claim that for a geometric point $\bar{s} \to \operatorname{Spec}(\mathbb{Z}_{(p)})$, the $G_{\bar{s}}$-orbits of $Z_{\bar{s}}$ are the connected components of $Z_{\bar{s}}$. To see this, take a finite $\mathbb{Z}_{(p)}$-algebra A which is an integral domain, s.t. $Z(A) \neq \emptyset$. For $z \in Z(A)$, the morphism

$$\begin{aligned} \cdot z : G_{\operatorname{Spec}(A)} &\longrightarrow Z_{\operatorname{Spec}(A)} \\ G(T) \ni g &\mapsto g \cdot z, \ \forall \ A\text{-scheme } T. \end{aligned}$$

is smooth by Corollary 3.3 of [9] Chapter IX. In particular, it is open. Note that one $G_{\bar{s}}$-orbit is the complement of the union of all other obits, so each orbit is open and closed. But each orbit is also connected, as $G_{\bar{s}}$ is connected. So the claim is proved.

Let R be the functor

$$\begin{aligned} ((\text{Schemes over } \operatorname{Spec}(\mathbb{Z}_{(p)}))) &\longrightarrow ((\text{Sets})) \\ T \mapsto \{(x,y) \mid (x,y) &\in \operatorname{Mor}_{\mathbb{Z}_{(p)}}(T,Z) \times \operatorname{Mor}_{\mathbb{Z}_{(p)}}(T,Z), \\ &\text{s.t. for any geometric point } \bar{t} \text{ of } T, \ x(\bar{t}) \text{ and } y(\bar{t}) \\ &\text{are in the same connected component of } Z_{\bar{t}}\}. \end{aligned}$$

Then R is represented by an open subscheme of $Z \times_{\mathbb{Z}_{(p)}} Z$ (see [17] 15.6.5). The scheme R gives an equivalence relation on Z, and the quotient of Z by R is denoted by $\pi_0(Z/\operatorname{Spec}(\mathbb{Z}_{(p)}))$. By [30] 6.8.1, $\pi_0(Z/\operatorname{Spec}(\mathbb{Z}_{(p)}))$ is always an algebraic space which is étale and locally separated over $\operatorname{Spec}(\mathbb{Z}_{(p)})$. And $\pi_0(Z/\operatorname{Spec}(\mathbb{Z}_{(p)}))$ is represented by a separated $\mathbb{Z}_{(p)}$-scheme if and only if R is closed in $Z \times_{\mathbb{Z}_{(p)}} Z$. The proof of the previous result implies that R is closed in $Z \times_{\mathbb{Z}_{(p)}} Z$.

Proposition 2.2.4. *Let $\kappa = O_{E,(v)}/(v)$, and $W = W(\kappa) \cong O_{E,v}$ be the ring of Witt vectors. Then the Shimura datum (G,X) determines one connected component $C \cong \operatorname{Spec}(O_{E,(p)})$ of the quotient $G_{\mathbb{Z}_{(p)}}\backslash Z$. Moreover, the inverse image of* $\operatorname{Spec}(W) \to C$ *in Z has a W-point.*

Proof. The Shimura datum (G, X) gives a $G(\mathbb{R})$-orbit X of the real manifold $\mathbf{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})(\mathbb{R})$. For any $x \in X$, it is a homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$. And we have a cocharacter $\mu_x : \mathbb{C}^\times \to G_{\mathbb{C}}(\mathbb{C})$ induced by h_x as follows. For any $G_{\mathbb{R}}$-representation U, there is a decomposition $U_{\mathbb{C}} = \oplus U^{p,q}$ with $h_x(z)u = z^{-p}\bar{z}^{-q}u, \ \forall\ u \in U^{p,q}$. Then the cocharacter μ_x is defined s.t. $\mu_x(z)u = z^{-p}u, \ \forall\ u \in U^{p,q}$. This gives us

$$X \subseteq \mathbf{Hom}_{\mathbb{C}}(\mathbb{S}_{\mathbb{C}}, G_{\mathbb{C}}) = \mathbf{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}}) \to \mathbf{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$$

where the last morphism is given by pre-composing the morphism

$$\mathrm{id} \times 0 : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}.$$

Note that the image of X is contained in one $G(\mathbb{C})$-orbit of $\mathbf{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$. In other words, X gives a $\mathbb{C}$-point $[\mu]$ of $G_{\mathbb{Z}_{(p)}} \backslash Z$, and hence gives a connected component $C \simeq \operatorname{Spec} O_{E,(p)}$.

For the second statement, consider the fiber product

$$\begin{tikzcd}
Z^0 \arrow[rr] \arrow[d, "f"] & & Z \arrow[d] \\
\operatorname{Spec}(W) \arrow[r] & C \arrow[r] & G_{\mathbb{Z}_{(p)}} \backslash Z.
\end{tikzcd}$$

By Theorem 2 of [29], the special fiber of f has a rational point. So Z^0 has a W-point by smoothness. □

2.2.5 An easy lemma

To get started, we need one preparation, namely the next lemma. It tells us how to construct an ascending filtration using the Hodge filtration (see 2.2.10). But to state it and to prove it, we need to fix some notations. Let k be a finite field of characteristic p, A be an abelian scheme over $W(k)$, σ be the ring automorphism $W(k) \to W(k)$ which lifts the p-Frobenius isomorphism on k. Denote by M the module $\mathrm{H}^1_{\mathrm{dR}}(A/W(k)) \cong \mathrm{H}^1_{\mathrm{cris}}(A_k/W(k))$ (see [20] 3.4.b, and this isomorphism is canonical). Then the absolute Frobenius on A_k induces a σ-linear map $\varphi : M \to M$ (see [20] 2.5.3, 3.4.2) whose linearization will be denoted by φ^{lin}. Let $M \supseteq M^1$ be the Hodge filtration. We know that M^1 is a direct summand of M, and its reduction modulo p gives the kernel of Frobenius $\bar{\varphi}$ on $\mathrm{H}^1_{\mathrm{dR}}(A \otimes k/k)$. This implies that $\varphi(M^1) \subseteq pM$, and hence $\varphi/p : M^1 \to M$ is well-defined. The following lemma is classical, but we still give a proof.

Lemma 2.2.6. *For any splitting $M = M^0 \oplus M^1$, the linear map*

$$\alpha : M^{(\sigma)} := M \otimes_{W(k),\sigma} W(k) = M^{0(\sigma)} \oplus M^{1(\sigma)} \xrightarrow{\varphi^{\mathrm{lin}}|_{M^{0(\sigma)}} + (\frac{\varphi}{p})^{\mathrm{lin}}|_{M^{1(\sigma)}}} M$$

is an isomorphism.

Proof. By [27] Proposition 5.2, for a smooth projective scheme over $W(k)$ with torsion free first de Rham cohomology, the filtered F-crystal structure on its first de Rham cohomology is strongly divisible. In particular, for abelian schemes as in our lemma, it means that the natural map

$$\varphi(M^1)/p\varphi(M^1) \to \big(pM \cap \varphi(M)\big)/p\varphi(M)$$

is an isomorphism of k-vector spaces (see [27] 3.1).

Note that $pM \subseteq \varphi(M)$, so $pM \cap \varphi(M) = pM$. Composing the previous isomorphism with the isomorphism $pM/p\varphi(M) \xrightarrow{p^{-1}} M/\varphi(M)$, we see that $\alpha|_{M^1}$ induces an isomorphism

$$M^{1(\sigma)}/pM^{1(\sigma)} \to M/\big(\varphi(M)\big).$$

But by our construction, M_0 modulo p is mapped surjectively to the image of $\bar{\varphi}$. So $\alpha|_{M^{0(\sigma)}}$ induces an isomorphism $M^{0(\sigma)}/pM^{0(\sigma)} \to \varphi(M)/p\varphi(M)$. And hence α induces an isomorphism modulo p. But by Nakayama's lemma, this means that α is surjective, and hence an isomorphism. $\square$

Notations 2.2.7. Now we will fix some notations that will be used later. By Proposition 2.2.4, $Z^0(W) \neq \emptyset$. We will take once and for all a cocharacter $\mu : \mathbb{G}_{m,W} \to G_{\mathbb{Z}_p} \otimes W$ in Z^0. Since we are interested in reductions of integral canonical models, we will work either over W or over κ. So we will simply write $\mathscr{S}$ for $\mathscr{S}_K(G,X)\otimes_{O_{E,(v)}} W$, and $\mathscr{S}_0$ for the special fiber of $\mathscr{S}$. We will write $\mathcal{A}$ for the pull back to $\mathscr{S}$ of the universal abelian scheme on $\mathscr{A}_{g,d,K'}$. We will still denote by $\mathcal{V}^\circ$ (resp s_{dR}) the pullback to $\mathscr{S}$ of $\mathcal{V}^\circ$ (resp. s_{dR}) on $\mathscr{S}_K(G,K)$ as in Theorem 2.1.2. And we will write $\overline{\mathcal{V}^\circ}$ (resp. $\overline{s_{\mathrm{dR}}}$) for the pull back to $\mathscr{S}_0$ of $\mathcal{V}^\circ$ (resp. s_{dR}) on $\mathscr{S}$.

2.2.8 Basic properties of $s_{\mathrm{dR},\widetilde{x}}$ and φ

Now we will discuss some basic properties of $s_{\mathrm{dR},\widetilde{x}}$ related to the Frobenius on $\mathcal{V}^\circ_{\widetilde{x}}$ and the filtration on $\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$ induced by the Hodge filtration. We will keep the notations as in 3.b) of Theorem 2.1.2. In particular, there exists an element $t\in \mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p}\otimes W(k(x)), s\otimes 1),(\mathcal{V}^\circ_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\big)(W(k(x)))$. The element t will be fixed once and for all in our discussion. Also, we will introduce some new notations as follows. Let

$$\mu' : \mathbb{G}_{m,W(k(x))} \to G_{\mathbb{Z}_p}\otimes W(k(x))$$

be a cocharacter s.t. $\mu'_t := t\mu' t^{-1}$ induces the the Hodge filtration on $\mathcal{V}^\circ_{\widetilde{x}}$. Note that μ' is a $W(k(x))$-point of C which is defined at the end of 2.2.4, as the Hodge filtration is always induced by a cocharacter that is $G_{\mathbb{Z}_p}$-conjugate to μ. We will write φ for the Frobenius on $\mathcal{V}^\circ_{\widetilde{x}}$ and $\mathcal{V}^\circ_{\widetilde{x}}=(\mathcal{V}^\circ_{\widetilde{x}})^0\oplus(\mathcal{V}^\circ_{\widetilde{x}})^1$ for the splitting induced by μ'_t. The filtration on $\mathcal{V}^\circ_{\widetilde{x}}$ induces a filtration on $\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$ by the constructions at the beginning of Section 1.1. And there is a Frobenius which is not defined on $\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$, but on $(\mathcal{V}^\circ_{\widetilde{x}}[\frac{1}{p}])^\otimes$ as follows. It is the tensor product of φ on $\mathcal{V}^\circ_{\widetilde{x}}[\frac{1}{p}]$ and

$$^\vee\varphi : \big(\mathcal{V}^\circ_{\widetilde{x}}[1/p])\big)^\vee \to \big(\mathcal{V}^\circ_{\widetilde{x}}[1/p]\big)^\vee, \quad f\mapsto \sigma(f\circ\varphi^{-1}), \quad \forall\, f\in \mathcal{V}^{\circ\vee}_{\widetilde{x}}$$

on $(\mathcal{V}^\circ_{\widetilde{x}}[1/p]))^\vee$. The induced Frobenius on $(\mathcal{V}^\circ_{\widetilde{x}}[\frac{1}{p}])^\otimes$ will still be denoted by φ. It is known that $s_{\mathrm{dR},\widetilde{x}}\in\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$ actually lies in $\mathrm{Fil}^0\,\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$ and and that $s_{\mathrm{dR},\widetilde{x}}$ is φ-invariant ([23] 1.3.3, and we view $s_{\mathrm{dR},\widetilde{x}}$ as an element in $(\mathcal{V}^\circ_{\widetilde{x}}[\frac{1}{p}])^\otimes$ when considering the φ-action).

We have the following better description.

Proposition 2.2.9. *The Frobenius* φ *takes integral value on* $\mathrm{Fil}^0 \mathcal{V}_{\widetilde{x}}^{\circ\otimes}$. *If we denote by* $(\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^0 \subseteq \mathcal{V}_{\widetilde{x}}^{\circ\otimes}$ *the submodule s.t.* $\mu_t'(\mathbb{G}_m)$ *acts trivially, then* $s_{\mathrm{dR},\widetilde{x}} \in (\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^0$.

Proof. We use notations from 2.2.8. To see the first statement, note that we have

$$\mathrm{Fil}^0(\mathcal{V}_{\widetilde{x}}^{\circ\otimes}) = \oplus_{i\geq 0}(\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^i$$

where $(\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^i$ is the submodule whose elements are of weight i with respect to the cocharacter μ_t'. And elements in $(\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^i$ are sums of elements from

$$((\mathcal{V}_{\widetilde{x}}^{\circ})^0)^{\otimes a} \otimes ((\mathcal{V}_{\widetilde{x}}^{\circ})^1)^{\otimes b} \otimes ((\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1})^{\otimes c} \otimes ((\mathcal{V}_{\widetilde{x}}^{\circ\vee})^0)^{\otimes d}$$

s.t. $b - c = i$. The σ-linear map φ induces well defined σ-linear maps

$$\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^0} : (\mathcal{V}_{\widetilde{x}}^{\circ})^0 \to \mathcal{V}_{\widetilde{x}}^{\circ} \quad \text{and} \quad {}^{\vee}\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ\vee})^0} : ((\mathcal{V}_{\widetilde{x}}^{\circ\vee})^0 \to \mathcal{V}_{\widetilde{x}}^{\circ\vee}.$$

But

$$(\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^1})^{\otimes b} \otimes ({}^{\vee}\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1}})^{\otimes c} : ((\mathcal{V}_{\widetilde{x}}^{\circ})^1)^{\otimes b} \otimes ((\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1})^{\otimes c} \to (\mathcal{V}_{\widetilde{x}}^{\circ})^{\otimes b} \otimes (\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{\otimes c}$$

is also defined, as

$$(\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^1})^{\otimes b} \otimes ({}^{\vee}\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1}})^{\otimes c} = p^{b-c} \cdot (\frac{\varphi}{p}|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^1})^{\otimes b} \otimes (p \cdot {}^{\vee}\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1}})^{\otimes c},$$

while $\frac{\varphi}{p}|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^1}$ and $p \cdot {}^{\vee}\varphi|_{(\mathcal{V}_{\widetilde{x}}^{\circ\vee})^{-1}}$ are well-defined. So φ is defined on $\mathrm{Fil}^0(\mathcal{V}_{\widetilde{x}}^{\circ\otimes})$.

To see that $s_{\mathrm{dR},\widetilde{x}} \in (\mathcal{V}_{\widetilde{x}}^{\circ\otimes})^0$, one only needs to use the fact that $s \in V_{\mathbb{Z}_p}^{\otimes}$ is $G_{\mathbb{Z}_p}$-invariant, and hence $s_{\mathrm{dR},\widetilde{x}}$ is also $G_{\mathbb{Z}_p}$-nvariant via t. In particular, it is of weight 0 with respect to the cocharacter μ_t'. □

2.2.10 Constructing some torsors over $W(k(x))$

Now we will show that using the Frobenius φ and the splitting induced by μ_t' (see 2.2.8 for the definition of μ' and μ_t'), we can get an element g_t of $G_{\mathbb{Z}_p}(W(k(x)))$. The element g_t is crucial in the study of the ascending filtration and the $\varphi_\bullet$ of an F-zip associated to a Hodge type Shimura variety.

Construction 2.2.11. Let $\sigma : W(k(x)) \to W(k(x))$ be as in 2.2.5, and ξ be the $W(k(x))$-linear isomorphism $V_{\mathbb{Z}_p} \otimes W(k(x)) \to \big(V_{\mathbb{Z}_p} \otimes W(k(x))\big)^{(\sigma)}$ given by $v \otimes w \mapsto v \otimes 1 \otimes w$ and $t^{(\sigma)}$ be the pull back of

$$t \in \mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\big)(W(k(x)))$$

via σ. Let $\xi_t = t^{(\sigma)} \circ \xi$, and g be the $W(k(x))$-linear map

$$\mathcal{V}^{\circ(\sigma)}_{\widetilde{x}} = (\mathcal{V}^{\circ}_{\widetilde{x}})^{0(\sigma)} \oplus (\mathcal{V}^{\circ}_{\widetilde{x}})^{1(\sigma)} \xrightarrow{\varphi^{\mathrm{lin}}|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^{0(\sigma)}} + (\frac{\varphi}{p})^{\mathrm{lin}}|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^{1(\sigma)}}} \mathcal{V}^{\circ}_{\widetilde{x}} \ .$$

We define g_t to be the composition $t^{-1} \circ g \circ \xi_t$, and $(\mathcal{V}^{\circ}_{\widetilde{x}})_0$ (resp. $(\mathcal{V}^{\circ}_{\widetilde{x}})_1$) to be the sub $W(k(x))$-module of $\mathcal{V}^{\circ}_{\widetilde{x}}$ generated by $\varphi((\mathcal{V}^{\circ}_{\widetilde{x}})^0)$ (resp. $\frac{\varphi}{p}((\mathcal{V}^{\circ}_{\widetilde{x}})^1)$).

We have the following

Proposition 2.2.12.

1) The linear map g_t is an element of $G_{\mathbb{Z}_p}(W(k(x)))$.

2) The splitting

$$V_{\mathbb{Z}_p} \otimes W(k(x)) = t^{-1}\big((\mathcal{V}^{\circ}_{\widetilde{x}})_0\big) \oplus t^{-1}\big((\mathcal{V}^{\circ}_{\widetilde{x}})_1\big)$$

is induced by a cocharacter $\nu = g_t \mu'^{(\sigma)} g_t^{-1}$ of $G_{\mathbb{Z}_p} \otimes W(k(x))$.

Proof. By Lemma 2.2.6, $g_t \in \mathrm{GL}(V_{\mathbb{Z}_p})(W(k(x)))$. So, to prove 1), it suffices to check that $g_t^{\otimes} : \big(V_{\mathbb{Z}_p} \otimes W(k(x))\big)^{\otimes} \to \big(V_{\mathbb{Z}_p} \otimes W(k(x))\big)^{\otimes}$ maps $s \otimes 1$ to itself. Now we compute $g_t(s \otimes 1)$. First, $\xi(s \otimes 1) = s \otimes 1 \otimes 1$ and $t^{(\sigma)}(s \otimes 1 \otimes 1) = s_{\mathrm{dR},\widetilde{x}} \otimes 1$. We decompose $\mathcal{V}^{\circ\otimes}_{\widetilde{x}} = \oplus_i (\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^i$ via the weights of the cocharacter μ'_t introduced before. Then $(\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^{(\sigma)} = \oplus_i \big((\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^i\big)^{(\sigma)}$. And $s_{\mathrm{dR},\widetilde{x}} \in (\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^0$ by Proposition 2.2.9. So

$$g^{\otimes} = \sum_i p^{-i} (\varphi^{\mathrm{lin}})^{\otimes}|_{(\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^i} : \bigoplus_i \big((\mathcal{V}^{\circ\otimes}_{\widetilde{x}})^i\big)^{(\sigma)} \to \mathcal{V}^{\circ\otimes}_{\widetilde{x}}$$

maps $s_{\mathrm{dR},\widetilde{x}} \otimes 1$ to $s_{\mathrm{dR},\widetilde{x}}$, as it is φ-invariant. And hence

$$g_t(s \otimes 1) = t^{-1} \circ g \circ \xi_t(s \otimes 1) = s \otimes 1,$$

as t^{-1} takes $s_{\mathrm{dR},\widetilde{x}}$ to $s\otimes 1$. This proves 1).

