A priori bounds for superlinear problems involving the N-Laplacian

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Abstract

In this paper we establish a priori bounds for positive solution of the equation

$$-\Delta_N u = f(u), \quad u \in H^1_0(\Omega)$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, and the nonlinearity $f$ has at most exponential growth. The techniques used in the proofs are a generalization of the methods of Brezis-Merle to the $N$-Laplacian, in combination with the Trudinger-Moser inequality, the Moving Planes method and a Comparison Principle for the $N$-Laplacian.

Keywords and phrases: a priori bounds; moving planes; Trudinger-Moser inequality.

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1 Introduction

This paper is concerned with a priori bounds for positive solutions of equations involving the N-Laplacian and superlinear nonlinearities in bounded domains in $\mathbb{R}^N$. More precisely, we consider

$$\begin{cases}
-\Delta_N u = f(u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(1.1)

where $\Omega$ is a strictly convex, bounded and smooth domain in $\mathbb{R}^N$, and $\Delta_N u = \text{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator. On the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ we assume that it is a locally Lipschitz function satisfying the following hypotheses:

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\[(f_1) \quad f(s) \geq 0, \text{ for all } s \geq 0,\]

and either

\[(f_2) \quad \text{there exists a positive constant } d \text{ such that} \]
\[\liminf_{s \to +\infty} \frac{f(s)}{s^{N-1+d}} > 0\]

and

\[(f_3) \quad \text{there exist constants } c, s_0 \geq 0 \text{ and } 0 < \alpha < 1 \text{ such that} \]
\[f(s) \leq ce^{s^\alpha}, \text{ for all } s \geq s_0,\]

or

\[(f_4) \quad \text{there exist constants } c_1, c_2 > 0 \text{ and } s_0 > 0 \text{ such that} \]
\[c_1 e^s \leq f(s) \leq c_2 e^s, \text{ for all } s \geq s_0.\]

The main result is the following

**Theorem 1.1 (A priori bound).** Under the assumptions \((f_1)\) and either \((f_2)\) and \((f_3)\) (subcritical case) or \((f_4)\) (critical case) there exists a constant \(C > 0\) such that every weak solution \(u \in W^{1,N}_0(\Omega) \cap C^1(\Omega)\) of Equation \((1.1)\) satisfies

\[\|u\|_{L^\infty(\Omega)} \leq C.\]  \[(1.2)\]

A priori bounds for superlinear elliptic equations have been a focus of research in recent years. On the one hand, such results give interesting qualitative information on the positive solutions of such equations; on the other hand they are also useful to obtain existence results via degree theory.

It seems that the first general result for a priori bounds for superlinear elliptic equations is due to Brezis-Turner \([5]\), 1977. They considered the equation

\[
\begin{cases}
-\Delta u = g(x,u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]  \[(1.3)\]

and proved an a priori bound under the (main) hypothesis

\[0 \leq g(x,s) \leq cs^p, p < \frac{N+1}{N-1}.\]

Their method is based on the **Hardy-Sobolev inequality**.

In 1981, Gidas and Spruck \([8]\) considered Equation \((1.3)\) under the assumption

\[\lim_{s \to \infty} \frac{g(x,s)}{s^p} = a(x) > 0 \quad \text{in } \overline{\Omega},\]
and proved a priori estimates under the condition
\[ 1 < p < \frac{N + 2}{N - 2} = 2^* - 1, \]
using blow-up techniques and Liouville theorems on \( \mathbb{R}^N \).

In 1982, De Figueiredo - P.L. Lions - Nussbaum [9] obtained a priori estimates under the assumptions that \( \Omega \) is convex, and \( g(s) \) is superlinear at infinity and satisfies
\[ g(s) \leq cs^p, \quad 1 < p < \frac{N + 2}{N - 2}, \quad \text{(and some technical conditions).} \]
Their method relied on the moving planes technique, see [7], to obtain estimates near the boundary, and on Pohozaev-type identities.

Due to the results by Gidas-Spruck and De Figueiredo-Lions-Nussbaum it was generally believed that the result of Brezis-Turner was not optimal. But surprisingly, Quittner-Souplet [14] showed in 2004 that under the general hypotheses of Brezis-Turner their result is optimal; in fact, they give a counterexample with a \( g(x,s) \) with strong \( x \)-dependence.

Concerning to the \( m \)-Laplace case, Azizieh-Clement [3] studied the problem
\[
\begin{aligned}
-\Delta_m u &= g(x,u) \quad &\text{in } \Omega, \\
u &> 0 \quad &\text{in } \Omega, \\
u &= 0 \quad &\text{on } \partial\Omega.
\end{aligned}
\]
They obtain a priori estimates for the particular case \( 1 < m < 2 \), assuming \( g(x,u) = g(u) \), with \( C_1 u^p \leq g(u) \leq C_2 u^p \), where \( 1 < p < N(m - 1)/(N - m) \) and \( \Omega \) is bounded and convex.

