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SARA CONFALONIERI

THE TELLING OF THE UNATTAINABLE ATTEMPT TO AVOID THE *CASUS IRREDUCIBILIS* FOR CUBIC EQUATIONS: CARDANO'S *DE REGULA ALIZA*.

WITH A COMPARED TRANSCRIPTION OF THE 1570 AND 1663 EDITIONS AND A PARTIAL ENGLISH TRANSLATION

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JURY

M. Paolo D'ALESSANDRO
M. Massimo GALUZZI, Co-directeur
M^{me} Veronica GAVAGNA, Rapporteur
M. Jeffrey OAKS
M. Marco PANZA, Directeur
M^{me} Sabine ROMMEVAUX, Président
M. Klaus VOLKERT, Rapporteur

To be wanted

(`xkcd.com`, comic number 372)

Acknowledgements

This work – as it frequently happens to me – is born by chance. It is by chance that, what now seems an awful lot of time ago, I have firstly met Marco Panza, by chance I have lingered on one standing page in the unintelligible mess of the *De regula aliza*, by the merest chance I happened to find out the position in Wuppertal that also gave me the possibility to accomplish this dissertation.

But, all in all, it has been a work of hard perseverance. I have learnt (and I am still learning) my way in doing something that I do not know how to do. Little by little, working on my dissertation, I have come to some precious lessons: I would like to thank everybody that let me gt to this point.

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I can no more recognise the (hardly French-speaking) student that seven years ago entered for the first time the seminar room at the first floor of the Tour

Montréal in the Dalles les Olympiades. I grew up in a considerable way, and this has only been possible thanks to the friendly and exceptionally stimulating atmosphere that I have found at the REHSEIS, subsequently SPHERE, group in Paris. It has been (and it is) a great pleasure and honour for me to be a part of it.

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Contents

Introduction

Solving cubic equations by a formula that involves only the elementary operations of sum, product, and exponentiation of the coefficients is one of the greatest results in the 16$^{\text{th}}$ century mathematics. At that time, the solution of quadratic equations was a well-mastered issue and the attention was driven on searching a generalisation to the cubic case. This was achieved by Girolamo Cardano's *Ars magna* in 1545. Still, a deep, substantial difference between the quadratic and the cubic formulae exists: while the quadratic formulae only involve imaginary numbers when all the solutions are imaginary too, it may happen that the cubic formulae contain imaginary numbers, even when the three solutions are all real (and different). This means that one could stumble upon numerical cubic equations of which he already knew three (real) solutions, but its cubic formula actually contains some square roots of negative numbers. This will be later called the '*casus irreducibilis*'.

In my dissertation, I will give an insight on another of Cardano's works, the *De regula aliza* (1570), in which he tried to overcome this problem.

The present work is organised in two parts. The first part is the out-and-out dissertation, while the second part contains the Latin transcription of the *De regula aliza* and a partial translation in English.

In the first part, I will comply with the following plan. First of all, in the preliminary chapter entitled 'Notations and more', I will explain the notations that I have used in order to account for Cardano's mathematics. The role of this short chapter is to justify the gap that is originated by my choice to employ a certain amount of formalism, while Cardano mainly spoke mathematics in full words. I will also seize the day to clarify what a cubic equation was for Cardano.

In Chapter 1, I will contextualise from both a non-mathematical and a mathematical viewpoint Cardano's mathematical works, with closer attention to the ones that deal with cubic equations. In particular, I will introduce the *Aliza*, which is a puzzling and controversial treatise under many respects. Then, I will try to give a first estimate in time of the book and report on its readers all along the centuries. Regarding to the mathematical side, I will retrace how we solve nowadays quadratic, cubic, and quartic equations. Concerning in particular cubic equations, I will also discuss the *casus irreducibilis* using Galois theory.

Due to the exceptionally problematic nature of the *Aliza*, I consider fully worthwhile to try to relieve at most the interpretative effort while dealing with this book. This will be achieved by a previous analysis of the topics related to cubic equations that can be found in Cardano's other mathematical works. This will also help in setting the mathematical context in which the *Aliza* fits. In particular, I will firstly analyse the most advanced results in the *Ars magna* (1545, 1570) and then I will go and search for their origins in the *Practica arithmeticæ* (1539) and in the *Ars magna arithmeticæ* (probably composed between 1539 and 1542). This will be of the utmost importance in decoding afterwards the *Aliza*. In Chapter 2, I will discuss in detail all the solving methods in the *Ars magna* for cubic equations. This will give us a quite accurate picture of their mutual inter-dependencies, which is fundamental in order to understand the impact of the problem set by the *casus irreducibilis*. Moreover, I will discuss a bunch of fringe topics, like transformations of equations and particular solving methods, which help in completing the frame. I will also take into account Cardano and Ferrari's treatment of quartic equation.

The main annoyance with Cardano's solving methods (or, better, with Cardano's enhancement of Tartaglia's poem) is that they are completely opaque concerning the way in which they had been discovered. They work, but their proofs belong to that varied family of proofs that let glimpse no hints on how one could have possibly come across them. Then, if we want to know how Cardano used to deal with equations, we must consider some other results, like the fringe topics that I have mentioned above. Some of these topics and others (and also a few solving methods for cubic equations) can be found in two of Cardano's previous works. This is why in Chapter 3 I will analyse some parts of the *Practica arithmeticæ* and the *Ars magna arithmeticæ*.

As said, all this work is necessary – in my opinion – to set the mathematical environment in which the *Aliza* is located. On the one hand, the study of the other mathematical treatises by Cardano is useful to get oriented while reading the *Aliza*, in particular when the guide of a solid, general architecture is missing – and this will frequently happen. On the other hand, we obviously need to know the contents of those treatises – especially of the *Ars magna arithmeticæ* and of the *Ars magna* – in order to fully grasp the explicit (and implicit) references that are made to them in the *Aliza*. But there is more. The targeted analysis of the other mathematical treatises by Cardano has the valuable effect to show the overall development trend of his inquiries on equations. The *Practica arithmeticæ* is an early work in the *abaco* old-school tradition, where only a few pages out of the whole book are devoted to equations. These are substantially seen as a side-effect of proportions. In the *Ars magna arithmeticæ* the situation begins to change. We still have a consistent part that deals with irrational numbers in the Euclidean style of *Elements*, Book X – the so-called 'arithmetisation' of Book X. Nevertheless, these results on irrational numbers are used to study the shape of the irrational solutions of some cubic equations, and gradually the equations become a subject of inquiry by themselves. In the *Ars magna arithmeticæ*, we also find a handful of results on equations that will later appear in the *Ars magna*, but they are much less refined. At the end of this development trend, the *Ars magna* is completely devoted to equations. They are the subject of the whole book, from the beginning to the end – and this will also be the case of the *Aliza*. In the *Ars magna*, complete and coherent methods to deal with quadratic, cubic, and quartic equations are expounded. Cubic equations are, in particular, systematically treated: a geometrical proof, an algorithmic rule (that is possibly translated into a formula), some examples, and sometimes other collateral rules are given. In this treatise, there is a well-developed context (I am referring, for instance, to the transformation rules) in which these equations are immersed. Little by little we are led to conceive the picture of the inter-dependency of all sorts of equations one on another and, in this way, the main problem of Cardano's construction arises: it is entailed by the *casus irreducibilis*. At this point, we will be ready to plunge into the *Aliza*.

In Chapter 4, I will propose my interpretation of this book. Though as accurate as possible, my account is often tentative, since Cardano's treatise is itself quite obscure in many parts. I hope, in any case, that my effort to understand his arguments, though possibly unsuccessful here and there, will be a useful basis for future interpretations. I have made a choice of the most meaningful chapters of the *Aliza*, where 'meaningful' sometimes poorly means 'understandable'. Once abandoned the (vain) hope to coherently interpret the whole book as it were a well-structured treatise (which is likely not, in fact), my aim has been, more reasonably, that of finding some common threads capable of guiding the interpretation of these parts of it, at least. These threads will be joined together by the shared purpose to overcome the problem entailed by the *casus irreducibilis*.

Then, some appendices follow. I will firstly relate on Cardano's life, and in particular on the controversy with Tartaglia. Afterwards, I will gather a short vocabulary of the mathematical terms in the *Aliza*. Finally, I will provide some lists, namely of all the internal and external references in and to the *Aliza*, of the numerical equations that Cardano considers there as examples, and of his results on cubic equations in the *Practica arithmeticæ*, *Ars magna arithmeticæ*, *Ars magna*, and *Aliza*.

In the second part of this work, I will firstly provide a compared Latin transcription of the 1570 and 1663 editions of the *De regula aliza*. Then, I will give a partial English translation of it, namely of all the chapters that I have taken into account in Chapter 4.

Far from offering an ultimate account of the *De regula aliza*, from both the editorial and interpretative viewpoint, my work – I hope – is a first step towards a better understanding of a book of one of the most outstanding scholars of the 16$^{\text{th}}$ century.

Notations and more

For a modern reader it is almost impossible – or at least it is really hard-working – to account for Cardano's time mathematics without reverting to some sort of formalism. This is why I choose to rewrite it using a certain amounts of symbols that he did not employ. On a side, these symbols betrays – as each translation – Cardano's text, since this goes more or less far away from the codified language of that time. On the other side, I believe that the advantages for the reader in terms of precisely grasping its contents are to be preferred over the ultimate fidelity to printed words.

Nevertheless, pay attention! Choosing to use the appearance of our mathematics to talk about Cardano's one does not mean at all that even the least part of our understanding belongs to Cardano. I will only employ our notations as a good stenography, or as a "convenient linguistic trick",[1] to avoid cumbersome and sometimes ambiguous expressions. As such, they are not essential. In other words, I do not believe in that "sartorics of mathematics"[2] that consists in clothing a unique, timeless mathematics in the fashionable idioms of a historical period. Cardano's mathematics and ours are different – they belong to completely different universes. It is true that our mathematics also enable us to understand the mathematical content of Cardano's arguments, but still Cardano's mathematics must be considered as a whole and contains subtleties that must be understood in their own context.[3]

Nevertheless, notations are not only a clothing, but they have a value. This cannot be prescriptive, but rather practical. We have a mathematical object and

[1]See [Panza 2010, page 207].

[2]See [Netz 2004, page 3].

[3]As Karine Chemla says, "*resituer l'objet dans le contexte ds procédures qui le mettent en œuvre s'avère essentiel pour en déterminer la nature [. . .]. Ces éléments sont à considérer dans les ensemble qu'ils forment, et non pas uniquement à l'état de pièces détachées sous le prétexte que seules ces pièces seraient pour nous significatives*", see [Chemla 1992, page 102].

then we point at it in more or less effective ways – good or bad notations, where ‘good notation’ means that it makes easier (or, at least, not more difficult) to handle that object. Then,the effectiveness of the following notations in rephrasing Cardano’s mathematics (and only its effectiveness!) is the reason for which I choose to use it.

A noteworthy part of Cardano’s mathematics that we will analyse in the following concerns equations. I will call ‘equation’ what we now call ‘equation’ – basically a standardised way of writing that involves an equality and that has a certain shape, depending on one or more unknown quantities and on their powers.[4] For instance, a cubic equation is an equality such as ‘$x^3 = a_1x + a_0$’, where x varies and a_1, a_0 are constant. Or one can also say ‘a cube equal to some things and a number’, as it was the case in Cardano’s mathematics. Similar expressions are used for linear, quadratic, quartic, or whatever else degree equations.

Concerning geometrical objects, I will use a double notation (except for points, which will be denoted by capital letters). According to the first notation, ‘AB’ stands for the segment delimited by the points A and B. In the same way, ‘AB^2’ stands for the square the side of which is AB, ‘AB^3’ for the cube the side of which is AB, ‘$AB\,CD$’ for the parallelogram the sides of which are AB and CD, and ‘$AB\,CD\,EF$’ for the parallelepiped the sides of which are AB, CD, and EF. When needed, I write ‘(A)’ for a segment (and in this case the segment is identified by the diagram), ‘$(AB)^2$’ for the square identified by its diagonal AB, ‘$(AB)^3$’ for the cube identified by the diagonal AB of one of its faces, and ‘(AB)’ for the parallelogram identified by its diagonal AB. Finally, ‘$(A\,B\,\ldots\,N)$’ stands for the gnomon identified by the points $A, B, \ldots, N$. I will use this standard

[4]Note that varied definitions has been given by historians and philosophers of mathematics. For instance, Jens Høyrup maintains in [Høyrup 2002b, page 84] that, generally speaking,

> [a]n “equation” is the statement that some complex quantity (for instance, the area A of a square) defined in terms of one or more simple quantities (in the example, the side s), or the measure of this complex quantity, equals a certain number or (the measure of) another quantity.

Concerning instead Arabic-speaking algebra, Albrecht Heeffer argues in [Rahman et al. 2008, page 89] and [Heeffer 2010, pages 58-59] that an “equation is *the act of keeping related polynomials equal*. Guglielmo de Lunis and Robert of Chester have a special term for this: *coaequare*”, from which the term ‘coequal polynomials’.

notation when, respect to the purpose of Cardano's text, one essentially needs to characterise the geometrical object at issue by its position compared to the one of other geometrical objects.

Note that it can happen that the relative positions of geometrical objects play a role, still without being essential. This happens, for instance, when a diagram is used to fix the reference[5] of the relevant object, in order to apply to it the results that we already have at disposal. A good way to check if the relative position of geometrical objects covers an essential role (or not) is to reputedly change the position of the involved objects and verify whether the argument still maintains its validity. If the answer is 'yes' – that is, when the position is not essential and only the relative sizes of the objects matter – I will use a different notation. In fact, in this case we better refer to the measure of the considered object, not to the object itself. I write '$\overline{AB}$' for the measure of the segment delimited by the points A and B, '$\overline{AB}^2$' for the measure of the square constructed on the segment that measures $\overline{AB}$, '$\overline{AB}^3$' for the measure of the cube constructed on the segment that measures $\overline{AB}$, '$\overline{AB}\,\overline{CD}$' for the measure of the parallelogram the sides of which respectively measure $\overline{AB}$ and $\overline{CD}$, and '$\overline{AB}\,\overline{CD}\,\overline{EF}$' for the measure of the parallelepiped the sides of which respectively measure $\overline{AB}, \overline{CD}$, and $\overline{EF}$. When needed, I write '$\overline{(A)}$', '$\overline{(AB)}^2$', '$\overline{(AB)}^3$', and '$\overline{(AB)}$ for the measures of the above objects. This happens, for instance, when a surface is not named after its vertexes, but is denoted only by a letter, or when a square or a parallelepiped is named after its diagonal. In this context, 'A' simply stands for the point A since we do not speak of the measure of a point. Quantities of such a kind can appear as coefficients in equations.

Concerning operations, I will use the standard, ambiguous (on a certain amount) notation. I will write '$\ldots + \ldots$' for the sum, '$\ldots - \ldots$' for the difference, nothing for the product, '$\frac{\ldots}{\ldots}$' for the division, '$(\ldots)^n$' for the n^{th}-power, and '$\sqrt[n]{\ldots}$' for the n^{th}-roots. This notation is ambiguous, since it denotes operations that are different according to which kind of objects one is dealing with. This means that we are not allowed to sum, subtract, multiply, or divide two objects of

[5]"Very often – most often – letters are not completely specified. So how do we know what they stand for? Very simple: we see this in the diagram", see [Netz 1999, pages 19-26].

different kinds, such as numbers and geometrical objects. So, as long as one do not dispose of a conceptual environment in which he can perform operations on quantities in general,[6] distinguishing different kinds of objects becomes of the utmost importance.

In addition, note that Cardano never multiplies or divide a geometrical object by another one. He only multiplies or divides geometrical objects by numbers. Note moreover that, always concerning geometrical objects, he takes the n^{th}-powers and n^{th}-roots only up to $n = 3$.

Concerning proportions, I will write '$a : b = c : d$' or '$\frac{a}{b} = \frac{c}{d}$', where a, b, c, d are quantities. In particular, I will take care to use '$a : b = c : d$' whenever a, b, c, d are geometrical quantities. In this case, '$a : b$' should not be intended as the result of the division of a by b. If a, b, c, d are numbers or non-positional, geometrical objects I will like better '$\frac{a}{b} = \frac{c}{d}$' to lighten the outfit of the text.

Using this notation, I will account for some of Cardano's proofs. My main concern being not to betray the spirit of the text, I will put into square brackets the additions and comments that are mine.

To refer to propositions, rules, or proofs in Cardano's texts, I will write '(⟨*Book title in short*⟩ ⟨*Chapter*⟩.⟨*Paragraph, when needed*⟩)'. Usually, I write the book title in capital letters, the chapter in Roman numerals, and the paragraph in Arabic numerals. For example, (A I.1) refers to the first proposition of the Chapter I of the *De regula aliza*.

All the examples, except when specified, are by Cardano himself. At page 369, there is a list of all the cubic equations in one unknown considered by Cardano in the selection of his works that that we will analyse.

So far concerning my notations in the present work. But what about Cardano's own notations?

As it is well known, Cardano has neither a stable nor an overall formalism. He employs some stenographic expressions, which still represent the language in which the statements are expressed. This is why it has been sometimes addressed

[6]See [PANZA 2005, pages 17-30] and [PANZA 2007].

as 'syncopated algebra'.[7] Cardano's stenography boils down to a few amount of symbols. First of all, we have operations: '*p:*' or simply 'and [*et*]' stand for the sum [*plus*] as well as '*m:*' stands for the difference [*minus*]. Note that, according to a common agreement of his time, Cardano writes, for instance, '$1\frac{1}{2}$' to mean '$1+\frac{1}{2}$'. '*R*' usually stands for the square root [*radix*] and '*Rcu:*' stands for the cubic root [*radix cubica*]. '*RV:*' stands for the "universal root [*radix universalis*]", which is the (usually, square or cubic) root of the sum or difference of two terms, like "RV: cub. 1 + R 2" or $\sqrt[3]{1+\sqrt{2}}$. Then, we have unknowns. Cardano sometimes denotes the first degree unknown [*positio*] by '*pos.*' , even if he more often calls it '*res*'. The second degree unknown [*quadratus*] is written '*quad.*' or '*q.*' and the third degree unknown [*cubus*] is written '*cub.*' or '*cu.*'. Higher unknowns do not usually appear in the *Aliza*, except a few occurrences of the fourth degree unknown '*quad. quad.*' and of the sixth degree unknown '*cu. quad.*'.

Sometimes, we find a completely different kind of stenography. At the end of the *Aliza* (in Chapters LIII and LVII), referring for instance to $x^3 = 20x + 32$, Cardano writes '*20 d. p. R p. 32*' to mean '20 divided in the part and the root that produce 32 [*divisum in partem et radicem producentes 32*]', '*32 p. 20 cum p. 32*' to mean 'what produces 20 with what produces 32 [*producentis 20 cum producente 32*]', '*Ag. R p: 20 p: n: 16*' to mean 'the aggregate of the roots of the parts of 20 that reciprocally multiplied produce 16 [*aggregatum radicum partium 20, quæ mutuo ductæ producunt 16*]' (or the half of 32), '*32 p: 20 c. p. 32*' to mean 'what produces 20 with what produces 32 [*producens 20 cum producente 32*]', '*R 20 p: d. 32*' to mean 'the root of 20 plus 32 divided by the same root [*R 20 p: diviso 32 per ipsam radicem*]', and finally '*R 20 f. 32*' to mean 'the root of 20 with a fragment of 32 [*R 20 cum fragmento 32*]' (a 'fragment [*fragmentum*]' is 'what comes forth from a division [*quod ex divisione prodit*]'). For more details, see below, at page 304.

Finally, a short issue concerning Cardano's equations, and in particular cubic equations. As said, he tells an equation in words, employing the common Latin terminology of the time. The coefficient of the term of degree zero is a "number

[7]See [Boyer 1991, page 180]. See also Albrecht Heeffer's criticism of this terminology in [Heeffer last checked January 22, 2014].

[*numerus*]" and the unknown is a "thing [*res*]" or a "position [*positio*]". Concerning the higher degrees of the unknown, we have the "squares [*quadratus*]" or "*census*" for degree two, the "cube [*cubus*]" for degree three, the "first related [*relatus primus*]" for degree five, the "second related [*relatus secundus*]" for degree seven. The non-prime degrees are obtained by multiplication of prime factors, such as the "square square [*quadratus quadratus*]" for degree four, the "square cube [*quadratus cubus*]" for degree six, the "square square square [*quadratus quadratus quadratus*]" for degree eight, the "cube cube [*cubus cubus*]" for degree nine, and so on as usual. When the unknown goes with a coefficient, Cardano uses the plural, for instance "some things [*res*]" or "some squares [*quadrati*]".

What is fundamental in order to get an understanding of Cardano's equations is to identify the quantities that are equated. Cardano's coefficients are non zero, positive numbers. In the greatest majority of his examples – as said – they are rational (even natural) numbers, but there are a handful of cases in which he also allows irrational numbers (see the table in Appendix D, at page 369). Whenever we would write a negative coefficient, we find that in Cardano's equations they have been moved on the other side of the equal. Moreover, Cardano only considers monic equations. These are quite standard criteria to write equations at the time. Cardano is interested in finding the real solutions. He has a strong preference for positive solutions (which he calls "true [*vera*]"), but sometimes, especially when dealing substitutions, he also needs to consider the negative ones (which he calls "fictitious [*ficta*]").

Concerning in particular cubic equations, we remark that Cardano takes into account only cubic equations that do not lower in degree *via* a substitution and that cannot be solved by taking the third root (that is, he does not include the two-terms equations like $x^3 = a_0$). We have then the following classification for cubic equations (the terminology is obviously mine):

- equations lacking in the second degree term (also called 'depressed'):

$$x^3 + a_1x = a_0 \quad \text{or} \quad \textit{cubus et res æquales numero},$$
$$x^3 = a_1x + a_0 \quad \text{or} \quad \textit{cubus æqualis rebus et numero},$$
$$x^3 + a_0 = a_1x \quad \text{or} \quad \textit{cubus et numerus æquales rebus};$$

- equations lacking in the first degree term:

$x^3 = a_2x^2 + a_0$ or *cubus æqualis quadratis et numero*,

$x^3 + a_2x^2 = a_0$ or *cubus et quadrati æquales numero*,

$x^3 + a_0 = a_2x^2$ or *cubus et numerus æquales quadratis*;

- complete equations:

$x^3 + a_2x^2 + a_1x = a_0$ or *cubus quadrati et res æquales numero*,

$x^3 + a_1x = a_2x^2 + a_0$ or *cubus et res æquales quadratis et numero*,

$x^3 + a_2x^2 = a_1x + a_0$ or *cubus et quadrati æquales rebus et numero*,

$x^3 = a_2x^2 + a_1x + a_0$ or *cubus æqualis quadratis rebuss et numero*,

$x^3 + a_0 = a_2x^2 + a_1x$ or *cubus et numerus æqualis quadratis et rebus*,

$x^3 + a_1x + a_0 = a_2x^2$ or *cubus res et numerus æquales quadratis*,

$x^3 + a_2x^2 + a_0 = a_1x$ or *cubus quadrati et numerus æquales rebus.*

Moreover, since Cardano only considers equations with positive (real) solutions – or, at least, he has a very strong preference for them[8] – the equations $x^3+a_1x+a_0 = 0$, $x^3+a_2x+a_0 = 0$, and $x^3+a_2x^2+a_1x+a_0 = 0$ do not appear in the classification. Sometimes – and this often happens in the *De regula aliza* – Cardano writes an equation such as $x^3 = a_2x^2 + a_0$ under the shape $x^2(x - a_2) = a_0$. We remark that this does not mean that he allows negative coefficients, since $a_0 > 0$ implies that $x - a_2 > 0$ (it would have been so if he had written $x^3 - a_2x^2 = a_0$, but it is not the case).

In the end, there are thirteen families of cubic equations. Such a classification provides an effective way to distribute cubic equations in families so that it is easier to speak about them. We remark that a classification in the same vein can already be found in al-Khayyām's algebra.

Recall that I use our formalism only as long as it is a more effective stenography. In particular, by writing the equations as in the above table, I do not obviously mean that Cardano is interested in the zeros of the function associated to the appropriated polynomial (intended, in modern terms, as an infinite, eventually

[8]To this end, see here, Section 2.1.1 at page 61 and Section 2.1.2 at page 65. It must be said in fact that, even though Cardano sticks to the tradition concerning negative numbers (see for instance "but the subtraction is made only of the smaller from the bigger, in fact it is entirely impossible to subtract a bigger number from a smaller [*detractio autem non fit nisi minoris a maiore: nam maiorem numerum, a minore, detrahere omnino est impossibile*]" in [Cardano 1663i, Chapter XI, page 18]).

zero sequence of coefficients) – it would simply sound ridiculous. A polynomial is a mere formal writing, and even if one wanted to allow operations like the product and the sum in it, they would be at large operations between real numbers, which – if one wants to be fully pedantic – are not exactly the same operations that Cardano uses, since he does not have 'the field of the real numbers' available.[9] He only has a kind of a system of numbers, which is by the way not close. More precisely, Cardano takes the freedom to consider that, when he performs some algebraic operations on rational numbers, other numbers that he knows how to deal with are produced.[10] I rather I agree with [Oaks 2007] and [Oaks 2009] (transposed from Arabic algebra to Cardano's case), for whom equations are equality between collections of certain objects taken a certain number of times.[11] In this way, a family of equations bears by itself a certain structure and it is not only a formal writing that must be filled in with the coefficients. More precisely,

[9]Under the same viewpoint and for the sake of brevity, I will sometimes use some notations concerning equations, especially the 'discriminant Δ_3', referring to Cardano's equations. In this case, the discriminant is not to be intended as the symmetric function $\alpha_3^3(x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2$ of the roots x_1, x_2, x_3 of a polynomial $\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$. Rather, it is a stenographic writing for the number that is under the square root in the formula (see here, Section 1.5 at page 38).

[10]For instance, the operation of taking the square root applied to 2 gives $\sqrt{2}$, and Cardano knows how to further perform algebraic operations on it.

This remark opens a great scoped issue, which I am not going to study here: how can we characterise Cardano's numbers?

[11]"[T]he two sides of an equation in the medieval Arabic algebra are aggregations of the algebraic "numbers" (powers) with no operations present. Unlike an expression such as our $3x + 4$, the Arabic polynomial "three things and four dirhams" is merely a collection of seven objects of two different types. Ideally, the two sides of an equation were polynomials so the Arabic algebraists preferred to work out all operations of the enunciation to a problem before stating an equation", see [Oaks 2009, page 169], or "Medieval algebraists conceived of polynomials differently than we do today. For us, a polynomial is constructed from the powers of x with the operations of scalar multiplication and addition/subtraction. In other words, it is a linear combination of the powers. By contrast, Arabic polynomials contain no operations at all. In the expression "four things", the "four" is not multiplied by "things". Instead, it merely indicates how many things are present. Think of "four things" like "four bottles". Further, the phrase "four things and five dirhams" entails no addition, but is a collection of nine items of two different kinds. Think of it like "four bottles and five cans". The wa ("and") connecting the things and the dirhams is the common conjunction. It does not take the meaning of the modern word "plus"", see [Oaks 2007, page 548].

In Cardano's case, since he allows irrational numbers to be coefficients, there could be a little inconsistency in taking an irrational number of times a certain object. Anyway, this could be also seen as s first step toward a further development and, as such, inconsistent under certain respect with the usual way (see [Oaks 2013b]).

the fact that a certain term is on a side or the other of the equal depends on the inner structure of a family of equations, and not on the sign of the coefficients (since they are always supposed to be positive).

CHAPTER 1

Putting in context Cardano's works and mathematics

This chapter is an introduction to Cardano's works from both a non-mathematical and a mathematical viewpoint.

In the following, I will give a brief overview of Cardano's mathematical treatises, including the few manuscripts still available, and discussing in particular the writings in which cubic equations are dealt with. At a second stage, I will introduce the *Aliza*. It is nevertheless worthwhile to address from now a few issues concerning this very controversial book, as the meaning of the title and its supposed aim. Moreover, I will try for a first estimate in time of the *Aliza*. Its detailed analysis will be delayed in Chapter 4. Finally, I will account in this section for the secondary literature concerning the *Aliza*. Being such an unfamiliar book as it is, it had a handful of readers during the four and a half centuries that have passed. The most of them only take episodically into account a very few topics. Pietro Cossali, instead, made a quite detailed analysis at the end of the 18$^{\text{th}}$ century, which I will discuss.

Finally, in order to fully grasp Cardano's reasoning, I will recall what our mathematics can nowadays say about quadratic, quartic, and especially cubic equations.

1.1. Cardano as a mathematical writer

Cardano has been a very prolific writer on a great variety of topics – borderline to compulsiveness. Although on two occasions he destroyed 129 books in all, a consistent number of treatises and manuscripts has come to us. In the latest version of his autobiography, which was completed in 1576, a few months before his death, he lists 131 published treatises and 111 manuscripts. At Cardano's death, they were left[1] to Rodolfo Silvestri, one of his good friends in Rome and a former pupil of him. Subsequently, Cardano's writings were handed to another of

[1] See [Baldi and Canziani 1999b].

his pupils, Fabrizio Cocanaro, and, in 1654, the majority of the manuscripts ended up to a Parisian bookseller. Eventually, they were bequeathed to Marc-Antoine Ravaud and Jean-Antoine Huguetan, who, with the help of the physician Charles Spon, published in 1663 in Lyon Cardano's *Opera omnia*. It consists of ten volumes and 127 treatises in all.

