Multiplicity of solutions for a superlinear p-Laplacian equation

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Abstract

We consider quasi-linear elliptic equations involving the p-Laplacian with nonlinearities which interfere asymptotically with the spectrum of the differential operator. We show that such equations have for certain forcing terms at least two solutions. Such equations are of so-called Ambrosetti-Prodi type. In particular, our theorem is a partial generalization of corresponding results for the semi-linear case by Ruf-Srikanth (1986) and de Figueiredo (1988).

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1 Introduction

In this paper we are interested in certain quasi-linear elliptic equations with nonlinearities which interfere asymptotically with the spectrum of the quasi-linear differential operator. The prototype for such problems is the famous result of Ambrosetti-Prodi [2] which says that for a semi-linear equation with a nonlinearity which crosses asymptotically the first eigenvalue of the differential operator there exist, in dependence of the forcing term, either zero, one or two solutions. In a dual situation, where the nonlinearity crosses all but the first eigenvalues, Ruf-Srikanth proved in [18] that again there are forcing terms for which there exist at least two solutions. Actually, it was conjectured by Lazer-McKenna [13] that for nonlinearities which cross infinitely many eigenvalues there exist, for any given number k, forcing terms with at least k solutions. This conjecture was recently confirmed by Dancer-Yan [8] in a surprising way by constructing solutions with an arbitrary number of peaks attached to a negative solution.

To be more precise: we consider elliptic equations of the following type

$$\begin{cases} -\Delta v = f(v) + \tau\varphi_1 + \psi \quad , & \text{in } \Omega \\ v = 0 \quad , & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\tau \in \mathbb{R}$ is a parameter, $\psi \in L^2(\Omega)$ is a fixed function, φ_1 is the first eigenfunction of $-\Delta$ corresponding to the first eigenvalue μ_1, and the continuous functions f satisfies $-\infty \leq \lim_{v\to-\infty} \frac{f(v)}{v} < \lim_{v\to+\infty} \frac{f(v)}{v} \leq +\infty$. Such equations are called problems with *jumping nonlinearities* or problems of *Ambrosetti-Prodi type*. In the pioneering work of *Ambrosetti* and *Prodi* [2] and other related articles $\big($*Berger* and *Podolak* [6], *Kazdan* and *Warner* [12], *Dancer* [7], *Amann* and *Hess* [1]$\big)$ it was shown that under the hypothesis

$$-\infty < \lim_{v\to-\infty} \frac{f(v)}{v} < \mu_1 < \lim_{v\to+\infty} \frac{f(v)}{v} < +\infty \tag{2}$$

there exists a constant τ_0 (depending on ψ) such that problem (1) has two solutions for $\tau < \tau_0$, one solution for $\tau = \tau_0$, and no solution for $\tau > \tau_0$.

On the other hand, *Ruf - Srikanth* [18] and *de Figueiredo* [9] considered instead of (2) the following assumptions on g

$$\mu_k < \lim_{v\to-\infty} \frac{f(v)}{v} < \mu_{k+1} \quad , \quad \lim_{v\to+\infty} \frac{f(v)}{v} = +\infty \quad ,$$

that is g interacts with all the spectrum of the Laplacian except the first k eigenvalues $\mu_1, \ldots, \mu_k$; they obtained the existence of at least two solutions for equation (1) provided that τ is sufficiently large.

For quasi-linear equations in which the Laplacian is replaced by the p-Laplacian, there are only few related results available. In particular, we mention the result of Arcoya-Ruiz [5] concerning a version of the Ambrosetti-Prodi result for the p-Laplacian operator. They considered the problem

$$\begin{cases} -\Delta_p w &= g(w) + \tau\phi_1 + \psi \quad , \quad \text{in } \Omega \\ \quad w &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} , \tag{3}$$

where $\tau \in \mathbb{R}$ is a parameter, $\psi \in L^\infty(\Omega)$ is a fixed function, ϕ_1 is the first eigenfunction of $-\Delta_p$ corresponding to the first eigenvalue λ_1. Assuming the analogous condition to (2), namely

$$-\infty < \lim_{w\to-\infty} \frac{g(w)}{|w|^{p-2}w} < \lambda_1 < \lim_{w\to+\infty} \frac{g(w)}{|w|^{p-2}w} < +\infty \,,$$

they proved $\big($with a mixed topological degree and sub-super solution method$\big)$ the existence of τ_* , τ^* , $-\infty < \tau_* \le \tau^* < +\infty$, such that (3) has at least two solutions if $\tau < \tau_*$, at least one solution if $\tau \le \tau^*$, no solutions if $\tau > \tau^*$. Under additional conditions (among which $p \ge 2$) they also showed $\tau_* = \tau^*$.

In this paper we give an extension of the result by Ruf-Srikanth and de Figueiredo to the case of the p-Laplacian. More precisely, we give a multiplicity result for the following problem

$$\begin{cases} -\Delta_p w &= \lambda|w|^{p-2}w + (w^+)^{q-1} + \tau\phi_1^{p-1} + H \quad , \quad \text{in } \Omega \\ \quad w &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} ; \tag{4}$$

here $\Omega \subset \mathbb{R}^N$ is a bounded domain, $-\Delta_p$ is the degenerate p-Laplacian operator $\big($with $p > 2\big)$, ϕ_1 is the positive first eigenfunction of the p-Laplacian, $H \in L^\infty(\Omega)$ is such that $\int_\Omega H\phi_1 dx = 0$ and $\lambda \in (\lambda_1, \lambda_2)$, $\tau > 0$, $q \in (p, p^*)$ are fixed real parameters, where λ_1, λ_2 are the first and the second eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, and p^* is the critical exponent of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Note that the nonlinearity $g(s) := \lambda\,|s|^{p-2}s + (s^+)^q$ satisfies

$$\lambda_1 < \lim_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} = \lambda < \lambda_2 \ , \quad \lim_{s\to+\infty} \frac{g(s)}{|s|^{p-2}s} = +\infty \ ; \tag{5}$$

hence g crosses (in a $(p-1)-linear$ sense) all but the first eigenvalue of the p-Laplacian.

The main result of this paper is

Theorem 1.1. *There exists $\Lambda \in (\lambda_1, \lambda_2]$ such that for any $\lambda \in (\lambda_1, \Lambda)$ problem (4) has two solutions when $\tau > 0$ is sufficiently large.*

Remark 1.1. Note that we require that $\lim_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s}$ is larger than and close to λ_1. This restriction is caused by technical problems due to the linearization of the p-Laplacian. It remains an open problem whether such a restriction is necessary, as well as the generalization to the case $\lambda_k < \lambda < \lambda_{k+1}$.

The proof of Theorem 1.1 proceeds as follows: by the rescaling $u = \left(\frac{1}{\tau}\right)^{\frac{1}{p-1}} w$, problem (4) is equivalent to

$$\left\{ \begin{array}{rcll} -\Delta_p u &=& \lambda|u|^{p-2}u + \tau^{\frac{q-p}{p-1}}\left(u^+\right)^{q-1} + \phi_1^{p-1} + \frac{1}{\tau}H & , \quad \text{in } \Omega \\ u &=& 0 & , \quad \text{on } \partial\Omega \end{array} \right. . \tag{6}$$

Any weak solution $u \in W_0^{1,p}(\Omega)$ of equation (6) must satisfy

$$\int_\Omega \left|\nabla u\right|^{p-2} \left\langle \nabla u \ , \ \nabla v \right\rangle_{\mathbb{R}^N} dx - \lambda \int_\Omega \left|u\right|^{p-2} uv \ \ dx - \tau^{\frac{q-p}{p-1}} \int_\Omega (u^+)^{q-1} v \ \ dx$$
$$- \int_\Omega \phi_1^{p-1} v \ \ dx - \frac{1}{\tau} \int_\Omega Hv \ \ dx = 0 \ , \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

namely $\left\langle J'_{\lambda,\tau}(u), v\right\rangle = 0$ for all $v \in W_0^{1,p}(\Omega)$, where $J_{\lambda,\tau} : W_0^{1,p}(\Omega) \to \mathbb{R}$ is the following functional

$$J_{\lambda,\tau}(u) = \frac{1}{p} \int_\Omega \left|\nabla u\right|^p dx - \frac{\lambda}{p} \int_\Omega \left|u\right|^p dx - \frac{\tau^{\frac{q-p}{p-1}}}{q} \int_\Omega (u^+)^q dx - \int_\Omega \phi_1^{p-1} u dx - \frac{1}{\tau} \int_\Omega Hu dx \, .$$

First, we prove that for $\lambda > \lambda_1$ and close to λ_1 and for τ sufficiently large there exists a *negative solution* $\phi_{\lambda,\tau}$ of equation (6). An important step consists in showing that the functional $J_{\lambda,\tau}$ restricted to the subspace $\langle \phi_{\lambda,\tau}^{p-1} \rangle^\perp$ has a strict local minimum in $\phi_{\lambda,\tau}$. This requires delicate estimates on the second derivative of J in $\phi_{\lambda,\tau}$. For the second solution we can then rely on the *Generalized Mountain Pass Theorem* by *P.H. Rabinowitz* [17]. In fact, we will show that the functional $J_{\lambda,\tau}(u)$ has a linking structure around the first solution $\phi_{\lambda,\tau}$. This then yields the second solution.

2 Preliminary results

2.1 Lindqvist's inequality in $\mathbb{R}^N$

Lemma 2.1 ([15], *Lemma 4.2*)**.** *Let $p \geq 2$, $N \in \mathbb{N}$. Then for any $w_1\ ,\ w_2 \in \mathbb{R}^N$,*

$$\big|w_2\big|^p \geq \big|w_1\big|^p + p\big|w_1\big|^{p-2}\big\langle w_1\ ,\ w_2 - w_1\big\rangle_{\mathbb{R}^N} + \frac{\big|w_2 - w_1\big|^p}{2^{p-1}-1}\ .$$

2.2 The p-Laplacian operator

Proposition 2.2 ([16] , *Theorem A.0.6*)**.** *The p-Laplacian operator $-\Delta_p\ :\ W_0^{1,p}(\Omega)\ \rightarrow\ W^{-1,p'}(\Omega)$ defined by*

$$\big\langle -\Delta_p(u), w\big\rangle = \int_\Omega \big|\nabla u\big|^{p-2}\big\langle \nabla u, \nabla w\big\rangle_{\mathbb{R}^{\mathbb{N}}} dx \quad , \quad \textit{for all } w \in W_0^{1,p}(\Omega)\ ,$$

is continuous and one-to-one. The inverse $-\Delta_p^{-1}\ :\ W^{-1,p'}(\Omega)\ \rightarrow\ W_0^{1,p}(\Omega)$ is also continuous.