The more general case \( 1 < m \leq 2 \) was considered by Ruiz [16]; he studied problem (1.4) where \( g \) is as in Azizieh-Clement but depends on \( x \); also, he does not need \( \Omega \) convex. In these two works, a blow-up argument together with a non existence result of positive super solutions, due to Mitidieri-Pohozaev [13], are used.

Recently, Lorca-Ubilla [12] obtained a priori estimates for solutions of (1.4) for more general nonlinearities \( g \). They only require \( 0 \leq g(x,u) < C_1 u^p \), \( 1 < p < N(m - 1)/(N - m) \), together with a superlinearity assumption at infinity. In this case the blow-arguments used by Azizieh-Clement and by Ruiz are not sufficient to obtain a contradiction. However using an adaptation of Ruiz’s argument, which consists in a combination of Harnack inequalities and local \( L^q \) estimates, it is possible to get the a priori estimate.

The above mentioned results are for \( N > 2 \); for \( N = 2 \) one has the embedding \( H^1_0(\Omega) \subset L^p \), for all \( p > 1 \), but easy examples show that \( H^1_0(\Omega) \not\subset L^\infty(\Omega) \). Thus, one may ask for the maximal growth function \( g(s) \) such that \( \int_\Omega g(u) < \infty \) for \( u \in H^1_0(\Omega) \). This maximal possible growth was determined independently by Yudovich, Pohozaev and Trudinger, leading to what is now called the Trudinger inequality: it says that for \( u \in H^1_0(\Omega) \) one has \( \int_\Omega e^{u^2}dx < +\infty \).
So, one can ask whether in dimension $N = 2$ one can prove a priori estimates for nonlinearities with growth up to the Trudinger-Moser growth. This is not the case, however some interesting result for equations with exponential growth have been proved in recent years. First, we mention the result of Brezis-Merle [4] who proved in 1991 that under the growth restriction

$$c_1 e^s \leq g(x,s) \leq c_2 e^s$$

one has: if $\int_{\Omega} g(x,u) dx \leq c$, for all $u > 0$ solution of Equation (1.1), then there exists $C > 0$ such that

$$\|u\|_{\infty} \leq C$$

for all positive solutions.

This is not quite an a priori result yet; however, from the boundary estimates of De Figueiredo - Lions - Nussbaum one obtains, assuming that $\Omega$ is convex (and adding some technical assumptions) that the condition $\int_{\Omega} g(x,u) \leq c$ of Brezis-Merle is satisfied. Hence, on convex domains the Brezis-Merle result yields indeed the desired a priori bounds. We note also that Brezis-Merle give examples of nonlinearities $g(x,s) = h(x)e^{\alpha s}$ with $\alpha > 1$ for which there exists a sequence of unbounded solutions.

Our Theorem 1.1 is motivated by the result of Brezis-Merle. We recall that in dimension $N$ the Trudinger inequality gives as maximal growth $g(s) \leq e^{\frac{|s|N}{N-1}}$, while our result shows that for a priori bounds it is again the exponential growth $g(s) \sim e^s$ which is the limiting growth to obtain a priori bounds.

The paper is organized as follows: in section 2 we obtain uniform bounds near the boundary $\partial \Omega$, using results of Damascelli-Sciunzi [6]. In section 3 we show that the boundary estimates yield easily a uniform bound on $\int_{\Omega} g(x,u)$. In section 4 we discuss the "subcritical case", i.e. when assumptions (f2) and (f3) hold, while in section 5 we prove the a priori bounds in the "critical case", i.e. under assumption (f4).

2 The boundary estimate

In this section we obtain a priori estimates on a portion of $\Omega$ including the boundary.

**Proposition 2.1** Assume (f2) or the left inequality in (f4). Then there exist positive constants $r,C$ such that every weak solution $u \in W^{1,N}_0(\Omega) \cap C^1(\Omega)$ of Equation (1.1) verifies

$$u(x) \leq C \text{ and } |\nabla u(x)| \leq C, x \in \Omega_r,$$

where $\Omega_r = \{ x \in \Omega : d(x,\partial \Omega) \leq r \}$.

**Proof.** For $x \in \partial \Omega$, let $\eta(x)$ denote the outward normal vector to $\partial \Omega$ in $x$. By Damascelli-Sciunzi [6], Theorem 1.5, there exists $t_0 > 0$ such that $u(x - t\eta(x))$ is nondecreasing for $t \in [0,t_0]$ and for $x \in \partial \Omega$. Note that $t_0$ depends only on the
geometry of $\Omega$. Following the ideas of de Figueiredo, Lions and Nussbaum’s paper [9] one now shows that there exists $\alpha > 0$, depending only on $\Omega$, such that

$$u(z - t\sigma) \text{ is nondecreasing for all } t \in [0, t_1],$$

where $|\sigma| = 1$, $\sigma \in \mathbb{R}^N$ verifies $\sigma \cdot \eta(z) \geq \alpha$, $z \in \partial \Omega$,

and $t_1 > 0$ depends only on $\Omega$.

