Multiplicity of solutions for a superlinear p-Laplacian equation

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Abstract

We consider quasi-linear elliptic equations involving the p-Laplacian with nonlinearities which interfere asymptotically with the spectrum of the differential operator. We show that such equations have for certain forcing terms at least two solutions. Such equations are of so-called Ambrosetti-Prodi type. In particular, our theorem is a partial generalization of corresponding results for the semi-linear case by Ruf-Srikanth (1986) and de Figueiredo (1988).

Keywords: p-Laplacian, Ambrosetti-Prodi problem, multiple solutions, linking theorem

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1 Introduction

In this paper we are interested in certain quasi-linear elliptic equations with nonlinearities which interfere asymptotically with the spectrum of the quasi-linear differential operator. The prototype for such problems is the famous result of Ambrosetti-Prodi [2] which says that for a semi-linear equation with a nonlinearity which crosses asymptotically the first eigenvalue of the differential operator there exist, in dependence of the forcing term, either zero, one or two solutions. In a dual situation, where the nonlinearity crosses all but the first eigenvalues, Ruf-Srikanth proved in [18] that again there are forcing terms for which there exist at least two solutions. Actually, it was conjectured by Lazer-McKenna [13] that for nonlinearities which cross infinitely many eigenvalues there exist, for any given number \( k \), forcing terms with at least \( k \) solutions. This conjecture was recently confirmed by Dancer-Yan [8] in a surprising way by constructing solutions with an arbitrary number of peaks attached to a negative solution.

To be more precise: we consider elliptic equations of the following type

\[
\begin{cases}
-\Delta v &= f(v) + \tau \varphi_1 + \psi \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega
\end{cases}
\]

where \( \tau \in \mathbb{R} \) is a parameter, \( \psi \in L^2(\Omega) \) is a fixed function, \( \varphi_1 \) is the first eigenfunction of \( -\Delta \) corresponding to the first eigenvalue \( \mu_1 \), and the continuous functions \( f \) satisfies \(-\infty \leq \lim_{v \to -\infty} \frac{f(v)}{v} < \lim_{v \to +\infty} \frac{f(v)}{v} \leq +\infty \). Such equations are called problems
with jumping nonlinearities or problems of Ambrosetti-Prodi type. In the pioneering work of Ambrosetti and Prodi [2] and other related articles (Berger and Podolak [6], Kazdan and Warner [12], Dancer [7], Amann and Hess [1]) it was shown that under the hypothesis

\[ -\infty < \lim_{v \to -\infty} \frac{f(v)}{v} < \mu_1 < \lim_{v \to +\infty} \frac{f(v)}{v} < +\infty \]  

(2)

there exists a constant \( \tau_0 \) (depending on \( \psi \)) such that problem (1) has two solutions for \( \tau < \tau_0 \), one solution for \( \tau = \tau_0 \), and no solution for \( \tau > \tau_0 \).

On the other hand, Ruf - Srikanth [18] and de Figueiredo [9] considered instead of (2) the following assumptions on \( g \)

\[ \mu_k < \lim_{v \to -\infty} \frac{f(v)}{v} < \mu_{k+1} , \quad \lim_{v \to +\infty} \frac{f(v)}{v} = +\infty , \]

that is \( g \) interacts with all the spectrum of the Laplacian except the first \( k \) eigenvalues \( \mu_1, \ldots, \mu_k \); they obtained the existence of at least two solutions for equation (1) provided that \( \tau \) is sufficiently large.

For quasi-linear equations in which the Laplacian is replaced by the \( p \)-Laplacian, there are only few related results available. In particular, we mention the result of Arcoya-Ruiz [5] concerning a version of the Ambrosetti-Prodi result for the \( p \)-Laplacian operator. They considered the problem

\[
\begin{align*}
-\Delta_p w &= g(w) + \tau \phi_1 + \psi , \quad \text{in } \Omega \\
w &= 0 , \quad \text{on } \partial \Omega ;
\end{align*}
\]

(3)

where \( \tau \in \mathbb{R} \) is a parameter, \( \psi \in L^\infty(\Omega) \) is a fixed function, \( \phi_1 \) is the first eigenfunction of \( -\Delta_p \) corresponding to the first eigenvalue \( \lambda_1 \). Assuming the analogous condition to (2), namely

\[-\infty < \lim_{w \to -\infty} \frac{g(w)}{|w|^{p-2}w} < \lambda_1 < \lim_{w \to +\infty} \frac{g(w)}{|w|^{p-2}w} < +\infty , \]

they proved (with a mixed topological degree and sub-super solution method) the existence of \( \tau_* , \tau^* \), \( -\infty < \tau_* \leq \tau^* < +\infty \), such that (3) has at least two solutions if \( \tau < \tau_* \), at least one solution if \( \tau \leq \tau^* \), no solutions if \( \tau > \tau^* \). Under additional conditions (among which \( p \geq 2 \)) they also showed \( \tau_* = \tau^* \).

In this paper we give an extension of the result by Ruf-Srikanth and de Figueiredo to the case of the \( p \)-Laplacian. More precisely, we give a multiplicity result for the following problem

\[
\begin{align*}
-\Delta_p w &= \lambda |w|^{p-2}w + (w^+)^{q-1} + \tau \phi_1^{p-1} + H , \quad \text{in } \Omega \\
w &= 0 , \quad \text{on } \partial \Omega ;
\end{align*}
\]

(4)

here \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( -\Delta_p \) is the degenerate \( p \)-Laplacian operator (with \( p > 2 \)), \( \phi_1 \) is the positive first eigenfunction of the \( p \)-Laplacian, \( H \in L^\infty(\Omega) \) is such that \( \int_{\Omega} H \phi_1 \, dx = 0 \) and \( \lambda \in (\lambda_1, \lambda_2) \), \( \tau > 0 \), \( q \in (p, p^*) \) are fixed real parameters,
where \( \lambda_1, \lambda_2 \) are the first and the second eigenvalue of \(-\Delta_p \) with Dirichlet boundary conditions, and \( p^* \) is the critical exponent of the Sobolev embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^{q}(\Omega) \). Note that the nonlinearity \( g(s) := \lambda |s|^{p-2} s + (s^+)^q \) satisfies

\[
\lambda_1 < \lim_{s \to -\infty} \frac{g(s)}{|s|^{p-2}} = \lambda < \lambda_2, \quad \lim_{s \to +\infty} \frac{g(s)}{|s|^{p-2}} = +\infty ;
\]

hence \( g \) crosses (in a \((p-1)-linear\) sense) all but the first eigenvalue of the \( p \)-Laplacian.

The main result of this paper is

**Theorem 1.1.** There exists \( \Lambda \in (\lambda_1, \lambda_2) \) such that for any \( \lambda \in (\lambda_1, \Lambda) \) problem (4) has two solutions when \( \tau > 0 \) is sufficiently large.

**Remark 1.1.** Note that we require that \( \lim_{s \to -\infty} \frac{g(s)}{|s|^{p-2}} \) is larger than and close to \( \lambda_1 \). This restriction is caused by technical problems due to the linearization of the \( p \)-Laplacian. It remains an open problem whether such a restriction is necessary, as well as the generalization to the case \( \lambda_k < \lambda < \lambda_{k+1} \).

The proof of Theorem 1.1 proceeds as follows: by the rescaling \( u = \left( \frac{1}{\tau} \right)^{\frac{1}{p-1}} w, \) problem (4) is equivalent to

\[
\begin{align*}
-\Delta_p u & = \lambda |u|^{p-2} u + \tau \frac{q-p}{p} (u^+)^{q-1} + \phi_1^{p-1} + \frac{1}{\tau} H , & \text{in } \Omega, \\
0 & = \phi_1^{p-1} v - \frac{1}{\tau} \int \phi_1^{p-1} v dx - \frac{1}{\tau} \int Hv dx = 0 , & \text{for all } v \in W^{1,p}_0(\Omega),
\end{align*}
\]

Any weak solution \( u \in W^{1,p}_0(\Omega) \) of equation (6) must satisfy

\[
\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle_{\mathbb{R}^N} dx - \lambda \int_{\Omega} |u|^{p-2} uv dx - \frac{q-p}{p} \int_{\Omega} (u^+)^{q-1} v dx \\
- \int_{\Omega} \phi_1^{p-1} uv dx - \frac{1}{\tau} \int_{\Omega} Hv dx = 0 , \quad \text{for all } v \in W^{1,p}_0(\Omega),
\]

namely \( \langle J_{\lambda,\tau}(u), v \rangle = 0 \) for all \( v \in W^{1,p}_0(\Omega) \), where \( J_{\lambda,\tau} : W^{1,p}_0(\Omega) \to \mathbb{R} \) is the following functional

\[
J_{\lambda,\tau}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^{p} dx - \frac{\lambda}{p} \int_{\Omega} |u|^{p} dx - \frac{\tau}{q} \int_{\Omega} (u^+)^{q} dx - \frac{1}{\tau} \int_{\Omega} \phi_1^{p-1} u dx - \frac{1}{\tau} \int_{\Omega} Hv dx.
\]

