ISOMETRIC EMBEDDINGS OF KÄHLER-RICCI SOLITONS IN
THE COMPLEX PROJECTIVE SPACE

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Abstract. We prove that a compact complex manifold endowed with a non-trivial Kähler-Ricci soliton cannot be isometrically embedded in the Fubini-Study complex projective space as a complete intersection.

INTRODUCTION

A Kähler metric $g$ on a complex manifold $M$ is said to be a Kähler Ricci soliton if there exists a holomorphic vector field $V$ on $M$ such that

$$\text{Ric}(g) = \lambda g + \mathcal{L}_V g,$$

where $\lambda$ is a real constant. Kähler Ricci solitons have been extensively studied in recent years mainly because they provide self-similar solutions to the Kähler Ricci flow which was introduced as a mean for finding Kähler-Einstein metrics. Kähler Ricci solitons are indeed a generalization of Kähler-Einstein metrics (taking $V = 0$ in (0.1) we get the Einstein equation) but they are alternative to them because the presence of a Kähler Ricci soliton with nontrivial $V$ is an obstruction to the existence of a Kähler-Einstein metric on a compact complex manifold with positive first Chern class (The Futaki invariant with respect to the real part of $V$ is nonzero). In fact it is a deep result proved by Tian and Zhu [10] that a compact Fano manifold can admit at most one Kähler Ricci soliton, including trivial ones.

The first nontrivial examples of compact Kähler Ricci solitons were found by Koiso: in [5] he proved the existence of a KRS on any Fano manifold admitting a cohomogeneity one action of a compact semisimple Lie group of isometries with two complex singular orbits. After that, Wang and Zhu [11] proved the existence of KRS on any compact toric Fano manifold and this result was later generalized in [8] to toric bundles over generalized flag manifolds. Since all compact KRS are Fano and can be holomorphically embedded in the complex projective space $\mathbb{CP}^m$, it is natural to ask whether a Kähler Ricci soliton may be induced by the Fubini-Study metric of $\mathbb{CP}^m$.

In this note we prove the following negative result. Recall that a smooth codimension $r$ subvariety of $\mathbb{CP}^m$ is a complete intersection if its ideal is generated by $r$ elements or equivalently if it may be described as the transverse intersection of $r$ algebraic hypersurfaces.
Theorem 0.1. Let $M$ be a closed complex submanifold of $\mathbb{CP}^m$ such that the metric induced on $M$ by the Fubini-Study metric $\omega_{FS}$ is a Kähler-Ricci soliton. If $M$ is a complete intersection then the Kähler-Ricci soliton is trivial and $M$ is a linear subspace or a smooth quadric subvariety of some linear subspace.

Our result may be thought as a generalization of the main theorem of [3] where the classification of Kähler Einstein manifolds isometrically embedded in $(\mathbb{CP}^n, \omega_{FS})$ as complete intersections is given. For general smooth subvarieties, beside the homogeneous case of flag manifolds (see [9] for the classification), no example of positive Kähler Einstein metric induced by $\omega_{FS}$ is known. On the other hand a Kähler Einstein submanifold of $(\mathbb{CP}^n, \omega_{FS})$ has necessarily positive scalar curvature by a result of Hulin [4].

1. Proof of the theorem

1.1. Kähler Ricci solitons. Let $M$ be a complex manifold and denote by $J$ its complex structure. Rephrasing (0.1) in terms of 2-forms, a Kähler Ricci soliton on $M$ is a Kähler metric $g$ whose associated, Ricci and Kähler form $\rho$ and $\omega = g(J\cdot, \cdot)$ respectively satisfy

\[ \rho = \lambda \omega + \mathcal{L}_V \omega \]

for some holomorphic vector field $V = X - iJX$, where $J$ is the complex structure. We will say that the Kähler Ricci soliton is trivial if $V = 0$, i.e. $(M, g)$ is Kähler-Einstein.