Since $u(z - t\sigma)$ is nondecreasing in $t$ for $z$ and $\sigma$ as above, for all $x \in \Omega_\epsilon$ we find a measure set $I_x$, and positive numbers $\gamma$ and $\epsilon$ (depending only on $\Omega$) such that

(i) $|I_x| \geq \gamma$

(ii) $I_x \subset \{x \in \Omega : d(x, \partial \Omega) \geq \frac{\epsilon}{2} \}$

(iii) $u(y) \geq u(x)$, for all $y \in I_x$.

We now use Piccone’s identity (see [2]), which says that if $v$ and $u$ are $C^1$ functions with $v \geq 0$ and $u > 0$ in $\Omega$, then

$$|\nabla v|^N \geq |\nabla u|^{N-2} \nabla \left( \frac{v^N}{u^{N-1}} \right) \nabla u.$$  

We apply this inequality with $v = e_1$, the first (positive) eigenfunction of the $N$-Laplacian on $\Omega$, and $u > 0$ a (weak) solution of $-\Delta_N u = f(u)$. We assume that $e_1$ is normalized, i.e. $\int_{\Omega} e_1^N = 1$. Then we have (observe that $\frac{e_1^N}{u^{N-1}}$ belongs to $W^{1,N}_0(\Omega)$ since $u$ is positive in $\Omega$ and has nonzero outward derivative on the boundary because of Hopf’s lemma, see [17])

$$c \geq \int_{\Omega} |\nabla e_1|^N dx \geq \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla \frac{e_1^N}{u^{N-1}} = \int_{\Omega} \frac{f(u) e_1^N}{u^{N-1}}.$$  

Thus condition $(f_2)$ (or condition $(f_4)$) implies $\int_{\Omega} u^d e_1^N \leq \tilde{C}$, and so

$$\eta^N \int_{\Omega \setminus \Omega^{\frac{\epsilon}{2}}} u^d \leq \tilde{C}$$

where $e_1(z) \geq \eta > 0$, $z \in \Omega \setminus \Omega^{\frac{\epsilon}{2}}$. By $(ii)$, given $x \in \Omega_\epsilon$, we have

$$\eta^N \int_{I_x} u^d \leq \tilde{C}.$$  

Now since $u^d(x)|I_x| \leq \int_{I_x} u^d$ by $(i)$ and $(ii)$, we have $u^d(x) \leq \frac{\tilde{C}}{\eta^N}$, and so $u(x) \leq C'$, for all $x \in \Omega_\epsilon$. Finally by Lieberman [11] (see also Azizieh and Clément [3]) we have

$$u \in C^{1,\alpha}(\Omega^{\frac{\epsilon}{2}}) \text{ with } \|u\|_{C^{1,\alpha}(\Omega^{\frac{\epsilon}{2}})} \leq C'.$$  

(2.1)
3 Uniform bound on $\int_\Omega f(u)$

In this section we show that the boundary estimates yield easily a bound on the term $\int_\Omega f(u)dx$, for all positive solutions of Equation (1.1).

**Proposition 3.1** Suppose estimate (2.1) holds. Then there exists a positive constant $C$ such that for every weak solution of Equation (1.1) we have

$$\int_\Omega f(u) \leq C.$$  \hspace{1cm} (3.1)

**Proof.** Let $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ on $\Omega \setminus \Omega_2$. We have

$$\int_\Omega |\nabla u|^{N-2}\nabla u \nabla \psi = \int_\Omega f(u)\psi$$ \hspace{1cm} (3.2)

Using

$$\int_{\Omega \setminus \Omega_2} f(u) \leq \int_\Omega f(u)\psi$$

and the a priori estimates in $\Omega_2$, see (2.1), we get

$$\int_{\Omega \setminus \Omega_2} f(u) \leq \int_\Omega |\nabla u|^{N-2}\nabla u \nabla \psi = \int_{\Omega_2^c} |\nabla u|^{N-2}\nabla u \nabla \psi \leq C.$$  

Hence the estimate (3.1) is proved.  \hfill \blacksquare

We also state here an adaptation of Theorems 2 and 6 in [15] to the $N$-Laplace operator $\Delta_N$ which will be useful in the sequel.

**Lemma 3.2** Let $u \in W^{1,N}_{loc}(\Omega)$ be a solution of

$$-\Delta_N u = h(x) \text{ in } \Omega,$$

where $h \in L^p(\Omega)$, $p > 1$. Let $B_{2R} \subset \Omega$. Then

$$\|u\|_{L^\infty(B_R)} \leq CR^{-1}(\|u\|_{L^N(B_{2R})} + RK)$$

where $C = C(N,p)$ and $K = R^{N(p-1)/p}(\|h\|_{L^p(\Omega)})^{1/(N-1)}$.

4 Subcritical Case

In this section, we prove Theorem 1.1 under the assumptions ($f_1$), ($f_2$) and ($f_3$), i.e. in the subcritical case.

The proof will be based on Hölder’s inequality in Orlicz spaces (cf. [1]): Let $\psi$ and $\widetilde{\psi}$ be two complementary $N$-functions. Then

$$\left| \int_\Omega h g \right| \leq 2\|h\|_\psi\|g\|_{\widetilde{\psi}},$$ \hspace{1cm} (4.1)

where $\|h\|_\psi$ and $\|g\|_{\widetilde{\psi}}$ denote the Luxemburg (or gauge) norms.