First, we prove that for \( \lambda > \lambda_1 \) and close to \( \lambda_1 \) and for \( \tau \) sufficiently large there exists a negative solution \( \phi_{\lambda,\tau} \) of equation (6). An important step consists in showing that the functional \( J_{\lambda,\tau} \) restricted to the subspace \( \langle \phi_{\lambda,\tau}^{-1}\rangle \) has a strict local minimum in \( \phi_{\lambda,\tau} \). This requires delicate estimates on the second derivative of \( J \) in \( \phi_{\lambda,\tau} \). For the second solution we can then rely on the Generalized Mountain Pass Theorem by P.H. Rabinowitz [17]. In fact, we will show that the functional \( J_{\lambda,\tau}(u) \) has a linking structure around the first solution \( \phi_{\lambda,\tau} \). This then yields the second solution.
2 Preliminary results

2.1 Lindqvist’s inequality in $\mathbb{R}^N$

Lemma 2.1 ([15], Lemma 4.2). Let $p \geq 2$, $N \in \mathbb{N}$. Then for any $w_1, w_2 \in \mathbb{R}^N$,

$$|w_2|^p \geq |w_1|^p + p|w_1|^{p-2}\langle w_1, w_2 - w_1 \rangle_{\mathbb{R}^N} + \frac{|w_2 - w_1|^p}{2^{p-1} - 1}.$$

2.2 The p-Laplacian operator

Proposition 2.2 ([16], Theorem A.0.6). The p-Laplacian operator $-\Delta_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by

$$\langle -\Delta_p(u), w \rangle = \int_{\Omega} |\nabla u|^{p-2}\langle \nabla u, \nabla w \rangle_{\mathbb{R}^N} dx,$$

is continuous and one-to-one. The inverse $-\Delta_p^{-1} : W^{-1,p'}(\Omega) \rightarrow W^{1,p}_0(\Omega)$ is also continuous.

In what follows, we will always assume

**Hypothesis 1.** If $N \geq 2$, then $\Omega$ is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1, \alpha}$ for some $\alpha \in (0, 1)$, and $\Omega$ satisfies also the interior sphere condition at every point of $\partial\Omega$. If $N = 1$ then $\Omega$ is a bounded open interval in $\mathbb{R}$.

The eigenvalues of the p-Laplacian $-\Delta_p$ namely the values $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases}
-\Delta_p \phi &= \lambda \phi^{p-1}, &\text{in } \Omega \\
\phi &= 0, &\text{on } \partial\Omega
\end{cases}$$

admits nontrivial solutions, form a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$, $\lambda_n \to +\infty$. For a proof of the existence of these eigenvalues we refer to [16], Chapter 1. The first eigenvalue $\lambda_1$ is given by the formula

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W^{1,p}_0(\Omega), \int_{\Omega} |u|^p dx = 1 \right\}.$$  \hspace{1cm} (7)

Moreover $\lambda_1$ is simple and the corresponding first eigenfunction $\phi_1 \in W^{1,p}_0(\Omega)$ has constant sign, by a result due to Anane ([4], Théorème 1, p.727) and later generalized in Lindqvist ([15], Theorem 1.3, p.157). We have $\phi_1 \in L^\infty(\Omega)$ by another result of Anane ([3], Théorème A.1, p.96). Consequently, under Hypothesis 1 on $\Omega$ we get even

$$\phi_1 \in C^{1, \beta}(\overline{\Omega})$$

for some $\beta \in (0, \alpha)$, by a regularity result due to Tolksdorf ([20], Theorem 1, p.127) (interior regularity), and to Lieberman ([14], Theorem 1, p.1203) (regularity near the boundary). Finally, the Hopf maximum principle ([21], Theorem 5, p.200) can be applied to obtain

$$\phi_1 > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial \phi_1}{\partial \nu} < 0 \text{ on } \partial\Omega,$$

where $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative on $\partial\Omega$.  


2.3 Compactness of the operator $u \rightarrow |u|^{p-2}u$

Lemma 2.3. The operator $\psi_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined as

$$\langle \psi_p(u), w \rangle = \int_{\Omega} |u|^{p-2}uw \, dx,$$

for all $w \in W^{1,p}_0(\Omega)$, is compact.

2.4 A norm depending on $\phi_1$

Proposition 2.4 ([19], Lemma 4.2). The functional

$$\|u\|_{\phi_1} := \left( \int_{\Omega} |\nabla \phi_1|^{p-2}|\nabla u|^2 \, dx \right)^{1/2}$$

is a norm on $W^{1,p}_0(\Omega)$. Moreover, if we denote with $W^{1,2}_{\phi_1}(\Omega)$ the completion of $W^{1,p}_0(\Omega)$ with respect to this norm, then

$$W^{1,2}_{\phi_1}(\Omega) \hookrightarrow L^2(\Omega),$$

and the embedding is compact.

2.5 The Fréchet derivative of the p-Laplacian operator

We define

$$F(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx$$

The Fréchet derivative $F'(u)$ of $F$ at $u \in W^{1,p}_0(\Omega)$ is given by $F'(u) = -\Delta_p(u)$ in $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This follows from

$$\langle F'(u), \phi \rangle = \int_{\Omega} |\nabla u|^{p-2}\langle \nabla u, \nabla \phi \rangle_{\mathbb{R}^N} \, dx$$

Moreover, the second Fréchet derivative $F''(u)$ of $F$ at $u \in W^{1,p}_0(\Omega)$ is given by

$$\langle F''(u)\psi, \phi \rangle = (p-2) \int_{\Omega} |\nabla u|^{p-4}\langle \nabla u, \nabla \psi \rangle_{\mathbb{R}^N} \langle \nabla u, \nabla \phi \rangle_{\mathbb{R}^N} \, dx + \int_{\Omega} |\nabla u|^{p-2}\langle \nabla \psi, \nabla \phi \rangle_{\mathbb{R}^N} \, dx.$$

The matrix representation of this derivative is the following:

$$\langle F''(u)\psi, \phi \rangle = \int_{\Omega} |\nabla u|^{p-2} \left( \mathbb{I}_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \langle \nabla \psi, \nabla \phi \rangle_{\mathbb{R}^N \times \mathbb{R}^N} \, dx,$$

where the tensor product $\otimes$ is defined by $a \otimes b = \left[ a_i b_j \right]_{i,j=1}^N$. For $a \in \mathbb{R}^N$ we introduce the abbreviation

$$\Theta(a) := \begin{cases} |a|^{p-2} \left( \mathbb{I}_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2) \frac{a \otimes a}{|a|^2} \right), & a \neq 0 \\ 0, & a = 0. \end{cases}$$
The matrix $a \otimes a$ represents the orthogonal projection on the one-dimensional space spanned by $a$, so we have

$$\Theta(a) b = \frac{|a|^{p-2} b}{|a|^{p-2}}, \quad \text{for } b \in \mathbb{R}^N \text{ with } \langle b, a \rangle_{\mathbb{R}^N} = 0,$$

$$\Theta(a) a = (p-1)|a|^{p-2} a.$$ 

The spectrum of the matrix $Id_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2)\frac{a \otimes a}{|a|^2}$ consists of the eigenvalues 1 and $p-1$; $p-1$ is simple with the eigenspace spanned by the eigenvector $a$.

For all $a, b \in \mathbb{R}^N \setminus \{0\}$ we thus obtain

$$1 \leq \frac{\langle \Theta(a) b, b \rangle_{\mathbb{R}^N}}{|a|^{p-2} |b|^2} \leq p-1. \quad (8)$$

As a direct consequence, we have the following

**Lemma 2.5.** For any $u \in W^{1,2}_{\phi_1}(\Omega)$,

$$\|u\|_{\phi_1}^2 \leq \int_\Omega \langle \Theta(\nabla \phi_1) \nabla u, \nabla u \rangle_{\mathbb{R}^N} dx \leq (p-1)\|u\|_{\phi_1}^2.$$

**2.5.1 A weak lower semicontinuity result**

**Lemma 2.6.** Let $v_n \rightharpoonup^{n \to \infty} \bar{v}$, weakly in $W^{1,2}_{\phi_1}(\Omega)$. Then

$$\liminf_{n \to \infty} \left( \int_\Omega \langle \Theta(\nabla \phi_1) \nabla v_n, \nabla v_n \rangle_{\mathbb{R}^N} dx \right) \geq \int_\Omega \langle \Theta(\nabla \phi_1) \nabla \bar{v}, \nabla \bar{v} \rangle_{\mathbb{R}^N} dx. \quad \text{The same conclusion is true assuming } v_n \rightharpoonup^{n \to \infty} \bar{v}, \text{ weakly in } W^{1,p}_{0}(\Omega).$$