For 2), consider the cocharacter $\nu : \mathbb{G}_{m,W(k(x))} \to \mathrm{GL}(V_{\mathbb{Z}_p}\otimes W(k(x)))$ which acts trivially on $t^{-1}\big((\mathcal{V}^{\circ}_{\widetilde{x}})_0\big)$ and as linear multiplication on $t^{-1}\big((\mathcal{V}^{\circ}_{\widetilde{x}})_1\big)$. We claim that

$$\nu(m)(v) = g_t\sigma(\mu'(m))g_t^{-1}(v), \quad \forall\, m\in\mathbb{G}_m(W(k(x))), \forall\, v\in V_{\mathbb{Z}_p}\otimes W(k(x)).$$

To prove the claim, we look at the commutative diagram

$$\begin{tikzcd} V_{\mathbb{Z}_p}\otimes W(k(x)) \arrow[r, "t^{(\sigma)}\circ\xi"] \arrow[dr, "g_t"'] & (\mathcal{V}^{\circ}_{\widetilde{x}})^{(\sigma)} \arrow[r, equal] & (\oplus_i \mathcal{V}^{\circ}_{\widetilde{x}})^{i(\sigma)} \arrow[d, "\Sigma_i p^{-i}\varphi^{\mathrm{lin}}|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^{i(\sigma)}}"] \\ & V_{\mathbb{Z}_p}\otimes W(k(x)) & \mathcal{V}^{\circ}_{\widetilde{x}} \arrow[l, "t^{-1}"'] \end{tikzcd}.$$

It shows directly that

$$\nu(m)(v) = g_t\sigma(\mu'(m))g_t^{-1}(v), \quad \forall\, m\in\mathbb{G}_m(W(k(x))), \forall\, v\in V_{\mathbb{Z}_p}\otimes W(k(x)).$$

□

Corollary 2.2.13. *Let $\mu : \mathbb{G}_{m,W} \to G_{\mathbb{Z}_p}\otimes W$ be the cocharacter as in Notations 2.2.7. Let $C^\bullet$ be the descending filtration on $V_{\mathbb{Z}_p}\otimes W$ induced by μ with stabilizer P_+. Let P_- be the opposite parabolic of P_+ with respect to μ, and $D_\bullet$ be the ascending filtration given by $\mu^{(\sigma)}$. Let $I_{\widetilde{x}}$ be the trivial torsor $\mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p}\otimes W(k(x)), s\otimes 1), (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\big)$. Then*

1) The closed subscheme

$$I_{\widetilde{x},+} := \mathbf{Isom}_{W(k(x))}\Big((V_{\mathbb{Z}_p}\otimes W(k(x)), s\otimes 1, C^\bullet), \big(\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}}, \mathcal{V}^{\circ}_{\widetilde{x}}\supseteq(\mathcal{V}^{\circ}_{\widetilde{x}})^1\big)\Big)\subseteq I_{\widetilde{x}}$$

is a trivial P_+-torsor.

2) The closed subscheme

$$I_{\widetilde{x},-} := \mathbf{Isom}_{W(k(x))}\Big((V_{\mathbb{Z}_p}\otimes W(k(x)), s\otimes 1, D_\bullet), \big(\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}}, (\mathcal{V}^{\circ}_{\widetilde{x}})_0\subseteq\mathcal{V}^{\circ}_{\widetilde{x}}\big)\Big)\subseteq I_{\widetilde{x}}$$

is a trivial $P_-^{(\sigma)}$-torsor.

Proof. To prove 1), first take a $g_1\in G_{\mathbb{Z}_p}(W(k(x)))$ s.t. $g_1\big(\mu\otimes W(k(x))\big)g_1^{-1}=\mu'$. Then we have $I_{\widetilde{x},+} = t\cdot g_1\big(P_+\otimes W(k(x))\big)g_1^{-1}$. For 2), by Proposition 2.2.12, $I_{\widetilde{x},-} = tg_t\cdot\Big(g_1\big(P_-\otimes W(k(x))\big)g_1^{-1}\Big)^{(\sigma)}$. □

2.2.14 The G-zip attached to a filtered F-crystal

Now we will explain how to attach a G-zip to the filtered F-crystal $\mathcal{V}^{\circ}_{\widetilde{x}}$. Let $\overline{\mu}$ be the reduction of the cocharacter μ as in Corollary 2.2.13 and $\overline{G}$ be $G_{\mathbb{Z}_p} \otimes \mathbb{F}_p$. We also write $\overline{P_+}$, $\overline{P_-}$ and $\overline{L}$ for the special fiber of P_+, P_- and L which is the centralizer of μ in $G_{\mathbb{Z}_p} \otimes W$ respectively. First of all, the cocharacter $\overline{\mu}$ induces an F-zip structure on $V_{\mathbb{Z}_p} \otimes \kappa$ as follows. The cocharacter $\overline{\mu}$ gives a splitting $V_{\mathbb{Z}_p} \otimes \kappa = (V_{\mathbb{Z}_p} \otimes \kappa)^0 \oplus (V_{\mathbb{Z}_p} \otimes \kappa)^1$, and the reduction modulo p of the descending filtration $C^\bullet$ in the previous corollary is the base change to $k(x)$ of

$$C^\bullet(V_{\mathbb{Z}_p} \otimes \kappa) : V_{\mathbb{Z}_p} \otimes \kappa \supseteq (V_{\mathbb{Z}_p} \otimes \kappa)^1 \supseteq 0.$$

The cocharacter $\overline{\mu}^{(p)}$ induces the splitting

$$(V_{\mathbb{Z}_p} \otimes \kappa)^{(p)} = ((V_{\mathbb{Z}_p} \otimes \kappa)^0)^{(p)} \oplus ((V_{\mathbb{Z}_p} \otimes \kappa)^1)^{(p)}$$

viewed as a cocharacter of $\mathrm{GL}((V_{\mathbb{Z}_p} \otimes \kappa)^{(p)})$ and the splitting

$$(V_{\mathbb{Z}_p} \otimes \kappa) = (V_{\mathbb{Z}_p} \otimes \kappa)_0 \oplus (V_{\mathbb{Z}_p} \otimes \kappa)_1$$

viewed as a cocharacter of $\mathrm{GL}((V_{\mathbb{Z}_p} \otimes \kappa))$ via ξ. Note that the base change to $k(x)$ of

$$D_\bullet(V_{\mathbb{Z}_p} \otimes \kappa) : 0 \subseteq (V_{\mathbb{Z}_p} \otimes \kappa)_0 \subseteq V_{\mathbb{Z}_p} \otimes \kappa$$

gives the reduction of the $D_\bullet$ as in the previous corollary. Moreover, ξ induces isomorphisms

$$\phi_0 : (V_{\mathbb{Z}_p} \otimes \kappa)^{(p)}/((V_{\mathbb{Z}_p} \otimes \kappa)^1)^{(p)} \stackrel{\mathrm{pr}_2}{\to} ((V_{\mathbb{Z}_p} \otimes \kappa)^0)^{(p)} \xrightarrow{\xi^{-1}} (V_{\mathbb{Z}_p} \otimes \kappa)_0$$

and

$$\phi_1 : ((V_{\mathbb{Z}_p} \otimes \kappa)^1)^{(p)} \xrightarrow{\xi^{-1}} (V_{\mathbb{Z}_p} \otimes \kappa)_1 \simeq (V_{\mathbb{Z}_p} \otimes \kappa)/(V_{\mathbb{Z}_p} \otimes \kappa)_0,$$

and hence induces σ-linear maps φ'_0, φ'_1 after pre-composing the natural map $V_{\mathbb{Z}_p} \otimes \kappa \to (V_{\mathbb{Z}_p} \otimes \kappa)^{(p)}$. The tuple $(V_{\mathbb{Z}_p} \otimes \kappa, C^\bullet(V_{\mathbb{Z}_p} \otimes \kappa), D_\bullet(V_{\mathbb{Z}_p} \otimes \kappa), \varphi'_\bullet)$ is an F-zip. The $\overline{G}$-zip associated to $(\overline{G}, \overline{\mu})$ is isomorphic to $\underline{I}_{\mathrm{id}}$ (here we use notations as at the end of Section 1.2).

To get a $\overline{G}$-zip from $\mathcal{V}^\circ_{\widetilde{x}}$, one needs to "compare" the above F-zip constructed using $\overline{\mu}$ and the one coming from $\mathcal{V}^\circ_{\widetilde{x}}$. Let $\varphi_0 : \mathcal{V}^\circ_x/(\mathcal{V}^\circ_x)^1 \to (\mathcal{V}^\circ_x)_0$ be the reduction mod p of $\varphi|_{(\mathcal{V}^\circ_{\widetilde{x}})^0}$, and

$$\varphi_1 : (\mathcal{V}^\circ_x)^1 \to (\mathcal{V}^\circ_x)_1 \cong \mathcal{V}^\circ_x/(\mathcal{V}^\circ_x)_0$$

be the reduction mod p of $\frac{\varphi}{p}|_{(\mathcal{V}^\circ_{\widetilde{x}})^1}$. Let $I_{x,+}$, $I_{x,-}$ and I_x be the reduction mod p of $I_{\widetilde{x},+}$, $I_{\widetilde{x},-}$ and $I_{\widetilde{x}}$ respectively. For simplicity, we still write t, g_1, g_t for their reductions. Note that, by the proof of Corollary 2.2.13, $I_{x,+}$ and $I_{x,-}$ are equipped right $\overline{P_+}_{k(x)}$-action and right $\overline{P_-}^{(p)}_{k(x)}$-action respectively as follows. For any $k(x)$-algebra R and any $p \in \overline{P_+}_{k(x)}(R)$, p acts on $I_{x,+}(R)$ via right multiplication by $g_1 p g_1^{-1}$. And for any $q \in \overline{P_-}^{(p)}_{k(x)}(R)$, q acts on $I_{x,-}(R)$ via multiplying $g_t g_1^{(p)} p g_1^{(p),-1} g_t^{-1}$ on the right (see Corollary 2.2.13). For any $\beta \in I^{(p)}_{x,+}(R)$, it is an isomorphism

$$\beta : \big(V_{\mathbb{Z}_p} \otimes k(x), \overline{s} \otimes 1, C^\bullet(V_{\mathbb{Z}_p} \otimes k(x))\big)^{(p)} \otimes R \to \big(\mathcal{V}^\circ_x, \overline{s_{\mathrm{dR},x}}, C^\bullet(\mathcal{V}^\circ_x)\big)^{(p)} \otimes R.$$

And it induces an isomorphism

$$\oplus\, \mathrm{gr}^i_C((V_{\mathbb{Z}_p} \otimes k(x))^{(p)} \otimes R) \xrightarrow{\sim} \oplus\, \mathrm{gr}^i_C(\mathcal{V}^{\circ(p)}_x \otimes R)$$

which will still be denoted by β. Note that the $\overline{U_+}$-action on $I_{+,x}$ is induced by the isomorphism $\overline{U_+} \to g_1 \overline{U_+} g_1^{-1}$, while $(g_1 \overline{U_+} g_1^{-1})^{(p)}$ acts trivially on $\oplus\, \mathrm{gr}^i_C((\mathcal{V}^\circ_x \otimes R)^{(p)})$. So β is an element in $I^{(p)}_{x,+}/U^{(p)}_+(R)$. Moreover, any element of $I^{(p)}_{x,+}/U^{(p)}_+(R)$ is fppf locally constructed as above. Let $\iota : I^{(p)}_{x,+}/U^{(p)}_+ \to I_{x,-}/U^{(p)}_-$ be the morphism taking β to

$$\begin{tikzcd}
\oplus\, \mathrm{gr}^D_i((V_{\mathbb{Z}_p} \otimes k(x)) \otimes R) \arrow[d, "(\phi_0^{-1} \oplus \phi_1^{-1}) \otimes 1"] & \oplus\, \mathrm{gr}^D_i(\mathcal{V}^\circ_x \otimes R) \\
\oplus\, \mathrm{gr}^i_C((V_{\mathbb{Z}_p} \otimes k(x))^{(p)} \otimes R) \arrow[r, "\beta"] & \oplus\, \mathrm{gr}^i_C(\mathcal{V}^{\circ(p)}_x \otimes R). \arrow[u, "\varphi^{\mathrm{lin}}_\bullet \otimes 1"']
\end{tikzcd}$$

We claim that ι is an isomorphism of $\overline{L}^{(p)}$-torsors. First note that

$$\phi_0^{-1} \oplus \phi_1^{-1} : \oplus\, \mathrm{gr}^D_i(V_{\mathbb{Z}_p} \otimes k(x)) \to \oplus\, \mathrm{gr}^i_C((V_{\mathbb{Z}_p} \otimes k(x))^{(p)})$$

and $\varphi_\bullet^{\mathrm{lin}} : \oplus \operatorname{gr}_C^i(\mathcal{V}_x^{\circ(p)}) \to \oplus \operatorname{gr}_i^D(\mathcal{V}_x^\circ)$ are isomorphisms, and so are their base changes to R. This implies that ι is an isomorphism. We only need to show that ι is $L^{(p)}$-equivariant. To do this, first note that $\beta = t^{(p)} g_1^{(p)} p g_1^{(p),-1}$ for some $p \in \overline{P_+}^{(p)}_{k(x)}(R)$ and

$$\iota(\beta) = \varphi_\bullet^{\mathrm{lin}} \circ t^{(p)} g_1^{(p)} p g_1^{(p),-1} = t g_t g_1^{(p)} p g_1^{(p),-1} g_t^{-1}.$$

For any $l \in \overline{L}^{(p)}_{k(x)}(S)$, we have

$$\beta \cdot l = t^{(p)} g_1^{(p)} p g_1^{(p),-1} \cdot l = t^{(p)} g_1^{(p)} p l g_1^{(p),-1}.$$

And hence we have

$$\iota(\beta) \cdot l = t g_t g_1^{(p)} p g_1^{(p),-1} g_t^{-1} \cdot l = t g_t g_1^{(p)} p l g_1^{(p),-1} g_t^{-1} = \iota(\beta \cdot l).$$

This proves that ι is $\overline{L}^p$-equivariant. So the tuple $(I_x, I_{x,+}, I_{x,-}, \iota)$ is a $G_{\mathbb{Z}_p} \otimes \kappa$-zip of type μ over $k(x)$. Using notations and constructions in the discussion after Theorem 1.2.2, we have $(I_x, I_{x,+}, I_{x,-}, \iota) \cong \underline{I}_{g_1^{-1} g_t g_1^{(p)}}$.

Remark 2.2.15. In this subsection, we do NOT follow [44] strictly. As they use left G-action on G/P, while we use conjugation action. But there is no difference, as P is its own's normalizer.

2.3 Construction of the G-zip over a complete local ring

We want to globalize the above point-wise results to $\mathscr{S}_0$. But to do so, we need first to work at completions of stalks at closed points. And to study the G-zip structure at the complete local rings, we need Faltings's deformation theory.

2.3.1 Faltings's deformation theory and complete local rings of the integral model

Now we will describe Faltings's deformation theory for p-divisible groups following [37] 4.5 and its relation with Shimura varieties following [23] 1.5, 2.3.

Let k be a perfect field of characteristic p, and $W(k)$ be the ring of Witt vectors. Let H be a p-divisible group over $W(k)$ with special fiber H_0. The formal deformation functor for H_0 is represented by a ring R of formal power series over $W(k)$. More precisely, let (M_0, M_0^1, φ_0) be the filtered Dieudonné module associated to H, and L be a Levi subgroup of $P = \mathrm{stab}(M_0 \supseteq M_0^1)$. Let U be the opposite unipotent of P, then R is isomorphic to the completion at the identity section of U. Let u be the universal element in $U(R)$, and $\sigma : R \to R$ be the homomorphism which is the Frobenius on $W(k)$ and p-th power on variables, then the filtered Dieudonné module of the universal p-divisible group over R is the tuple $(M, M^1, \varphi, \nabla)$, where $M = M_0 \otimes R$, $M^1 = M_0^1 \otimes R$, $\varphi = u \cdot (\varphi_0 \otimes \sigma)$, and ∇ is an integrable connection which we don't want to specify, but just refer to [37] Chapter 4.

More generally, let $G \subseteq \mathrm{GL}(M)$ be a reductive group defined by a tensor $s \in \mathrm{Fil}^0(M^{\otimes}) \subseteq M^{\otimes}$ which is φ_0-invariant. Assume that the filtration $M_0 \supseteq M_0^1$ is induced by a cocharacter μ of G. Let R_G be the completion along the identity section of the opposite unipotent of $P_G = \mathrm{stab}_G(M_0 \supseteq M_0^1) \subseteq G$ with respect to μ, and u_G be the universal element in $U_G(R_G)$ which is also the pull back to R_G of u. Then R_G parametrizes deformations of H s.t. the horizontal continuation of s remains a Tate tensor (see [37] Proposition 4.9).

For any closed point $x \in \mathscr{S}_0$, let $\widehat{O_{\mathscr{S}_0,x}}$ and $\widehat{O_{\mathscr{S},x}}$ be the completions of $O_{\mathscr{S}_0,x}$ and $O_{\mathscr{S},x}$ with respect to the maximal ideals defining x respectively. Clearly $\widehat{O_{\mathscr{S},x}}/p\widehat{O_{\mathscr{S},x}} = \widehat{O_{\mathscr{S}_0,x}}$. Let $\widetilde{x}$ be a $W(k(x))$-point of $\mathscr{S}$ lifting x, and μ' be a cocharacter of $G_{\mathbb{Z}_p} \otimes W(k(x))$ as in the proof of Proposition 2.2.12, which induces the Hodge filtration on $\mathcal{V}^{\circ}_{\widetilde{x}}$ via t as introduced in Theorem 2.1.2 3.b). Let R_G be as above and $\sigma : R_G \to R_G$ be the morphism which is Frobenius on $W(k(x))$ and p-th power on variables. We will simply write u for u_G. Then by the proof of [23] Proposition 2.3.5, the p-divisible group $\mathcal{A}[p^\infty]|_{\widehat{O_{\mathscr{S},x}}}$ gives a formal deformation of $\mathcal{A}[p^\infty]|_x$, and induces an isomorphism $R_G \to \widehat{O_{\mathscr{S},x}}$. Moreover, if we take the Frobenius on $\widehat{O_{\mathscr{S},x}}$ to be the one on R_G, then the Dieudonné module of $\mathcal{A}[p^\infty]|_{\widehat{O_{\mathscr{S},x}}}$ is of the form

$\big(\mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}}, (\mathcal{V}^{\circ}_{\widetilde{x}})^1\otimes \widehat{O_{\mathscr{S},x}}, \varphi, \nabla\big)$, where φ is the composition

$$\mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}} \xrightarrow{\varphi\otimes\sigma} \mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}} \xrightarrow{u_t} \mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}}$$

with $u_t = tut^{-1}$, and ∇ is given by restricting the connection on the universal deformation to the closed sub formal scheme $\mathrm{Spf}(\widehat{O_{\mathscr{S},x}})$ (see [37] 4.5). Note that by [23] 1.5.4 and the proof of Corollary 2.3.9, $s_{\mathrm{dR},\widetilde{x}}\otimes 1 = s_{\mathrm{dR}}\otimes 1$ in $\mathcal{V}^{\circ\otimes}\otimes \widehat{O_{\mathscr{S},x}}$.

Lemma 2.3.2.

1) The scheme

$$\mathbb{I} := \mathbf{Isom}_{\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}\otimes W, s)\otimes_W \widehat{O_{\mathscr{S},x}},\ (\mathcal{V}^{\circ}, s_{\mathrm{dR}})\otimes \widehat{O_{\mathscr{S},x}}\big)$$

is a trivial $G_{\mathbb{Z}_p}$-torsor over $\widehat{O_{\mathscr{S},x}}$.

2) The closed subscheme $\mathbb{I}_+\subseteq \mathbb{I}$ defined by

$$\mathbb{I}_+ = \mathbf{Isom}_{\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}\otimes W, C^{\bullet}, s)\otimes_W \widehat{O_{\mathscr{S},x}},\ (\mathcal{V}^{\circ}, \mathcal{V}^{\circ}\supseteq(\mathcal{V}^{\circ})^1, s_{\mathrm{dR}})\otimes \widehat{O_{\mathscr{S},x}}\big)$$

is a trivial P_+-torsor over $\widehat{O_{\mathscr{S},x}}$.