We first prove the following inequality:
Lemma 4.1 Let $\gamma > 0$; then

$$st \leq s(\log(s + 1))^{1/\gamma} + t(e^{t\gamma} - 1), \text{ for all } s, t \geq 0$$

Proof. Consider for fixed $t > 0$

$$\max_{s \geq 0} \{st - s(\log(s + 1))^{1/\gamma}\}$$

In the maximum point $s_t$ we have

$$t = (\log(s_t + 1))^{1/\gamma} + \frac{s_t}{\gamma}(\log(s_t + 1))^{\frac{1}{\gamma} - 1} \geq (\log(s_t + 1))^{1/\gamma}$$

and hence $e^{t\gamma} \geq s_t + 1$. Thus

$$\max_{s \geq 0} \{st - s(\log(s + 1))^{1/\gamma}\} = s_t t - s_t(\log(s_t + 1))^{1/\gamma} \leq s_t t \leq t(e^{t\gamma} - 1).$$

Note that for the $N$-function $\psi(s) = s(\log(s + 1))^{1/\gamma}$, the complementary $N$-function $\tilde{\psi}(t)$ is by definition given by

$$\tilde{\psi}(t) = \max_{s \geq 0} \{st - s(\log(s + 1))^{1/\gamma}\}.$$

The above Lemma shows that $\varphi(t) := t(e^{t\gamma} - 1) \geq \tilde{\psi}(t)$, for all $t \geq 0$, and hence $\|g\|_{\tilde{\psi}} \leq \|g\|_{\varphi}$, and so the Hölder inequality (4.1) is valid also for the gauge norm $\varphi$ in place of $\tilde{\psi}$:

$$\left| \int_{\Omega} hg \right| \leq 2\|h\|_{\psi}\|g\|_{\varphi}, \quad (4.2)$$

Let now $u \in W^{1,N}_0(\Omega)$ be a weak solution of (1.1), denote

$$\gamma = \frac{N}{N - 1} - \alpha, \text{ and } \beta = \frac{\alpha}{\gamma},$$

and consider

$$\int_{\Omega} |\nabla u|^N = \int_{\Omega} f(u)u = \int_{\Omega} \frac{f(u)}{u^\beta} u^{1+\beta} \leq \int_{\Omega} \frac{f(u)}{u^\beta} \chi_u u^{1+\beta} + c, \quad (4.3)$$

where $\chi_u$ is the characteristic function of the set $\{x \in \Omega : u(x) \geq 1\}$. By (4.2) we conclude that

$$\int_{\Omega} |\nabla u|^N \leq 2\|u^{1+\beta}\|_{\varphi} \left\| \frac{f(u)}{u^\beta} \chi_u \right\|_{\psi} + c. \quad (4.4)$$

We now estimate the two Orlicz-norms in (4.4):
First note that there exists $d_\gamma > 0$ such that $\varphi(t) = t \left( e^{\gamma t} - 1 \right) \leq e^{d_\gamma t} - 1$, and hence
\[
\|u^{1+\beta}\|_{\varphi} = \inf \left\{ k > 0 : \int_{\Omega} \varphi \left( \frac{u^{1+\beta}}{k} \right) \leq 1 \right\} \\
\leq \inf \left\{ k > 0 : \int_{\Omega} \left( e^{d_\gamma \left( \frac{u^{1+\beta}}{k} \right)^\gamma} - 1 \right) \leq 1 \right\} \\
= \inf \left\{ k > 0 : \int_{\Omega} \left( e^{d_\gamma \frac{N-1}{k}} - 1 \right) \leq 1 \right\}
\tag{4.5}
\]
since $(1 + \beta)\gamma = \gamma + \alpha = N/(N-1)$. Now recall the Trudinger-Moser inequality which says that
\[
\sup_{\|u\|_{W^{1,N}} \leq 1} \int_{\Omega} e^{\alpha \|u\|^N/(N-1)} dx < +\infty, \quad \text{if } \alpha \leq \alpha_N, \tag{4.6}
\]
where $\alpha_N = N \omega_N^{1/(N-1)}$, and $\omega_N$ is the measure of the unit sphere in $\mathbb{R}^N$. Thus, if we take $k^\gamma = \frac{d_\gamma}{\alpha_N} \|\nabla u\|_{L^N(\Omega)}$ in (4.5), we see that the last integral in (4.5) is finite, and it becomes smaller than 1 if we choose $k^\gamma = c \|\nabla u\|_{L^N(\Omega)}^{N-1}$, for $c > 0$ suitably large, since $\varphi$ is a convex function. Thus, we get
\[
\|u^{1+\beta}\|_{\varphi} \leq c \|\nabla u\|_{L^N(\Omega)}^{1/\gamma} \frac{N}{N-1}.
\]