The *Opera omnia* includes thirteen mathematical treatises, some of which are there published for the first time, while others had a previous edition. They are[2] (chronologically listed according to the date of the first publication, then in the order in which they appear in the *Opera omnia*):

- the *Practica arithmeticæ* (Milan 1539, Lyon 1663),
- the *Ars magna* (Nuremberg 1545, Basel 1570, Lyon 1663),
- the *De proportionibus* (Basel 1570, Lyon 1663),
- the *De regula aliza* (Basel 1570, Lyon 1663),
- the *De ludo aleæ* (Lyon 1663),
- the *De numerorum proprietatibus* (Lyon 1663),
- the *Libellus qui dicitur computus minor* (Lyon 1663),
- the *Ars magna arithmeticæ* (Lyon 1663),
- the *Sermo de plus et minus* (Lyon 1663),
- the *Encomium geometriæ* (Lyon 1663),
- the *Exæreton mathematicorum* (Lyon 1663),
- the *De artis arithmeticæ tractatus de integris* (Lyon 1663),
- the *De mathematicis quæsitis (paralipomena)* (Lyon 1663).

They are all contained in the fourth volume of the *Opera omnia*, which is devoted to mathematics, except the *De ludo aleæ*, which is in the first volume, and the *De artis arithmeticæ tractatus de integris* together with the *De mathematicis quæsitis (paralipomena)*, which are in the tenth volume. We remark that in the fourth volume of the *Opera omnia* two spurious writings have also been included. They are the *Operatione della linea* and the *Della natura de principij e regole musicali*.[3]

[2]See [www.cardano.unimi.it last checked January 22, 2014].

[3]Contrary to all the others, they are written in Vernacular. The *Operatione della linea* is actually a work by Galileo Galilei contained in his *Le operazioni del compasso geometrico e militare* (1606). Concerning the *Della natura de principij e regole musicali*, see [Sabaino 2003,

Moreover, we have three mathematical manuscripts available. One is the manuscript Plimpton 510/1700 at the Columbia University Library in New York, which is an Italian translation of the *Ars magna* entitled *L'algebra*. Another is the manuscript N 187 at the Biblioteca Trivulziana in Milan, which was firstly entitled *Supplementum practicæ Hieronimi Castillionei Cardani medici mediolanensis in arithmetica* and then *Hieronimi Castillionei Cardani medici mediolanensis in libruum suum artis magnæ sive quadraginta capitulorum et quadraginta quæstionum*. It partially attests to the *Ars magna arithmeticæ*. Finally, there is the manuscript Lat. 7217, called *Commentaria in Euclidis Elementa*, at the Bibliothèque Nationale de France in Paris. It is a treatise in nine books on geometry, which has many common points with one of Cardano's unfinished projects, the *Nova geometria*, which should have likely completed and enhanced the *Elements*.[4]

We must also consider another work that Cardano would have liked to write, but in the end he never wrote. It is the *Opus arithmeticæ perfectum*, probably conceived between 1530s and 1560s. It should have been an encyclopedic, arithmetical work in fourteen books, a sort of counterpart to the *Nova geometria* (for more details, see below, Section 1.3 at page 26).

Finally, there are some non-mathematical treatises, which anyway contain some mathematics as examples. They are the logical treatise *Dialectica* (Basel 1566, Lyon 1663) and the philosophical writing *De subtilitate* (Lyon 1550, then fifteen editions more).

Among this huge amount of pages, we will only consider the works that deal with equations and, more precisely, principally with cubic equations. As said, they are the *Practica arithmeticæ*, the *Ars magna* (of which we have also available an English translation made by Richard Witmer in 1968), the *Ars magna arithmeticæ*, and – obviously – the *De regula aliza*. Except the *Ars magna arithmeticæ*, they were all published during Cardano's lifespan. The *Ars magna* also had a second edition, and all the mathematical treatises were posthumously republished in the *Opera omnia*.

page 97]. Also Ian MacLean includes them in a list of manuscripts written in Vernacular and not acknowledged by Cardano himself, see [Baldi and Canziani 2009, page 60].

[4] For more details, see [Gavagna 2003].

During the 14th and 15th centuries in the region of northern Italy (where Cardano subsequently grew and lived for the first part of his life), a significant, even if modest, tradition of *scuole d'abaco* developed.[5] It is plausible[6] that Cardano had some contacts with this environment, since at that time these were the only places where one could learn algebra, or better what it was so called at that time. Anyway, there is no evidence that he had been himself an *abaco* master. It is moreover very likely[7] that Cardano pit himself against the more widespread Italian treatises, namely the inheritors of the *abaco* treatises like the *Nuovo lume* by Giovanni Sfortunati and the *Libro de abacho* by Pietro Borghi, and obviously the *Summa de arithmetica, geometria, proportioni et proportionalita* by Luca Pacioli. It indeed seems that the real link between Cardano and the *abaco* school was Gabriel Arator, one of Pacioli's students.

The general image of practical mathematics conveyed by these treatises is a double-sided one. On the one hand, there are miscellaneous of problems solved case by case where the calculation skills are overwhelming. On the other hand, there is an encyclopedic work mainly focused on theory, that is Pacioli's treatise. It is in this tradition that the 1539 *Practica arithmeticæ* – the first work in which Cardano deals with cubic equations – can be inserted. Even if it is written in Latin and not in Vernacular, it contains all the major features of an *abaco* treatise: topics, choice of problems, and also terminology. It sometimes provides original reworkings of some techniques of the *abaco* tradition. Moreover, Cardano himself makes the connection with Pacioli, even if in a conflicting way, which was a common publishing strategy after that the *Summa* appeared (given the turmoil that this work caused in the long term). In the *Practica arithmeticæ*, cubic equations are only one among various problems and are seen as tightly depending from proportions.

Then, according to the above chronological list, the *Ars magna* should follow. Actually, thanks to some recent reappraisals of Cardano's mathematical writings,[8]

[5]See [Gavagna 1999, page 273].

[6]See [Franci 1996]. Moreover, Cardano frequently mentions a certain Gabriel Arator, who assigns to himself the name of *magister*, see [Cardano 1539, Chapter LI, paragraph 17] and [Gavagna 1999, page 275].

[7]See [Gavagna 1999, page 276].

[8]See the edition project of Cardano's works by Maria Luisa Baldi and Guido Canziani [www.cardano.unimi.it last checked January 22, 2014] and the conference proceedings [Kessler

we have come to the knowledge that the *Ars magna arithmeticæ* preexisted to the *Ars magna*. For a long time, the *Ars magna arithmeticæ* has been considered as a minor work and in any case a late work.[9] Cardano himself did not mention this treatise in his autobiographies and never gave it to press. The (supposedly chronological) position of the published version in the framework of the 1663 *Opera omnia* contributed to the misunderstanding, since it was wrongly placed after the *Ars magna*.

Nowadays it is commonly agreed that *Ars magna arithmeticæ* has been conceived before – or at least at the same time than – the *Ars magna*. Massimo Tamborini suggests[10] that the *Ars magna arithmeticæ* and the *Ars magna* were at the beginning planned together, and then separated and revised, maybe in order to publish the *Ars magna* as soon as possible. In fact, in his 1544 autobiography, Cardano speaks of an "Ars Magna, which contains 67 chapters",[11] whereas both the three editions of the *Ars magna* and the *Ars magna arithmeticæ* published in 1663 both consist of only 40 chapters. In this context, Cardano is talking about his project of an *Opus arithmeticæ perfectum*, crediting this "*Ars Magna*" to be its tenth volume. We will take a more detailed look at the *Opus arithmeticæ perfectum* in Section 1.3.

Ian MacLean prefaces his edition of Cardano's autobiographies[12] by a commented chronology of Cardano's treatises. While speaking about the description of the *Opus arithmeticæ perfectum* given in Cardano's 1557 autobiography,[13] he identifies the tenth volume of the *Opus arithmeticæ perfectum* or the "*De regulis magnis, atque ideo ars magna vocatur*" with the *Ars magna arithmeticæ* itself. But MacLean's interpretation is confusing. In fact, if we go and look at the full 1557 quotation, it says: "De regulis magnis, and for that reason it is called ars magna. And only this one among the others is published",[14] whereas the *Ars*

1994], [Baldi and Canziani 1999a], [Baldi and Canziani 2003], and [Baldi and Canziani 2009].

[9] See [Bortolotti 1926], [Loria 1931, 1950, page 298], and [Cardano 1968, footnote 20, pages xvii-xviii].

[10] See [Tamborini 2003, pages 177-9].

[11] "*Ars magna, continet sexaginta septem capitula*", see [Cardano 2004, page 131].

[12] See [Cardano 2004].

[13] See [Cardano 2004, page 65].

[14] "*De regulis magnis, atque ideo ars magna vocatur: atque hic solus ex omnibus editus est*", see [Cardano 2004, page 185].

magna arithmeticæ has never been printed during Cardano's lifespan. Moreover, in the footnote to the full quotation, MacLean contradicts, and (fairly) identifies the same "De regulis magnis" with the *Ars magna*.

Finally, Veronica Gavagna argues[15] that the *Ars magna arithmeticæ* had been composed between 1539 and 1542. She argues as follows.

Firstly, the *Ars magna arithmeticæ* is dedicated to "Philippum Archintum, bishop of Borgo Santo Sepolcro", who was in office between 1539 and 1546.

Secondly, the manuscript N 187 (1r-62v) at the Biblioteca Trivulzian in Milan, which partially attests to the *Ars magna arithmeticæ*, should have been at the beginning an addition to the *Practica arithmeticæ*, as its first title '*Supplementum practicæ Hieronimi Castillionei Cardani medici mediolanensis in arithmetica*' suggests. Then, the title was modified into the final heading '*Hieronimi Castillionei Cardani medici mediolanensis in libruum suum artis magnæ sive quadraginta capitulorum et quadraginta quæstionum*'. When collated, the *Supplementum* and the *Ars magna arithmeticæ* show a word by word correspondence in entire chapters, especially the ones related to cubic equations. Then, it is quite likely that the latter originated from a reorganisation of the former. Moreover, in the *Practica arithmeticæ* we read this:

> in the book titled Supplementum Practicae, where I solved algebraic equations of whatever degree, possible and impossible, general and not; so in that book there is everything people want to know in algebra. I have added new algebraic cases, but I can't publish them in the Practica because is too large [...] furthermore Supplementum is the crowning achievement of the whole of algebra and it is inspired to book X of the Elements.[16]

And this:

[15]See [Gavagna 2012].

[16]This is Gavagna translation. "*[L]ibello qui dicit supplementum practice in quo ostendi omnia capitula algebræ possibilia et impossibilia usque in infinitum et quæ sint generalia et quæ non itaque non est aliquid desiderabile in tota arte quantacumque difficile quod non habeat radicem dantem cognitionem in illo libro et addidi plura capitula nova in ipso et non potui edere in ipsunt propter nimiam magnitudinem huius libelli eo quod est impressus in forma parva licet liber ille non transiendit tria aut quattuor folia et est consumatio totius artis et est exctractus ex decimo euclidis*", see [Cardano 1539, Chapter LXVIII, paragraph 18, no page numbering].

> there is a section lacking, that I can't add for the sake of brevity, because this volume is already too large: since this [new] book is a completion of algebra, its title will be Ars magna. Here you find the rules to solve all equations and all my outcomes obtained by the arithmetisation of book X. In particular, I have added some beautiful rules and two new cases.[17]

We will see in Section 3.2 that this description – a book that deals with cubic and quartic equations, which is entitled "Ars magna", and that contains the arithmetisation of *Elements*, Book X – fits the *Ars magna arithmeticæ*. Moreover, at the very beginning of the *Ars magna arithmeticæ*, we read this:

> [a]fter writing the Practica, it has seemed to me that it was necessary illustrating the cases, considered by the majority impossible, that I have found thanks to the proofs contained in our three books on Euclid, except for two rules,[18]

which is exactly the *incipit* of the manuscript N 187, except that here the words 'the former has been revealed by magister Nicolò Tartaglia, the latter by Ludovico Ferrari' are stroked through. Then, the *Supplementum* comes truly to be the link between the *Practica arithmeticæ* and the *Ars magna arithmeticæ*, which come one after the other in this order.

These two evidences would moreover place the minimal time limit of the composition of the *Ars magna arithmeticæ* on 1539. Gavagna's third remark is that Scipione del Ferro is never mentioned in the *Ars magna arithmeticæ*, whereas it is in the *Ars magna*. Since Cardano and Ferrari discovered Scipione's contribution in 1542 (see below, appendix A at page 347), this suggests to fix the maximum time limit on this year.

[17]This is Gavagna translation. "*[D]eest in opusculum quod ob exiguam formulam cum in nimiam liber hic auctus sit magnitudinem adiici non potuit, ad artis totius complementum hoc artis magne titulo dicat: in quo universorumque capitulorum algebræ usque in infinitum inveniendi formula descripta est et quod super euclidis decimum ad normam numero rerum reducti inveneram congessi*", see [Cardano 1539, *Ad lectorem*, no page numbering].

[18]This is Gavagna translation. "*Post compositionem Libri Practicæ visum est mihi necessarium ea ostendere, quæ a pluribus impossibilia iudicata sunt, quæ omnia nos invenimus ex demonstrationibus trium librorum a nobis supra Euclidem , exceptis duabus regulis harum*", see [Cardano 1663c, page 303]. The mentioned "three books on Euclid" are the *Nova geometria*, a treatise on geometry started in 1535, which should have been composed at the beginning by three, then seven, nine, and finally fifteen books.

But Gavagna goes further on. Her main thesis is that the *Ars magna arithmeticæ*, originally an addition to the *Practica arithmeticæ*, was the first writing that should have released the cubic and quartic formulae. The time passing by, this work became so considerably filled with results that it conflicted with the project of the *Opus arithmeticæ perfectum*. Thus, it should have been reorganised in two parts. The first part devoted to irrational numbers should have formed the Book III of the *Opus arithmeticæ perfectum*, while the second one on equations became the core of the *Ars magna*. This is indeed the structure of the *Ars magna arithmeticæ* as we know it nowadays.

We will consider then the *Ars magna arithmeticæ* as an intermediate step between the *Practica arithmeticæ* and the *Ars magna*. In the following, we will come across several links between the two treatises that go in this direction and, then, that confirm Gavagna's hypothesis .

Even though the *Ars magna arithmeticæ* is originated from the *Practica arithmeticæ*, a lot of work has been done in the meanwhile concerning cubic equations. As Gavagna indicates and as we will see, the *Ars magna arithmeticæ* marks the rupture with the *abaco* schools and opens the way to equations as a proper subject of inquiry.

Finally, in 1545 comes what has been considered Cardano's masterpiece, the *Ars magna*. There, the equations, and in particular the cubic ones, play the main role. One can only grasp the importance of the *Ars magna* in opposition to what came before. At last all the scattered methods, rules, and special cases that has been discovered up to that time give birth to a systematic treatment of cubic equations, with geometrical proofs of the solving methods.

1.2. A puzzling book

Nevertheless, in the *Ars magna* the things reveal to be not so plain as Cardano could have wished. As we will see, when a cubic equation has three real distinct solutions, it happens that some square roots of negative numbers appear in the formula – which is the *casus irreducibilis*, as it has been lately called (for the nowadays method for deriving the formula, see Appendix 1.5 at page 38).

Twenty-five years after the publication of the first edition of the *Ars magna*, comes the *De regula aliza*, in which Cardano conveys his hope to overcome the above problem. It is Cardano himself that, in the 1570 *Ars magna*, Chapter XII, provides the reader with this tip (for the full quotation see the next section at page 26). This is a key point, since my whole interpretation of the book starts from the link that is established there between the *Aliza* and the *casus irreducibilis*.

Unluckily, the *Aliza* is such a cryptic work, starting from the title. It immediately strikes no matter which reader who is trying to get acquainted with the book. What is the meaning of the term '*aliza*' in the title? This is neither a common Latin word nor it was one of the brand-new mathematical words loan-translated in Latin at that time. Moreover, the term does not seem to belong to Cardano's own mathematical terminology. As far as I know, only three other occurrences can be found in the whole of Cardano's mathematical writings, and only one of these refers to the book itself. In chronological order, the first reference is in the *Ars magna*, Chapter XII,[19] while speaking of the *casus irreducibilis*. There, we read of an "*aliza* problem" (in 1545) or of a "book of *aliza*" (in 1570 and 1663), which are supposed to amend the cubic formula in the case $\Delta_3 < 0$. The second occurrence steps outside the topic of cubic equations and is a mere mention. It is in the 1554 (and following) edition(s) of the *De subtilitate*, while Cardano deals with the "reflexive ratio [*proportio reflexa*]".[20] Nevertheless, I could not retrieve the corresponding passage in the *Aliza*. The last occurrence is in the *Sermo de plus et minus*, which contains a specific mention of *Aliza*, Chapter XXII (see here, Section 4.6 at page 337). Thereafter, except in the title page of the *Aliza*, the term is nowhere else mentioned. We also remark that Cardano does not pay any attention in trying to explain this so rare term.

It has been hinted in 1799 by Pietro Cossali[21] that the term '*aliza*' means 'unsolvable'. In 1892, Moritz Cantor[22] relates that Armin Wittstein suggests that

[19]See [Cardano 1545, Chapter XII, page 251] or see here, page 112.

[20]More precisely, Cardano makes reference to the *Ars magna* and to the *De regula aliza* ("*ex demonstratis in Arte magna et regula Aliza*") concerning the ratios of the side of a regular heptagon to any diagonal subtended to two or three sides, see [Cardano 1554, Book XVI, page 428].

[21]"De Regula Aliza*, cioè* De regula Irresolubili", see [Cossali 1799a, volume II, page 441] or [Cossali 1966, Chapter I, paragraph 2, page 26].

[22]"*[D]ie dritte führte den nie und nirgend erklärten Titel De regula Aliza, der durch unrichtige Trasscription aus dem arabischen Worte a'izzâ (schwierig anzustellen, mühselig, beschwerlich)*

this term comes from a wrong transcription of the Arabic word '*a'izzâ*' and means 'difficult to do', 'laborious', 'ardous'. In 1929 Gino Loria[23] advises as a common opinion that the term comes from a certain Arabic word that means 'difficult'. None of the above hypotheses are supported by a precise etymology.

Very recently, Paolo D'Alessandro has confirmed Cossali's hypothesis.[24] The term '*aliza*', or '*aluza*', is likely a misspelling based on the Byzantine pronunciation 'ɑ́liʤjɑ' of the Greek word 'ἀλυθεῖα', which is composed by the negative particle 'α' and by the feminine singular passive aorist participle of the verb 'λύω', 'to unbind', 'to unfasten', 'to loosen', 'to dissolve', 'to break up', 'to undo', 'to solve'.[25] Thus, '*aluza*' means 'non-solvable'. This etymology soundly agrees with Cardano's words in *Ars magna*, Chapter XII, where the *Aliza* and the *casus irreducibilis* are linked. Note that, if we understand '*irreducibilis*' as the fact that the cubic formulae (which should convey three real roots) cannot reduce to some real radicals, then in Cardano's mathematics '*irreducibilis*' completely overlaps with 'non-solvable'.

If the enigma of the title can be relatively easily fixed, the same cannot be said about other difficulties. Basically, there is one fundamental reason to which we may ascribe the charge of the high unfamiliarity of the *Aliza* for the modern readers. Spoiling the whole surprise, Cardano does not reach his aim (or at least the aim that concerns the *casus irreducibilis*) in the *Aliza*. In very truth, he could not reach it, but this is only proved three centuries later.[26] The book then tells of a failed attempt – rather, of an unattainable attempt – so that it is not surprising to find a good amount of looseness inside.

There are anyway other extrinsic reasons that contributed to worsen the situation. However common the typos and mistakes in calculations were at the

entstanden sein kann [Diese Vermuthung rührt von H. Armin Wittstein her.], und alsdann Regel der schwierigen Fälle bedeuten würde", see [Cantor 1892, page 532].

[23]"*[...] il titolo, sinora inesplicato* De Regula Aliza *(secondo alcuni* aliza *deriverebbe da una parola araba significante* difficile*)*", see [Loria 1931, 1950, p. 298].

[24]By personal communication.

[25]See [Liddel and Scott 1996] and [Panza 1997, page 367]. If it is so, then note that '*aluza*' and 'analysis' have a common origin.

[26]In fact, it has been proved that imaginary numbers have necessarily to appear in the cubic formula when the equation has three real, different roots by Pierre Laurent Wantzel in 1843, Vincenzo Mollame in 1890, Otto Hölder in 1891, and Adolf Kneser in 1892. Paolo Ruffini also provided an incomplete proof in 1799. See [Wantzel 1842], [Mollame 1890] and [Mollame 1892], [Hölder 1891], [Kneser 1892], [Gegenbauer 1893], and [Ruffini 1799] and [Ruffini 1813]. I plan to develop further this topic in a next work.

time, the book is particularly fulfilled with them. Sometimes the misprints concern the correspondence of the lettering between the diagram and the text, and then we can only hope that the text is sufficiently enough devoid of incongruities to manage to correct the wrong letter(s). Sometimes we find some lettering in the text and no diagram at all. In other cases, there must have been full lines that were skipped at the page break. Moreover – and to be kind – Cardano's peculiar style in writing in Latin is not the plainest one. Speaking the mathematics as Cardano and his contemporaries did without a complete and stable formalism causes as such a certain amount of ambiguity, and Cardano was not used to simplify the task to his readers – to say the least. The most favourable case is when a general speech is accompanied by numerical examples, so that we can untangle the text relying on calculations. Otherwise, Cardano's general observations are usually long, convoluted sentences with more than one possible interpretation. Unluckily, the 1663 edition cannot help the reader. In fact, Spon, who revised Cardano's writings, was not a mathematician and the best that he could do was to proofread the grammar and to correct here and there some clear mistakes.

The *Aliza*'s obscurity is displayed by the lack of any global ordering in the flowing of the chapters' topics. At the beginning of Chapter 4, I will provide a general overview of the kinds of issues that are dealt with in the *Aliza*. A same topic is usually scattered all along the book, with repetitions and logical gaps, so that in the end the reader feels highly disoriented. Under these conditions, the *Aliza* is more pragmatically to be intended as a miscellany rather than as a unitary book. To resume, in Pietro Cossali's words,

> [t]o the obscurity of that book caused by the difference of language, another [obscurity] is coupled, [which is the one] produced by an infinite, very pernicious mass of errors both in the numbers in calculations and in the letters in figures and related proofs. And another one, which is absolute and intrinsic, has to be added. It is the lack of order due to which it is very hard and uncomfortable to pull the strings, to see the result, and to evaluate the discovery. We spot at Cardano trying, opening

> new roads, retracing his steps, turning to one side or another according to his intellect's suggestions. In a word, the book is the action of attempting, not the ordering of the discoveries. And in the end it is one of Cardano's books, that his contemporaneous Bombelli called obscure in the telling.[27]

It is exactly because of these long-range incongruities that one needs to master as good as possible the contents of the other treatises by Cardano that deal with equations, in order to try to understand it, or a part of it, at least. It is an obviously recommend practice to contextualise a work in the frame of the other writings of the same author. But more than that, in this particular case where we miss the guide of a general structure, we will need all the small pieces of coherent mathematics that can be gathered from Cardano's other works (and in particular from the *Practica arithmeticæ*, the *Ars magna arithmeticæ*, and the *Ars magna*) to get oriented in reading the *Aliza*. This is why we will devote two whole chapters to those works and other contextual matters.

1.3. From the "*aliza* problem" to the *De Regula Aliza*: changes in the editorial plan of the *Opus Arithmeticæ Perfectum* and an attempt to loosely assign a date

Before addressing the mathematical details, it is fully worthwhile to spend a few words to clarify as much as possible the chronological boundaries of the *Aliza*. In fact, this work appears to be a miscellany of mathematical writings, notes, remarks, and observations that shine out neither for consistency nor for cohesion – at least at first sight. Then, the question that one immediately wonders is whether it is possible to date these writings and, if it is so, to when – even if approximately. If one takes a look at the (supposedly chronological) order in

[27]"*Alla tenebria di quel libro a noi cagionata dalla diversità del linguaggio se ne accoppia un'altra prodotta da un'infinita perversissima folla di errori, e di numeri ne' calcoli e di lettere si' nelle figure, si' nelle relative dimostrazioni; e bisogna eziandio aggiungersene altra assoluta, ed intrinseca proveniente da mancanza di ordine, per la quale riesce cosa faticosissima, e malegevolissima l'unire le fila, vedere il risultato, valutare il discoprimento. Si scorge Cardano, che tenta, che si apre nuove strade, che ritorna sulle battute, che si volge or da un lato, or dall'altro seguendo i suggerimenti varij dell'ingegno; in una parola il libro è l'atto del tentare, non un ordine delle scoperte; e finalmente è un libro di Cardano, che da Bombelli pur suo coetaneo fu denominato nel dire oscuro*", see [Cossali 1799a, volume II, page 442] e [Cossali 1966, page 27].

which the books are displayed in the fourth volume of the *Opera omnia*, he will notice that the *Aliza* comes immediately after the *Ars magna arithmeticæ*, which in turn follows the *Ars magna*. But this is completely ineffective, since – as we have seen in the preceding section – the *Ars magna arithmeticæ* has been very likely composed before the *Ars magna*.

It is a very hard issue to give a reasonable estimate in time of the *Aliza* starting from the internal relations, namely from the mutual relationship of the contents of the chapters. At present, I will content myself with checking the external references to this book.

Let us firstly address the references to the *Aliza* that appear in the other mathematical writings by Cardano. As already said, the term '*aliza*' appears only four times in the whole Cardano's mathematical writings, once in the title of the book itself, once in the *Ars magna*, Chapter XII, once in the *De subtilitate* (1550), and once in the *Sermo de plus et minus*. I leave aside the reference in the *De subtilitate* (see above, footnote 20 at page 23), since it is a very brief mentioning of a certain result from the *Aliza*, which is not better specified and that I could not retrieve. I will analyse the reference in the *Sermo de plus et minus* in Section 4.6 at page 337. Here follows the quotation from the 1545 *Ars magna*'s edition:[28]

> [w]hen the cube of one-third the coefficient of x is greater that the square of one-half the constant of the equation, which happens whenever the constant is less than three-fourths of this cube or when two-thirds the coefficient of x multiplied by the square root of one-third the same number is greater than the constant of the equation [that is, for short, when $\Delta_3 < 0$], then the solution of this [$x^3 = a_1x + a_0$] can be found by the aliza problem which is discussed in the book of the geometrical problems[29]

and from 1570 and 1663 editions:

[28]See also below, footnotes 53 and 55, page 112.

[29]"*At ubi cubus tertiæ partis numeri rerum excedat quadratum dimidii numeri æquationis, quod accidit quandocunque numerus æquationis est minor* $\frac{3}{4}$ *cubi illius, vel ubi ex* $\frac{2}{3}$ *numeri rerum producitur in R* $\frac{1}{3}$ *esiusedm numeri maior numerus numero æquationis, tunc hoc dissolvitur per quæstionem Alizam, de qua in libro de quæstionibus geometricis dictum est*", see [Cardano 1545, Chapter XII, page 62] or see [Cardano 1968, page 103].

> [w]hen the cube of one-third the coefficient of x is greater that the square of one-half the constant of the equation, which happens whenever the constant is less than three-fourths of this cube or when two-thirds the coefficient of x multiplied by the square root of one-third the same number is greater than the constant of the equation [that is, for short, when $\Delta_3 < 0$], then consult the Aliza book appended to this work.[30]

The quotations not only allows to fix the *Aliza*'s overall subject topic, but they also couple (in 1545) the name '*aliza*' with a certain "book of the geometrical problems". Let us store away this information for the following. Note moreover that in 1545 the "*aliza*" is merely a "problem", while in 1570 it has reached the extent of a whole "book" (and it must be so, since in the last quotation Cardano makes reference to one of the books appended to the new *Ars magna*'s edition). Therefore, the *Aliza* likely arises as a problem, more precisely as a "geometrical" problem, and then grows bigger and bigger until it becomes a book. This could (at least, partially) explain the lack of order in its structure. But, much more important, is the fact that the *Aliza* is firstly published together with a book – the *Ars magna* – that contains, even though in a few lines, the key to interpret it. This is fundamental in order to understand how Cardano possibly regarded to the *Aliza*, namely as the treatise that (hopefully) contains the solution to the problem entailed by the *casus irreducibilis*.

Concerning the other way round, that is the references to other Cardano's mathematical writing that appear in the *Aliza*, I do not have the least hope that they could help in locating the *Aliza* in time. In fact, it is absolutely possible – actually, likely – that Cardano subsequently added some references in a text that had already been written since a while. We should remark, in fact, that he mainly refers to the *De proportionibus* and *Ars magna*, which were published together with the *Aliza* in 1570.

[30]"*At ubi cubus tertiæ partis numeri rerum excedat quadratum dimidii numeri æquationis, quod accidit quandocunque numerus æquationis est minor* $\frac{3}{4}$ *cubi illius, vel ubi ex* $\frac{2}{3}$ *numeri rerum producitur in R* $\frac{1}{3}$ *esiusedm numeri maior numerus numero æquationis, tunc consules librum Alizæ hic adjectum*", see [Cardano 1570b, Chapter XII, page 31v] and [Cardano 1663b, Chapter XII, page 251] or see footnote 5 in [Cardano 1968, page 103].

