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SHARP ADAMS-TYPE INEQUALITIES IN $\mathbb{R}^n$

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ABSTRACT. Adams' inequality for bounded domains $\Omega \subset \mathbb{R}^4$ states that the supremum of $\int_\Omega e^{32\pi^2 u^2}\, dx$ over all functions $u \in W_0^{2,2}(\Omega)$ with $\|\Delta u\|_2 \leq 1$ is bounded by a constant depending on Ω only. This bound becomes infinite for unbounded domains and in particular for $\mathbb{R}^4$.

We prove that if $\|\Delta u\|_2$ is replaced by a suitable norm, namely $\|u\| := \| - \Delta u + u\|_2$, then the supremum of $\int_\Omega (e^{32\pi^2 u^2} - 1)\, dx$ over all functions $u \in W_0^{2,2}(\Omega)$ with $\|u\| \leq 1$ is bounded by a constant independent of the domain Ω.

Furthermore, we generalize this result to any $W_0^{m,\frac{n}{m}}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ and m an even integer less than n.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. The Sobolev embedding theorem asserts that

$$W_0^{1,p}(\Omega) \subset L^q(\Omega), \qquad 1 \leq q \leq \frac{np}{n-p}.$$

If we look at the limiting Sobolev case $p = n$, then

$$W_0^{1,n}(\Omega) \subset L^q(\Omega) \qquad \forall q \geq 1,$$

but it is well known that

$$W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega).$$

To fill in this gap, it is natural to look for the maximal growth function $g : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \leq 1} \int_\Omega g(u)\, dx < +\infty,$$

where $\|\nabla u\|_n^n = \int_\Omega |\nabla u|^n dx$ denotes the Dirichlet norm of u. S. I. Pohozaev [18] and N. S. Trudinger [23] proved independently that the maximal growth is of exponential type and more precisely that there exist constants $\alpha_n > 0$ and $C_n > 0$ depending only on n such that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx \leq C_n |\Omega|.$$

Later J. Moser in [16] found the best constant α_n and proved the following sharp result.

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Theorem 1.1 ([16], Theorem 1)**.** *There exists a constant $C_n > 0$ such that*

$$\sup_{u\in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx \le C_n |\Omega|, \tag{1.1}$$

where $\alpha_n := n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the surface measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Furthermore (1.1) *is sharp, i.e. if $\alpha > \alpha_n$, then the supremum in* (1.1) *is infinite.*

In the literature (1.1) is known under the name *Trudinger-Moser inequality.* In what follows we will refer to the sharpness of an inequality in the sense expressed in the second part of Theorem 1.1.

The supremum in (1.1) becomes infinite for domains Ω with $|\Omega| = +\infty$. However, in the case $n = 2$ (i.e., for $W_0^{1,\,2}(\Omega)$ with $\Omega \subseteq \mathbb{R}^2$), B. Ruf [19] showed that if the Dirichlet norm is replaced by the standard Sobolev norm, namely

$$\|u\|_{W^{1,\,n}} := \left(\|\nabla u\|_n^n + \|u\|_n^n\right)^{\frac{1}{n}},$$

then this supremum is uniformly bounded independently of the domain Ω:

Theorem 1.2 ([19], Theorem 1.1)**.** *There exists a constant $C > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^2$*

$$\sup_{u\in W_0^{1,\,2}(\Omega),\, \|u\|_{W^{1,\,2}} \le 1} \int_\Omega (e^{4\pi u^2} - 1)\, dx \le C,$$

and this inequality is sharp.

In [12], Y. Li and B. Ruf extended Theorem 1.2 to arbitrary dimensions $n > 2$, i.e., to $W_0^{1,n}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ not necessarily bounded and $n > 2$. Adimurthi and Y. Yang [2] proved an analogous inequality in $\mathbb{R}^n$ involving critical Trudinger-Moser nonlinearities with singular Hardy coefficients.

In 1988 D. R. Adams [1] obtained a generalized version of (1.1) for Sobolev spaces with higher order derivatives. For these spaces the Sobolev embedding theorem says that if $\Omega \subset \mathbb{R}^n$, then

$$W_0^{m,\,p}(\Omega) \subset L^{\frac{pn}{n-pm}}(\Omega),$$

and hence the limiting case is $p = \frac{n}{m}$. In the particular case that m is an even integer and $\Omega \subset \mathbb{R}^n$ with $m < n$, Adams' result can be stated as follows: for $u \in W^{m,\,p}(\Omega)$ with $1 \le p < +\infty$, we will denote by $\nabla^j u$, $j \in \{1,\, 2\, \ldots,\, m\}$, the jth order gradient of u, namely

$$\nabla^j u := \begin{cases} \Delta^{\frac{j}{2}} u & j \text{ even,} \\ \nabla \Delta^{\frac{j-1}{2}} u & j \text{ odd.} \end{cases}$$

Theorem 1.3 ([1], Theorem 3)**.** *Let m be an even integer, and let $\Omega \subset \mathbb{R}^n$ with $m < n$. There exists a constant $C_{m,\,n} > 0$ such that*

$$\sup_{u\in W_0^{m,\,\frac{n}{m}}(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,\,n} |\Omega|, \tag{1.2}$$

where

$$\beta_0 = \beta_0(m,\, n) := \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}}.$$

Furthermore inequality (1.2) *is sharp.*

As before, one notes that the bound in (1.2) becomes infinite for domains Ω with $|\Omega| = +\infty$. In the case that m is even, namely $m = 2k$, we will show that replacing the norm $\|\nabla^m u\|_{\frac{n}{m}}$ with the norm

$$\|u\|_{m,\,n} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{\frac{n}{m}} = \|(-\Delta + I)^k u\|_{\frac{n}{m}},$$

where I denotes the identity operator, the supremum in (1.2) is bounded by a constant independent of Ω.

Let

$$\phi(t) := e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!},$$

where

$$j_{\frac{n}{m}} := \min\left\{ j \in \mathbb{N} \;\middle|\; j \geq \frac{n}{m} \right\} \geq \frac{n}{m}.$$

Our main result is the following:

Theorem 1.4. *Let m be an even integer less than n. There exists a constant $C_{m,\,n} > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u \in W_0^{m,\,\frac{n}{m}}(\Omega),\, \|u\|_{m,\,n} \leq 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \leq C_{m,\,n}, \tag{1.3}$$

and this inequality is sharp.

In [11, Theorem 1.2], Kozono et al. explicitly exhibit a constant $\beta^*_{m,\,n} \leq \beta_0$, with $\beta^*_{m,\,2m} = \beta_0(m,\,2m)$, such that if $\beta < \beta^*_{m,\,n}$, then

$$\sup_{u \in W^{m,\,\frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{m,\,n} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta |u|^{\frac{n}{n-m}}\right) dx \leq C(\beta,\,m,\,n), \tag{1.4}$$

where $C(\beta,\,m,\,n) > 0$ is a constant depending on β, m and n, while if $\beta > \beta_0$ the supremum is infinite. To do this, they reduce the inequality to some equivalent form by means of Bessel potentials, then they apply techniques of symmetric decreasing rearrangements and, following a procedure similar to Adams', they make use of O'Neil's result [17] on the rearrangement of convolution functions. But with these arguments they did not answer the question of whether or not the uniform boundedness in (1.4) also holds for the limiting case $\beta = \beta_0$.

In the proof of Theorem 1.4 we will follow a different approach. The idea is to adapt the arguments in [19], but in order to do this, one encounters difficulties in the use of symmetrization techniques to reduce the general problem to the radial case. Indeed, this cannot be done directly as in [19], since one would have to establish inequalities between $\|\nabla^m u\|_{\frac{n}{m}}$ and $\|\nabla^m u^*\|_{\frac{n}{m}}$, where u^* denotes the symmetrized function of u, and such estimates are unknown in general for higher order derivatives. To get around this problem, the idea is to apply a suitable comparison principle. For example, in [4] and [5], the authors used the well-known Talenti comparison principle (see [20]). Under suitable assumptions, this comparison principle leads to compare a function u, not necessarily radial, with a radial function v in such a way that $\|\nabla^m u\|_p = \|\nabla^m v\|_p$ and $\|u\|_p \leq \|v\|_p$ for any $p \in [1,\,+\infty)$. Therefore, the Talenti comparison principle is a suitable tool if one works with the L^p-norm of the m-th order gradient. In our case, since we want to obtain an estimate independent of the domain, we need to replace the Dirichlet norm $\|\nabla^m u\|_{\frac{n}{m}}$

by a larger norm, and a natural choice is the norm

$$\|u\|_{m,\,n} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{\frac{n}{m}}.$$

It is easy to check that the norm $\|u\|_{m,\,n}$ is equivalent to the Sobolev norm

$$\|u\|_{W^{m,\,\frac{n}{m}}} := \left(\|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{m} \|\nabla^j u\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n}},$$

and in particular, if $u \in W_0^{m\,\frac{n}{m}}(\Omega)$ (or $u \in W^{m,\,\frac{n}{m}}(\mathbb{R}^n)$), then

$$\|u\|_{W^{m,\,\frac{n}{m}}} \le \|u\|_{m,\,n}. \tag{1.5}$$

But the Talenti comparison principle cannot be applied to the norm $\|u\|_{m,n}$ since it increases the $\|\cdot\|_{m,\,n}$-norm; however, the norm $\|u\|_{m,n}$ is well suited to apply (an iterated version of) a comparison principle due to G. Trombetti and J. L. Vázquez, which appears in [22], see also [6].