Next, we show that $\frac{d_\gamma}{\alpha_N} = \beta$ and (3.1) imply
\[
\| \frac{f(u)}{u^\beta} \chi u \|_{\psi} \leq \int_{\Omega} df(u) \leq C.
\]
Indeed, assumption $(f_3)$ implies
\[
\| \frac{f(u)}{u^\beta} \chi u \|_{\psi} = \inf \left\{ k > 0 : \int_{\Omega} \frac{f(u)}{k u^\beta} \chi u \left( \log \left( 1 + \frac{f(u)}{k u^\beta} \chi u \right) \right)^{\frac{1}{\gamma}} \leq 1 \right\} \\
\leq \inf \left\{ k > 1 : \int_{\Omega} \frac{f(u)}{k u^\beta} \chi u \left( \log \left( 1 + f(u) \right) \right)^{\frac{1}{\gamma}} \leq 1 \right\} \\
\leq \inf \left\{ k > 1 : \int_{\Omega} \frac{f(u)}{k u^\beta} \chi u \left( \log(c e^{\alpha u}) \right)^{\frac{1}{\gamma}} \leq 1 \right\} \\
\leq \inf \left\{ k > 1 : \int_{\Omega} \frac{f(u)}{k} \chi u^{\frac{N}{N-1} - \beta} \leq 1 \right\} \\
\leq \int_{\Omega} df(u) \leq C.
\]
Hence, joining these estimates, we conclude by (4.4) that
\[
\|\nabla u\|_{L^N(\Omega)}^{N} \leq C \|\nabla u\|_{L^N(\Omega)}^{\frac{N}{N-1}} + c.
\]
Finally, note that \( \alpha < 1 \) implies that

\[
\| \nabla u \|_{L^N(\Omega)} \leq C_N,
\]

for any solution positive \( u \in W^{1,N}(\Omega) \), with \( C_N \) depending only on \( N \) and \( \Omega \).

To obtain also a uniform \( L^\infty \)-bound, we proceed as follows: Let \( p > 1 \), then given \( \varepsilon > 0 \) there exists \( C(\varepsilon) \) such that

\[
p^s \alpha \leq \varepsilon s^{N/(N-1)} + C(\varepsilon).
\]

Thus we can estimate

\[
\int_\Omega |f(u)|^p \leq C_1(\varepsilon) \int_\Omega e^{\varepsilon \|u\|_{L^N(\Omega)}^{N/(N-1)}}.
\]

Now, choosing \( \varepsilon > 0 \) such that \( \varepsilon C_N^{N/(N-1)} \leq \alpha N \), the estimate (4.7) and the Trudinger–Moser inequality imply

\[
\int_\Omega |f(u)|^p \leq C_1(\varepsilon) \int_\Omega e^{C_N^{N/(N-1)}} \left\| \frac{u}{\|u\|_{L^N(\Omega)}} \right\|_{L^N(\Omega)}^{N/(N-1)} \leq C.
\]

And so, since \( \int_\Omega |f(u)|^p \leq C \), we have by Lemma 3.2 that \( \|u\|_{L^\infty(K)} \leq C = C(K) \) for every compact \( K \subset \subset \Omega \). We are finished, since in Section 3 we have proved a priori estimates near the boundary.

5 Critical Case

In this section, we will prove Theorem 1.1 under assumptions \((f_1)\) and \((f_4)\). It is convenient to introduce the following number

\[
d_N = \inf_{X \neq Y} \frac{|X|^{N-2}X - |Y|^{N-2}Y, X - Y}{|X - Y|^N}.
\]

By Proposition 4.6 of [10] we know that \( d_N \geq \left( \frac{2}{N} \right) \left( \frac{1}{2} \right)^{N-2} \). Also, by taking \( Y = 0 \) we see that \( d_N \leq 1 \).

We will use the following standard comparison result

**Lemma 5.1** Suppose that \( u, v \in W^{1,N}(\Omega) \cap C(\overline{\Omega}) \) verify \(-\Delta_N u \leq -\Delta_N v \) weakly in \( \Omega \), that is

\[
\int_\Omega |\nabla u|^{N-2} \nabla u - |\nabla v|^{N-2} \nabla v, \nabla \phi \leq 0,
\]

for all \( \phi \in W^{1,N}_0 \) such that \( \phi \geq 0 \) in \( \Omega \). If \( u \leq v \) on \( \partial \Omega \), then \( u \leq v \) in \( \Omega \).

**Proof.**

By taking \( \phi = (u - v)^+ \) we get

\[
d_N \int_{\{u \geq v\}} |\nabla (u - v)|^N \leq \int_{\{u \geq v\}} (|\nabla u|^{N-2} \nabla u - |\nabla v|^{N-2} \nabla v, \nabla (u - v)) \leq 0,
\]
where \( d_N \) is given by (5.1). This inequality implies \( u \leq v \) in \( \Omega \). 

We also need the following results by Ren and Wei [15] (Lemmas 4.1 and 4.3), which generalize the corresponding inequality for \( N = 2 \) of Brezis-Merle.