So far, so good, but we did not progress much. Cardano already had in mind a certain "*aliza* problem" around 1545 – which makes sense, since he already had to came across the *casus irreducibilis* developing his cubic formulae in the *Ars magna*. Afterwards, we can check for the references to the *Aliza* in Cardano's non-mathematical treatises, and there we are extremely lucky. In fact, we do not only have his autobiography *De libris propriis*, but, being the true graphomaniac that he was, he left us five versions of it, written over a period of more than thirty years. They are [Cardano 1544] (which is also in the *Opera omnia* in [Cardano 1663d]), [Cardano 1998] (which dates back to 1550), [Cardano 1557] (which is also in the *Opera omnia* in [Cardano 1663e]), [Cardano 1562] (which is also in the *Opera omnia* in [Cardano 1663f]), and [Cardano 1663g] (which dates back to 1576).[31] There, Cardano remembers not only his personal life, but also his career and his writings. Referring to his autobiography, Cardano says that

> [t]he book [the *De libris propriis*] was often modified [...]. Thus I set forth for myself in it a certain image of everything that I had written, not only as an aide-memoire, and a mean of selecting those books which I would finish and correct first, but also to set down why, when and in what order I wrote what I did [...]. Indeed I did not only set down here the titles of my books, but also their size, incipits, contents, order, the utility of the division and order of the books, and what they contained of importance.[32]

Each of these versions of Cardano's autobiography consecrates to the *Aliza* only a few lines – and the most of the time as a fringe topic – but we will see that we can anyway draw some conclusions.

[31] For more information on the chronology of Cardano's works, see Maclean's essay in the preface of [Cardano 2004].

[32] The English translation is by Maclean in [Cardano 2004, page 11], or "*[d]einde auxi, iterumque mutavi [...]. Iconem ergo quendam mihi in eo proposui omnium eorum quæ a me conscripta sunt, non solum ad memoriam confirmandam, eligendosque mihi libros quos prius absolverem et castigarem: sed ut doceam, quibus causis, temporibus, quoque ordine talia conscripserim, et ut vim numunis suo loco testarer [...]. Verum non hic solum docui librorum nomina, sed et magnitudinem, initium materiam pertractatam, ordinem divisionis, ordinisque librorum inter se utilitatem, quidque in se præcipuum continerent*" in [Cardano 1562, pages 23-24].

In the lists of his works that Cardano makes in the different versions of his autobiography, the *Aliza* as a published book is of course only mentioned once, in 1576.[33] The most of the information concerning the *Aliza* comes collaterally to the description of the *Opus arithmeticæ perfectum*. This[34] should have been a mathematical, encyclopedic work, composed by fourteen books, and probably conceived between 1530s and 1560s. Unluckily, it has never been accomplished. It is quite difficult to gather some consistent information on the supposed table of contents of the *Opus arithmeticæ perfectum*.[35] Anyway, it should have included calculations with integer, fractional, irrational, and denominated numbers, proportions, properties of numbers, commercial arithmetic, algebra, plane and solid geometry, and some arithmetical and geometrical problems. Veronica Gavagna[36] suggests that, though the general outfit follows the *Practica arithmeticæ*'s structure, it also reveals an evident symmetry with Cardano's unpublished comment to the *Elements* (which will be later referred to as the *Nova Geometria*), suggesting that the *Opus arithmeticæ perfectum* should be considered as a sort of arithmetical counterpart. Moreover, it is likely that some of Cardano's mathematical printed books that are nowadays available should have fill up some of the books, or part of them, in the table of contents of the *Opus arithmeticæ perfectum*. Namely,[37] it is the case for the *Tractatus de integris*, the *De proportionibus*, and the *De numerorum proprietatibus*, which can be found in the fourth and tenth volumes of the *Opera omnia*. The *Ars magna arithmeticæ*, the *Ars magna*, and the *De regula aliza* also play a role in this game, but the matter is more ticklish.

All the versions of Cardano's autobiography agree that the Book X of the *Opus arithmeticæ perfectum* should have been on algebra and equations. In 1544, he says that "the tenth [book] is entitled Ars magna, it contains 67 chapters",[38] whereas the Books XIII and XIV "are assigned to arithmetical and geometrical

[33] See [Cardano 1663g, page 40]. The other versions of Cardano's autobiography are all previous to 1570.
[34] See [Tamborini 2003], [Gavagna 2010], and [Gavagna 2012].
[35] For a detailed discussion, see [Cardano 2004, pages 64-66].
[36] See [Gavagna 2010, page 65].
[37] See below, footnote 43, page 32.
[38] "*[D]ecimus inscribitur Ars magna, continet sexaginta septem capitula*", see [Cardano 1544, page 426].

problems".[39] This quotation raises the incidental question to know what was that "Ars magna" of which Cardano is speaking, since all the editions that we have with that title (handwritten or printed) contain only 40 chapters. Massimo Tamborini in [Tamborini 2003, pages 178-179], Ian Maclean in [Cardano 2004, page 65], and Veronica Gavagna in [Gavagna 2012] report on this incongruity, and I refer to them for an accurate discussion. In a nutshell (despite Maclean's interpretation), it seems to me more likely that both the *Ars magna arithmeticæ* and the *Ars magna* (or, at least, their original common core on equations) should have concurred to the Book X of the *Opus arithmeticæ perfectum*.[40] We moreover recall that, in Chapter XII of the 1545 *Ars magna*, the "*aliza* problem" was coupled with a certain "book on geometrical problems" (see above, at page 28). Then, in 1544 the situation is the following. The Book X of the *Opus arithmeticæ perfectum* is devoted to equations and possibly consists of some parts of the *Ars magna arithmeticæ* and *Ars magna*. The *Aliza* is not explicitly mentioned, but is connected (at least, starting from 1545) to certain geometrical problems in the *Opus arithmeticæ perfectum*'s last book.

Later on, in 1550, Cardano devotes the Book X to "all the chapters on the square together with the Aliza rule". The Book XIV still concerns geometry, but is restricted to the "measure of figures".[41] Any reference to the "Ars magna" or to our *Ars magna* has disappeared. It really seems likely to me that now the *Aliza* is shifted from filling the last book to filling the tenth one, and that accordingly the last book of the *Opus arithmeticæ perfectum* is reduced in contents.

Afterwards, from 1557 on, the *Aliza* disappears again, coming back to the same situation as in 1544.[42] This time Cardano gives the *incipit* and the lengths of the books in the *Opus arithmeticæ perfectum* that he has already written. In

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[39]"*[T]ertiusdecimus ac quartusdecimus, quæstionibus Arithmeticis et Geometricis destinantur*", see [Cardano 1544, page 426].

[40]We also recall that, according to Veronica Gavagna, Book III of the *Opus arithmeticæ perfectum* (the *incipit* of which is "*Cum in radicibus quantitatum*") should have been covered by the first part of the *Ars magna arithmeticæ* on the arithmetisation of Book X of the *Elements*, see here page 22.

[41]"*In [decimo] omnia capitula supra quadratum cum regula Aliza*" and "*[i]n [quartodecimo] ad mensuram figurarum pertinentia, quæ geometrica vocantur*", see [Cardano 1998, page 9v].

[42]"*Decimus de regulis magnis, atque ideo ars magna vocatur: atque hic solus ex omnibus editus est*" and "*Tertiusdecimus quæstiones Arithmeticas, ut Quartusdecimus Geometricas*" in [Cardano 1557, pages 37-38] and [Cardano 1562, page 16].

particular, he says that the Book X begins with "*[h]æc ars olim a Mahomete*" and is made up of 83 *folia*. This enables[43] us to identify for sure the Book X with the *Ars magna*'s that we have nowadays available.

Finally, but almost twenty years later in 1576, Cardano tells that in 1568 he joined the *Aliza* and the *De proportionibus* to the *Ars magna* and passed them for press.[44]

We can now try to fix some temporal limits for the *Aliza*. An "*aliza*" is firstly mentioned in 1545 (but we can stretch this time limit of a year, up to 1544) and it sinks into oblivion at least from 1557. It reappears in 1568, but it is only a quick mention. We can therefore conjecture that the miscellany of writings that compose the *Aliza* (or a part of them) starts to be conceived at worst between 1544 and 1545. Then, its proofreading could have last at best until 1557, when Cardano had already lost interest in it. Afterwards, it is likely that during the 1560s, due to his personal misfortune, Cardano desists from the *Opus arithmeticæ perfectum* project and in general gives up with all of his mathematical projects. Moreover, regarding at least to the *casus irreducibilis*, the changes in the editorial project of the *Opus arithmeticæ perfectum* could also attest of Cardano's hope to find some new results. In particular, the lapse of time during which only the *Aliza* appears as a part of the *Opus arithmeticæ perfectum* could correspond to the lapse of time during which Cardano believes to manage to avoid the *casus irreducibilis*, namely for the few years from 1550 to (at best) 1557. But, when the *Aliza* is printed in 1570, its second title (just above 'Chapter I') recalls its origins as a problem, since it contains the word "*libellus*", which means 'pamphlet', 'booklet', 'small book'. No matter then to replace the *Ars magna*.

We can finally address the sources that mention the *Aliza*, but that were not written by Cardano himself. I am only aware of one,[45] which is in [Betti 2009, page 163]. Gian Luigi Betti relates about Ercole Bottrigari, printer, scholar, music theorist, and bibliophile, who lived in Bologna between 1531 and 1612. We do

[43]By the way, this also enables us to stick the *Tractatus de integris* to the first book, the *De proportionibus* to the fifth book, and the *De numerorum proprietatibus* to the sixth book of the *Opus arithmeticæ perfectum*, see [Gavagna 2012].

[44]"*De proportionibus, et Aliza regula addidi anno MDLXVIII ad librum Artis magnæ et edidi*" in [Cardano 1663g, page 41].

[45]I thank Veronica Gavagna for this reference.

not know when he met Cardano for the first time, but he wrote in his *La mascara overo della fabbrica de' teatri e dello apparato delle scene tragisatirocomiche*[46] that he questioned Cardano about the *Aliza* "two or three years" before 1570.[47] As Bottrigari testifies, Cardano avoids to answer, but in doing so he drops that the *Aliza* is going to be printed in Germany.[48] This is in accordance with Cardano's assertion in his 1576 autobiography.

Summing up, a certain "*aliza* problem", which, according to the *Ars magna*, concerns the *casus irreducibilis* and which should have originally been linked to geometry, is already there in 1544 or 1545. We get the strong impression that, from that moment on, the problem at issue grows more and more in importance and attains its best around 1550. The reference to the "*aliza*" disappears then in 1557. The *Aliza* (as we know it) is ready for press in 1568 and printed in 1570.

Therefore, the core of the book on the *casus irreducibilis* was already conceived a long time before its publication. At a certain point, a more or less sizeable number of pages on fringe topics could have been added and the name of the problem was handed on the whole miscellany. In this way, what at the beginning was nothing but an "*aliza* problem" could have become an entire book – our *De regula aliza*. Obviously, handing a title to a miscellany of heterogeneous chapters does not automatically mean to make the text consistent. Indeed, we will see in Section 4.1 at page 223 that the *Aliza* has not been proofread or proofread very quickly.

[46]This is one of Bottigari's late works. Gian Luigi Betti found the autographic manuscript MS B 45 in the library G. B. Martini at the Academy of music in Bologna.

[47]"*É invece certo che [Bottigari] gli [Cardano] abbia direttamente posto una questione 'sopra a quel suo capitolo Alizam algebratico [. . .] nella sua Arte magna, il qual da poi publicò due o tre anni, facendo ristampare essa Arte Magna insieme con il libro De propositionibus [sic in Betti] l'anno 1570'*" in [Betti 2009, page 163].

[48]"*La risposta ricevuta al quesito proposto fu invero piuttosto singolare, poiché il Cardano avrebbe affermato : 'ch'egli non lo sapea, affermandomi che il suo genio era stato e non egli che lo aveva scritto, con soggiungermi che per ciò spesse volte egli stesso non sapea quello che avesse scritto, et che leggendolo non lo intendeva. Cosa che veramente mi scandalizzò molto a prima faccia, poi mi diè che molto a meravigliarmi. Finalmente mi spedì dicendomi ch'io aspettassi che tosto (come ho detto, che poi veramente fu) ei verrebbe in istampa da Alemagna'*" in [Betti 2009, page 163] referring to the *Mascara*, page 84.

1.4. The readers of the *De Regula Aliza*

Due to the problems that I have detailed in the preceding section, the *Aliza* was, and still remains, a very unfamiliar book. During the centuries, it had nevertheless a handful of reader. In the following, I will give an overview on the studies that have already been done on this book. Note that I will not mention the authors who simply quote the title.

One of the very first readers of the *Aliza* is possibly Rafael Bombelli. Despite the last sentence of the above quotation by Cossali,[49] I did not manage to retrieve any comments in his *Algebra* (1572) about the *Aliza*, but we have an indirect link that deserve to be mentioned. In fact, it is Cardano himself that couples in the first paragraph of the *Sermo de plus et minus* a reference to the *Aliza* and Bombelli's name.[50] What we know for sure is that Bombelli read the *Ars magna*, since in his *Algebra*, in the preface to the readers, he criticises the obscure way in which the *Ars magna* was written.[51] This also shows that Cossali's last sentence in the above quotation is inaccurate.

Federico Commandino read for sure the *Aliza*, or at least some parts. In fact, in his edition of the *Elements*, he briefly mentions Cardano's alternative sign rule from *Aliza*, Chapter XXII (see below, Section 4.6 from page 331).[52]

Belonging to the next generation of mathematicians, Simon Stevin makes reference to the *Aliza* in his *Arithmétique*. While solving the equation $x^3 = 6x+40$, Stevin shortly says that Cardano also puts some examples in his *Aliza*.[53]

Also Thomas Harriot makes reference to the *Aliza*, and twice. We know that he read (at least) one chapter of the *Aliza*. In the British Library,[54] Rosalind

[49] See above, at page 25.

[50] For more details, see below, Section 4.6 at page 331.

[51] "*Hanno poi, e Barbari, e Italiani à nostri tempi scritto [...], oltre che il Cardano Mediolanese nella sua arte magna, ove di questa scientia assai disse, ma nel dire fu oscuro*", see [Bombelli 1572, Agli Lettori, no page number].

[52] "*[C]oguntur dicere si minus per minus multiplicetur produci plus. Quod verum non esse primus animadvertit Hieronymus Cardanus non solum mathematicus, sed et philosophus ac medicus prestantissimus, ut apparet in libro de regula aliza*", see [Commandino 1572, Book X, Proposition 34, Theorem III, page 149].

[53] "*Cardane met aussi en son* Aliza *quelques exemples, servans à ceste matiere, mais pas generaux*", see [Stevin 1585, Book II, page 309].

[54] It is the Add. Ms. 6783, folio 121. The page has no date and Tanner remarks that no sufficient indication can be found to place it in time. Anyway, she remarks that all the dated pages in the manuscript are between 1589 and 1619. Note that Tanner says that the folio 121

Tanner has found[55] the following note to one of Harriot's colleagues, Walter Warner.

> Although Cardane in the beginning the first Chapter (pag. 6)
> of his 10$^{\text{th}}$ booke of Arithmeticke wold have $\sqrt{9}$ to be $+3$ or -3,
> yet in his Aliza being a latter worke he was of another opinion.
> I prey read his 32 Chapter being at the 42 page.[56]

The chapter of the *Aliza* to which Harriot is referring is actually Chapter XXII, which fairly starts at the *folio* 42 of the 1570 edition. In her two articles,[57] Tanner establishes a relation between Harriot's remarks on the fact that there is a choice behind the sign rule and the new sign rule proposed by Cardano in the *Aliza* (see Section 4.6 at page 331).

As far as I know, there is only one scholar who studied the *Aliza* pretty in details. It is Pietro Cossali, born in Verona in 1748 and dead in Padua in 1815. He was a priest by the Teatini's order in Milan, calculus, physics, and astronomy professor at the university in Padua, member of the Società italiana delle scienze, and pensionnaire of the Reale istituto italiano di scienze, lettere ed arti.[58] He also was a historian of mathematics. In 1799 his history of algebra in the Renaissance Italy, the *Origine, trasporto in Italia, primi progressi in essa dell'Algebra*[59] was published in two volumes. Moreover, since 1966 we have available the commented transcription by Romano Gatto of Cossali's manuscript *Storia del caso irriducibile*,[60] which has a considerable number of pages in common with the history of algebra.

In the *Storia del caso irriducibile* and mostly in the second volume of the history of algebra, Cardano plays a central role and the *Aliza* is handled on

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is a stray sheet among a batch of unconnected papers, but in truth it is associated with other notes on the *Ars magna*, see [Schemmel and Stedall last checked January 22, 2014].
The second reference to the *Aliza* is in Add. Ms. 6785, folio 197. I thank Jackie Stedall for these connections.

[55]See [Tanner 1980b]

[56]I have picked the quotation up from [Tanner 1980b, page 144]. The '10$^{\text{th}}$ booke of Arithmeticke' is in truth the *Ars magna*, which at a certain point should have become Book X of the *Opus arithmeticæ perfectum*, see above, Section 1.3 at page 26.

[57]See [Tanner 1980b] and [Tanner 1980a].

[58]For a more detailed biography, see [Cossali 1966, page 3]

[59]See [Cossali 1799a].

[60]See [Cossali 1966].

the same footing as the *Ars magna*, as well as the *Ars magna arithmeticæ*. Nevertheless, it must be said that for the most of the time Cossali's accuracy as historian is not up to standards. He seldom gives the exact references, has no hesitation in integrating or completely rewriting Cardano's proofs, and pays no attention to the interconnections between Cardano's treatises. In short, he is doing mathematics from a historical starting point rather than history of mathematics.

Indeed, Cossali was a mathematician, and I believe that this could partially explain my above remarks on his work as historian. It is exactly in this period that the mathematicians start to study in depth the *casus irreducibilis* for cubic equations. In 1781 the Academy of Padua announced a competition to prove whether it is possible to free the cubic formula from imaginary numbers, but finally the prize was not assigned. Then, in the same 1799, Ruffini begins to get involved in the study of equations and in 1813 he publishes an incomplete proof of the fact that, when a cubic equation goes under the *casus irreducibilis*, each of its solution cannot be expressed only by real radicals.[61] Cossali was very interested in the topic of the *casus irreducibilis*. His very first work, the *Particularis methodi de cubicarum æquationum solutione a Cardano luci traditæ. Generalis posteriorum analystarum usus ex cap. I De Regula Aliza ipsius Cardani vitio luculentissime evictus. Atque mysterium casus irreducibilis post duo sæcula prorsus retecta causa sublatum specimen analyticum primum* (1799) is devoted to it, and the *Disquisizione sui varj metodi di eliminazione con il componimento di uno nuovo* (1813) ranges around the same issue.[62] He wanted to take part in the 1781 competition, but he could not finish on time his *Sul quesito analitico proposto all'Accademia di Padova per il premio dell'anno 1781 di una assoluta dimostrazione della irriducibilità del binomio cubico*.[63] There, he also started a long lasting controversy with Anton Maria Lorgna on some mistakes that he did while dealing with the *casus irreducibilis*. Briefly, the mathematician Cossali worked pretty much on equations, and in particular on cubic equations. Cossali's

[61]See [Ruffini 1799] and [Ruffini 1813].
[62]See [Cossali 1799b] and [Cossali 1813].
[63]See [Cossali 1782].

strong interest toward the *casus irreducibilis* is the reason why – I suggest – he troubled with the *Aliza* under a mainly mathematical viewpoint.

In our chronological review of the *Aliza*'s readers, we stumble at this point upon some scholars who write an overall history of mathematics and limit themselves to chose and explain a bunch of tiny mathematical techniques from the *Aliza*. They are Charles Hutton (*Tracts on mathematical and philosophical subjects* (1812), at pages[64] 219-224), Moritz Cantor (*Vorlesungen über die Geschichte der Mathematik* (1892), at pages[65] 532-537), and Gino Loria (*Storia delle matematiche* (1931), at pages[66] 298-299). In particular, concerning the overall flowing of the *Aliza*, Loria says that

> [w]hile [...] all of Cardano's writings do not shine out for clarity, this last [referring to the *Aliza*] is despairingly unclear. Nobody or a few people (one of these is the historian Cossali) struggled to understand it and its influence was poor or none. The one who goes through it follows Cardano while he tries one thousand ways to solve that distressing enigma. He sees him stopping, moving backwards, then resuming his way ahead, pushed by the longing to discover the limits of applicability of the solving method conceived by Tartaglia.[67]

Finally, we have the contemporary readers, like the already mentioned Rosalind Tanner (*The alien realm of the minus: deviatory mathematics in Cardano's writings* (1980), at pages[68] 166-168), Silvio Maracchia (*Storia dell'algebra* (2005), at pages[69] 277 and 331-335), and Jacqueline Stedall (*From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra* (2011), at page[70] 10).

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[64] See [Hutton 1812].

[65] See [Cantor 1892].

[66] See [Loria 1931, 1950].

[67] "*Mentre [...] tutti gli scritti del Cardano non brillano per chiarezza, quest'ultimo è di una disperante oscurità; nessuno o pochi (uno di questi è lo storico Cossali) si sforzarono di comprenderlo, ove scarsa per non dire nulla ne fu l'influenza. Chi lo percorre segue ll Cardano mentre tenta mille vie per scioglire quell'angoscioso enigma; lo vede arrestarsi, retrocedere per poi riprendere il cammino in avanti, sospinto dalla brama di scoprire i limiti di applicabilità del metodo di risoluzione ideato dal Tartaglia*", see [Loria 1931, 1950, page 298].

[68] See [Tanner 1980a].

[69] See [Maracchia 2005].

[70] See [Stedall 2011].

In a greater or lesser measure, they conform to their predecessors and give an episodic account of the *Aliza*.

It is also worthwhile to mention the new impulse that has been given in Milan from the early nineties to the studies of Cardano's writings thanks to the project of the edition of his works (see the webpage `http://www.cardano.unimi.it/`).

1.5. The mathematical context: how we deal with equations nowadays

As we will see in the following sections, Cardano is interested in the main in how the numbers can be expressed rather than in the numbers themselves. This is what we could call the 'algebraic shapes', that is, some expressions written in his stenography, like $\sqrt{a} + b$ or $\sqrt[3]{a} + \sqrt[3]{b}$ where a, b are rational numbers such that respectively their square or cubic roots are irrational. When he addresses the problem entailed by the *casus irreducibilis*, he will turn to these algebraic shapes rather than to the numerical values of the solutions, or to the structural relationships between them (as Lagrange, Ruffini, Abel, and Galois will subsequently do). In the end, he is trying for a general theory of the shapes under which certain numbers can be written. In fact, a recurring strategy in Cardano's mathematical treatises is to establish whether these shapes can convey a solution of an equation or not (see below, Section 3.2.4 at page 185 and Section 4.3.1 at page 256). As such, Cardano's attempt is doomed to failure. Nevertheless, this study of the algebraic shapes is very up-to-date and marks a trend that will go on for about two centuries up to the 18^{th} century.

Let us now try to understand why Cardano had to fail. In the following sections, I will recall our solving methods that lead to the formulae for quadratic, cubic, and quartic equations. In particular, I will deal with the cases in which the coefficients are real. Then, especially concerning cubic equations, I will explain, using Galois' theory, why imaginary numbers necessarily appear in the cubic formula when the equation falls into the *casus irreducibilis*. Finally, I will also provide some paradigmatic, cubic examples.

1.5.1. Solving quadratic equations. We consider the generic quadratic equation

$$\alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0, \tag{1.5.1}$$

with $\alpha_2, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\alpha_2 \neq 0$. We are looking for its solutions in $\mathbb{C}$.

Dividing by α_2, we get the monic equation

$$x^2 + \frac{\alpha_1}{\alpha_2}x + \frac{\alpha_0}{\alpha_2} = 0.$$

We rewrite the above equation in the following way

$$x^2 + \frac{\alpha_1}{\alpha_2}x + \left(\frac{(\alpha_1)^2}{4(\alpha_2)^2} - \frac{(\alpha_1)^2}{4(\alpha_2)^2}\right) = -\frac{\alpha_0}{\alpha_2}.$$

Thus, using the formula for the square of a binomial, we find a square on the left side of the equation

$$\left(x + \frac{\alpha_1}{2\alpha_2}\right)^2 = -\frac{\alpha_0}{\alpha_2} + \frac{(\alpha_1)^2}{4(\alpha_2)^2}.$$

Taking the square root and developing calculations, we obtain

$$x_1 = \frac{-\alpha_1 + \sqrt{(\alpha_1)^2 - 4\alpha_2\alpha_0}}{2\alpha_2} \quad \text{and} \quad x_2 = \frac{-\alpha_1 - \sqrt{(\alpha_1)^2 - 4\alpha_2\alpha_0}}{2\alpha_2}. \tag{1.5.2}$$

The relation (1.5.2) is called the 'quadratic formula' and $\Delta_2 = (\alpha_1)^2 - 4\alpha_2\alpha_0$ is called the 'discriminant' of a quadratic equation.

1.5.2. Quadratic equations with real coefficients. If α_2, α_1, α_0 in equation (1.5.1) are real, we can then discuss the nature of its solutions starting from the sign of its discriminant Δ_2. In the case $\Delta_2 > 0$, it is easily seen that we have two distinct real solutions. Analogously, in the case $\Delta_2 = 0$, we have two coincident real solutions. Finally, in the case $\Delta_2 < 0$, we have two complex conjugate solutions.

The main idea behind this method is the so-called 'completion of the square'. It is exposed above as an algebraic trick, but whenever the coefficients are real we can interpret it in a geometrical way, according to the following diagram.

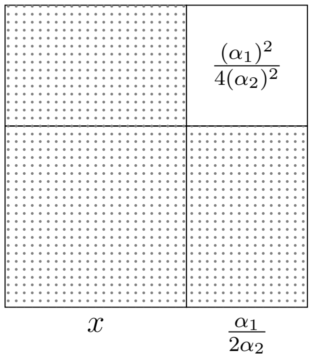


Figure 1.1 – Geometrical interpretation of the completion of the square for the equation $\alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0$ with real coefficients.

We consider the dotted part to be given. Then, if we add $\frac{(\alpha_1)^2}{4(\alpha_2)^2}$, we obtain a square. Through its side we can infer x. In this way, a support for intuition is provided.

1.5.3. Solving cubic equations. We consider the general cubic equation

$$\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0, \tag{1.5.3}$$

with $\alpha_3, \alpha_2, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\alpha_3 \neq 0$. We are looking for its solutions in $\mathbb{C}$.

The substitution $x = y - \frac{\alpha_2}{3\alpha_3}$ yields to the so-called depressed cubic equation, which is monic and lacks in the second degree term. It is

$$y^3 + py + q = 0 \tag{1.5.4}$$

with

$$p = \frac{3\alpha_3\alpha_1 - (\alpha_2)^2}{3(\alpha_3)^2}, \tag{1.5.5}$$
$$q = \frac{2(\alpha_2)^3 - 9\alpha_3\alpha_2\alpha_1 + 27(\alpha_3)^3\alpha_0}{27(\alpha_3)^3}.$$

We introduce two variables u, v such that $y = u + v$. Substituting in equation (1.5.4), we get

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0. \tag{1.5.6}$$

Let us notice that we can impose on u, v (which have been arbitrarily chosen) another condition, for instance $3uv + p = 0$ suggested by the previous equation.[71] Therefore, we have $u^3v^3 = -\frac{p^3}{27}$ and equation (1.5.6)

$$u^3 + v^3 + q = 0.$$

becomes simpler. Thus, if we manage to solve the system coming from the two conditions on u, v

$$\begin{cases} u^3 + v^3 = -q \\ u^3v^3 = -\frac{p^3}{27} \end{cases}, \tag{1.5.7}$$

we can also solve equation (1.5.3). Let us notice that this system has two solutions in u^3, v^3 and eighteen solutions in u, v. We are interested in the sum $u + v$, which will give the three solutions of cubic equation (1.5.4), then of equation (1.5.3).

We consider u^3 and v^3 as unknowns. Since we know their sum and their product, we associate[72] the system (1.5.7) with the equation

$$t^2 + qt - \frac{p^3}{27} = 0. \tag{1.5.8}$$

This is called the 'Lagrange resolvent' for the cubic equation (1.5.4). Now we use the quadratic formula to draw backwards in chain all the solutions of the other equations: we firstly find the two solutions t_1, t_2 of equation (1.5.8), from which we obtain the two couples of solutions

$$\begin{cases} u^3 = t_1 \\ v^3 = t_2 \end{cases} \quad \text{and} \quad \begin{cases} u^3 = t_2 \\ v^3 = t_1 \end{cases}$$

of system (1.5.7).

[71] In fact, the linear system given by

$$\begin{cases} y = u + v \\ 3uv + p = 0 \end{cases}, \quad \text{that is} \quad \begin{cases} y = u + v \\ uv = -\frac{p}{3} \end{cases},$$

always has solutions in $\mathbb{C}$.

[72] Since

$$(\alpha - t)(\beta - t) = t^2 - (\alpha + \beta)t + \alpha\beta,$$

if we search two numbers of which we know their sum S and product P, it is enough to solve the quadratic equation $t^2 - St + P = 0$.

We consider the first couple. We want to draw u, v. The three cubic roots of $t_1, t_2 \in \mathbb{C}$ are respectively

$$\sqrt[3]{t_1},\ \omega\sqrt[3]{t_1},\ \omega^2\sqrt[3]{t_1} \quad \text{and} \quad \sqrt[3]{t_2},\ \omega\sqrt[3]{t_2},\ \omega^2\sqrt[3]{t_2},$$

with ω a primitive third root of unity[73]. Thus, there are nine rearrangements for $u+v$. At first, we arbitrarily choose u. Since the condition $uv = -\frac{p}{3}$ must hold, it follows that only three rearrangements out of nine fit, which are:

$$y_1 = \sqrt[3]{t_1} + \sqrt[3]{t_2}, \quad \text{or} \quad y_2 = \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2}, \quad \text{or} \quad y_3 = \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2}.$$

These are the three solutions of equation (1.5.4).

Considering then the second couple of solutions of system (1.5.7), where the values for u^3, v^3 are exchanged, we find the same values for the symmetric expression $u+v$.