Proof. 1) follows from $(\mathcal{V}^{\circ}, s_{\mathrm{dR}})\otimes \widehat{O_{\mathscr{S},x}} \cong (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\otimes \widehat{O_{\mathscr{S},x}}$ and Theorem 2.1.2 3.a). And 2) follows from

$$\big(\mathcal{V}^{\circ}, \mathcal{V}^{\circ}\supseteq(\mathcal{V}^{\circ})^1, s_{\mathrm{dR}}\big)\otimes \widehat{O_{\mathscr{S},x}} \cong \big(\mathcal{V}^{\circ}_{\widetilde{x}}, \mathcal{V}^{\circ}_{\widetilde{x}}\supseteq(\mathcal{V}^{\circ}_{\widetilde{x}})^1, s_{\mathrm{dR},\widetilde{x}}\big)\otimes \widehat{O_{\mathscr{S},x}}.$$

and Corollary 2.2.13 1). $\square$

Let t be as in 2.3.1, and $\widehat{g}_t$ be the composition of

$$\begin{aligned}&\xi : V_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}} \to (V_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}})^{(\sigma)};\quad v\otimes s\mapsto v\otimes 1\otimes s,\\ &(t\otimes 1)^{(\sigma)} : (V_{\mathbb{Z}_p}\otimes W(k(x))\otimes \widehat{O_{\mathscr{S},x}})^{(\sigma)} \to (\mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}})^{(\sigma)},\end{aligned}$$

with

$$\widehat{g} : (\mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}})^{(\sigma)} = \big(((\mathcal{V}^{\circ}_{\widetilde{x}})^0\oplus(\mathcal{V}^{\circ}_{\widetilde{x}})^1)\otimes \widehat{O_{\mathscr{S},x}}\big)^{(\sigma)} \xrightarrow{u_t\circ\big(((\varphi|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^0}+\frac{\varphi}{p}|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^1})\otimes\sigma)^{\mathrm{lin}}\big)} \mathcal{V}^{\circ}_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}}$$

and $(t\otimes 1)^{-1}$. We have the following

Lemma 2.3.3. *The $O_{\widehat{\mathscr{S},x}}$-linear map $\widehat{g}_t$ is an element of $G_{\mathbb{Z}_p}(O_{\widehat{\mathscr{S},x}})$. Let $\widehat{\mathcal{V}^\circ}_0$ (resp. $\widehat{\mathcal{V}^\circ}_1$) be the module $\widehat{g}\big(((\mathcal{V}^\circ_{\widetilde{x}})^0 \otimes O_{\widehat{\mathscr{S},x}})^{(\sigma)}\big)$ (resp. $\widehat{g}\big(((\mathcal{V}^\circ_{\widetilde{x}})^1 \otimes O_{\widehat{\mathscr{S},x}})^{(\sigma)}\big)$), then the scheme $\mathbb{I}_-$ given by*

$$\mathbf{Isom}_{\mathrm{Spec}(O_{\widehat{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}\otimes W, D_\bullet, s)\otimes_W O_{\widehat{\mathscr{S},x}},\ (\mathcal{V}^\circ\otimes O_{\widehat{\mathscr{S},x}}, \widehat{\mathcal{V}^\circ}_0 \subseteq \mathcal{V}^\circ\otimes O_{\widehat{\mathscr{S},x}}, s_{\mathrm{dR}}\otimes 1)\big)$$

is a trivial $P_-^{(\sigma)}$-torsor over $O_{\widehat{\mathscr{S},x}}$.

Proof. To prove the first statement, we need to show that $\widehat{g}_t^{\otimes}(s\otimes 1) = s\otimes 1$. We have the following commutative diagram

$$\begin{tikzcd}
V_{\mathbb{Z}_p}\otimes O_{\widehat{\mathscr{S},x}} \arrow[r, "(t\otimes 1)^{(\sigma)}\circ\xi"] \arrow[dr, "g_t\otimes 1"] \arrow[ddr, "\widehat{g}_t"'] & (\mathcal{V}^\circ_{\widetilde{x}}\otimes_{W(k(x))} O_{\widehat{\mathscr{S},x}})^{(\sigma)} \arrow[r, equal] & \big((\oplus_i(\mathcal{V}^\circ_{\widetilde{x}})^i)\otimes O_{\widehat{\mathscr{S},x}}\big)^{(\sigma)} \arrow[d, "\Sigma_i(p^{-i}\varphi\otimes\sigma)^{\mathrm{lin}}"] \\
 & V_{\mathbb{Z}_p}\otimes O_{\widehat{\mathscr{S},x}} \arrow[d, "u"] & \mathcal{V}^\circ_{\widetilde{x}}\otimes O_{\widehat{\mathscr{S},x}} \arrow[l, "t^{-1}\otimes 1"'] \arrow[d, "u_t"] \\
 & V_{\mathbb{Z}_p}\otimes O_{\widehat{\mathscr{S},x}} & \mathcal{V}^\circ_{\widetilde{x}}\otimes O_{\widehat{\mathscr{S},x}}. \arrow[l, "t^{-1}\otimes 1"']
\end{tikzcd}$$

We know from Proposition 2.2.12 that $g_t^{\otimes}(s\otimes 1) = s\otimes 1$. But $u^{\otimes}(s\otimes 1) = s\otimes 1$ by definition. So $\widehat{g}_t^{\otimes}(s\otimes 1) = s\otimes 1$.

To prove the second statement, we use the same method as in the proof of Corollary 2.2.13. Let $g_1 \in G_{\mathbb{Z}_p}(W(k(x)))$ be such that $g_1\big(\mu\otimes W(k(x))\big)g_1^{-1} = \mu'$, and let $I_{\widetilde{x},+}$, $I_{\widetilde{x},-}$ be as in Corollary 2.2.13. Then by the proof of Lemma 2.3.2, we have

$$\mathbb{I}_+ = \Big(t\cdot g_1\big(P_+\otimes W(k(x))\big)g_1^{-1}\Big)\times \mathrm{Spec}(O_{\widehat{\mathscr{S},x}}) = I_{\widetilde{x},+}\times \mathrm{Spec}(O_{\widehat{\mathscr{S},x}}).$$

By the proof of Proposition 2.2.12 2) and the commutative diagram above, the splitting $V_{\mathbb{Z}_p}\otimes O_{\widehat{\mathscr{S},x}} = (t\otimes 1)^{-1}(\widehat{\mathcal{V}^\circ}_0) \oplus (t\otimes 1)^{-1}(\widehat{\mathcal{V}^\circ}_1)$ is induced by the cocharacter $u(\nu\otimes 1)u^{-1}$. So $\mathbb{I}_- = t\cdot u\big(t^{-1}I_{\widetilde{x},-}\times \mathrm{Spec}(O_{\widehat{\mathscr{S},x}})\big)u^{-1}$ and hence it is a trivial $P_-^{(\sigma)}$-torsor over $\mathrm{Spec}(O_{\widehat{\mathscr{S},x}})$. $\square$

2.3.4 Description of the G-zip over the complete local ring

Now we will describe the G-zip structure on $\mathrm{Spec}(O_{\mathscr{S}_0,x}^{\widehat{}})$. For simplicity, we will still write G, V, R_G, $\mathbb{I}$, $\mathbb{I}_+$, $\mathbb{I}_-$, U_+, U_-, L, g_1, u and g_t for their reduction mod p. And also, we will work on R_G instead of $O_{\mathscr{S}_0,x}^{\widehat{}}$ as they are isomorphic. Then to get a G-zip structure on R_G, we only need to construct an isomorphism $\iota : \mathbb{I}_+^{(p)}/U_+^{(p)} \to \mathbb{I}_-/U_-^{(p)}$ of $L^{(p)}$-torsors over R_G. The method here is precisely the same as that in 2.2.14.

Let
$$\phi_0 : (V\otimes\kappa)^{(p)}/((V\otimes\kappa)^1)^{(p)} \to (V\otimes\kappa)_0$$
and
$$\phi_1 : (V\otimes\kappa)^1)^{(p)} \to (V\otimes\kappa)/(V\otimes\kappa)_0$$
be ϕ_0 and ϕ_1 defined in 2.2.14 respectively. For any R_G-algebra A and any section $\beta \in \mathbb{I}_+^{(p)}(A)$, β is an isomorphism
$$\beta : \big(V\otimes k(x), \overline{s}\otimes 1, C^\bullet(V\otimes k(x)))\otimes R_G\big)^{(p)}\otimes A \to \big(\mathcal{V}^\circ_{R_G}, \overline{s_{\mathrm{dR},R_G}}, C^\bullet(\mathcal{V}^\circ_{R_G})\big)^{(p)}\otimes A.$$

And it induces an isomorphism
$$\oplus\,\mathrm{gr}^i_C((V\otimes R_G)^{(p)}\otimes A) \xrightarrow{\sim} \oplus\,\mathrm{gr}^i_C(\mathcal{V}^{\circ(p)}_{R_G}\otimes A)$$
which will still be denoted by β. This gives a section $\beta \in \mathbb{I}_+^{(p)}/U_+^{(p)}$. Note that β is of the form $\beta = t^{(p)} g_1^{(p)} p g_1^{(p),-1}$ for some $p \in P_+^{(p)}(A)$. Let $\iota(\beta)$ be the composition

$$\begin{tikzcd}
\oplus\,\mathrm{gr}^D_i((V\otimes R_G)\otimes A) \arrow[d, "(\phi_0^{-1}\oplus\phi_1^{-1})\otimes 1"] & \oplus\,\mathrm{gr}^D_i(\mathcal{V}^\circ_{R_G}\otimes A) \\
\oplus\,\mathrm{gr}^i_C((V\otimes R_G)^{(p)}\otimes A) \arrow[r, "\beta"] & \oplus\,\mathrm{gr}^i_C(\mathcal{V}^{(p)}_{R_G}\otimes A). \arrow[u, "\varphi^{\mathrm{lin}}_\bullet\otimes 1"']
\end{tikzcd}$$

Then $\iota(\beta) = tug_t \cdot g_1^{(p)} p g_1^{(p),-1}$. One uses the same computation as in 2.2.14 and shows that ι is an isomorphism of L^p-torsors. We have an isomorphism of G-zips $(\mathbb{I}, \mathbb{I}_+, \mathbb{I}_-, \iota) \cong \underline{I}_{ug_t}$.

2.4 Construction of the G-zip on the reduction of the integral model

We will now explain how to get a G-zip on $\mathscr{S}_0$. Let $V_\kappa = V_{\mathbb{Z}_p} \otimes \kappa$, $G_\kappa = G_{\mathbb{Z}_p} \otimes \kappa$ and $\bar{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $\mathscr{S}_0$. Then $\bar{\mathcal{A}}[p]$ is a finite flat group scheme over $\mathscr{S}_0$. Moreover, if we denote by $F : \bar{\mathcal{A}}[p] \to \bar{\mathcal{A}}[p]^{(p)}$ and $V : \bar{\mathcal{A}}[p]^{(p)} \to \bar{\mathcal{A}}[p]$ the Frobenius and Verschiebung on it respectively, then the sequences

$$\bar{\mathcal{A}}[p] \xrightarrow{F} \bar{\mathcal{A}}[p]^{(p)} \xrightarrow{V} \bar{\mathcal{A}}[p] \qquad \text{and} \qquad \bar{\mathcal{A}}[p]^{(p)} \xrightarrow{V} \bar{\mathcal{A}}[p] \xrightarrow{F} \bar{\mathcal{A}}[p]^{(p)}$$

are exact. After applying the contravariant Dieudonné functor, we get exact sequences

$$\overline{\mathcal{V}^\circ} \xrightarrow{V} \overline{\mathcal{V}^\circ}^{(p)} \xrightarrow{F} \overline{\mathcal{V}^\circ} \qquad \text{and} \qquad \overline{\mathcal{V}^\circ}^{(p)} \xrightarrow{F} \overline{\mathcal{V}^\circ} \xrightarrow{V} \overline{\mathcal{V}^\circ}^{(p)}.$$

Denote by $\delta : \overline{\mathcal{V}^\circ} \to \overline{\mathcal{V}^\circ}^{(p)}$ the Frobenius semi-linear map $v \mapsto v \otimes 1$, let $C^\bullet$ be the descending filtration

$$C^0 := \overline{\mathcal{V}^\circ} \supseteq C^1 := \operatorname{kernel}(F \circ \delta) \supseteq C^2 := 0,$$

and $D_\bullet$ be the ascending filtration

$$D_{-1} := 0 \subseteq D_0 := \operatorname{image}(F) \subseteq D_1 := \overline{\mathcal{V}^\circ}.$$

Let $\varphi_0 : C^0/C^1 \to D_0$ be the natural map induced by $F \circ \delta$. Note that V induces an isomorphism $\overline{\mathcal{V}^\circ}/\operatorname{image}(F) \xrightarrow{\simeq} \operatorname{kernel}(F)$, whose inverse will be denoted by V^{-1}. Let $\varphi_1 : C^1 \to D_1/D_0$ be the map $V^{-1} \circ (\delta|_{C^1})$, then the tuple $(\overline{\mathcal{V}^\circ}, C^\bullet, D_\bullet, \varphi_\bullet)$ is an F-zip over $\mathscr{S}_0$.

Before stating the next theorem, we will fix some notations. We will write G, V, s and μ for their reduction mod p. We will denote by P_+, P_- and L the parabolics and Levi subgroup induced by μ as at the beginning of 1.2.

Theorem 2.4.1.

1) Let $I \subseteq \mathbf{Isom}_{\mathscr{S}_0}(V_\kappa \otimes O_{\mathscr{S}_0}, \overline{\mathcal{V}^\circ})$ be the closed subscheme defined as

$$I := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s) \otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}})\big).$$

Then I is a G_κ-torsor over $\mathscr{S}_0$.

2) Let $I_+ \subseteq I$ be the closed subscheme

$$I_+ := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s, C_\mu^\bullet)\otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, C^\bullet)\big).$$

Then I_+ is a P_+-torsor over $\mathscr{S}_0$.

3) Let $I_- \subseteq I$ be the closed subscheme

$$I_- := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s, D_\bullet^{\mu^{(p)}})\otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, D_\bullet)\big).$$

Then I_- is a $P_-^{(p)}$-torsor over $\mathscr{S}_0$.

4) The σ-linear maps φ_0 and φ_1 induce an isomorphism

$$\iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$$

of $L^{(p)}$-torsors over $\mathscr{S}_0$.

Hence the tuple (I, I_+, I_-, ι) is a G-zip over $\mathscr{S}_0$.

Proof. To prove 1), it suffices to show that I is smooth over $\mathscr{S}_0$ with non-empty fibers. The non-emptyness of I_x for a closed point $x \in \mathscr{S}_0$ follow from Theorem 2.1.2 3.a). For smoothness, by Lemma 2.3.2 1), $I \to \mathscr{S}_0$ is smooth after base-change to the complete local rings at stalks of closed points. And hence $I \to \mathscr{S}_0$ is smooth at the stalk of each closed point of $\mathscr{S}_0$. But this implies that it is smooth at an open neighborhood for each closed point, and hence smooth.

And 2) follows from Corollary 2.2.13 1) and Lemma 2.3.2 2) using the same strategy.

To prove 3), we also use the same strategy. Take a point $x \in \mathscr{S}_0$, we consider $I_- \times_{\mathscr{S}_0} \mathrm{Spec}(\widehat{O_{\mathscr{S}_0,x}})$. We claim that

$$I_- \times_{\mathscr{S}_0} \mathrm{Spec}(\widehat{O_{\mathscr{S}_0,x}}) \cong \mathbb{I}_- \otimes_{W(k(x))} k(x).$$

To see this, using notations in Lemma 2.3.3, we only need to show that

$$(\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, D_\bullet)\otimes \widehat{O_{\mathscr{S}_0,x}} \cong (\mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S},x}}, s_{\mathrm{dR}}\otimes 1, \widehat{\mathcal{V}^\circ}_0 \subseteq \mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S},x}})\otimes k(x).$$

But by our construction, $\widehat{\mathcal{V}^\circ}_0 \subseteq \mathcal{V}^\circ \otimes O_{\widehat{\mathscr{S},x}}$ is the submodule generated by $\varphi(\widehat{\mathcal{V}^\circ}^0)$, and the composition

$$\widehat{\mathcal{V}^\circ}^0 \otimes O_{\widehat{\mathscr{S}_0,x}} \subseteq \mathcal{V}^\circ \otimes O_{\widehat{\mathscr{S}_0,x}} \twoheadrightarrow (\mathcal{V}^\circ \otimes O_{\widehat{\mathscr{S}_0,x}})/(\widehat{\mathcal{V}^\circ}^1 \otimes O_{\widehat{\mathscr{S}_0,x}})$$

is an isomorphism, as it has an inverse pr_1. So $\widehat{\mathcal{V}^\circ}_0 \otimes O_{\widehat{\mathscr{S}_0,x}} = \mathrm{Im}(\varphi)$ in $\mathcal{V}^\circ \otimes O_{\widehat{\mathscr{S}_0,x}}$. And this proves 3).

To prove 4), we need to construct a morphism $I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$ which is an isomorphism of $L^{(p)}$-torsors over $\mathscr{S}_0$. Let S be a scheme over κ, for an $f \in I_+^{(p)}(S)$, it is an isomorphism

$$\big((V_\kappa, s, C_\mu^\bullet) \otimes O_S\big)^{(p)} \to \big((\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, C^\bullet) \otimes O_S\big)^{(p)},$$

and hence induces an isomorphism

$$\mathrm{gr}(f) : \oplus\, \mathrm{gr}_C^i \left((V_\kappa \otimes_\kappa O_S)^{(p)}\right) \to \oplus\, \mathrm{gr}_C^i \left((\overline{\mathcal{V}^\circ} \otimes_S O_S)^{(p)}\right).$$

Note that $U_+^{(p)}$ acts trivially on $\oplus\, \mathrm{gr}_C^i \left((V_\kappa \otimes_\kappa O_S)^{(p)}\right)$, so it is an element in $I_+^{(p)}/U_+^{(p)}(S)$. Denote by $(V_\kappa, C^\bullet(V_\kappa), D_\bullet(V_\kappa), \varphi_\bullet(V_\kappa))$ the F-zip determined by μ introduced in 2.2.14. If we still denote by ϕ_0 and ϕ_1 the isomorphisms

$$(V_\kappa \otimes O_S)^{(p)}/((V_\kappa \otimes O_S)^1)^{(p)} \xrightarrow{\mathrm{pr}_2} ((V_\kappa \otimes O_S)^0)^{(p)} \xrightarrow{\xi^{-1}} (V_\kappa \otimes O_S)_0$$

and

$$((V_\kappa \otimes O_S)^1)^{(p)} \xrightarrow{\xi^{-1}} (V_\kappa \otimes O_S)_1 \simeq (V_\kappa \otimes O_S)/(V_\kappa \otimes O_S)_0$$

respectively, then the composition

$$\begin{tikzcd}
\oplus\, \mathrm{gr}_i^D(V_\kappa \otimes O_S) \arrow[d, "\phi_0^{-1} \oplus \phi_1^{-1}"] & \oplus\, \mathrm{gr}_i^D(\overline{\mathcal{V}^\circ} \otimes O_S) \\
\oplus\, \mathrm{gr}_C^i((V_\kappa \otimes O_S)^{(p)}) \arrow[r, "\mathrm{gr}\,(f)"] & \oplus\, \mathrm{gr}_C^i((\overline{\mathcal{V}^\circ} \otimes O_S)^{(p)}) \arrow[u, "\oplus \varphi_i^{\mathrm{lin}}"']
\end{tikzcd}$$

is an element of $I_-/U_-^{(p)}(S)$. So the morphism $\iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$ is defined to be the natural morphism induced by the morphism of pre-sheaves

$$I_+^{(p)} \to I_-/U_-^{(p)}; \quad f \mapsto \varphi_i^{\mathrm{lin}} \circ \mathrm{gr}\,(f) \circ (\phi_0^{-1} \oplus \phi_1^{-1}), \ \ \forall\ S/k \text{ and } \forall\ f \in I_+^{(p)}(S).$$

The morphism ι is clearly an isomorphism. To see that it is $L^{(p)}$-equivariant, we use the same strategy as at the end of 2.2.14. Over an étale covering S' of S, there is a $g_1 \in G(S')$ s.t. $g_1 \mu g_1^{-1}$ induces the Hodge filtration on $\overline{\mathcal{V}^\circ}_{S'}$ via t, where

$$t \in \mathbf{Isom}\big(((V_\kappa, s, C^\bullet(V_\kappa)) \otimes O_{S'}),\ \big((\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, C^\bullet) \otimes O_{S'}\big)\big)(S').$$

Then $I_{+,S'} = tg_1 P_{+,S'} g_1^{-1}$, and $I_{-,S'} = tg_t g_1^{(p)} P_{-,S'}^{(p)} g_1^{(p),-1} g_t^{-1}$ for some $g_t \in G(S')$ constructed using the same method as before Lemma 2.3.3. For any $\beta \in I_{+,S'}^{(p)}$, we have $\beta = t^{(p)} g_1^{(p)} p g_1^{(p),-1}$ for some $p \in P_+^{(p)}(S')$ and $\iota(\beta) = tg_t g_1^{(p)} p g_1^{(p),-1} g_t^{-1}$. So for any $l \in L^{(p)}(S')$, we have

$$\beta \cdot l = t^{(p)} g_1^{(p)} p g_1^{(p),-1} \cdot l = t^{(p)} g_1^{(p)} p l g_1^{(p),-1}.$$

And hence,

$$\iota(\beta) \cdot l = tg_t g_1^{(p)} p g_1^{(p),-1} g_t^{-1} \cdot l = tg_t g_1^{(p)} p l g_1^{(p),-1} g_t^{-1} = \iota(\beta \cdot l).$$

This proves that ι is $L^{(p)}$-equivariant. $\square$

3 Ekedahl-Oort Strata for Hodge type Shimura varieties

3.1 Basic properties of Ekedahl-Oort strata

In this section, we will define Ekedahl-Oort strata for Hodge Type Shimura varieties and study their basic properties. Before stating the next theorem, we will fix some notations. Since in this chapter we will simply work on κ, we will write G, V and μ for their reduction mod p, and we will denote by P_+, P_- and L the parabolics and Levi subgroup induced by μ as at the beginning of 1.2.