**Lemma 5.2** Let \( u \in W^{1,N}(\Omega) \) verifying \(-\Delta_N u = h\) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), where \( h \in L^1(\Omega) \cap C^0(\Omega) \) is nonnegative. Then, for every \( \delta \) with \( 0 < \delta < N \omega_N^{1-\frac{1}{N}} \)

\[
\int_{\Omega} e^{\frac{(N\omega_N^{1-\frac{1}{N}} - \delta) |u|}{\int_{\Omega} h^{\frac{1}{N-1}}}} \leq \frac{N \omega_N^{\frac{1}{N-1}} |\Omega|}{\delta},
\]

where \( \omega_N \) denotes the surface measure of the unit sphere in \( \mathbb{R}^N \).

**Lemma 5.3** Let \( u \in W^{1,N}(\Omega) \) verifying \(-\Delta_N u = h\) in \( \Omega \) and \( u = g \) on \( \partial \Omega \), where \( h \in L^1(\Omega) \cap C^0(\Omega) \) and \( g \in L^\infty(\Omega) \). Let \( \phi \in W^{1,N}(\Omega) \) such that \( \Delta_N \phi = 0 \) in \( \Omega \) and \( \phi = g \) on \( \partial \Omega \). Then, for every \( \delta \) with \( 0 < \delta < N \omega_N^{1-\frac{1}{N}} \)

\[
\int_{\Omega} e^{\frac{(N\omega_N^{1-\frac{1}{N}} - \delta) d_N^{\frac{1}{N-1}}}{\int_{\Omega} h^{\frac{1}{N-1}}}} |u - \phi| \leq \frac{N \omega_N^{\frac{1}{N-1}} |\Omega|}{\delta}.
\]

**Proof of Theorem 1.1 (critical case)**

Suppose by contradiction that there is no a priori estimate, then there would exist a sequence \( \{u_n\}_n \subset W^{1,N}(\Omega) \cap C^{1,\alpha}(\bar{\Omega}) \) of weak solutions of (1.1) such that \( \|u_n\|_{L^\infty(\Omega)} \to \infty \). Observe that by Proposition 3.1 there exists a constant \( C \) such that

\[
\int_{\Omega} f(u_n) \leq C.
\]

We may assume that \( f(u_n) \) converges in the sense of measures on \( \Omega \) to some nonnegative bounded measure \( \mu \), that is

\[
\int_{\Omega} f(u_n) \psi \to \int_{\Omega} \psi \, d\mu, \text{ for all simple functions } \psi.
\]

As in [4], let us introduce the concept of *regular point*. We say that \( x_0 \in \Omega \) is a regular point with respect to \( \mu \) if there exists an open neighborhood \( V \subset \Omega \) of \( x_0 \) such that

\[
\int_{\Omega} \chi_V \, d\mu < N^{N-1} \omega_N.
\]

Next, we define the set \( A \) as follows: \( x \in A \) if and only if there exists an open neighborhood \( U \subset \Omega \) of \( x \) such that

\[
\int_{\Omega} \chi_U \, d\mu < N^{N-1} \omega_N \, d_N,
\]

where \( d_N \) is the constant introduced in (5.1).

Because \( d_N \leq 1 \), we have that the set \( A \) contains only regular points. Also, note that there is only a finite number of points \( x \in \Omega \setminus A \); in fact, if \( x \in \Omega \setminus A \) then

\[
\int_{B_R(x)} \, d\mu \geq N^{N-1} \omega_N \, d_N, \text{ for all } R > 0 \text{ such that } B_R(x) \subset \Omega,
\]
which implies $\mu(\{x\}) \geq N^{N-1} \omega_N d_N$. Hence, since
\[
\sum_{x \in \Omega \setminus A} \mu(\{x\}) \leq \mu(\Omega) = \int_{\Omega} d\mu \leq C,
\]
the set of points in $\Omega \setminus A$ is finite.

Before finishing the proof we need two claims.

**Claim 1.** Let $x_0$ be a regular point, then there exist $C$ and $R$ such that for all $n \in \mathbb{N}$
\[
\|u_n\|_{L^\infty(B_R(x_0))} \leq C
\]

**Proof of Claim 1.** We divide the proof into two cases.

**Case 1:** $x_0 \in A$

By the definitions of the set $A$ and the measure $\mu$, there exist $R, \delta$ and $n_0 > 0$ such that for all $n > n_0$ we have
\[
\left(\int_{B_R(x_0)} f(u_n)\right)^{\frac{1}{N-1}} < \left(N^{\frac{1}{N-1}} - \delta\right) d_N^{\frac{1}{N-1}}.
\]

Let $\phi_n$ be satisfying
\[
\begin{cases}
-\Delta_N \phi_n = 0 & \text{in } B_R \\
\phi_n = u_n & \text{on } \partial B_R.
\end{cases}
\]

Then $\phi_n \leq u_n$ in $B_R$ by Lemma 5.1. Since $c \geq \int_{\Omega} f(u_n) \geq c_1 \int_{\Omega} e^{u_n}$ by $(f_4)$, we have $\int_{\Omega} u_n^N < C'$ and thus $\int_{\Omega} \phi_n^N < C'$. Now, by using Lemma 3.2 we have
\[
\|\phi_n\|_{L^\infty(B_R^2)} \leq CR^{-1}(\|\phi_n\|_{L^N(B_R^2)} + c) \leq C''.
\]