Having reduced the problem to the radial case, in order to prove Theorem 1.4, we will show that the supremum of

$$\int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx$$

over all radial functions with homogeneous Navier boundary conditions belonging to the unit ball of

$$\left(W_{N,\,\mathrm{rad}}^{m,\,\frac{n}{m}}(B_R) := W_N^{m,\,\frac{n}{m}}(B_R) \cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_R),\ \|\cdot\|_{W^{m,\,\frac{n}{m}}}\right)$$

is bounded by a constant independent of $R > 0$. Here and below, $B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$ is the ball of radius $R > 0$, and

$$\begin{aligned} W_N^{m,\,\frac{n}{m}}(B_R) &:= \left\{u \in W^{n,\,\frac{n}{m}}(B_R) \;\middle|\; \Delta^j u|_{\partial B_R} = 0 \right. \\ &\qquad\qquad \left. \text{in the sense of traces for } 0 \le j < \frac{m}{2}\right\} \\ W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_R) &:= \left\{u \in W^{m,\,\frac{n}{m}}(B_R) \mid u(x) = u(|x|) \ \text{ a.e. in } B_R\right\} \end{aligned}$$

are respectively the space of $W^{m,\,\frac{n}{m}}(B_R)$*-functions with homogeneous Navier boundary conditions* and the space of *radial* $W^{m,\,\frac{n}{m}}(B_R)$*-functions*. This result is expressed in the following.

Proposition 1.1. *Let m be an even integer less than n. There exists a constant $C_{m,\,n} > 0$ such that*

$$\sup_{u \in W_{N,\,rad}^{m,\,\frac{n}{m}}(B_R),\, \|u\|_{W^{m,\,n/m}} \le 1} \int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \le C_{m,\,n} \tag{1.6}$$

independently of $R > 0$, and this inequality is sharp.

This ends an outline of the proof of Theorem 1.4.

We point out that, as $W_0^{m,\,\frac{n}{m}}(\Omega) \subset W_N^{m,\,\frac{n}{m}}(\Omega)$, we have

$$\begin{aligned} &\sup_{u \in W_0^{m,\,\frac{n}{m}}(\Omega),\, \|u\|_{m,\,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \\ &\qquad \le \sup_{u \in W_N^{m,\,\frac{n}{m}}(\Omega),\, \|u\|_{m,\,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx, \end{aligned}$$

and actually we will also prove the following stronger version of the Adams-type inequality (1.3).

Proposition 1.2. *Let m be an even integer less than n. There exists a constant $C_{m,n} > 0$ such that for any bounded domain $\Omega \subset \mathbb{R}^n$*

$$\sup_{u \in W_N^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \leq C_{m,n}, \tag{1.7}$$

and this inequality is sharp.

Comparing this last result with Theorem 1.4, in the case of bounded domains, it is remarkable that the sharp exponent β_0 does not depend on all the traces but only on the zero Navier boundary conditions. This is not obvious, as shown by A. Cianchi in [7] in the case of first order derivatives: with zero Neumann boundary conditions (i.e., in $W^{1,n}(\Omega)$ instead of $W_0^{1,n}(\Omega)$) the sharp exponent α_n in Theorem 1.1 strictly decreases.

This paper is organized as follows. In Section 2 we recall the comparison principle of G. Trombetti and J. L. Vázquez, and we introduce an iterated version of it. In the following sections (Sections 3 and 4), we first prove that the supremum of

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx$$

over all radial functions belonging to the unit ball of $(W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{W^{m,\frac{n}{m}}})$ is bounded:

Theorem 1.5. *Let m be an even integer less than n. There exists a constant $C_{m,n} > 0$ such that*

$$\sup_{u \in W_{rad}^{m,\frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{W^{m,n/m}} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \leq C_{m,n}, \tag{1.8}$$

where

$$W_{rad}^{m,\frac{n}{m}}(\mathbb{R}^n) := \left\{ u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \mid u(x) = u(|x|) \quad a.e.\ in\ \mathbb{R}^n \right\}.$$

Furthermore this inequality is sharp.

Second we will see that the proof of Theorem 1.5 can be easily adapted to prove Proposition 1.1. To make transparent the main ideas of the proof, in Section 3 we prove Theorem 1.5 and Proposition 1.1 in the simplest case $m = 2$, $n = 4$, and we give a general proof for $m \geq 2$ even and $n > m$ in Section 4. In Section 5 we prove the main theorem (Theorem 1.4), and we end the section with the proof of Proposition 1.2. The proof of the sharpness of (1.3), (1.6), (1.7), and (1.8) is given in Section 6.

2. An iterated comparison principle

A crucial tool for the proof of Theorem 1.4 in the case $m = 2$ is the following comparison principle of G. Trombetti and J. L. Vázquez which we state only for balls $B_R \subset \mathbb{R}^n$, $n \geq 2$, in order to simplify the notation and as this is the case of our main interest. We will denote by $|B_R|$ the Lebesgue measure of B_R, namely $|B_R| := \sigma_n R^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Let $u : B_R \to \mathbb{R}$ be a measurable function. The *distribution funtion* of u is defined by

$$\mu_u(t) := |\{x \in B_R \mid |u(x)| > t\}| \qquad \forall t \geq 0.$$

The *decreasing rearrangement* of u is defined by

$$u^*(s) := \inf\{t \geq 0 \mid \mu_u(t) < s\} \qquad \forall s \in [0, |B_R|],$$

and the *spherically symmetric decreasing rearrangement* of u by

$$u^\sharp(x) := u^*(\sigma_n |x|^n) \qquad \forall x \in B_R.$$

The function $u^\sharp$ is the unique nonnegative integrable function which is radially symmetric, nonincreasing, and has the same distribution function as $|u|$.

Let u be a weak solution of

$$\begin{cases} -\Delta u + u = f & \text{in } B_R, \\ u \in W_0^{1,2}(B_R), \end{cases} \tag{2.1}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$.

Proposition 2.1 ([22], Inequality (2.20))**.** *If u is a nonnegative weak solution of* (2.1)*, then*

$$-\frac{du^*}{ds}(s) \leq \frac{1}{n^2 \sigma_n^{\frac{2}{n}}}\, s^{\frac{2}{n}-2} \int_0^s (f^* - u^*)\, d\tau \qquad \forall s \in (0, |B_R|). \tag{2.2}$$

We now consider the problem

$$\begin{cases} -\Delta v + v = f^\sharp & \text{in } B_R, \\ v \in W_0^{1,2}(B_R). \end{cases} \tag{2.3}$$

Due to the radial symmetry of the equation, the unique solution v of (2.3) is radially symmetric, and it is easy to see that

$$-\frac{d\hat{v}}{ds}(s) = \frac{1}{n^2 \sigma_n^{\frac{2}{n}}}\, s^{\frac{2}{n}-2} \int_0^s (f^* - \hat{v})\, d\tau \qquad \forall s \in (0, |B_R|), \tag{2.4}$$

where $\hat{v}(\sigma_n |x|^n) := v(x) \quad \forall x \in B_R$.

The maximum principle, together with inequalities (2.2) and (2.4), leads as proved in [22] to the following *comparison of integrals in balls*:

Proposition 2.2 ([22], Theorem 1)**.** *Let u, v be weak solutions of* (2.1) *and* (2.3)*, respectively. For every $r \in (0, R)$ we have*

$$\int_{B_r} u^\sharp \, dx \leq \int_{B_r} v \, dx.$$

We are now interested in obtaining a *comparison principle* for the polyharmonic operator, which will allow us to reduce the proof of Theorem 1.4 to the radial case. To this aim let $m = 2k$ with k a positive integer, and let $u \in W^{m,2}(B_R)$ be a weak solution of

$$\begin{cases} (-\Delta + I)^k u = f & \text{in } B_R, \\ u \in W_N^{m,2}(B_R), \end{cases} \tag{2.5}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$. If we consider the problem

$$\begin{cases} (-\Delta + I)^k v = f^\sharp & \text{in } B_R, \\ v \in W_N^{m,2}(B_R), \end{cases} \tag{2.6}$$

then the following *comparison of integrals in balls* holds.

Proposition 2.3. *Let u, v be weak solutions of the polyharmonic problems* (2.5) *and* (2.6)*, respectively. For every $r \in (0, R)$, we have*

$$\int_{B_r} u^\sharp \, dx \leq \int_{B_r} v \, dx.$$

Proof. Since equations in (2.5) and (2.6) are considered with homogeneous Navier boundary conditions, they may be rewritten as second order systems:

$$(P_1)\begin{cases} -\Delta u_1 + u_1 = f & \text{in } B_R, \\ u_1 \in W_0^{1,2}(B_R), \end{cases} \qquad (P_i)\begin{cases} -\Delta u_i + u_i = u_{i-1} & \text{in } B_R, \\ u_i \in W_0^{1,2}(B_R), \end{cases}$$

$$(\overline{P}_1)\begin{cases} -\Delta v_1 + v_1 = f^\sharp & \text{in } B_R, \\ v_1 \in W_0^{1,2}(B_R), \end{cases} \qquad (\overline{P}_i)\begin{cases} -\Delta v_i + v_i = v_{i-1} & \text{in } B_R, \\ v_i \in W_0^{1,2}(B_R), \end{cases}$$

for $i \in \{2, 3, \ldots, k\}$, where $u_k = u$ and $v_k = v$. Thus we have to prove that for every $r \in (0, R)$,

$$\int_{B_r} u_k^\sharp \, dx \leq \int_{B_r} v_k \, dx. \tag{2.7}$$

When $k = 1$, inequality (2.7) is the inequality in Proposition 2.2. When $k \geq 2$, we proceed by finite induction, proving that

$$\int_{B_r} u_i^\sharp \, dx \leq \int_{B_r} v_i \, dx \tag{2.8}$$

holds for every $i \in \{1, 2, \ldots, k\}$. By Proposition 2.2 it follows that if $i = 1$, then (2.8) holds. Now, assuming that inequality (2.8) has been proved for some $i \in \{1, 2, \ldots, k-1\}$, we show that

$$\int_{B_r} u_{i+1}^\sharp \, dx \leq \int_{B_r} v_{i+1} \, dx. \tag{2.9}$$

Without loss of generality we may assume that $u_{i+1} \geq 0$. In fact, let $\overline{u}_{i+1}$ be a weak solution of

$$\begin{cases} -\Delta \overline{u}_{i+1} + \overline{u}_{i+1} = |u_i| & \text{in } B_R, \\ \overline{u}_{i+1} \in W_0^{1,2}(B_R). \end{cases}$$

Then by the maximum principle $\overline{u}_{i+1} \geq 0$ and $\overline{u}_{i+1} \geq u_{i+1}$ in B_R.