Definition 3.1.1. The G-zip (I, I_+, I_-, ι) on $\mathscr{S}_0$ induces a morphism of smooth algebraic stacks $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}^{\mu}_{\kappa}$. For a point x in the topological space of $G\text{-}\mathtt{Zip}^{\mu}_{\kappa} \otimes \bar{\kappa}$, the Ekedahl-Oort stratum in $\mathscr{S}_0 \otimes \bar{\kappa}$ associated to x is defined to be $\zeta^{-1}(x)$.

Now we will state our main result.

Theorem 3.1.2. *The morphism $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}^{\mu}_{\kappa}$ is smooth.*

Proof. By Theorem 1.2.7, $G_\kappa \to G\text{-}\mathtt{Zip}^{\mu}_{\kappa}$ is an $E_{G,\mu}$-torsor. To prove that $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}^{\mu}_{\kappa}$ is smooth, it suffices to prove that in the cartesian diagram

$$\begin{tikzcd} \mathscr{S}_0^{\#} \arrow[r] \arrow[d, "\zeta^{\#}"] & \mathscr{S}_0 \arrow[d, "\zeta"] \\ G_\kappa \arrow[r] & G\text{-}\mathtt{Zip}^{\mu}_{\kappa}, \end{tikzcd}$$

the morphism $\zeta^{\#}$ is smooth. Note that $\mathscr{S}_0^{\#}$ and G_κ are both smooth over κ, so to show that $\zeta^{\#}$ is smooth, it suffices to show that the tangent map at each closed point is surjective (see [18] Chapter 3, Theorem 10.4).

Let $x^{\#} \in \mathscr{S}_0^{\#}$ be a closed point, its image in $\mathscr{S}_0$ is denoted by x which is also a closed point. Let R_G be as in 2.3.4 which is actually the modulo p of the universal deformation ring at x. Consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & \mathrm{Spec}(R_G) \arrow[d] \\ G_\kappa \arrow[r] & G\text{-}\mathtt{Zip}^\mu_\kappa. \end{tikzcd}$$

The morphism $X \to \mathrm{Spec}(R_G)$ is a trivial $E_{G,\mu}$-torsor by our construction at the very end of 2.3.4: the G-zip over R_G is isomorphic to $\underline{I}_{ug_t}$ (see Construction 1.2.4). The R_G-point ug_t of G_κ gives a trivialization of the $E_{G,\mu}$-torsor X over R_G. This trivialization induces an isomorphism from $\mathrm{Spec}(R_G) \times_\kappa E_{G,\mu}$ to X that translates α into the morphism from $\mathrm{Spec}(R_G) \times_\kappa E_{G,\mu}$ to G_κ that sends, for any κ-scheme T, a point (u, l, u_+, u_-) to $lu_+ug_t(l^{(p)}u_-)^{-1}$ (see Equation 1.2.6 and the line following it, and note that as κ-scheme, $E_{G,\mu} = L \times U_+ \times U_-^{(p)}$, and that R_G is the complete local ring of U_- at the origin). It follows that the tangent space at the origin of $\mathrm{Spec}(R_G) \times L \times U_+$ maps surjectively to that of G_κ. $\square$

3.1.3 Dimension and closure of a stratum

Thanks to Theorem 3.1.2, the combinatory description for the topological space of $[E_{G,\mu}\backslash G_\kappa]$ developed in [43] can be used to describe Ekedahl-Oort strata for reduction of a Hodge type Shimura variety, and gives dimension formula and closure for each stratum. We will first present some notations and technical results following [53] and [43], and then state how to use them.

We will first collect some basic facts about Weyl groups following the appendix of [53]. Let G be a reductive group over $k = \bar{k}$. Take a maximal torus T and a Borel subgroup B, s.t. $T \subseteq B \subseteq G$. The centralizer of T is T, the Weyl group $W(T) := \mathrm{Norm}_G(T)(k)/T(k)$ is a finite group. It admits a Coxeter group structure, i.e. one can take the generating set of simple reflections $I(B, T)$ to be the set of simple reflections defined by B. A priori, these data depend on the pair (B, T), but by remark 3.1.4 below, any other such pair (B', T') is obtained by conjugating (B, T) by some element $g \in G(k)$ which is unique up to right multiplication by an element in $T(k)$. Thus conjugation by g induces isomorphisms $W(T) \to W(T')$ and $I(B,T) \to I(B',T')$ that are independent of the choice of g. Especially, the isomorphisms associated to any three such pairs are compatible with each other. So $(W,I) := (W(T), I(B,T))$ for any choice of (B,T) can be viewed as "the" Weyl group and "the" set of simple reflections, in the sense that the pair depends only on G up to a unique isomorphism.

Remark 3.1.4. Fixing $T \subseteq B \subseteq G$, there are several interpretations for the Weyl group as a set

1) ([50], corollary 6.4.12) $W \to \{\text{Borel subgroups containing } T\}$, $w \mapsto wBw^{-1}$ is bijective.

2) ([38], 3.2) Let $\mathscr{B}$ be the set of Borels of G. The map $W \to G\backslash(\mathscr{B} \times \mathscr{B})$, $w \mapsto G$-orbit of (B, wBw^{-1}) is bijective. This is actually an restatement of the previous one, using the fact that the intersection of two Borel subgroups contains a maximal torus (not necessarily T).

Let $J \subseteq I$ be a subset, we denote by W_J the subgroup of W generated by J, and W^J (resp. ${}^JW$) be the set of elements w s.t. w is the element of minimal length in some coset $w'W_J$ (resp. W_Jw'). Note that there is a unique element in $w'W_J$ (resp. W_Jw') of minimal length, and each $w \in W$ can be uniquely written as $w = w^J w_J = w'_J {}^J w$ with $w_J, w'_J \in W_J, w^J \in W^J$, and ${}^Jw \in {}^JW$. In particular, W^J and ${}^JW$ are systems of representatives of W/W_J and $W_J\backslash W$.

Furthermore, if K is a second subset of I, then for each w, there is a unique element in W_JwW_K which is of minimal length. We will denote by ${}^JW^K$ the set of element of minimal length. Then ${}^JW^K = {}^JW \cap W^K$, and ${}^JW^K$ is a set of representatives of $W_J\backslash W/W_K$.

For a parabolic subgroup $P \subseteq G$, let B be a Borel subgroup contained in P and $T \subseteq B$ be a maximal torus of G. We denote by U_P the unipotent radical of P (i.e. the maximal connected unipotent normal smooth subgroup of P). There is a unique reductive subgroup (called Levi subgroup) $L \subseteq P$ containing T, s.t. the natural map $L \to P/U_P$ is an isomorphism. One can view I as the set $\{gT(k) \mid g \in I \subseteq W\}$, then P gives a subset $J \subseteq I$, which is defined to be the subset $\{g \in I \mid gT(k) \subseteq P(k)\} \subseteq I$, called the type of P. Here, we first work with $W(T)$ and $I(B,T)$, and then use the canonical identification to see that J is independent of the choice of (B,T). All the parabolics form a smooth projective scheme on k, and two parabolics are conjugate if and only if they have the same type.

There is another way to define the type of a parabolic subgroup P. Using notations as in the previous paragraph, attached to the chosen pair $(B,T) \subseteq P$, there are unique collections of positive roots and simple root. We can define I to be the set of all the simple roots, and the type J of P to be the set of simple roots whose inverse are roots of P.

Assume that G is a reductive group over $\mathbb{F}_p$. We will explain how to relate the set ${}^JW$ to the quotient stack $[E_{G,\mu}\backslash G_\kappa]$ following [43] and [44].

Choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq G$ s.t. $T \subseteq B$. Note that by [29] Theorem 2, such a B exists. And such a T also exists by [50] Theorem 13.3.6 and Remark 13.3.7. By what we have seen, such a pair $T \subseteq B$ gives the Weyl group W together with a set of simple reflections I. Let φ be the Frobenius on G given by the p-th power. It induces an isomorphism $(W,I) \to (W,I)$ of Coxeter systems, which will still be denoted by φ. The cocharacter $\mu : \mathbb{G}_m \to G_\kappa$ gives a parabolic subgroup P (see the construction of P_- at the beginning of 1.2), and hence gives a subset $J \subseteq I$ by taking the type of P.

Let ω_0 be the element of maximal length in W, set $K := {}^{\omega_0}\sigma(J)$. Here we write ${}^gJ$ for gJg^{-1}. Let $x \in {}^KW^{\sigma(J)}$ be the element of minimal length in $W_K\omega_0W_{\sigma(J)}$. Then x is the unique element of maximal length in ${}^KW^{\sigma(J)}$ (see [53] 4.2). There is a partial order $\preceq$ on ${}^JW$, defined by $w' \preceq w$ if and only if there exists $y \in W_J$, $yw'x\varphi(y^{-1})x^{-1} \leq w$ (see [53] Definition 4.8, Proposition 4.9). Here $\leq$ is the Bruhat order (see A.2 of [53] for the definition). The partial order $\preceq$ makes ${}^JW$ into a topological space.

Now we can state the the main result of Pink-Wedhorn-Ziegler that gives a combinatory description of the topological space of $[E_{G,\mu}\backslash G_\kappa]$ (and hence G-$\mathtt{Zip}^\mu_\kappa$).

Theorem 3.1.5. *For $w \in {}^JW$, take any $T' \subseteq B' \subseteq G_{\overline{\kappa}}$ with T' (resp. B') a maximal torus (resp. Borel) of $G_{\overline{\kappa}}$, s.t. $T' \subseteq L_{\overline{\kappa}}$ and $B' \subseteq P_{-,\overline{\kappa}}$. Let $g, \dot{w} \in \mathrm{Norm}_{G_{\overline{\kappa}}}(T)$ be a representative of $\varphi^{-1}(x)$ and w respectively, and $G^w \subseteq G_{\overline{\kappa}}$ be the $E_{G,\mu}$-orbit of $gB'\dot{w}B'$. Then*

1) The orbit G^w does not depends on the choices of $\dot{w}$, T', B' or g.

2) The orbit G^w is a locally closed smooth subvariety of $G_{\overline{\kappa}}$. Its dimension is $\dim(P) + l(w)$. Moreover, G^w consists of only one $E_{G,\mu}$-orbit. So G^w is actually the orbit of $g\dot{w}$.

3) Denote by $\big|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\big|$ the topological space of $[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}$, and still write ${}^JW$ for the topological space induced by the partial order $\preceq$. Then the association $w \mapsto G^w$ induces an homeomorphism ${}^JW \to \big|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\big|$.

Proof. The first statement is Proposition 5.8 of [43]. The second statement is [43] Theorem 1.3, Proposition 7.3 and Theorem 7.5. And the third statement is [43] Theorem 1.4. □

The next statement (including its proof) is a word by word adaptation of results in [53] (to be more precise, Proposition 4.7, Theorem 6.1 and Corollary 9.2). But note that it is actually Theorem 2.4.1 and Theorem 3.1.2 that make it works. So it is not simply implied by [53].

Proposition 3.1.6. *Let J be the type of P_+, then the Ekedahl-Oort strata are listed by the finite set ${}^JW$. For $w \in {}^JW$, the stratum $\mathscr{S}_0^w$ is smooth and equi-dimensional of dimension $l(w)$ if $\mathscr{S}_0^w \neq \emptyset$. Moreover, the closure of $\mathscr{S}_0^w$ is the union of $\mathscr{S}_0^{w'}$ s.t. $w' \preceq w$.*

Proof. The first statement follows from our definition of Ekedahl-Oort strata and Theorem 3.1.5 3). For the second one, note that by Theorem 3.1.5 2), each G_w is equi-dimensional of codimension $\dim(U_-) - l(w)$ in G_κ, so each $\mathscr{S}_0^w$ is equi-dimensional of codimension $\dim(U_-) - l(w)$ in $\mathscr{S}_{0,\overline{\kappa}}$, as ζ is smooth by Theorem 3.1.2. So the dimension of $\mathscr{S}_0^w$ is $l(w)$, as $\dim(\mathscr{S}_0) = \dim(U_-)$. The smoothness of each stratum follows from a direct adaption of the proof of Proposition 10.3 of [53]. For the last statement, by Theorem 3.1.5 3), the closure of $\{w\}$ in $\big|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\big|$ is $\{w' \mid w' \preceq w\}$. So $\overline{\mathscr{S}_0^w} = \zeta^{-1}(\overline{w})$ by the universally-openness of ζ. □

Remark 3.1.7. By [44] Lemma 12.14, Theorem 12.17 and [40] 3.26, there is a unique open dense stratum corresponding to the unique maximal element in ${}^JW$. This stratum will be called the ordinary stratum. And there is also a unique minimal element in ${}^JW$, namely the element 1. Its corresponding stratum is called the superspecial stratum. We expect that it is non-empty (but we can not prove it now). And the non-emptiness of the superspecial stratum implies that every stratum is non-empty, as ζ is a open map by Theorem 3.1.2.

3.2 On extra structures on F-zips attached to $\mathscr{S}_0$

In this section, we will give some remarks and comments concerning extra structures on F-zips associated to reductions of Hodge type Shimura varieties. The structure of this part is as follows. We list some compatibilities between s_{dR} and Dieudonné theory in 3.2.1 and 3.2.2. Then we describe the extra structures on F-zips attached to reduction of a Hodge type Shimura variety in 3.2.3. In 3.2.4 we prove that two points lie in the same Ekedahl-Oort stratum if and only if after passing to a common extension of the residue fields of those two points, there is an isomorphism of their F-zips respecting extra structures.

We will first give some technical remarks here. Let $G_{\mathbb{Z}_p}$, $V_{\mathbb{Z}_p}$ and s be as in 2.1. Let W and $C^\bullet$ be as in Corollary 2.2.13. If we denote by $(\mathcal{V}^\circ)^1 \subseteq \mathcal{V}^\circ$ the Hodge filtration on $\mathcal{V}^\circ$, then by Lemma 2.3.2 (and also use notations there), we know that

$$I := \mathbf{Isom}_{\mathscr{S}}\big((V_{\mathbb{Z}_p} \otimes W, s) \otimes_W O_{\mathscr{S}},\ (\mathcal{V}^\circ, s_{\mathrm{dR}})\big)$$

is a right $G_{\mathbb{Z}_p} \otimes W$-torsor, and

$$I_+ := \mathbf{Isom}_{\mathscr{S}}\big((V_{\mathbb{Z}_p} \otimes W, C^\bullet, s) \otimes_W O_{\mathscr{S}},\ (\mathcal{V}^\circ, \mathcal{V}^\circ \supseteq (\mathcal{V}^\circ)^1, s_{\mathrm{dR}})\big)$$

is a right P_+-torsor.

3.2.1 Compatibility of s_{dR} with Frobenius and connection

Let $\mathcal{A}/\mathscr{S}$ be the abelian scheme as at the beginning of 2.2.5. We are going to state a certain compatibilities between $s_{\mathrm{dR}} \in \mathcal{V}^{\circ\otimes}$ and the Dieudonné crystal (in sense of [21] Definition 2.3.2) $\mathbb{D}(\mathcal{A}_{\mathscr{S}_0}[p^\infty])$.

Let $S = \operatorname{Spec} R \subset \mathscr{S}$ be an open affine subscheme. Let $\hat{R}$ be the p-adic completion of R with any lift of Frobenius σ. Here we remark that there always exists a lifting of the Frobenius to $\hat{R}$. To see this, let $A_n = \hat{R}/(p^n)$ and $A = \hat{R}/(p)$. Let $I_{n+1} = \ker(A_{n+1} \to A_n)$, then I_{n+1} is an A-module. Now we can use notations and methods as in [21] Lemma 1.1.2 and Lemma 1.2.2. The obstruction to lift $\sigma : A_n \to A_n$ to A_{n+1} lies in $\mathrm{Ext}^1_{A_n}(\mathrm{L}_{A_n/\mathbb{Z}/(p^n)} \otimes_\sigma A_n, I_{n+1})$. The quasi-isomorphism $\mathrm{L}_{A_n/\mathbb{Z}/(p^n)} \otimes_\sigma A_n \otimes A \cong \Omega^1_{A/\mathbb{F}_p} \otimes_\sigma A$ induces an isomorphism

$$\mathrm{Ext}^1_{A_n}(\mathrm{L}_{A_n/\mathbb{Z}/(p^n)} \otimes_\sigma A_n, I_{n+1}) \cong \mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{F}_p} \otimes_\sigma A, I_{n+1}).$$

But $\mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{F}_p} \otimes_\sigma A, I_{n+1})$ is trivial as $\Omega^1_{A/\mathbb{F}_p}$ is projective.

By evaluating $\mathbb{D}(\mathcal{A}_{\mathscr{S}_0}[p^\infty])$ at $(\hat{R}, \sigma)$, we get a tuple (M, ∇, φ) where M is a locally free $\hat{R}$-module, $\nabla : M \to M \otimes_{\hat{R}} \hat{\Omega}^1_{\hat{R}}$ is the integral and topologically quasi-nilpotent (see [21] Remark 2.2.4 c for the definition) connection coming from the descent data (see [21] Remark 2.2.4 d), and $\varphi : M \to M$ is a σ-linear map which is horizontal with respect to ∇ (see [21] Definition 2.3.4, and the paragraph after it). More precisely, let $\hat{R}(1)$ be the p-adic completion of the PD-envelope of $\hat{R}\hat{\otimes}\hat{R} \to \overline{R}$, the crystal structure gives an isomorphism $\varepsilon : \mathrm{pr}_2^*(M) \cong \mathrm{pr}_1^*(M)$. And $\nabla(x)$ is defined to be $\theta(x) - x \otimes 1$, with $\theta : M \to \mathrm{pr}_1^*(M)$, $x \mapsto \varepsilon(1 \otimes x)$.

For any p-adically complete and p-torsion free W-algebra R' equipped with a lift of Frobenius σ' and a homomorphism of W-algebras $\iota : \hat{R} \to R'$, we get a triple (M', ∇', φ') as follows. Take $M' = M \otimes_{\hat{R}} R'$, $\nabla' = \nabla \otimes 1$, and φ' be the σ'-linear map whose linearization is

$$\sigma'^*(M') = \sigma'^* \iota^* M \xrightarrow{\varepsilon'} \iota^* \sigma'^* M \to \iota^* M = M'.$$

Here ε' is the base-change to R' of ε via the homomorphism $\hat{R}(1) \to R'$ induced by $\sigma'\iota \cdot \iota\sigma$. If R' is s.t. R'/pR' is a completion of a smooth κ-algebra, then the evaluation of $\mathbb{D}(\mathcal{A}_{\mathscr{S}_0}[p^\infty])$ at R' is the triple (M', ∇', φ').

Take a closed point $x \in \mathrm{Spec}(R/pR)$, denote by $\hat{R}_x$ the completion of R with respect to the maximal ideal defining x. Then $\hat{R}_x$ is also p-adically complete, and there is a natural injective homomorphism $\hat{R} \to \hat{R}_x$. Moreover, $\hat{R}_x$ is also equipped with an endomorphism σ_0 lifting the Frobenius on its reduction modulo p. We will take σ_0 to be the one described in [23] 1.5. Then $\mathbb{D}(\mathcal{A}_{\overline{R}}[p^\infty])(\hat{R}_x)$ gives the triple $(M' = M \otimes \hat{R}_x, \nabla' = \nabla \otimes 1, \varphi')$. Here $\varphi' : M' \to M'$ is the σ_0-linear map whose linearization is $\iota^* \varphi^{\mathrm{lin}} \circ (\sigma_0 \iota \cdot \iota\sigma)^* \varepsilon$.