**Proof.** Consider the bilinear form $B : W^{1,2}_{\phi_1}(\Omega) \times W^{1,2}_{\phi_1}(\Omega) \to \mathbb{R}$ defined as

$$B(w_1, w_2) := \int_\Omega \langle \Theta(\nabla \phi_1) \nabla w_1, \nabla w_2 \rangle_{\mathbb{R}^N} dx$$

$$= \int_U |\nabla \phi_1|^{p-2} \left( (p-2) \langle \nabla \phi_1, \nabla w_1 \rangle_{\mathbb{R}^N} \langle \nabla \phi_1, \nabla w_2 \rangle_{\mathbb{R}^N} + \langle \nabla w_1, \nabla w_2 \rangle_{\mathbb{R}^N} \right) dx,$$

where $U := \{ x \in \Omega : |\nabla \phi_1| \neq 0 \}$. Using the Cauchy-Schwarz inequality it is easy to see that the operator $L_\tau : W^{1,2}_{\phi_1}(\Omega) \to \mathbb{R}$

$$L_\tau(u) := B(\tau, u), \quad u \in W^{1,2}_{\phi_1}(\Omega),$$

belongs to the dual space of $W^{1,2}_{\phi_1}(\Omega)$. In particular, by the weak convergence of $v_n$ to $\bar{v}$,

$$\int_\Omega \langle \Theta(\nabla \phi_1) \nabla \tau, \nabla v_n \rangle_{\mathbb{R}^N} dx = L_\tau(v_n) \rightharpoonup^{n \to \infty} L_\tau(\bar{v}) = \int_\Omega \langle \Theta(\nabla \phi_1) \nabla \tau, \nabla \bar{v} \rangle_{\mathbb{R}^N} dx. \quad (9)$$
Now obviously \( B(w, w) \geq 0 \), for all \( w \in W^{1,2}_{\phi_1}(\Omega) \), hence
\[
0 \leq \liminf_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla(v_n - \tau) \rangle dx \right)
\]
\[
= \liminf_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla v_n \rangle dx - 2\int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla \tau, \nabla v_n \rangle dx + \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla \tau, \nabla \tau \rangle dx \right)
\]
\[
= \liminf_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla v_n, \nabla v_n \rangle dx \right) - \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla \tau, \nabla \tau \rangle dx.
\]
From \( W^{1,p}_0(\Omega) \hookrightarrow W^{1,2}_{\phi_1}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,2}_{\phi_1}(\Omega))' \hookrightarrow W^{-1,p'}(\Omega) \) we infer \( L_\tau \in W^{-1,p'}(\Omega) \).
Therefore (9) is true also if \( v_n \stackrel{n \to \infty}{\rightharpoonup} \tau \), weakly in \( W^{1,p}_0(\Omega) \), and we get the same conclusion.

\[
\square
\]

2.5.2 Another variational characterization of \( \lambda_1 \)

We are now stating a variational formula for \( \lambda_1 \) different from (7) which will be crucial in our next considerations. To do this an additional hypothesis on \( \Omega \) is needed, if \( \partial \Omega \) is not connected:

**Hypothesis 2.** If \( N \geq 2 \) and \( \partial \Omega \) is not connected, then there is no function \( v \in W^{1,2}_{\phi_1}(\Omega) \) with the following properties:

(i) \( v = \phi_1 \chi_S \) a.e. in \( \Omega \), where \( S \subset \Omega \) is a Lebesgue measurable set such that \( 0 < |S|_N < |\Omega|_N \).

(ii) \( S \) is connected and \( \overline{S} \cap \partial \Omega \neq \emptyset \).

(iii) Every connected component of \( U := \{ x \in \Omega : \nabla \phi_1(x) \neq 0 \} \) is entirely contained either in \( S \) or else in \( \Omega \setminus S \).

(iv) \( \partial S \cap \Omega \subset \Omega \setminus U \).

(v) \( \int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla v, \nabla v \rangle dx - (p - 1)\lambda_1 \int_{\Omega} \phi_1^{p-2}v^2 dx = 0. \)

**Proposition 2.7** ([19], Lemma 4.1, Proposition 4.4). Suppose that Hypothesis 1 on \( \Omega \) holds and that either \( \partial \Omega \) is connected or Hypothesis 2 is satisfied. Then
\[
\inf_{w \in W^{1,2}_{\phi_1}(\Omega)} \frac{\int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla w, \nabla w \rangle dx}{(p - 1) \int_{\Omega} \phi_1^{p-2}|w|^2 dx} = \lambda_1,
\]
and \( w \) is a minimizer of (10) if and only if \( w = \kappa \phi_1 \) for some constant \( \kappa \in \mathbb{R} \).

**Lemma 2.8.** Let
\[
\Lambda_\infty := \inf_{h \in W^{1,2}_{\phi_1}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \langle \Theta(\nabla \phi_1)\nabla h, \nabla h \rangle dx}{(p - 1) \int_{\Omega} \phi_1^{p-2}|h|^2 dx}.
\]

Then \( \Lambda_\infty > \lambda_1 \).
Moreover, Lemma 2.6 and Proposition 2.7. In fact, if \( h \in L^1(\Omega) \) can assume \( h_n \to h^\ast \), weakly in \( W^{1,2}_{\phi_1}(\Omega) \). In particular
\[
\int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx = \lim_{n \to \infty} \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx = 0.
\]

Moreover, Lemma 2.6 yields
\[
\liminf_{n \to \infty} \int_{\Omega} \langle \theta(\nabla \phi_1), \nabla h_n^\ast, \nabla h_n^\ast \rangle \, dx \geq 
\int_{\Omega} \langle \theta(\nabla \phi_1), \nabla h^\ast, \nabla h^\ast \rangle \, dx,
\]
while, by the compactness of the immersion \( W^{1,2}_{\phi_1}(\Omega) \to L^2(\Omega) \), \( h_n \to h^\ast \), strongly in \( L^2(\Omega) \) and therefore we deduce
\[
\int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx \to \infty \int_{\Omega} \phi_1^{-2} |h^\ast|^2 \, dx.
\]

Now we claim that
\[
h^\ast \neq 0.
\]

In fact, if \( \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx \to 0 \) we get
\[
\lambda_1 = \lim_{n \to \infty} \int_{\Omega} \langle \theta(\nabla \phi_1), \nabla h_n, \nabla h_n \rangle \, dx
\]
\[
\geq \frac{1}{(p-1) \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx}
\]
\[
\geq \frac{1}{(p-1) \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx},
\]
a contradiction. Then
\[
\lambda_1 = \liminf_{n \to \infty} \int_{\Omega} \langle \theta(\nabla \phi_1), \nabla h_n, \nabla h_n \rangle \, dx
\]
\[
\geq \frac{1}{(p-1) \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx}
\]
\[
\geq \frac{1}{(p-1) \int_{\Omega} \phi_1^{-2} |h_n|^2 \, dx},
\]
and Proposition 2.7 forces \( h^\ast = \kappa \phi_1 \), in contradiction with (11), (14). \( \Box \)
2.5.3 Uniform continuity on \( C^{1,\beta}(\overline{\Omega}) \)

The proof of the following lemma is easily achieved by Lebesgue’s dominated convergence theorem, Cauchy-Schwarz and Hölder inequalities.

**Lemma 2.9.** Let \( \psi_n \xrightarrow{n \to \infty} \psi \), in \( C^{1,\beta}(\Omega) \). Then

\[
\sup_{\|h\|_{1,p} = 1} \left| \int_{\Omega} \langle \Theta(\nabla \psi_n) \nabla h, \nabla h \rangle_{\mathbb{R}^N} dx - \int_{\Omega} \langle \Theta(\nabla \psi) \nabla h, \nabla h \rangle_{\mathbb{R}^N} dx \right| \leq \delta_n \xrightarrow{n \to \infty} 0.
\]

3 Existence of a first negative solution \( \phi_{\lambda,\tau} \)

In this section we obtain the existence of a first solution of (6). First we recall some known results.

**Proposition 3.1 ([10], Theorem 12.26).** For any \( \lambda \in (\lambda_1, \lambda_2) \), \( f^* \in W^{-1,p'}(\Omega) \) the problem

\[
\begin{cases}
 -\Delta_p u - \lambda |u|^{p-2}u = f^* , & \text{in } \Omega \\
 u = 0 , & \text{on } \partial \Omega
\end{cases}
\]  

is solvable in \( W^{1,p}_0(\Omega) \).

**Definition 3.1.** For any \( \lambda \in (\lambda_1, \lambda_2) \), \( \tau > 0 \), we denote with \( \phi_{\lambda,\tau} \) an arbitrary solution of

\[
\begin{cases}
 -\Delta_p u - \lambda |u|^{p-2}u = \phi_1^{p-1} + \tau^{-1}H , & \text{in } \Omega \\
 u = 0 , & \text{on } \partial \Omega
\end{cases}
\]

**Proposition 3.2 ([11], Theorem 4.1).** Let \( \{\lambda_n\}, \{\tau_n\} \subset \mathbb{R}^+ \) be sequences with \( \lambda_n \xrightarrow{n \to \infty} \lambda_1^+ \), \( \tau_n \xrightarrow{n \to \infty} +\infty \). Then we have the following asymptotic estimate:

\[
\phi_{\lambda_n,\tau_n} = \frac{\phi_1 + v_n^\perp}{-(\lambda_n - \lambda_1)^{\frac{1}{p-1}} + o\left((\lambda_n - \lambda_1)^{\frac{1}{p-1}}\right)},
\]

where \( v_n^\perp \in \langle \phi_1^{p-1}\rangle \perp, \ x_n \xrightarrow{n \to \infty} 0, \) strongly in \( C^{1,\beta}(\overline{\Omega}) \), for some \( \beta > 0 \) (up to a subsequence). In particular \( \phi_{\lambda_n,\tau_n} < 0 \), in \( \Omega \), for any \( n \) sufficiently large.

**Corollary 3.3.** There is \( \Lambda^1 \in (\lambda_1, \lambda_2] \) such that for any \( \lambda \in (\lambda_1, \Lambda^1) \) there exists \( \tau_\lambda^1 > 0 \) such that for any \( \tau > \tau_\lambda^1 \)

\[
\phi_{\lambda,\tau} < 0 , \text{ in } \Omega.
\]

For such values of \( \lambda, \tau \), since \( \phi_{\lambda,\tau} \) is a negative solution of problem (16), \( \phi_{\lambda,\tau} \) solves also problem (6).