In what follows, we will always assume

Hypothesis 1. *If $N \geq 2$, then Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, and Ω satisfies also the interior sphere condition at every point of $\partial\Omega$. If $N = 1$ then Ω is a bounded open interval in $\mathbb{R}$.*

The eigenvalues of the p-Laplacian $-\Delta_p$ namely the values $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases} -\Delta_p\phi &= \lambda\phi^{p-1} \quad , \quad \text{in } \Omega \\ \quad\ \phi &= \quad 0 \qquad , \quad \text{on } \partial\Omega \end{cases}$$

admits nontrivial solutions, form a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 ...\quad ,\quad \lambda_n \overset{n\to\infty}{\rightarrow} +\infty$. For a proof of the existence of these eigenvalues we refer to [16], *Chapter 1*. The first eigenvalue λ_1 is given by the formula

$$\lambda_1 = \inf\left\{ \int_\Omega \big|\nabla u\big|^p dx\ :\ u \in W_0^{1,p}(\Omega)\ ,\ \int_\Omega |u|^p dx\ =\ 1 \right\}. \tag{7}$$

Moreover λ_1 is simple and the corresponding first eigenfunction $\phi_1 \in W_0^{1,p}(\Omega)$ has constant sign, by a result due to *Anane* $\big($[4], *Théorème 1, p.727*$\big)$ and later generalized in *Lindqvist* $\big($[15], *Theorem 1.3, p.157*$\big)$. We have $\phi_1 \in L^\infty(\Omega)$ by another result of *Anane* $\big($[3], *Théorème A.1, p.96*$\big)$. Consequently, under Hypothesis 1 on Ω we get even

$$\phi_1 \in C^{1,\beta}(\overline{\Omega}) \quad \text{for some } \beta \in (0,\alpha)\ ,$$

by a regularity result due to *Tolksdorf* $\big($[20], *Theorem 1, p.127*$\big)$ (interior regularity), and to *Lieberman* $\big($[14], *Theorem 1, p.1203*$\big)$ (regularity near the boundary). Finally, the *Hopf maximum principle* $\big($[21], *Theorem 5, p.200*$\big)$ can be applied to obtain

$$\phi_1 > 0 \quad \text{in } \Omega \qquad , \qquad \frac{\partial\phi_1}{\partial\nu} < 0 \quad \text{on } \partial\Omega\ ,$$

where $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative on $\partial\Omega$.

2.3 Compactness of the operator $u \to |u|^{p-2}u$

Lemma 2.3. *The operator* $\psi_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ *defined as*

$$\langle \psi_p(u), w \rangle = \int_\Omega |u|^{p-2} uw \ dx \ , \ \textit{for all } w \in W_0^{1,p}(\Omega) \ ,$$

is compact.

2.4 A norm depending on ϕ_1

Proposition 2.4 ([19], *Lemma 4.2*)**.** *The functional*

$$\|u\|_{\phi_1} := \left(\int_\Omega |\nabla \phi_1|^{p-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \quad , \quad u \in W_0^{1,p}(\Omega) \quad ,$$

is a norm on $W_0^{1,p}(\Omega)$*. Moreover, if we denote with* $W_{\phi_1}^{1,2}(\Omega)$ *the completion of* $W_0^{1,p}(\Omega)$ *with respect to this norm, then*

$$W_{\phi_1}^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \ ,$$

and the embedding is compact.

2.5 The Fréchet derivative of the p-Laplacian operator

We define

$$\mathcal{F}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p dx \quad , \quad u \in W_0^{1,p}(\Omega).$$

The Fréchet derivative $\mathcal{F}'(u)$ of $\mathcal{F}$ at $u \in W_0^{1,p}(\Omega)$ is given by $\mathcal{F}'(u) = -\Delta_p(u)$ in $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This follows from

$$\langle \mathcal{F}'(u), \phi \rangle = \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle_{\mathbb{R}^N} dx \quad , \quad \phi \in W_0^{1,p}(\Omega).$$

Moreover, the second Fréchet derivative $\mathcal{F}''(u)$ of $\mathcal{F}$ at $u \in W_0^{1,p}(\Omega)$ is given by

$$\langle \mathcal{F}''(u)\psi, \phi \rangle = (p-2) \int_\Omega |\nabla u|^{p-4} \langle \nabla u, \nabla \psi \rangle_{\mathbb{R}^N} \langle \nabla u, \nabla \phi \rangle_{\mathbb{R}^N} \ dx + \int_\Omega |\nabla u|^{p-2} \langle \nabla \psi, \nabla \phi \rangle_{\mathbb{R}^N} dx.$$

The matrix representation of this derivative is the following:

$$\langle \mathcal{F}''(u)\psi, \phi \rangle = \int_\Omega |\nabla u|^{p-2} \left\langle Id_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \ , \ \nabla \phi \otimes \nabla \psi \right\rangle_{\mathbb{R}^N \times \mathbb{R}^N} dx \ ,$$

where the tensor product $\otimes$ is defined by $a \otimes b = \left[a_i b_j \right]_{i,j=1}^N$. For $a \in \mathbb{R}^N$ we introduce the abbreviation

$$\Theta(a) := \begin{cases} |a|^{p-2} \left(Id_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2) \dfrac{a \otimes a}{|a|^2} \right) & , \quad a \neq 0 \\ 0 & , \quad a = 0 \end{cases} .$$

The matrix $\frac{a \otimes a}{|a|^2}$ represents the orthogonal projection on the one-dimensional space spanned by a, so we have

$$\begin{aligned} \Theta(a)b &= |a|^{p-2}b \qquad , \quad \text{for } b \in \mathbb{R}^N \text{ with } \langle b, a\rangle_{\mathbb{R}^N} = 0, \\ \Theta(a)a &= (p-1)|a|^{p-2}a \ . \end{aligned}$$

The spectrum of the matrix $Id_{\mathbb{R}^N \times \mathbb{R}^N} + (p-2)\frac{a \otimes a}{|a|^2}$ consists of the eigenvalues 1 and $p-1$; $p-1$ is simple with the eigenspace spanned by the eigenvector a.
For all $a, b \in \mathbb{R}^N \setminus \{0\}$ we thus obtain

$$1 \le \frac{\langle \Theta(a)b, b\rangle_{\mathbb{R}^N}}{|a|^{p-2}|b|^2} \le p-1 \ . \tag{8}$$

As a direct consequence, we have the following

Lemma 2.5. *For any* $u \in W^{1,2}_{\phi_1}(\Omega)$,

$$\|u\|^2_{\phi_1} \le \int_\Omega \langle \Theta(\nabla\phi_1)\nabla u, \nabla u\rangle_{\mathbb{R}^N} dx \le (p-1)\|u\|^2_{\phi_1} \ .$$

2.5.1 A weak lower semicontinuity result

Lemma 2.6. *Let* $v_n \overset{n\to\infty}{\rightharpoonup} \overline{v}$*, weakly in* $W^{1,2}_{\phi_1}(\Omega)$*. Then*

$$\liminf_{n\to\infty} \left(\int_\Omega \langle \Theta(\nabla\phi_1)\nabla v_n, \nabla v_n\rangle_{\mathbb{R}^N} dx \right) \ge \int_\Omega \langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla \overline{v}\rangle_{\mathbb{R}^N} dx \ .$$

The same conclusion is true assuming $v_n \overset{n\to\infty}{\rightharpoonup} \overline{v}$*, weakly in* $W^{1,p}_0(\Omega)$.

Proof. Consider the bilinear form $B : W^{1,2}_{\phi_1}(\Omega) \times W^{1,2}_{\phi_1}(\Omega) \to \mathbb{R}$ defined as

$$\begin{aligned} B(w_1, w_2) &:= \int_\Omega \langle \Theta(\nabla\phi_1)\nabla w_1, \nabla w_2\rangle_{\mathbb{R}^N} dx \\ &= \int_U |\nabla\phi_1|^{p-2}\left((p-2)\Big\langle \frac{\nabla\phi_1}{|\nabla\phi_1|}, \nabla w_1\Big\rangle_{\mathbb{R}^N} \Big\langle \frac{\nabla\phi_1}{|\nabla\phi_1|}, \nabla w_2\Big\rangle_{\mathbb{R}^N} + \langle \nabla w_1, \nabla w_2\rangle_{\mathbb{R}^N} \right) dx \ , \end{aligned}$$

where $U := \{x \in \Omega \ : \ |\nabla\phi_1| \neq 0\}$. Using the *Cauchy-Schwarz* inequality it is easy to see that the operator $L_{\overline{v}} : W^{1,2}_{\phi_1}(\Omega) \to \mathbb{R}$

$$L_{\overline{v}}(u) := B(\overline{v}, u) \quad , \quad u \in W^{1,2}_{\phi_1}(\Omega) \ ,$$

belongs to the dual space of $W^{1,2}_{\phi_1}(\Omega)$. In particular, by the weak convergence of v_n to $\overline{v}$,

$$\int_\Omega \langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla v_n\rangle_{\mathbb{R}^N} dx = L_{\overline{v}}(v_n) \overset{n\to\infty}{\to} L_{\overline{v}}(\overline{v}) = \int_\Omega \langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla \overline{v}\rangle_{\mathbb{R}^N} dx \ . \tag{9}$$

Now obviously $B\big(w,w\big) \geq 0$, for all $w \in W^{1,2}_{\phi_1}(\Omega)$, hence

$$0 \leq \liminf_{n\to\infty} \left(\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla(v_n - \overline{v}), \nabla(v_n - \overline{v}) \big\rangle_{\mathbb{R}^N} dx \right)$$

$$= \liminf_{n\to\infty} \left(\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla v_n, \nabla v_n \big\rangle_{\mathbb{R}^N} dx - 2\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla v_n \big\rangle_{\mathbb{R}^N} dx + \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla \overline{v} \big\rangle_{\mathbb{R}^N} dx \right)$$

$$\overset{(9)}{=} \liminf_{n\to\infty} \left(\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla v_n, \nabla v_n \big\rangle_{\mathbb{R}^N} dx \right) - \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{v}, \nabla \overline{v} \big\rangle_{\mathbb{R}^N} dx\,.$$

From $W^{1,p}_0(\Omega) \hookrightarrow W^{1,2}_{\phi_1}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \big(W^{1,2}_{\phi_1}(\Omega)\big)' \hookrightarrow W^{-1,p'}(\Omega)$ we infer $L_{\overline{v}} \in W^{-1,p'}(\Omega)$. Therefore (9) is true also if $v_n \overset{n\to\infty}{\rightharpoonup} \overline{v}$, weakly in $W^{1,p}_0(\Omega)$, and we get the same conclusion. □

2.5.2 Another variational characterization of λ_1

We are now stating a variational formula for λ_1 different from (7) which will be crucial in our next considerations. To do this an additional hypothesis on Ω is needed, if $\partial\Omega$ is not connected:

Hypothesis 2. *If $N \geq 2$ and $\partial\Omega$ is not connected, then there is no function $v \in W^{1,2}_{\phi_1}(\Omega)$ with the following properties:*

(i) $v = \phi_1\chi_S$ a.e. in Ω, where $S \subset \Omega$ is a Lebesgue measurable set such $0 < \big|S\big|_N < \big|\Omega\big|_N$.