Since u_{i+1} is a nonnegative weak solution of (P_{i+1}) then (2.2) holds, and since v_{i+1} is a weak solution of $(\overline{P}_{i+1})$ also an analogue of (2.4) holds, namely

$$-\frac{du_{i+1}^*}{ds}(s) \leq \frac{1}{n^2 \sigma_n^{\frac{2}{n}}} \, s^{\frac{2}{n}-2} \int_0^s (u_i^* - u_{i+1}^*) \, d\tau \qquad \forall s \in (0, |B_R|),$$

$$-\frac{d\hat{v}_{i+1}}{ds}(s) = \frac{1}{n^2 \sigma_n^{\frac{2}{n}}} \, s^{\frac{2}{n}-2} \int_0^s (\hat{v}_i - \hat{v}_{i+1}) \, d\tau \qquad \forall s \in (0, |B_R|).$$

Therefore for any $s \in (0,\,|B_R|)$,

$$\frac{d\hat{v}_{i+1}}{ds}(s)-\frac{du^*_{i+1}}{ds}(s)-\frac{1}{n^2\sigma_n^{\frac{2}{n}}}\ s^{\frac{2}{n}-2}\int_0^s(\hat{v}_{i+1}-u^*_{i+1})\,d\tau\leq\frac{1}{n^2\sigma_n^{\frac{2}{n}}}\ s^{\frac{2}{n}-2}\int_0^s(u^*_i-\hat{v}_i)\,d\tau.$$

But as a consequence of the fact that inequality (2.8) holds for i, we have that

$$\int_0^s(u^*_i-\hat{v}_i)\,d\tau\leq 0\quad\forall s\in(0,\,|B_R|),$$

and we get

$$\frac{d\hat{v}_{i+1}}{ds}(s)-\frac{du^*_{i+1}}{ds}(s)-\frac{1}{n^2\sigma_n^{\frac{2}{n}}}\ s^{\frac{2}{n}-2}\int_0^s(\hat{v}_{i+1}-u^*_{i+1})\,d\tau\leq 0\quad\forall s\in(0,\,|B_R|).$$

We can now proceed as in [22], setting

$$y(s):=\int_0^s(\hat{v}_{i+1}-u^*_{i+1})\quad\forall s\in(0,\,|B_R|)$$

so that

$$\begin{cases}y''-\frac{1}{n^2\sigma_n^{\frac{2}{n}}}\ y\leq 0 & \text{in } (0,\,|B_R|),\\ y(0)=y'(|B_R|)=0,\end{cases}$$

and the maximum principle leads us to conclude that $y\geq 0$ which is equivalent to (2.9). □

Actually, in the proof of Theorem 1.4, we will not directly use the *comparison of integrals in balls*, Proposition 2.3, to reduce the problem to the radial case; in fact we will apply a corollary of it. As stated in [22], a well-known direct consequence of Proposition 2.3 is the following *comparison principle*:

Proposition 2.4 ([22], Corollary 1)**.** *Let u, v be weak solutions of* (2.5) *and* (2.6)*, respectively. For every convex nondecreasing function $\phi:\,[0,+\infty)\to[0,+\infty)$, we have*

$$\int_{B_R}\phi(|u|)\,dx\leq\int_{B_R}\phi(v)\,dx.$$

Remark 2.1. It is easy to adapt the previous arguments to obtain a result for general bounded domains. Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$ be a bounded domain. We consider the problems:

$$\begin{cases}(-\Delta+I)^k u=f & \text{in } \Omega,\\ u\in W_N^{m,2}(\Omega),\end{cases}\qquad\begin{cases}(-\Delta+I)^k v=f^\sharp & \text{in } \Omega^\sharp,\\ v\in W_N^{m,2}(\Omega^\sharp),\end{cases}$$

where $f\in L^{\frac{2n}{n+2}}(\Omega)$ and $\Omega^\sharp$ is the ball in $\mathbb{R}^n$ centered at $0\in\mathbb{R}^n$ with the same measure as Ω. Then for every convex nondecreasing function $\phi:\,[0,+\infty)\to[0,+\infty)$, we have

$$\int_\Omega\phi(|u|)\,dx\leq\int_{\Omega^\sharp}\phi(v)\,dx.$$

Remark 2.2. We can now explain how this last proposition may be used in the proof of Theorem 1.4. Let $m=2k<n$ with k a positive integer. Let $u\in\mathcal{C}_0^\infty(B_R)$ with $B_R\subset\mathbb{R}^n$, and define $f:=(-\Delta+I)^k u$ in B_R. By construction u is the unique

solution of (2.5). Let v be the unique radial solution of (2.6), then by Proposition 2.4 it follows that

$$\int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \int_{B_R} \phi\left(\beta_0 |v|^{\frac{n}{n-m}}\right) dx.$$

Since $f \in L^{\frac{n}{m}}(B_R)$, we have that $f^\sharp \in L^{\frac{n}{m}}(B_R)$ and thus $v \in W^{m,\frac{n}{m}}_{N,\,\mathrm{rad}}(B_R)$. Furthermore,

$$\|v\|_{m,\,n} = \|(-\Delta + I)^k v\|_{\frac{n}{m}} = \|f^\sharp\|_{\frac{n}{m}} = \|f\|_{\frac{n}{m}} = \|(-\Delta + I)^k u\|_{\frac{n}{m}} = \|u\|_{m,\,n}.$$

This means that, starting with a function $u \in C_0^\infty(B_R)$, we can always consider a radial function $v \in W^{m,\frac{n}{m}}_{N,\,\mathrm{rad}}(B_R)$ which increases the integral we are interested in and which has the same $\|\cdot\|_{m,\,n}$-norm as u.

3. An Adams-type inequality for radial functions in $W^{2,2}(\mathbb{R}^4)$

In this section we will prove the first part of Theorem 1.5 in the case $m = 2$ and $n = 4$, namely we will prove the existence of a constant $C > 0$ such that

$$\sup_{u \in W^{2,2}_{\mathrm{rad}}(\mathbb{R}^4),\, \|u\|_{W^{2,2}} \le 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx \le C. \tag{3.1}$$

To do this, we follow the techniques adopted in [19] for the proof of Theorem 1.2, and the key to adapt these arguments to the case of second order derivatives is the following stronger version of Adams' inequality:

Theorem 3.1 ([21])**.** *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\sup_{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\, dx \le C|\Omega|,$$

and this inequality is sharp.

Remark 3.1. We point out that Adams' inequality, in its original form, deals with functions in $W^{2,2}_0(\Omega)$ (see Theorem 1.3) which is the closure of the space of smooth compactly supported functions. Note that $W^{2,2}_0(\Omega)$ is strictly contained in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and

$$\sup_{u \in W^{2,2}_0(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\, dx \le \sup_{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\, dx,$$

therefore Theorem 3.1 improves Adams' inequality showing that the sharp exponent $32\pi^2$ does not depend on all the traces.

In [21] C. Tarsi obtained more general embeddings in Zygmund spaces, and Theorem 3.1 is a particular case of these results. For the convenience of the reader, we give here an alternative proof (see also C. S. Lin and J. Wei [13]). To do this, we will follow an argument introduced by H. Brezis and F. Merle (see the proof of Theorem 1 in [3]), who construct an auxiliary function written in Riesz potential form, and we will apply to this auxiliary function the following theorem due to D. R. Adams:

Theorem 3.2 ([1], Theorem 2). *For $1 < p < +\infty$, there is a constant $c_0 = c_0(p)$ such that for all $f \in L^p(\mathbb{R}^n)$ with support contained in Ω, $|\Omega| < +\infty$,*

$$\int_\Omega e^{\frac{n}{\omega_{n-1}}\left|\frac{I_\alpha * f(x)}{\|f\|_p}\right|^{p'}}\,dx \le c_0,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, ω_{n-1} is the surface measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and

$$I_\alpha * f(x) := \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy$$

is the Riesz potential of order $\alpha := \frac{n}{p}$.

Proof of Theorem 3.1. Let

$$C_D^\infty(\Omega) := \left\{u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\right\}.$$

By density arguments, it suffices to prove that

$$\sup_{u \in C_D^\infty(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\,dx \le C|\Omega|.$$

Let $u \in C_D^\infty(\Omega)$ be such that $\|\Delta u\|_2 \le 1$, and set $f := \Delta u$ in Ω, so that u is a solution of the Dirichlet boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

We extend f to be zero outside Ω

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^4 \setminus \Omega, \end{cases}$$

and we define

$$\overline{u} := \left(\frac{4}{\omega_3 32\pi^2}\right)^{\frac{1}{2}} I_2 * |\overline{f}| \qquad \text{in } \mathbb{R}^4$$

so that $-\Delta\overline{u} = |\overline{f}|$ in $\mathbb{R}^4$. By construction $\overline{u} \ge 0$ in $\mathbb{R}^4$ and from the maximum principle, it follows that $\overline{u} \ge |u|$ in Ω. Furthermore

$$32\pi^2\overline{u}^2 \le \frac{4}{\omega_3}\left(\frac{I_2 * |\overline{f}|}{\|\overline{f}\|_2}\right)^2 \qquad \text{in } \mathbb{R}^4.$$

Therefore

$$\int_\Omega e^{32\pi^2 u^2}\,dx \le \int_\Omega e^{32\pi^2\overline{u}^2}\,dx \le \int_\Omega e^{\frac{4}{\omega_3}\left(\frac{I_2 * |\overline{f}|(x)}{\|\overline{f}\|_2}\right)^2}\,dx,$$

and the last integral is bounded by a constant which depends on Ω only as a consequence of Theorem 3.2 with $n = 4$ and $p = 2$. □

We can now begin the proof of (3.1). Let $u \in W^{2,2}_{\mathrm{rad}}(\mathbb{R}^4)$ be such that $\|u\|_{W^{2,2}} \le 1$. Fix $r_0 > 0$ and set

$$I_1 := \int_{B_{r_0}} (e^{32\pi^2 u^2} - 1)\,dx, \qquad I_2 := \int_{\mathbb{R}^4 \setminus B_{r_0}} (e^{32\pi^2 u^2} - 1)dx$$

so that

$$\int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx = I_1 + I_2.$$

During the proof, we will show that it is possible to choose a suitable $r_0 > 0$ independent of u such that I_1 and I_2 are bounded by a constant which depends on r_0 only, and so we can conclude that (3.1) holds.