There is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d] & M \otimes \hat{\Omega}_{\hat{R}} \arrow[d] \\ M \otimes \hat{R}_x \arrow[r, "\nabla\otimes 1"] & M \otimes \hat{\Omega}_{\hat{R}_x} \end{tikzcd}$$

with injective vertical homomorphisms. But $\nabla \otimes 1(s_{\mathrm{dR}} \otimes 1) = 0$, so $\nabla(s_{\mathrm{dR}} \otimes 1) = 0$. Moreover, we know that $\varepsilon = \nabla + \mathrm{id} \otimes 1$ and $\varphi'(s_{\mathrm{dR}} \otimes 1) = s_{\mathrm{dR}} \otimes 1$ (note that φ' is actually defined on $M'^{\otimes}[1/p]$). So s_{dR} is φ-invariant, as

$$\varphi'(s_{\mathrm{dR}} \otimes 1) = \iota^* \varphi^{\mathrm{lin}} \circ (\sigma_0 \iota \cdot \iota\sigma)^* (\nabla + \mathrm{id} \otimes 1)(s_{\mathrm{dR}} \otimes 1) = s_{\mathrm{dR}} \otimes 1.$$

The same computation shows that for any (R', σ') as before s.t R'/pR' is a completion of a smooth κ-algebra, we have $s_{\mathrm{dR}} \otimes 1 \in \mathcal{V}_{R'}^{\circ\otimes}$ is φ'-invariant, and $\nabla'(s_{\mathrm{dR}} \otimes 1) = 0$.

3.2.2 A better description for Frobenius invariance of s_{dR}

Let $T = \mathrm{Spec}(A)$ be s.t. $T \to \mathscr{S}$ is étale, and $I(T) \neq \emptyset$. Here I is the $G_{\mathbb{Z}_p} \otimes W$-torsor over $\mathscr{S}$ defined at the beginning of 3.2. Write $\overline{A}$ and $\hat{A}$ for the reduction modulo p and p-adic completion of A respectively. We will assume that the image of T in $\mathscr{S}$ is an open affine subscheme of $\mathscr{S}$, denoted by $S = \mathrm{Spec}(R)$. Clearly, one can always find pairs like (T, S), and one can choose finitely many $\{(T_\alpha, S_\alpha)\}_{\alpha\in\Delta}$ s.t. $\cup S_\alpha = \mathscr{S}$. Note that $\hat{A}$ is also equipped with a lift of Frobenius σ', but we DO NOT assume that the diagram

$$\begin{tikzcd} \hat{R} \arrow[r, "\sigma"] \arrow[d] & \hat{R} \arrow[d] \\ \hat{A} \arrow[r, "\sigma'"] & \hat{A} \end{tikzcd}$$

commutes.

For any $t \in I(T)$, it induces an $\hat{A}$-point of I, which will still be denoted by t. By 3.2.1, we have

$$\mathbb{D}(\mathcal{A}_{\mathscr{S}_0}[p^\infty])(\hat{A}, \sigma') = (M', \nabla', \varphi')$$

as before, and that $s_{\mathrm{dR}} \otimes 1 \in M'^{\otimes}$ is φ'-invariant. The cocharacter μ and $t \in I(\hat{A})$ induces a splitting $M' = M'^0 \oplus M'^1$, and the same argument as in the proof of Proposition 2.2.9 shows that φ' is defined on $(M'^{\otimes})^0$, and that $s_{\mathrm{dR}} \otimes 1 \in (M'^{\otimes})^0$. In particular, the σ'-linear map $M' = M'^0 \oplus M'^1 \xrightarrow{\varphi' + \frac{\varphi'}{p}} M'$ takes $s_{\mathrm{dR}} \otimes 1$ to itself.

3.2.3 Description of the extra structures

Now we will describe what are the extra structures on F-zips attached to reduction of a Hodge type Shimura variety. We will simply write σ for the p-Frobenius on $\mathscr{S}_0$, and φ for the Frobenius on $\overline{\mathcal{V}^\circ}$. We will write (G, V, μ, s) for the reduction modulo p of $(G_{\mathbb{Z}_p}, V_{\mathbb{Z}_p}, \mu : \mathbb{G}_{m,W} \to G_W, s)$. The residue field of W is denoted by κ. We will write $C^\bullet(V_\kappa)$ for the descending filtration induced by μ and $D_\bullet(V_\kappa)$ for the ascending filtration induced by $\mu^{(p)}$. There are several conditions that the tuple $(\overline{\mathcal{V}^\circ}, C^\bullet, D_\bullet, \varphi_\bullet)$ has to satisfy.

1) As we have seen in Theorem 2.4.1,

$$I := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s) \otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}})\big)$$

is a right G_κ-torsor over $\mathscr{S}_0$,

$$I_+ := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s, C^\bullet(V_\kappa)) \otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, C^\bullet)\big)$$

is a right P_+-torsor over $\mathscr{S}_0$, and

$$I_- := \mathbf{Isom}_{\mathscr{S}_0}\big((V_\kappa, s, D_\bullet(V_\kappa)) \otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}, D_\bullet)\big)$$

is a right $P_-^{(p)}$-torsor over $\mathscr{S}_0$.

2) The last two sentences of 3.2.2 imply that $\overline{s_{\mathrm{dR}}} \in \overline{\mathcal{V}^{\circ\otimes}}$ gives a Tate sub F-zip of weight zero of $\overline{\mathcal{V}^{\circ\otimes}}$, as φ' and $\frac{\varphi'}{p}$ gives φ_0 and φ_1 (see the proof of 2.4.1). In particular, étale locally, $\oplus\varphi_\bullet$ is induced by an element g of $G(\mathscr{S}_0)$ via t which is a section of I (see Lemma 2.3.3. And one can use the construction of $\widehat{g}_t$ to define g. Everything works if we change $O_{\mathscr{S},x}^{\widehat{}}$ to the p-adic completion of a W-algebra which is étale over $\mathscr{S}$).

The second condition in 1) is equivalent to that étale locally, the Hodge filtration on $\overline{\mathcal{V}^\circ}$ is induced by a cocharacter of $G_{\mathscr{S}_0}$ conjugate to μ, and the third condition in 1) is equivalent to that étale locally, the ascending filtration on $\overline{\mathcal{V}^\circ}$ is induced by a cocharacter of $G_{\mathscr{S}_0}$ conjugate to $\mu^{(p)}$. Now we will define what are the extra structures on F-zips attached to reduction of a Hodge type Shimura variety, and show how to attach G-zips to them.

Definition 3.2.4. Let (G, V, μ, s) be as at the beginning of 3.2.3. Let S be a locally Noetherian scheme over κ. By an F-zip with a Tate class s_{dR} over S, we mean an F-zip $(\mathcal{V}, C^\bullet, D_\bullet, \varphi_\bullet)$ over S equipped with a section s_{dR} of $\mathcal{V}^\otimes$, s.t.

1) The scheme $I := \mathbf{Isom}_S((V_\kappa, s) \otimes O_S,\ (\mathcal{V}, s_{\mathrm{dR}}))$ is a G_κ-torsor over S.

2) There is an étale covering S' of S, s.t. $I(S') \neq \emptyset$. For any $t \in I(S')$, the filtration $C^\bullet_{S'}$ is induced by a cocharacter of $G_{S'}$ via t which is $G_\kappa(S')$ conjugate to μ.

3) Étale locally, the ascending filtration $D_\bullet$ is induced by a cocharacter of G_S.

4) The O_S-submodule $O_S \hookrightarrow \mathcal{V}^\otimes$ corresponding to the section s_{dR} is a Tate sub F-zip of weight zero.

Remark 3.2.5. The Tate sub F-zip $O_S \hookrightarrow \mathcal{V}^{\otimes}$ in the above definition is a locally direct summand. As $O_S \hookrightarrow \mathcal{V}^{\otimes}$ remains injective at the residue field of each closed point, so Theorem 22.5 of [31] implies that it is a locally direct summand. Moreover, the embedding $O_S \hookrightarrow \mathcal{V}^{\otimes}$ is admissible in the sense of Definition 1.1.5.

We will first show that, étale locally, $D_\bullet$ is induced by a cocharacter conjugate to $\mu^{(p)}$.

Lemma 3.2.6. *Let $(\mathcal{V}, C^\bullet, D_\bullet, \varphi_\bullet)$ be an F-zip with a Tate class s_{dR} over S. Let S' and t be as in the definition. For simplicity and without loss of generality, let's assume that μ induces a splitting $\mathcal{V}_{S'} = \mathcal{V}^0 \oplus \mathcal{V}^1$ of $C^\bullet$ via t. Let $\xi : V \otimes S' \to (V \otimes S')^{(p)}$ be given by $v \otimes s \mapsto v \otimes 1 \otimes s$, $\forall\ v \in V$ and $\forall\ s \in O_S$, and $\mathcal{V}_{S'} = \mathcal{V}_0 \oplus \mathcal{V}_1$ be the splitting of $D_{\bullet,S'}$ induced by a cocharacter ν of $G_{S'}$ via t. We still write φ_0 for $\mathcal{V}^0 \to \mathcal{V}_{S'}/\mathcal{V}^1 \to \mathcal{V}_0$ and φ_1 for $\mathcal{V}^1 \to \mathcal{V}_{S'}/\mathcal{V}_0 \stackrel{\mathrm{pr}_2}{\to} \mathcal{V}_1$. Then the composition*

$$V \otimes S' \xrightarrow{\xi} (V \otimes S')^{(p)} \xrightarrow{t^{(p)}} \mathcal{V}_{S'}^{(p)} = \mathcal{V}^{0(p)} \oplus \mathcal{V}^{1(p)} \xrightarrow{\oplus\varphi_\bullet^{\mathrm{lin}}} \mathcal{V}_{S'} \xrightarrow{t^{-1}} V \otimes S'$$

is an element g_t in $G(S')$. Moreover, the cocharacter ν equals to $g_t \mu^{(p)} g_t^{-1}$ via t.

Proof. For the first statement, we only need to prove that $g_t^{\otimes}(s \otimes 1) = s \otimes 1$. And to do so, we only need to check that $(\oplus\varphi_\bullet^{\mathrm{lin}})^{\otimes} : \mathcal{V}_{S'}^{\otimes} \to \mathcal{V}_{S'}^{\otimes}$ maps s_{dR} to itself. Let $(\mathcal{V}_{S'}^{\otimes})^0$ (resp. $(\mathcal{V}_{S'}^{\otimes})_0$) be the subspace of weight zero in $\mathcal{V}_{S'}^{\otimes}$ with respect to the cocharacter μ via t (resp. the cocharacter ν via t). Then $s_{\mathrm{dR}} \in (\mathcal{V}_{S'}^{\otimes})^0$, and condition 3) in Definition 3.2.4 implies that $(\oplus\varphi_\bullet^{\mathrm{lin}})^{\otimes}(s_{\mathrm{dR}}) = s_{\mathrm{dR}} + d_{-1}$, here d_{-1} is an element of weight ≤ -1 with respect to ν. But $(\oplus\varphi_\bullet^{\mathrm{lin}})^{\otimes}(s_{\mathrm{dR}})$ lies in $(\mathcal{V}_{S'}^{\otimes})_0$ and hence is of weight zero with respect to ν. While s_{dR} is $G(S')$-invariant via t, so $d_{-1} = 0$, and $(\oplus\varphi_\bullet^{\mathrm{lin}})^{\otimes}(s_{\mathrm{dR}}) = s_{\mathrm{dR}}$. The second statement is clear. □

Remark 3.2.7. There is a G-zip attached to an F-zip $(\mathcal{V}, C^\bullet, D_\bullet, \varphi_\bullet)$ with a Tate class s_{dR}. By Definition 3.2.4 and Lemma 3.2.6, we only need to construct an isomorphism $\iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^p$ of $L^{(p)}$-torsors. But this follows from the proof of Theorem 2.4.1 word by word. Conversely, we can get an F-zip with a Tate class s_{dR} from a G-zip by taking contract products. More precisely, given a G-zip (I, P_+, P_-, ι) over S, we take $\mathcal{V} = I \times^G V_S$, $C^1 = I_+ \times^{P_+} C^1(V_\kappa)_S$, $D_0 = I_- \times^{P_-^{(p)}} D_0(V_\kappa)_S$. The Tate class is the image of $I \times \{s\}$ in $\mathcal{V}^\otimes = I \times^G (V_S^\otimes)$, and $\oplus\varphi_i : \oplus\mathrm{gr}_C^i \to \oplus\mathrm{gr}_i^D$ is the σ-linear map whose linearization is the morphism

$$\iota \times (\phi_0 \oplus \phi_1) : I_+^{(p)}/U_+^{(p)} \times^{L^{(p)}} \oplus\mathrm{gr}_C^i((V_\kappa)_S^{(p)}) \to I_-/U_-^{(p)} \times^{L^{(p)}} \oplus\mathrm{gr}_i^D(V_\kappa)_S.$$

Here ϕ_0 and ϕ_1 are as in 2.2.14. The condition that ι is $L^{(p)}$-equivariant implies that $\iota \times (\phi_0 \oplus \phi_1)$ is well defined. And that the submodule generated by s_{dR} is a Tate sub F-zip of $(\mathcal{V}, C^\bullet, D_\bullet, \varphi_\bullet)$ is straightforward.

3.2.8 Defining Ekedahl-Oort strata using F-zips

In this section, we will follow the construction in [40] and [53] to show that the Ekedahl-Oort strata defined using G-zips are the same as those defined using F-zips with a Tate class. The main technical tool is still [43]. We will write (G, V, μ, s) for the reduction modulo p of $(G_{\mathbb{Z}_p}, V_{\mathbb{Z}_p}, \mu : \mathbb{G}_{m,W} \to G_W, s)$. Fix the datum (G, V, μ, s), we consider the following functor Z_μ which associates a κ-scheme S the set of F-zip structures $(C^\bullet, D_\bullet, \varphi_\bullet)$ on V_S with Tate class $s \otimes 1$. Clearly, Z_μ is representable. And it is an F-zip interpretation of C_Z constructed in [43] Section 12.

Now we will construct a morphism $Z_\mu \to [E_{G,\mu}\backslash G_\kappa]$. By definition, to give such a morphism is the same as to give an $E_{G,\mu}$-torsor H over Z_μ, equipped with an $E_{G,\mu}$-equivariant morphism $H \to G_\kappa$.

There is a distinguished element $(V_\kappa, C^\bullet(V_\kappa), D_\bullet(V_\kappa), \phi_\bullet)$ in $Z_\mu(\kappa)$ attached to the datum (G, μ) constructed as in 2.2.14. One checks easily that the line generated by $s \in V^\otimes$ is a Tate sub F-zip of weight 0. Using the proof of [43] Lemma 12.5, the group $G_\kappa \times G_\kappa$ acts on Z_μ transitively via

$$(g, h) \cdot (C^\bullet, D_\bullet, \varphi_\bullet) = (gC^\bullet, hD_\bullet, h\varphi_\bullet g^{-1})$$

where $h\varphi_i g^{-1}$ is the composition

$$g(C^i)/g(C^{i+1}) \xrightarrow{g^{-1}} C^i/C^{i+1} \to D_i/D_{i-1} \xrightarrow{h} h(D_i)/h(D_{i-1}).$$

Under the above action, the stabilizer of $(V_\kappa, C^\bullet(V_\kappa), D_\bullet(V_\kappa), \phi_\bullet)$ is $E_{G,\mu}$ (still use the proof of [43] Lemma 12.5), and hence the action induces an $E_{G,\mu}$-torsor $G_\kappa \times G_\kappa \to Z_\mu$ which is G_κ-equivariant with respect to the diagonal action on $G_\kappa \times G_\kappa$ and the restriction to diagonal on Z_μ. The morphism $m : G_\kappa \times G_\kappa \to G_\kappa$, $(g, h) \mapsto g^{-1}h$ is a G_κ-torsor which is $E_{G,\mu}$-equivariant. By the same reason as in [43] Theorem 12.7, we get an isomorphism of stacks $\beta : [G_\kappa\backslash Z_\mu] \simeq [E_{G,\mu}\backslash G]$ after passing to quotients.

The G_κ-torsor $I = \mathbf{Isom}_{\mathscr{S}_0}((V_\kappa, s) \otimes O_{\mathscr{S}_0},\ (\overline{\mathcal{V}^\circ}, \overline{s_{\mathrm{dR}}}))$ induces a morphism $\zeta' : \mathscr{S}_0 \to [G_\kappa\backslash Z_\mu]$, while our Ekedahl-Oort strata are defined by the morphism $\zeta : \mathscr{S}_0 \to [E_{G,\mu}\backslash G_\kappa]$ defined in subsection 3.1. By what we have seen, one can identify $[G_\kappa\backslash Z_\mu]$ with $[E_{G,\mu}\backslash G]$ via β. So it is natural to ask whether they induce the same theory of Ekedahl-Oort strata. One can prove that two points lie in the same Ekedahl-Oort stratum defined using ζ if and only if after passing to a common extension of the residue fields of those two points, there is an isomorphism of their F-zips respecting the Tate F-zip. The following more conceptual statement holds.

Proposition 3.2.9. *We have an equality $\beta \circ \zeta' = \zeta$.*

Proof. By [43] 12.6, there is a cartesian diagram

$$\begin{tikzcd} G_\kappa \times G_\kappa \arrow[r, "m"] \arrow[d, "n"] & G_\kappa \arrow[d] \\ Z_\mu \arrow[r] & {[E_{G,\mu}\backslash G_\kappa]} \end{tikzcd}$$

with vertical arrows G_κ-equivariant $E_{G,\mu}$-torsors and horizontal arrows $E_{G,\mu}$-equivariant G_κ-torsors. One only needs to check that the pull back to $G_\kappa \times G_\kappa$ of $\mathscr{S}^\# \to G_\kappa$ and $I \to Z_\mu$ are $G_\kappa \times E_{G,\mu}$-equivariantly isomorphic over $G_\kappa \times G_\kappa$.

Let $\widetilde{\mathscr{S}_0}$ be the pull back

$$\begin{tikzcd} \widetilde{\mathscr{S}_0} \arrow[r] \arrow[d] & \mathscr{S}_0^{\#} \arrow[d] \\ G_\kappa \times G_\kappa \arrow[r, "m"] & G_\kappa. \end{tikzcd}$$

For any T/κ,

$$\widetilde{\mathscr{S}_0}(T) = \{(g_1, g_2, a, b) \mid g_i \in G_\kappa(T), (a, d) \in \mathscr{S}_0(T) \text{ s.t. } g_1^{-1}g_2 = a^{-1}b\}.$$

For any $(g, p_1, p_2) \in G_\kappa \times E_{G_\kappa,\mu}(T)$, the action is given by

$$(g, p_1, p_2) \cdot (g_1, g_2, a, b) = (gg_1p_1^{-1}, gg_2p_2^{-1}, ap_1^{-1}, bp_2^{-1}).$$

Let $\widetilde{I}$ be the pull back

$$\begin{tikzcd} \widetilde{I} \arrow[r] \arrow[d] & I \arrow[d] \\ G_\kappa \times G_\kappa \arrow[r, "n"] & Z_\mu. \end{tikzcd}$$

For any T/κ,

$$\begin{aligned} \widetilde{I}(T) = \{(g_1, g_2, t) \mid g_i \in G_\kappa(T), t \in I(T) \text{ s.t. } &(g_1C^\bullet(V_\kappa)_T, g_2D_\bullet(V_\kappa)_T, g_2\phi_\bullet g_1^{-1}) \\ &= t^{-1}(C^\bullet(\overline{\mathcal{V}^\circ}_T), D_\bullet(\overline{\mathcal{V}^\circ}_T), \varphi_\bullet)\}. \end{aligned}$$

For any $(g, p_1, p_2) \in G_\kappa \times E_{G,\mu}(T)$, the action is given by

$$(g, p_1, p_2) \cdot (g_1, g_2, t) = (gg_1p_1^{-1}, gg_2p_2^{-1}, g \cdot t).$$

There is a $G_\kappa \times G_\kappa$-morphism $\widetilde{\mathscr{S}_0} \to \widetilde{T}$ mapping (g_1, g_2, a, b) to (g_1, g_2, ag_1^{-1}). This is clearly an isomorphism. One also checks easily that it is $G_\kappa \times E_{G,\mu}$-equivariant. $\square$

4 Ekedahl-Oort strata for CSpin-varieties

4.1 Orthogonal groups

Let V be a $n+2$-dimensional $\mathbb{Q}$-vector space with basis $\{e_1, e_2, \cdots, e_{n+2}\}$, equipped with a non-degenerate quadratic form

$$Q = -a_1x_1^2 - a_2x_2^2 + a_3x_3^2 + \cdots a_{n+2}x_{n+2}^2, \qquad a_i > 0 \text{ and square free}$$

under the above basis. Denote by $\langle -, - \rangle = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$ the associated bilinear form, and by

$$I_Q = \begin{pmatrix} -a_1 & & & & \\ & -a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_{n+2} \end{pmatrix}$$

the matrix corresponding to Q.