**Proof.** If the assertion is not true, we can take two sequences \( \{\lambda_n\}_{n \in \mathbb{N}}, \{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \), with \( \lambda_n \xrightarrow{n \to \infty} \lambda_1^+ \), \( \tau_n \xrightarrow{n \to \infty} +\infty \) such that for any \( n \) there is \( x_\lambda \in \Omega \) such that \( \phi_{\lambda_n,\tau_n}(x_\lambda) \geq 0 \). This contradicts Proposition 3.2, which states that \( \phi_{\lambda_n,\tau_n} < 0 \) in \( \Omega \), as \( n \to \infty \).  \( \square \)
Moreover, according to Proposition 3.2, we get $\phi$ with $\lambda$
If the assertion is not true, we can take two sequences $\{\lambda_n\}_{n \in \mathbb{N}}$, $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, with $\lambda_n \underset{n \to \infty}{\rightarrow} \lambda_1^+$, $\tau_n \underset{n \to \infty}{\rightarrow} +\infty$ such that
\[
\lim_{n \to \infty} J_{\lambda_n, \tau_n}(\phi_{\lambda_n, \tau_n}) \leq 0,
\] (17)
Using $\phi_{\lambda_n, \tau_n}$ as test function in $-\Delta_p(\phi_{\lambda_n, \tau_n}) - \lambda_n|\phi_{\lambda_n, \tau_n}|^{p-2}\phi_{\lambda_n, \tau_n} = \phi_1^{p-1} + \tau_n^{-1}H$
we get
\[
\left\| \phi_{\lambda_n, \tau_n} \right\|_{1,p}^p - \lambda_n \left\| \phi_{\lambda_n, \tau_n} \right\|_p^p = \int_{\Omega} \left( \phi_1^{p-1} + \tau_n^{-1}H \right) \phi_{\lambda_n, \tau_n} \, dx.
\] (18)
Moreover, according to Proposition 3.2,
\[
\phi_{\lambda_n, \tau_n} = -\frac{\phi_1 + v_n^-}{\mu_n},
\] (19)
where $\int_{\Omega} \phi_1^{p-1} v_n^\perp \, dx = 0$, $v_n^\perp \underset{n \to \infty}{\rightarrow} 0$, strongly in $C^{1,\beta}(\Omega)$, $\mu_n \underset{n \to \infty}{\rightarrow} +\infty$.
Assume that $n$ is sufficiently large to have $\phi_{\lambda_n, \tau_n} < 0$ in $\Omega$. Then, recalling also that $\int_{\Omega} H \phi_1 \, dx = 0$,
\[
J_{\lambda_n, \tau_n}(\phi_{\lambda_n, \tau_n}) \underset{n \to \infty}{\rightarrow} 0 \quad \frac{1}{p} \left\| \phi_{\lambda_n, \tau_n} \right\|_{1,p}^p - \frac{\lambda}{p} \left\| \phi_{\lambda_n, \tau_n} \right\|_p^p = \int_{\Omega} \left( \phi_1^{p-1} + \tau_n^{-1}H \right) \phi_{\lambda_n, \tau_n} \, dx
\] (18) \[
= \int_{\Omega} \phi_1^{p-1} + \frac{H}{\tau_n^n} \phi_{\lambda_n, \tau_n} \, dx \overset{\text{(19)}}{=} \frac{1}{\mu_n} \left( 1 - \frac{1}{p} \right) \int_{\Omega} \phi_1^{p-1} + \frac{H}{\tau_n} \, dx
\]
\[
= \frac{1}{\mu_n} \left( 1 - \frac{1}{p} \right) \left( \int_{\Omega} \phi_1^p + \frac{1}{\tau_n} \int_{\Omega} H v_n^\perp \, dx \right) \geq \frac{1}{\mu_n} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{\|H\|_{1,p}}{\tau_n} \right) \overset{n \to \infty}{\rightarrow} +\infty,
\]
a contradiction with (17).

\[
\square
\]

3.1 Behavior of $J_{\lambda, \tau}$ near $\phi_{\lambda, \tau}$

**Proposition 3.5.** There is $\Lambda^3 \in (\lambda_1, \lambda_2)$ such that for any $\lambda \in (\lambda_1, \Lambda^3)$ there exists $\tau_\lambda^3 > 0$ such that for any $\tau > \tau_\lambda^3$
\[
\inf_{\|h\|_{1,p} = 1} J_{\lambda, \tau}(\phi_{\lambda, \tau} + \tau^{\frac{1}{p-1}} h) > J_{\lambda, \tau}(\phi_{\lambda, \tau}) + \xi \tau^{-\frac{p}{p-1}},
\]
where $\xi > 0$ is a fixed constant.

**Proof.** If the assertion is not true, we can suppose the existence of three sequences $\{\lambda_n\}_{n \in \mathbb{N}}$, $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{h_n\}_{n \in \mathbb{N}} \subset \phi_{\lambda_n, \tau_n}^\perp$, such that $\|h_n\|_{1,p} = 1$ for all $n$, $\lambda_n \underset{n \to \infty}{\rightarrow} \lambda_1^+$, $\tau_n \underset{n \to \infty}{\rightarrow} +\infty$,
\[
\lim_{n \to \infty} J_{\lambda_n, \tau_n}(\phi_{\lambda_n, \tau_n} + \tau_n h_n) - J_{\lambda_n, \tau_n}(\phi_{\lambda_n, \tau_n}) \leq 0
\] (20)
where for all \( n \) we have set \( r_n := \tau_n^{-\frac{1}{p-1}} \), and \( \phi_{n, \tau_n} \) is for all \( n \) an arbitrary solution of

\[
\begin{cases}
-\Delta_p u - \lambda_n |u|^{p-2} u = \phi_n^{p-1} + \tau_n^{-1} H, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

To simplify the notations, we put \( \phi_n := \phi_{n, \tau_n} \), for all \( n \in \mathbb{N} \). Using Proposition 3.2 we can assume that \( n \) is sufficiently large to have \( \phi_n < 0 \), in \( \Omega \). Moreover, since \( \|h_n\|_{1,p} = 1 \) for all \( n, h_n \xrightarrow{n \to \infty} H \), weakly in \( W_0^{1,p}(\Omega) \) (up to a subsequence).

Now we give some estimates of

\[
(21)
\]

Next, we can assume that

\[
(22)
\]

and, recalling the definition \( r_n = \tau_n^{-\frac{1}{p-1}} \),

\[
(23)
\]

and, recalling the definition \( r_n = \tau_n^{-\frac{1}{p-1}} \),

\[
(24)
\]

Joining (21), (22), (23) and (24) one obtains

\[
(25)
\]
Now by definition
\[
J_{\lambda_n, \tau_n}(\phi_n) = \frac{1}{p} \left\| \phi_n \right\|_{1,p}^p - \frac{\lambda_n}{p} \left\| \phi_n \right\|_p^p - \int_\Omega \left( \phi_n^{p-1} + \tau_n^{-1} H \right) \phi_n \, dx .
\] (26)

Moreover, since \( \phi_n \) is for all \( n \) a solution of
\[
\begin{cases}
- \Delta_p u - \lambda_n |u|^{p-2} u = \phi_n^{p-1} + \tau_n^{-1} H, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and \( h_n \in \langle \phi_n^{p-1} \rangle_\perp \), we have also
\[
\int_\Omega |\nabla \phi_n|^{p-2} \langle \nabla \phi_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx - \int_\Omega \left( \phi_n^{p-1} + \tau_n^{-1} H \right) h_n \, dx = \lambda_n \int_\Omega |\phi_n|^{p-2} \phi_n h_n \, dx = 0 .
\] (27)

Hence from (25) we infer
\[
\lim_{n \to \infty} \frac{J_{\lambda_n, \tau_n}(\phi_n + r_n h_n) - J_{\lambda_n, \tau_n}(\phi_n)}{r_n^p} \geq \frac{1}{p} \frac{1}{2^{p-1} - 1} > 0 ,
\]
in contradiction with (20).