First, we write I_2 using the power series expansion of the exponential function

$$I_2 = \sum_{k=1}^{+\infty} \frac{(32\pi^2)^k}{k!}\, I_{2,k}\,, \qquad I_{2,k} := \int_{\mathbb{R}^4 \setminus B_{r_0}} |u|^{2k}\, dx.$$

We estimate the single terms $I_{2,k}$ applying the following radial lemma.

Lemma 3.1 ([9], Lemma 1.1, Chapter 6)**.** *If $u \in W^{1,2}_{\mathrm{rad}}(\mathbb{R}^4)$, then*

$$|u(x)| \le \frac{1}{\sqrt{\omega_3}} \frac{1}{|x|^{3/2}}\, \|u\|_{W^{1,2}}$$

for a.e. $x \in \mathbb{R}^4$, where $\omega_3 = 2\pi^2$ is the surface measure of the unit sphere $S^3 \subset \mathbb{R}^4$.

Hence for $k \ge 2$, we obtain

$$I_{2,k} \le \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^k}\, \omega_3 \int_{r_0}^{+\infty} \frac{1}{\rho^{3k}}\, \rho^3\, d\rho = \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^{k-1}} \cdot \frac{{r_0}^{4-3k}}{3k-4} < \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^{k-1}}\, {r_0}^{4-3k}.$$

This implies that

$$I_2 \le 32\pi^2 \|u\|_2^2 + \omega_3 r_0^4 \sum_{k=2}^{+\infty} \frac{1}{k!} \left(\frac{32\pi^2 \|u\|^2_{W^{1,2}}}{\omega_3 {r_0}^3} \right)^k \le c(r_0),$$

where the constant $c(r_0) > 0$ depends only on r_0 since by assumption $\|u\|_2 \le 1$ and $\|u\|_{W^{1,2}} \le 1$.

To estimate I_1, the idea is to use Theorem 3.1, and in order to do this we have to associate to $u \in W^{2,2}(B_{r_0})$ an auxiliary function $w \in W^{2,2}(B_{r_0}) \cap W^{1,2}_0(B_{r_0})$ such that $\|\Delta w\|_2 \le 1$. Recalling that $u \in W^{2,2}_{\mathrm{rad}}(\mathbb{R}^4)$, we define a radial function $v = v(|x|)$ as

$$v(|x|) =: u(|x|) - u(r_0) \qquad \text{for } 0 \le |x| \le r_0,$$

and we can notice that $v \in W^{2,2}(B_{r_0}) \cap W^{1,2}_0(B_{r_0})$. Again applying the radial lemma, we get for $0 < |x| \le r_0$

$$\begin{aligned} u^2(|x|) &= v^2(|x|) + 2v(|x|)u(r_0) + u^2(r_0) \le v^2(|x|) + [v^2(|x|)u^2(r_0) + 1] + u^2(r_0) \\ &\le v^2(|x|) + v^2(|x|) \left[\frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \right] + 1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \\ &\le v^2(|x|) \left[1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \right] + d(r_0). \end{aligned}$$

Now we define

$$w(|x|) := v(|x|) \sqrt{1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}}} \qquad \text{for all } \ 0 \le |x| \le r_0$$

so that $w \in W^{2,2}(B_{r_0}) \cap W_0^{1,2}(B_{r_0})$ and

$$u^2(|x|) \le w^2(|x|) + d(r_0) \qquad \text{for all } \ 0 < |x| \le r_0. \tag{3.2}$$

By construction

$$\int_{B_{r_0}} (\Delta v)^2\, dx = \int_{B_{r_0}} (\Delta u)^2\, dx \le \|\Delta u\|_2^2 \le 1 - \|u\|^2_{W^{1,2}},$$

and hence

$$\begin{aligned}\int_{B_{r_0}} (\Delta w)^2\, dx &= \int_{B_{r_0}} \left[\Delta\left(v\sqrt{1 + \frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|^2_{W^{1,2}}}\right)\right]^2 dx \\ &= \left(1 + \frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|^2_{W^{1,2}}\right)\int_{B_{r_0}} (\Delta v)^2\, dx \\ &\le \left(1 + \frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|^2_{W^{1,2}}\right)(1 - \|u\|^2_{W^{1,2}}) \\ &\le 1 - \left(1 - \frac{1}{2\pi^2}\frac{1}{{r_0}^3}\right)\|u\|^2_{W^{1,2}} \le 1,\end{aligned}$$

provided that ${r_0}^3 \ge \frac{1}{2\pi^2}$. From (3.2) it follows that

$$I_1 \le e^{32\pi^2 d(r_0)} \int_{B_{r_0}} e^{32\pi^2 w^2}\, dx,$$

and if $r_0 \ge \sqrt[3]{\frac{1}{2\pi^2}}$, then the right-hand side of this last inequality is bounded by a constant which depends on r_0 only, as a consequence of Theorem 3.1. This ends the proof of the first part of Theorem 1.5 in the case $m = 2$ and $n = 4$; for sharpness see Section 6.

Remark 3.2. In the estimate of I_1 we might expect to apply Adams' inequality (1.2). But to do this, one would need to construct an auxiliary function w which is in $W_0^{2,2}(B_{r_0})$, and this is not an easy task. However in view of Theorem 3.1 it is sufficient that $w \in W^{2,2}(B_{r_0}) \cap W_0^{1,2}(B_{r_0})$, $\|\Delta w\|_2 \le 1$ to conclude that $\int_{B_{r_0}} (e^{32\pi^2 w^2} - 1)\, dx$ is bounded by a constant which depends on r_0 only.

We can easily adapt the arguments above to obtain a proof of Proposition 1.1 in the case $m = 2$ and $n = 4$.

Proof of Proposition 1.1 *in the case $m = 2$ and $n = 4$.* Fix $R > 0$, and let

$$u \in W^{2,2}_{N,\,\mathrm{rad}}(B_R)$$

be radial and such that $\|u\|_{W^{2,2}} \le 1$. First of all we recall that

$$W_N^{2,2}(B_R) = W^{2,2}(B_R) \cap W_0^{1,2}(B_R),$$

and so $u \in W^{2,2}_{\mathrm{rad}}(B_R) \cap W_0^{1,2}(B_R)$. To prove Proposition 1.1, we have to show that there exists a constant $C > 0$ independent of R and u such that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx \le C. \tag{3.3}$$

We have two alternatives:

(I) $R \le \sqrt[3]{\frac{1}{2\pi^2}}$. As in particular $\|\Delta u\|_2^2 \le 1$, we can apply Theorem 3.1 obtaining that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx \le C\, |B_R| \le C \left| B_{\sqrt[3]{\frac{1}{2\pi^2}}} \right|.$$

(II) $R > \sqrt[3]{\frac{1}{2\pi^2}}$. In this case we set

$$I_1 := \int_{B_{r_0}} (e^{32\pi^2 u^2} - 1)\, dx, \qquad I_2 := \int_{B_R \setminus B_{r_0}} (e^{32\pi^2 u^2} - 1)\, dx,$$

where $\sqrt[3]{\frac{1}{2\pi^2}} \le r_0 < R$, so that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx = I_1 + I_2.$$

To estimate I_1 and I_2 with a constant independent of R and u, we can use the same arguments as in the proof of Theorem 1.5. It suffices to notice that the radial lemma (Lemma 3.1) holds for any radial function in $W^{1,2}(\mathbb{R}^4)$ and, as $u \in W_0^{1,2}(B_R)$, we can extend u to be zero outside the ball B_R obtaining that $u \in W^{1,2}(\mathbb{R}^4)$, furthermore,

$$\|u\|_{W^{1,2}(\mathbb{R}^4)} = \|u\|_{W^{1,2}(B_R)}$$

□

4. An Adams-type inequality for radial functions in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$

In this section we will prove the first part of Theorem 1.5 in the case $m = 2k$ with k a positive integer and $m < n$. To this aim a crucial tool is the following extension of Adams' inequality to functions with homogeneous Navier boundary conditions.

Theorem 4.1 ([21])**.** *Let $m = 2k$ with k a positive integer, and let $\Omega \subset \mathbb{R}^n$, with $m < n$, be a bounded domain. There exists a constant $C_{m,n} > 0$ such that*

$$\sup_{u \in W_N^{m,\frac{n}{m}}(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,n} |\Omega|,$$

and this inequality is sharp.