(ii) $\overline{S}$ is connected and $\overline{S} \cap \partial\Omega \neq \varnothing$.

(iii) Every connected component of $U := \big\{x \in \Omega \ : \ \nabla\phi_1(x) \neq 0\big\}$ is entirely contained either in S or else in $\Omega \setminus S$.

(iv) $\partial S \cap \Omega \ \subset\ \Omega \setminus U$.

(v) $\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla v\ ,\ \nabla v \big\rangle_{\mathbb{R}^N} dx - (p-1)\lambda_1 \int_\Omega \phi_1^{p-2} v^2 dx = 0.$

Proposition 2.7 ([19], *Lemma 4.1* , *Proposition 4.4*)**.** *Suppose that Hypothesis 1 on Ω holds and that either $\partial\Omega$ is connected or Hypothesis 2 is satisfied. Then*

$$\inf_{w \in W^{1,2}_{\phi_1}(\Omega)} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla w, \nabla w \big\rangle_{\mathbb{R}^N} dx}{\displaystyle(p-1)\int_\Omega \phi_1^{p-2}|w|^2 dx} = \lambda_1\,, \tag{10}$$

and w is a minimizer of (10) if and only if $w = \kappa\phi_1$ for some constant $\kappa \in \mathbb{R}$.

Lemma 2.8. *Let*

$$\Lambda_\infty := \inf_{\substack{h \in W^{1,2}_{\phi_1}(\Omega) \setminus \{0\} \\ \int_\Omega \phi_1^{p-1} h dx = 0}} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h, \nabla h \big\rangle_{\mathbb{R}^N} dx}{\displaystyle(p-1)\int_\Omega \phi_1^{p-2}|h|^2 dx}\,.$$

Then $\Lambda_\infty > \lambda_1$.

Proof. Suppose to the contrary that for a sequence $\{h_n\}_{n\in\mathbb{N}} \subset W^{1,2}_{\phi_1}(\Omega)$, $\int_\Omega \phi_1^{p-1} h_n dx = 0$,

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n|^2 dx} = \lambda_1 \, .$$

Let $h_n^* = \frac{h_n}{\|h_n\|_{\phi_1}}$, for any $n \in \mathbb{N}$. The sequence $\{h_n^*\}_{n\in\mathbb{N}}$ is bounded in $W^{1,2}_{\phi_1}(\Omega)$ so we can assume $h_n^* \overset{n\to\infty}{\rightharpoonup} h^*$, weakly in $W^{1,2}_{\phi_1}(\Omega)$. In particular

$$\int_\Omega \phi_1^{p-1} h^* dx = \lim_{n\to\infty} \int_\Omega \phi_1^{p-1} h_n^* dx \overset{h_n^* \in \langle \phi_1^{p-1}\rangle^\perp}{=} 0 \, . \tag{11}$$

Moreover, *Lemma 2.6* yields

$$\liminf_{n\to\infty} \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n^*, \nabla h_n^*\big\rangle_{\mathbb{R}^N} dx \geq \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h^*, \nabla h^*\big\rangle_{\mathbb{R}^N} dx \, , \tag{12}$$

while, by the compactness of the immersion $W^{1,2}_{\phi_1}(\Omega) \hookrightarrow L^2(\Omega)$, $h_n^* \overset{n\to\infty}{\to} h^*$, strongly in $L^2(\Omega)$ and therefore we deduce

$$\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx \overset{n\to\infty}{\to} \int_\Omega \phi_1^{p-2}|h^*|^2 dx \, . \tag{13}$$

Now we claim that

$$h^* \neq 0 \, . \tag{14}$$

In fact, if $\displaystyle\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx \overset{n\to\infty}{\to} 0$ we get

$$\lambda_1 = \lim_{n\to\infty} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n|^2 dx} = \lim_{n\to\infty} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n^*, \nabla h_n^*\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx} \geq$$

$$\overset{\textit{Lemma 2.5}}{\geq} \frac{\big\|h_n^*\big\|^2_{\phi_1}}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx} = \frac{1}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx} \overset{n\to\infty}{\to} +\infty \, ,$$

a contradiction. Then

$$\lambda_1 = \liminf_{n\to\infty} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n|^2 dx} = \liminf_{n\to\infty} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n^*, \nabla h_n^*\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h_n^*|^2 dx} \geq$$

$$\overset{(12)\,,\,(13)}{\geq} \frac{\displaystyle\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h^*, \nabla h^*\big\rangle_{\mathbb{R}^N} dx}{(p-1)\displaystyle\int_\Omega \phi_1^{p-2}|h^*|^2 dx} \overset{\textit{Proposition 2.7}}{\geq} \lambda_1 \, ,$$

and *Proposition 2.7* forces $h^* = \kappa\phi_1$, in contradiction with (11) , (14). $\square$

2.5.3 Uniform continuity on $C^{1,\beta}(\overline{\Omega})$

The proof of the following lemma is easily achieved by *Lebesgue*'s dominated convergence theorem, *Cauchy-Schwarz* and *Hölder* inequalities.

Lemma 2.9. *Let $\psi_n \overset{n\to\infty}{\to} \psi$, in $C^{1,\beta}(\overline{\Omega})$. Then*

$$\sup_{\|h\|_{1,p}=1} \left| \int_\Omega \langle \Theta(\nabla\psi_n)\nabla h, \nabla h \rangle_{\mathbb{R}^N} dx - \int_\Omega \langle \Theta(\nabla\psi)\nabla h, \nabla h \rangle_{\mathbb{R}^N} dx \right| \leq \delta_n \overset{n\to\infty}{\to} 0\ .$$

3 Existence of a first negative solution $\phi_{\lambda,\tau}$

In this section we obtain the existence of a first solution of (6). First we recall some known results.

Proposition 3.1 ([10], *Theorem 12.26*)**.** *For any $\lambda \in (\lambda_1, \lambda_2)$, $f^* \in W^{-1,p'}(\Omega)$ the problem*

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u &= f^* \quad , \quad \text{in } \Omega \\ \qquad u &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} \tag{15}$$

is solvable in $W_0^{1,p}(\Omega)$.

Definition 3.1. *For any $\lambda \in (\lambda_1, \lambda_2)$, $\tau > 0$, we denote with $\phi_{\lambda,\tau}$ an arbitrary solution of*

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u &= \phi_1^{p-1} + \tau^{-1}H \quad , \quad \text{in } \Omega \\ \qquad u &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} . \tag{16}$$

Proposition 3.2 ([11], *Theorem 4.1*)**.** *Let $\{\lambda_n\}$, $\{\tau_n\} \subset \mathbb{R}_+$ be sequences with $\lambda_n \overset{n\to\infty}{\to} \lambda_1^+$, $\tau_n \overset{n\to\infty}{\to} +\infty$. Then we have the following asymptotic estimate:*

$$\phi_{\lambda_n,\tau_n} = \frac{\phi_1 + v_n^\perp}{-(\lambda_n - \lambda_1)^{\frac{1}{p-1}} + o\Big((\lambda_n - \lambda_1)^{\frac{1}{p-1}}\Big)}\ ,$$

where $v_n^\perp \in \langle \phi_1^{p-1} \rangle^\perp$, $v_n^\perp \overset{n\to\infty}{\to} 0$, strongly in $C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$ (up to a subsequence). In particular $\phi_{\lambda_n,\tau_n} < 0$, in Ω, for any n sufficiently large.

Corollary 3.3. *There is $\Lambda^1 \in \big(\lambda_1, \lambda_2\big]$ such that for any $\lambda \in \big(\lambda_1, \Lambda^1\big)$ there exists $\tau_\lambda^1 > 0$ such that for any $\tau > \tau_\lambda^1$*

$$\phi_{\lambda,\tau} < 0\ , \ \text{in } \Omega\ .$$

For such values of λ, τ, since $\phi_{\lambda,\tau}$ is a negative solution of problem (16), $\phi_{\lambda,\tau}$ solves also problem (6).

Proof. If the assertion is not true, we can take two sequences $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $\lambda_n \overset{n\to\infty}{\to} \lambda_1^+$, $\tau_n \overset{n\to\infty}{\to} +\infty$ such that for any n there is $x \in \Omega$ such that $\phi_{\lambda_n,\tau_n}(x) \geq 0$. This contradicts *Proposition 3.2*, which states that $\phi_{\lambda_n,\tau_n} < 0$ in Ω, as $n \to \infty$. □

Corollary 3.4. *There is* $\Lambda^2 \in \big(\lambda_1, \lambda_2\big]$ *such that for any* $\lambda \in \big(\lambda_1, \Lambda^2\big)$ *there exists* $\tau_\lambda^2 > 0$ *such that for any* $\tau > \tau_\lambda^2$

$$J_{\lambda,\tau}\big(\phi_{\lambda,\tau}\big) > 0\,,\ \textit{in}\ \Omega\ .$$

Proof. If the assertion is not true, we can take two sequences $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $\lambda_n \stackrel{n\to\infty}{\to} \lambda_1^+$, $\tau_n \stackrel{n\to\infty}{\to} +\infty$ such that

$$\lim_{n\to\infty} J_{\lambda_n,\tau_n}\big(\phi_{\lambda_n,\tau_n}\big) \le 0\,, \tag{17}$$

Using ϕ_{λ_n,τ_n} as test function in $-\Delta_p\big(\phi_{\lambda_n,\tau_n}\big) - \lambda_n\big|\phi_{\lambda_n,\tau_n}\big|^{p-2}\phi_{\lambda_n,\tau_n} = \phi_1^{p-1} + \tau_n^{-1}H$ we get

$$\Big\|\phi_{\lambda_n,\tau_n}\Big\|_{1,p}^p - \lambda_n\Big\|\phi_{\lambda_n,\tau_n}\Big\|_p^p = \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)\phi_{\lambda_n,\tau_n}dx\ . \tag{18}$$

Moreover, according to *Proposition 3.2*,

$$\phi_{\lambda_n,\tau_n} = -\frac{\phi_1 + v_n^\perp}{\mu_n}\ , \tag{19}$$

where $\displaystyle\int_\Omega \phi_1^{p-1}v_n^\perp dx = 0$, $v_n^\perp \stackrel{n\to\infty}{\to} 0$, strongly in $C^{1,\beta}(\overline{\Omega})$, $\mu_n \stackrel{n\to\infty}{\to} +\infty$.