**Step 2.** Therefore we may assume that \( h_n \overset{n \to \infty}{\longrightarrow} \bar{h} \neq 0 \), weakly in \( W_0^{1,p}(\Omega) \). In particular \( \|h_n\|_q^q < K \), and, since \( r_n = \tau_n^{1/p} \) we get
\[
\frac{\tau_n^{q/p-1}}{q} \| (\phi_n + r_n h_n) + \| q \phi_n < 0 \|_{\tau_n^{q/p-1}} \| r_n h_n \|_{\tau_n^{q/p-1}}^q \leq \frac{\tau_n}{q} K \overset{p=2}{=} o(r_n^2) .
\] (28)

Due to the asymptotic estimate in Proposition 3.2,
\[
\phi_n = \frac{\psi_n}{\mu_n}
\] (29)

where
\[
\mu_n = (\lambda_n - \lambda_1)^{\frac{1}{p-1}} + o((\lambda_n - \lambda_1)^{\frac{1}{p-1}}) , \quad \psi_n \overset{n \to \infty}{\rightharpoonup} \phi_1, \text{ strongly in } C^{1,\beta}(\bar{\Omega}) .
\]

Observe that \( \mu_n \overset{n \to \infty}{\rightharpoonup} 0^+ \), since \( \lambda_n \overset{n \to \infty}{\rightharpoonup} \lambda_1^+ \). By (29)
\[
\| \phi_n + r_n h_n \|_{1,p}^p = \mu_n^{-p} \| \psi_n + \mu_n r_n h_n \|_{1,p}^p,
\] (30)

We now make second order Taylor expansions as follows:
\[
\| \psi_n + \mu_n r_n h_n \|_{1,p}^p = \| \psi_n \|_{1,p}^p + (\mu_n r_n) p \int_\Omega |\nabla \psi_n|^{p-2} \langle \nabla \psi_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx
\]
\[
+ (\mu_n r_n)^2 p \int_\Omega \langle \Theta(\nabla \psi_n) \nabla h_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx + o((\mu_n r_n)^2) ,
\] (31)
Now we claim that, as \( n \to \infty \),

\[
\left\| \psi_n + \mu_n r_n h_n \right\|_p^p = \left\| \psi_n \right\|_p^p + (\mu_n r_n)p \int_\Omega |\psi_n|^{p-2} \psi_n h_n \, dx \\
+ (\mu_n r_n)^2 p(p-1) \int_\Omega |\psi_n|^{p-2} |h_n|^2 \, dx + o((\mu_n r_n)^2) .
\] (32)

Inserting (30), (31), (32) and (28) in (21) one gets

\[
J_{\lambda_n, r_n} \left( \phi_n + r_n h_n \right) - J_{\lambda_n, r_n} (\phi_n) \\
\geq \frac{r^2_n}{\mu_n^{p-2}} \left( \int_\Omega \Theta (\nabla \psi_n) \nabla h_n, \nabla h_n \right)_{\mathbb{R}^N} \, dx - \lambda_n (p-1) \int_\Omega |\psi_n|^{p-2} |h_n|^2 \, dx \right) + o(r^2_n) \mu_n^{-p-2} \\
+ r_n \left( \frac{p}{p-1} \int_\Omega |h_n|^{p-2} \nabla \phi_n \cdot \nabla h_n \right)_{\mathbb{R}^N} \, dx - \int_\Omega \Theta \left( \phi_n \right) \nabla h_n \cdot \nabla h_n \, dx \right) \\
+ \left( \frac{1}{p} \left\| \phi_n \right\|_{1,p}^p - \frac{\lambda_n}{p} \left\| \phi_n \right\|_p^p \right) \left( \frac{p}{p-1} \int_\Omega (\phi_n^{p-1} + \tau_n^{-1} H) h_n \, dx - J_{\lambda_n, r_n} (\phi_n) \right),
\]

that is, from (26), (27),

\[
J_{\lambda_n, r_n} \left( \phi_n + r_n h_n \right) - J_{\lambda_n, r_n} (\phi_n) \\
\geq \frac{r^2_n}{\mu_n^{p-2}} \left( \int_\Omega \Theta (\nabla \psi_n) \nabla h_n, \nabla h_n \right)_{\mathbb{R}^N} \, dx - \lambda_n (p-1) \int_\Omega |\psi_n|^{p-2} |h_n|^2 \, dx \right) + o(r^2_n) \mu_n^{-p-2} .
\] (33)

**Step 3.** Now we claim that, as \( n \to \infty \),

\[
\int_\Omega \Theta (\nabla \psi_n) \nabla h_n, \nabla h_n \right)_{\mathbb{R}^N} \, dx - \lambda_n (p-1) \int_\Omega |\psi_n|^{p-2} |h_n|^2 \, dx > \kappa > 0 .
\] (34)

This, according to (33), would carry

\[
\frac{J_{\lambda_n, r_n} \left( \phi_n + r_n h_n \right) - J_{\lambda_n, r_n} (\phi_n)}{r^2_n} \geq (r_n \mu_n)^{2-p} \left( \kappa + o(1) \right)^{n \to \infty} + \infty ,
\] (35)

a contradiction with (20). Since \( \psi_n \overset{n \to \infty}{\rightharpoonup} \phi_1 \), strongly in \( C^{1,\beta} (\overline{\Omega}) \), we can apply Lemma 2.9, getting

\[
\int_\Omega \Theta (\nabla \psi_n) \nabla h_n, \nabla h_n \right)_{\mathbb{R}^N} \, dx - \lambda_n (p-1) \int_\Omega |\psi_n|^{p-2} |h_n|^2 \, dx \\
\geq \int_\Omega \Theta (\nabla \phi_1) \nabla h_n, \nabla h_n \right)_{\mathbb{R}^N} \, dx - \lambda_n (p-1) \int_\Omega \phi_1^{p-2} |h_n|^2 \, dx \geq \delta_n,
\] (36)

where \( \delta_n \overset{n \to \infty}{\to} 0^+ \). Now from the weak convergence of \( h_n \) to \( h \) in \( W_0^{1,p} (\Omega) \) we infer \( h_n \overset{n \to \infty}{\rightharpoonup} h \) strongly in \( L^2 (\Omega) \) and, consequently,

\[
\lim_{n \to \infty} \int_\Omega \phi_1^{p-2} |h_n|^2 \, dx = \int_\Omega \phi_1^{p-2} |h|^2 \, dx .
\] (37)
Moreover, by Lemma 2.6,

$$\lim_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \phi_1) \nabla h_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx \right) \geq \int_{\Omega} \langle \Theta(\nabla \phi_1) \nabla \overline{h}, \nabla \overline{h} \rangle_{\mathbb{R}^N} \, dx .$$

(38)

Joining (36), (37) and (38), and recalling that \( \lambda_n \xrightarrow{n \to \infty} \lambda_1^+ \) by hypothesis, we obtain

$$\lim_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \psi_n) \nabla h_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx - \lambda_n(p-1) \int_{\Omega} |\psi_n|^{p-2} |h_n|^2 \, dx \right)$$

$$\geq \int_{\Omega} \langle \Theta(\nabla \phi_1) \nabla \overline{h}, \nabla \overline{h} \rangle_{\mathbb{R}^N} \, dx - \lambda_1(p-1) \int_{\Omega} |\phi_1|^{p-2} |\overline{h}|^2 \, dx .$$

(39)

As in (29) we now rewrite \( \phi_n = \frac{\psi_n}{\mu_n} \), where \( \psi_n \xrightarrow{n \to \infty} -\phi_1 \) in \( C^1,\beta(\Omega) \), \( \{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \). Since \( h_n \in \langle \phi_1^{p-1} \rangle_\perp \) for all \( n \), in particular we must have \( \int_\Omega |\psi_n|^{p-2} \psi_n h_n \, dx = 0 \), for all \( n \). As a consequence

$$\left| \int_{\Omega} \phi_1^{p-1} \overline{h} \, dx \right| \xrightarrow{n \to \infty} \lim_{n \to \infty} \left| \int_{\Omega} \left( \phi_1^{p-1} + |\psi_n|^{p-2} \psi_n \right) h_n \, dx \right| = 0 ,$$

namely \( \overline{h} \in \langle \phi_1^{p-1} \rangle_\perp \). Then we obtain from Lemma 2.8 that

$$\int_{\Omega} \langle \Theta(\nabla \phi_1) \nabla \overline{h}, \nabla \overline{h} \rangle_{\mathbb{R}^N} \, dx \geq \Lambda_\infty(p-1) \int_{\Omega} |\phi_1|^{p-2} |\overline{h}|^2 \, dx ,$$

(40)

where \( \Lambda_\infty > \lambda_1 \). Inserting (40) in (39), one finally gets

$$\lim_{n \to \infty} \left( \int_{\Omega} \langle \Theta(\nabla \psi_n) \nabla h_n, \nabla h_n \rangle_{\mathbb{R}^N} \, dx - \lambda_n(p-1) \int_{\Omega} |\psi_n|^{p-2} |h_n|^2 \, dx \right)$$

$$\geq (p-1)(\Lambda_\infty - \lambda_1) \int_{\Omega} |\phi_1|^{p-2} |\overline{h}|^2 \, dx > \kappa > 0 ,$$

since \( \phi_1 > 0 \) in \( \Omega \) and, by Step 1, \( \overline{h} \neq 0 \). That is, claim (34) is proved, and then we deduce (35), which contradicts (20). This completes the proof. \( \square \)

### 3.2 Properties of the solution \( \phi_{\lambda, \tau} \)

**Proposition 3.6.** There is \( \Lambda \in (\lambda_1, \lambda_2) \) such that for any \( \lambda \in (\lambda_1, \Lambda) \) there exists \( \tau_\lambda > 0 \) such that for any \( \tau > \tau_\lambda \) problem (6) admits a solution \( \phi_{\lambda, \tau} \) with the following properties:

(i) \( \phi_{\lambda, \tau} < 0 \), in \( \Omega \).

(ii) \( J_{\lambda, \tau}(\phi_{\lambda, \tau}) > 0 \).

(iii) There exists \( r > 0 \) such that \( \inf_{h \in \langle \phi_1^{p-1} \rangle_\perp \, \text{and} \, \|h\|_{1,p} = 1} J_{\lambda, \tau}(\phi_{\lambda, \tau} + rh) > J_{\lambda, \tau}(\phi_{\lambda, \tau}) \).