We give an alternative proof, following the idea of the proof of Theorem 3.1:

Proof of Theorem 4.1. By density arguments, it suffices to prove that

$$\sup_{u \in C_N^\infty(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,n} |\Omega|,$$

where

$$C_N^\infty(\Omega) := \{ u \in C^\infty(\Omega) \cap C^{m-2}(\overline{\Omega}) \;\big|\; u|_{\partial\Omega} = \Delta^j u|_{\partial\Omega} = 0\,,\ 1 \le j < k \}.$$

Let $u \in C^\infty_N(\Omega)$ be such that $\|\nabla^m u\|_{\frac{n}{m}} = \|\Delta^k u\|_{\frac{n}{m}} \le 1$, and set $f := \Delta^k u$ in Ω, so that u is a solution of the Navier boundary value problem

$$\begin{cases} \Delta^k u = f & \text{in } \Omega, \\ u = \Delta^j u = 0 & \text{on } \partial\Omega \quad \forall j \in \{1, 2, \dots, k-1\}. \end{cases}$$

We extend f by zero outside Ω,

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and we define

$$\overline{u} := \left(\frac{n}{\omega_{n-1}\beta_0} \right)^{\frac{n-m}{n}} I_m * |\overline{f}| \qquad \text{in } \mathbb{R}^n,$$

so that $(-1)^k \Delta^k \overline{u} = |\overline{f}|$ in $\mathbb{R}^n$. By construction $\overline{u} \ge 0$ in $\mathbb{R}^n$ and

$$\beta_0 |\overline{u}|^{\frac{n}{n-m}} \le \frac{n}{\omega_{n-1}} \left(\frac{I_m * |\overline{f}|}{\|\overline{f}\|_{\frac{n}{m}}} \right)^{\frac{n}{n-m}} \qquad \text{in } \mathbb{R}^n.$$

To end the proof, it suffices to show that $\overline{u} \ge |u|$ in Ω. Indeed, if $\overline{u} \ge |u|$ in Ω, then

$$\int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le \int_\Omega e^{\beta_0 |\overline{u}|^{\frac{n}{n-m}}}\, dx \le \int_\Omega e^{\frac{n}{\omega_{n-1}} \left(\frac{I_m * |\overline{f}|}{\|\overline{f}\|_{\frac{n}{m}}} \right)^{\frac{n}{n-m}}}\, dx,$$

and the last integral is bounded by a constant depending on Ω only, as a consequence of Theorem 3.2 with $p = \frac{n}{m} > 1$.

To see that $\overline{u} \ge |u|$, consider the systems,

$$\begin{cases} \Delta u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u_i = u_{i-1} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \quad i \in \{2, \dots, k\},$$

$$\begin{cases} \Delta \overline{u}_1 = (-1)^k |\overline{f}| & \text{in } \Omega, \\ \overline{u}_1 = \Delta^{k-1} \overline{u} & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta \overline{u}_i = \overline{u}_{i-1} & \text{in } \Omega, \\ \overline{u}_i = \Delta^{k-i} \overline{u} & \text{on } \partial\Omega, \end{cases} \quad i \in \{2, \dots, k\},$$

where obviously $u_k = u$ and $\overline{u}_k = \overline{u}$ in Ω. Since for $i \in \{1, 2, \dots, k-1\}$ we have

$$(-1)^k \Delta^{k-i} \overline{u} \begin{cases} \ge 0 & i \text{ even}, \\ \le 0 & i \text{ odd}, \end{cases} \qquad \text{in } \mathbb{R}^n$$

by finite induction, and with the aid of the maximum principle we can conclude that $\overline{u} \ge |u|$ in Ω and this ends the proof. $\square$

Now we begin the proof of the first part of Theorem 1.5. Let $u \in W^{m, \frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ be such that $\|u\|_{W^{m, \frac{n}{m}}} \le 1$. Fix $r_0 > 0$, and set

$$I_1 := \int_{B_{r_0}} \phi\left(\beta_0 |u|^{\frac{n}{n-m}} \right) dx, \qquad I_2 := \int_{\mathbb{R}^n \setminus B_{r_0}} \phi\left(\beta_0 |u|^{\frac{n}{n-m}} \right) dx,$$

so that

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}} \right) dx = I_1 + I_2.$$

We can notice that the starting point is the same as in the proof of the case $m = 2$, $n = 4$ and, as before, we will show that it is possible to choose a suitable

$r_0 > 0$ independent of u such that I_1 and I_2 are bounded by a constant which depends on r_0 only.

In the estimate of I_2 there are no substantial differences to the case $m = 2$ and $n = 4$, we first need a suitable radial lemma, namely an adaptation of [9], Lemma 1.1, Chapter 6:

Lemma 4.1. *If $u \in W^{1,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$, then*

$$|u(x)| \leq \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|u\|_{W^{1,\frac{n}{m}}}$$

for a.e. $x \in \mathbb{R}^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Applying this radial lemma and using the power series expansion of the exponential function, we get

$$\begin{aligned} I_2 \quad &\leq \quad \frac{\beta_0^{j_{\frac{n}{m}}-1}}{(j_{\frac{n}{m}}-1)!} \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)}\, dx \\ &+ \quad \frac{n^2(m-1)}{n-m} \sigma_n r_0^n \sum_{j=j_{\frac{n}{m}}}^{+\infty} \frac{1}{j!} \left(\frac{\beta_0 \|u\|_{W^{1,\frac{n}{m}}}^{\frac{n}{n-m}}}{(m\sigma_n)^{\frac{m}{n-m}} r_0^{\frac{n-1}{n-m}m}} \right)^j \\ &\leq \quad \frac{\beta_0^{j_{\frac{n}{m}}-1}}{(j_{\frac{n}{m}}-1)!} \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)}\, dx + c(m,\, n,\, r_0). \end{aligned}$$

To estimate the first term on the right-hand side of this last inequality, we need the continuity of the embedding of $W^{m,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ in suitable L^q-spaces:

Lemma 4.2 ([14], Théorème II.1)**.** *The embedding $W^{m\,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ is continuous for $\frac{n}{m} \leq q < +\infty$.*

Now it suffices to notice that $\frac{n}{n-m}\left(j_{\frac{n}{m}} - 1\right) \geq \frac{n}{m}$ to conclude that $I_2 \leq \tilde{c}(m, n, r_0)$.

To estimate I_1 we apply, as in the case $m = 2$ and $n = 4$, Theorem 4.1 to an auxiliary radial function $w \in W^{m,\frac{n}{m}}_N(B_{r_0})$ with $\|\nabla^m w\|_{\frac{n}{m}} \leq 1$ which increases the integral we are interested in. But the construction of this auxiliary function is rather difficult with respect to the case $m = 2$ and $n = 4$. In fact, in the case of second order derivatives, we only need to construct an auxiliary radial function which is zero on the boundary of B_{r_0}, while when dealing with mth order derivatives, with $m > 2$, the auxiliary radial function has to be zero on the boundary of B_{r_0} together with its jth order Laplacian for any $j \in \{1, 2, \ldots, k-1\}$.

If $m = 2k > 2$, then for each $i \in \{1, 2, \ldots, k-1\}$ we define

$$g_i(|x|) := |x|^{m-2i} \qquad \forall x \in B_{r_0}$$

so that $g_i \in W^{m,\frac{n}{m}}_{\mathrm{rad}}(B_{r_0})$ and

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j\, |x|^{m-2(i+j)} & \text{for } j \in \{1, \ldots, k-i\}, \\ 0 & \text{for } j \in \{k-i+1, \ldots, k\}, \end{cases} \qquad \forall x \in B_{r_0},$$

where

$$c_i^j := \prod_{h=1}^{j} [n + m - 2(h+i)]\, [m - 2(i+h-1)] \qquad \forall j \in \{1, 2, \ldots, k-i\}.$$

These functions will be helpful in the construction of the auxiliary radial function w. A similar device was used in [8] to prove an embedding result for higher order Sobolev spaces, but with another aim, namely to show that a radial function defined in a ball may be extended to the whole space without increasing the Dirichlet norm while increasing the L^p-norm.

Let

$$v(|x|) := u(|x|) - \sum_{i=1}^{k-1} a_i g_i(|x|) - a_k \qquad \forall x \in B_{r_0},$$

where

$$\begin{aligned} a_i &:= \frac{\Delta^{k-i}u(r_0) - \sum_{j=1}^{i-1} a_j \Delta^{k-i} g_j(r_0)}{\Delta^{k-i} g_i(r_0)}, \qquad \forall i \in \{1, 2, \ldots, k-1\}, \\ a_k &:= u(r_0) - \sum_{i=1}^{k-1} a_i g_i(r_0). \end{aligned}$$

We point out that if $m = 2k = 2$, namely when we deal with second order derivatives, then v reduces to

$$v(|x|) := u(|x|) - u(r_0) \qquad \forall x \in B_{r_0}.$$

By construction $v \in W_N^{m, \frac{n}{m}}(B_{r_0}) \cap W_{\mathrm{rad}}^{m, \frac{n}{m}}(B_{r_0})$ and $\Delta^k v = \Delta^k u$ in B_{r_0} or equivalently $\nabla^m v = \nabla^m u$ in B_{r_0}. Furthermore

Lemma 4.3. *For $0 < |x| \le r_0$ we have*

$$\begin{aligned} \Big|u(|x|)\Big|^{\frac{n}{n-m}} &\le \Big|v(|x|)\Big|^{\frac{n}{n-m}} \Bigg(1 + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{r_0^{2j\frac{n}{m}-1}} \|\Delta^{k-j} u\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} \\ &+ \frac{c_{m,n}}{r_0^{n-1}} \|u\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\Bigg)^{\frac{n}{n-m}} + d(m, n, r_0), \end{aligned}$$

where $c_{m,n} > 0$ depends only on m and n and $d(m, m, r_0) > 0$ depends only on m, n and r_0.

Proof. To simplify the notation, let

$$g(|x|) := \sum_{i=1}^{k-1} a_i g_i(|x|) + a_k \qquad \forall x \in B_{r_0}$$

so that $v(|x|) = u(|x|) - g(|x|)$ for all $x \in B_{r_0}$. Fix $0 < |x| \le r_0$, and set $r := |x|$ so $0 < r \le r_0$.

