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ISOMETRIC EMBEDDINGS OF KÄHLER-RICCI SOLITONS IN THE COMPLEX PROJECTIVE SPACE

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ABSTRACT. We prove that a compact complex manifold endowed with a nontrivial Kähler-Ricci soliton cannot be isometrically embedded in the Fubini-Study complex projective space as a complete intersection.

INTRODUCTION

A Kähler metric g on a complex manifold M is said to be a Kähler Ricci soliton if there exists a holomorphic vector field V on M such that

$$\mathrm{Ric}(g) = \lambda g + \mathcal{L}_V g\,, \tag{0.1}$$

where λ is a real constant. Kähler Ricci solitons have been extensively studied in recent years mainly because they provide self-similar solutions to the Kähler Ricci flow which was introduced as a mean for finding Kähler-Einstein metrics. Kähler Ricci solitons are indeed a generalization of Kähler-Einstein metrics (taking $V = 0$ in (0.1) we get the Einstein equation) but they are alternative to them because the presence of a Kähler Ricci soliton with nontrivial V is an obstruction to the existence of a Kähler-Einstein metric on a compact complex manifold with positive first Chern class (The Futaki invariant with respect to the real part of V is nonzero). In fact it is a deep result proved by Tian and Zhu [10] that a compact Fano manifold can admit at most one Kähler Ricci soliton, including trivial ones.

The first nontrivial examples of compact Kähler Ricci solitons were found by Koiso: in [5] he proved the existence of a KRS on any Fano manifold admitting a cohomogeneity one action of a compact semisimple Lie group of isometries with two complex singular orbits. After that, Wang and Zhu [11] proved the existence of KRS on any compact toric Fano manifold and this result was later generalized in [8] to toric bundles over generalized flag manifolds. Since all compact KRS are Fano and can be holomorphically embedded in the complex projective space $\mathbb{CP}^m$, it is natural to ask whether a Kähler Ricci soliton may be induced by the Fubini-Study metric of $\mathbb{CP}^m$.

In this note we prove the following negative result. Recall that a smooth codimension r subvariety of $\mathbb{CP}^m$ is a complete intersection if its ideal is generated by r elements or equivantly if it may be described as the transverse intersection of r algebraic hypersurfaces.

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Theorem 0.1. *Let M be a closed complex submanifold of $\mathbb{CP}^m$ such that the metric induced on M by the Fubini-Study metric ω_{FS} is a Kähler-Ricci soliton. If M is a complete intersection then the Kähler-Ricci soliton is trivial and M is a linear subspace or a smooth quadric subvariety of some linear subspace.*

Our result may be thought as a generalization of the main theorem of [3] where the classification of Kähler Einstein manifolds isometrically embedded in $(\mathbb{CP}^n, \omega_{FS})$ as complete intersections is given. For general smooth subvarieties, beside the homogeneous case of flag manifolds (see [9] for the classification), no example of positive Kähler Einstein metric induced by ω_{FS} is known. On the other hand a Kähler Einstein submanifold of $(\mathbb{CP}^n, \omega_{FS})$ has necessarily positive scalar curvature by a result of Hulin [4].

1. Proof of the theorem

1.1. **Kähler Ricci solitons.** Let M be a complex manifold and denote by J its complex structure. Rephrasing (0.1) in terms of 2-forms, a Kähler Ricci soliton on M is a Kähler metric g whose associated, Ricci and Kähler form ρ and $\omega = g(J\cdot, \cdot)$ respectively satisfy

$$\rho = \lambda\omega + \mathcal{L}_V \omega \tag{1.1}$$

for some holomorphic vector field $V = X - iJX$, where J is the complex structure. We will say that the Kähler Ricci soliton is *trivial* if $V = 0$, i.e. (M, g) is Kähler-Einstein.
Note that $\mathcal{L}_X J = 0$ because V is holomorphic and equation (1.1) implies that $\mathcal{L}_{JX}\omega = 0$, i.e. JX preserves ω, hence g because it also preserves J. Note also that (1.1) implies

$$\rho = \lambda\omega + \mathcal{L}_X \omega\,. \tag{1.2}$$

The fact that ∇X is g-self adjoint means that the 1-form dual to X is closed; since a KRS may exist only on Fano manifolds and these are simply connected (Kobayashi's theorem), we see that X is the gradient with respect to g of some smooth function f. We will indicate $\nabla f := \mathrm{grad}_g(f)$. This implies that

$$\mathcal{L}_X\omega = \mathcal{L}_{\nabla f}\omega = d\iota_{\nabla f}\omega = d\iota_{J\nabla f}\omega = dd^c f\,.$$

Recalling[1] that $\partial = \frac{1}{2}(d + id^c)$ and $\bar\partial = \frac{1}{2}(d - id^c)$, equation (1.2) turns out to be equivalent to

$$\rho = \lambda\omega + 2i\partial\bar\partial f\,. \tag{1.3}$$

Indeed the previous computation shows also that the function f indeed admits another useful interpretation. Since $\iota_{JX}\omega = -df$ the function f is, up to a constant multiple, a *moment map* for the infinitesimal action of the Killing vector field JX on M, or more precisely it is the projection along JX of a moment map μ for the Hamiltonian action of $\mathrm{Iso(M, g)}$ on M. (Recall that since M is simply connected every symplectic action on M is Hamiltonian)

[1] We are using the convention according to which $d^c h(Y) = Jdh(Y) = dh(-JY)$.