Assume that n is sufficiently large to have $\phi_{\lambda_n,\tau_n} < 0$ in Ω. Then, recalling also that $\displaystyle\int_\Omega H\phi_1 dx = 0$,

$$J_{\lambda,\tau}\big(\phi_{\lambda_n,\tau_n}\big) \stackrel{\phi_{\lambda_n,\tau_n}<0}{=} \tfrac{1}{p}\Big\|\phi_{\lambda_n,\tau_n}\Big\|_{1,p}^p - \tfrac{\lambda_n}{p}\Big\|\phi_{\lambda_n,\tau_n}\Big\|_p^p - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)\phi_{\lambda_n,\tau_n}dx$$

$$\stackrel{(18)}{=} \Big(\frac{1}{p} - 1\Big)\int_\Omega \Big(\phi_1^{p-1} + \frac{H}{\tau_n}\Big)\phi_{\lambda_n,\tau_n}dx \stackrel{(19)}{=} \frac{1}{\mu_n}\Big(1 - \frac{1}{p}\Big)\int_\Omega \Big(\phi_1^{p-1} + \frac{H}{\tau_n}\Big)\big(\phi_1 + v_n^\perp\big)dx$$

$$= \frac{1}{\mu_n}\Big(1 - \frac{1}{p}\Big)\Big(\int_\Omega \phi_1^p dx + \frac{1}{\tau_n}\int_\Omega Hv_n^\perp dx\Big) \ge \frac{1}{\mu_n}\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{\|H\|_{-1,p'}\|v_n^\perp\|_{1,p}}{\tau_n}\Big) \stackrel{n\to\infty}{\to} +\infty,$$

a contradiction with (17). ☐

3.1 Behavior of $J_{\lambda,\tau}$ near $\phi_{\lambda,\tau}$

Proposition 3.5. *There is* $\Lambda^3 \in \big(\lambda_1, \lambda_2\big]$ *such that for any* $\lambda \in \big(\lambda_1, \Lambda^3\big)$ *there exists* $\tau_\lambda^3 > 0$ *such that for any* $\tau > \tau_\lambda^3$

$$\inf_{\|h\|_{1,p}=1\ ,\ h\in\langle\phi_{\lambda,\tau}^{p-1}\rangle^\perp} J_{\lambda,\tau}\Big(\phi_{\lambda,\tau} + \tau^{-\frac{1}{p-1}}h\Big) > J_{\lambda,\tau}\big(\phi_{\lambda,\tau}\big) + \xi\tau^{-\frac{p}{p-1}}\ ,$$

where $\xi > 0$ *is a fixed constant.*

Proof. If the assertion is not true, we can suppose the existence of three sequences $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{h_n\}_{n\in\mathbb{N}} \subset \big\langle\phi_{\lambda_n,\tau_n}^{p-1}\big\rangle^\perp$, such that $\big\|h_n\big\|_{1,p} = 1$ for all n, $\lambda_n \stackrel{n\to\infty}{\to} \lambda_1^+$, $\tau_n \stackrel{n\to\infty}{\to} +\infty$,

$$\lim_{n\to\infty}\frac{J_{\lambda_n,\tau_n}\Big(\phi_{\lambda_n,\tau_n} + r_nh_n\Big) - J_{\lambda_n,\tau_n}\big(\phi_{\lambda_n,\tau_n}\big)}{r_n^p} \le 0 \quad , \tag{20}$$

where for all n we have set $r_n := \tau_n^{-\frac{1}{p-1}}$, and ϕ_{λ_n,τ_n} is for all n an arbitrary solution of

$$\begin{cases} -\Delta_p u - \lambda_n |u|^{p-2}u &= \phi_1^{p-1} + \tau_n^{-1}H \quad , \quad \text{in } \Omega \\ u &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} .$$

To simplify the notations, we put $\phi_n := \phi_{\lambda_n,\tau_n}$, for all $n \in \mathbb{N}$. Using *Proposition 3.2* we can assume that n is sufficiently large to have $\phi_n < 0$, in Ω. Moreover, since $\|h_n\|_{1,p} = 1$ for all n, $h_n \overset{n\to\infty}{\rightharpoonup} \overline{h}$, weakly in $W_0^{1,p}(\Omega)$ (up to a subsequence).
Now we give some estimates of

$$J_{\lambda_n,\tau_n}\Big(\phi_n + r_n h_n\Big) - J_{\lambda_n,\tau_n}(\phi_n) = \frac{1}{p}\Big\|\phi_n + r_n h_n\Big\|_{1,p}^p - \frac{\lambda_n}{p}\Big\|\phi_n + r_n h_n\Big\|_p^p$$
$$-\frac{\tau_n^{\frac{q-p}{p-1}}}{q}\Big\|\big(\phi_n + r_n h_n\big)^+\Big\|_q^q - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)\big(\phi_n + r_n h_n\big)dx - J_{\lambda_n,\tau_n}(\phi_n) \ , \tag{21}$$

which will be in contradiction with (20).
Step 1. $\overline{h} \neq 0$. In fact, suppose on the contrary that $h_n \overset{n\to\infty}{\rightharpoonup} 0$, weakly in $W_0^{1,p}(\Omega)$.
Using *Lindqvist's inequality* $\big($*Lemma 2.1*$\big)$ with $w_2 = \nabla\big(\phi_n + r_n h_n\big)$, $w_1 = \nabla\phi_n$, we get

$$\Big\|\phi_n + r_n h_n\Big\|_{1,p}^p$$
$$\overset{\|h_n\|_{1,p}=1}{\geq} \Big\|\phi_n\Big\|_{1,p}^p + r_n p\int_\Omega \big|\nabla\phi_n\big|^{p-2}\big\langle\nabla\phi_n \ , \ \nabla h_n\big\rangle_{\mathbb{R}^N} dx + \frac{r_n^p}{2^{p-1}-1} \ . \tag{22}$$

On the other hand, by the *Rellich-Kondrachov theorem*, $h_n \overset{n\to\infty}{\to} 0$, strongly in $L^p(\Omega)$, $L^q(\Omega)$. Therefore

$$\Big\|\phi_n + r_n h_n\Big\|_p^p = r_n^p\Big\|\frac{\phi_n}{r_n} + h_n\Big\|_p^p = r_n^p\left(\Big\|\frac{\phi_n}{r_n}\Big\|_p^p + o(1)\right) = \Big\|\phi_n\Big\|_p^p + o(r_n^p) \ , \tag{23}$$

and, recalling the definition $r_n = \tau_n^{-\frac{1}{p-1}}$,

$$\frac{\tau_n^{\frac{q-p}{p-1}}}{q}\Big\|\big(\phi_n + r_n h_n\big)^+\Big\|_q^q \overset{\phi_n<0}{\leq} \frac{\tau_n^{\frac{q-p}{p-1}}}{q}\Big\|r_n h_n\Big\|_q^q = \frac{r_n^p}{q}\Big\|h_n\Big\|_q^q = o(r_n^p) \ . \tag{24}$$

Joining (21), (22), (23) and (24) one obtains

$$J_{\lambda_n,\tau_n}\Big(\phi_n + r_n h_n\Big) - J_{\lambda_n,\tau_n}(\phi_n) \geq \frac{1}{p}\frac{r_n^p}{2^{p-1}-1} + o(r_n^p)$$
$$+r_n\left(\int_\Omega \big|\nabla\phi_n\big|^{p-2}\big\langle\nabla\phi_n \ , \ \nabla h_n\big\rangle_{\mathbb{R}^N} dx - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)h_n dx\right) \tag{25}$$
$$+\left(\frac{1}{p}\Big\|\phi_n\Big\|_{1,p}^p - \frac{\lambda_n}{p}\Big\|\phi_n\Big\|_p^p - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)\phi_n dx - J_{\lambda_n,\tau_n}(\phi_n)\right) .$$

Now by definition

$$J_{\lambda_n,\tau_n}(\phi_n) = \frac{1}{p}\Big\|\phi_n\Big\|_{1,p}^p - \frac{\lambda_n}{p}\Big\|\phi_n\Big\|_p^p - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)\phi_n dx\,. \tag{26}$$

Moreover, since ϕ_n is for all n a solution of

$$\begin{cases} -\Delta_p u - \lambda_n |u|^{p-2}u &= \phi_1^{p-1} + \tau_n^{-1}H \quad , \quad \text{in } \Omega \\ \qquad u &= \qquad 0 \qquad\quad , \quad \text{on } \partial\Omega \end{cases},$$

and $h_n \in \big\langle \phi_n^{p-1}\big\rangle^\perp$, we have also

$$\int_\Omega \big|\nabla\phi_n\big|^{p-2}\big\langle\nabla\phi_n\,,\,\nabla h_n\big\rangle_{\mathbb{R}^N} dx - \int_\Omega \big(\phi_1^{p-1} + \tau_n^{-1}H\big)h_n dx = \lambda_n \int_\Omega \big|\phi_n\big|^{p-2}\phi_n h_n dx = 0\,. \tag{27}$$

Hence from (25) we infer

$$\lim_{n\to\infty} \frac{J_{\lambda_n,\tau_n}\Big(\phi_n + r_n h_n\Big) - J_{\lambda_n,\tau_n}(\phi_n)}{r_n^p} \geq \frac{1}{p}\frac{1}{2^{p-1}-1} > 0\,,$$

in contradiction with (20).