Let $\mathrm{SO}(V)$ be the group scheme over $\mathbb{Q}$, whose R-valued points are

$$\{g \in \mathrm{GL}(V)(R) \mid g^t I_Q g = I_Q, \det(g) = 1\}$$

for all $\mathbb{Q}$-algebra R. Consider the morphism $h : \mathbb{S} \to \mathrm{SO}(V)_{\mathbb{R}}$ given by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto \begin{pmatrix} \frac{a^2-b^2}{a^2+b^2} & \frac{2ab}{a^2+b^2}\sqrt{\frac{a_2}{a_1}} & & & \\ -\frac{2ab}{a^2+b^2}\sqrt{\frac{a_1}{a_2}} & \frac{a^2-b^2}{a^2+b^2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Lemma 4.1.1. *The morphism h gives a Hodge structure of type $(-1,1)+(0,0)+(1,-1)$ on V with $\dim V^{-1,1} = 1$. Moreover, h gives a Shimura datum.*

Proof. Direct computation shows that $h(z)$ acts on $e_1+i\sqrt{\frac{a_1}{a_2}}e_2$ (resp. $e_1-i\sqrt{\frac{a_1}{a_2}}e_2$) in $V_{\mathbb{C}}$ as multiplication by $\frac{z}{\bar{z}}$ (resp. $\frac{\bar{z}}{z}$), and trivially on $\langle e_3, \cdots, e_{n+2}\rangle$. Hence h induces a Hodge structure of type $(-1,1)+(0,0)+(1,-1)$ on V with $\dim V^{-1,1}=1$.

Now we will check that it gives a Shimura datum.

(SV1) Under the basis $\{i\frac{e_1}{\sqrt{2a_1}}-\frac{e_2}{\sqrt{2a_2}}, i\frac{e_1}{\sqrt{2a_1}}+\frac{e_2}{\sqrt{2a_2}}, \frac{e_3}{\sqrt{a_3}}, \cdots, \frac{e_{n+2}}{\sqrt{a_{n+2}}}\}$, I_Q becomes

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

And

$$\begin{aligned} \mathrm{Lie}(\mathrm{SO}(V)_{\mathbb{C}}) &= \{M \in \mathrm{M}_{(n+2)\times(n+2)} | M^t J = -JM\} \\ &= \{\begin{pmatrix} A_{2\times 2} & B \\ C & D \end{pmatrix} | a_{11} = -a_{22}, a_{12} = a_{21} = 0, \\ &\qquad b_{2i} = -c_{1i}, b_{1i} = -c_{i2}, D = -D^t\}. \end{aligned}$$

Denote by δ_{ij} be the matrix whose elements are zero except the one at the i-th row and j-th collum which is 1. In our case, $\mathrm{Lie}(\mathrm{SO}(2,n)_{\mathbb{C}})$ has basis $\{\delta_{11}-\delta_{22}, \delta_{2i}-\delta_{i1}, \delta_{1j}-\delta_{j2}, \delta_{st}-\delta_{ts}\}$ for $i,j,s,t>2$. The conjugation action of h, or equivalently, the conjugate action of $\begin{pmatrix} \frac{c_1}{c_2} & 0 & 0 \\ 0 & \frac{c_2}{c_1} & 0 \\ 0 & 0 & I_n \end{pmatrix}$ has eigenvalue c_1/c_2 (resp. has eigenvalue c_2/c_1, resp. act trivially) on $\delta_{1j}-\delta_{j2}$ (resp. $\delta_{2i}-\delta_{i1}$, resp. $\delta_{11}-\delta_{22}$ and $\delta_{st}-\delta_{ts}$). And hence (SV1) holds.

(SV2) $\mathrm{Inn}(h(i))$ gives a Cartan involution on $\mathrm{SO}(V)^{\mathrm{ad}}_{\mathbb{R}}$. By [7] Lemma 2.8, this is the same as to find a bilinear form ψ on $V_{\mathbb{R}}$ which is $\mathrm{SO}(\mathbb{R})$-invariant, and such that $\psi(u,h(i)v)$ is symmetric with $\psi(v,h(i)v)$ positive definite. But one can just take ψ to be $\langle -,-\rangle_{\mathbb{R}}$ defined at the beginning of this section.

(SV3) $\mathrm{SO}(V)^{\mathrm{ad}}$ is simple, so it has no simple factor defined over $\mathbb{Q}$ onto which h has trivial projection. □

One can also determine the reflex field of this Shimura datum. First note that the torus $\mathbb{S}$ is actually defined over $\mathbb{Z}[\frac{1}{2}]$, namely, one can take $\mathcal{S} = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}][x,y,z]/((x^2+y^2)z-1))$, with co-multiplication given by

$$\begin{aligned} x &\mapsto x \otimes x - y \otimes y \\ y &\mapsto x \otimes y + y \otimes x \\ z &\mapsto z \otimes z. \end{aligned}$$

The morphism $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z, 1)$ is defined over $\mathbb{Z}[\frac{i}{2}]$. To be precise, the morphism $\mathbb{G}_{m,\mathbb{Z}[\frac{i}{2}]} \to \mathcal{S}_{\mathbb{Z}[\frac{i}{2}]}$, $t \mapsto \begin{pmatrix} \frac{t+1}{2} & \frac{-it+i}{2} \\ \frac{it-i}{2} & \frac{t+1}{2} \end{pmatrix}$ is a model of it, which will also be denoted by μ.

Lemma 4.1.2. *If $n > 0$, then the reflex field of the above Shimura datum is $\mathbb{Q}$.*

Proof. The cocharacter $h \circ \mu$ of $\mathrm{SO}(V)$, given by

$$t \mapsto \begin{pmatrix} \frac{t^2+1}{2t} & -i\frac{t^2-1}{2t}\sqrt{\frac{a_2}{a_1}} & & & \\ i\frac{t^2-1}{2t}\sqrt{\frac{a_1}{a_2}} & \frac{t^2+1}{2t} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

is clearly defined over $\mathbb{Q}(i\sqrt{\frac{a_2}{a_1}})$, but the action of the non-trivial element in $\mathrm{Gal}(\mathbb{Q}(i\sqrt{\frac{a_2}{a_1}})/\mathbb{Q})$ on $h \circ \mu$ is the same as the conjugation action of

$$\begin{pmatrix} 1 & 0 & & & & \\ 0 & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in \mathrm{SO}(V).$$

So the reflex field of the associated Shimura datum is $\mathbb{Q}$. $\square$

Remark 4.1.3. One sees from the proof easily that if $n = 0$, the reflex field is $\mathbb{Q}(i\sqrt{\frac{a_2}{a_1}})$=$\mathbb{Q}(i\sqrt{a_1a_2})$.

4.2 Clifford algebras and CSpin groups

Definition 4.2.1. Let V and Q be as in 4.1, then the Clifford algebra $C(V)$ is the $\mathbb{Q}$-algebra with a map $i : V \to C(V)$ which is universal with respect to all the maps $\alpha : V \to A$, with A a R-algebra, satisfying $\alpha(v)^2 = Q(v), \ \forall \ v \in V$.

Remark 4.2.2. The Clifford algebra $C(V)$ can be constructed as follows. Let $T(V)$ be the tensor algebra of V, let I be the two-side ideal of $T(V)$ generated by elements of the form $v \otimes v - Q(v), v \in V$. Then $C(V) \cong T(V)/I$. And $C(V)$ is finite dimensional with basis $\{e_{i_1}e_{i_2}\cdots e_{i_r}\}_{i_1<i_2<\cdots i_r, 0\leq r\leq n+2}$. Here $r = 0$ means the element $1 \in \mathbb{Q}$.

Remark 4.2.3. $C(V)$ will be viewed as a representable ring functor:

$$((\mathbb{Q}\text{-schemes})) \to ((\text{rings})), \qquad T \mapsto C(V) \otimes_{\mathbb{Q}} \Gamma(T).$$

This functor is represented by

$$\underline{C}(V) = \mathrm{Spec}(\mathrm{Sym}_{\mathbb{Q}}(C(V)^{\vee})) \cong \mathrm{Spec}(\mathbb{Q}[\{X_{i_1,\cdots,i_r}\}_{i_1<i_2<\cdots i_r, 0\leq r\leq n+2}]).$$

The co-addition and co-multiplication

$$\mathbb{Q}[\{X_{i_1,\cdots,i_r}\}_{i_1<i_2<\cdots i_r,0\leq r\leq n+2}] \to \mathbb{Q}[\{X_{i_1,\cdots,i_r}\}_{i_1<i_2<\cdots i_r,0\leq r\leq n+2}]^{\otimes 2}$$

are given respectively by

$$X_{i_1,\cdots,i_r} \mapsto X_{i_1,\cdots,i_r} \otimes 1 + 1 \otimes X_{i_1,\cdots,i_r}, \quad \forall i_1 < i_2 < \cdots i_r, 0 \leq r \leq n+2$$

and

$$X_{i_1,\cdots,i_r} \mapsto f_{i_1,\cdots,i_r}(\{X_{i_1,\cdots,i_s}\otimes 1\}_{i_1<i_2<\cdots i_s,0\leq s\leq n+2}, \{1\otimes X_{i_1,\cdots,i_t}\}_{i_1<i_2<\cdots i_t,0\leq t\leq n+2}).$$

Here $f_{i_1,\cdots,i_r}$ is given by

$$\sum_{i_1,\cdots,i_s} x_{i_1,\cdots,i_s} e_{i_1,\cdots,i_s} \cdot \sum_{i_1,\cdots,i_t} y_{i_1,\cdots,i_t} e_{i_1,\cdots,i_t} = \sum_{i_1,\cdots,i_r} f_{i_1,\cdots,i_r} e_{i_1,\cdots,i_r},$$

which is a polynomial in $x_{i_1,\cdots,i_s}$ and $y_{i_1,\cdots,i_t}$.

Remark 4.2.4. We only defined Clifford algebras over $\mathbb{Q}$, but in fact, they can be defined in a much more general setting. For our purpose, it will be enough to know that when there is a lattice $V_{\mathbb{Z}} \subseteq V$, s.t. Q takes integral value on it, one can also define $C(V_{\mathbb{Z}})$ using the universal property. And it is a free $\mathbb{Z}$-module with basis $\{t_{i_1}t_{i_2}\cdots t_{i_r}\}_{i_1<i_2<\cdots i_r, 0\leq r\leq n}$, where t_is form a basis of $V_{\mathbb{Z}}$. We refer to [3] Chapter IX §9 for more details.

There is an involution τ on $C(V)$ given by $e_{i_1}\cdots e_{i_r} \mapsto e_{i_r}\cdots e_{i_1}$. Here $e_{i_1},\cdots,e_{i_r}$ are in the chosen basis as in 4.1, and they are different from each other. We will write $C^+(V)$ (resp $C^-(V)$) for the even (resp. odd) part of $C(V)$. Note that $C^+(V)$ is an algebra, while $C^-(V)$ is just a $C^+(V)$-module. As we are more interested in $C^+(V)$, a structure theorem will be given here.

Theorem 4.2.5. *([15], Theorem 7.7) Let Q, V be as at the beginning of 4.1. Then we have:*

1) If $n+2 = 2m$, let $d := (-1)^m a_1 \cdots a_{n+2}$. Then the even Clifford algebra $C^+(V)$ is isomorphic to one of the following two algebras

(a) if $\sqrt{d} \in \mathbb{Q}$, then $C^+(V) = M_{2^{m-2}}(D) \times M_{2^{m-2}}(D)$ with a quaternion algebra D over $\mathbb{Q}$.

(b) if $\sqrt{d} \notin \mathbb{Q}$, then $C^+(V) = M_{2^{m-2}}(D)$ with a quaternion algebra D over $\mathbb{Q}(\sqrt{d})$.

2) if $n+2 = 2m+1$, then $C^+(V) = M_{2^{m-1}}(D)$ for a quaternion algebra D over $\mathbb{Q}$.

Remark 4.2.6. There is a "reduction mod p" version of the above theorem. That is, let V be a $n+2$ dimensional $\mathbb{F}_p$-vector space ($p \geq 3$), Q be a non-degenerate quadratic form on V. Then precisely the same statement holds, if one use $\mathbb{F}_p$ instead of $\mathbb{Q}$. One can use precisely the same proof of [15], except that one should use Waring-Chevally instead of Meyer's theorem on page 17 of [15].

Remark 4.2.7. ([26], Proposition 8.4) One can also see how the involutions on $C^+(V)$ look like. The list is as follows.

1) If $n \equiv 0 \bmod 4$, then the involution is unitary.

2) If $n \equiv 2 \bmod 4$, then the involution fixes elements in center of $C^+(V)$. And the involution is orthogonal if $n \equiv 6 \bmod 8$, symplectic if $n \equiv 2 \bmod 8$.

3) If $n \equiv 5, 7 \bmod 8$, then the involution is orthogonal.

4) If $n \equiv 1, 3 \bmod 8$, then the involution is symplectic.

By the previous theorem, the functor whose R-points are invertible elements in $C^+(V)$ is represented by a group scheme $C^+(V)^\times$ which is an open subscheme of $C^+(V)$. The group scheme $\mathrm{CSpin}(V)$ is defined to be the scheme representing the functor

$$\mathrm{CSpin}(V)(R) = \{g \in C^+(V)(R)^\times \mid gV_Rg^{-1} = V_R\},$$

or in other words, the stabilizer of $V \subseteq C(V)$ under the conjugation action of $C^+(V)^\times$. One can show that $\mathrm{CSpin}(V) \subseteq G$, where

$$G(R) = \{g \in C^+(V)(R) \mid \tau(g)g \in R^\times\}.$$

See Proposition 4 of [3] Chapter IX, 9.5 for more details.

Remark 4.2.8. For simplicity, let's assume that $k = \bar{k}$, and that $q = \sum_{i=1}^{n+2} x_i^2$. Then $\mathrm{Lie}(G)$ and $\mathrm{Lie}(\mathrm{CSpin}(V))$ can be naturally viewed as subspaces of $C^+(V)$. As by definition, $\mathrm{Lie}(G)$ is $\{r \in C^+(V) | (1 + r\varepsilon)(1 + \tau(r)\varepsilon) \in 1 + k\varepsilon\}$, which is the same as the subspace of $C^+(V)$ s.t. $\tau(r) + r \in k$. One can see easily that it is the subspace of $C^+(V)$ with basis 1 and $\{e_I\}_{\#I \equiv 2 \bmod 4}$. And

$$\mathrm{Lie}(\mathrm{CSpin}(V)) = \{r \in \mathrm{Lie}(G) \mid (1 + r\varepsilon)V \otimes_k k[\varepsilon](1 + \tau(r)\varepsilon) \in V \otimes_k k[\varepsilon]\},$$

which equals to $\{r \in C^+(V) \mid \tau(r) + r \in k, \text{ and } re_i + e_i\tau(r) \in V, \forall\, i\}$. Clearly, it is the subspace of $C^+(V)$ with basis 1 and $\{e_ie_j\}$. This means that G and $\mathrm{CSpin}(V)$ are different once $\dim(V) \geq 6$.

4.3 The Kuga-Satake construction

Let $\mathbb{S}, V, Q, h$ be as in 4.1, then one can find an $\widetilde{h} : \mathbb{S} \to \mathrm{CSpin}(V_\mathbb{R})$, s.t.

$$\begin{tikzcd} & \mathrm{CSpin}(V_\mathbb{R}) \arrow[d] \\ \mathbb{S} \arrow[ur, "\widetilde{h}"] \arrow[r, "h"'] & \mathrm{SO}(V_\mathbb{R}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{R}} \arrow[r, "w"] \arrow[d, equal] & \mathbb{S} \arrow[r, "\det"] \arrow[d, "\widetilde{h}"'] & \mathbb{G}_{m,\mathbb{R}} \arrow[d, equal] \\ \mathbb{G}_{m,\mathbb{R}} \arrow[r] & \mathrm{CSpin}(V_\mathbb{R}) \arrow[r, "\mathrm{norm}"] & \mathbb{G}_{m,\mathbb{R}} \end{tikzcd}$$

are commutative (see [7] 4.2).

We will need an explicit version of the above construction which is given in [15] 5.6. Take $J := \frac{e_1 e_2}{\sqrt{a_1 a_2}} \in \mathrm{CSpin}(V)$, then a lifting $\widetilde{h} : \mathbb{S} \to \mathrm{CSpin}(V)_\mathbb{R}$ is given by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a - bJ.$$

Theorem 4.3.1. *([7] Proposition 4.5) The morphism $\widetilde{h}$ induces a Hodge structure of type $(-1,0)+(0,-1)$ on $C^+(V)$ via the embedding*

$$\mathrm{CSpin}(V)_\mathbb{R} \hookrightarrow \mathrm{GL}(C^+(V)_\mathbb{R}), \quad g \mapsto g \cdot .$$

We also want to study polarizations on $C^+(V)$.

Proposition 4.3.2. *The correspondence $x \mapsto (e_1e_2)^{-1}\tau(x)(e_1e_2)$ is a positive involution on $C^+(V)$, i.e. the bilinear form $tr_{C^+(V)/\mathbb{Q}}((e_1e_2)^{-1}\tau(x)(e_1e_2)y)$ is symmetric and positive definite.*

Proof. See [15] Proposition 5.9, or [47] Proposition 2. $\square$

Let $\lambda = e_1e_2$, $\langle x, y\rangle_\lambda = tr_{C^+(V)/\mathbb{Q}}(\lambda\tau(x)y)$. The previous proposition and the sentence before Remark 4.2.8 show that $\mathrm{CSpin}(V) \subseteq \mathrm{GSp}(C^+(V), \langle,\rangle_\lambda)$ which implies the following statement.

Corollary 4.3.3. *The morphism $\widetilde{h}$ gives a Hodge type Shimura datum.*

Denote by B the opposite algebra of $C^+(V)$ with multiplication denoted by $*$, $C^+(V)$ is a left B-module induced by the right multiplication of $C^+(V)$. There is an involution on B

$$x \mapsto x^\lambda := \lambda^{-1} * \tau(x) * \lambda = \lambda\tau(x)\lambda^{-1}$$

Lemma 4.3.4. *The involution $x \mapsto x^\lambda$ is a positive involution on B, and the B-module structure on $C^+(V)$ satisfies $\langle b * u, v\rangle_\lambda = \langle u, b^\lambda * v\rangle_\lambda$. The group scheme G defined before Remark 4.2.8 is the subgroup of $\mathrm{GSp}(C^+(V), \langle,\rangle_\lambda)$ that commutes with the action of B.*

Proof. First, note that under the basis $\{e_I\}_{\#I \text{ even}}$, we have

$$tr_{B/\mathbb{Q}}(e_I) = tr_{C^+(V)/\mathbb{Q}}(e_I) = \begin{cases} 0 & \text{, if } I \neq \emptyset, \text{ i.e. } e_I \neq 1 \\ 2^{n+1} & \text{, if } I = \emptyset, \text{ i.e. } e_I = 1 \end{cases}$$

This simply means that $tr_{B/\mathbb{Q}}(c) = tr_{C^+(V)/\mathbb{Q}}(c), \forall c \in B$.

For the first statement, one needs to show that $tr_{B/\mathbb{Q}}(x^\lambda * x)$ is positive definite. We have

$$\begin{aligned} tr_{B/\mathbb{Q}}(x^\lambda * x) &= tr_{B/\mathbb{Q}}(xx^\lambda) = tr_{B/\mathbb{Q}}(x\lambda\tau(x)\lambda^{-1}) \\ tr_{C^+(V)/\mathbb{Q}}(\lambda^{-1}\tau(x)\lambda x) &= tr_{C^+(V)/\mathbb{Q}}(x\lambda^{-1}\tau(x)\lambda). \end{aligned}$$

But $tr_{C^+(V)/\mathbb{Q}}(\lambda^{-1}\tau(x)\lambda x)$ is positive definite by Proposition 4.3.2, so $tr_{B/\mathbb{Q}}(x^\lambda * x)$ is also positive definite noting that $\lambda^{-1} = -\frac{\lambda}{a_1^2 a_2^2}$.