Note that $\mathcal{L}_X J = 0$ because $V$ is holomorphic and equation (1.1) implies that $\mathcal{L}_{JX} \omega = 0$, i.e. $JX$ preserves $\omega$, hence $g$ because it also preserves $J$. Note also that (1.1) implies

\[ \rho = \lambda \omega + \mathcal{L}_X \omega. \]

The fact that $\nabla X$ is $g$-self adjoint means that the 1-form dual to $X$ is closed; since a KRS may exist only on Fano manifolds and these are simply connected (Kobayashi’s theorem), we see that $X$ is the gradient with respect to $g$ of some smooth function $f$. We will indicate $\nabla f := \text{grad}_g(f)$. This implies that

\[ \mathcal{L}_X \omega = \mathcal{L}_{\nabla f} \omega = d\nabla f \omega = dJ\nabla f \omega = dd^c f. \]

Recalling\(^1\) that $\partial = \frac{1}{2}(d + id^c)$ and $\bar{\partial} = \frac{1}{2}(d - id^c)$, equation (1.2) turns out to be equivalent to

\[ \rho = \lambda \omega + 2i\bar{\partial} \partial f. \]

Indeed the previous computation shows also that the function $f$ indeed admits another useful interpretation. Since $i_{JX} \omega = -df$ the function $f$ is, up to a constant multiple, a moment map for the infinitesimal action of the Killing vector field $JX$ on $M$, or more precisely it is the projection along $JX$ of a moment map $\mu$ for the Hamiltonian action of $\text{Iso}(M, g)$ on $M$. (Recall that since $M$ is simply connected every symplectic action on $M$ is Hamiltonian)

\(^1\)We are using the convention according to which $d^c h(Y) = Jdh(Y) = dh(-JY)$. 

1.2. proof of the theorem. Let \( n \) be the complex dimension of \( M \) and \( r = m - n \) the codimension. Denote also by \( i : M \to \mathbb{C}P^m \) the inclusion and simply by \( \omega \) the restriction \( i^*\omega_{FS} \). By hypothesis \( \omega \) satisfies (1.3) where \( f \) is the potential of the holomorphic vector field \( X = \nabla f \). We suppose that \( M \) is embedded in \( \mathbb{C}P^m \) as a complete intersection. Namely \( M \) is assumed to admit \( r \) homogeneous polynomials \( P_1, P_2, \ldots, P_r \) on \( \mathbb{C}^{n+1} \) which define \( M \) as their zero locus and generate the ideal associated to \( M \).

It is a direct consequence of the adjunction formula that the canonical line bundle \( K_M = \Lambda^{n,0} \) of \( M \) is the restriction of a line bundle on \( \mathbb{C}P^m \), more precisely

\[
K_M = i^*\mathcal{O}(d - m - 1)
\]

where \( d = \sum_{j=1}^{r} \deg P_j \). Since the Chern class of \( K_M \) is represented by \( \frac{1}{2\pi} \) times the Ricci form, the constant \( \lambda \) in (1.3) is forced to be equal to \( m + 1 - d > 0 \).

It is well known Hermitian metrics \( h \) on \( K_M^* \) correspond bijectively to positive volumes (nowhere vanishing real \( 2n \)-forms) \( v \) of \( M \), the correspondence being given by

\[
\langle v, (-2)^m(\sqrt{-1})^m x \wedge \bar{x} \rangle = h(x, x)
\]

for \( x \in K_M^* \). Let \( V \) be the volume of \( M \) corresponding to the fibre metric on \( K_M^* \), whose Chern curvature form is exactly \((m + 1 - d)\omega\). In [3] (proposition 2) it is computed explicitly the real positive function \( \phi \) such that \( \omega^n = \phi V \) in the case where \( M \) is a complete intersection. More precisely, recalling that the Chern curvature form of the fibre metric induced by \( \omega \) on \( K_M^* \) is exactly the Ricci form \( \rho \) (see [1], p.82), we have the following