Step 2. Therefore we may assume that $h_n \overset{n\to\infty}{\rightharpoonup} \overline{h} \neq 0$, weakly in $W_0^{1,p}(\Omega)$. In particular $\big\|h_n\big\|_q^q < K$, and, since $r_n = \tau_n^{-\frac{1}{p-1}}$ we get

$$\frac{\tau_n^{\frac{q-p}{p-1}}}{q}\Big\|\big(\phi_n + r_n h_n\big)^+\Big\|_q^q \overset{\phi_n<0}{\leq} \frac{\tau_n^{\frac{q-p}{p-1}}}{q}\Big\|r_n h_n\Big\|_q^q \leq \frac{r_n^p}{q}K \overset{p\geq 2}{=} o(r_n^2)\,. \tag{28}$$

Due to the asymptotic estimate in *Proposition 3.2*,

$$\phi_n = \frac{\psi_n}{\mu_n} \tag{29}$$

where

$$\mu_n = \ (\lambda_n - \lambda_1)^{\frac{1}{p-1}} + o\Big((\lambda_n-\lambda_1)^{\frac{1}{p-1}}\Big) \quad , \quad \psi_n \overset{n\to\infty}{\to} -\phi_1, \text{ strongly in } C^{1,\beta}(\overline{\Omega})\,.$$

Observe that $\mu_n \overset{n\to\infty}{\to} 0^+$, since $\lambda_n \overset{n\to\infty}{\to} \lambda_1^+$. By (29)

$$\begin{aligned} \Big\|\phi_n + r_n h_n\Big\|_{1,p}^p &= \mu_n^{-p}\Big\|\psi_n + \mu_n r_n h_n\Big\|_{1,p}^p \\ \Big\|\phi_n + r_n h_n\Big\|_p^p &= \mu_n^{-p}\Big\|\psi_n + \mu_n r_n h_n\Big\|_p^p\,. \end{aligned} \tag{30}$$

We now make second order Taylor expansions as follows:

$$\begin{aligned} \Big\|\psi_n + \mu_n r_n h_n\Big\|_{1,p}^p = \Big\|\psi_n\Big\|_{1,p}^p + (\mu_n r_n)p\int_\Omega \big|\nabla\psi_n\big|^{p-2}\big\langle\nabla\psi_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx \\ +(\mu_n r_n)^2 p\int_\Omega \big\langle\Theta(\nabla\psi_n)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx + o\big((\mu_n r_n)^2\big)\,, \end{aligned} \tag{31}$$

$$\left\|\psi_n+\mu_n r_n h_n\right\|_p^p=\left\|\psi_n\right\|_p^p+(\mu_n r_n)p\int_\Omega \left|\psi_n\right|^{p-2}\psi_n h_n dx$$
$$+(\mu_n r_n)^2 p(p-1)\int_\Omega \left|\psi_n\right|^{p-2}\left|h_n\right|^2 dx+o\big((\mu_n r_n)^2\big)\ . \tag{32}$$

Inserting (30), (31), (32) and (28) in (21) one gets

$$J_{\lambda_n,\tau_n}\Big(\phi_n+r_nh_n\Big)-J_{\lambda_n,\tau_n}(\phi_n)$$
$$\geq \frac{r_n^2}{\mu_n^{p-2}}\left(\int_\Omega \big\langle\Theta(\nabla\psi_n)\nabla h_n,\nabla h_n\big\rangle_{\mathbb{R}^N}dx-\lambda_n(p-1)\int_\Omega \left|\psi_n\right|^{p-2}\left|h_n\right|^2dx\right)+\frac{o(r_n^2)}{\mu_n^{p-2}}$$
$$+r_n\left(\int_\Omega \left|\nabla\phi_n\right|^{p-2}\big\langle\nabla\phi_n\ ,\ \nabla h_n\big\rangle_{\mathbb{R}^N}dx-\int_\Omega \big(\phi_1^{p-1}+\tau_n^{-1}H\big)h_n dx\right)$$
$$+\left(\frac{1}{p}\left\|\phi_n\right\|_{1,p}^p-\frac{\lambda_n}{p}\left\|\phi_n\right\|_p^p-\int_\Omega \big(\phi_1^{p-1}+\tau_n^{-1}H\big)\phi_n dx-J_{\lambda_n,\tau_n}(\phi_n)\right)\ ,$$

that is, from (26), (27),

$$J_{\lambda_n,\tau_n}\Big(\phi_n+r_nh_n\Big)-J_{\lambda_n,\tau_n}(\phi_n)$$
$$\geq \frac{r_n^2}{\mu_n^{p-2}}\left(\int_\Omega \big\langle\Theta(\nabla\psi_n)\nabla h_n,\nabla h_n\big\rangle_{\mathbb{R}^N}dx-\lambda_n(p-1)\int_\Omega \left|\psi_n\right|^{p-2}\left|h_n\right|^2dx\right)+\frac{o(r_n^2)}{\mu_n^{p-2}}\ . \tag{33}$$

Step 3. Now we claim that, as $n\to\infty$,

$$\int_\Omega \Big\langle\Theta\big(\nabla\psi_n\big)\nabla h_n,\nabla h_n\Big\rangle_{\mathbb{R}^N}dx-\lambda_n(p-1)\int_\Omega \left|\psi_n\right|^{p-2}\left|h_n\right|^2dx>\kappa>0\ . \tag{34}$$

This, according to (33), would carry

$$\frac{J_{\lambda_n,\tau_n}\Big(\phi_n+r_nh_n\Big)-J_{\lambda_n,\tau_n}(\phi_n)}{r_n^p}\geq (r_n\mu_n)^{2-p}\Big(\kappa+o(1)\Big)\overset{n\to\infty}{\to}+\infty\ , \tag{35}$$

a contradiction with (20). Since $\psi_n\overset{n\to\infty}{\to}-\phi_1$, strongly in $C^{1,\beta}(\overline{\Omega})$, we can apply *Lemma 2.9*, getting

$$\int_\Omega \Big\langle\Theta\big(\nabla\psi_n\big)\nabla h_n,\nabla h_n\Big\rangle_{\mathbb{R}^N}dx-\lambda_n(p-1)\int_\Omega \left|\psi_n\right|^{p-2}\left|h_n\right|^2dx$$
$$\geq \int_\Omega \Big\langle\Theta\big(\nabla\phi_1\big)\nabla h_n,\nabla h_n\Big\rangle_{\mathbb{R}^N}dx-\lambda_n(p-1)\int_\Omega \phi_1^{p-2}\left|h_n\right|^2dx-\delta_n\ , \tag{36}$$

where $\delta_n\overset{n\to\infty}{\to}0^+$. Now from the weak convergence of h_n to $\overline{h}$ in $W_0^{1,p}(\Omega)$ we infer $h_n\overset{n\to\infty}{\to}\overline{h}$ strongly in $L^2(\Omega)$ and, consequently,

$$\lim_{n\to\infty}\int_\Omega \phi_1^{p-2}\left|h_n\right|^2dx=\int_\Omega \phi_1^{p-2}\left|\overline{h}\right|^2dx\ . \tag{37}$$

Moreover, by *Lemma 2.6*,

$$\liminf_{n\to\infty}\left(\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx\right) \geq \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{h}, \nabla \overline{h}\big\rangle_{\mathbb{R}^N} dx\ . \tag{38}$$

Joining (36), (37) and (38), and recalling that $\lambda_n \overset{n\to\infty}{\to} \lambda_1^+$ by hypothesis, we obtain

$$\begin{aligned}\liminf_{n\to\infty}&\left(\int_\Omega \big\langle \Theta(\nabla\psi_n)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx - \lambda_n(p-1)\int_\Omega |\psi_n|^{p-2}|h_n|^2 dx\right)\\ &\geq \int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{h}, \nabla \overline{h}\big\rangle_{\mathbb{R}^N} dx - \lambda_1(p-1)\int_\Omega |\phi_1|^{p-2}|\overline{h}|^2 dx\ .\end{aligned} \tag{39}$$

As in (29) we now rewrite $\phi_n = \frac{\psi_n}{\mu_n}$, where $\psi_n \overset{n\to\infty}{\to} -\phi_1$ in $C^{1,\beta}(\overline{\Omega})$, $\{\mu_n\}_{n\in\mathbb{R}} \subset \mathbb{R}$. Since $h_n \in \big\langle \phi_n^{p-1}\big\rangle^\perp$ for all n, in particular we must have $\int_\Omega |\psi_n|^{p-2}\psi_n h_n dx = 0\ ,$ for all $n\ .$ As a consequence

$$\left|\int_\Omega \phi_1^{p-1}\overline{h}dx\right| \overset{h_n\rightharpoonup\overline{h}}{=} \lim_{n\to\infty}\left|\int_\Omega \Big(\phi_1^{p-1} + |\psi_n|^{p-2}\psi_n\Big)h_n dx\right| = 0\ ,$$

namely $\overline{h} \in \big\langle \phi_1^{p-1}\big\rangle^\perp$. Then we obtain from *Lemma 2.8* that

$$\int_\Omega \big\langle \Theta(\nabla\phi_1)\nabla \overline{h}, \nabla \overline{h}\big\rangle_{\mathbb{R}^N} dx \geq \Lambda_\infty(p-1)\int_\Omega |\phi_1|^{p-2}|\overline{h}|^2 dx\ , \tag{40}$$

where $\Lambda_\infty > \lambda_1$. Inserting (40) in (39), one finally gets

$$\begin{aligned}\liminf_{n\to\infty}&\left(\int_\Omega \big\langle \Theta(\nabla\psi_n)\nabla h_n, \nabla h_n\big\rangle_{\mathbb{R}^N} dx - \lambda_n(p-1)\int_\Omega |\psi_n|^{p-2}|h_n|^2 dx\right)\\ &\geq (p-1)(\Lambda_\infty - \lambda_1)\int_\Omega |\phi_1|^{p-2}|\overline{h}|^2 dx > \kappa > 0\ ,\end{aligned}$$

since $\phi_1 > 0$ in Ω and, by *Step 1*, $\overline{h} \neq 0$. That is, claim (34) is proved, and then we deduce (35), which contradicts (20). This completes the proof. □

3.2 Properties of the solution $\phi_{\lambda,\tau}$

Proposition 3.6. *There is $\Lambda \in \big(\lambda_1, \lambda_2\big]$ such that for any $\lambda \in \big(\lambda_1, \Lambda\big)$ there exists $\tau_\lambda > 0$ such that for any $\tau > \tau_\lambda$ problem (6) admits a solution $\phi_{\lambda,\tau}$ with the following properties:*

(i) $\phi_{\lambda,\tau} < 0\ ,$ *in* $\Omega\ .$

(ii) $J_{\lambda,\tau}\big(\phi_{\lambda,\tau}\big) > 0\ .$

(iii) *There exists $r > 0$ such that* $\displaystyle\inf_{\|h\|_{1,p}=1\ ,\ h\in\langle\phi_{\lambda,\tau}^{p-1}\rangle^\perp} J_{\lambda,\tau}\Big(\phi_{\lambda,\tau} + rh\Big) > J_{\lambda,\tau}\Big(\phi_{\lambda,\tau}\Big)\ .$

Proof. This is a direct consequence of *Corollary 3.3*, *Corollary 3.4* and *Proposition 3.5*. □

4 The linking structure

From now and on we will suppose $\lambda \in \big(\lambda_1, \Lambda\big)$ and $\tau > \tau_\lambda$ to be fixed, where Λ and τ_λ are those defined in *Proposition 3.6*. Moreover we will simplify the notations as follows:

$$\varphi = \phi_{\lambda,\tau} \quad , \quad \theta = \tau^{\frac{q-p}{p-1}} \quad , \quad F = \phi_1^{p-1} + \frac{1}{\tau}H \quad , \quad J = J_{\lambda,\tau} \; .$$

Then

$$J(u) = \frac{1}{p}\int_\Omega \big|\nabla u\big|^p dx - \frac{\lambda}{p}\int_\Omega \big|u\big|^p dx - \frac{\theta}{q}\int_\Omega (u^+)^q dx - \int_\Omega F u dx \quad , \quad u \in W_0^{1,p}(\Omega) \; . \tag{41}$$

According to *Proposition 3.6*, φ is a negative solution to the problem

$$\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u + \theta(u^+)^{q-1} + F \quad , \quad \text{in } \Omega \\ \quad u &= 0 \quad , \quad \text{on } \partial\Omega \end{cases} \; , \tag{42}$$

$J\big(\varphi\big) > 0$ and there exists $r > 0$ such that

$$\inf_{h \in \partial B_r(0) \cap \langle \varphi^{p-1} \rangle^\perp} J\big(\varphi + h\big) > J\big(\varphi\big) \; , \tag{43}$$

that is, property **[A]** of the *Generalized Mountain Pass Theorem* is fulfilled. Now we will deal with property **[B]**.