There are also equalities

$$\begin{aligned} \langle b * u, v\rangle_\lambda &= \langle ub, v\rangle_\lambda = tr_{C^+(V)/\mathbb{Q}}(\lambda\tau(b)\tau(u)v) \\ \langle u, b^\lambda * v\rangle_\lambda &= \langle u, v\lambda\tau(b)\lambda^{-1}\rangle_\lambda = tr_{C^+(V)/\mathbb{Q}}(\lambda\tau(u)v\lambda\tau(b)\lambda^{-1}) \\ &= tr_{C^+(V)/\mathbb{Q}}(\lambda\tau(b)\tau(u)v). \end{aligned}$$

So $\langle b * u, v\rangle_\lambda = \langle u, b^\lambda * v\rangle_\lambda$.

For the last statement, clearly $G \subseteq \mathrm{GSp}(C^+(V), \langle,\rangle_\lambda) \bigcap \mathbf{End}_B(C^+(V))$. But by [15] Lemma 6.5, we have $C^+(V) = \mathbf{End}_B(C^+(V))$. For any $g \in C^+(V)^\times$ s.t. $tr(\lambda\tau(u)\tau(g)gv) = r \cdot tr(\lambda\tau(u)gv)$, $\forall\, u, v \in C^+(V)$, we can take $\tau(u) = \lambda^{-1}$ and $v = 1$, then the equality becomes $tr(\tau(g)g) = r \cdot tr(1)$, hence $\tau(g)g$ has to be a scalar. $\square$

Remark 4.3.5. The composition $\mathbb{S} \xrightarrow{\widetilde{h}} \mathrm{CSpin}(V)_\mathbb{R} \subseteq G_\mathbb{R}$ gives a Shimura datum of PEL-type.

Lemma 4.3.6. *If $n > 0$, the reflex fields of the Shimura data given by* $\mathrm{CSpin}(V)$ *and G are both equal to $\mathbb{Q}$.*

Proof. By a theorem of Deligne (see, for example [34] Remark 12.3 c), we only need to show that the reflex field of $\mathrm{CSpin}(V)$ Shimura datum is $\mathbb{Q}$.

The cocharacter of $\widetilde{h} \circ \mu$ given by

$$t \mapsto \frac{t+1}{2} + i\frac{t-1}{2}J$$

is defined over $\mathbb{Q}(\sqrt{-a_1a_2})$, and the action on it by the non-trivial element in $\mathrm{Gal}(\mathbb{Q}(\sqrt{-a_1a_2})/\mathbb{Q})$ is the same as the conjugation action of $e_2e_3 \in \mathrm{CSpin}(V)$. Hence the reflex field of the Shimura datum is $\mathbb{Q}$. $\square$

Remark 4.3.7. One can also compute the reflex field for G directly, when G is of type A or C. According to [5] page 1, the reflex field is generated by $\{tr_{C_0}(b) \mid b \in B\}$ over $\mathbb{Q}$. Here C_0 is the subspace of $C^+(V)_\mathbb{C}$ where $\mathbb{C}^\times$ acts trivially via μ. Denote by $\Delta_\mathrm{e} = \{e_I | I \subseteq \{3, 4, \cdots, n+2\}, \#I \text{ is even}\}$ and $\Delta_\mathrm{o} = \{e_I | I \subseteq \{3, 4, \cdots, n+2\}, \#I \text{ is odd}\}$. Then the disjoint union $\Delta_\mathrm{e} \coprod e_1e_2\Delta_\mathrm{e} \coprod e_1\Delta_\mathrm{o} \coprod e_2\Delta_\mathrm{o}$ form a basis of $C^+(V)$, and $\{e_I - iJe_I\}_{e_I \in \Delta_\mathrm{e} \coprod e_1\Delta_\mathrm{o}}$ form a $\mathbb{C}$-basis of C_0. Note that the B-action is induced by right multiplication of $C^+(V)$, so for an element e_I in the basis $\{e_I\}_{\#I \text{ even}}$ of B, it has non-trivial trace on V_0 if and only if $e_I = 1$.

Remark 4.3.8. Let a_i be coefficients of Q as in 4.1. Assume that all the a_is are square free integers. Then for a prime p s.t. $(p, 2\prod_i a_i) = 1$, the polarization

$\langle , \rangle_\lambda$ is of degree prime to p. As

$$\langle e_I, e_J\rangle_\lambda = tr_{C^+(V)/\mathbb{Q}}(\lambda\tau(e_I)e_J),$$

and it is non-zero if and only if $e_J = e_I\lambda$ up to scalar. So under basis $\{1, e_1e_2, \cdots, e_I, e_I\lambda, \cdots\}$, the matrix of $\langle , \rangle_\lambda$ is of the form

$$\begin{pmatrix} 0 & a & & & & & \\ -a & 0 & & & & & \\ & & 0 & b & & & \\ & & -b & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & c \\ & & & & & -c & 0 \end{pmatrix},$$

with all the none-zero entries products of 2^{n+1} and a_i. So the degree, which is the determinant of the above matrix, is prime to p.

4.4 Ekedahl-Oort strata for CSpin-varieties

Let $p > 2$ be a prime number and n be a positive integer. Let V be a $\mathbb{Q}$-vector space with basis $e_1, e_2, \cdots, e_{n+2}$. Let Q be a quadratic form on V, s.t. under basis $e_1, e_2, \cdots, e_{n+2}$, $Q = \sum_{i=1}^{n+2} a_i x_i^2$ with $a_i \in \mathbb{Q}_{>0} \bigcap \mathbb{Z}_{(p)}$ and square free. In this case, if we denote by $V_{\mathbb{Z}_p}$ the $\mathbb{Z}_p$-lattice of $V \otimes \mathbb{Q}_p$ generated by $e_1, e_2, \cdots, e_{n+2}$, then $\mathrm{SO}(V_{\mathbb{Z}_p})$ is reductive, and so is $\mathrm{CSpin}(V_{\mathbb{Z}_p})$. In particular, $K_p = \mathrm{CSpin}(V_{\mathbb{Z}_p})(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathrm{CSpin}(V)(\mathbb{Q}_p)$. Let $K^p \subseteq \mathrm{CSpin}(V)(\mathbb{A}_f^p)$ be a compact open subgroup which is small enough. Then the $\mathbb{Q}$-variety $\mathrm{Sh}_{K_pK^p}(\mathrm{CSpin}(V), X)$ extends to a smooth quasi-projective $\mathbb{Z}_{(p)}$-scheme $\mathscr{S}_{K_pK^p}(\mathrm{CSpin}(V), X)$ by Corollary 4.3.3 and Theorem 2.1.2. Let $\mathscr{S}_0$ the special fiber of the scheme $\mathscr{S}_{K_pK^p}(\mathrm{CSpin}(V), X)$, then our results in the 3.1 work for $\mathscr{S}_0$.

For simplicity of notations, we will take m s.t. $n+2 = 2m$ when n is even, and $n+2 = 2m+1$ when n is odd. And as we only work with reductions, the

$\mathbb{F}_p$-vector space $V_{\mathbb{Z}_p} \otimes \mathbb{F}_p$ will be denoted by V. Note that m is the dimension of a maximal torus in $\mathrm{SO}(V)$.

Proposition 4.4.1. *There are at most* $2m$ *Ekedahl-Oort strata for* $\mathscr{S}_0$.

Proof. Let W be the Weyl group of $\mathrm{CSpin}(V)$, which is the same as the one for $\mathrm{SO}(V)$. By Theorem 3.1.5, all the possible Ekedahl-Oort strata are parameterized by the finite set ${}^JW$. We claim that $\#({}^JW) = 2m$. To compute $\#({}^JW)$, we will work with $\mathrm{SO}(V)$.

1) Assume that n is odd. After passing to $k = \overline{\mathbb{F}_p}$, one can multiply the basis $\{e_1, \cdots, e_{n+2}\}$ by scalars s.t. $Q = \sum_i (-1)^{i+1} x_i^2$. We still denote this basis by $\{e_1, \cdots, e_{n+2}\}$. Let $f_1 = \frac{1}{2}(e_1+e_2)$, $f_{n+2} = \frac{1}{2}(e_1-e_2)$, $f_2 = \frac{1}{2}(e_3+e_4)$, $f_{n+1} = \frac{1}{2}(e_3-e_4), \cdots, f_m = \frac{1}{2}(e_n+e_{n+1})$, $f_{m+2} = \frac{1}{2}(e_n-e_{n+1})$, $f_{m+1} = e_{n+2}$. Then under basis $\{f_1, f_2, \ldots, f_{n+2}\}$, Q becomes

$$y_1 y_{n+2} + y_2 y_{n+1} + \cdots + y_m y_{m+2} + y_{m+1}^2.$$

Consider the diagonal maximal torus and the the lower triangle Borel subgroup of $SO(V)$. The cocharacter μ (see the construction before Lemma 4.1.2) is given by

$$t \mapsto \begin{pmatrix} t & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & t^{-1} \end{pmatrix}.$$

Then the descending filtration induced by μ is

$$V \supseteq \langle f_2, \cdots, f_{n+2} \rangle \supseteq \langle f_{n+2} \rangle \supseteq 0.$$

And its stabilizer is the parabolic

$$\begin{pmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{pmatrix}.$$

It is easy to compute that $W \cong \{\pm 1\}^m \rtimes S_m$. And W is generated by simple refections $\{s_i\}_{i=1,\cdots,m}$ where

$$s_i = \begin{cases} (i, i+1)(n-i+2, n-i+3), \text{ for } i = 1, \cdots, m-1; \\ (m, m+2), \text{ for } i = m. \end{cases}$$

The subgroup W_J is generated by $\{s_i\}_{i=2,\cdots,m}$, moreover, $W_J \cong \{\pm 1\}^{m-1} \rtimes S_{m-1}$. So $\#(W_J \backslash W) = 2m$.

2) When n is even, one uses the above proof word by word, except that $f_m = \frac{1}{2}(e_{n+1}+e_{n+2})$, $f_{m+1} = \frac{1}{2}(e_{n+1}-e_{n+2})$ and $s_m = (m-1, m+1)(m, m+2)$. In this case, $W \cong \{\pm 1\}_0^m \rtimes S_m$ and $W_J \cong \{\pm 1\}_0^{m-1} \rtimes S_{m-1}$. Here $\{\pm 1\}_0^i$ means the set of elements in $\{\pm 1\}^i$ with even number of -1s. $\square$

Now we will compute the dimension and describe the Zariski closure of each stratum (assuming that it is non-empty). We will first study the scheme of types for parabolic subgroups of a reductive group. This is necessary for the study of general Hodge type Shimura varieties, but not really necessary for CSpin-varieties. So people who only care about Ekedahl-Oort strata for CSpin-varieties can go to 4.4.5 and 4.4.7 directly.

4.4.2 The scheme of types

Here we give an effective version of A.5 of [53]. Let G be a reductive group (NOT necessarily quasi-split) over a perfect field k, denote by $\overline{k}$ the algebraic closure of k. Take any pair (B, T) with B a Borel subgroup of $G_{\overline{k}}$, $T \subseteq B$ a maximal torus of $G_{\overline{k}}$. For a $\sigma \in Gal(\overline{k}/k)$, there is a unique $g_\sigma \in G(\overline{k})$ up to right multiplication by $\sigma T(\overline{k})$, s.t. ${}^{g_\sigma}(\sigma B, \sigma T) = (B, T)$. So one can define an action $*$ of $Gal(\overline{k}/k)$ on B by $\sigma * b = {}^{g_\sigma}\sigma(b)$, which takes T to T. Note that the formula $g_{\sigma\tau} = g_\sigma \sigma(g_\tau)$ implies that it is an action. This new action takes positive roots (resp. simple roots) to positive root (resp. simple roots). Denote by $S := S(B, T)$ the set of simple roots, and P_S the power set of S. Then we have an action of $Gal(\overline{k}/k)$ on S and hence on P_S, which gives a finite étale scheme $\mathcal{P}$ over k.

The scheme $\mathcal{P}$ is independent of choices of (B, T) in the sense that for a different (B', T'), denote by g'_σ (resp. g) the unique element in $G(\overline{k})$ s.t. ${}^{g'_\sigma}(\sigma B', \sigma T') = (B', T')$ (resp. ${}^{g}(B, T) = (B', T')$), $I' = I(B', T')$ and $*'$ be the action of $Gal(\overline{k}/k)$ on B' given by $b \mapsto {}^{g'_\sigma}\sigma(b)$, then the diagram

$$\begin{array}{ccc} I & \xrightarrow{g\cdot} & I' \\ {\scriptstyle \sigma*}\downarrow & & \downarrow{\scriptstyle \sigma*'} \\ I & \xrightarrow[g\cdot]{} & I' \end{array}$$

is commutative. This follows from that $g_\sigma = g^{-1} g'_\sigma \sigma(g)$.

As $\mathcal{P}$ is independent of choices of (B, T), we can choose T to be a maximal torus defined over k. Then g_σ is a unique element in $W(T)$, denoted by w_σ.

Now we will show that $\mathcal{P}ar_G/G$ is isomorphic to $\mathcal{P}$. To do this, we only need to fix a pair (B, T), and construct a $Gal(\overline{k}/k)$ equivariant bijection $\mathcal{P}ar_G(\overline{k})/G(\overline{k}) \to P_S$ with respect to the ordinary action on $\mathcal{P}ar_G(\overline{k})/G(\overline{k})$ and the $*$ action on P_S. In each $G(\overline{k})$-orbit O of $Par_G(\overline{k})$, there is a unique parabolic P containing B, and hence gives a subset of $J \subseteq I$. To see that it is Galois equivariant, note that $\sigma(O) = \{\sigma(g)\sigma(P)\sigma(g^{-1})\}$, and $\sigma(P)$ is the unique element in $\sigma(O)$ containing $\sigma(B)$. So ${}^{w_\sigma}\sigma(P)$ is the unique element in $\sigma(O)$ containing B, and gives a subset ${}^{w_\sigma}\sigma(J)$. This mean that the map is Galois equivariant, and $\mathcal{P}ar_G/G \cong \mathcal{P}$.

As main examples, we compute the scheme of types for orthogonal groups. And the perfect field k will be of characteristic > 2. Let V be a k-vector space with basis $\{e_1, \cdots, e_n\}$. Let $q(x) = \sum_i a_i x_i^2$ be a non-degenerate quadratic form, where $x = \sum_i x_i e_i$, and $\mathrm{SO}(V, q)$ be the special orthogonal group fixing q.

Example 4.4.3. Let $2m+1=n$. Consider the maximal torus

$$T^0 = \prod_{i \leqslant m} T_i \subseteq \mathrm{SO}(V,q),$$

where

$$T_i = \{g \in SL_2 \mid g^t \begin{pmatrix} a_i & \\ & a_{n+1-i} \end{pmatrix} g = \begin{pmatrix} a_i & \\ & a_{n+1-i} \end{pmatrix}\}.$$

Let $b_i \in \overline{k}$ be such that $b_i^2 = -\dfrac{a_i}{a_{n+1-i}}$ for $i \leqq m$. Then under basis

$$\{e_1 + b_1 e_n, e_2 + b_2 e_{n-1}, \cdots, e_m + b_m e_{m+2}, 2e_{m+1}, e_m - b_m e_{m+2},$$
$$\cdots, e_2 - b_2 e_{n-1}, e_1 - b_1 e_n\},$$

the quadratic form q becomes $4x_{m+1}^2 + \sum_{i \leqslant m} 4a_i x_i x_{n+1-i}$. Note that under this basis, T_0 becomes

$$T = \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_m & & & & \\ & & & 1 & & & \\ & & & & t_m^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix},$$

which is a split maximal torus of $\mathrm{SO}(V,q)_{\overline{k}}$.

The group of characters $X^*(T)$ of T has a basis $\{\chi_i\}_{i \leq m}$. Where χ_i is the character

$$\begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_m & & & & \\ & & & 1 & & & \\ & & & & t_m^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \to t_i.$$

We will use the upper triangle Borel and T to describe positive roots. In this case, the positive roots are χ_i, $i \leq m$ and $\chi_i \pm \chi_j$, $i < j$; and the simple roots are $\chi_1 - \chi_2, \chi_2 - \chi_3, \cdots, \chi_{m-1} - \chi_m, \chi_m$. The action of the Galois group $Gal(\overline{k}/k)$ on $X^*(T)$ factor through the finite quotient $Gal(k'/k)$, where k' is the field $k(b_i)_{i\leq m}$. To be more precise, we have $\sigma(\chi_i) = \frac{\sigma(b_i)}{b_i}\chi_i$.

Let

$$\mu_B = \sum_{i=1}^{m}(m+1-i)\chi_i^\vee = m\chi_1^\vee + \cdots + \chi_m^\vee.$$

Then the set of positive (resp. negative) roots are the roots s.t. $\langle \mu_B, \alpha\rangle > 0$ (resp. $\langle \mu_B, \alpha\rangle < 0$). And hence the set of Borels containing T is in bijection with the $W(T)$-orbit of μ_B. So w_σ is the unique element in $W(T)$ s.t. $w_\sigma(\sigma(\mu_B)) = \mu_B$. But

$$\sigma(\mu_B) = \sigma(\sum_{i=1}^{m}(m+1-i)\chi_i^\vee) = \sum_{i=1}^{m}(m+1-i)\frac{\sigma(b_i)}{b_i}\chi_i^\vee,$$

So $w_\sigma = (\frac{\sigma(b_1)}{b_1}, \cdots, \frac{\sigma(b_m)}{b_m})$. Here we use the identification $W(T) \cong \{\pm 1\}^m \rtimes S_m$. And the Galois action $*$ is always trivial on the set of simple roots. $\square$

Example 4.4.4. If $n = 2m$, consider the torus $T^0 = \prod_{i\leqslant m} T_i \subseteq \mathrm{SO}(V, q)$, where

$$T_i = \{g \in SL_2 \mid g^t \begin{pmatrix} a_i & \\ & a_{n+1-i} \end{pmatrix} g = \begin{pmatrix} a_i & \\ & a_{n+1-i} \end{pmatrix}\}.$$

Let $b_i \in \overline{k}$ be such that $b_i^2 = -\dfrac{a_i}{a_{n+1-i}}$ for $i \leqq m$. Then under basis

$$\{e_1+b_1e_n, e_2+b_2e_{n-1}, \cdots, e_m+b_me_{m+1}, e_m-b_me_{m+1}, \cdots, e_2-b_2e_{n-1}, e_1-b_1e_n\},$$

q becomes $\sum_{i\leqslant m} 4a_ix_ix_{n+1-i}$. Note that under this basis, T_0 becomes

$$T = \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \ddots & & \\ & & & t_2^{-1} & \\ & & & & t_1^{-1} \end{pmatrix},$$

which is a split maximal torus of $\mathrm{SO}(V,q)_{\overline{k}}$.

The group of characters $X^*(T)$ of T has a basis $\{\chi_i\}_{i\leq m}$. Where χ_i is the character

$$\begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \ddots & & \\ & & & t_2^{-1} & \\ & & & & t_1^{-1} \end{pmatrix} \to t_i.$$

We will use the upper triangle Borel and T to describe positive roots. In this case, the positive roots are $\chi_i \pm \chi_j$, $i<j$ and the simple roots are $\chi_1-\chi_2$, $\chi_2-\chi_3$, $\cdots$, $\chi_{m-1}-\chi_m$, $\chi_{m-1}+\chi_m$. The action of the Galois group $Gal(\overline{k}/k)$ on $X^*(T)$ factor through the finite quotient $Gal(k'/k)$, where k' is the field $k(b_i)_{i\leq m}$. To be more precise, we have $\sigma(\chi_i) = \frac{\sigma(b_i)}{b_i}\chi_i$.

Next, we will compute w_σ. To do this, let

$$\mu_B = \sum_{i=1}^{m}(m-i)\chi_i^\vee = (m-1)\chi_1^\vee + \cdots + \chi_{m-1}^\vee.$$

Then the set of positive (resp. negative) roots are the roots s.t. $\langle \mu_B, \alpha\rangle > 0$ (resp. $\langle \mu_B, \alpha\rangle < 0$). And hence the set of Borels containing T is in bijection with the $W(T)$-orbit of μ_B. So w_σ is the unique element in $W(T)$ s.t. $w_\sigma(\sigma(\mu_B)) = \mu_B$. But

$$\sigma(\mu_B) = \sigma(\sum_{i=1}^{m}(m-i)\chi_i^\vee) = \sum_{i=1}^{m}(m-i)\frac{\sigma(b_i)}{b_i}\chi_i^\vee,$$

so

$$\begin{aligned} w_\sigma &= (\frac{\sigma(b_1)}{b_1}, \cdots, \frac{\sigma(b_{m-1})}{b_{m-1}}, \frac{\sigma(b_1\cdots b_{m-1})}{b_1\cdots b_{m-1}}) \\ &= (\frac{\sigma(b_1)}{b_1}, \cdots, \frac{\sigma(b_m)}{b_m}) \cdot (1, \cdots, 1, \frac{\sigma(b_1\cdots b_m)}{b_1\cdots b_m}). \end{aligned}$$

Here we use the identification $W(T) \cong \{\pm 1\}_0^m \rtimes S_m$, where $\{\pm1\}_0^m$ means the subset of elements in $\{\pm 1\}^m$ whose product of all factors are $+1$.