Proposition 1.1 (Hano [3]). Let \( M \) be a complete intersection in \( \mathbb{C}P^m \) defined by the polynomials \( P_1, \ldots, P_r \). Denote by \( d = \sum_i \deg P_i \) and by \( \rho \) the Ricci form of the metric \( \omega \) induced by \( \omega_{FS} \). Then we have

\[
\rho = (m + 1 - d)\omega + i\partial\bar{\partial} \log \phi, \quad \text{with} \quad \phi = \frac{\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2}{\|z\|^{2(d-r)}}.
\]

Here \( \phi \) is expressed in terms of unitary homogeneous coordinates of \( \mathbb{C}P^m \) and \( \|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2 = \sum |P_{\lambda_1 \ldots \lambda_r}|^2 \) where \( dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r = \sum P_{\lambda_1 \ldots \lambda_r} dz_{\lambda_1} \wedge \cdots \wedge dz_{\lambda_r} \). Note also that the previous expression of \( \phi \) is invariant under any unitary coordinate transformation.

Combining (1.4) with the Kähler-Ricci soliton equation we get

\[
\partial\bar{\partial} \log \phi = 2\partial\bar{\partial} f.
\]

so that

\[
\phi = C \cdot e^{2f}
\]

for some constant \( C \in \mathbb{R} \). Now the key fact is that we can find an explicit expression also for \( f \) in terms of homogeneous coordinates of \( \mathbb{C}P^m \). Indeed, as already remarked, \( f \) is a moment map for the action of the 1-parameter group of isometries generated by \( JX \) and this enables us to write it down in suitable coordinates.

To start with, by a famous result of Calabi [2] the Killing vector field \( JX \) can be extended to a Killing vector field of \((\mathbb{C}P^m, \omega_{FS})\) so that with respect to an appropriate system of unitary homogeneous coordinates it can be written in diagonal form \( \text{diag}(i\lambda_0, \ldots, i\lambda_m) \) as an element of \( \text{su}(m+1) \).
Thus a moment map for the Hamiltonian action of the 1-parameter group \( \{ \exp tJX \} \) on \( \mathbb{C}P^m \) is
\[
\mu_{JX} = \frac{1}{2} \sum_{j=0}^{m} \lambda_j |z_j|^2
\]
and \( f \) is nothing but the restriction of \( \mu_{JX} \) to \( M \). So there exists a constant \( C \in \mathbb{R} \) such that on \( M \) one has
\[
\frac{\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|}{\|z\|^{2(d-r)}} = Ce \frac{\sum \lambda_j |z_j|^2}{\Sigma |z_j|^2}.
\]
We claim that (1.5) holds if and only if \( f(z, \bar{z}) \) is constant. Let \( p \) and \( q \) be any two points of \( M \). Since \( M \) is Fano, by a Theorem of Kollár Miyaoka and Mori [6] there exists a rational curve passing through \( p \) and \( q \), say \( F: \mathbb{C}P^1 \to M \subseteq \mathbb{C}P^m \) defined by \( F([s : t]) = [F_0(s, t) : \ldots : F_m(s, t)] \) where the functions \( F_m(s, t) \) are homogeneous polynomials of degree \( \delta \) in \( s \) and \( t \).

Evaluating (1.5) at \( F(\mathbb{C}P^1) \) we get
\[
\frac{\|dP_1(F([s : t])) \wedge \cdots \wedge dP_r(F([s : t]))\|^2}{(\sum_j |F_j(s, t)|^2)^{(d-r)}} = Ce \frac{\sum \lambda_j |F_j(s, t)|^2}{\Sigma |F_j(s, t)|^2}
\]
for every \( [s : t] \in \mathbb{C}P^1 \) and this is clearly impossible unless \( f \) is constant on \( F(\mathbb{C}P^1) \), otherwise the right hand side of (1.6) would not be a rational function of \( s \) and \( t \). Since \( p \) and \( q \) are arbitrary, \( f \) must be constant on all of \( M \): this means that \( X = \nabla f \) vanishes and the Kähler-Ricci soliton is trivial, i.e. \( \omega \) is Kähler-Einstein. According to Hano [3] this happens only if \( M \) is a linear subspace or it is a smooth quadric subvariety of some linear subspace.

**References**

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