Step 1. We want to dominate $|u(r)|^{\frac{n}{n-m}}$ with $|v(r)|^{\frac{n}{n-m}}$ up to multiplicative and additive constants depending only on m, n, r_0 and $g(r)$, and more precisely we will prove that

(4.1)

$$|u(r)|^{\frac{n}{n-m}} \le |v|^{\frac{n}{n-m}} \left(1 + \frac{m}{n-m} 2^{\frac{m}{n-m}} |g(r)|^{\frac{n}{m}}\right) + 2^{\frac{m}{n-m}} \left(1 + \frac{n}{n-m} |g(r)|^{\frac{n}{n-m}}\right).$$

To this aim we recall that the binomial estimate

$$(a+b)^q \le a^q + q2^{q-1}(a^{q-1}b + b^q)$$

is valid for $q \geq 1$ and $a, b \geq 0$. Using the definition of v and applying this binomial estimate we get

$$|u(r)|^{\frac{n}{n-m}} \leq |v(r)|^{\frac{n}{n-m}} + \frac{n}{n-m} 2^{\frac{m}{n-m}} \left(|v(r)|^{\frac{m}{n-m}} |g(r)| + |g(r)|^{\frac{n}{n-m}}\right). \tag{4.2}$$

As Young's inequality says that

$$ab \leq \frac{m}{n}(ab)^{\frac{n}{m}} + \frac{n-m}{n}$$

provided that $ab \geq 0$, we can estimate

$$|v(r)|^{\frac{m}{n-m}} |g(r)| \leq \frac{m}{n} |v(r)|^{\frac{n}{n-m}} |g(r)|^{\frac{n}{m}} + \frac{n-m}{n}, \tag{4.3}$$

and this together with inequality (4.2) gives (4.1).

Step 2. We have to obtain a suitable estimate for $|g(r)|^\alpha$ and in particular we are interested in the cases $\alpha = \frac{n}{m}$ and $\alpha = \frac{n}{n-m}$, so we will assume that $\alpha > 1$. By convexity arguments

$$|g(r)|^\alpha \leq 2^{k(\alpha-1)+1} \sum_{i}^{k-1} |a_i|^\alpha g_i^\alpha(r_0) + 2^{\alpha-1} |u(r_0)|^\alpha.$$

We will prove in Step 3 below that

$$|a_i|^\alpha \leq \overline{c}_i \sum_{j=1}^{i} r_0^{2\alpha(i-j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha \qquad \forall i \in \{1, 2, \ldots, k-1\}, \tag{4.4}$$

where the constants $\overline{c}_i > 0$ depend on m and n only. As a consequence of (4.4) we get

$$\begin{aligned} |g(r)|^\alpha &\leq 2^{k(\alpha-1)+1} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \overline{c}_i r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha + 2^{\alpha-1} |u(r_0)|^\alpha \\ &= 2^{k(\alpha-1)+1} \sum_{j=1}^{k-1} \left(r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha \sum_{i=j}^{k-1} \overline{c}_i \right) + 2^{\alpha-1} |u(r_0)|^\alpha \\ &= 2^{k(\alpha-1)+1} \sum_{j=1}^{k-1} \tilde{c}_j r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha + 2^{\alpha-1} |u(r_0)|^\alpha \end{aligned}$$

with

$$\tilde{c}_j := \sum_{i=j}^{k-1} \overline{c}_i \qquad \forall j \in \{1, 2, \ldots, k-1\}.$$

Now the radial lemma, Lemma 4.1, leads to

$$\begin{aligned} |g(r)|^\alpha \leq 2^{k(\alpha-1)+1} &\left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}\alpha} \sum_{j=1}^{k-1} \tilde{c}_j r_0^{\alpha(m-2j-\frac{n-1}{n}m)} \|\Delta^{k-j} u\|^\alpha_{W^{1,\frac{n}{m}}} \\ &+ 2^{\alpha-1} \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}\alpha} \frac{1}{r_0^{\frac{n-1}{n}m\alpha}} \|u\|^\alpha_{W^{1,\frac{n}{m}}}. \end{aligned} \tag{4.5}$$

Step 3. We have to show that (4.4) holds. We proceed by finite induction on i. When $i = 1$, by the definition of a_1 and g_1 we have

$$|a_1|^\alpha = \left| \frac{\Delta^{k-1} u(r_0)}{\Delta^{k-1} g_1(r_0)} \right|^\alpha = \frac{1}{(c_1^{k-1})^\alpha} \left|\Delta^{k-1} u(r_0)\right|^\alpha,$$

which is nothing but (4.4) provided that $\overline{c}_1 := (c_1^{k-1})^{-\alpha}$. We now assume that (4.4) holds for any $j \in \{1, 2, \ldots, i\}$ with $i \in \{1, 2, \ldots, k-2\}$, and we show that

$$|a_{i+1}|^\alpha \leq \overline{c}_{i+1} \sum_{j=1}^{i+1} r_0^{2\alpha(i+1-j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha.$$

Using the definition of a_{i+1} and g_{i+1}, we get

$$|a_{i+1}|^\alpha \leq \frac{2^{\alpha-1}}{(c_{i+1}^{k-i-1})^\alpha} \left|\Delta^{k-i-1} u(r_0)\right|^\alpha + \frac{2^{i(\alpha-1)}}{(c_{i+1}^{k-i-1})^\alpha} \sum_{j=1}^{i} |a_j|^\alpha \left|\Delta^{k-i-1} g_j(r_0)\right|^\alpha.$$

By finite induction assumption and by definition of g_j with $j \in \{1, 2, \ldots, i\}$, we can estimate

$$\begin{aligned}
\sum_{j=1}^{i} a_j^\alpha \left(\Delta^{k-i-1} g_j(r_0)\right)^\alpha &\leq \sum_{j=1}^{i} \sum_{h=1}^{j} \overline{c}_j (c_j^{k-i-1})^\alpha r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^\alpha \\
&= \sum_{h=1}^{i} r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^\alpha \left(\sum_{j=h}^{i} \overline{c}_j (c_j^{k-i-1})^\alpha \right) \\
&= \sum_{h=1}^{i} \hat{c}_h r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^\alpha
\end{aligned}$$

with

$$\hat{c}_h := \sum_{j=h}^{i} \overline{c}_j (c_j^{k-i-1})^\alpha.$$

In conclusion

$$\begin{aligned}
|a_{i+1}|^\alpha &\leq \frac{2^{\alpha-1}}{(c_{i+1}^{k-i-1})^\alpha} \left|\Delta^{k-i-1} u(r_0)\right|^\alpha + \frac{2^{i(\alpha-1)}}{(c_{i+1}^{k-i-1})^\alpha} \sum_{h=1}^{i} \hat{c}_h r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^\alpha \\
&\leq \overline{c}_{i+1} \sum_{h=1}^{i+1} r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^\alpha.
\end{aligned}$$

Step 4. Combining (4.1) and inequality (4.5) with $\alpha = \frac{n}{n-m}$, we obtain that

$$|u(r)|^{\frac{n}{n-m}} \leq |v|^{\frac{n}{n-m}} \left(1 + \frac{m}{n-m} 2^{\frac{m}{n-m}} |g(r)|^{\frac{n}{m}}\right) + d(m,\, n,\, r_0),$$

as $\|\Delta^{k-j} u\|_{W^{1,\, \frac{n}{m}}} \leq 1$ for $j \in \{1, \ldots, k-1\}$ and $\|u\|_{W^{1,\, \frac{n}{m}}} \leq 1$. Now, a further application of inequality (4.5) with $\alpha = \frac{n}{m}$ leads to

$$\begin{aligned}
|u(r)|^{\frac{n}{n-m}} \leq |v(r)|^{\frac{n}{n-m}} &\left(1 + c_{m,\, n} \sum_{j=1}^{k-1} \frac{1}{r_0^{2j\frac{n}{m}-1}} \|\Delta^{k-j} u\|_{W^{1,\, \frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,\, n}}{r_0^{n-1}} \|u\|_{W^{1,\, \frac{n}{m}}}^{\frac{n}{m}} \right) \\
&+ d(m,\, n,\, r_0),
\end{aligned}$$

which easily implies the inequality expressed by the lemma. □

Now we define

$$w(|x|) := v(|x|)\left(1+c_{m,\,n}\sum_{j=1}^{k-1}\frac{1}{r_0^{2j\frac{n}{m}-1}}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}+\frac{c_{m,\,n}}{r_0^{n-1}}\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right)\quad \forall x\in B_{r_0}.$$

As $v\in W_N^{m,\,\frac{n}{m}}(B_{r_0})\cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_{r_0})$, we have that $w\in W_N^{m,\,\frac{n}{m}}(B_{r_0})\cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_{r_0})$ and from Lemma 4.3 it follows that

$$\Big|u(|x|)\Big|^{\frac{n}{n-m}}\le \Big|w(|x|)\Big|^{\frac{n}{n-m}}+d(m,\,n,\,r_0)\qquad \forall 0<|x|\le r_0.$$

Since

$$\|\nabla^m v\|_{\frac{n}{m}}=\|\nabla^m u\|_{\frac{n}{m}}\le\left(1-\sum_{j=1}^{k-1}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}-\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right)^{\frac{m}{n}},$$

and the inequality

$$(1-A)^q\le 1-qA$$

holds for $0\le A\le 1$ and for $0<q\le 1$, we have that

$$\|\nabla^m v\|_{\frac{n}{m}}\le\left(1-\frac{m}{n}\sum_{j=1}^{k-1}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}-\frac{m}{n}\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right).$$

Therefore

$$\begin{aligned}\|\nabla^m w\|_{\frac{n}{m}} &= \|\nabla^m v\|_{\frac{n}{m}}\left(1+c_{m,\,n}\sum_{j=1}^{k-1}\frac{1}{r_0^{2j\frac{n}{m}-1}}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}+\frac{c_{m,\,n}}{r_0^{n-1}}\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right)\\ &\le \left(1-\frac{m}{n}\sum_{j=1}^{k-1}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}-\frac{m}{n}\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right)\\ &\quad\cdot\left(1+c_{m,\,n}\sum_{j=1}^{k-1}\frac{1}{r_0^{2j\frac{n}{m}-1}}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}+\frac{c_{m,\,n}}{r_0^{n-1}}\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\right),\end{aligned}$$

and in conclusion

(4.6)

$$\|\nabla^m w\|_{\frac{n}{m}}\le 1+\sum_{j=1}^{k-1}\left(\frac{c_{m,\,n}}{r_0^{2j\frac{n}{m}-1}}-\frac{m}{n}\right)\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}+\left(\frac{c_m}{r_0^{n-1}}-\frac{m}{n}\right)\|u\|^{\frac{n}{m}}_{W^{1,\,\frac{n}{m}}}\le 1,$$

provided that $r_0>0$ is sufficiently large: this is our choice of $r_0>0$. In conclusion

$$I_1\le e^{\beta_0 d(m,\,n,\,r_0)}\int_{B_{r_0}}e^{\beta_0|w|^{\frac{n}{n-m}}}\,dx,$$

and the right-hand side of this inequality is bounded by a constant depending on r_0 only as a consequence of Theorem 4.1.