**Proof.** This is a direct consequence of Corollary 3.3, Corollary 3.4 and Proposition 3.5. \( \square \)
4 The linking structure

From now on we will suppose \( \lambda \in (\lambda_1, \Lambda) \) and \( \tau > \tau_\lambda \) to be fixed, where \( \Lambda \) and \( \tau_\lambda \) are those defined in Proposition 3.6. Moreover we will simplify the notations as follows:

\[
\varphi = \phi_{\lambda, \tau}, \quad \theta = \tau^{\frac{q}{p-2}} \quad F = \varphi_{p-1}^p + \frac{1}{\tau}H, \quad J = J_{\lambda, \tau}.
\]

Then

\[
J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{p} \int_\Omega |u|^p dx - \frac{\theta}{q} \int_\Omega (u^+)^q dx - \int_\Omega Fudx, \quad u \in W_0^{1,p}(\Omega).
\]

(41)

According to Proposition 3.6, \( \varphi \) is a negative solution to the problem

\[
\begin{cases}
- \Delta_p u = \lambda |u|^{p-2}u + \theta (u^+)^{q-1} + F, & \text{in } \Omega \\
0 & \text{on } \partial \Omega,
\end{cases}
\]

(42)

\[
J(\varphi) > 0 \quad \text{and there exists } r > 0 \text{ such that } \inf_{h \in \partial B_r(0) \cap \langle \phi \rangle} J(\varphi + h) > J(\varphi),
\]

(43)

that is, property [A] of the Generalized Mountain Pass Theorem is fulfilled. Now we will deal with property [B].

First of all we compute the value \( J(\varphi) > 0 \). Since \( \varphi \) satisfies

\[
- \Delta_p u = \lambda |u|^{p-2}u + \theta (u^+)^{q-1} + F
\]

one obtains

\[
0 < \left( \frac{p}{p-1} \right) J(\varphi) = - \left( \|\|\phi\|_p^p - \lambda \|\|\phi\|_p^p \right) - \int_\Omega F\varphi dx \quad \text{(44)}
\]

4.1 Behavior of \( J \) in the subspace \( \langle \varphi \rangle \)

**Proposition 4.1.** \( \varphi \) is a strict maximum point for the functional \( J \) on \( \langle \varphi \rangle \) namely

\[
J(t\varphi) < J(\varphi), \quad \text{for any } t \neq 1.
\]

**Proof.** Denoting with \( t^- = \max(-t, 0) \),

\[
J(t\varphi) = \left( \frac{p}{p-1} \right) J(\varphi) - \left( \lambda \|\|\phi\|_p^p - \lambda \|\|\phi\|_p^p \right) - t \int_\Omega F\varphi dx \quad \text{(44)}
\]

Define \( f(t) := \left( t - \frac{|t|^p}{p} \right) \), \( t \in \mathbb{R} \). Then clearly \( t = 1 \) is a strict maximum point for \( f \).

As a consequence, for all \( t \neq 1 \),

\[
J(t\varphi) \leq f(t) \left( \left( \frac{p}{p-1} \right) J(\varphi) \right) < f(0) \left( \left( \frac{p}{p-1} \right) J(\varphi) \right) = J(\varphi).
\]

\[\square\]
4.2 Behavior of $J$ far away from $\varphi$

**Definition 4.1.** For any $\varepsilon \in [0, \varepsilon_\Omega]$ define the two following subsets of $\Omega$

$$
\Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \},
$$

$$
\Omega'_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon \},
$$

and then

$$
\mathcal{O}_\varepsilon := \left\{ \psi \in W_0^{1,p}(\Omega) : \psi(x) = 0, \text{ for all } x \in \Omega_\varepsilon \right\}, \int_\Omega |\varphi|^{p-1}\psi dx = \int_\Omega |\varphi|^p dx,
$$

where $\varepsilon_\Omega$ is supposed to be small enough to have $\mathcal{O}_{\varepsilon_\Omega} \neq \emptyset$.

For any $\varepsilon \in [0, \varepsilon_\Omega]$ $\mathcal{O}_\varepsilon$ is a closed convex subset of $W_0^{1,p}(\Omega)$, and as a consequence there exists $\psi_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$
\|\varphi - \psi_\varepsilon\|_{1,p} = \min_{\psi \in \mathcal{O}_\varepsilon}\|\varphi - \psi\|_{1,p}.
$$

Observe that $\psi_\varepsilon \neq \varphi$ for any $\varepsilon \in (0, \varepsilon_\Omega]$, since $|\varphi| > 0$ in $\Omega_\varepsilon$. On the other hand

$$
\left\|\varphi - \psi_\varepsilon\right\|_{1,p} = \min_{\psi \in \mathcal{O}_\varepsilon}\left\|\varphi - \psi\right\|_{1,p} \xrightarrow{\varepsilon \to 0} \min_{\psi \in \mathcal{O}_0}\left\|\varphi - \psi\right\|_{1,p} \equiv 0. \ (45)
$$

**Lemma 4.2.** For any $\varepsilon \in [0, \varepsilon_\Omega]$ define

$$
h_\varepsilon(x) := |\varphi(x)| - \psi_\varepsilon(x), \ x \in \Omega.
$$

Then

[a] $h_\varepsilon \in \langle \varphi^{p-1} \rangle^\perp$.

[b] $\|h_\varepsilon\|_{1,p} \xrightarrow{\varepsilon \to 0} 0$.

[c] $h_\varepsilon(x) = |\varphi(x)|$, for all $x \in \Omega_\varepsilon$.

**Proof.** Property [a] is given by

$$
-\int_\Omega |\varphi|^{p-2}\varphi h_\varepsilon dx \xRightarrow{\varepsilon \to 0} \int_\Omega |\varphi|^{p-1} h_\varepsilon dx = \int_\Omega |\varphi|^p dx - \int_\Omega |\varphi|^{p-1} \psi_\varepsilon dx \psi_\varepsilon \leq \mathcal{O}_\varepsilon 0,
$$

[b] follows from (45) and [c] is true by construction, since $\psi_\varepsilon \equiv 0$ in $\Omega_\varepsilon$.

**Proposition 4.3.** There exists $\varepsilon \in (0, \varepsilon_\Omega]$ such that for any $\mu \in \mathbb{R}$

$$
J\left(t(\mu \varphi + h_\varepsilon)\right) \xrightarrow{t \to +\infty} -\infty.
$$

**Proof.** By definition, for any $\varepsilon \in (0, \varepsilon_\Omega]$, $t > 0$,

$$
J\left(t(\mu \varphi + h_\varepsilon)\right) = \frac{1}{p}\left\|t(\mu \varphi + h_\varepsilon)\right\|_{1,p}^p - \frac{\lambda}{p}\left\|t(\mu \varphi + h_\varepsilon)\right\|_p^p
$$

$$
-\frac{\theta}{q}\int_\Omega \left((t(\mu \varphi + h_\varepsilon))^+\right)_dx - t\int_\Omega F(\mu \varphi + h_\varepsilon)dx
$$

$$
\leq t^p\left(\frac{1}{p}\|\mu \varphi + h_\varepsilon\|_{1,p}^p - \frac{\lambda}{p}\|\mu \varphi + h_\varepsilon\|_p^p\right) - t^q\left(\frac{\theta}{q}\int_\Omega ((\mu \varphi + h_\varepsilon)^+)_dx\right) + t\Delta_{\mu,\varepsilon},
$$

(46)
where we have denoted $D_{\mu,\varepsilon} := \|F\|_{-1,p'} \left( \|\mu\|_1 \|\varphi\|_{1,p} + \|h_\varepsilon\|_{1,p} \right)$.

**First case:** $\mu > \frac{1}{2}$. It follows from Lemma 4.2 (property [b]) that $\|h_\varepsilon\|_{1,p} \xrightarrow{\varepsilon \to 0} 0$. Hence by the continuity of the norm

$$
\|\mu \varphi + h_\varepsilon\|_{1,p}^p - \lambda \|\mu \varphi + h_\varepsilon\|_p^p = \|\mu \varphi\|_{1,p}^p - \lambda \|\mu \varphi\|_p^p + \kappa(\varepsilon)
$$

where $A = \left(\frac{p}{p-1}\right) J(\varphi)$ is a positive constant, $\kappa(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$.

Choosing $\varepsilon$ such that $|\kappa(\varepsilon)| < \frac{1}{4p} A$, we get

$$
J\left(t(\mu \varphi + h_\varepsilon)\right) \leq \frac{p}{2} \left(-|\mu|^p A + \kappa(\varepsilon)\right) + tD_{\mu,\varepsilon} \xrightarrow{\mu \to \frac{1}{2}} \frac{p}{2} \left(-\frac{1}{2p} A + \kappa(\varepsilon)\right) + tD_{\mu,\varepsilon} \xrightarrow{\varepsilon \to +\infty} -\infty.
$$

**Second case:** $\mu \leq \frac{1}{2}$. It follows from Lemma 4.2 (property [c]) that

$$(\mu \varphi + h_\varepsilon)^+ > \left( \frac{1}{2} \varphi + |\varphi| \right)^+ > 0 \quad \text{in} \ \Omega_\varepsilon.$$ 

As a consequence

$$
-t^q \left( \frac{\theta}{q} \int_{\Omega} \left( (\mu \varphi + h_\varepsilon)^+ \right)^q dx \right) < -t^q \left( \frac{\theta}{q} \frac{1}{2p} \int_{\Omega} |\varphi|^q dx \right) = -t^q K_\varepsilon, \quad (47)
$$

where $K_\varepsilon$ is a positive constant. Setting

$$
C_{\mu,\varepsilon} = \frac{1}{p} \int_{\Omega} |\nabla (\mu \varphi + h_\varepsilon)|^p dx - \frac{\lambda}{p} \int_{\Omega} |\mu \varphi + h_\varepsilon|^p dx, \quad (48)
$$

we obtain, inserting (47) and (48) in (46),

$$
J\left(t(\mu \varphi + h_\varepsilon)\right) \leq -t^q K_\varepsilon + t^p C_{\mu,\varepsilon} + tD_{\mu,\varepsilon} \xrightarrow{t \to +\infty} -\infty,
$$

since $q > p > 1$.