The Galois action $*$ on χ_i is give by

$$\sigma * \chi_i = w_\sigma \cdot \sigma(\chi_i) = \begin{cases} \chi_i \quad , \text{if } i \neq m \\ \dfrac{\sigma(\sqrt{(-1)^m d})}{\sqrt{(-1)^m d}} \chi_i, \text{ if } i = m \end{cases}$$

Here $d = \prod_{i=1}^n a_i$. Now we can describe the Galois action on P_S. Let $C = \{\chi_{m-1} - \chi_m, \chi_{m-1} + \chi_m\}$. Denote by $s(J) = (\frac{\sigma(\sqrt{(-1)^m d})}{\sqrt{(-1)^m d}})^{\#(J \cap C)}$ for a subset $J \subseteq I$. Then J is Galois invariant if and only if $s(J) = 1$, and $\sigma(J) = (J - C) \cup (C - (J \cap C))$ if $s(J) = -1$. So if we denote by k' the field extension $k(\sqrt{(-1)^m d})$ of k, then

$$P \cong \begin{cases} \coprod_{2^m} \operatorname{Spec}(k), & \text{if } (-1)^m d \text{ is a square;} \\ (\coprod_{2^{m-1}} \operatorname{Spec}(k)) \coprod (\coprod_{2^{m-2}} \operatorname{Res}_{k'/k} \operatorname{Spec}(k')), & \text{if } (-1)^m d \text{ is not a square.} \end{cases}$$

□

4.4.5 Ekedahl-Oort strata for odd dimensional CSpin-varieties

We will describe the dimension and Zariski closure of an Ekedahl-Oort stratum. The key points are our Proposition 3.1.6 and that the partial order $\preceq$ on ${}^J W$ is finer than the Bruhat order (see [44] 1.5). Keep notations as what we did before Proposition 4.4.1. We will compute the case when n is odd here, and when n is even in 4.4.7.

Proposition 4.4.6. *Let V be of dimension $n+2$ with n odd. Then for any integer $0 \leq i \leq n$, there is at most one stratum $\mathscr{S}_0^i$ s.t. $\dim(\mathscr{S}_0^i) = i$. And these are all the Ekedahl-Oort strata on $\mathscr{S}_0$. Moreover, the Zariski closure of $\mathscr{S}_0^i$ is the union of all the $\mathscr{S}_0^{i'}$, s.t. $i' \leq i$.*

Proof. We use notations as in the proof of Proposition 4.4.1. Under basis $\{f_1, \cdots, f_{n+2}\}$, the largest element $\omega \in W$ has a reduced expression

$$\omega = s_m(s_{m-1}s_m s_{m-1})(s_{m-2}s_{m-1}s_m s_{m-1}s_{m-2}) \cdots (s_1 \cdots s_{m-1}s_m s_{m-1} \cdots s_1)$$

by [1] page 15, Table 1. Direct computation shows that

$$\omega = (1, n+2)(2, n+1)\cdots(m, m+2),$$

which is a central element of order 2. One can also deduce this from [6] Remark 13.1.8, which says that for an algebraic group of type B_m, the largest element in its Weyl group is the central element

$$-1 = (-1, -1, \cdots, -1) \in \{\pm 1\}^m \rtimes S_m.$$

So in this case, $K := {}^{\omega}\varphi(J) = J$, and $W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega$. The last equality is because of that w_0 is central and that W_J is a group. To get the shortest element x in $W_K \omega W_{\varphi(J)} = W_J \omega$, we only need to find the largest element w_0' in W_J. As then $x = w_0' \omega$. By viewing W_J as the Weyl group of $\mathrm{SO}(V|_{\langle f_2, \cdots f_{n+1}\rangle})$ and using the above argument, we see that $w_0' = (2, n+1)(3, n)\cdots(m, m+2)$, and hence $x = (1, n+2)$.

Note that ${}^J W$ has the following description: viewing elements of W as permutations of $n+2$ letters, then

$${}^J W = \{w \in W \mid w(m+1) = m+1, w^{-1}(2) < w^{-1}(3) < \cdots < w^{-1}(n+1)\}.$$

Elements in ${}^J W$ are uniquely determined by $w^{-1}(1)$. If $w^{-1}(1) = n+2$, then $w(i) + w(n+3-i) = n+3$ implies that $w(1) = n+2$. And hence $w = (1, n+2)$, denoted by w_n. Clearly, $w_n = s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_1$ is a reduced expression, so the Ekedahl-Oort corresponding to w_n is of dimension $2m-1 = n$. If $w^{-1}(1) = 1$, then $w(1) = 1$ and $w(n+2) = n+2$. And so $w(i) = i$ for all i, which mean that $w = id$, denoted by w_0. In general, if $w^{-1}(1) = i$ for a i s.t. $n+2 > i > m+1$, then $w(i) = 1$ and $w(n+3-i) = n+2$. Denote this element by w_{i-2}, then

$$(w_{i-2}^{-1}(2), \cdots, w_{i-2}^{-1}(n+1)) = (1, \cdots, n+2-i, n+4-i, \cdots, i-1, i+1, \cdots, n+2).$$

Claim that $w_{i-2} = s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_{n+1-i}$. To see this, first note that w_{n-1} (i.e. when $i = n+1$) is obtained from w_n by interchanging the roles of $1, n+2$ with those of $2, n+1$. And hence $w_{n-1} = w_n s_1$. But w_i is obtained from w_{i+1} by interchanging the roles of $n-i, i+3$ with those of $n-i+1, i+2$, so we get the claim inductively. We also see that the expression $w_{i-2} = s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_{n+3-i}$ is reduced, and hence the length of w_{i-2} is $i-2$ for $i > m+1$. If $1 < i < m+1$, write w_{i-1} for the element s.t. $w^{-1}(1) = i$. Then by the same method, $w_{i-1} = s_1 \cdots s_{i-1}$ which is of length $i-1$. This proves that for each $0 \leq i \leq n$, there is at most one stratum with dimension i, and that there are all the strata.

To see the Zariski closure of a stratum, we note that the partial order $\preceq$ is finer than the Bruhat order which is a total order. So the partial order $\preceq$ has to coincide with the Bruhat order. This proves the last statement. $\square$

4.4.7 Ekedahl-Oort strata for even dimensional CSpin-varieties

Now we turn to the cases when n is even and positive. We have the following result.

Proposition 4.4.8. *Let V be of dimension $n+2$ with n even and positive. Then for any integer $0 \leq i \leq n$ and $i \neq n/2$, there is at most one stratum $\mathscr{S}_0^i$ s.t. $\dim(\mathscr{S}_0^i) = i$. There are at most 2 strata of dimension $n/2$. And these are all the Ekedahl-Oort strata on $\mathscr{S}_0$. Moreover, the Zariski closure of the stratum $\mathscr{S}_0^w$ is the union of $\mathscr{S}_0^w$ with all the strata whose dimensions are smaller than $\dim(\mathscr{S}_0^w)$.*

Proof. We still use the notations as in the proof of Proposition 4.4.1. In this case, the reductive group is of type D_m, and the largest element $\omega \in W$ has a reduced expression

$$\omega = s_{m-1} s_m (s_{m-2} s_{m-1} s_m s_{m-2})(s_{m-3} s_{m-2} s_{m-1} s_m s_{m-2} s_{m-3}) \\ \cdots (s_1 \cdots s_{m-2} s_{m-1} s_m \cdots s_1)$$

by [1] page 15, Table 1. Direct computation shows that

$$\omega = (1, n+2)(2, n+1) \cdots (m, m+1)$$

when m is even, and $\omega = (1, n+2)(2, n+1)\cdots(m-1, m+2)$ when m is odd. One can also deduce this from [6] Remark 13.1.8, which says that for an algebraic group of type D_m, the largest element in its Weyl group is the central element $-1 = (-1, -1, \cdots, -1) \in \{\pm 1\}^m \rtimes S_m$ when m is even, and it is $-1 = (-1, -1, \cdots, -1) \in \{\pm 1\}^m \rtimes S_m$ multiplied by the order 2 automorphism of the Coxeter diagram (i.e. interchanging the role of s_{m-1} and s_m, which means mapping χ_m to $-\chi_m$, or multiplying by $(m, m+1)$ in language of permutations). The subgroup W_J is generated by $\{s_i\}_{i=2,\cdots,m}$.

Let's first assume that m is even. In this case, $K := {}^{\omega}\varphi(J) = J$, and $W_K \omega W_{\varphi(J)} = W_J \omega$ as in the previous case. The largest element in W_J is $w' = (2, n+1)(3, n)\cdots(m-1, m+2)$, and so the shortest element in $W_J\omega$ is $(1, n+2)(m, m+1)$, with a reduced expression $s_1 \cdots s_{m-1} s_m s_{m-2} \cdots s_1$. And ${}^J W = W' \coprod W''$. Here

$$
\begin{aligned}
W' = \{w \in W \mid w^{-1}(1) \geq m+1, w^{-1}(2) < \cdots < w^{-1}(m-1) < w^{-1}(m+1) \\
< w^{-1}(m) < w^{-1}(m+2) < \cdots < w^{-1}(n+1)\}; \\
W'' = \{w \in W \mid w^{-1}(1) \leq m, w^{-1}(2) < \cdots < w^{-1}(n+1)\}.
\end{aligned}
$$

For $i > m+1$ (resp. $i < m$), we denote by w_{i-2} (resp. w_{i-1}) the element s.t. $w^{-1}(1) = i$. And we write w'_{m-1} (resp. w''_{m-1}) for the element in W' (resp. W'') s.t. $w^{-1}(1) = m+1$ (resp. $w^{-1}(1) = m$).

We see that $w_n = (1, n+2)(m, m+1)$. Claim that

$$
w_i = \begin{cases} s_1 \cdots s_m s_{m-2} \cdots s_{n+1-i}, & \text{if } i > m-1 \\ s_1 \cdots s_i, & \text{if } i < m-1 \end{cases}
$$

and $w'_{m-1} = s_1 \cdots s_{m-2} s_m$, $w''_{m-1} = s_1 \cdots s_{m-1}$. To see this, by our construction,

$$
\begin{aligned}
&\big(w_i^{-1}(2), \cdots, w_i^{-1}(m-1), w_i^{-1}(m+1), w_i^{-1}(m), w_i^{-1}(m+2), \\
&\cdots, w_i^{-1}(n+1)\big) \\
=&(1, \cdots, n-i, n+2-i, \cdots, i+1, i+3, \cdots, n+2)
\end{aligned}
$$

for $i > m-1$, and

$$\begin{aligned}&\big(w'^{-1}_{m-1}(2), \cdots, w'^{-1}_{m-1}(m-1), w'^{-1}_{m-1}(m+1), w'^{-1}_{m-1}(m), w'^{-1}_{m-1}(m+2),\\&\cdots, w'^{-1}_{m-1}(n+1))\\=&(1, \cdots, m-1, m+2, \cdots, n+2).\end{aligned}$$

Note that $w_i(m) = m+1$, $w_i(m+1) = m$, and similar method as in the proof of Proposition 4.4.6 shows that $w_{i-1} = w_i s_{n+1-i}$ for $i > m-1$. Moreover, $w'_{m-1} = w_m s_m$. And this proves the claim for $w_i, i > m-1$ and w'_{m-1}. The other half works similarly.

For m odd, the longest element $\omega \in W$ is no longer central. But $\varphi(J) = J$ and ${}^{\omega}J = J$ still hold. This is because J is given by a cocharacter defined over $\mathbb{F}_p$, and J is the subset $\{s_2, s_3, \cdots, s_m\}$ of I, while ${}^{\omega}s_i = s_i$ for $1 < i < m-1$, and ${}^{\omega}s_{m-1} = s_m$, ${}^{\omega}s_m = s_{m-1}$. So $W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega$ as before. Using the same method as in the case when m is even, we conclude that the largest element in ${}^{J}W$ is w_n with a reduced expression $s_1 \cdots s_{m-1} s_m s_{m-2} \cdots s_1$. And

$$w_i = \begin{cases} s_1 \cdots s_m s_{m-2} \cdots s_{n+1-i}, & \text{if } i > m-1 \\ s_1 \cdots s_i, & \text{if } i < m-1, \end{cases}$$

$w'_{m-1} = s_1 \cdots s_{m-2} s_m$, $w''_{m-1} = s_1 \cdots s_{m-1}$.

Now it is clear that there is at most one stratum of dimension i for $i \neq n/2$, and that there are almost 2 strata of dimension $n/2$. These are clearly all the possible strata. To describe the Zariski closure of a stratum, we still use the fact that the partial order $\preceq$ is finer than the Bruhat order. It simply implies that for $i \neq n/2$, the Zariski closure of the i-dimensional stratum is the union of all the strata with dimension $\leq i$. But the Zariski closure of a $n/2$-dimensional stratum can not contain other strata of dimension $\geq n/2$, and so the last statement holds. $\square$

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Summary

Let (G, X) be a Shimura datum of Hodge type, and E be its reflex field. Let $p > 2$ be a prime such that (G, X) has good reduction. Let v be a place of E over p. Let $K_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup, and $K^p \subseteq G(\mathbb{A}_f^p)$ be a compact open subgroup which is small enough. Let $K = K_p K^p$. By works of Deligne, we know that the smooth complex variety

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

has a canonical model $\mathrm{Sh}_K(G, X)$ over E. By recent works of Vasiu and Kisin, the E-variety has an integral canonical model $\mathscr{S}_K(G, X)$ over $O_{E,v}$. The scheme $\mathscr{S}_K(G, X)$ is smooth over $O_{E,v}$, and uniquely determined by a certain extension property.

Let $\kappa = O_{E,v}/(v)$ and $\mathscr{S}_0$ be the special fiber of $\mathscr{S}_K(G, X)$. The goal of this thesis is to develop a theory of Ekedahl-Oort stratification for $\mathscr{S}_0$, generalizing known theory for PEL Shimura varieties developed by Oort, Moonen, Wedhorn, Viehmann...

Thanks to works of Pink, Wedhorn and Ziegler on G-zips, we have the definition and technical tools for such a theory. Fixing a symplectic embedding, our first main result is the construction of a G-zip over $\mathscr{S}_0$. This induces a morphism $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}_\kappa^\mu$, where $G\text{-}\mathtt{Zip}_\kappa^\mu$ is the stack of G-zips of type μ constructed by Pink, Wedhorn and Ziegler. Fibers of ζ are defined to be Ekedahl-Oort strata.

Our second main result is that ζ is smooth. One can then transfer knowledge about geometry of $G\text{-}\mathtt{Zip}_\kappa^\mu$ to results about Ekedahl-Oort strata. In particular, we have a dimension formula for each non-empty stratum, and we know which strata lie in the closure of a given stratum.

Samenvatting

Laat (G, X) een Shimura datum van Hodge type zijn, en E zijn reflexlichaam. Laat $p > 2$ een priemgetal zijn waar (G, X) goede reductie heeft. Laat v een plaats van E boven p zijn. Laat $K_p \subseteq G(\mathbb{Q}_p)$ een hyperspeciale ondergroep zijn, en $K^p \subseteq G(\mathbb{A}_f^p)$ een voldoend kleine compacte open ondergroep. Laat $K = K_p K^p$. Deligne heeft laten zien dat de gladde complexe variëteit

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} := G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)$$

een canoniek model $\mathrm{Sh}_K(G, X)$ over E heeft. Volgens recent werk van Vasiu en Kisin heeft deze E-variëteit een canoniek model $\mathscr{S}_K(G, X)$ over $O_{E,v}$. Het schema $\mathscr{S}_K(G, X)$ is glad over $O_{E,v}$ en is uniek bepaald door een uitbreidingseigenschap.

Laat $\kappa = O_{E,v}/(v)$ en $\mathscr{S}_0$ de speciale vezel van $\mathscr{S}_K(G, X)$ zijn. Het doel van dit proefschrift is een theorie van Ekedahl-Oort stratificatie voor $\mathscr{S}_0$ te ontwikkelen, die theorie ontwikkeld door Oort, Moonen, Wedhorn, Viehmann... voor PEL Shimura variëteiten generaliseert.

Dankzij werk van Pink, Wedhorn en Ziegler over G-zips hebben we geschikte definities en gereedschappen voor zo'n theorie. We kiezen een symplectische inbedding. Ons eerste belangrijke resultaat is de constructie van een G-zip over $\mathscr{S}_0$. Deze induceert een morfisme $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}_\kappa^\mu$, waarin $G\text{-}\mathtt{Zip}_\kappa^\mu$ de stack van G-zips van type μ is, geconstrueerd door Pink, Wedhorn en Ziegler. De vezels van ζ zijn per definitie de Ekedahl-Oort strata van $\mathscr{S}_0$.

Ons tweede belangrijke resultaat is dat ζ glad is. Daarmee kan men kennis over de meetkunde van $G\text{-}\mathtt{Zip}_\kappa^\mu$ overbrengen naar kennis over Ekedahl-Oort strata. In het bijzonder krijgen we een dimensie-formule voor de niet-lege strata, en weten we welke strata in de afsluiting van een stratum zetten.

Sommario

Sia (G, X) uno Shimura datum del tipo di Hodge, e sia E il suo campo riflesso. Sia $p > 2$ un primo di buona riduzione per (G, X). Sia v un luogo di E presso p. Sia $K_p \subseteq G(\mathbb{Q}_p)$ un gruppo iperspeciale e sia $K^p \subseteq G(\mathbb{A}_f^p)$ un sottogruppo aperto e compatto abbastanza piccolo. Sia $K = K_p K^p$. Dai lavori di Deligne sappiamo che la varietà liscia e complessa

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

ammette un modello canonico $\mathrm{Sh}_K(G, X)$ su E. Vasiu e Kisin mostrano nei loro lavori che tale varietà su E ammette un modello canonico integrale $\mathscr{S}_K(G, X)$ su $O_{E,v}$. Lo schema $\mathscr{S}_K(G, X)$ è liscio su $O_{E,v}$ ed unicamente determinato da alcune proprietà d'estensione.

Sia $\kappa = O_{E,v}/(v)$ e $\mathscr{S}_0$ la fibra speciale di $\mathscr{S}_K(G, X)$. L'obiettivo di questa tesi è sviluppare la teoria della stratificazione di Ekedahl-Oort per $\mathscr{S}_0$, generalizzando i noti teoremi per le varietà PEL di Shimura, ottenuti da Oort, Moonen, Wedhorn, Viehmann e altri.

Grazie ai lavori di Pink, Wedhorn e Ziegler sulle G-zips, abbiamo oggi a disposizione il definizione e i giusti strumenti per una tale teoria. Una volta fissata un' immersione simplettica, il nostro primo risultato importante è la costruzione di una G-zip su $\mathscr{S}_0$. Questa induce un morfismo $\zeta : \mathscr{S}_0 \to G\text{-}\mathtt{Zip}_\kappa^\mu$, ove $G\text{-}\mathtt{Zip}_\kappa^\mu$ è la pila di G-zips di tipo μ costruita da Pink, Weidhorn e Ziegler. Le fibre di ζ sono costruite in modo da essere Ekedahl-Oort strata.

Il nostro secondo importante risultato è la dimostrazione che la mappa ζ è liscia. Si possono allora trasferire le conoscenze riguardo la geometria di $G\text{-}\mathtt{Zip}_\kappa^\mu$ in risultati riguardo gli strata di Ekedahl-Oort. In particolare, abbiamo una formula per la dimensione di ogni stratum non vuoto, e, dato uno stratum, siamo in grado di determinare quali strata sono nella sua chiusura.

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Curriculum Vitae

Chao Zhang was born on 17th December 1984, in Taian, Shandong, China. After finishing his senior middle school in his hometown, he went to Qingdao and started his undergraduate study in September 2003.

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He mainly worked on algebraic geometry during his graduate study. He wrote his master thesis "On superspecial locus of moduli spaces" in 2010 under the supervision of Prof. Ke-Zheng Li in Capital Normal University.

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