Hence, the solutions of equation (1.5.3) are

$$x_i = y_i - \frac{\alpha_2}{3\alpha_3} \quad \text{for} \quad i = 1, 2, 3,$$

that is,

$$\begin{aligned} x_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \\ x_2 &= \omega\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^2\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \\ x_3 &= \omega^2\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \end{aligned} \tag{1.5.9}$$

explicitly written as function of the coefficients p, q in (1.5.5). This is called the 'cubic formula' for equation (1.5.3) and

$$\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}, \tag{1.5.10}$$

$$\text{where } p = \frac{3\alpha_3\alpha_1 - (\alpha_2)^2}{3(\alpha_3)^2}, q = \frac{2(\alpha_2)^3 - 9\alpha_3\alpha_2\alpha_1 + 27(\alpha_3)^3\alpha_0}{27(\alpha_3)^3},$$

[73] That is, $\omega^3 = 1$ e $\omega^k \neq 1$ per $k = 1, 2$. In particular, if ω is a primitive third root of unity, then $\omega = -\frac{1}{2} + \imath\frac{\sqrt{3}}{2}$ e $\omega^2 = -\frac{1}{2} - \imath\frac{\sqrt{3}}{2}$.

is called[74] 'discriminant' of the cubic equation (1.5.3).

The main idea behind this method is to reveal the structure of a cubic equation introducing two new variables u, v linked by the linear condition $u + v = y$, and then to exploit the degree of freedom obtained in this way by imposing a further condition on the product uv. The method works because it takes advantage of symmetries in a fundamental way. In fact, the number of solutions of the system (1.5.7) decreases thanks to the symmetry in the expression $u + v$.

1.5.4. Cubic equations with real coefficients. We first remark that, if α_3, α_2, α_1, α_0 in equation (1.5.3) are real, then p, q in (1.5.5) are real too.

We can then discuss the nature of the solutions of equation (1.5.3) starting from the sign of its discriminant Δ_3. We know that each odd degree equation with real coefficients has at least one real solution, for instance as a simple consequence of the Intermediate values theorem.

It is convenient to neglect the term $\frac{\alpha_2}{3\alpha_3}$, which has no effect since it is real, and discuss the nature of the solutions of equation (1.5.4). In the case $\Delta_3 > 0$, we have one real solution and two complex conjugate solutions. In fact, we get that $\sqrt[3]{t_1}$, $\sqrt[3]{t_2}$ are real. Thus,

$$
\begin{aligned}
y_1 &= \sqrt[3]{t_1} + \sqrt[3]{t_2} \in \mathbb{R}, \\
y_2 &= \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2} = -\frac{1}{2}(\sqrt[3]{t_1} + \sqrt[3]{t_2}) + \imath\frac{\sqrt{3}}{2}(\sqrt[3]{t_1} - \sqrt[3]{t_2}) \in \mathbb{C}, \\
y_3 &= \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2} = -\frac{1}{2}(\sqrt[3]{t_1} + \sqrt[3]{t_2}) - \imath\frac{\sqrt{3}}{2}(\sqrt[3]{t_1} - \sqrt[3]{t_2}) \in \mathbb{C},
\end{aligned}
$$

and y_2, y_3 are complex conjugate.

[74] Note that, usually, the discriminant is defined from a higher viewpoint by $(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$, where x_1, x_2, x_3 are the roots of the cubic polynomial (see, for instance, [Lang 2002, Chapter IV, Sections 6 and 8].). My definition differs from the usual one by the factor $-\frac{1}{108}$. More precisely, if Δ'_3 is the usual discriminant, we have $\Delta_3 = -1/108\Delta'_3$. Nevertheless, I choose to use this definition, since in Cardano's text we explicitly find the value $\frac{q^2}{4} + \frac{p^3}{27}$ without the factor $-\frac{1}{108}$.

In the case $\Delta_3 = 0$, we have three real solutions, two of which coincide. In fact, we get $\sqrt[3]{t_1} = \sqrt[3]{-\frac{q}{2}} = \sqrt[3]{t_2}$. Thus,

$$\begin{aligned}
y_1 &= 2\sqrt[3]{t_1} = -2\sqrt[3]{\frac{q}{2}} \in \mathbb{R}, \\
y_2 &= (\omega + \omega^2)\sqrt[3]{t_1} = -\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{\frac{q}{2}} = (\omega^2 + \omega)\sqrt[3]{t_1} = y_3 \in \mathbb{R}.
\end{aligned}$$

Finally, let $\Delta_3 < 0$. Then, we have three distinct real solutions. In fact, we get $\sqrt[3]{t_1}, \sqrt[3]{t_2} \in \mathbb{C}$. But, since the equality $\sqrt[3]{t_1}\sqrt[3]{t_2} = -\frac{p}{3} \in \mathbb{R}$ holds, $\sqrt[3]{t_1}, \sqrt[3]{t_2}$ are complex conjugate. Thus,

$$\begin{aligned}
y_1 &= \sqrt[3]{t_1} + \sqrt[3]{t_2} = \sqrt[3]{t_1} + \overline{\sqrt[3]{t_1}} \in \mathbb{R}, \\
y_2 &= \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2} = \omega\sqrt[3]{t_1} + \overline{\omega\sqrt[3]{t_1}} \in \mathbb{R}, \\
y_3 &= \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2} = \omega^2\sqrt[3]{t_1} + \overline{\omega^2\sqrt[3]{t_1}} \in \mathbb{R}.
\end{aligned}$$

We observe that $y_2 + y_3 = (\omega + \omega^2)\sqrt[3]{t_1} + (\overline{\omega} + \overline{\omega^2})\overline{\sqrt[3]{t_1}} = -(\sqrt[3]{t_1} + \sqrt[3]{t_2}) = -y_1$.

More precisely (but only for equation (1.5.4)), when we write t_1, t_2 in trigonometric form, we have

$$\begin{aligned}
\sqrt[3]{t_1} &= \sqrt[3]{\rho}\sqrt[3]{\cos\theta + \imath\sin\theta} = \sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \imath\sin\frac{\theta}{3}\right), \\
\sqrt[3]{t_2} &= \overline{\sqrt[3]{t_1}} = \sqrt[3]{\rho}\left(\cos\frac{\theta}{3} - \imath\sin\frac{\theta}{3}\right),
\end{aligned}$$

with $\rho \in [0, 2\pi)$ and $\theta \in [0, +\infty)$, and then

$$\begin{aligned}
y_1 &= \sqrt[3]{t_1} + \overline{\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\frac{\theta}{3}, \\
y_2 &= \omega\sqrt[3]{t_1} + \overline{\omega\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\left(\frac{\theta}{3} + \frac{2}{3}\pi\right), \\
y_3 &= \omega^2\sqrt[3]{t_1} + \overline{\omega^2\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\left(\frac{\theta}{3} + \frac{4}{3}\pi\right).
\end{aligned}$$

We remark that the three angles in the cosine function argument are at the distance $\frac{2}{3}\pi$ one from the other, so that one of them necessarily lays in the first or fourth quadrant where the cosine function is positive. Another of them necessarily lays in the second or third quadrant where the cosine function is negative. On the contrary, the remaining one varies in such a way that one cannot establish *a* *priori* the sign of its cosine. Hence, we obtain a positive and a negative solution, whereas the sign of the third solution is changeable.

This last case – when a cubic equation has three distinct real solutions, or $\Delta_3 < 0$ – is called the '*casus irreducibilis*'.[75] Note that $\Delta_3 < 0$ means that imaginary numbers appear in the cubic formula (see below, Section 1.5.6 at page 46).

Let us notice that, in the cubic case with real coefficients, no geometric interpretation provides a good support for intuition as the completion of the square was in the quadratic case. In other words, the completion of the cube does not have an analogous heuristic value. Actually, it does not work at all. In fact, if we write equation (1.5.3) as

$$x^3 + \frac{\alpha_2}{\alpha_3}x^2 + \frac{\alpha_1}{\alpha_3}x = -\frac{\alpha_0}{\alpha_3}$$

and we complete the cube, we find the condition $\alpha_1 = \frac{(\alpha_2)^2}{3}$.

Suppose then $\alpha_1 \neq \frac{(\alpha_2)^2}{3}$. We can try to write equation (1.5.3) as

$$x^3 + \frac{\alpha_2}{\alpha_3}x^2 = -\frac{\alpha_1}{\alpha_3}x - \frac{\alpha_0}{\alpha_3},$$

and add and subtract the quantity $\frac{\alpha_2^2}{\alpha_3^2}\frac{x}{3} + \frac{\alpha_2^3}{27\alpha_3^3}$ in order to obtain the cube of a binomial

$$\left(x + \frac{\alpha_2}{3}\right)^3 = -\frac{\alpha_1}{\alpha_3}x - \frac{\alpha_0}{\alpha_3} + \frac{\alpha_2^2}{\alpha_3^2}\frac{x}{3} + \frac{\alpha_2^3}{27\alpha_3^3}.$$

But then we can in no way write the right side of the equation as a cube of a polynomial one term of which is x, since x^3 does not appear on the right side. Then, one needs to follows another way and introduce the new variables u, v.

1.5.5. Cubic equations with real coefficients solved in a trigonometrical way. In the preceding section, we have remarked that, when a cubic equation has three distinct real solutions, imaginary numbers appear in the cubic formula, since $\Delta_3 < 0$ and the discriminant is under square roots. Nevertheless, there is a way to obtain a real expression for the solutions. One can achieve this

[75]This is a late terminology that we do not find it in Cardano's works. Up to now, the earliest reference to the *casus irreducibilis* that I have found is in the 1781 contest by the Accademia in Padova, where they asked to prove that Cardano's formula necessary entails imaginary numbers when the equation falls into the *casus irreducibilis*.

by exploiting the triple-angle formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Consider the depressed equation 1.5.4 and set $y = u\cos\theta$. Then, $u^3\cos^3\theta + pu\cos\theta + q = 0$. We want to choose u in order to compare the equation with the triple-angle formula. We multiply both sides of the equation by $\frac{4}{u^3}$ and we have $4\cos^3\theta + \frac{4p}{u^2}\cos\theta + \frac{4q}{u^3} = 0$. There, we are allowed to choose $u = \sqrt{-\frac{4}{3}p}$, so that we get $4\cos^3\theta - 3\cos\theta + \frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}} = 0$. Note that u is well-defined, since $\Delta_3 < 0$ implies that $p < 0$. Then, by comparison, we have

$$-\cos 3\theta = \frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}$$

and, since $y = u\cos\theta$, we finally have that

$$\begin{aligned}
y_1 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right)\right),\\
y_2 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right) + \frac{2}{3}\pi\right),\\
y_3 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right) + \frac{4}{3}\pi\right).
\end{aligned}$$

The arccosine is defined only between -1 and 1, which gives the condition on the discriminant $27q^2 + 4p^3 \leq 0$. We remark that, as a matter of fact, the above formulae do not contain imaginary numbers. Anyway, this formulae are not as interesting as the algebraic ones in Section 1.5.3, since they do not give as much information. Indeed, it is still out-of-reach to exactly calculate the cosine and arccosine functions. The above formulae look very easy, but in fact they hide the complex structure of the equation in an artificially introduced function (and its almost-inverse function) that has a particular relationship with a certain cubic polynomial (the triple-angle formula). Moreover, when $\Delta_3 > 0$, one is compelled to use hyperbolic functions.

1.5.6. What Galois' theory can say about cubic equations. There[76] is a deeper reason that explains why imaginary numbers necessarily appear in the cubic formula, and it surfaces using Galois' theory. Galois' original definition

[76] I thank Massimo Galuzzi for this section.

of the group of an equation easily leads to impracticable calculations when the degree of the equation is greater than four. Nevertheless, his approach has the advantage of giving a very simple proof of the impossibility of overcoming the use of imaginary numbers to solve an irreducible cubic equation with real coefficients that falls into the *casus irreducibilis*.

Firstly, I will quickly sum up how to calculate the Galois group in the general case of a n^{th} degree equation. Let $f(x) = 0$ be the equation and denote its (simple) solutions by $x_1, x_2, \ldots, x_n$. Then, Galois states[77] – as obvious – that it is possible to find some integers $A_1, A_2, \ldots, A_n$ such that the function of the solutions (the so-called Galois resolvent)

$$V(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = A_1 x_{i_1} + A_2 x_{i_2} + \ldots + A_n x_{i_n}$$

takes $n!$ different values on the set of the permutations of the solutions.

Once that a particular value of this function, for instance

$$V_1 = A_1 x_1 + A_2 x_2 + \ldots + A_n x_n$$

obtained by the identical permutation, is given, we have, in modern terms, a primitive element. All the solutions of $f(x) = 0$ may be expressed as rational functions of V_1, with the coefficients that are in the same field as $f(x)$.

In force of the Fundamental theorem of symmetric polynomials, the coefficients of the polynomial

$$\varphi(X) = \prod_{i_1, i_2, \ldots, i_n} (X - (A_1 x_{i_1} + A_2 x_{i_2} + \ldots + A_n x_{i_n}))$$

are in the same field as $f(x)$. Let $\Phi(X)$ be the irreducible factor of $\varphi(X)$ that contains the root[78] V_1. The Galois group is given by the permutations that correspond to the roots of $\Phi(X)$.

What does it mean, in this approach, to solve an equation? It amounts to "reduce its group successively to the point where it does not contain more than a single permutation".[79] This reduction is, of course, accomplished by adjoining quantities that can be supposed as known.

[77]See [NEUMANN 2011, page 110]. I have slightly modified Galois' original notation.
[78]Of course, in the general case, we have $\Phi(X) = \varphi(X)$. The polynomial $\Phi(x)$ is the minimal polynomial of V_1.
[79]See [NEUMANN 2011, page 121].

Let us now detail this approach in the particular case of cubic equations. We consider the irreducible equation

$$x^3 + px + q = 0, \tag{1.5.11}$$

with p, q real, and let x_1, x_2, x_3 be its solutions. In this case, a Galois resolvent may be given by[80]

$$V(x_i, x_j, x_k) = x_i - x_k.$$

By means of this resolvent, the polynomial $\varphi(X)$ becomes

$$\varphi(X) = \prod_{x_i \neq x_j} (X - (x_i - x_j)) = X^6 + 6pX^4 + 9p^2X^2 + 4p^3 + 27q^2,$$

or

$$\varphi(X) = (X^3 + 3pX)^2 + 4p^3 + 27q^2.$$

In this calculation, we remark that the quantity $4p^3 + 27q^2$ has a real structural value and is not a mere device to makes the demonstration work.

The result, namely that imaginary numbers are indispensable tools to solve a cubic equation when it falls into the *casus irreducibilis*, has been proved, during the 19th century, by Paolo Ruffini in 1813 (see [Ruffini 1799] for an incomplete proof, and [Ruffini 1813]), Pierre Laurent Wantzel in 1843 (see [Wantzel 1842]), Vincenzo Mollame in 1890 (see [Mollame 1890] and [Mollame 1892]), Otto Hölder in 1891 (see [Hölder 1891]), Adolf Kneser in 1892 (see [Kneser 1892]), and Leopold Gegenbauer in 1893 (see [Gegenbauer 1893]). I plan to develop the study of this topic in a forthcoming work.

1.5.7. Paradigmatic examples for cubic equations. We have already touched on the fact that Cardano is interested in the algebraic shapes for the solutions of a cubic equation. In particular, let us have a look at the family of equations $x^3 = a_1x + a_0$, which is fundamental in Cardano's arguments. We will try to deduce some information on the shapes of its solutions.

Since for the most of the time Cardano uses rational coefficients and since in his treatises there is the well-identifiable topic of equations with rational coefficients, let us assume that a_1, a_0 are rational. In this case, consider a, b

[80]There are two kinds of tests to make. First, if $x_1 - x_2 = x_2 - x_1$, then $x_1 = x_2$. Second, if $x_1 - x_2 = x_2 - x_3$, then $2x_2 = x_1 + x_3$, that is $3x_2 = x_1 + x - 2 + x_3 = 0$. The other cases are similar.

positive, rational. Cardano states that the (positive) irrational solutions go either under the shape $\sqrt{a}+b$, with a such that $\sqrt{a}$ is irrational, or under the shape $\sqrt[3]{a}+\sqrt[3]{b}$, with a,b such that $\sqrt[3]{a}$, $\sqrt[3]{b}$ are irrational and $\sqrt[3]{ab}$ is rational. Indeed, it turns out that these two shapes can never be solution of the same cubic equation, since, if a rational equation has a solution of the first shape, then it has $\Delta_3 < 0$ and, if it has a solution of the second shape, then it has $\Delta_3 > 0$.

In fact, we firstly consider a rational, irreducible polynomial of degree three and we suppose that it exists a,b positive, rational such that $x_1 = \sqrt{a}+b$ is irrational and is one of its roots. Then, by Galois theory, we have that the polynomial is irreducible over $\mathbb{Q}$, that is, if x_2 is another root of the polynomial, then it is rational. Moreover, $x_3 = -\sqrt{a}+b$. Since we get three real roots, the discriminant is negative.

We consider then a rational, irreducible polynomial of degree three and we suppose that it exists a,b positive, rational such that $x_1 = \sqrt[3]{a}+\sqrt[3]{b}$ is irrational and is one of its roots. Then, again by Galois theory, we have that, if x_2 is another root of the polynomial, then $x_2 = \omega^r\sqrt[3]{a}+\omega^s\sqrt[3]{b}$, with ω a primitive third root of unity and $r,s\in\mathbb{Z}/3\mathbb{Z}$ such that $r,s\not\equiv 0 \bmod 3$. If we moreover assume that $\sqrt[3]{ab}$ is rational, then $r+s\equiv 0 \bmod 3$. We can assume that $r=1$ and $s=2$ and $x_3=\omega\sqrt[3]{a}+\omega^2\sqrt[3]{b}$ (up to interchanging x_2 and x_3). It is easily seen that x_2, x_3 cannot be real, so that the discriminant of the equation is in the end positive.

For instance, consider the equation $x^3 = 6x+4$, which is in *Aliza*, Chapters XXIV and LIX. It has $\Delta_3 < 0$ and its real solutions are $x_1 = 1+\sqrt{3}$, $x_2=-2$, and $x_3 = 1-\sqrt{3}$. Instead, the equation $x^3 = 6x+6$ (which is in *Ars magna arithmeticæ*, Chapters XXIII, XXVII, XXXII, in *Ars magna*, Chapter XII, and in *Aliza*, Chapters XVIII, XIX, XXV, LVIII) has $\Delta_3 > 0$ and it real solution is $\sqrt[3]{4}+\sqrt[3]{2}$.

Moreover, there is the case in which an equation of the family $x^3 = a_1x+a_0$ has $\Delta_3 > 0$ and the real solution is rational. For instance, 4 is the real solution of $x^3 = 6x+40$, which is in *Ars magna*, Chapter XII and *Aliza*, Chapters XLIX, LIII, LIX.

A consequence of the method in Section 1.5.3 is that solving cubic equations is often brought back to cubic root calculations, which is not always easy to carry out. For instance, when we want to draw a solution of $x^3+6x = 40$ using the cubic formula, $\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}$ is returned. In truth, this solution is 4, since the terms under the cubic roots are two cubes, namely $20+\sqrt{392} = (2+\sqrt{2})^3$ and $20-\sqrt{392} = (2-\sqrt{2})^3$, but this is not evident at all.[81] Indeed, in *Ars magna*, Chapter XII Cardano does not know that the terms under the cubic roots are two cubes. In *Aliza*, Chapters XLIX and LIII he knows that a solution is 4, but there is no mention of the cubic formula. Instead, in *Aliza*, Chapter LIX he knows it, and he explicitly states it in the third corollary.

Much more interesting – in the context of Cardano's inquiries – is the case in which we use the cubic formula to try to solve $x^3 = 6x + 4$. In fact, the formula returns $\sqrt[3]{2+\sqrt{-4}}+\sqrt[3]{2-\sqrt{-4}}$, which in truth is real and is $1+\sqrt{3}$, but Cardano can in no way remark the equality.

1.5.8. Solving quartic equations. We consider the general quartic equation

$$\alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0, \tag{1.5.12}$$

with $\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\alpha_4 \neq 0$. We are looking for its solutions in $\mathbb{C}$.

The substitution $x = y - \frac{\alpha_3}{4\alpha_4}$ yields to the so-called depressed quartic equation, which is monic and lacks in the third degree term

$$y^4 + py^2 + qy + r = 0, \tag{1.5.13}$$

with

$$\begin{aligned} p &= \frac{-3(\alpha_3)^2 + 8\alpha_4\alpha_2}{8(\alpha_4)^2}, \\ q &= \frac{(\alpha_3)^3 - 4\alpha_4\alpha_3\alpha_2 + 8(\alpha_4)^2\alpha_1}{8(\alpha_4)^3}, \\ r &= \frac{-3(\alpha_3)^4 + 16\alpha_4(\alpha_3)^2\alpha_2 - 64(\alpha_4)^2\alpha_3\alpha_1 + 256(\alpha_4)^3\alpha_0}{256(\alpha_4)^4}. \end{aligned} \tag{1.5.14}$$

[81] In short, if one wants to know if $\sqrt{a}+b$, with a, b rational, is a cube in $\mathbb{Q}(\sqrt{a})$, one has to search for two rational x, y such that $\sqrt[3]{\sqrt{a}+b} = \sqrt{x}+y$ holds. For more details about the calculation, see for instance [Cossali 1799a, Capo V.4, p. 291] or [Maracchia 2005, pages 242-244].

We rewrite equation (1.5.13) in the following way

$$y^4 + py^2 = -qy - r,$$

in order to obtain two perfect squares on the left and right side of the equation.

First, we exploit the formula to expand the square of a binomial. Adding the identity $(y^2+p)^2 - y^4 - 2py^2 = p^2$ to equation (1.5.13) yields to

$$(y^2 + p)^2 = py^2 - qy + p^2 - r. \tag{1.5.15}$$

Then, in short, we introduce a variable u such that the perfect square on the left side of equation (1.5.15) is preserved (but it will be the square of a trinomial rather than of a binomial). We will impose then a further condition on u in order to have a perfect square on the right side. We will finally exploit the formula to expand the square of a trinomial. Following this plan, we add the identity $(y^2+p+u)^2 - (y^2+p)^2 = 2uy^2 + 2pu + u^2$ to equation (1.5.15). This yields to

$$(y^2 + p + u)^2 = (p + 2u)y^2 - qy + (p^2 + 2pu + u^2 - r). \tag{1.5.16}$$

We choose a value for u such that the right side of equation (1.5.16) becomes a perfect square. For that, we consider the second degree polynomial in y on the right side of equation (1.5.16), and we force its discriminant

$$\Delta_2 = q^2 - 4(p + 2u)(p^2 + 2pu + u^2 - r)$$

to be zero.

In this way, we want to solve the nested cubic equation in u

$$u^3 + \frac{5}{2}pu^2 + (2p^2 - r)u + (\frac{1}{2}p^3 - \frac{1}{2}pr - \frac{1}{8}q^2) = 0 \tag{1.5.17}$$

in order to solve equation (1.5.16). We apply to it the method in Section 1.5.3 and we get the depressed nested cubic equation

$$v^3 + Pv + Q = 0, \tag{1.5.18}$$

with

$$P = -\frac{p^2}{12} - r,$$
$$Q = -\frac{p^3}{108} + \frac{pr}{3} - \frac{q^2}{8},$$

that is $P, Q \in \mathbb{C}$ depend on the coefficients $\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0$ of equation (1.5.12). The solutions of equation (1.5.17) are

$$\begin{aligned} u_1 &= \sqrt[3]{-\frac{Q}{2}+\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}+\sqrt[3]{-\frac{Q}{2}-\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}-\frac{5}{6}p,\\ u_2 &= \omega\sqrt[3]{-\frac{Q}{2}+\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}+\omega^2\sqrt[3]{-\frac{Q}{2}-\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}-\frac{5}{6}p,\\ u_3 &= \omega^2\sqrt[3]{-\frac{Q}{2}+\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}+\omega\sqrt[3]{-\frac{Q}{2}-\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}-\frac{5}{6}p. \end{aligned}$$

Each one of the above solutions gives a polynomial with a double root on the right side of equation (1.5.16), allowing us to factorise it in the following way.

Using such a u, we have that

$$(p+2u)y^2 - qy + (p^2 + 2pu + u^2 - r)$$

is a perfect square of the form

$$(\alpha y + \beta)^2 = \alpha^2 y^2 + 2\alpha\beta y + \beta^2.$$

By comparison, we get $\alpha = \sqrt{p+2u}$ and $\beta = \frac{-q}{2\sqrt{p+2u}}$. Equation (1.5.16) can be then rewritten as

$$(y^2+p+u)^2 = \left((\sqrt{p+2u})y + \frac{-q}{2\sqrt{p+2u}}\right)^2.$$

Taking the square root and collecting like powers of y, we obtain

$$y^2 + (\mp\sqrt{p+2u})y + \left(p+u \pm \frac{-q}{2\sqrt{p+2u}}\right) = 0. \tag{1.5.19}$$

These are two quadratic equations. We apply the method in Section 1.5.1 to get the solutions

$$\begin{aligned} y_1, y_2 &= \frac{\sqrt{p+2u} \pm \sqrt{(p+2u) - 4\left(p+u+\frac{-q}{2\sqrt{p+2u}}\right)}}{2},\\ y_1', y_2' &= \frac{-\sqrt{p+2u} \pm \sqrt{(p+2u) - 4\left(p+u-\frac{-q}{2\sqrt{p+2u}}\right)}}{2}. \end{aligned}$$

The solutions of equation (1.5.12) are then

$$x_i = y_i - \frac{\alpha_3}{4\alpha_4} \quad \text{with} \quad i = 1, 2, 3, 4,$$

that is

$$\begin{aligned}
(1.5.20) \qquad x_1 &= \frac{\sqrt{p+2u} + \sqrt{(p+2u) - 4\left(p+u+\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4},\\
x_2 &= \frac{\sqrt{p+2u} - \sqrt{(p+2u) - 4\left(p+u+\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4},\\
x_3 &= \frac{-\sqrt{p+2u} + \sqrt{(p+2u) - 4\left(p+u-\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4} = x_1',\\
x_4 &= \frac{-\sqrt{p+2u} - \sqrt{(p+2u) - 4\left(p+u-\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4} = x_2',
\end{aligned}$$

explicitly written as function of the coefficients p, q, r in (1.5.14).

1.5.9. Quartic equations with real coefficients. We firstly remark that, if α_4, α_3, α_2, α_1, α_0 in equation (1.5.12) are real, then p, q, r in (1.5.14) are real too, that is the nested cubic equation (1.5.17) has real coefficients.

We know that equation (1.5.17) has at least one real solution. We take u to be such a real solution.

We can assume that $u > -\frac{p}{2}$, so that $p + 2u > 0$ and $\sqrt{p+2u} \in \mathbb{R}$. We evaluate the polynomial $R(v) = v^3 + Pv + Q$ on the right side of equation (1.5.18) at $v = \frac{p}{3}$ and we get

$$R\left(\frac{p}{3}\right) = -\frac{q^2}{8} < 0.$$

But

$$\lim_{v \to +\infty} R(v) = +\infty.$$

This means that equation (1.5.18) has a real solution $v > \frac{p}{3}$, so that equation (1.5.17) has a real solution $u > v - \frac{5}{6}p = -\frac{p}{2}$.

Using such a u in equation (1.5.19), we get $\sqrt{p+2u} \in \mathbb{R}$ and the two quadratic equations have real coefficients.

Therefore, if α_4, α_3, α_2, α_1, α_0 in equation (1.5.12) are real, we can choose u such that equation (1.5.19) has real coefficients.

Afterwards, let us consider equation (1.5.12). For the sake of ease, we take $\alpha_4 = 1$, since we can always easily reduce any equation to its monic form

$$x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0. \tag{1.5.21}$$

The coefficients of the depressed quartic equation (1.5.13) become then

$$\begin{aligned} p &= \frac{-3(\alpha_3)^2 + 8\alpha_2}{8}, \\ q &= \frac{(\alpha_3)^3 - 4\alpha_3\alpha_2 + 8\alpha_1}{8}, \\ r &= \frac{-3(\alpha_3)^4 + 16(\alpha_3)^2\alpha_2 - 64\alpha_3\alpha_1 + 256\alpha_0}{256}. \end{aligned}$$

Developing calculations in (1.5.20), we get

$$\begin{aligned} x_1, x_2 =& \frac{1}{2}\left(+\sqrt{\frac{\alpha_3^2}{4} - \frac{2\alpha_2}{3} + \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} + \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}}}\right. \\ & \left.\pm\sqrt{\frac{\alpha_3^2}{2} - \frac{4\alpha_2}{3} - \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} - \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}} + \frac{-\alpha_3^3 + 4\alpha_3\alpha_2 - 8\alpha_1}{4\sqrt{\frac{\alpha_3^2}{4} - \frac{2\alpha_2}{3} + \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} + \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}}}}}\right) \\ & - \frac{\alpha_3}{4}, \\ x_1', x_2' =& \frac{1}{2}\left(-\sqrt{\frac{\alpha_3^2}{4} - \frac{2\alpha_2}{3} + \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} + \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}}}\right. \\ & \left.\pm\sqrt{\frac{\alpha_3^2}{2} - \frac{4\alpha_2}{3} - \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} - \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}} - \frac{-\alpha_3^3 + 4\alpha_3\alpha_2 - 8\alpha_1}{4\sqrt{\frac{\alpha_3^2}{4} - \frac{2\alpha_2}{3} + \frac{\sqrt[3]{2}B}{3\sqrt[3]{A+\sqrt{\Delta_4}}} + \sqrt[3]{\frac{A+\sqrt{\Delta_4}}{54}}}}}\right) \\ & - \frac{\alpha_3}{4}, \end{aligned}$$

with

$$
\begin{aligned}
A =& 2\alpha_2^3 - 9\alpha_3\alpha_2\alpha_1 + 27\alpha_1^2 + 27\alpha_3^2\alpha_0 - 72\alpha_2\alpha_0,\\
B =& \alpha_2^2 - 3\alpha_3\alpha_1 + 12\alpha_0,\\
\Delta_4 =& -4\Big(\alpha_2^2 - 3\alpha_3\alpha_1 + 12\alpha_0\Big)^3 + \Big(2\alpha_2^3 - 9\alpha_3\alpha_2\alpha_1 + 27\alpha_1^2 + 27\alpha_3^2\alpha_0 - 72\alpha_2\alpha_0\Big)^2\\
=& -4B^3 + A^2.
\end{aligned}
$$

In this way, the so-called discriminant Δ_4 of a quartic equation is highlighted. The above formulae are called 'quartic formula'.