First of all we compute the value $J\big(\varphi\big) > 0$. Since φ satisfies $-\Delta_p\big(\varphi\big) - \lambda\big|\varphi\big|^{p-2}\varphi = F$ one obtains

$$0 < \Big(\frac{p}{p-1}\Big) J\big(\varphi\big) = -\Big(\big\|\varphi\big\|_{1,p}^p - \lambda\big\|\varphi\big\|_p^p\Big) = -\int_\Omega F\varphi dx \; . \tag{44}$$

4.1 Behavior of J in the subspace $\langle\varphi\rangle$

Proposition 4.1. *φ is a strict maximum point for the functional J on $\big\langle\varphi\big\rangle$ namely*

$$J\big(t\varphi\big) < J\big(\varphi\big) \; , \textit{ for any } t \neq 1 \; .$$

Proof. Denoting with $t^- = \max(-t, 0)$,

$$\begin{aligned} J\big(t\varphi\big) &= \tfrac{|t|^p}{p}\int_\Omega \big|\nabla\varphi\big|^p dx - \tfrac{\lambda|t|^p}{p}\int_\Omega \big|\varphi\big|^p dx - (t^-)^q\frac{\theta}{q}\int_\Omega \varphi^q dx - t\int_\Omega F\varphi dx \\ &\leq \tfrac{|t|^p}{p}\Big(\big\|\varphi\big\|_{1,p}^p - \lambda\big\|\varphi\big\|_p^p\Big) - t\int_\Omega F\varphi dx \overset{(44)}{=} \Big(t - \tfrac{|t|^p}{p}\Big)\Big(\Big(\tfrac{p}{p-1}\Big)J\big(\varphi\big)\Big) \; . \end{aligned}$$

Define $f(t) := \Big(t - \frac{|t|^p}{p}\Big)$, $t \in \mathbb{R}$. Then clearly $t = 1$ is a strict maximum point for f. As a consequence, for all $t \neq 1$,

$$J\big(t\varphi\big) \leq f(t)\Big(\Big(\tfrac{p}{p-1}\Big)J\big(\varphi\big)\Big) < f(1)\Big(\Big(\tfrac{p}{p-1}\Big)J\big(\varphi\big)\Big) = J\big(\varphi\big) \; .$$

$\square$

4.2 Behavior of J far away from φ

Definition 4.1. *For any $\varepsilon \in [0, \varepsilon_\Omega]$ define the two following subsets of Ω*

$$\Omega_\varepsilon := \left\{ x \in \Omega \ : \ dist(x, \partial\Omega) < \varepsilon \right\} ,$$
$$\Omega'_\varepsilon := \left\{ x \in \Omega \ : \ dist(x, \partial\Omega) \geq \varepsilon \right\} ,$$

and then

$$\mathcal{O}_\varepsilon := \left\{ \psi \in W_0^{1,p}(\Omega) \ : \ \psi(x) = 0 \, , \ \text{for all } x \in \Omega_\varepsilon \ , \ \int_\Omega |\varphi|^{p-1} \psi dx = \int_\Omega |\varphi|^p dx \right\} ,$$

where ε_Ω is supposed to be small enough to have $\mathcal{O}_{\varepsilon_\Omega} \neq \varnothing$.

For any $\varepsilon \in [0, \varepsilon_\Omega]$ $\mathcal{O}_\varepsilon$ is a closed convex subset of $W_0^{1,p}(\Omega)$, and as a consequence there exists $\psi_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\left\| |\varphi| - \psi_\varepsilon \right\|_{1,p} = \min_{\psi \in \mathcal{O}_\varepsilon} \left\| |\varphi| - \psi \right\|_{1,p} .$$

Observe that $\psi_\varepsilon \neq |\varphi|$ for any $\varepsilon \in (0, \varepsilon_\Omega]$, since $|\varphi| > 0$ in Ω_ε. On the other hand

$$\left\| |\varphi| - \psi_\varepsilon \right\|_{1,p} = \min_{\psi \in \mathcal{O}_\varepsilon} \left\| |\varphi| - \psi \right\|_{1,p} \overset{\varepsilon \to 0}{\to} \min_{\psi \in \mathcal{O}_0} \left\| |\varphi| - \psi \right\|_{1,p} \overset{|\varphi| \in \mathcal{O}_0}{=} 0 . \tag{45}$$

Lemma 4.2. *For any $\varepsilon \in [0, \varepsilon_\Omega]$ define*

$$h_\varepsilon(x) := |\varphi(x)| - \psi_\varepsilon(x) \ \ , \ x \in \Omega .$$

Then

[a] $\quad h_\varepsilon \in \langle \varphi^{p-1} \rangle^\perp$.

[b] $\quad \|h_\varepsilon\|_{1,p} \overset{\varepsilon \to 0}{\to} 0$.

[c] $\quad h_\varepsilon(x) = |\varphi(x)|$ *, for all $x \in \Omega_\varepsilon$.*

Proof. Property **[a]** is given by

$$-\int_\Omega |\varphi|^{p-2} \varphi h_\varepsilon dx \overset{\varphi \leq 0}{=} \int_\Omega |\varphi|^{p-1} h_\varepsilon dx = \int_\Omega |\varphi|^p dx - \int_\Omega |\varphi|^{p-1} \psi_\varepsilon dx \overset{\psi_\varepsilon \in \mathcal{O}_\varepsilon}{=} 0,$$

[b] follows from (45) and **[c]** is true by construction, since $\psi_\varepsilon \equiv 0$ in Ω_ε. ☐

Proposition 4.3. *There exists $\epsilon \in (0, \varepsilon_\Omega]$ such that for any $\mu \in \mathbb{R}$*

$$J\Big(t(\mu\varphi + h_\epsilon)\Big) \overset{t \to +\infty}{\to} -\infty .$$

Proof. By definition, for any $\varepsilon \in (0, \varepsilon_\Omega]$, $t > 0$,

$$J\Big(t(\mu\varphi + h_\varepsilon)\Big) = \frac{1}{p} \left\| t(\mu\varphi + h_\varepsilon) \right\|_{1,p}^p - \frac{\lambda}{p} \left\| t(\mu\varphi + h_\varepsilon) \right\|_p^p$$
$$-\frac{\theta}{q} \int_\Omega \Big(\big(t(\mu\varphi + h_\varepsilon) \big)^+ \Big)^q dx - t \int_\Omega F(\mu\varphi + h_\varepsilon) dx$$
$$\leq t^p \left(\frac{1}{p} \left\| \mu\varphi + h_\varepsilon \right\|_{1,p}^p - \frac{\lambda}{p} \left\| \mu\varphi + h_\varepsilon \right\|_p^p \right) - t^q \left(\frac{\theta}{q} \int_\Omega \Big((\mu\varphi + h_\varepsilon)^+ \Big)^q dx \right) + t D_{\mu,\varepsilon}, \tag{46}$$

where we have denoted $D_{\mu,\varepsilon} := \|F\|_{-1,p'}\Big(|\mu|\|\varphi\|_{1,p} + \|h_\varepsilon\|_{1,p}\Big)$.

First case: $\mu > \frac{1}{2}$. It follows from *Lemma 4.2* $\big($property **[b]**$\big)$ that $\|h_\varepsilon\|_{1,p} \overset{\varepsilon\to 0}{\to} 0$. Hence by the continuity of the norm

$$\Big\|\mu\varphi + h_\varepsilon\Big\|_{1,p}^p - \lambda\Big\|\mu\varphi + h_\varepsilon\Big\|_p^p = \Big\|\mu\varphi\Big\|_{1,p}^p - \lambda\Big\|\mu\varphi\Big\|_p^p + \kappa(\varepsilon)$$

$$= |\mu|^p\left(\|\varphi\|_{1,p}^p - \lambda\|\varphi\|_p^p\right) + \kappa(\varepsilon) \overset{(44)}{=} -|\mu|^p\Big(\frac{p}{p-1}\Big)J\big(\varphi\big) + \kappa(\varepsilon) = -|\mu|^p A + \kappa(\varepsilon)\ ,$$

where $A = \Big(\frac{p}{p-1}\Big)J\big(\varphi\big)$ is a positive constant , $\kappa(\varepsilon) \overset{\varepsilon\to 0}{\to} 0$.
Choosing ϵ such that $\big|\kappa(\epsilon)\big| < \frac{1}{2^p}A$, we get

$$J\Big(t\big(\mu\varphi + h_\epsilon\big)\Big) \overset{(46)}{\leq} \frac{t^p}{p}\Big(-|\mu|^p A + \kappa(\epsilon)\Big) + tD_{\mu,\epsilon} \overset{\mu>\frac{1}{2}}{<} \frac{t^p}{p}\Big(-\frac{1}{2^p}A + \kappa(\epsilon)\Big) + tD_{\mu,\epsilon} \overset{t\to+\infty}{\to} -\infty\ .$$

Second case: $\mu \leq \frac{1}{2}$. It follows from *Lemma 4.2* $\big($property **[c]**$\big)$ that

$$\big(\mu\varphi + h_\epsilon\big)^+ > \big(\tfrac{1}{2}\varphi + |\varphi|\big)^+ > 0 \quad \text{in } \Omega_\epsilon\ .$$