By applying Lemma 5.3, we get
\[
\int_{B_{R^2}} e^{q|u_n - \phi_n|} \leq \int_{B_{R^2}} e^{q|u_n - \phi_n|} < K.
\]

By (5.3) we conclude that $\int_{B_{R^2}} e^{q|u_n - \phi_n|} \leq CR^{-1}(\|u_n\|_{L^N(B_{R^2})} + RK)$

Again by Lemma 3.2 we infer
\[
\|u_n\|_{L^\infty(B_{R^2})} \leq CR^{-1}
\]

\[
\leq K_1,
\]
where \( K_1 = K \left( R, \| u_n \|_{L^N(B_R)}, \| f(u_n) \|_{L^q(B_R)} \right) \)

**Case 2:** \( x_0 \notin A \)

Since \( \Omega \setminus A \) is finite we can choose \( R > 0 \) such that \( \partial B_R(x_0) \subset A \). Taking \( x \in \partial B_R(x_0) \), by case 1 there is \( r = r(x) \) such that for all \( n \in \mathbb{N} \)

\[
\| u_n \|_{L^\infty(B_r(x_0))} \leq c(x).
\]

This implies by compactness, for some \( k \in \mathbb{N} \)

\[
\partial B_R \subseteq \bigcup_{i=1}^{k} B_{r(x_i)}(x_i).
\]

Now, if \( y \in \partial B_R \), then \( y \in B_{r(x_i)}(x_i) \), for some \( 1 \leq i \leq k \). Hence

\[
\| u_n \|_{L^\infty(\partial B_R)} \leq \max_{i=1, \ldots, k} C(x_i) =: K \text{ for all } n \in \mathbb{N}.
\]

Let \( U_n \) be the solution of

\[
\begin{cases}
-\Delta_N U_n &= f(u_n) \text{ in } B_R \\
U_n &= K \text{ on } \partial B_R,
\end{cases}
\]

which is equivalent to

\[
\begin{cases}
-\Delta_N (U_n - K) &= f(u_n) \text{ in } B_R \\
U_n - K &= 0 \text{ on } \partial B_R.
\end{cases}
\]

Therefore

\[
U_n \geq u_n, \text{ on } B_R,
\]

by Lemma 5.1. Thus by applying Lemma 5.2 we have

\[
\int_{B_R} e^{\frac{1}{\|u_n\|_{L^1}^{n-1}}} |U_n - K| \leq N \omega_N^{\frac{1}{n}} C R^N
\]

for any \( \delta' \in (0, N \omega_N^{1/(N-1)}) \).

Since \( x_0 \) is a regular point, there exist \( R_1 < R \) and \( n_0 \in \mathbb{N} \) such that for every \( n > n_0 \) we have for some \( \delta > 0 \)

\[
\left( \int_{B_{R_1}(x_0)} f(u_n) \right)^{\frac{1}{n-1}} < N \omega_N^{\frac{1}{n-1}} - \delta.
\]

Taking \( \delta' > 0 \) sufficiently small, we have

\[
1 < q = \frac{N \omega_N^{\frac{1}{n-1}} - \delta'}{N \omega_N^{\frac{1}{n-1}} - \delta} < \frac{N \omega_N^{\frac{1}{n-1}} - \delta'}{\| f(u_n) \|_{L^q}^{\frac{1}{q-1}}}.
\]
and hence by (5.4)
\[
\int_{B_{R_1}} e^{q|u_n-K|} < C , \quad \text{and then} \quad \int_{B_{R_1}} e^{qu_n} < K' ;
\]
this implies
\[
\int_{B_{R_1}} e^{qu_n} \leq K'' .
\]
and therefore by \((f_4)\)
\[
\int_{B_{R_1}} f(u_n)^q \leq K(q) , \quad \text{and also} \quad \|u_n\|_{L^N(B_{R_1})} \leq C .
\]
Hence, by Lemma 4.1
\[
\|u_n\|_{L^\infty(B_{R_1})} \leq C R_1^{-1}\left(\|u_n\|_{L^N(B_{R_1})} + C\|f(u_n)\|_{L^q(B_{R_1})}\right) < K''' .
\]
This finishes the proof of Claim 1.

Next, we define
\[
\Sigma = \{x \in \Omega : x \text{ is not regular for } \mu\} .
\]
We note that \(\Sigma \subset \Omega \setminus A\) where \(A\) is defined in the proof of Theorem 1.1. Hence, also \(\Sigma\) has finitely many elements.