As in the case of quadratic and cubic equations, we would like to discuss the nature of the solutions of equation (1.5.12) starting from the sign of its discriminant Δ_4.

Let us consider the quartic monic polynomial on the left side of equation (1.5.21)

$$x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0. \tag{1.5.22}$$

The discriminant Δ_4 of a quartic polynomial is defined as the square of the product of the differences of the roots.[82] Let x_i with $i = 1, 2, 3, 4$ be the roots. Then

$$\Delta_4 = (x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2.$$

The discriminant is well defined, in the sense that it does not depend on the order of the roots. Indeed, rearranging the x_i boils down to a permutation of the factors of the product, since the differences are raised to the square.

Using this formula, it is easily seen that if (at least) two of the roots coincide, then $\Delta_4 = 0$.

Let us suppose now that all the roots are distinct. Since the polynomial factors as the product of two quadratic polynomials, there are three possibilities for its roots. We can have four real roots, or two real roots and two complex conjugate roots, or four complex roots conjugate in pairs.

If the four roots are real, it is easily seen that $\Delta_4 > 0$, since it is the product of squares of real numbers.

[82] See [Lang 2002, Chapter IV, Sections 6 and 8].

Let us suppose that two roots are distinct and real, say $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, and two roots are complex conjugate, say $x_3, x_4 \in \mathbb{C}$ with $x_4 = \overline{x_3}$. We write $x_3 = a + \imath b$ and $x_4 = a - \imath b$. Then, the factor $(x_1 - x_2)^2 > 0$, since it is the square of a real number. We remark that four differences out of six occur as pairs of complex conjugate numbers, so that their products are real, and thus positive. In fact, considering for example $(x_1 - x_3)^2(x_1 - x_4)^2$, we get $\left((x_1 - a - \imath b)(x_1 - a + \imath b)\right)^2$. The remaining factor is $x_3 - x_4 = a + \imath b - (a - \imath b) = 2\imath b$, the square of which is negative. In this way, we get $\Delta_4 < 0$.

Finally, let us suppose that there are two pairs of complex conjugate roots, say x_1, x_2, x_3, $x_4 \in \mathbb{C}$ with $x_2 = \overline{x_1}$, and $x_4 = \overline{x_3}$. We write $x_1 = \alpha_1 + \imath b_1$, $x_2 = \alpha_1 - \imath b_1$, $x_3 = \alpha_3 + \imath b_3$, and $x_4 = \alpha_3 - \imath b_3$. As before, four of the six differences occur as pairs of complex conjugate numbers, so that their products are real, and then positive. The two remaining factors are $x_1 - x_2 = \alpha_1 + \imath b_1 - (\alpha_1 - \imath b_1) = 2\imath b_1$ and $x_3 - x_4 = \alpha_3 + \imath b_3 - (\alpha_3 - \imath b_3) = 2\imath b_3$, the square of which are negative. In this way, we get $\Delta_4 > 0$.

In particular, we see that $\Delta_4 < 0$ if and only if the polynomial (1.5.22) has exactly two real roots and two complex conjugate roots, and $\Delta_4 > 0$ if and only if the polynomial (1.5.22) has four real roots or four complex roots that are not real. This result can also be obtained by brute force calculations using the above formula for the roots x_1, x_2, x_3, x_4.

Therefore, we observe that in the quartic case the sign of the discriminant does not provide complete information on the nature of the roots. In particular, if $\Delta_4 \geq 0$, there is some ambiguity.

It is worth noting that if we have a polynomial of degree five or higher it is impossible to have an algebraic formula for the roots. This is a deep result of Galois theory.[83]

[83]See [Herstein 1975, Theorem 5.7.3].

CHAPTER 2

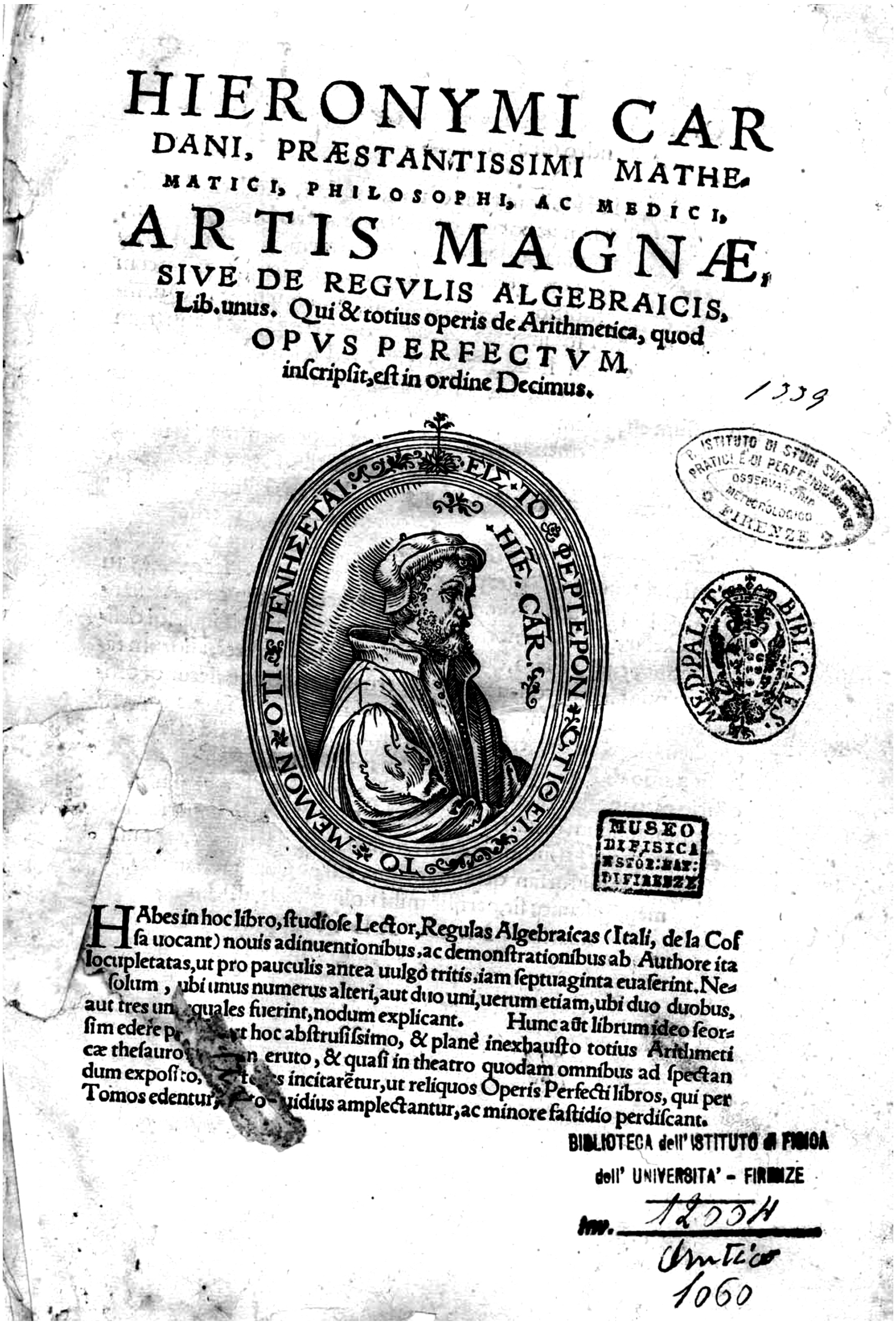
Cardano solves equations in the *Ars Magna*

In this section, I will give an insight of the *Ars magna*. It is well-known that this is Cardano's most noteworthy work, since it contains the most advanced results that he reached concerning equations – which were also the most advanced for that time.

There are[1] three editions of the *Hieronymi Cardani, præstantissimi mathematici, philosophi, ac medici, Artis magnae sive de regulis algebraicis, lib. unus, qui et totius operis de arithmetica, quod opus perfectum inscripsit, est in ordine decimus*, also shortly called *Ars magna*. The first one dates back to 1545 in Nuremberg by Iohannes Petreium. The second one is a joint publication of the *Ars magna* together with the *De Proportionibus* and the *De regula aliza* in 1570 in Basel by Oficina Henricpetrina. This edition was made during Cardano's lifespan and contains some additions. The third one is posthumous and is contained in the fourth volume of the 1663 edition of Cardano's *Opera omnia*, printed in Lyon by Ioannis Antonii Huguetan and Marci Antonii Ravaud. For an extensive explanation of the reference to the *Opus Perfectum* in the *Ars magna*'s title, see above Section 1.3, page 26. Moreover, there is a German translation *Roth, Petrus, Arithmetica philosophica, oder schöne künstliche Rechnung der Coss oder Algebrae. Im 1. Theil werden Herrn D. H. Cardani dreyzehn Reguln esetzt. Dessgleichen noch drey andere Reguln, zu den ersten drey cubiccosischen aequationibus gehörig*, made in Nuremberg in 1608 by Peter Roth. We also have available the manuscript Plimpton 510/1700: s.a., which is entitled *L'algebra*, at the Columbia University Library in New York.

The structure of the *Ars magna* is tricky. Unlikely the algebra treatises of the time, Cardano does not start with the usual considerations on the nature of numbers, but rather with some miscellaneous considerations on the structure of

[1] See [www.cardano.unimi.it last checked January 22, 2014].

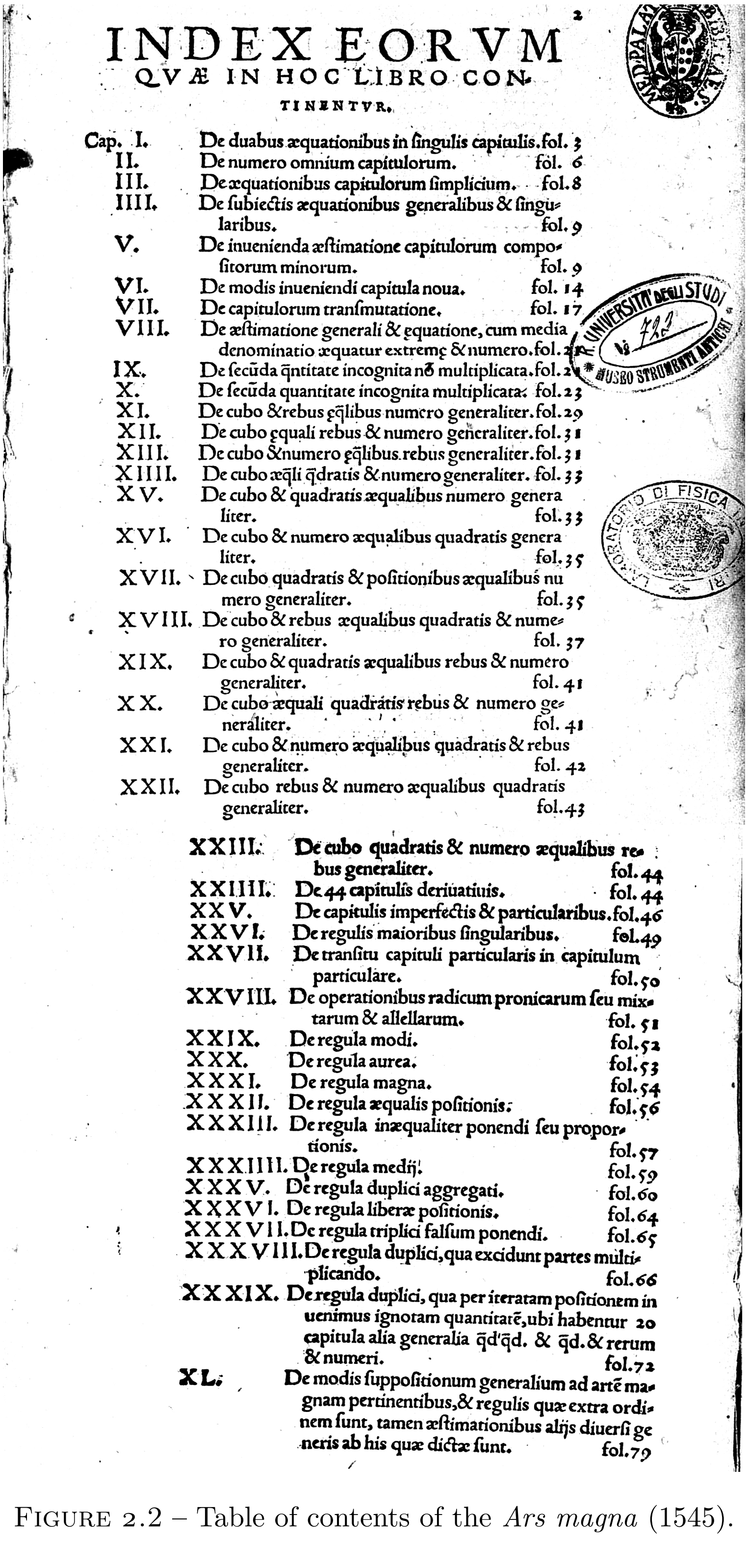


HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

1339

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres un[illegible]quales fuerint, nodum explicant. Hunc aūt librum ideo seorsim edere p[illegible]t hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro [illegible]n eruto, & quasi in theatro quodam omnibus ad spectandum exposito, [illegible]s incitaretur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, [illegible]uidius amplectantur, ac minore fastidio perdiscant.

FIGURE 2.1 – Title page of the *Ars magna* (1545).



INDEX EORVM QVÆ IN HOC LIBRO CONTINENTVR.

FIGURE 2.2 – Table of contents of the *Ars magna* (1545).

equations and on relations between them. These last from Chapter I to Chapter X. This is a first noteworthy clue on the novelty of this treatise. Then, we find a block constituted by Chapters XI-XXIII, where Cardano deals uniquely with cubic equations. Finally, in Chapters XXIV-XL the usual algebraic rules, such as the golden rule or the rule of equal position together with the cubic formulae are applied to solve problems.

In the following, my main concern will be to analyse the methods, rules, techniques, and procedures at Cardano's disposal to deal with equations.

I will firstly present in Section 2.1 a selection of topics from the chapters in the first and last block of the *Ars magna*. Note that they do not specially concern cubic equations.

In Section 2.1.1, we will follow Cardano while solving linear, quadratic, and other simple kind of equations. One the one hand, this recalls the usual way to approach these subjects at that time. On the other hand, it witnesses for Cardano's early encounter with the so-called "sophistic" quantities or square roots of negative quantities.

Then, Cardano shows some techniques to transform equations (see Section 2.1.2). Thanks to them, one can establish some relationships between two equations by the means of their coefficients and solutions. In this context, it sometimes happens that Cardano also considers negative solutions. These transformations of equations will prove to be one of the preferred and most useful methods to enquire on equations.

All along the *Ars magna*, we find the opposition between "particular" and "general", either concerning the solutions themselves or the methods employed. I will report on this in Section 2.1.4. Cardano's efforts are obviously "general"-oriented. Anyway, we find some "particular" techniques to put a patch where the (expected) "general" method turns out to fail. The "general" method provided by Cardano to translate the problem of solving the cross-party class of equations $x^q + a_0 = a_p x^p$, where $q > p$, into the problem of solving a certain system is particularly noteworthy (see Section 2.1.3). Cardano also states the formula for the cube of a binomial (see Section 2.1.5).

Proving this last formula, Cardano makes some important remarks about geometry. While analysing these passages – as for all the followings, I will particularly take care of the way in which Cardano proves his statements. In the *Ars magna*, the most widespread pattern is to settle first of all a geometrical environment, where one can interpret the given equation or the equation which comes from the given problem. In this environment one can also draw some inferences exploiting the known results on geometrical quantities.

In Sections 2.2, 2.3, and 2.4, we will analyse the solving formulae for all the families of cubic equations. The proofs of these formulae follow the usual pattern. Moreover, in eleven cases out of thirteen, they rely on drawing them back to an already solved equation, sometimes using the transformations of which Cardano already spoke or sometimes using some new transformations. In the flowing of these proofs, the arguments employed become the more and more repetitive to reach the point where they loose their link with the possible geometrical arguments that underlie their justification.

Cardano soon runs into the major inconvenient of these formulae, the so-called *casus irreducibilis*, which prevents them from being as general as Cardano hoped. I will show in Section 2.5, that the solving methods for the cubic equations are eventually all linked one to another, even if sometimes in a twisted way.

Finally, in Section 2.6, I will give an overview of Cardano and Ferrari's method to solve quartic equations.

Since an English translation of this work is available, I will make reference in the following to Witmer's edition [CARDANO 1968].

2.1. A miscellaneous on equations

2.1.1. Solving linear, quadratic, and other simple equations. Coming across the square roots of negative numbers. Starting from the very beginning, I will spend a few words on the solution of linear, quadratic, and other simple kind of equations. The *Ars magna* aiming to be – in my opinion – a compendium on equations, we obviously find an explanation on these topics in it.

In **Chapter I** "On double solutions in certain types of cases [*De duabus æquationibus in singulis capitulis*]", after some introductory considerations, Cardano starts recalling briefly how to solve equations of type $x^n = a_0$ by taking the (real) n^{th}-roots, with n a natural. We pay particular attention to the fact that Cardano also accepts negative solutions, when he says that the values of x in $x^2 = 9$ are 3 and -3 or that the values of x in $x^4 = 81$ are 3 and -3.

Then, Cardano speaks about the so-called "derivative cases [*derivativa capitula*]", where one can use substitutions such as $y = x^n$ to go back to equations that one already know how to solve. To quote an example, we mention $x^4 + 3x^2 = 28$.

In **Chapter V** "On finding the solution for equations composed of minors [*Ostendit æstimationem capitulorum compositorum minorum, quæ sunt quadratorum, numeri, et rerum*]", Cardano solves the quadratic equations.

AM V.4-6.
- *If* $x^2 + a_1x = a_0$, *then* $x = \sqrt{a_0 + \left(\frac{a_1}{2}\right)^2} - \frac{a_1}{2}$.
- *If* $x^2 + a_0 = a_1x$ *and* $\left(\frac{a_1}{2}\right)^2 - a_0 \geq 0$, *then* $x = \frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$.
- *If* $x^2 = a_1x + a_0$, *then* $x = \sqrt{a_0 + \left(\frac{a_1}{2}\right)^2} + \frac{a_1}{2}$.

For each one of the above rules, Cardano gives a demonstration. We will consider only the first one, since the others are similar.

AM V.1 — *Proof*. Cardano[2] shows that, given the equation $x^2 + 6x = 91$, then a solution is $x = \sqrt{91 + \left(\frac{6}{2}\right)^2} - \frac{6}{2} = 7$.

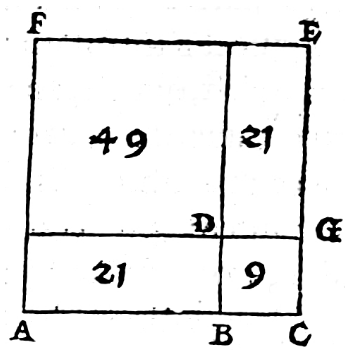


FIGURE 2.3 – *Ars magna* V.1.

[2]See [CARDANO 1545, Chapter V, paragraph 1, pages 9v-10r].

Cardano takes $\overline{AB} = x$ and $\overline{BC} = \frac{6}{2} = 3$. Then, he considers $\overline{AB} + \overline{BC}$ and draws the square on it, getting in this way four objects, two of which are equal by *Elements* I.43.[3] They are $\overline{AB}^2$, $\overline{BC}^2$, and $\overline{AB}\,\overline{BD} = \overline{DG}\,\overline{GE}$.

By the given equation, Cardano gets then the equality $\overline{AB}^2 + 2(\overline{AB}\,\overline{BD}) = 91$. Completing the square, he obtains

$$\overline{AB}^2 + 2(\overline{AB}\,\overline{BD}) + \overline{BC}^2 = 91 + 9, \tag{$\star$}$$

that is $\overline{AC}^2 = 100$, then $\overline{AC} = 10$ [that is $x = 7$]. □

We observe that Cardano presents the very standard technique of the completion of the square to solve quadratic equations. He provides a geometrical environment where he interprets the given equation, that is where he identifies the quantities x and $\frac{a_1}{2}$ with the measure of some segments. This interpretation enables him to use some known results on geometrical quantities such as *Elements* I.43. Then, he only needs to exploit the relationships between the objects in his model to be done. In term of the general equation $x^2 + a_1 x = a_0$, the relation ($\star$) says that $\left(x + \frac{a_1}{2}\right)^2 = a_0 + \left(\frac{a_1}{2}\right)^2$, which gives the formula once that the square root is taken.

By the way, a short remark on what edition of the *Elements* Cardano could have read follows. In the short preface to the manuscript *Commentaria in Euclidis Elementa*, Cardano explicitly mentions[4] the edition by Jacques Lefèvre d'Étaples or Jacopus Faber Stapulensis, printed in Paris in 1516. This was one of the most popular editions during the 16$^{\text{th}}$ century. Anyway, despite his ambition to overtake the misinterpretations by Campano da Novara and Bartolomeo Zamberti, Lefèvre d'Étaples confined himself to uncritically juxtapose the two writings.

In **Chapter XXXVII** "On the rule for assuming a false [condition] [*De regula falsum ponendi*]" Cardano deals again with the solutions of a quadratic equations. There are three ways to "assume a false": either admitting negative solutions ("*purum m:*"), or admitting solutions where the square root of a negative quantity appears ("*R m:* or *sophisticum m:*"), or finally admitting negative solutions where

[3] *Elements* I.43: "In any parallelogram the complement of the parallelograms about the diameter are equal to one another", see [Heath 1956a, page 340].

[4] See [Gavagna 2003, page 134].

the square root of a negative quantity appears. In particular, the second and third ways occur for the equation $x^2 + a_0 = a_1 x$ when $\left(\frac{a_1}{2}\right)^2 - a_0 < 0$.

Concerning the first way, Cardano says that he will look for the solutions "which can at last be verified in the positive".[5] In fact, he remarks that the solutions of $x^2 = a_1 x + a_0$ are opposite in sign to the solutions of $x^2 + a_1 x = a_0$. Then, a possible negative solution is in some sense admissible, since it corresponds to a positive solution of another family of equations where the coefficient of the first degree term is opposite in sign. Indeed, Cardano says that "[i]f the case is impossible [to solve] with either a positive or a negative, the problem is a false one. If it is a true [problem] with a positive solution in one, it will be a true [problem] with a negative solution in the other".[6]

Then, there are the "completely false [*omnino falsa*]" cases. Concerning the second way, Cardano only suggests one example, namely to solve the system

$$\begin{cases} 10 = x + y \\ 40 = xy \end{cases},$$

which is equivalent to the equation $x^2 + 40 = 10x$. Its solutions are $x = 5 \pm \sqrt{-15}$. It is easily seen that, in this way, we get two couples (x, y) that are both solutions, but by the symmetry of the system they reduce to

$$\begin{cases} x = 5 + \sqrt{-15} \\ y = 5 - \sqrt{-15} \end{cases}$$

alone. Cardano tries to justify the expressions $5 \pm \sqrt{-15}$ using the geometrical representation of the quadratic formula. But then he would need to pass throughout a negative area to represent $\sqrt{-15}$. Hence, he verifies that $5 \pm \sqrt{-15}$ are the real demanded quantities, multiplying them as follows

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40.$$

[5]See [Cardano 1968, page 217] or "*Primo igitur quærimus quæstionum solutiones, quæ per p: verificari minime possunt*", see [Cardano 1545, Chapter XXXVII, rule I, page 65r].

[6]See [Cardano 1968, page 219], or "*si casus est impossibilis, in utroque quæstio falsa est, per p: et per m: et si vera est, per p: in uno, erit vera per m: in alio*" see [Cardano 1545, Chapter XXXVII, rule I, page 65v].

There, Cardano is compelled to admit that $(\sqrt{-15})^2 = -15$. Concerning the third way, the only example provided by Cardano undergoes to a similar shortcoming.

Cardano says that "this [referring to the geometrical quantity that represents $\sqrt{-15}$] is truly sophisticated, since with it one cannot carry out the operations one can in the case of a pure negative and other [numbers]".[7] For example – he continues – one cannot use the quadratic formula. Note that Cardano does not add any further ontological remark concerning these "sophistic" quantities. He simply tries for an arithmetic of square roots of negative quantities, which would require that $(\sqrt{-a})^2 = -a$.

2.1.2. Transformations of equations. Let us now turn to cubic equations. One of the most used methods, which fills inside out the whole of Cardano's research on equations, is what I call 'transformation' of equations. With this term I mean that one finds a way to establish a relationship between two equations, either by a substitution that involves some of the coefficients of the two equations or by a substitution that involves one of their solutions.

In **Chapter I** Cardano explains some relations that hold between the solutions of some equations, depending on the sign of the discriminant. This leads to an investigation of the number of "true [*vera*]" (that is, positive) and "fictitious [*ficta*]" (that is, negative) solutions of an equation.

We will only examine the parts concerning cubic equations.

AM I.4. *Consider the equation $x^3 + a_1x = a_0$. Then, there is only a positive [real] solution.*

More generally, Cardano observes that "[w]here several [odd] powers, or even a thousand of them, are compared with a number, there will be one true solution and none that is fictitious".[8]

AM I.5-6. *Consider the equation $x^3 + a_0 = a_1x$. Then, its solutions correspond up to the sign to the solutions of $y^3 = a_1y + a_0$.*

[7]"*[Q]uævere est sophistica, quoniam per eam, non ut in puro m: nec in alijs operatione exercere licet*", see [Cardano 1545, Chapter XXXVII, rule II, proof, page 66r].

[8]See [Cardano 1968, page 11] or "*ubi plures denominationes numero comparantur, etiamsi mille forent, una erit æstimatio rei vera, et nulla ficta*", see [Cardano 1545, Chapter I, paragraph 4, page 4r].

More precisely,

- *if [a condition*[9] *equivalent to* $\Delta_3 = 0$ *holds],* $x_1 < 0$ *and* $x_2 = x_3 > 0$ *are solutions of* $x^3 + a_0 = a_1 x$*. Moreover,* $x_2 = \sqrt{\frac{a_1}{3}}$ *and* $x_1 = -(x_2 + x_3) = -2\sqrt{\frac{a_1}{3}}$*. Then* $y_1 = -x_1 > 0$ *is a solution of* $y^3 = a_1 y + a_0$*;*
- *if [a condition equivalent to* $\Delta_3 < 0$ *holds],* $x_1 < 0$ *and* $x_2, x_3 > 0$ *are solutions of* $x^3 + a_0 = a_1 x$*. Moreover,* $x_1 = -(x_2 + x_3)$*. Then* $y_1 = -x_1 > 0$ *is a solution of* $y^3 = a_1 y + a_0$*;*
- *If [a condition equivalent to* $\Delta_3 > 0$ *holds],* $x_1 < 0$ *is a solution of* $x^3 + a_0 = a_1 x$*. Then* $y_1 = -x_1 > 0$ *is a solution of* $y^3 = a_1 y + a_0$*.*

In Cardano's explanation of the first point, we find *in nuce* the idea of multiple solutions, when he says that "the remaining fictitious solution of which we spoke in the other example is the sum of two true ones, but since the true ones are equal to each other, the fictitious one is twice the true one".[10]

Since the positive solutions are to be preferred to the negative ones, Cardano does not feel the need to draw the negative solutions of $y^3 = a_1 y + a_0$. Anyway, for symmetry reasons we can complete the above proposition as follows.

AM I.5-6 supplemented. *Consider the equation* $x^3 + a_0 = a_1 x$*. Then, its solutions correspond up to the sign to the solutions of* $y^3 = a_1 y + a_0$*.*
More precisely,

- *if [a condition equivalent to* $\Delta_3 = 0$ *holds],* $x_1 < 0$ *and* $x_2 = x_3 > 0$ *are solutions of* $x^3 + a_0 = a_1 x$*. Moreover,* $x_2 = \sqrt{\frac{a_1}{3}}$ *and* $x_1 = -(x_2 + x_3) = -2\sqrt{\frac{a_1}{3}}$*. Then* $y_1 = -x_1 > 0$ *and* $y_2 = y_3 = -x_2 < 0$ *are solutions of* $y^3 = a_1 y + a_0$*;*
- *if [a condition equivalent to* $\Delta_3 < 0$ *holds],* $x_1 < 0$ *and* $x_2, x_3 > 0$ *are solutions of* $x^3 + a_0 = a_1 x$*. Moreover,* $x_1 = -(x_2 + x_3)$*. Then*

[9] In all these cases, Cardano discuss the sign of $\frac{2}{3} a_1 \sqrt{\frac{a_1}{3}} - a_0$, which is clearly equivalent to discussing the sign of $\Delta_3 = \frac{\alpha_0^2}{4} + \frac{\alpha_1^3}{27}$. We remark that the condition on the discriminant Δ_3 is explicitly given by Cardano in the particular case $\alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_1 = \pm a_1$, and $\alpha_0 = \pm a_0$ (see here page 40).