**Corollary 4.4.** If $c_1, c_2 \in \mathbb{R}$, $c_1 \leq c_2$, then

$$
\sup_{\rho \in [c_1,c_2]} J\left(t(\mu \varphi + h_\varepsilon)\right) \xrightarrow{t \to +\infty} -\infty.
$$

**Proposition 4.5.** There exists $\varrho > 0$ such that, for any $\rho \geq \varrho$,

1. $J\left(\varphi + \rho \varphi + \zeta \ h_\varepsilon\right) \leq J(\varphi)$, for all $\zeta \in [0, \rho]$.
2. $J\left(\varphi - \rho \varphi + \zeta \ h_\varepsilon\right) \leq J(\varphi)$, for all $\zeta \in [0, \rho]$.
3. $J\left(\varphi + \zeta \varphi + \rho \ h_\varepsilon\right) \leq J(\varphi)$, for all $\zeta \in [-\rho, \rho]$.
Proof. **Step 1.** We prove (i) (the proof of (ii) is basically the same). By contradiction, suppose that there exists a sequence \( \{(R_n, \zeta_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) with

\[
R_n \to +\infty , \quad \zeta_n \in [0, R_n] , \quad (49)
\]

\[J(\varphi + R_n \varphi + \zeta_n h_\epsilon) > J(\varphi) , \text{ for all } n \in \mathbb{N}. \quad (50)
\]

Dividing (50) by \( R_n^p \) and taking the limit as \( n \to \infty \) we obtain

\[
limit_{n \to +\infty} \frac{J(\varphi + R_n \varphi + \zeta_n h_\epsilon) - J(\varphi)}{R_n^p} \geq 0 . \quad (51)
\]

Now suppose that \( \frac{\zeta_n}{R_n} \to 0 \). In view of (41),

\[
\frac{J(\varphi + R_n \varphi + \zeta_n h_\epsilon) - J(\varphi)}{R_n^p} \leq \frac{1}{p} \left( \left\| \varphi + \left( \frac{1}{R_n} \varphi + \frac{\zeta_n}{R_n} h_\epsilon \right) \right\|^p_1 - \frac{\lambda}{p} \left\| \varphi + \left( \frac{1}{R_n} \varphi + \frac{\zeta_n}{R_n} h_\epsilon \right) \right\|^p_p \right)
\]

\[
- \int_\Omega F\left(1 + \frac{R_n}{R_n^p} \varphi + \frac{\zeta_n}{R_n^p} h_\epsilon \right) dx - \frac{J(\varphi)}{R_n^p}. \quad (52)
\]

Taking the limit as \( n \to \infty \) in (52) we get, by the continuity of the norm,

\[
\lim_{n \to +\infty} \frac{J(\varphi + R_n \varphi + \zeta_n h_\epsilon) - J(\varphi)}{R_n^p} = \frac{1}{p} \left( \left\| \varphi \right\|^p_p - \lambda \left\| \varphi \right\|^p_p \right) = -\left( 1 - p \right) J(\varphi) < 0 ,
\]

which contradicts (51). As a consequence it must be \( \frac{\zeta_n}{R_n} \not\to 0 \), and therefore we infer the existence of \( \alpha_1, \alpha_2 \in \mathbb{R} \) such that

\[
0 < \alpha_1 = \liminf_{n \to +\infty} \frac{\zeta_n}{1 + R_n} \leq \limsup_{n \to +\infty} \frac{\zeta_n}{1 + R_n} = \alpha_2 \in [0, R_n] , \quad \alpha_2 \leq 1 , \quad (53)
\]

\[
1 \leq \frac{1}{\alpha_2} = \liminf_{n \to +\infty} \frac{1 + R_n}{\zeta_n} \leq \limsup_{n \to +\infty} \frac{1 + R_n}{\zeta_n} = \frac{1}{\alpha_1} < +\infty .
\]

Observe that, in particular, \( \frac{\zeta_n}{R_n} \to +\infty \). Now, joining (50), (53) and **Corollary 4.4** we get

\[
0 \leq \lim_{n \to +\infty} J(\varphi + R_n \varphi + \zeta_n h_\epsilon) - J(\varphi) = \lim_{n \to +\infty} J\left( \frac{1 + R_n}{\zeta_n} \varphi + h_\epsilon \right) - J(\varphi) \quad (50)
\]

\[
\leq \lim_{n \to +\infty} \left\{ \sup_{\mu \in \left[ \frac{1}{\alpha_1^2}, \frac{1}{\alpha_1} \right]} J\left( \frac{1 + R_n}{\zeta_n} \varphi + h_\epsilon \right) - J(\varphi) \right\} \quad \text{**Corollary 4.4**} \leq -\infty ,
\]

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which is a contradiction. Then the sequence \( \{ (R_n, \zeta_n) \}_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) cannot exist.

**Step 2.** Proof of (iii). By contradiction, suppose that there exists a sequence \( \{ (\zeta_n, R_n) \}_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) with

\[
R_n \xrightarrow{n \to \infty} +\infty, \quad \zeta_n \in [-R_n, R_n],
\]

\[
J(\varphi + \zeta_n \varphi + R_n h_\epsilon) > J(\varphi), \quad \text{for all } n \in \mathbb{N}.
\]

Dividing (55) by \( R_n^p \) and taking the limit as \( n \to \infty \) we obtain

\[
\lim_{n \to +\infty} \frac{J(\varphi + \zeta_n \varphi + R_n h_\epsilon) - J(\varphi)}{R_n^p} \geq 0.
\]

Now suppose that \( \frac{|\zeta_n|}{R_n} \xrightarrow{n \to +\infty} 0 \). By Lemma 4.2 (property [c])

\[
h_\epsilon \equiv |\varphi|, \quad \text{in } \Omega_\epsilon,
\]

therefore we may assume

\[
\left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon > \frac{1}{2} |\varphi|, \quad \text{in } \Omega_\epsilon,
\]

for any \( n \) sufficiently large. In view of (41) one has

\[
J\left( \varphi + \zeta_n \varphi + R_n h_\epsilon \right) - J(\varphi) = \frac{1}{p} \left\| \left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon \right\|_p^p - \frac{\lambda}{p} \left\| \left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon \right\|_p^p
\]

\[-R_n^{q-p} \frac{\theta}{q} \int_\Omega \left( \left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon \right)^q dx - \int_\Omega F\left( \frac{1 + \zeta_n}{R_n} \varphi + \frac{1}{R_n^p} h_\epsilon \right) dx - \frac{J(\varphi)}{R_n^p},
\]

and, inserting (57) in (58), we get

\[
J\left( \varphi + \zeta_n \varphi + R_n h_\epsilon \right) - J(\varphi) \leq \frac{1}{p} \left\| \left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon \right\|_p^p - \frac{\lambda}{p} \left\| \left( \frac{1 + \zeta_n}{R_n} \right) \varphi + h_\epsilon \right\|_p^p
\]

\[-R_n^{q-p} \frac{\theta}{q} \frac{1}{2^q} \int_\Omega |\varphi|^q dx - \int_\Omega F\left( \frac{1 + \zeta_n}{R_n} \varphi + \frac{1}{R_n^p} h_\epsilon \right) dx - \frac{J(\varphi)}{R_n^p}.
\]

Taking the limit as \( n \to \infty \) in (59) we obtain

\[
\lim_{n \to +\infty} \frac{J\left( \varphi + \zeta_n \varphi + R_n h_\epsilon \right) - J(\varphi)}{R_n^p}
\]

\[\leq \frac{1}{p} \left( \|h_\epsilon\|_p^p - \lambda \|h_\epsilon\|_p^p \right) - \lim_{n \to +\infty} R_n^{q-p} \frac{\theta}{q} \frac{1}{2^q} \int_\Omega |\varphi|^q dx = -\infty,
\]

in contradiction with (56). As a consequence it must be \( \frac{|\zeta_n|}{R_n} \xrightarrow{n \to +\infty} 0 \), that is

\[-1 \leq \gamma_1 = \inf_{n \to +\infty} \frac{1 - |\zeta_n|}{R_n} \leq \sup_{n \to +\infty} \frac{1 + |\zeta_n|}{R_n} = \gamma_2 \leq 1.\]
Observe that, in particular, $|ζ_n| \to +\infty$. Using (55), (60) and Corollary 4.4 we get

$$0 \leq \lim_{n \to \infty} J(φ + ζ_n φ + R_n h_n) - J(φ) = \lim_{n \to \infty} J(R_n \left(\frac{1 + ζ_n}{R_n} φ + h_n\right)) - J(φ) \tag{55}$$

and

$$\leq \lim_{n \to \infty} \left\{ \sup_{μ \in [γ_1, γ_2]} J(R_n (μ φ + h_n)) - J(φ) \right\} \tag{60}$$

Corollary 4.4

which is a contradiction. Then the sequence $\{(R_n, ζ_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^2$ cannot exist. This completes the proof. □

5 Conclusion

5.1 The Palais-Smale condition

Proposition 5.1. Let $p > 2$. The functional $J : W_0^{1,p}(Ω) \to \mathbb{R}$ satisfies the Palais-Smale condition, namely if $\{u_n\}_{n \in \mathbb{N}} \subset W_0^{1,p}(Ω)$ is a sequence such that

$$|J(u_n)| < M \tag{61}$$

$$J'(u_n) \rightharpoonup 0 \tag{62}$$

then $\{u_n\}_{n \in \mathbb{N}}$ has a convergent subsequence.