The second claim is

**Claim 2.** \(\Sigma = \emptyset\) .

**Proof of Claim 2.** Arguing by contradiction, let us assume that there exists \(x_0 \in \Sigma\) and \(R > 0\) such that
\[
B_R(x_0) \cap \Sigma = \{x_0\} .
\]
We recall that \(u_n\) verifies
\[
\begin{cases}
-\Delta_N u_n = f(u_n) & \text{in } B_R(x_0) \\
u_n > 0 & \text{on } \partial B_R(x_0) .
\end{cases}
\]
By the previous claim and because all the points are regular in \(B_R(x_0) \setminus \{x_0\}\), passing to a subsequence we can assume that \(u_n \to u\) \(C^1\)-uniformly on compact subsets of \(B_R(x_0) \setminus \{x_0\}\). Consider the function \(w(x) = N \log \frac{R}{|x-x_0|}\), which satisfies
\[
\begin{cases}
-\Delta_N w = N^{N-1} |x-x_0|^{N-2} & \text{in } B_R(x_0) \\
w = 0 & \text{on } \partial B_R(x_0) .
\end{cases}
\]
For \(k > 0\), and define the functions
\[
T_k(s) = \begin{cases}
0 & \text{if } s < 0 , \\
s & \text{if } 0 \leq s \leq k , \\
k & \text{if } k < s .
\end{cases}
\]
Consider now the functions given by $z_n^{(k)} = T_k(w - u_n)$; because the functions $u_n$ are positive we have that $z_n^{(k)} \in W_0^{1,N}(B_R)$, and $z_n^{(k)}(x_0) = k$, for all $n \in \mathbb{N}$. Also

$$z_n^{(k)} \rightarrow z^{(k)} = \begin{cases} T_k(w - u), & \text{if } x \neq x_0 \\ k, & \text{if } x = x_0. \end{cases}$$

Note that $z^{(k)}$ is a measurable function. We have

$$\int_{B_R} \left( |\nabla w|^{N-2} \nabla w - |\nabla u_n|^{N-2} \nabla u_n \right) \nabla z_n^{(k)} = N^{N-1} \omega_N k - \int_{B_R} f(u_n) z_n^{(k)}. \quad (5.5)$$

Now set $d\mu_n = f(u_n)dx$; then we may apply the following Proposition which is a generalization of Fatou’s Lemma (see e.g. Royden, Real Analysis, Proposition 11.17):

**Proposition:** Suppose that $\mu_n$ is a sequence of (positive) measures which converges to $\mu$ setwise, and $g_n$ is a sequence of measurable, nonnegative functions that converge pointwise to $g$. Then

$$\liminf_{n \to \infty} \int g_n d\mu_n \geq \int g d\mu$$

Hence, we can write

$$\int_{B_R} f(u_n)z_n^{(k)}dx = \int z_n^{(k)}d\mu_n$$

and conclude that

$$\liminf_{n \to \infty} \int_{B_R} f(u_n)z_n^{(k)} = \liminf_{n \to \infty} \int z_n^{(k)}d\mu_n$$

$$\geq \int z^{(k)}d\mu$$

$$\geq \int \{x_0\} z^{(k)}d\mu$$

$$\geq N^{N-1} \omega_N k,$$

where we have used that $z^{(k)}(x_0) = k$ and $\mu(x_0) \geq N^{N-1} \omega_N$, because $x_0 \in \Sigma$.

Thus we obtain from (5.5) that for all $k \in \mathbb{N}$

$$\int_{B_R} \left( |\nabla w|^{N-2} \nabla w - |\nabla u|^{N-2} \nabla u \right) \nabla z^{(k)} \leq 0 ,$$

that is

$$\int_{B_R \cap \{0 \leq w - u \leq k\}} \left( |\nabla w|^{N-2} \nabla w - |\nabla u|^{N-2} \nabla u \right) \nabla (w - u) \leq 0 , \quad k \in \mathbb{N} .$$

By inequality (5.1) we obtain

$$d_N \int_{B_R \cap \{0 \leq w - u \leq k\}} |\nabla (w - u)|^N \leq 0 , \quad k \in \mathbb{N} .$$
Finally, letting $k \to \infty$, we conclude that
\[
d_N \int_{B_R} |\nabla (w - u)^+|^N \leq 0.
\]
Because we know that $(w - u)^+ \leq 0$ on $\partial B_R$, the above inequality implies that $w \leq u$ in $W^{1,N}_0(B_R)$, and therefore we conclude that
\[
\liminf_{n \to +\infty} \int_{B_R} f(u_n) \geq \liminf_{n \to +\infty} \int_{B_R} c_1 e^{u_n} \\
\geq c_1 \int_{B_R} e^u \\
\geq \int_{B_R} \frac{C}{|x - x_0|^N} = +\infty
\]
This is a contradiction and the proof of Claim 2 is complete.

To finish the proof of Theorem 1.1, we observe that there exists a sequence $x_n$ of points in $\Omega$ such that $u_n(x_n) = \|u_n\|_{L^\infty(\Omega)}$ and we can assume that $x_n \to x_0$. Because we have an a priori estimate near the boundary of $\Omega$, we have $x_0 \in \Omega$. It is easy to see that for all $R > 0$ we have
\[
\lim_{n \to +\infty} \|u_n\|_{L^\infty(B_R)} = +\infty.
\]
By Claim 1, we conclude that $x_0$ is not a regular point, but this is impossible by Claim 2. Hence there are no blow-up points. □

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