As a consequence

$$-t^q\left(\frac{\theta}{q}\int_\Omega \Big(\big(\mu\varphi + h_\epsilon\big)^+\Big)^q dx\right) < -t^q\left(\frac{\theta}{q}\frac{1}{2^q}\int_{\Omega_\epsilon}|\varphi|^q dx\right) = -t^q K_\epsilon\ , \tag{47}$$

where K_ϵ is a positive constant. Setting

$$C_{\mu,\epsilon} = \frac{1}{p}\int_\Omega \big|\nabla\big(\mu\varphi + h_\epsilon\big)\big|^p dx - \frac{\lambda}{p}\int_\Omega \big|\mu\varphi + h_\epsilon\big|^p dx\ , \tag{48}$$

we obtain, inserting (47) and (48) in (46),

$$J\Big(t\big(\mu\varphi + h_\epsilon\big)\Big) \leq -t^q K_\epsilon + t^p C_{\mu,\epsilon} + tD_{\mu,\epsilon} \overset{t\to+\infty}{\to} -\infty\ ,$$

since $q > p > 1$. □

Corollary 4.4. *If $c_1, c_2 \in \mathbb{R}$, $c_1 \leq c_2$, then*

$$\sup_{\mu\in[c_1,c_2]} J\Big(t\big(\mu\varphi + h_\epsilon\big)\Big) \overset{t\to+\infty}{\to} -\infty\ .$$

Proposition 4.5. *There exists $\varrho > 0$ such that, for any $\rho \geq \varrho$,*

$$(i)\quad J\Big(\varphi + \rho\ \varphi + \zeta\ h_\epsilon\Big) \ \leq\ J\big(\varphi\big)\quad ,\qquad \textit{for all } \zeta \in [0,\rho]\ .$$

$$(ii)\quad J\Big(\varphi - \rho\ \varphi + \zeta\ h_\epsilon\Big) \ \leq\ J\big(\varphi\big)\quad ,\qquad \textit{for all } \zeta \in [0,\rho]\ .$$

$$(iii)\quad J\Big(\varphi + \zeta\ \varphi + \rho\ h_\epsilon\Big) \ \leq\ J\big(\varphi\big)\quad ,\qquad \textit{for all } \zeta \in [-\rho,\rho]\ .$$

Proof. **Step 1.** We prove *(i)* $\big($the proof of *(ii)* is basically the same$\big)$. By contradiction, suppose that there exists a sequence $\big\{\big(R_n,\zeta_n\big)\big\}_{n\in\mathbb{N}}\subset\mathbb{R}^2$ with

$$R_n \overset{n\to\infty}{\to} +\infty \quad , \quad \zeta_n\in[0,R_n] \quad , \tag{49}$$

$$J\Big(\varphi+R_n\varphi+\zeta_n h_\epsilon\Big) > J\big(\varphi\big) \text{ , for all } n\in\mathbb{N}. \tag{50}$$

Dividing (50) by R_n^p and taking the limit as $n\to\infty$ we obtain

$$\lim_{n\to+\infty}\frac{J\Big(\varphi+R_n\varphi+\zeta_n h_\epsilon\Big)-J\big(\varphi\big)}{R_n^p}\geq 0\ . \tag{51}$$

Now suppose that $\frac{\zeta_n}{R_n}\overset{n\to+\infty}{\to}0$. In view of (41),

$$\begin{aligned}&\frac{J\Big(\varphi+R_n\varphi+\zeta_n h_\epsilon\Big)-J\big(\varphi\big)}{R_n^p}\\ \leq\ &\frac{1}{p}\left\|\varphi+\Big(\frac{1}{R_n}\varphi+\ \frac{\zeta_n}{R_n}h_\epsilon\Big)\right\|_{1,p}^p-\frac{\lambda}{p}\left\|\varphi+\Big(\frac{1}{R_n}\varphi+\ \frac{\zeta_n}{R_n}h_\epsilon\Big)\right\|_p^p\\ &-\int_\Omega F\Big(\frac{1+R_n}{R_n^p}\varphi+\frac{\zeta_n}{R_n^p}h_\epsilon\Big)dx-\ \frac{J(\varphi)}{R_n^p}\ .\end{aligned} \tag{52}$$

Taking the limit as $n\to\infty$ in (52) we get, by the continuity of the norm,

$$\lim_{n\to+\infty}\frac{J\Big(\varphi+R_n\varphi+\zeta_n h_\epsilon\Big)-J\big(\varphi\big)}{R_n^p}=\frac{1}{p}\bigg(\big\|\varphi\big\|_{1,p}^p-\lambda\big\|\varphi\big\|_p^p\bigg)\overset{(44)}{=}-\Big(\frac{1}{p-1}\Big)J\big(\varphi\big)<0\ ,$$

which contradicts (51). As a consequence it must be $\frac{\zeta_n}{R_n}\overset{n\to+\infty}{\not\to}0$, and therefore we infer the existence of $\alpha_1,\alpha_2\in\mathbb{R}$ such that

$$\begin{aligned}&0<\alpha_1=\liminf_{n\to\infty}\ \frac{\zeta_n}{1+R_n}\leq\limsup_{n\to\infty}\ \frac{\zeta_n}{1+R_n}=\alpha_2\overset{\zeta_n\in[0,R_n]}{\leq}1,\\ &1\leq\ \frac{1}{\alpha_2}=\liminf_{n\to\infty}\ \frac{1+R_n}{\zeta_n}\ \leq\limsup_{n\to\infty}\ \frac{1+R_n}{\zeta_n}\ =\ \frac{1}{\alpha_1}<+\infty\ .\end{aligned} \tag{53}$$

Observe that, in particular, $\zeta_n\overset{n\to\infty}{\to}+\infty$. Now, joining (50), (53) and *Corollary 4.4* we get

$$\begin{aligned}0&\overset{(50)}{\leq}\lim_{n\to\infty}J\Big(\varphi+R_n\varphi+\zeta_n h_\epsilon\Big)-J\big(\varphi\big)=\lim_{n\to\infty}J\left(\zeta_n\Big(\frac{1+R_n}{\zeta_n}\varphi+h_\epsilon\Big)\right)-J(\varphi)\\ &\overset{(53)}{\leq}\lim_{n\to\infty}\left\{\sup_{\mu\in\left[\frac{1}{\alpha_2},\frac{1}{\alpha_1}\right]}J\left(\zeta_n\Big(\mu\varphi+h_\epsilon\Big)\right)-J(\varphi)\right\}\overset{\textit{Corollary 4.4}}{=}-\infty\ ,\end{aligned}$$

which is a contradiction. Then the sequence $\{(R_n,\zeta_n)\}_{n\in\mathbb{N}}\subset\mathbb{R}^2$ cannot exist.
Step 2. Proof of (iii). By contradiction, suppose that there exists a sequence $\{(\zeta_n,R_n)\}_{n\in\mathbb{N}}\subset\mathbb{R}^2$ with

$$R_n \overset{n\to\infty}{\to} +\infty \quad , \quad \zeta_n\in[-R_n,R_n] \quad , \tag{54}$$

$$J\Big(\varphi+\zeta_n\varphi+R_nh_\epsilon\Big) > J(\varphi)\text{ , for all } n\in\mathbb{N}. \tag{55}$$

Dividing (55) by R_n^p and taking the limit as $n\to\infty$ we obtain

$$\lim_{n\to+\infty}\frac{J\Big(\varphi+\zeta_n\varphi+R_nh_\epsilon\Big)-J(\varphi)}{R_n^p}\geq 0\ . \tag{56}$$

Now suppose that $\frac{|\zeta_n|}{R_n}\overset{n\to+\infty}{\to}0$. By *Lemma 4.2* (property **[c]**)

$$h_\epsilon\equiv\big|\varphi\big|\ \text{ , in }\Omega_\epsilon\ \text{ ,}$$

therefore we may assume

$$\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon>\frac{1}{2}\big|\varphi\big| \quad , \quad \text{in }\Omega_\epsilon\ \text{ ,} \tag{57}$$

for any n sufficiently large. In view of (41) one has

$$\frac{J\Big(\varphi+\zeta_n\varphi+R_nh_\epsilon\Big)-J(\varphi)}{R_n^p}=\frac{1}{p}\Bigg\|\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon\Bigg\|_{1,p}^p-\frac{\lambda}{p}\Bigg\|\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon\Bigg\|_p^p$$
$$-R_n^{q-p}\,\frac{\theta}{q}\int_\Omega\bigg(\Big(\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon\Big)^+\bigg)^q dx-\int_\Omega F\Big(\frac{1+\zeta_n}{R_n^p}\varphi+\frac{1}{R_n^{p-1}}h_\epsilon\Big)dx-\ \frac{J(\varphi)}{R_n^p}\ , \tag{58}$$

and, inserting (57) in (58), we get

$$\frac{J\Big(\varphi+\zeta_n\varphi+R_nh_\epsilon\Big)-J(\varphi)}{R_n^p}\leq\frac{1}{p}\Bigg\|\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon\Bigg\|_{1,p}^p-\frac{\lambda}{p}\Bigg\|\Big(\frac{1+\zeta_n}{R_n}\Big)\varphi+h_\epsilon\Bigg\|_p^p$$
$$-R_n^{q-p}\,\frac{\theta}{q}\frac{1}{2^q}\int_{\Omega_\epsilon}\big|\varphi\big|^q dx-\int_\Omega F\Big(\frac{1+\zeta_n}{R_n^p}\varphi+\frac{1}{R_n^{p-1}}h_\epsilon\Big)dx-\ \frac{J(\varphi)}{R_n^p}\ . \tag{59}$$

Taking the limit as $n\to\infty$ in (59) we obtain

$$\lim_{n\to+\infty}\frac{J\Big(\varphi+\zeta_n\varphi+R_nh_\epsilon\Big)-J(\varphi)}{R_n^p}$$
$$\leq\frac{1}{p}\bigg(\big\|h_\epsilon\big\|_{1,p}^p-\lambda\big\|h_\epsilon\big\|_p^p\bigg)-\lim_{n\to\infty}R_n^{q-p}\bigg(\frac{\theta}{q}\frac{1}{2^q}\int_{\Omega_\epsilon}\big|\varphi\big|^q dx\bigg)=-\infty\ ,$$

in contradiction with (56). As a consequence it must be $\frac{|\zeta_n|}{R_n}\overset{n\to+\infty}{\not\to}0$, that is

$$-1\overset{\zeta_n\in[-R_n,R_n]}{\leq}\gamma_1=\liminf_{n\to\infty}\frac{1-|\zeta_n|}{R_n}\leq\limsup_{n\to\infty}\frac{1+|\zeta_n|}{R_n}=\gamma_2\overset{\zeta_n\in[-R_n,R_n]}{\leq}1\ . \tag{60}$$