We end this section with the

Proof of Proposition 1.1. We want to adapt the above arguments to obtain a proof of Proposition 1.1. The idea is to proceed exactly as in the case $m=2$ and $n=4$, but for this we have to specify

- how the radial lemma (Lemma 4.1) can be used to obtain pointwise estimates for u and $\Delta^j u$ with $j\in\{1,\,2,\,\ldots,\,k-1\}$; and

- how to modify the argument (Lemma 4.2) used in the estimate of I_2 to obtain an upper bound for the integral

$$\int_{B_R \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)} \, dx \tag{4.7}$$

  independent of u and R.

Let $u \in W^{m,\frac{n}{m}}_{N,\mathrm{rad}}(B_R)$ with $B_R \subset \mathbb{R}^n$. Since

$$W^{m,\frac{n}{m}}_{N}(B_R) \subset W^{1,\frac{n}{m}}_{0}(B_R),$$

we may extend u by zero outside B_R, and obtain $u \in W^{1,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ with

$$\|u\|_{W^{1,\frac{n}{m}}(\mathbb{R}^n)} = \|u\|_{W^{1,\frac{n}{m}}(B_R)}.$$

Thus we can apply Lemma 4.1 to u.

Similarly, for fixed $j \in \{1, 2, \ldots, k-1\}$, we have

$$W^{m-2j,\frac{n}{m}}_{N}(B_R) \subset W^{1,\frac{n}{m}}_{0}(B_R),$$

and since $\Delta^j u \in W^{m-2j,\frac{n}{m}}_{N,\mathrm{rad}}(B_R)$, we have in particular $\Delta^j u \in W^{1,\frac{n}{m}}_{0}(B_R)$. We extend $\Delta^j u$ to be zero outside B_R

$$f_j := \begin{cases} \Delta^j u & \text{in } B_R, \\ 0 & \text{in } \mathbb{R}^n \setminus B_R. \end{cases}$$

As $\Delta^j u \in W^{1,\frac{n}{m}}_{0}(B_R)$ is radial, we have that $f_j \in W^{1,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ and f_j satisfies the assumption of Lemma 4.1. Therefore, for a.e. $x \in B_R$, we have

$$\begin{aligned} |\Delta^j u(x)| &= |f_j(x)| \le \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|f_j\|_{W^{1,\frac{n}{m}}(\mathbb{R}^n)} \\ &= \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|\Delta^j u\|_{W^{1,\frac{n}{m}}(B_R)}. \end{aligned}$$

It remains only to specify how to obtain an upper bound independent of u and R for the integral (4.7). Let $u \in W^{m,\frac{n}{m}}_{N,\mathrm{rad}}(B_R)$ be such that $\|u\|_{W^{m,\frac{n}{m}}} \le 1$. As $u \in W^{1,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$, from Lemma 4.1, it follows that there exists $r_1 = r_1(m, n) > 0$ independent of u and R such that

$$|u(x)| < 1 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus B_{r_1}.$$

Therefore for $R > r_1$, we can choose $0 < r_1 \le r_0 < R$ so that

$$|u(x)| < 1 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus B_{r_0},$$

and since

$$\frac{n}{n-m}\left(j_{\frac{n}{m}} - 1\right) \ge \frac{n}{m},$$

we obtain that

$$\int_{B_R \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)} dx \le \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)} dx \le \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{m}} dx \le 1.$$

To conclude, we can argue as in the proof of Proposition 1.1 in the case $m = 2$ and $n = 4$, but now the two alternatives that we have to distinguish are $R < \tilde{R}$ and $R \ge \tilde{R}$ with $\tilde{R} > r_1$ and such that (4.6) holds. □

5. Proof of the main theorem (Theorem 1.4)

Let $m = 2k$ with k a positive integer, let $m < n$ and let $\Omega \subseteq \mathbb{R}^n$ be a domain. Since any function $u \in W_0^{m,\frac{n}{m}}(\Omega)$ can be extended to be zero outside Ω obtaining a function in $(W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{m,n})$, we have that

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{m,n} \le 1} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx,$$

and the proof of the first part of Theorem 1.4 reduces to the inequality

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \qquad \forall u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\ \|u\|_{m,n} = 1 \tag{5.1}$$

for some constant $C_{m,n} > 0$.

Let $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$ be such that $\|u\|_{m,n} = 1$. Then there exists $\{u_j\}_{j \ge 1} \subset C_0^\infty(\mathbb{R}^n)$ such that $u_j \to u$ in $(W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{m,n})$ and $\|u_j\|_{m,n} = 1\ \forall j \ge 1$. Therefore $u_j \to u$ a.e. in $\mathbb{R}^n$, up to subsequences, and by Fatou's lemma

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \liminf_{j \to +\infty} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx.$$

But, for each fixed $j \ge 1$, there exists $R_j > 0$ such that supp $u_j \subset B_{R_j}$, so

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx = \int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx.$$

It is clear that if we can bound the integral on the right-hand side of this last equality with a constant independent of j, then the proof of (5.1) is completed and hence Theorem 1.4 is thus proved. So it suffices to show that there exists a constant $C_{m,n} > 0$ independent of j such that

$$\int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \qquad \forall j \ge 1. \tag{5.2}$$

To this aim, for fixed $j \ge 1$, we define

$$f_j := (-\Delta + I)^k u_j,$$

and consider the problem

$$\begin{cases} (-\Delta + I)^k v_j = f_j^\sharp & \text{in } B_{R_j}, \\ v_j \in W_N^{m,2}(B_{R_j}). \end{cases} \tag{5.3}$$

We now apply Proposition 2.4, which leads to a comparison between the integral in (5.2) and an analogous one involving v_j, as pointed out in Remark 2.2. In this way we obtain the estimate

$$\int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx \le \int_{B_{R_j}} \phi\left(\beta_0 |v_j|^{\frac{n}{n-m}}\right) dx.$$

This estimate reduces the proof of (5.2) to the inequality

$$\int_{B_{R_j}} \phi\left(\beta_0 |v_j|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \tag{5.4}$$

for some constant $C_{m,n} > 0$ independent of j. But, as already noticed in Remark 2.2, $v_j \in W^{m,\frac{n}{m}}_{N,\,\mathrm{rad}}(B_R)$ and by (1.5)

$$\|v_j\|_{W^{m,\frac{n}{m}}} \le \|v_j\|_{m,n} = \|u_j\|_{m,n} = 1.$$

Thus (5.4) is a consequence of Proposition 1.1. □

We end the section with the proof of Proposition 1.2.

Proof of Proposition 1.2. As in the proof of Theorem 4.1, by density arguments it suffices to prove that (1.7) holds for functions in

$$C^\infty_N(\Omega) := \left\{ u \in C^\infty(\Omega) \cap C^{m-2}(\overline{\Omega}) \;\middle|\; u|_{\partial\Omega} = \Delta^j u|_{\partial\Omega} = 0\,,\ 1 \le j < k := \frac{m}{2} \right\}.$$

Let $u \in C^N_\infty(\Omega)$ be such that $\|u\|_{m,n} \le 1$. We define

$$f := (-\Delta + I)^k u,$$

and we consider the problem

$$\begin{cases} (-\Delta + I)^k v = f^\sharp & \text{in } \Omega^\sharp, \\ v \in W^{m,2}_N(\Omega^\sharp), \end{cases} \tag{5.5}$$

where $\Omega^\sharp$ is the ball in $\mathbb{R}^n$ centered at $0 \in \mathbb{R}^n$ with the same measure as Ω. Thus, as Ω is a bounded domain, we can apply the iterated version of the Trombetti-Vazquez comparison principle (see Remark 2.1) obtaining that

$$\int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \int_{\Omega^\sharp} \phi\left(\beta_0 |v|^{\frac{n}{n-m}}\right) dx,$$

and the last integral is bounded by a constant $C_{m,n} > 0$ independent of the domain Ω as a consequence of Proposition 1.1. □

6. Sharpness

We have already mentioned in the Introduction that Kozono et al. ([11], Corollary 1.3) proved that the supremum

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{m,n} \le 1} \int_{\mathbb{R}^n} \phi\left(\beta |u|^{\frac{n}{n-m}}\right) dx$$

is infinite for $\beta > \beta_0$. To do this, they argue by contradiction using Bessel potentials and the sharpness of Adams' inequality (1.2), while here we will exhibit a sequence of test functions for which the integral in (1.3) can be made arbitrarily large, if the exponent β_0 is replaced by a number $\beta > \beta_0$.

In the case $m = 2$ and $n = 4$, we will consider a sequence of test functions that was used in [15] to prove a generalized version of Adams' inequality for bounded domains in $\mathbb{R}^4$. The following proposition gives the sharpness of inequality (1.3) in the case $m = 2$ and $n = 4$.