[10] See [Cardano 1968, page 12] or "*et ita relique ficta, de qua diximus, in alio exemplo, aggregatur ex duabus veris, sed quia vere sunt invicem æquales, ideo ficta semper dupla est veræ*", see [Cardano 1545, Chapter I, paragraph 5, page 4r].

$y_1 = -x_1 > 0$, $y_2 = -x_2 < 0$, and $y_3 = -x_3 < 0$ are solutions of $y^3 = a_1 y + a_0$;

- *If [a condition equivalent to $\Delta_3 > 0$ holds], $x_1 < 0$ is a solution of $x^3 + a_0 = a_1 x$. Then $y_1 = -x_1 > 0$ is a solution of $y^3 = a_1 y + a_0$.*

Note that obviously only the real solutions are taken into account.

Concerning cubic equations lacking in the first degree term, Cardano observes the following.

AM I.8 i. *Consider the equation $x^3 = a_2 x^2 + a_0$. Then, there is only a positive [real] solution.*

AM I.8 ii. *Consider the equation $x^3 + a_2 x^2 = a_0$. Then, its solutions correspond up to the sign to the solutions of $y^3 + a_0 = a_2 y^2$.*
More precisely,

- *if [a condition[11] equivalent to $\Delta_3 < 0$ holds], $x_1 > 0$ and $x_2, x_3 < 0$ are solutions of $x^3 + a_2 x^2 = a_0$. Moreover, $(x_2 + x_3) - x_1 = a_2$. Then $y_1 = -x_1 < 0$ and $y_2 = -x_2, y_3 = -x_3 > 0$ are solutions of $y^3 + a_0 = a_2 y^2$;*
- *If [a condition equivalent to $\Delta_3 > 0$ holds], $x_1 > 0$ is a solution of $x^3 + a_2 x^2 = a_0$. Then $y_1 = -x_1 < 0$ is a solution of $y^3 + a_0 = a_2 y^2$.*

Cardano does not consider at all the case $\Delta_3 = 0$.

Then, Cardano moves on studying the complete cubic equations in a similar way.

AM I.9 i. *Consider the equation $x^3 + a_1 x = a_2 x^2 + a_0$. Then, there are no negative [real] solutions.*

Cardano rather affirms that in this case “there may be three solutions, all true and none fictitious”. Anyway, his justification says that “the cube of a minus quantity is a minus, and thus [if there were a negative solution] a minus would equal a

[11]We remark as before that Cardano discuss the sign of $\frac{a_2}{3}\left(\frac{2}{3}a_2\right)^2 - a_0$, which is equivalent to a condition imposed on $\Delta_3 = \frac{\alpha_0}{108}\left(27\alpha_0 + 4\alpha_2^3\right)$. For instance, in the case $x^3 + a_2 x^2 = a_0$, we get

$$\Delta_3 > 0 \Longleftrightarrow \frac{-a_0}{108}\left(-27a_0 + 4a_2^3\right) < 0 \Longleftrightarrow -27a_0 + 4a_2^3 < 0 \Longleftrightarrow \frac{4}{27}a_2^3 < a_0,$$

which is Cardano’s condition. The condition on the discriminant Δ_3 is explicitly given by Cardano in the particular case $\alpha_3 = 1$, $\alpha_2 = \pm a_2$, $\alpha_1 = 0$, and $\alpha_0 = \pm a_0$ (see above, at page 40).

plus, which cannot be",[12] which shows that the equation $x^3 + a_1x = a_2x^2 + a_0$ cannot have negative solutions.[13]

AM I.9 ii. *Consider the equation* $x^3 + a_2x^2 + a_1x = a_0$.

- *If* $x_1 \geq 0$, $x_2, x_3 \leq 0$ *are its [real] solutions, then* $y_1 = -x_1 \leq 0$ *and* $y_2 = x_2, y_3 = x_3 \leq 0$ *are solutions of* $y^3 + a_1y + a_0 = a_2y^2$.
- *If* $x_1 \geq 0$ *is its [real] solution, then* $y_1 = -x_1 \leq 0$ *is a solution of* $y^3 + a_1y + a_0 = a_2y^2$.

The vice versa *also holds.*

AM I.10. *Consider the equation* $x^3 = a_2x^2 + a_1x + a_0$.

- *If* $x_1 \geq 0$, $x_2, x_3 \leq 0$ *are its [real] solutions, then* $y_1 = -x_1 \leq 0$ *and* $y_2 = x_2, y_3 = x_3 \leq 0$ *are solutions of* $y^3 + a_2y^2 + a_0 = a_1y$.
- *If* $x_1 \geq 0$ *is its [real] solution, then* $y_1 = -x_1 \leq 0$ *is a solution of* $y^3 + a_2y^2 + a_0 = a_1y$.

The vice versa *also holds.*

AM I.11. *Consider the equation* $x^3 + a_2x^2 = a_1x + a_0$.

- *If* $x_1 \geq 0$, $x_2, x_3 \leq 0$ *are its [real] solutions, then* $y_1 = -x_1 \leq 0$ *and* $y_2 = x_2, y_3 = x_3 \leq 0$ *are solutions of* $y^3 + a_0 = a_2y^2 + a_1y$.
- *If* $x_1 \geq 0$ *is its [real] solution, then* $y_1 = -x_1 \leq 0$ *is a solution of* $y^3 + a_0 = a_2y^2 + a_1y$.

The vice versa *also holds.*

No mention of $x^3 + a_1x = a_2x^2 + a_0$ is done.

We remark that Cardano often gives no justification of his statements, or at best he sometimes adds a numerical example.

As a matter of fact, Cardano's results on the relations between equations through their solutions also provide an overview of his knowledge on the number

[12]See [Cardano 1968, page 17] or "*[i]n his autem capitulis, quæ duplici denominatione, impari et una pari ac numero constant, si cubus et res, æquales sint, quadratis et numero, æquationes possunt esse tres, et omnes veræ, et nulla ficta, quia ut dictum est, minus cum ad solidus deducitur, fit minus, et ita minus æquales esset plus, quod esse non potest*", see [Cardano 1545, Chapter I, paragraph 9, page 5v].

[13]Actually, no cubic equation with real coefficients can have three positive solutions, see here Section 1.5.4.

of (real) solutions of a cubic equation. This is particularly relevant in the case of depressed equations and of equations lacking in the first degree term, where he manages to list (almost) all the possibilities depending on the discriminant. In order to do that, Cardano necessarily has to mention negative solutions. This chapter is one of the few places where it happens. So, despite the bad reputation that the negative, "fictitious" solutions have in Cardano's text (and his terminology reflects this aspect), it is anyway a fact that he knows and uses them.

Up to now, Cardano gave no proofs of his statements. But at the end of the chapter, he says that "it is meet that we should show these very wonderful things by a demonstration, as we will do throughout this whole book, so that, beyond mere experimental knowledge, reasoning may reinforce belief in them".[14] Then, we find a proof for one of the above cases, namely for the first point in (AM I.11).

AM I.13 — *Proof.* Cardano[15] consider the equation $x^3+a_0 = a_2x^2+a_1x$, where $x^3 = \overline{AB}$, $a_0 = \overline{BC}$, $a_2x^2 = \overline{DE}$, and $a_1x = \overline{EF}$. He takes $\overline{(H)} > 0$ to be a solution [and wants to show that $-\overline{(H)}$ is a solution of $x^3 + a_2x^2 = a_1x + a_0$.]

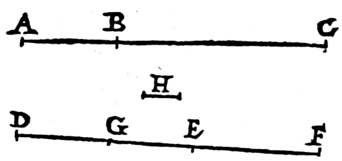


FIGURE 2.4 – *Ars magna* I.13.

The equation gives the equality $\overline{AB} + \overline{BC} = \overline{DE} + \overline{EF}$. Since $\overline{AB} + \overline{BC} = \overline{AC}$ and $\overline{DE} + \overline{EF} = \overline{DF}$, Cardano gets $\overline{AC} = \overline{DF}$. Then, he marks G on DF such that $\overline{AB} = \overline{DG}$ [and as a consequence $\overline{BC} = \overline{GF}$]. Then, the following equalities hold

$$\begin{cases} \overline{DE} = \overline{AB} + \overline{GE} \\ \overline{BC} = \overline{GE} + \overline{EF} \end{cases},$$

[14]See [CARDANO 1968, page 20] or "*[e]t iam oportunum est, ut ostendamus hæc demonstratione, quod etiam in toto hoc libro facturi sumus, ut rebus tam admirabilibus, ultra experientiam, fidei ratio accedat*", see [CARDANO 1545, Chapter I, paragraph 13, page 6v].

[15]See [CARDANO 1545, Chapter I, paragraph 13, page 6v].

that is

$$\begin{cases} \overline{DE} - \overline{AB} = \overline{GE} \\ \overline{BC} - \overline{FE} = \overline{GE} \end{cases}.$$

Taking $-\overline{(H)} < 0$ instead of $\overline{(H)}$, we have $x^3 = -\overline{AB} < 0$ and $a_1 x = -\overline{EF} < 0$, but $a_0 = \overline{BC} > 0$ and $a_2 x^2 = \overline{DE} > 0$. Then

$$\begin{cases} \overline{DE} - (-\overline{AB}) = \overline{GE} \\ \overline{BC} - (-\overline{FE}) = \overline{GE} \end{cases},$$

that is $\overline{DE} + \overline{AB} = \overline{BC} + \overline{EF}$, that is $x^3 + a_2 x^2 = a_1 x + a_0$. □

We observe that this is the common way to show that a certain quantity is the solution of an equation. One takes the hypothesis on the coefficients and on the solution, and shows by substitution that the equality is verified. Cardano settles a geometrical environment where he interprets the hypotheses as geometrical objects. This fixation of reference enables him to deal with these objects as if they were numbers, since he only considers their measures. We moreover observe that no dimensional considerations enter at this step, as for instance x^3 is represented by the segment AB. Then, Cardano has only to take care to move on the other side of the equal the negative quantities and to translate back from the geometrical language to the language of equations.

In **Chapter VI** "On the methods for solving new cases [*De modi inveniendi capitula nova*]", paragraph 3, Cardano tells us that there is a method

> which is called [the method of] similitude. It is four fold: [1] From the nature of the equation, as when the [solution in the] case of the cube equal to the first power and number is derived from the [solution in the] case of the cube and first power equal to the number. [2] From an augmentation of solutions, and thus we discovered the nonuniversal case of the fourth power, first power and number. [3] By the conversion of equations into [others] equivalent in nature, as we will show hereafter. [4] By the method of proceeding to solutions by the creations of cube and squares, or by proportion (as half or double), or by addition

and subtraction, these being three variant methods within a whole.[16]

We will see later on more in details what is the relationship between $x^3 = a_1x + a_0$ in (AM XII) and $x^3 + a_1x = a_0$ in (AM XI). It is instead not clear what the second and fourth fold is.

In the immediately following paragraph, we have an hint on the third fold, namely on the "conversion [*conversio*]"[17] of equations into others equivalent in nature. There, Cardano says that there is also a "way of transmutation" by which he "discovered many general rules before [their] demonstration"[18] and among others those of $x^3 = a_2x^2 + a_0$ and $x^3 + a_2x^2 = a_0$. Cardano suggests an example. He consider the problem to "find two numbers the sum of which is equal to the square of the second and the product of which is 8 [*duos invenias numeros, quoru aggregatum æquale sit alterius quadrato, et ex uno in alterum ducto, producatur 8*]", that is to find two real numbers x, y such that the system

$$\begin{cases} x + y = y^2 \\ xy = 8 \end{cases}$$

holds. Depending on the substitution in the second line that Cardano chooses, he obtains two cubic equations

$$y^3 = y^2 + 8,$$

taking $x = \frac{8}{y}$, or

$$x^3 + 8x = 64,$$

taking $y = \frac{8}{x}$. Then, being able to solve one of the two equations, Cardano will also know the solution of the other *via* one of the substitutions $x = \frac{8}{y}$ or $y = \frac{8}{x}$.

---

[16]See [Cardano 1968, page 51] or "*qui similitudinis dicitur, atque hic quadruplex. A natura æquationis, velut cum capitulum cubi æqualis rebus et numero, extrahitur ex capitulo cubi et rerum æqualium numero. Ab augmentis æquationum, sicque capitulo non universalia invenimus quod quadrati, rerum, ac numeri. A conversione æquationum in natura ei æquivalentem, ut exponemus infra. A modo pro ædendi ad æquationes per cuborum vel quadratorum generationem, aut per proportionem ut dupli vel dimidij, aut per additionem vel diminutionem, tres enim sunt modi variandi in universum*", see [Cardano 1545, Chapter VI, paragraph 3, page 15v].

[17]This is what I call 'transformation'.

[18]See [Cardano 1968, page 51] or "*[e]st etiam transmutationis via, qua ante demonstrationem universalia capitula multa inveni*", see [Cardano 1545, Chapter VI, paragraph 4, pages 15v-16r].

Jacqueline Stedall comments on this point (in her own translation of Cardano's text).

> This is the first example we have in the history of algebra of the transformation of an equation by an operation on the roots. It is of fundamental importance. Cardano's optimism at this point shines through in his writing: "Transform problems that are by some ingenuity understood to those that are not understood", he wrote, "and there will be no end to the discovery of rules".[19]

As a matter of fact, this passage could give a hint on how Cardano had the idea of the series of transformations that we have seen in Chapter I and that we will see in Chapter VII.

In **Chapter VII** "On the transformation of equations [*De capitulorum transmutatione*]", Cardano gives two rules about the relation that holds between the solutions of some equations, in a way similar to what he did in Chapter I. But despite of Chapter I, he no more considers the sign of the discriminant, rather allowing some coefficients to change. He says that these rule are "general [*generalis*]". The first rule is two-folded.

AM VII.2-3.

$$x^q + a_0 = a_p x^p \xrightarrow{y=\frac{(\sqrt[q]{a_0})^2}{x}} y^q + a_0 = a_{p'} y^{p'},$$
$$x^q = a_p x^p + a_0 \xrightarrow[y=\frac{(\sqrt[q]{a_0})^2}{x}]{} y^q + a_{p'} y^{p'} = a_0$$

with q, p, p' *naturals such that* $p' = q - p$ *and*

$$a_{p'} = \begin{cases} \left(\sqrt[q]{a_0}\right)^{p'-p} \frac{1}{a_p}, & \textit{if } p < p' < q \\ \left(\sqrt[q]{a_0}\right)^{p-p'} a_p, & \textit{if } p' < p < q \end{cases}$$

The arrow means 'correspond to', that is, if x is the given solution of the equation on the left side of the arrow and if we take such a coefficient $a_{p'}$ as above, then

[19]See [Stedall 2011, page 9]. In the quotation we find Stedall's own translation of Cardano's text: "*[q]uæstiones igitur alio ingenio cognitas ad ignotas transfer positiones, nec capitulorum inventio finem est habitura*", see [Cardano 1545, Chapter VI, paragraph 3, page 16r].

$y = \frac{(\sqrt[q]{a_0})^2}{x}$ is the solution of the equation on the right side. Obviously, inverting the substitution we can go in the other way round.

In particular, let us firstly take $q = 3$ and $p = 1$. Then, we get

$$\text{(AM XIII) } x^3 + a_0 = a_1 x \xrightarrow{y=\frac{(\sqrt[3]{a_0})^2}{x}} y^3 + a_0 = \left(\sqrt[3]{a_0}\frac{1}{a_1}\right) y^2 \text{ (AM XVI)}$$
$$\text{(AM XII) } x^3 = a_1 x + a_0 \xrightarrow[y=\frac{(\sqrt[3]{a_0})^2}{x}]{} y^3 + \left(\sqrt[3]{a_0}\frac{1}{a_1}\right) y^2 = a_0 \text{ (AM XV).}$$

Taking then $q = 3$ and $p = 2$, we get

$$\text{(AM XVI) } x^3 + a_0 = a_2 x^2 \xrightarrow{y=\frac{(\sqrt[3]{a_0})^2}{x}} y^3 + a_0 = (\sqrt[3]{a_0} a_2)\, y \text{ (AM XIII)}$$
$$\text{(AM XIV) } x^3 = a_2 x^2 + a_0 \xrightarrow[y=\frac{(\sqrt[3]{a_0})^2}{x}]{} y^3 + (\sqrt[3]{a_0} a_2)\, y = a_0 \text{ (AM XI).}$$

Cardano provides a demonstration for the first line of (AM VII.2-3) in the particular case $q = 3$ and $p = 2$. I will only sketch the proof.

AM VII.5 (idea) — *Proof*. Cardano[20] wants to show that $x^3 + a_0 = a_2 x^2$ corresponds to $y^3 + a_0 = a_1 y$, provided that $a_1 = \sqrt[3]{a_0}\, a_2$ and $y = \frac{(\sqrt[3]{a_0})^2}{x}$. First of all and as usual, Cardano translate his problem in a geometrical fashion. He assumes $a_2 = \overline{AM}$ and $x = \overline{AD}$ to be given. Rewriting the equation under the form $a_0 = x^2(a_2 - x)$, he draws a_0 depending on a_2 and x, that is $a_0 = \overline{AD}^2\, \overline{DM}$.

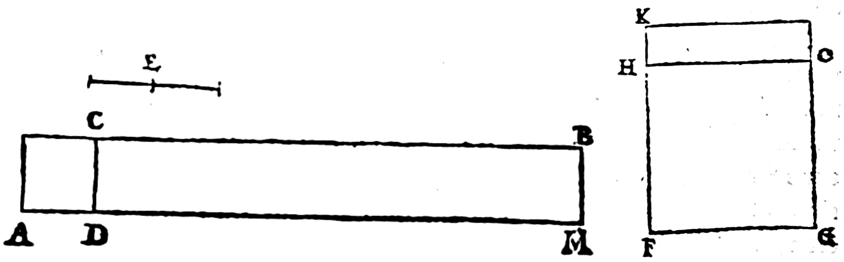


FIGURE 2.5 – *Ars magna* VII.5.

[20]See [CARDANO 1545, Chapter VII, paragraph 5, pages 18v-19r].

Then, he assumes that there exist $\overline{FH}[= y]$, $\overline{(E)}[= \sqrt[3]{a_0}]$, and $\overline{FK} = \overline{FH} + \overline{HK}$ such that

$$\frac{\overline{DM}}{\overline{FH}} = \frac{\overline{FH}}{\overline{(E)}} = \frac{\overline{(E)}}{\overline{AD}} \quad [\text{that is, } y = \frac{\left(\sqrt[3]{a_0}\right)^2}{x}]$$

and

$$\overline{FH}\,\overline{FK} = \overline{(E)}\,\overline{AM} \quad [\text{that is, } a_1 = \sqrt[3]{a_0}a_2].$$

Now, Cardano has only to work out the relations given by the above proportions. [This corresponds to draw the value of x from $y = \frac{(\sqrt[3]{a_0})^2}{x}$, substitute it in the equation $y^3 + a_0 = a_1 y$, and verify to have obtained the good value for a_1]. □

We remark that, if we complete the bi-dimensional diagrams with the appropriate heights, we get a geometrical model, into which Cardano interprets the given hypothesis. As a result, he is enabled to use the language of proportions and their properties to derive the solution. More precisely, in the end, Cardano gets to the relation $\overline{FH}^3 + \overline{FH}^2\,\overline{HK} = (\overline{FK}\,\overline{FH})\overline{FH}$, which represents the equation $y^3 + a_0 = a_1 y$. The core of the proof, where the proportions are used as a calculation tool, is to show that $\overline{FH}^2\,\overline{HK} = \overline{AD}^2\,\overline{DM}$. Note that Cardano's assumptions respect a homogeneity bond concerning dimension, that is only bodies of the same dimension are summed. In fact, a_2 in the equation $x^3 + a_0 = a_2 x^2$ has dimension one (it is the segment AM) and a_1 in the equation $y^3 + a_0 = a_1 y$ has dimension two (it is the parallelogram $FK\,FH$) such that both $a_2 x^2$ and $a_1 y$ has dimension three. Moreover, a_0 has always dimension three (it is the parallelepipeds $AD^2 DM$ and $FH^2 HK$).

The second rule is also two-folded.

AM VII.6-7.

$$x^q + a_0 = a_p x^p \xrightarrow{y=\sqrt[p]{a_p - x^{p'}}} y^q + a_0' = a_p y^{p'},$$

with $p' = q - p$ *and* $a_0' = \left(\sqrt[p]{a_0}\right)^{p'}$.

$$x^q = a_p x^p + a_0 \xrightarrow[y=\sqrt[p]{x^{p'} - a_p}]{} y^q + a_p y^{p'} = a_0',$$

with q, p, p' *naturals such that* $p' = q - p$ *and* $a_0' = \left(\sqrt[p]{a_0}\right)^{p'}$.

Here, the coefficient that changes is no more the coefficient of the term of degree p, but rather the coefficient of degree zero.

In particular, considering firstly $q = 3$ and $p = 1$, we get

$$\text{(AM XIII) } x^3 + a_0 = a_1 x \xrightarrow{y=a_1-x^2} y^3 + a_0^2 = a_1 y^2 \text{ (AM XVI)}$$
$$\text{(AM XII) } x^3 = a_1 x + a_0 \xrightarrow[y=x^2-a_1]{} y^3 + a_1 y^2 = a_0^2 \text{ (AM XV)}.$$

Considering then $q = 3$ and $p = 2$, we get

$$\text{(AM XVI) } x^3 + a_0 = a_2 x^2 \xrightarrow{y=\sqrt{a_2-x}} y^3 + \sqrt{a_0} = a_2 y \text{ (AM XIII)}$$
$$\text{(AM XIV) } x^3 = a_2 x^2 + a_0 \xrightarrow[y=\sqrt{x-a_2}]{} y^3 + a_2 y = \sqrt{a_0} \text{ (AM XI)}.$$

Cardano gives a demonstration for the first line of (AM VII.6-7) in the particular case $q = 3$ and $p = 1$. I will skip it, since it is similar to the preceding one.

At the end of this chapter, Cardano gives two other rules without any further explication. We will only glance at them.

AM VII.12.

$$x^4 + a_2 x^2 + a_0 = a_3 x^3 \xrightarrow[y=\frac{\sqrt{a_0}}{x}]{} y^4 + a_2 y^2 + a_0 = (\sqrt{a_0}\, a_3)\, y.$$

AM VII.13.

$$x^3 + a_0 = a_2 x^2 \longrightarrow x^3 + \left(\frac{a_2}{2}\right)^2 x = a_2 x^2 + \frac{a_0}{8}.$$

2.1.3. Cardano solves "middle power equal to the highest power and a number" in "general". The following chapter provides, on the one hand, some clarifications on the opposition "particular" *versus* "general" (see below, Section 2.1.4, page 81) and, on the other hand, gives a method to find (or better to guess) a solution for a large class of equations (not only the cubic ones).

In **Chapter VIII** "On the solution (general) for a middle power equal to the highest power and a number [*Docetur æquatio generaliter mediæ denominationis æqualis extremæ et numero*]", Cardano gives the following rule.

AM VIII.2. *Let $x^q + a_0 = a_p x^p$, with $0 < p < q$, and assume that two positive, real numbers f, g exist such that*

$$\begin{cases} a_p = f + g \\ a_0 = f\left(\sqrt[q-p]{g}\right)^p \end{cases}.$$

Then $x = \sqrt[q-p]{g}$.

We remark that solving the above system is equivalent either to solve an equation of degree $\frac{q}{p}$ or an equation of degree $\frac{q}{q-p}$. In fact, the system is equivalent either to the following

$$\begin{cases} g = a_p - f \\ g = \left(\frac{a_0}{f}\right)^{\frac{q-p}{p}} \end{cases} \iff \begin{cases} g = a_p - f \\ f^{\frac{q}{p}} - a_p f^{\frac{q-p}{p}} + a_0^{\frac{q-p}{p}} = 0 \end{cases}$$

or to

$$\begin{cases} f = a_p - g \\ f = \frac{a_0}{g^{\frac{p}{q-p}}} \end{cases} \iff \begin{cases} f = a_p - g \\ g^{\frac{q}{q-p}} - a_p g^{\frac{p}{q-p}} + a_0 = 0 \end{cases}$$

In the particular case $q = 3$ and $p = 1$, that is when $x^3 + a_0 = a_1 x$, we want to solve the system

$$\begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g} \end{cases}$$

in order to find a solution. Then $x = \sqrt{g}$. Let us consider Cardano's example $x^3 + 3 = 10x$. Taking $f = 1$ and $g = 9$, a solution is $x = 3$. The above example is the only cubic equation that Cardano suggests in this chapter. We add that in the case $q = 3$ and $p = 2$, that is when $x^3 + a_0 = a_2 x^2$, we want to solve the system

$$\begin{cases} a_2 = f + g \\ a_0 = fg^2 \end{cases}$$

in order to find a solution of the considered equation. Then $x = g$. We remark that, in both cases, solving the system is equivalent to solve some cubic equations. In fact, in the first case, we get $f^3 - a_1 f^2 + a_0^2 = 0$ and $g\sqrt{g} - a_1\sqrt{g} + a_0 = 0$. In the second case, we get $g^3 - a_2 g^2 + a_0 = 0$ and $f\sqrt{f} - a_2\sqrt{f} + \sqrt{a_0} = 0$. By a simple substitution and considering only positive solutions, we remark that the above equations, which are not of degree three, can be reduced to cubic equations.

Here follows a remark from a modern viewpoint to make Cardano's proposition more understandable. Consider $x^3 + \alpha_1 x + \alpha_0 = 0$, with α_0, α_1 real, and let x_1, x_2, x_3 be its solutions. Then, by Vieta's formulae,

$$\begin{cases} 0 = -x_1 - x_2 - x_3 \\ \alpha_1 = x_1x_2 + x_2x_3 + x_1x_3 \\ \alpha_0 = -x_1x_2x_3 \end{cases},$$

that is

$$\begin{cases} x_3 = -x_1 - x_2 \\ \alpha_1 = -x_1^2 - x_2^2 - x_1x_2 \\ \alpha_0 = x_1x_2(x_1 + x_2) \end{cases}.$$

Let us call $G = x_1$ and $F = x_1x_2 + x_2^2$. Then, the above system is equivalent to

(AM VIII.2 i) $$\begin{cases} -\alpha_1 = F + G^2 \\ \alpha_0 = FG \end{cases}.$$

Note that, given the system (AM VIII.2 i), we obtain the equivalent system

$$\begin{cases} F = -\alpha_1 - G^2 \\ G^3 + \alpha_1 G + \alpha_0 = 0 \end{cases}.$$

Therefore, solving the system is completely equivalent to solve the equation $x^3 + \alpha_1 x + \alpha_0 = 0$, The solutions are $x_1 = G$, together with the solutions of $x_2^2 + x_1x_2 - F = 0$. In the end, we get

$$x_1 = G, \quad x_2 = -\frac{G}{2} + \sqrt{\frac{G^2}{4} + F}, \quad \text{and} \quad x_3 = -\frac{G}{2} - \sqrt{\frac{G^2}{4} + F}.$$

Note that if x_1, x_2, x_3 are all real, that is $\Delta_3 < 0$, we find that $\frac{G^2}{4} + F$ must be positive. We moreover remark that Cardano takes $g = G^2$, that is $G = \pm\sqrt{g}$, and $f = F$. From a modern viewpoint, the substitution $g = G^2$ is not a real change of variable, since the map $z \mapsto z^2$ is not a bijection. However, in the simple cases considered by Cardano, we have that each equation, or each system in G, is equivalent to two equations, or to two systems in g, where G is replaced by $\sqrt{g}$ or $-\sqrt{g}$. Namely, the system (AM VIII.2 i) is equivalent to either

$$\begin{cases} -\alpha_1 = f + g \\ \alpha_0 = f\sqrt{g} \end{cases} \quad \text{or} \quad \begin{cases} -\alpha_1 = f + g \\ -\alpha_0 = f\sqrt{g} \end{cases},$$

and the associated solutions are

$$x_1 = \pm\sqrt{g}, \quad x_2 = \mp\frac{\sqrt{g}}{2} + \sqrt{\frac{g}{4} + f}, \quad \text{and} \quad x_3 = \mp\frac{\sqrt{g}}{2} - \sqrt{\frac{g}{4} + f},$$

where the sign $+$ or $-$ is chosen accordingly to the fact that (g, f) is a solution either of the first or of the second system. The first system has positive, real solutions f, g if and only if the equation $x^3 + \alpha_1 x + \alpha_0 = 0$ has a positive, real solution x_1. The second system has positive, real solutions f, g if and only if the equation $x^3 + \alpha_1 x + \alpha_0 = 0$ has a negative, real solution x_1. When the discriminant of $x^3 + \alpha_1 x + \alpha_0 = 0$ is negative, both systems admit a solution, since there is always a positive solution of the equation and a negative one. Moreover,[21] x_1 is opposite in sign either to x_2 or to x_2. The sign of x_2, respectively of x_3, depends instead on the coefficients (see above, Section 1.5.4, page 43). Depending on the sign of x_1, Cardano sometimes chooses $G = \sqrt{g}$, sometimes $G = -\sqrt{g}$. Remark that, if we take $G = \sqrt{g}$, $F = f$, $\alpha_1 = -a_1$, and $\alpha_0 = a_0$, with a_1, a_0 positive real, we get the first system, which is Cardano's system in (AM VIII.2) for $x^3 + a_0 = a_1 x$.