1.2. **proof of the theorem.** Let n be the complex dimension of M and $r = m-n$ the codimension. Denote also by $i : M \to \mathbb{CP}^m$ the inclusion and simply by ω the restriction $i^*\omega_{FS}$. By hypothesis ω satisfies (1.3) where f is the potential of the holomorphic vector field $X = \nabla f$. We suppose that M is embedded in $\mathbb{CP}^m$ as a complete intersection. Namely M is assumed to admit r homogeneous polynomials $P_1, P_2, \dots, P_r$ on $\mathbb{C}^{m+1}$ which define M as their zero locus and generate the ideal associated to M.
It is a direct consequence of the adjunction formula that the canonical line bundle $K_M = \Lambda^{n,0}M$ of M is the restriction of a line bundle on $\mathbb{CP}^m$, more precisely

$$K_M = i^*\mathcal{O}(d-m-1)$$

where $d = \sum_{j=1}^r \deg P_j$. Since the Chern class of K_M is represented by $\frac{1}{2\pi}$ times the Ricci form, the constant λ in (1.3) is forced to be equal to $m+1-d > 0$.
It is well known Hermitian metrics h on K_M^* correspond bijectively to positive volumes (nowhere vanishing real $2n$-forms) v of M, the correspondence being given by

$$\langle v, (-2)^m(\sqrt{-1})^{m^2} x \wedge \bar{x}\rangle = h(x,x)$$

for $x \in K_M^*$. Let V be the volume of M corresponding to the fibre metric on K_M^* whose Chern curvature form is exactly $(m+1-d)\omega$. In [3] (proposition 2) it is computed explicitly the real positive function ϕ such that $\omega^n = \phi V$ in the case where M is a complete intersection. More precisely, recalling that the Chern curvature form of the fibre metric induced by ω on K_M^* is exactly the Ricci form ρ (see [1], p.82), we have the following

Proposition 1.1 (Hano [3])**.** Let M be a complete intersection in $\mathbb{CP}^m$ defined by the polynomials $P_1, \dots, P_r$. Denote by $d = \sum_i \deg P_i$ and by ρ the Ricci form of the metric ω induced by ω_{FS} . Then we have

$$\rho = (m+1-d)\omega + i\partial\bar{\partial}\log\phi\,, \quad \text{with} \quad \phi = \frac{\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2}{\|z\|^{2(d-r)}}\,. \tag{1.4}$$

Here ϕ is expressed in terms of unitary homogeneous coordinates of $\mathbb{CP}^m$ and $\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2 = \sum |P_{\lambda_1 \dots \lambda_r}|^2$ where $dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r = \sum P_{\lambda_1 \dots \lambda_r} dz_{\lambda_1} \wedge \dots \wedge dz_{\lambda_r}$. Note also that the previous expression of ϕ is invariant under any unitary coordinate transformation.
Combining (1.4) with the Kähler-Ricci soliton equation we get

$$\partial\bar{\partial}\log\phi = 2\partial\bar{\partial}f\,.$$

so that

$$\phi = C \cdot e^{2f}$$

for some constant $C \in \mathbb{R}$. Now the key fact is that we can find an explicit expression also for f in terms of homogeneous coordinates of $\mathbb{CP}^m$. Indeed, as already remarked, f is a moment map for the action of the 1-parameter group of isometries generated by JX and this enables us to write it down in suitable coordinates. To start with, by a famous result of Calabi [2] the Killing vector field JX can be extended to a Killing vector field of $(\mathbb{CP}^m, \omega_{FS})$ so that with respect to an appropriate system of unitary homogeneous coordinates it can be written in diagonal form $\mathrm{diag}(i\lambda_0, \dots, i\lambda_m)$ as an element of $\mathfrak{su}(m+1)$.

Thus a moment map for the Hamiltonian action of the 1-parameter group $\{\exp tJX\}$ on $\mathbb{CP}^m$ is

$$\mu_{JX} = \frac{1}{2}\frac{\sum_{j=0}^{m}\lambda_j|z_j|^2}{\sum_{j=0}^{m}|z_j|^2},$$

and f is nothing but the restriction of μ_{JX} to M. So there exists a constant $C \in \mathbb{R}$ such that on M one has

$$\frac{\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2}{\|z\|^{2(d-r)}} = Ce^{\frac{\sum \lambda_j|z_j|^2}{\sum |z_j|^2}}. \tag{1.5}$$

We claim that (1.5) holds if and only if $f(z,\bar{z})$ is constant. Let p and q be any two points of M. Since M is Fano, by a Theorem of Kollár Miyaoka and Mori [6] there exists a rational curve passing through p and q, say $F : \mathbb{CP}^1 \to M \subseteq \mathbb{CP}^m$ defined by $F([s:t]) = [F_0(s,t) : \ldots : F_m(s,t)]$ where the functions $F_m(s,t)$ are homogeneous polynomials of degree δ in s and t.
Evaluating (1.5) at $F(\mathbb{CP}^1)$ we get

$$\frac{\|dP_1(F([s:t])) \wedge \cdots \wedge dP_r(F([s:t]))\|^2}{(\sum_j |F_j(s,t)|^2)^{(d-r)}} = Ce^{\frac{\sum \lambda_j|F_j(s,t)|^2}{\sum |F_j(s,t)|^2}} \tag{1.6}$$

for every $[s:t] \in \mathbb{CP}^1$ and this is clearly impossible unless f is constant on $F(\mathbb{CP}^1)$, otherwise the right hand side of (1.6) would not be a rational function of s and t. Since p and q are arbitrary, f must be constant on all of M: this means that $X = \nabla f$ vanishes and the Kähler-Ricci soliton is trivial, i.e. ω is Kähler-Einstein. According to Hano [3] this happens only if M is a linear subspace or it is a smooth quadric subvariety of some linear subspace.

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