Proposition 6.1. *Assume that $\beta > 32\pi^2$. Then for any domain $\Omega \subseteq \mathbb{R}^4$*

$$\sup_{u \in W^{2,2}_0(\Omega),\, \|u\|_{2,4} \le 1} \int_\Omega (e^{\beta u^2} - 1)\, dx = +\infty.$$

Proof. Without loss of generality we assume that the unit ball $B_1 \subset \Omega$. For $\varepsilon > 0$, we define

$$u_\varepsilon(x) := \begin{cases} \sqrt{\frac{1}{32\pi^2}\log\frac{1}{\varepsilon}} - \frac{|x|^2}{\sqrt{8\pi^2\varepsilon\log\frac{1}{\varepsilon}}} + \frac{1}{\sqrt{8\pi^2\log\frac{1}{\varepsilon}}} & |x| \le \sqrt[4]{\varepsilon}, \\ \frac{1}{\sqrt{2\pi^2\log\frac{1}{\varepsilon}}}\log\frac{1}{|x|} & \sqrt[4]{\varepsilon} < |x| \le 1, \\ \eta_\varepsilon & |x| > 1, \end{cases} \tag{6.1}$$

where $\eta_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ is such that $\eta_\varepsilon|_{\partial B_1} = \eta_\varepsilon|_{\partial\Omega} = 0\,,\ \frac{\partial\eta_\varepsilon}{\partial\nu}|_{\partial B_1} = \frac{1}{\sqrt{2\pi^2\log\frac{1}{\varepsilon}}}\,,\ \frac{\partial\eta_\varepsilon}{\partial\nu}|_{\partial\Omega} = 0$ and η_ε, $|\nabla\eta_\varepsilon|$, $\Delta\eta_\varepsilon$ are all $O\left(1/\sqrt{\log\frac{1}{\varepsilon}}\right)$. If $0 < \varepsilon < 1$, then we have that $u_\varepsilon \in W_0^{2,\,2}(\Omega)$, and easy computations give

$$\|u_\varepsilon\|_2^2 = o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big),\ \|\nabla u_\varepsilon\|_2^2 = o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big),\ \|\Delta u_\varepsilon\|_2^2 = 1 + o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)$$

and $\|u_\varepsilon\|_{2,\,4} = \Big(\|\Delta u_\varepsilon\|_2^2 + 2\|\nabla u_\varepsilon\|_2^2 + \|u_\varepsilon\|_2^2\Big)^{1/2} \to 1$ as $\varepsilon \to 0^+$. Now we normalize u_ε, setting

$$\tilde{u}_\varepsilon := \frac{u_\varepsilon}{\|u_\varepsilon\|_{2,\,4}} \in W_0^{2,\,2}(\Omega)$$

for $\varepsilon > 0$ sufficiently small. Since

$$\tilde{u}_\varepsilon \ge \frac{1}{\|u_\varepsilon\|_{2,\,4}}\sqrt{\frac{1}{32\pi^2}\log\frac{1}{\varepsilon}} \quad \text{on } B_{\sqrt[4]{\varepsilon}}\,,$$

we have

$$\begin{aligned} \sup_{u\in W_0^{2,\,2}(\Omega),\,\|u\|_{2,\,4}\le 1}\int_\Omega (e^{\beta u^2} - 1)\,dx &\ge \lim_{\varepsilon\to 0^+}\int_{B_{\sqrt[4]{\varepsilon}}} (e^{\beta\tilde{u}_\varepsilon^2} - 1)\,dx \\ &\ge \lim_{\varepsilon\to 0^+} 2\pi^2\Big(e^{\frac{1}{\|u_\varepsilon\|^2}\frac{\beta}{32\pi^2}\log\frac{1}{\varepsilon}} - 1\Big)\left[\frac{r^4}{4}\right]_0^{\sqrt[4]{\varepsilon}} \\ &= +\infty. \end{aligned}$$

□

The test functions u_ε with $\varepsilon > 0$ defined in (6.1) of the above proof also give the sharpness of inequalities (1.6), (1.7), and (1.8) in the case $m = 2$ and $n = 4$.

We now consider the general case $m = 2k < n$ with k a positive integer. In this case the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains in [1] also gives the sharpness of Adams' inequality in unbounded domains.

Proposition 6.2. *Assume that $\beta > \beta_m$. Then, for any domain $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u\in W_0^{m,\,\frac{n}{m}}(\Omega),\,\|u\|_{m,\,n}\le 1}\int_\Omega \phi\left(\beta|u|^{\frac{n}{n-m}}\right)\,dx = +\infty.$$

Proof. Without loss of generality we assume that the unit ball $B_1 \subset \Omega$. Let $\phi \in C^\infty([0,\,1])$ be such that

$$\begin{aligned} \phi(0) &= \phi'(0) = \cdots = \phi^{m-1}(0) = 0, \\ \phi(1) &= \phi'(1) = 1, \quad \phi''(1) = \cdots = \phi^{(m-1)}(1) = 0. \end{aligned}$$

For $0<\varepsilon<\frac{1}{2}$ we set

$$H(t):=\begin{cases}\varepsilon\phi\left(\frac{t}{\varepsilon}\right) & 0<t\le\varepsilon,\\ t & \varepsilon<t\le 1-\varepsilon,\\ 1-\varepsilon\phi\left(\frac{1-t}{\varepsilon}\right) & 1-\varepsilon<t\le 1,\\ 1 & 1<t,\end{cases}$$

and the choice of $0<\varepsilon<\frac{1}{2}$ will be made during the proof. We introduce Adams' test functions

$$\psi_r(|x|):=H\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{r}}\right)\qquad\forall x\in\mathbb{R}^n\setminus\{0\}.$$

By construction, for $r>0$ sufficiently small, $\psi_r\in W_0^{m,\frac{n}{m}}(\Omega)$, $\psi(|x|)=1$ for $x\in B_r\setminus\{0\}$, and Adams in [1] proved that

$$\|\nabla^m\psi_r\|_{\frac{n}{m}}^{\frac{n}{m}}\le\omega_{n-1}a(m,n)^{\frac{n}{m}}\left(\log\frac{1}{r}\right)^{1-\frac{n}{m}}A_r,$$

where

$$a(m,n):=\frac{\beta_0^{\frac{n-m}{n}}}{n\sigma_n^{\frac{m}{n}}},\qquad A_r=A_r(m,n):=\left[1+2\varepsilon\Big(\|\psi'\|_\infty+O\big((\log 1/r)^{-1}\big)\Big)^{\frac{n}{m}}\right].$$

Easy computations also give that for $r>0$ sufficiently small

$$\begin{aligned}\|\psi_r\|_{\frac{n}{m}}^{\frac{n}{m}}&=o\Big(\big(\log\frac{1}{r}\big)^{-\frac{n-m}{m}}\Big),\\ \|\nabla^j\psi_r\|_{\frac{n}{m}}^{\frac{n}{m}}&=o\Big(\big(\log\frac{1}{r}\big)^{-\frac{n-m}{m}}\Big)\quad\forall j\in\{1,2,\dots,m-1\}.\end{aligned}$$

Now we define

$$u_r(|x|):=\left(\log\frac{1}{r}\right)^{\frac{n-m}{n}}\cdot\psi_r(|x|)\qquad\forall x\in\mathbb{R}^n\setminus\{0\}.$$

We can notice that for $r>0$ sufficiently small $u_r\in W_0^{m,\frac{n}{m}}(\Omega)$, $u_r(|x|)=\left(\log\frac{1}{r}\right)^{\frac{n-m}{n}}$ for $x\in B_r\setminus\{0\}$ and

$$\begin{aligned}\|u_r\|_{m,n}^{\frac{n}{m}}&\le\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+c_{m,n}\Big(\|u_r\|_{\frac{n}{m}}^{\frac{n}{m}}+\sum_{j=1}^{m-1}\|\nabla^j u_r\|_{\frac{n}{m}}^{\frac{n}{m}}\Big)\\ &\le\omega_{n-1}a^{\frac{n}{m}}(m,n)\big(A_r+o(1)\big),\end{aligned}$$

so in particular

$$\|u_r\|_{m,n}^{\frac{n}{n-m}}\le\omega_{n-1}^{\frac{m}{n-m}}a^{\frac{n}{n-m}}(m,n)\big(A_r+o(1)\big)^{\frac{m}{n-m}}=\frac{\beta_0}{n}\big(A_r+o(1)\big)^{\frac{m}{n-m}}.$$

Therefore, for $r > 0$ sufficiently small, we have

$$\begin{aligned}\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\,\|u\|_{m,n}\le 1}\int_\Omega \phi\left(\beta|u|^{\frac{n}{n-m}}\right)dx &\ge \lim_{r\to 0^+}\int_{B_r}\phi\left(\beta\left(\frac{|u_r|}{\|u_r\|_{m,n}}\right)^{\frac{n}{n-m}}\right)dx\\ &\ge \lim_{r\to 0^+}\sigma_n\phi\left(\frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\log\frac{1}{r}\right)r^n\\ &\ge \lim_{r\to 0^+}\sigma_n e^{\log r\left(n-\frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right)}.\end{aligned}$$

If we choose $0 < \varepsilon < \frac{1}{2}$ so that

$$\beta_0 < \beta_0\left(1+2\varepsilon\|\phi'\|_\infty^{\frac{n}{m}}\right)^{\frac{m}{n-m}} < \beta,$$

then

$$\lim_{r\to 0^+}\left(n-\frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right)\le n\left(1-\frac{\beta}{\beta_0(1+2\varepsilon\|\phi'\|_\infty^{\frac{n}{m}})^{\frac{m}{n-m}}}\right)<0$$

and

$$\lim_{r\to 0^+}\sigma_n e^{\log r\left(n-\frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right)} = +\infty$$

□

The same proof also gives the sharpness of inequalities (1.6), (1.7), and (1.8) in the general case $m = 2k < n$ with k a positive integer.

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