Proof. We have

$$J(u_n) = \frac{1}{p} \int_Ω |∇u_n|^p dx - \frac{λ}{p} \int_Ω |u_n|^p dx - \frac{θ}{q} \int_Ω (u_n^+)^q dx - \int_Ω F u_n dx \tag{62}$$

$$\langle J'(u_n), v \rangle = \int_Ω |∇u_n|^{p-2} (∇u_n, v) dx \geq -\lambda \int_Ω |u_n|^{p-2} u_n v dx - \theta \int_Ω (u_n^+)^{q-2} u_n v dx \tag{63}$$

The condition $J'(u_n) \rightharpoonup 0$ means that for any $ε > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$|\langle J'(u_n), v \rangle| < ε \|v\|_{1,p}, \text{ for all } v \in W_0^{1,p}(Ω), n \geq N_0. \tag{64}$$

Hence the hypotheses (61) on $\{u_n\}_{n \in \mathbb{N}}$ imply

$$M + ε \|u_n\|_{1,p} \geq J(u_n) - \frac{1}{p} \langle J'(u_n), u_n \rangle \stackrel{(62),(63)}{=} \theta \left(1 - \frac{1}{q}\right) \int_Ω (u_n^+)^q dx. \tag{65}$$

Now we claim that $\|u_n\|_{1,p}$ is bounded.

To prove this, assume on the contrary that $\|u_n\|_{1,p} \to +\infty$. Dividing inequality (65) by $\|u_n\|_{1,p}^{q-1}$, and noting that $q > p > 2 \Rightarrow \frac{q(q-1)}{q-1} > 1$

$$M + ε \|u_n\|_{1,p} \geq M + \frac{ε}{\|u_n\|_{1,p}^{1-q}}, \text{ for all } n \geq N_0 \tag{66}$$

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we find
\[ \int_{\Omega} \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right)^{\frac{1}{q-1}} \, dx \xrightarrow{n \to \infty} 0, \]
which is the same as
\[ \int_{\Omega} \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right)^{\frac{1}{q-1}} \, dx \xrightarrow{n \to \infty} 0, \quad \text{that is,} \quad \left\| (u_n^+)^{q-1} \right\|_{\frac{p}{q-1}} \xrightarrow{n \to \infty} 0. \] (66)

From \( W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) = (L^{\frac{q}{q-1}}(\Omega))' \) it follows \( L^{\frac{q}{q-1}}(\Omega) \hookrightarrow (W_0^{1,p}(\Omega))' = W^{-1,p'}(\Omega) \), and then by (66)
\[ \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right)^{\frac{1}{q-1}} \xrightarrow{n \to \infty} 0, \text{ strongly in } W^{-1,p'}(\Omega). \] (67)

By assumption
\[ -\Delta_p(u_n) - \lambda \| u_n \|_{1,p}^{p-2} u_n - \theta (u_n^+)^{q-1} - F = J'(u_n) \xrightarrow{n \to \infty} 0, \text{ in } W^{-1,p'}(\Omega). \] (68)

Dividing (68) by \( \| u_n \|_{1,p}^{p-1} \) and writing \( v_n = \frac{u_n}{\| u_n \|_{1,p}} \) we get
\[ -\Delta_p v_n - \lambda \| v_n \|_{1,p}^{p-2} v_n - \theta \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right) - \| F \|_{1,p}^{p-1} = \| J'(u_n) \|_{1,p}^{p-1}. \] (69)

\( \{ v_n \}_{n \in \mathbb{N}} \) is bounded, so \( v_n \xrightarrow{n \to \infty} v \) weakly in \( W_0^{1,p}(\Omega) \). Lemma 2.3 gives
\[ -\lambda \| v_n \|_{1,p}^{p-2} v_n \xrightarrow{n \to \infty} -\lambda \| v \|_{1,p}^{p-2} v, \text{ in } W^{-1,p'}(\Omega), \] (70)
and from (67), (68), we infer
\[ \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right) \xrightarrow{n \to \infty} 0, \quad \| J'(u_n) \|_{1,p}^{p-1} \xrightarrow{n \to \infty} 0, \quad \| F \|_{1,p}^{p-1} \xrightarrow{n \to \infty} 0, \text{ in } W^{-1,p'}(\Omega). \] (71)

Therefore it follows from (70), (71), that also
\[ -\Delta_p v_n \overset{(69)}{=} \lambda \| v_n \|_{1,p}^{p-2} v_n + \theta \left( \frac{(u_n^+)^{q-1}}{\| u_n \|_{1,p}^{p-1}} \right) + \frac{F}{\| u_n \|_{1,p}^{p-1}} + \frac{J'(u_n)}{\| u_n \|_{1,p}^{p-1}} \]
converges in \( W^{-1,p'}(\Omega) \). By the continuity of \( -\Delta_p^{-1} \), \( v_n \xrightarrow{n \to \infty} v \), strongly in \( W_0^{1,p}(\Omega) \).

Passing to the limit in (69) and using the continuity of \( -\Delta_p \) we get
\[ -\Delta_p v - \lambda \| v \|_{1,p}^{p-2} v = 0, \]
a contradiction, since \( \| v_n \|_{1,p} = 1 \) for all \( n \) gives \( \| v \|_{1,p} = 1 \) and \( \lambda \) is not an eigenvalue of \( -\Delta_p \).

Hence \( \{ u_n \}_{n \in \mathbb{N}} \) is bounded in \( W_0^{1,p}(\Omega) \) and we can suppose \( u_n \xrightarrow{n \to \infty} u \) weakly in
$W^{1,p}_0(\Omega)$, $u_n \xrightarrow{n \to \infty} u$ strongly in $L^\alpha(\Omega)$, for any $\alpha \in \left(1, \frac{Np}{N-p}\right)$, by the Rellich-Kondrachov theorem. As a consequence $(u_n^\pm)^{q-1}$ and $|u_n|^{p-2}u_n$ converge in $W^{-1,p'}(\Omega)$, namely

$$-\Delta_p u_n = \lambda |u_n|^{p-2}u_n + \theta (u_n^\pm)^{q-1} + F$$

converges strongly in $W^{-1,p'}(\Omega)$, and, by the continuity of $-\Delta_p^{-1}$, $u_n \xrightarrow{n \to \infty} u$ strongly in $W^{1,p}_0(\Omega)$.

5.2 Proof of the main result

We consider the decomposition

$$W^{1,p}_0(\Omega) = \langle \phi \rangle \oplus \langle \phi^{p-1} \rangle_{-}.$$  

By (43) there exists $r > 0$ such that

$$\inf_{h \in \partial B_r(0) \cap \langle \phi^{p-1} \rangle_{-}} J(\phi + h) > J(\phi).$$

(72)

Taking $\epsilon$ as in Proposition 4.3, $h_\epsilon$ as in Lemma 4.2 ($h_\epsilon \in \langle \phi^{p-1} \rangle_{-}$ by property [a]) and $\varrho$ as in Proposition 4.5 (any $\rho \geq \varrho$ is good here, so we can assume $\varrho > r$), if

$$Q := \{ \phi + t\varphi : t \in [-\varrho, \varrho] \} \oplus \{ sh_\epsilon : 0 \leq s \leq \varrho \}$$

then $\partial Q = \bigcup_{i=1}^{4} \Sigma_i$, where

$$\Sigma_1 = \{ \varphi + s\varphi : -\varrho \leq s \leq \varrho \}, \quad \Sigma_2 = \{ \varphi + \varphi + sh_\epsilon : 0 \leq s \leq \varrho \}, \quad \Sigma_3 = \{ \varphi - \varrho \varphi + sh_\epsilon : 0 \leq s \leq \varrho \}, \quad \Sigma_4 = \{ \varphi + s\varphi + gh_\epsilon : -\varrho \leq s \leq \varrho \}.$$  

and it follows from Proposition 4.5 and Proposition 4.1 that

$$\sup_{v \in \partial Q} J(v) \leq J(\varphi).$$  

(73)

Therefore, in view of (72) and (73), hypotheses [A] and [B] of the Generalized Mountain Pass Theorem are fulfilled by $J$, which satisfies also the Palais-Smale condition, by Proposition 5.1. As a consequence we obtain a critical point $\omega$ for $J$ such that $J(\varphi) < J(\omega)$, that is, $\omega$ and $\varphi$ are two distinct weak solutions of problem (42).

References


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