Similarly, consider $x^3 + \alpha_2 x^2 + \alpha_0 = 0$, with α_0, α_2 real, and let x_1, x_2, x_3 be its solutions. Then,

$$\begin{cases} \alpha_2 = -x_1 - x_2 - x_3 \\ 0 = x_1 x_2 + x_2 x_3 + x_1 x_3 \\ \alpha_0 = -x_1 x_2 x_3 \end{cases},$$

[21]Namely, $\begin{cases} -\frac{G}{2} + \sqrt{\frac{G^2}{4} + F} > 0 & \text{if and only if } G < 0 \text{ or } G > 0 \text{ and } F > 0 \\ -\frac{G}{2} - \sqrt{\frac{G^2}{4} + F} > 0 & \text{if and only if } G < 0 \text{ and } F < 0 \end{cases}.$

that is

$$\begin{cases} \alpha_2 = -x_1 - (x_2 + x_3) \\ x_2 x_3 = -x_1(x_2 + x_3) \\ \alpha_0 = x_1^2(x_2 + x_3) \end{cases},$$

that is

$$\begin{cases} \alpha_2 = -x_1 - \frac{x_2^2}{x_1+x_2} \\ x_3 = -\frac{x_1 x_2}{x_1+x_2} \\ \alpha_0 = x_1^2\left(\frac{x_2^2}{x_1+x_2}\right) \end{cases}.$$

Let us call $G = x_1$ and $F = \frac{x_2^2}{x_1+x_2}$. Then, the above system is equivalent to

(AM VIII.2 ii)
$$\begin{cases} -\alpha_2 = F + G \\ \alpha_0 = FG^2 \end{cases}.$$

The solutions of $x^3 + \alpha_2 x^2 + \alpha_0 = 0$ are $x_1 = G$, together with the solutions of $x_2^2 - Fx_2 - Fx_1 = 0$. In the end, we get

$$x_1 = G, \quad x_2 = \frac{F}{2} + \sqrt{\frac{F^2}{4} + FG}, \quad \text{and} \quad x_3 = \frac{F}{2} - \sqrt{\frac{F^2}{4} + FG}.$$

Similarly to above, note that if x_1, x_2, x_3 are all real, that is $\Delta_3 < 0$, we find that $\frac{F^2}{4} + FG$ must be positive.[22] We moreover remark that, if we take $f = F$, $g = G$, $\alpha_2 = -a_2$, $\alpha_0 = a_0$, with a_2, a_0 positive real, we get Cardano's system in (AM VIII.2) for $x^3 + a_0 = a_2 x^2$.

In the end, we observe that solving the systems is not simpler than finding the solutions of the corresponding equations, since the systems are nothing else but another way to rewrite these equations. Nevertheless, my opinion is that rewriting the equations under the system form is to be considered as a way to help intuition to guess a solution. We will see in detail how the systems can be exploited in a very special case (where guessing a solution is of the utmost

[22] About the sign of the solutions, we can only say that

$$\begin{cases} \frac{F}{2} + \sqrt{\frac{F^2}{4} + FG} > 0 & \text{if and only if } F > 0 \text{ or } F < 0 \text{ and } G < 0 \\ \frac{F}{2} - \sqrt{\frac{F^2}{4} + FG} > 0 & \text{if and only if } F > 0 \text{ and } G < 0 \end{cases}.$$

importance, since the cubic formula runs into troubles) dealing with Chapter XXV, at page 85.

Cardano gives a demonstration of this proposition in the case $q = 6$ and $p = 1$.

AM VIII.1 (idea) — *Proof.* He[23] considers the equation $x^6 + a_0 = a_1 x$ such that

$$\begin{cases} a_1 = f + g \\ a_0 = f\left(\sqrt[5]{g}\right) \end{cases}$$

holds. [We need to assume f, g real.]

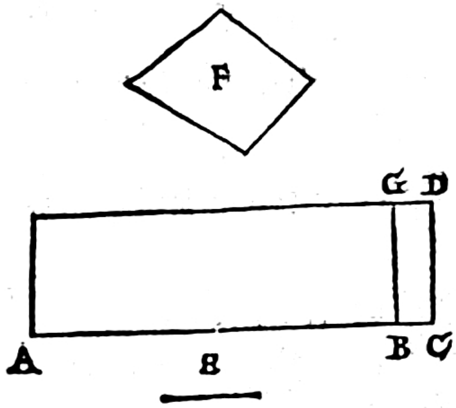


FIGURE 2.6 – *Ars magna* VIII.1.

Cardano wants to show that $x = \sqrt[5]{g}$.

Again, he translates the problem in a geometrical fashion, taking $a_1 = (\overline{AB} + \overline{BC})\,\overline{CD}$, $f = \overline{AB}\,\overline{CD}$, $g = \overline{BC}\,\overline{CD}$, and calling $a_0 = \overline{(F)}$. Then, he only has to check that the equation $x^6 + a_0 = a_1 x$ holds when the above values for the coefficients are substituted and one takes $x = \sqrt[5]{g} = \overline{(E)}$. □

This proof clearly stands aside from the others. In fact, it is obvious that no dimensional argument can hold, as in the equation appears x^6 and the sixth dimension cannot be coherently represented in the same way as Cardano did before with the first three dimensions. Then, he has trickily to take $\overline{(E)}$ to represent the fifth root of a certain quantity and $\overline{(F)}$ to represent the product

[23]See [CARDANO 1545, Chapter VII, paragraph 1, page 21r].

of two quantities one of which is the fifth root of a certain quantity. As it will become more and more clear in the following, the distinctive feature of Cardano's geometry is not to use in an essential way the relationships given by the relative positions of the geometrical objects involved, but rather only to respect some dimensional constraints. In the most of the cases, there is a natural way to respect these constraints, which is implicit in the problem (for instance, when a cubic equation is involved). Then, Cardano's geometry can be seen as an almost one-to-one translation from arithmetic. But this cannot be the case here. It is indeed where the geometrical environment fails to imitate the arithmetical steps (as in the proof of (AM VIII.1)) that the real extent of Cardano's geometry comes to light.

We observe that the word 'general' does not appear nowhere else but in the title of the chapter. We remark that Cardano's proposition is really general in our sense. In fact, the conditions in the hypotheses on the coefficients are non-special ones, since it is always possible to find such f, g (by solving a cubic equation). Nevertheless, we will see that the sense of 'general' used by Cardano does not always match ours. I refer then to the next section (especially while commenting Chapter XXV) for further explanations.

Cardano ends this chapter observing that in the case $x^q + a_0 = a_p x^p$ "there are necessarily two solutions in all [cases] and most obviously so [in those] with a large constant".[24] No further explanation is given.

2.1.4. "Particular" *versus* "general". In the *Ars magna*, the first time that Cardano introduces us to the contrast between "particular" and "general" is in **Chapter IV** "On the general and particular solutions that follow [*De subiectis æquationibus generalibus et singularibus*]". Cardano begins by stating that

> [s]olutions by which no class of cases can be completely resolved are called particular. Such are whole numbers or fractions or roots (square, cube, or what you will) of any number or, as I will put it, any simple quantity. Likewise, all *constantes* made up of two roots, either of which is a square root, a fourth root or, generally, any even root, and, therefore, those consisting

[24]See [Cardano 1968, page 70] or "*in omnibus, præterque in maximo numero, duas æstimationes necessario habet*", see [Cardano 1545, Chapter VIII, paragraph 3, page 21v].

> of two [such] terms and their *apotomes* (or, as they are called, *recisa*) of the third and sixth classes are not suitable for a general solution.[25]

A[26] "*constans*" or *binomium* is a (positive, irrational) quantity that can be written in the form $\alpha+\beta$, where $\alpha^2>\beta^2$. Moreover, when $\sqrt{\alpha^2-\beta^2}$ is "commensurable [*commensurabilis*]" with α, Cardano says that it is of the

1st type if $\alpha=a$ is rational and $\beta=\sqrt{b}$ is irrational, that is $\alpha+\beta=a+\sqrt{b}$,
2nd type if $\alpha=\sqrt{a}$ is irrational and $\beta=b$ is rational, that is $\alpha+\beta=\sqrt{a}+b$,
3rd type if $\alpha=\sqrt{a}$ is irrational and $\beta=\sqrt{b}$ is irrational, that is $\alpha+\beta=\sqrt{a}+\sqrt{b}$,

and, when $\sqrt{\alpha^2-\beta^2}$ is "incommensurable [*incommensurabilis*]" with α, we say that it is of the

4th type if $\alpha=a$ is rational and $\beta=\sqrt{b}$ is irrational, that is $\alpha+\beta=a+\sqrt{b}$,
5th type if $\alpha=\sqrt{a}$ is irrational and $\beta=b$ is rational, that is $\alpha+\beta=\sqrt{a}+b$,
6th type if $\alpha=\sqrt{a}$ is irrational and $\beta=\sqrt{b}$ is irrational, that is $\alpha+\beta=\sqrt{a}+\sqrt{b}$,

where the square roots are irrational. We say that two real quantities x,y are "commensurable" if $\frac{x}{y}$ is rational and "incommensurable" if $\frac{x}{y}$ is irrational.[27] Likewise for an "*apotome*" or *recisum*, which is written in the form $\alpha-\beta$. Note that only positive numbers are taken into account (for instance, $a-\sqrt{b}$ has no place in this classification when $a<\sqrt{b}$).

Then, Cardano calls "particular [*particularis*]" solutions:

- a natural a,
- a positive rational a,

[25]See [Cardano 1968, page 30] or "*[s]ingulares dicuntur æquationes, in quibus nullum capitulum perfecte potest absolvi, et tales sunt numerus integer, vel fractus, latus etiam omne numeri, su quadratum seu cubicum vel alterius generis, atque ut ita dicam, omnis simplex quantitas, item constantes ex duabus radicibus omnes, quarum altera sit quadrata, vel RR et generaliter radix par, unde quæ ex duobus constant nominibus, et apotome seu ut dicunt recisa tertij ac sexti generis, non apta sunt æquationi generali*", see [Cardano 1545, Chapter IV, paragraph 1, page 9r].

[26]Cardano does not specify the definition here. Nevertheless, we find a detailed explanation in the *Ars magna arithmeticæ*, see [Cardano 1663c, Chapter III, pages 307-308]. This comes from Euclid, see [Heath 1956c, Book X, definitions II.1-6, pages 101-102]. See also [Cardano 1570a, Chapter IV, pages 8-13], where Cardano moreover explains some properties of the *binomia* and *recisa* and gives some rules to make calculations with them.

[27]See *Elements* X.5-8 in [Heath 1956c, pages 24-28].

- an irrational $\sqrt[n]{a}$, with n natural,
- an irrational $\sqrt[n]{a}\pm\sqrt[n]{b}$, with n even natural (when $n=2$, it is a "*binomium* or *recisum* of the 3rd or 6th type").

As a brief intermezzo, Cardano observes that

AM IV.2. *Every complete monic cubic equation can be reduced to another one by the substitution $y = x \pm \frac{a_2}{3}$, where a_2 is the coefficient of the second degree term.*

We remark that the reduced equation will be a depressed cubic equation.

Afterwards, Cardano observes that

AM IV.5. *A cubic* binomium *or* recisum *of the 3rd or 6th kind $x = \sqrt[3]{a} \pm \sqrt[3]{b}$ can be a "general" solution for $x^3 = a_1x + a_0$ and $x^3 + a_1x = a_0$ respectively.*

In particular, this means that, if a cubic *binomium* is a solution of $x^3 = a_1x + a_0$, then its *recisum* is a solution of $x^3 + a_1x = a_0$.

Finally, Cardano observes that

AM IV.6. *The equation $x^3 = a_2x^2+a_0$ has a solution of the form $x = \sqrt[3]{a}+c+\sqrt[3]{b}$, where $\sqrt[3]{a} : c = c : \sqrt[3]{b}$, [that is $\sqrt[3]{ab} = c^2$].*

The equation $x^3 + a_2x^2 = a_0$ has a solution of the form $x = \sqrt[3]{a} - c + \sqrt[3]{b}$, where $\sqrt[3]{a} : c = c : \sqrt[3]{b}$, [that is $\sqrt[3]{ab} = c^2$].

In short, Cardano calls "general [*generalis*]" solutions:

- $\sqrt[3]{a} \pm \sqrt[3]{b}$,
- $\sqrt[3]{a} \pm c + \sqrt[3]{b}$, where $\sqrt[3]{ab} = c^2$.

These (up to the signs) indeed are the shapes under which a real solution of a cubic equation can go, assuming that a, b are real. These shapes can be easily gathered simply by looking at the cubic formulae that Cardano provides further on. This chapter is quite short and contains no further explanations. It gives the impression of being a sort of an *a posteriori* melange between a condensed version of the study of the shapes of the irrational solutions of a rational cubic equation inspired by *Elements*, Book X (as it can be found in the *Ars magna arithmeticæ* or in the *De regula aliza*, see below, Sections 3.2.4 and 4.3.1 respectively at pages 185 and 256) and of the new formulae in the *Ars magna*.

Let us now turn back to the first quotation. A first approximation of what Cardano means could be to think of "general solutions" as the most general shape under which the solution of an equation can go. In other words, this is the natural meaning that we would give to the term 'general' nowadays. This would imply that "general" or "particular" solutions should concern class of equations, namely equations of a same degree. Then, for example, a "general" solution for a cubic equation is its cubic formula. This explains why the "simple quantities" that Cardano lists are "particular" in the sense that they are not "general" or obtained by further calculations starting from a "general" solution. Obviously, this does not mean that no solution of a cubic equation is among those listed in the above quotation. If "general" is referred to the shape under which the solution of an equation always go, then by opposition "particular" should be used for some special shapes under which the solution of an equation sometimes goes. Anyway, this interpretation does not explain why Cardano calls "general" solutions the quantities $\sqrt[3]{a} \pm \sqrt[3]{b}$ and $\sqrt[3]{a} \pm c + \sqrt[3]{b}$, where $\sqrt[3]{ab} = c^2$. We should then look for a weaker version.

In **Chapter XXV** "On imperfect and particular rules [*De capitulis imperfectis et particularibus*]", after having stated all the cubic formulae in Chapters XI-XXIII as we will see later on, Cardano harks back to the opposition "particular" *versus* "general" saying that

> [t]he [foregoing] rules are called general for two reasons. The first is because the method itself is general, even though it is repugnant to the nature of a solution that it should be universal. Thus, if someone should say that the product of every number that is multiplied by itself is a square, that is a general rule. But it does not follow from this rule that I will know the square of every number, since it is impossible to know every number that is produced by multiplying another number by itself. A rule may also be called general because it exhausts a universal type of solution, although a solution does not exhaust the rule.

> Nevertheless there are also particular rules [and they are so called] since we cannot solve every given problem by them.[28]

We find again the terms "general" and "particular", but referred to rules. Here, there are two possible meanings for "general" rules and, as a consequence, for "particular" rules. The second one seems to refer back to Chapter IV, since it concerns the shape under which a solution of an equation can go. A rule is "general" if it "exhausts a universal [that is, "general"] type of solution", and "particular" if not. Saying it again, a "general" rule is a rule that leads back to a "general" solution, otherwise it is a "particular" rule. The other meaning, instead, concerns the methods employed to find the solution of an equation. A rule is "general" if "the method itself is general", and "particular" if "we cannot solve every given problem" by it. Saying it again, a "general" rule is a rule that employs a "general" method, where "general" method is supposedly a method by which every problem can be solved. Otherwise, it is a "particular" rule.

These words by Cardano introduce a series of "particular" rules. Regarding to the comment at the end of Chapter XII (see below, at page 112) and regarding to the fact that almost all the examples by Cardano[29] have $\Delta_3 < 0$, we will assume that this rules should be especially useful in the case $\Delta_3 < 0$. Then, all the equations will have three real solutions. We take f, g, h to be positive real.

AM XXV.1. *Consider* $x^3 = a_1x + a_0$.
If

$$\begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g} \end{cases},$$

then $x = \sqrt{f + \frac{g}{4}} + \frac{\sqrt{g}}{2}$.

[28]See [Cardano 1968, page 160] or "*[r]egulæ hæ, dicuntur generales, et hoc duabus de causis, prima, qua modus in se generalis est, quamque repugnet naturæ æstimationis, ut sit universalis, velut si quis dicat, omnis numerus productus ex aliquo in se ducto, quadratus est, regula est generalis, nec tamen sequitur, quod per hanc regulam, cognoscam omnem numerum quadratum, quia non licet cognoscere omnem numerum, qui e alio in se ducto producitur. Dicitur et generalis regula, quia exhaurit æstimationis genus universum, quamque æstimatio non exhauriat regulam particulares tamen sunt regulæ, quia non omnem propositam quæstionem per illa solvere posumus*", see [Cardano 1545, Chapter XXV, page 46r].

[29]Except in (AM XXV.3), (AM XXV.5), and (AM XXV.6) in the second example.

We remark that this is the system (AM VIII.2 i), with $G = -\sqrt{g}$, $F = f$, $\alpha_1 = -a_1$ and $\alpha_0 = -a_0$. The third solution $-\sqrt{f + \frac{g}{4}} + \frac{\sqrt{g}}{2}$ is always negative.

AM XXV.2. *Consider* $x^3 = a_1 x + a_0$.
If

$$\begin{cases} g = \sqrt{a_1 + f} \\ a_0 = fg \end{cases}, \ \textit{[that is} \begin{cases} a_1 = g^2 - f \\ a_0 = fg \end{cases}, \textit{]}$$

then $x = \sqrt{a_1 + f}$ $[= g = \sqrt{a_1 + \frac{a_0}{x}}]$.

Note that my addition into square brackets is necessary in order to preserve the truth of the statement. In fact, if one only considers the positive square root (as Cardano does), it is not true that $x = \sqrt{a_1 + \frac{a_0}{x}}$ follows from $x^3 = a_1 x + a_0$ (rather, the valid implication is the one on the opposite direction).

It is easy to observe that this proposition gives the negative solution $-\sqrt{g}$ in (AM XXV.1), and that the solution which was positive in (AM XXV.1) is now negative. We remark in fact that the system in the hypothesis of (AM XXV.2) is the system (AM VIII.2 i) with $G = g$, $F = -f$, $\alpha_1 = -a_1$ and $\alpha_0 = -a_0$. The other solutions $-\frac{g}{2} \pm \sqrt{f + \frac{g}{4}}$ are always negative.

In 1570 and 1663 editions, a few words are added to justify some of the rules. In particular, in this case, Cardano observes that, if the hypothesis in (AM XXV.2) holds, then

$$g^3 = g^2 g = (a_1 + f)g, \quad \text{that is} \quad g^3 = a_1 g + a_0.$$

AM XXV.3. *Consider* $x^3 = a_1 x + a_0$.
If

$$\begin{cases} a_1 = f + g \\ \frac{a_0}{2} = \sqrt{f}\, g + f\sqrt{g} \end{cases}, \ \textit{[that is} \begin{cases} a_1 = f + g \\ a_0 = 2\sqrt{f}\, g + 2f\sqrt{g} \end{cases}, \textit{]}$$

then $x = \sqrt{f} + \sqrt{g}$.

We remark that this is the system (AM VIII.2 i), with $G = \sqrt{f} + \sqrt{g}$, $F = -2\sqrt{fg}(\sqrt{f} + \sqrt{g})$, $\alpha_1 = -a_1$ and $\alpha_0 = -a_0$. The other solutions are always negative.

In 1570 and 1663 editions, the proposition is justified remarking that

$$\frac{y^3+z^3+y^2z+y\,z^2}{2yz(y+z)}=\frac{y^2+z^2}{2y\,z},$$

with y, z positive real. We notice that this means that

$$y^3+z^3+y^2z+y\,z^2=(y^2+z^2)(y+z).$$

If we take $y=\sqrt{f}$ and $z=\sqrt{g}$, then we can rewrite the hypothesis in (AM XXV.3) as follows

(AM XXV.3 bis) $$\begin{cases} a_1=y^2+z^2 \\ \frac{a_0}{2}=y^2z+y\,z^2 \end{cases}.$$

In this case, we get $y^3+z^3+y^2z+y\,z^2=a_1(y+z)$. Assuming moreover $x=y+z$, we verify that the equation $x^3=a_1x+a_0$ holds, that is

$$(y+z)^3=(y^3+y^2z+y\,z^2+z^3)+(2y^2z+2y\,z^2).$$

Then, besides (AM VIII.2 i), we remark that it exists a second access point to this rule, namely the substitution $x=y+z$.

AM XXV.4. *Consider $x^3=a_1x+a_0$.*
If

$$\begin{cases} a_1=f+g+h \\ f:g=g:h \\ a_0=g(\sqrt{f}+\sqrt{h}) \end{cases}, \ \textit{[that is} \begin{cases} a_1=\left(\frac{f+g}{\sqrt{f}}\right)^2-g \\ a_0=g\left(\frac{f+g}{\sqrt{f}}\right) \end{cases},\textit{]}$$

then $x=\sqrt{f}+\sqrt{h}\,[=\frac{f+g}{\sqrt{f}}]$.

We remark that this is the system (AM VIII.2 i), with $G=\frac{f+g}{\sqrt{f}}$, $F=-g$, $\alpha_1=-a_1$ and $\alpha_0=-a_0$. The other solutions are always negative.

In 1570 and 1663 editions, the proposition is justified remarking that

$$\frac{y^3+z^3+2y^2z+2y\,z^2}{yz(y+z)}=\frac{y^2+z^2+yz}{yz} \quad (\text{and} \quad \frac{y^2}{yz}=\frac{yz}{z^2})$$

with y, z positive real. We notice that this means that

$$y^3+z^3+2y^2z+2y\,z^2=(y^2+z^2+yz)(y+z).$$

If we take $\sqrt{f} = y$ and $\sqrt{h} = z$, then $g = \sqrt{fh} = yz$. Then, we can rewrite the hypothesis in (AM XXV.4) as follows

$$\text{(AM XXV.4 bis)} \qquad \begin{cases} a_1 = y^2 + z^2 + yz \\ a_0 = y^2 z + y\, z^2 \end{cases}.$$

In this case, we get $y^3 + z^3 + 2y^2 z + 2y\, z^2 = a_1(y + z)$. Assuming moreover $x = y + z$ to be a solution, we verify that the equation $x^3 = a_1 x + a_0$ holds, that is

$$(y + z)^3 = (y^3 + 2y^2 z + 2y\, z^2 + z^3) + (y^2 z + y\, z^2).$$

Then, besides (AM VIII.2 i), we remark that it exists a second access point to this rule, namely the substitution $x = y + z$.

AM XXV.5. *Consider* $x^3 = a_1 x + a_0$.
If

$$\begin{cases} f^2 + g^2 = a_1 + fg \\ \frac{a_0}{3} = fg(f + g) \end{cases}, \ \textit{[that is} \begin{cases} a_1 = (f + g)^2 - 3fg \\ a_0 = 3f^2 g + 3f\, g^2 \end{cases}, \textit{]}$$

then $x = f + g$.

We remark that this is the system (AM VIII.2 i), with $G = f + g$, $F = -3fg$, $\alpha_1 = -a_1$ and $\alpha_0 = -a_0$. The other solutions are always negative.

In 1570 (and 1663) editions, Cardano affirms that it is “the converse of a general rule”. Consider the following diagram and let us call $y = \overline{AB}$ and $z = \overline{BC}$.

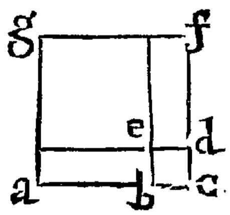


FIGURE 2.7 – *Ars magna* XXV.5.

Taking $f = y$ and $g = z$, Cardano rewrite the hypothesis in (AM XXV.4) as follows

$$\text{(AM XXV.5 bis)} \qquad \begin{cases} a_1 = (y - z)^2 + yz \\ a_0 = 3z\, y^2 + 3y\, z^2 \end{cases}.$$

We notice that, assuming $x = f + g$, this means that

$$a_1 x = (f^2 + g^2 - fg)(f + g) = f^3 + g^3.$$

So, Cardano verifies that the equation $x^3 = a_1 x + a_0$, that is

$$(f + g)^3 = (f^3 + g^3) + (3f^2 g + 3f g^2)$$

holds. Then, besides (AM VIII.2 i), we remark that it exists a second access point to this rule, namely the substitution $x = y + z$.

AM XXV.6. *Consider* $x^3 = a_1 x + a_0$.
If $f^3 + a_0 = a_1 f$, *then* $x + f$ *divides* $x^3 + f^3$ *and* $a_1 x + a_0 + f^3$.
If $f^3 - a_0 = a_1 f$ *[with* $f^3 > a_0$*], then* $x - f$ *divides* $x^3 - f^3$ *and* $a_1 x + a_0 - f^3$.

We will try to get a deeper insight of this proposition. If we can find a common factor in both sides of the equation, then we manage to lower the degree of the equation. The easiest case is to consider a common factor of degree one $x - \alpha$, with α real. The only α such that $x - \alpha$ divides x^3 is 0. But, since $a_0 \neq 0$, x does not divide $a_1 x + a_0$. Then, we add to both sides of the equation a constant such that the left side always turns out to be divisible by $x - \alpha$. In this way, we will only have to check the divisibility on the right side. Let us take $-\alpha^3$. Then,

$$\frac{x^3 - \alpha^3}{x - \alpha} = \frac{a_1 x + a_0 - \alpha^3}{x - \alpha}.$$

Since

$$a_1 x + a_0 - \alpha^3 = a_1(x - \alpha) + (a_1 \alpha + a_0 - \alpha^3),$$

we force $a_1 \alpha + a_0 - \alpha^3 = 0$. By the way, we remark that this means that α is a solution of $x^3 = a_1 x + a_0$ (this is nothing else but Ruffini's theorem). Performing the division, we get the quadratic equation $x^2 + \alpha x + \alpha^2 - a_1 = 0$. Its solutions are $-\frac{\alpha}{2} \pm \sqrt{a_1 - \frac{3}{4}\alpha^2}$. We conclude that the solutions of the equation $x^3 = a_1 x + a_0$ are

$$\alpha, \quad -\frac{\alpha}{2} + \sqrt{a_1 - \frac{3}{4}\alpha^2} \quad \text{and} \quad -\frac{\alpha}{2} - \sqrt{a_1 - \frac{3}{4}\alpha^2}.$$

We know[30] that, when the coefficients are real and $\Delta_3 < 0$, the equation $x^3 = a_1x + a_0$ always has one positive and one negative solution. Cardano takes $\alpha = \pm f$.

We remark then that, once that one already knows no matter how a solution of the cubic equation, this proposition helps in finding the two remaining real solutions. In this case, then, we can avoid to use the cubic formula, which will instead contain imaginary numbers.

We moreover remark that this proposition is a another access point to (AM XXV.1) and (AM XXV.2).

AM XXV.7. *Consider $x^3 = a_1x + a_0$.*
Then $a_0 = \left(\frac{\alpha}{2} + \sqrt{a_1 - \frac{3}{4}\alpha^2}\right)\left(\frac{\alpha}{2} - \sqrt{a_1 - \frac{3}{4}\alpha^2}\right)^2 + \left(\frac{\alpha}{2} - \sqrt{a_1 - \frac{3}{4}\alpha^2}\right)\left(\frac{\alpha}{2} + \sqrt{a_1 - \frac{3}{4}\alpha^2}\right)^2$, *where α is a solution of the considered equation.*

Let us try to understand where this proposition comes from. We know by (AM VIII.2 i) that the solutions of $x^3 = a_1x + a_0$ are G and $-\frac{G}{2} \pm \sqrt{\frac{G^2}{4} + F}$ where F, G satisfy the system

$$\begin{cases} a_1 = F + G^2 \\ -a_0 = FG \end{cases}.$$

Since $F = a_1 - G$, we can rewrite the solutions as G and $-\frac{G}{2} \pm \sqrt{a_1 - \frac{3}{4}G^2}$. Or else we can write $-\left(\left(-\frac{G}{2} + \sqrt{a_1 - \frac{3}{4}G^2}\right) + \left(-\frac{G}{2} - \sqrt{a_1 - \frac{3}{4}G^2}\right)\right)$ (see also Cardano's remark after proposition (AM I.5-6), above at page 65) and $-\frac{G}{2} \pm \sqrt{a_1 - \frac{3}{4}G^2}$. Since we suppose $\Delta_3 < 0$, all these solutions are real. Moreover, we know that a_0 is the product of the three solutions, so that

$$a_0 = \left(-\frac{G}{2} + \sqrt{a_1 - \frac{3}{4}G^2}\right)\left(-\frac{G}{2} - \sqrt{a_1 - \frac{3}{4}G^2}\right) \left(\left(\frac{G}{2} - \sqrt{a_1 - \frac{3}{4}G^2}\right) + \left(\frac{G}{2} + \sqrt{a_1 - \frac{3}{4}G^2}\right)\right),$$

which gives the equality in (AM XXV.7) taking $G = \alpha$. Then, this proposition basically derives from the fact that a_0 is the product of the three solutions of $x^3 = a_1x + a_0$ and again from (AM VIII.2 i).

[30]See above, at page 43.

AM XXV.8. *Consider* $x^3 = a_1x + a_0$.

Then $\frac{a_0}{2} = \left(\frac{\alpha}{2} + \sqrt{\frac{a_0}{2\alpha} + a_1 - \frac{3}{4}\alpha^2}\right)\left(\frac{\alpha}{2} - \sqrt{\frac{a_0}{2\alpha} + a_1 - \frac{3}{4}\alpha^2}\right)^2$
$+\left(\frac{\alpha}{2} - \sqrt{\frac{a_0}{2\alpha} + a_1 - \frac{3}{4}\alpha^2}\right)\left(\frac{\alpha}{2} + \sqrt{\frac{a_0}{2\alpha} + a_1 - \frac{3}{4}\alpha^2}\right)^2$, *where* α *is a solution of the considered equation.*

We remark that this proposition is a variation on the above one, thus also deriving from the fact that a_0 is the product of the three solutions of $x^3 = a_1x + a_0$ and from (AM VIII.2 i).