Observe that, in particular, $\left|\zeta_n\right| \overset{n\to\infty}{\to} +\infty$. Using (55), (60) and *Corollary 4.4* we get

$$
\begin{aligned}
0 &\overset{(55)}{\leq} \lim_{n\to\infty} J\Big(\varphi + \zeta_n\varphi + R_n h_\epsilon\Big) - J(\varphi) = \lim_{n\to\infty} J\left(R_n\Big(\frac{1+\zeta_n}{R_n}\varphi + h_\epsilon\Big)\right) - J(\varphi) \\
&\overset{(60)}{\leq} \lim_{n\to\infty}\left\{\sup_{\mu\in\left[\gamma_1,\gamma_2\right]} J\left(R_n\Big(\mu\varphi + h_\epsilon\Big)\right) - J(\varphi)\right\} \overset{\textit{Corollary 4.4}}{=} -\infty \ ,
\end{aligned}
$$

which is a contradiction. Then the sequence $\left\{\left(R_n, \zeta_n\right)\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ cannot exist. This completes the proof. $\square$

5 Conclusion

5.1 The Palais-Smale condition

Proposition 5.1. *Let $p > 2$. The functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition, namely if $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ is a sequence such that*

$$
\begin{aligned}
&\left|J(u_n)\right| < M \\
&J'\big(u_n\big) \overset{W^{-1,p'}(\Omega)}{\to} 0 \ ,
\end{aligned}
\tag{61}
$$

then $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof. We have

$$
J\big(u_n\big) = \frac{1}{p}\int_\Omega \left|\nabla u_n\right|^p dx - \frac{\lambda}{p}\int_\Omega \left|u_n\right|^p dx - \frac{\theta}{q}\int_\Omega \big(u_n^+\big)^q dx - \int_\Omega F u_n dx \ , \tag{62}
$$

$$
\begin{aligned}
\big\langle J'(u_n), v\big\rangle &= \int_\Omega \left|\nabla u_n\right|^{p-2}\big\langle\nabla u_n \ , \ \nabla v\big\rangle_{\mathbb{R}^N} dx - \lambda\int_\Omega \left|u_n\right|^{p-2} u_n v dx \\
&-\theta\int_\Omega \big(u_n^+\big)^{q-1} v dx - \int_\Omega F v dx \quad , \quad \text{for all } v \in W_0^{1,p}(\Omega) \ .
\end{aligned}
\tag{63}
$$

The condition $J'\big(u_n\big) \overset{n\to\infty}{\to} 0$ means that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$
\left|\big\langle J'\big(u_n\big), v\big\rangle\right| < \varepsilon\left\|v\right\|_{1,p} , \text{ for all } v \in W_0^{1,p}(\Omega), n \geq N_0. \tag{64}
$$

Hence the hypotheses (61) on $\{u_n\}_{n\in\mathbb{N}}$ imply

$$
M + \varepsilon\left\|u_n\right\|_{1,p} \overset{(64)}{>} J\big(u_n\big) - \tfrac{1}{p}\big\langle J'\big(u_n\big), u_n\big\rangle \overset{(62),(63)}{=} \theta\Big(\tfrac{1}{p} - \tfrac{1}{q}\Big)\int_\Omega \big(u_n^+\big)^q dx \ . \tag{65}
$$

Now we claim that $\left\|u_n\right\|_{1,p}$ is bounded.
To prove this, assume on the contrary that $\left\|u_n\right\|_{1,p} \overset{n\to\infty}{\to} +\infty$. Dividing inequality (65) by $\left\|u_n\right\|_{1,p}^{\frac{q(p-1)}{q-1}}$, and noting that $\Big(q > p > 2 \Rightarrow \frac{q(p-1)}{q-1} > 1\Big)$

$$
\frac{M + \varepsilon\left\|u_n\right\|_{1,p}}{\left\|u_n\right\|_{1,p}^{\frac{q(p-1)}{q-1}}} = \frac{M}{\left\|u_n\right\|_{1,p}^{\frac{q(p-1)}{q-1}}} + \frac{\varepsilon}{\left\|u_n\right\|_{1,p}^{\frac{q(p-1)}{q-1}-1}} \overset{n\to\infty}{\to} 0 \ ,
$$

we find

$$\frac{\int_\Omega (u_n^+)^q dx}{\|u_n\|_{1,p}^{\frac{q(p-1)}{q-1}}} \overset{n\to\infty}{\to} 0 \,,$$

which is the same as

$$\int_\Omega \left(\frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} \right)^{\frac{q}{q-1}} dx \overset{n\to\infty}{\to} 0 \;\;, \quad \text{that is} \quad , \;\; \left\| \frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} \right\|_{\frac{q}{q-1}} \overset{n\to\infty}{\to} 0 \,. \tag{66}$$

From $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) = \left(L^{\frac{q}{q-1}}(\Omega)\right)'$ it follows $L^{\frac{q}{q-1}}(\Omega) \hookrightarrow \left(W_0^{1,p}(\Omega)\right)' = W^{-1,p'}(\Omega)$, and then by (66)

$$\frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\to} 0 \text{ , strongly in } W^{-1,p'}(\Omega). \tag{67}$$

By assumption

$$-\Delta_p(u_n) - \lambda |u_n|^{p-2} u_n - \theta (u_n^+)^{q-1} - F = J'(u_n) \overset{n\to\infty}{\to} 0 \text{ , in } W^{-1,p'}(\Omega) \;. \tag{68}$$

Dividing (68) by $\|u_n\|_{1,p}^{p-1}$ and writing $v_n = \frac{u_n}{\|u_n\|_{1,p}}$ we get

$$-\Delta_p v_n - \lambda |v_n|^{p-2} v_n - \theta \frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} - \frac{F}{\|u_n\|_{1,p}^{p-1}} = \frac{J'(u_n)}{\|u_n\|_{1,p}^{p-1}} \,. \tag{69}$$

$\{v_n\}_{n\in\mathbb{N}}$ is bounded, so $v_n \overset{n\to\infty}{\rightharpoonup} v$ weakly in $W_0^{1,p}(\Omega)$. *Lemma 2.3* gives

$$-\lambda |v_n|^{p-2} v_n \overset{n\to\infty}{\to} -\lambda |v|^{p-2} v \text{ , in } W^{-1,p'}(\Omega) \,, \tag{70}$$

and from (67), (68), we infer

$$\frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\to} 0 \,, \quad \frac{J'(u_n)}{\|u_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\to} 0 \,, \quad \frac{F}{\|u_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\to} 0 \,, \text{ in } W^{-1,p'}(\Omega). \tag{71}$$

Therefore it follows from (70), (71), that also

$$-\Delta_p v_n \overset{(69)}{=} \lambda |v_n|^{p-2} v_n + \theta \frac{(u_n^+)^{q-1}}{\|u_n\|_{1,p}^{p-1}} + \frac{F}{\|u_n\|_{1,p}^{p-1}} + \frac{J'(u_n)}{\|u_n\|_{1,p}^{p-1}}$$

converges in $W^{-1,p'}(\Omega)$. By the continuity of $-\Delta_p^{-1}$, $v_n \overset{n\to\infty}{\to} v$, strongly in $W_0^{1,p}(\Omega)$. Passing to the limit in (69) and using the continuity of $-\Delta_p$ we get

$$-\Delta_p v - \lambda |v|^{p-2} v = 0 \,,$$

a contradiction, since $\|v_n\|_{1,p} = 1$ for all n gives $\|v\|_{1,p} = 1$ and λ is not an eigenvalue of $-\Delta_p$.

Hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and we can suppose $u_n \overset{n\to\infty}{\rightharpoonup} u$ weakly in

$W_0^{1,p}(\Omega)$, $u_n \overset{n\to\infty}{\to} u$ strongly in $L^\alpha(\Omega)$, for any $\alpha \in \left(1, \frac{Np}{N-p}\right)$, by the *Rellich-Kondrachov theorem*. As a consequence $\left(u_n^+\right)^{q-1}$ and $\left|u_n\right|^{p-2}u_n$ converge in $W^{-1,p'}(\Omega)$, namely

$$-\Delta_p u_n = \lambda\left|u_n\right|^{p-2}u_n + \theta\left(u_n^+\right)^{q-1} + F$$

converges strongly in $W^{-1,p'}(\Omega)$, and, by the continuity of $-\Delta_p^{-1}$, $u_n \overset{n\to\infty}{\to} u$ strongly in $W_0^{1,p}(\Omega)$. $\square$

5.2 Proof of the main result

We consider the decomposition $W_0^{1,p}(\Omega) = \langle\varphi\rangle \oplus \langle\varphi^{p-1}\rangle^\perp$. By (43) there exists $r > 0$ such that

$$\inf_{h\in\partial B_r(0)\cap\langle\varphi^{p-1}\rangle^\perp} J\big(\varphi + h\big) > J\big(\varphi\big)\ . \tag{72}$$

Taking ϵ as in *Proposition 4.3*, h_ϵ as in *Lemma 4.2* $\big(h_\epsilon \in \langle\varphi^{p-1}\rangle^\perp$ by property **[a]**$\big)$ and ϱ as in *Proposition 4.5* $\big($any $\rho \geq \varrho$ is good here, so we can assume $\varrho > r\big)$, if

$$Q := \Big\{\varphi + t\varphi \quad : \quad t \in \big[-\varrho, \varrho\big]\Big\} \oplus \Big\{sh_\epsilon \quad : \quad 0 \leq s \leq \varrho\Big\}$$

then $\partial Q = \cup_{i=1}^4 \Sigma_i$, where

$$\Sigma_1 = \left\{ \quad \varphi + s\varphi \quad : \quad -\varrho \leq s \leq \varrho \quad \right\},\ \Sigma_2 = \left\{ \quad \varphi + \varrho\varphi + sh_\epsilon \quad : \quad 0 \leq s \leq \varrho \quad \right\},$$
$$\Sigma_3 = \left\{ \quad \varphi - \varrho\varphi + sh_\epsilon \quad : \quad 0 \leq s \leq \varrho \quad \right\},\ \Sigma_4 = \left\{ \quad \varphi + s\varphi + \varrho h_\epsilon \quad : \quad -\varrho \leq s \leq \varrho \quad \right\},$$

and it follows from *Proposition 4.5* and *Proposition 4.1* that

$$\sup_{v\in\partial Q} J\big(v\big) \leq J\big(\varphi\big)\ . \tag{73}$$

Therefore, in view of (72) and (73), hypotheses **[A]** and **[B]** of the *Generalized Mountain Pass Theorem* are fulfilled by J, which satisfies also the *Palais-Smale* condition, by *Proposition 5.1*. As a consequence we obtain a critical point ω for J such that $J\big(\varphi\big) < J\big(\omega\big)$, that is, ω and φ are two distinct weak solutions of problem (42).

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