AM XXV.9. *Consider* $x^3 + a_0 = a_1x$.
If

$$\begin{cases} [a_1 = -f + (a_1 + f)] \\ a_0 = f\sqrt{a_1 + f} \end{cases},$$

then $x = \frac{\sqrt{a_1+f}}{2} \pm \sqrt{a_1 - \frac{3}{4}(a_1 + f)}$.

Note that in this case both solutions are positive, since $\frac{\sqrt{a_1+f}}{2} \geq \sqrt{a_1 - \frac{3}{4}(a_1 + f)}$.

We remark that this is the system (AM VIII.2 i), with $G = -\sqrt{a_1 + f}$, $F = -f$, $\alpha_1 = -a_1$ and $\alpha_0 = a_0$.

Moreover, we remark that this proposition corresponds to (AM XXV.1). In fact, if we write a'_1, a'_0 for the coefficients and f', g' for the functions of the coefficients in (AM XXV.1), then it is enough to take $f' = -f$ and $\sqrt{g'} = \sqrt{a_1 + f}$ in order to verify that $a_1 = a'_1$ and $a_0 = -a'_0$.

AM XXV.10. *Consider* $x^3 + a_0 = a_1x$.
If $f = a_1\sqrt[3]{f - a_0}$, *then* $x - \sqrt[3]{f - a_0}$ *divides* $x^3 + a_0 - f$ *and* $a_1x - f$.

We remark that a reasoning similar to (AM XXV.6) is present in this proposition. In this case, Cardano takes $\alpha = \sqrt[3]{a_0 - f}$. We remark that, if it holds $f = a_1\sqrt[3]{f - a_0}$, then $\sqrt[3]{f - a_0}$ is a solution of $x^3 + a_0 = a_1x$. I do not have any hints on why Cardano does not also consider the divisibility by $x + \sqrt[3]{f - a_0}$.

We moreover remark that this proposition is a another access point to (AM XXV.9).

AM XXV.11. *Consider* $x^3 + a_0 = a_1x$.

If

$$\begin{cases} a_1 = f+g+h \\ f:g = g:h \\ \frac{a_0}{3} = g(\sqrt{f}-\sqrt{h}) \end{cases}, \text{ [that is } \begin{cases} a_1 = \left(\frac{f-g}{\sqrt{f}}\right)^2 + 3g \\ a_0 = 3g\left(\frac{f-g}{\sqrt{f}}\right) \end{cases}, \text{]}$$

then $x = \sqrt{f} - \sqrt{h}\,[= \frac{f-g}{\sqrt{f}}]$.

Note that $f > g$, since we assume that Cardano is admitting only positive solutions.

We remark that this is the system (AM VIII.2 i), with $G = \frac{f-g}{\sqrt{f}} > 0$, $F = 3g$, $\alpha_1 = -a_1$ and $\alpha_0 = a_0$. Among the other solutions, there is a negative one and a positive one, which is $x = \frac{g-f}{2\sqrt{f}} + \sqrt{\frac{(f-g)^2}{4f} + 3g}$.

Cardano remarks that this proposition corresponds to (AM XXV.4). We observe in fact that, if we write a'_1, a'_0 for the coefficients and f', g' for the functions of the coefficients in (AM XXV.4), then we can express f', g' depending on f, g (even if this cannot be done in an easy way) in order to verify that $a_1 = a'_1$ and $a_0 = -a'_0$. Moreover, we can try to justify (AM XXV.11) in a way similar to Cardano's 1570 (and 1663) editions. Let us call $y = \sqrt{f}$ and $z = \sqrt{h}$. Then, we can rewrite the hypothesis in (AM XXV.11) as follows

(AM XXV.11 bis)
$$\begin{cases} a_1 = y^2 + z^2 + yz \\ a_0 = 3y^2 z - 3yz^2 \end{cases}.$$

In this case, we get $y^3 - z^3 = a_1(y-z)$. Assuming moreover $x = y - z$ to be a solution, we verify that the equation $x^3 + a_0 = a_1 x$ holds, that is

$$(y-z)^3 + (3y^2 z - 3yz^2) = y^3 - z^3.$$

Then, besides (AM VIII.2 i), we remark that it could exist a second access point to this rule, namely the substitution $x = y + z$.

AM XXV.12. *Consider* $x^3 + a_0 = a_1 x$.
If

$$\begin{cases} \frac{a_1}{\sqrt[3]{a_0}} = f + g \\ a_0 = fg^2 \end{cases},$$

then $\sqrt[3]{a_0} : x = x : f$, *[that is* $x = \sqrt{f\sqrt[3]{a_0}} = \sqrt[6]{f^4 g^2}$*].*

We remark that this is the system (AM VIII.2 i), with $G = \sqrt{f\sqrt[3]{f\,g^2}}$, $F = g\sqrt[3]{f\,g^2}$, $\alpha_1 = -a_1$ and $\alpha_0 = a_0$. Among the other solutions, there is a negative one and a positive one, which is $x = \frac{\sqrt{5}-1}{2}\sqrt[6]{f^4 g^2}$.

AM XXV.13. *Consider* $x^3 + a_0 = a_1 x$.
If

$$\begin{cases} \frac{a_1}{3} = f + g \\ \frac{a_0}{2} = f\sqrt{f} + g\sqrt{g} \end{cases}, \ \textit{[that is} \begin{cases} a_1 = 3f + 3g \\ a_0 = 2f\sqrt{f} + 2g\sqrt{g} \end{cases}, \textit{]}$$

then $x = \sqrt{f} + \sqrt{g}$.

We remark that this is the system (AM VIII.2 i), with $G = \sqrt{f} + \sqrt{g}$, $F = \frac{2f\sqrt{f}+2g\sqrt{g}}{\sqrt{f}+\sqrt{g}}$, $\alpha_1 = -a_1$ and $\alpha_0 = a_0$. Among the other solutions, there is a negative one and a positive one, which is $x = -\frac{\sqrt{f}+\sqrt{g}}{2} + \sqrt{9f + 6\sqrt{f}\sqrt{g} + 9g}$.

Cardano remarks that this proposition corresponds to (AM XXV.3). We observe in fact that, if we write a'_1, a'_0 for the coefficients and f', g' for the functions of the coefficients in (AM XXV.3), then we can express f', g' depending on f, g (even if this cannot be done in an easy way) in order to verify that $a_1 = a'_1$ and $a_0 = -a'_0$. Moreover, we can try to justify (AM XXV.13) in a way similar to Cardano's 1570 (and 1663) editions. Let us call $y = \sqrt{f}$ and $z = \sqrt{g}$. Then, we can rewrite the hypothesis in (AM XXV.13) as follows

(AM XXV.13 bis)
$$\begin{cases} a_1 = 3y^2 + 3z^2 \\ a_0 = 2y^3 + 2z^3 \end{cases}.$$

In this case, we get $3(y^3 + y^2 z + yz^2 + z^3) = a_1(y + z)$. Assuming moreover $x = y + z$ to be a solution, we verify that the equation $x^3 + a_0 = a_1 x$ holds, that is

$$(y + z)^3 + (2y^3 + 2z^3) = 3(y^3 + y^2 z + yz^2 + z^3).$$

Then, besides (AM VIII.2 i), we remark that it could exist a second access point to this rule, namely the substitution $x = y + z$.

AM XXV.17. *Consider* $x^3 + a_0 = a_1 x$.

If

$$\begin{cases} \sqrt{a_1} = f + g \\ a_0 = 2f\,g^2 + f^2 g \end{cases}, \ \textit{[that is} \begin{cases} a_1 = g^2 + 2fg + f^2 \\ a_0 = g(2fg + f^2) \end{cases}, \textit{]}$$

then $x = g$.

We remark that this proposition corresponds to (AM XXV.2). In fact, if we write a_1', a_0' for the coefficients and f', g' for the functions of the coefficients in (AM XXV.2), then it is enough to take $f' = -g$ and $g' = 2fg + f^2$ in order to verify that $a_1 = a_1'$ and $a_0 = -a_0'$.

Moreover, we remark that this is the system (AM VIII.2 i), with $G = g$, $F = 2fg + f^2$, $\alpha_1 = -a_1$ and $\alpha_0 = a_0$. Among the other solutions, there is a negative one and a positive one, which is $x = -\frac{g}{2} + \sqrt{\frac{g^2}{4} + 2fg + f^2}$.

AM XXV.14. *Consider* $x^3 + a_2x^2 = a_0$.
If

$$\begin{cases} a_2 = f + g \\ a_0 = fg^2 \end{cases},$$

then $x = \sqrt{f\left(g + \frac{f}{4}\right)} - \frac{f}{2}$.

We remark that this is the system (AM VIII.2 ii), with $G = -g$, $F = -f$, $\alpha_2 = a_2$ and $\alpha_0 = -a_0$. The third solution is always negative.

AM XXV.18. *Consider* $x^3 + a_2x^2 = a_0$.
If

$$\begin{cases} fg = \left(\frac{a_2}{3}\left(\sqrt[3]{f} - \sqrt[3]{g}\right)\right)^3 \\ a_0 = f - g \end{cases}, \ \textit{[that is} \begin{cases} a_2 = \frac{3\sqrt[3]{fg}}{\sqrt[3]{f} - \sqrt[3]{g}} \\ a_0 = f - g \end{cases}, \textit{]}$$

then $x = \sqrt[3]{f} - \sqrt[3]{g}$.

Note that $f > g$, since we assume that Cardano is only admitting positive solutions.

We remark that this is the system (AM VIII.2 ii), with $G = \sqrt[3]{f} - \sqrt[3]{g} > 0$, $F = -\frac{f-g}{\left(\sqrt[3]{f} - \sqrt[3]{g}\right)^2}$, $\alpha_2 = a_2$ and $\alpha_0 = -a_0$. The other solutions are always negative.

AM XXV.15. *Consider* $x^3 + a_0 = a_2 x^2$.
If

$$\begin{cases} \frac{a_2}{4} < f < \frac{a_2}{3} \\ \frac{a_0}{f} = g^2 \\ \frac{g}{2} + a_2 = 4f \end{cases}, \text{ [that is} \begin{cases} \frac{a_2}{4} < f < \frac{a_2}{3} \\ a_0 = f\,g^2 \\ a_2 = 4f - \frac{g}{2} \end{cases}, \text{]}$$

then $x = 2f \pm \sqrt{4f(a_2 - 3f)}$.

Note that the condition $f < \frac{a_2}{3}$ in the hypothesis grants that $4f(a_2 - 3f) > 0$, that is that the solutions are all real. The condition $\frac{a_2}{4} < f$ grants that $2f - \sqrt{4f(a_2 - 3f)} > 0$.

We remark that this proposition corresponds to (AM XXV.14). In fact, if we write a_2', a_0' for the coefficients and f', g' for the functions of the coefficients in (AM XXV.14), then it is enough to take $f' = -4f$ and $g' = \frac{g}{2}$ in order to verify that $a_2 = -a_2'$ and $a_0 = -a_0'$.

Moreover, we remark that this is the system (AM VIII.2 ii), with $G = -\frac{g}{2}$, $F = 4f$, $\alpha_2 = -a_2$ and $\alpha_0 = a_0$.

Cardano gives no further justifications for these rules, saying only that they are founded on the propositions in Chapter VI (on the cube of a binomial) and that they are self-evident to those familiar with his books on Euclid.[31]

Let the propositions (AM XXV.7), (AM XXV.8) stand aside. In the following I will refer only to the remaining ones. We observe that there are strong links between these propositions, as represented in the following table.

[31] See [Cardano 1968, page 168], [Cardano 1545, Chapter XXV, Note, page 48v]. The books on Euclid of which Cardano is speaking are the *Novæ Geometriæ*, see [Cardano 2004, page 58] and [MacLean 2007]. The *Novæ Geometriæ* was destroyed by Cardano himself in 1573.

$x^3 = a_1x + a_0$		$x^3 + a_0 = a_1x$	$x^3 + a_2x^2 = a_0$		$x^3 + a_0 = a_2x^2$
(AM XXV.1)	$\longleftrightarrow$	(AM XXV.9)	(AM XXV.14)	$\longleftrightarrow$	(AM XXV.15)
(AM XXV.2)	$\longleftrightarrow$	(AM XXV.17)	(AM XXV.18)		
(AM XXV.3)	$\longleftrightarrow$	(AM XXV.13)			
(AM XXV.4)	$\longleftrightarrow$	(AM XXV.11)			
(AM XXV.5)		(AM XXV.12)			
(AM XXV.6)	$\longleftrightarrow$	(AM XXV.10)			

TABLE 2.1 – Links between "particular" propositions in *Ars magna*, Chapter XXV.

Note that only (non-complete) equations where it can be $\Delta_3 < 0$ are involved.

Moreover, except for (AM XXV.6) and (AM XXV.10) which are instead directly linked respectively the first to (AM XXV.1) and (AM XXV.2), and the second to (AM XXV.9), they all can be drawn back either to (AM VIII.2 i) or to (AM VIII.2 ii), so that at the end they are all linked to each other. In some cases, I directly highlighted the connections that seemed to be the most natural.

From a modern viewpoint, none of the previous substitutions, which help to pass from a statement in Chapter XXV either to (AM VIII.2 i) or to (AM VIII.2 ii), or to another statement in the same Chapter XXV are real substitutions, since they are not invertible. But given a solution of the transformed system, one can always find at least one solution of the original system inverting in a naive way the substitution. The point is that we are interested in finding only one solution of a system (and not all the solutions), since it is enough to lower the degree of the equation. Anyway – as already observed, considering the systems in (AM VIII.2 i) or (AM VIII.2 ii) cannot help at all in finding the solutions of the corresponding equation, since finding the solution G of one of the system is perfectly equivalent (and as much difficult) to solve the corresponding cubic equation.

Let us now turn back to Cardano's (alleged) viewpoint. It is true that, even if (AM VIII.2) or one of the above equivalent propositions are not a step in finding a cubic formula that works at every turn, at least they move towards the direction of bypassing the problem set by the *casus irreducibilis*. In fact, once found no matter how a solution, one could lower the degree of the equation by polynomial division. By the way, this is why Cardano is probably not interested in listing all the positive, real solutions that he can deduce from the systems, but only one. Indeed, one solution is enough to lower the degree of the cubic equation. And once that one happens to fall down on a quadratic equation, he would no more occurs on imaginary quantities, since we are interested in the case $\Delta_3 < 0$, where we know that all the solutions are real and since quadratic equations do not bear the *casus irreducibilis*. The propositions (AM XXV.6) and (AM XXV.10) partially teach how to perform polynomial division.[32] Under this perspective, it seems natural to me to consider the other propositions as an aid to guess a solution. More precisely, if it is very hard to predict a solution as such looking only at the appearance of an equation, it could sometimes be more easy to guess f, g and from there to reconstruct a solution in the way taught by Cardano in his propositions. In this sense, the systems in the hypotheses of the above propositions provide assistance to intuition. Unluckily, a considerable amount of uncertainty still remains. The coefficients of the equation have to be simple enough, or one has to be smart enough, to guess one of its solutions. In short, the propositions in Chapter XXV do not provide a safe algorithmic way to bypass the *casus irreducibilis*. More work still need to be done in this direction. Moreover, there is another viewpoint from which the rules in Chapter XXV turn out to be useful. In fact, one may use them to inquire the shape of a solution of an equation of a certain family, choosing adapted coefficients and starting from their composition in terms of f, g, h.

Then, we can stretch our interpretation of "general" and "particular" rules as referred to "general" and "particular" methods (see above, at page 85) in order to include the above propositions in it as "particular". In fact, even if it is true that they work *a priori* for each equation (since in the end they are equivalent to solve a cubic equation),[33] in concrete terms they only work if one can find such f, g.

[32]Even if similar proposition for cubic equations lacking in the first degree term exist, Cardano does not enunciate them.

[33]Note that [Cossali 1799a] remarks that a "particular" rule is derived by a particular structure: "*oltre le generali regole per risolvere le equazioni di terzo grado insegna Cardano parecchie regole di scioglimenti speciali ricavati dai modi dalla struttura loro, dalla composizione dei termini ignoti, e dal termine noto, dai rapporti tra gli uni e l'altro*", see [Cossali 1799a, volume II, chapter II, Soluzioni speciali, page 166].

This means that the above propositions provide a solution of a cubic equation where $\Delta_3 < 0$ only if one already found another solution, which is intended to be discovered case-by-case, since the cubic formula leads to imaginary numbers. Saying it again, the above rules prove to be *a priori* "general", but in practice Cardano manages to apply them only in some cases, and thus they are "particular" in the sense of the opening quotation.

Under this interpretation, also (AM VIII.2) becomes "particular", contradicting the title of the chapter. Indeed, this is a real inconsistency in Cardano's text, and one can only explain it resorting to the changeableness of the terms 'general' and 'particular'.

We moreover observe that there are other links between the above propositions. In fact, there is a second access point to (AM XXV.1), (AM XXV.2), and (AM XXV.9) which is given by polynomial division in (AM XXV.6) and (AM XXV.10). There is also a second access point to (AM XXV.3), (AM XXV.4), (AM XXV.5), (AM XXV.11), and (AM XXV.13) which is given by the substitution $x = y + z$.

After the proposition (AM XXV.15), Cardano observes that "none of these rules can be [considered] general with respect to a solution" and that he will show this in one case for all.[34] This means that not only the above propositions are "particular" referred to the methods employed, but also referred to the solutions.

Cardano considers the rule given in (AM XXV.15), because – he says – this case is the more readily believable, since he has already obtained some results concerning the family of equations $x^3 + a_0 = a_2x^2$ (see, for instance, (AM I.8 ii), (AM VII.2-3), (AM VII.6-7), and (AM VIII.2)).

AM XXV.16 — *Proof***.** Cardano[35] wants to show that (AM XXV.15) is "particular" respect to solutions.

[34]See [CARDANO 1968, page 168] or "*Opræprecium fuerit nunc ostendere, quod hæ regulæ non possunt esse generales, respectu æstimationis, et modus in uno sufficiet ad ostendum in reliquis capitulis*", see [CARDANO 1545, Chapter XXV, paragraph 16, pages 48v-49r].
[35]See [CARDANO 1545, Chapter XXV, paragraph 16, page 48v-48r].

He draws[36] an equation of the family $x^3 + a_0 = a_2 x^2$, where a_0 is not integer and a_2 is integer, such that he knows a non-integer solution x. He uses this example to justify[37] the statement that, if a_2 is integer and x non-integer, then a_0 is not integer.

Afterwards, Cardano remarks that $a_0 = x^2(a_2 - x)$. We notice that this also comes from the hypothesis of (AM XXV.15). In fact, we get by the system that

$$a_0 = \left(-\tfrac{g}{2}\right)^2 \left(a_2 + \tfrac{g}{2}\right),$$

knowing by (AM VIII.2 ii) that $-\frac{g}{2}$ is a solution.

Then, Cardano assumes $a_2 = 7$ and moreover supposes x to be integer. He remarks that the integer part of the maximum of $x^2(a_2 - x)$ for $x \in [0, 7]$ is 50. Then, $a_0 = 6, 20, 36, 48, 50$. This means that there are only five equations of the family $x^3 + a_0 = a_2 x^2$, where $a_2 = 7$ and a_0 natural, the solution of which is $x \in [0, 7]$ integer.

Cardano ends saying that therefore the rule "is particular and, indeed, exceedingly particular".[38] □

Cardano's proof is riddled with oversights or sometimes with real mistakes. I did not manage to use this poof to help in clarifying Cardano's statements about the fact that the above propositions should be "particular" respect to a solution. By the way, [Cossali 1799a] is of the opinion that 'general' and 'particular' rules was quite a common terminology on which Cardano's contemporaries did not agree.[39]

2.1.5. Cardano's "royal road". In **Chapter VI** Cardano says that

[36]Namely, he considers the equation $x^3 + a_0 = 7x^2$, where a_0 is left at first undetermined. Then, he assumes that the solution goes under the form $x = x' + \sqrt{x''}$. Taking $x' = \frac{8}{3}$, he finds $x'' = 16$, which implies by substitution that $x = \frac{20}{3}$, and $a_0 = \frac{400}{27}$.

[37]Actually, this is true by Gauss's lemma (in this case, by the fact that each rational root of a monic polynomial with integer coefficients is integer).

[38]See [Cardano 1968, page 169] or "*particularis igitur est, ac valde etiam particularis*", see [Cardano 1545, Chapter XXV, paragraph 16, page 49r].

[39]"*Pare che l'idea delle accennate regole non fosse comunemente abbastanza chiara, e ben fissa, che per non distinguerne i diversi aspetti, chi errasse in voler loro negato ogni senso di generalità, e chi in attibuirlo alle medesime oltre dovere; e che Cardano mirato abbia a correggere e gli uni, e gli altri*", see [Cossali 1799a, volume II, chapter II, Soluzioni speciali, page 183].

> [w]hen, moreover, I understood that the rule that Niccolò Tartaglia handed to me had been discovered by him through a geometrical demonstration, I thought that this would be the royal road to pursue in all cases.[40]

Without interruption, Cardano puts forward three "highly useful propositions", as they were the prosecution of Tartaglia's geometrical "royal road". We will consider only the first one about the cube of a binomial, the others being variations on it.

AM VI.6. *If a quantity is divided into two parts, the cube of the whole is equal to the cubes of the two parts plus three times the products of each and the square of the other.*[41]

AM VI.6 — *Proof***.** [Let $\overline{AC}$ be divided in $\overline{AB}$ and $\overline{BC}$. Cardano wants to prove that $\left(\overline{AB}+\overline{BC}\right)^3 = \overline{AB}^3 + 3\overline{AB}^2\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3$.]

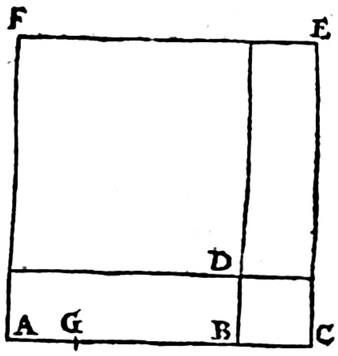


FIGURE 2.8 – *Ars magna* VI.6.

[40]See [CARDANO 1968, page 52] or "*[c]um autem intellexissem capitulum, quod Nicolaus Tartalea mihi tradiderat, ab eo fuisse demonstrationem inventum Geometrica, cogitavi eam viam esse regiam, ad omnia capitula venanda*", see [CARDANO 1545, Chapter VI, paragraph 5, page 16r].
[41]See [CARDANO 1968, page 52] or "*[s]i quantitas in duas partes dividatur, cubus totius aæqualis est, cubis ambarum partium, triploque productorum, uniuscuisque earum, vicissim in alterius quadratum*", see [CARDANO 1545, Chapter VI, paragraph 6, page 16r].

Cardano[42] draws the cube on AC, obtaining eight parts. He observes then that, by *Elements* II.4,[43] the following equalities hold

$$\overline{BC}^2 = \overline{(CD)}^2, \quad \overline{AB}^2 = \overline{(DF)}^2, \quad \text{and} \quad \overline{AB}\,\overline{BC} = \overline{(AD)} = \overline{(DE)}.$$

Then, since $\overline{AC}^2 = \overline{(CD)}^2 + \overline{(DF)}^2 + \overline{(AD)} + \overline{(DE)}$, it holds that

$$\overline{AC}^3 = \overline{AC}\,\overline{AC}^2 = \overline{AC}\left(\overline{(CD)}^2 + \overline{(DF)}^2 + \overline{(AD)} + \overline{(DE)}\right),$$

and, since $\overline{AC} = \overline{AB} + \overline{BC}$, it holds that

$$\overline{AC}^3 = \overline{AB}\left(\overline{(CD)}^2 + \overline{(DF)}^2 + \overline{(AD)} + \overline{(DE)}\right) + \overline{BC}\left(\overline{(CD)}^2 + \overline{(DF)}^2 + \overline{(AD)} + \overline{(DE)}\right).$$

But, since "the bases and heights being the same", by *Elements* I.43,[44] the following equalities hold

$$\overline{AB}\,\overline{(DF)}^2 = \overline{AB}^3, \quad \overline{BC}\,\overline{(CD)}^2 = \overline{BC}^3,$$
$$\overline{AB}\,\overline{(CD)}^2 = \overline{BC}\,\overline{(AD)} = \overline{BC}\,\overline{(DE)} = \overline{AB}\,\overline{BC}^2,$$
$$\overline{AB}\,\overline{(AD)} = \overline{AB}\,\overline{(DE)} = \overline{BC}\,\overline{(DF)}^2 = \overline{AB}^2\,\overline{BC}.$$

Then,

$$\overline{AC}^3 = \overline{AB}^3 + \overline{BC}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2.$$

□

The proof flows quite in an easy way, since it is mainly based on remarking some equalities between measures (of the parts of the big solid that Cardano supposes to draw). As usual, he provides a geometrical environment to interpret the problem. The diagram acts as fixation of reference for the objects involved, so that one can think about the quantities in the hypotheses as measures associated to some geometrical objects.The references being fixed by the diagram, Cardano is enabled to use all the known results which are stated in terms of geometrical

[42]See [Cardano 1545, Chapter VI, paragraph 6, pages 16r-16v].
[43]*Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangles contained by the segments", see [Heath 1956a, page 379].
[44]*Elements* I.43: "In any parallelogram the complement of the parallelograms about the diameter are equal to one another", see [Heath 1956a, page 340].

quantities. In particular, *Elements* II.4 and *Elements* I.43 are employed. Then, the amount and the kind (positional or not) of geometry in Cardano's proof depends in turn on the amount and the kind of geometry that one grants to those propositions in Euclid's *Elements*. In proofs of this kind, I have a penchant in bestowing an extremely narrow role to geometry, which means that I interpret the above proof mainly as an arithmetical calculation, once that the references are fixed by the diagram. In the end, if it is true that the relative position of the geometrical objects involved is for sure useful in identifying objects that have the same measure (and which are the same), but are called with different names, like $\overline{AB(DF)^2} = \overline{AB^3}$, it is in the end not essential.

We finally remark that this very common kind of diagram is recurring – probably the most exploited – in the *Ars magna*.

2.1.6. Summing up. We begin now to have a better overview of Cardano's knowledge on equations, in particular on cubic equations. The *Ars magna* is a treasury of miscellaneous observations, rules, methods, and special cases. In the main, the ordering of the material is not always easy flowing. My impression is that the arrangement of the chapters does not match the chronology of the discoveries such that we often find repetitions, summaries, and in general a non-linear course.

In this section I recall some of the methods at Cardano's disposal to handle with equations. According to the tradition, he knows how to solve linear, quadratic, and other simple equations such as $x^n = a_0$, with n natural. One of the most used method in the whole of Cardano's research on equations is the transformations of equations such as in Chapters I and VII. Outwardly, when Cardano wants to solve certain systems and finds more than one substitution that fits, he gets to different equations, which bear a relation to each other.

Cardano then shows a "general" way to deal with the cross-party family of equations $x^q + a_0 = a_p x^p$, with $q > p$. He is able to rewrite these equations under the form of certain systems. Anyway, finding the solutions of the systems is then equivalent to find the solutions of the equation (and as taught as it was for equations). This can nevertheless provide an aid to intuition to guess a solution when needed. In fact, this strategy is especially useful in the "particular" cases of

Chapter XXV, where the cubic formula contains imaginary numbers. All along this section, we have remarked Cardano's disposition towards the opposition "general" *versus* "particular". At best, Cardano's aim is to find "general" rules to provide "general" solutions for an equation. But, if this terminology is intended in our modern terms, it could be quite misleading. We will look then for a better understanding in the following. Moreover, there is a second sense of this opposition applied to methods instead to solutions. In this second sense, Cardano uses the term 'particular' to mean that a method cannot be applied in every case, where this impossibility is not only to be intended in a theoretical way, but also in practice.

Almost all the proofs follow a similar pattern. Basically, Cardano provides a geometrical environment where he interprets the problem. The diagrams act as fixation of reference. Then, Cardano can deal with the given quantities as if they were measures of certain geometrical quantities. In this way, he is enabled to apply the whole amount of known results that deal with geometrical quantities, such as the treasury of Euclid's *Elements*. Since the geometrical objects are considered only under their quantitative feature, their relative position becomes unimportant. This is a distinctive feature of Cardano's geometry. Then, the only constraint that one must respect in doing such a kind of geometry concerns the bonds established by the dimension. And it is exactly when Cardano needs to deal with dimensions higher than three that we can appreciate how much his geometry tends to mime arithmetic, and not the contrary.

2.2. Solving depressed cubic equations

In the middle of *Ars magna*, there is a compact block, which is composed by Chapters XI–XXIII. They are entirely devoted to the study of cubic equations. Cardano assigns one chapter to each one of the thirteen families of cubic equations (see above, at page 10).

All the chapters follow a similar pattern. They begin with a "proof [*demonstratio*]", from where is subsequently drawn a "rule [*regula*]" which is an algorithmic or step-by-step description of the operations that one has to perform on the coefficients of the given equation in order to get a solution. After that, we sometimes find an additional or alternative proof. Cardano always ends these chapters with a list of solved examples.

We will now deal with the so-called depressed cubic equations, that is with cubic equations lacking in the second degree term. In the following paragraphs (each one devoted to a family of cubic equations), I will rather start with the cubic formula followed by the proof and by my comments.

2.2.1. "On the cube and some things equal to a number". In **Chapter XI** Cardano deals with the equation $x^3 + a_1x = a_0$. He says that

> Scipio del Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult. My version of it follows.[45]

Cardano shows that:

AM XI. *Given the equation $x^3 + a_1x = a_0$, a solution is*

$$x = \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} + \frac{a_0}{2}} + \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} - \frac{a_0}{2}}.$$

We observe that in this case we always have $\Delta_3 > 0$. In fact, taking in the formula (1.5.10) at page 42 $\alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_1 = a_1 > 0$ and $\alpha_0 = -a_0 < 0$, we get $\Delta_3 = \left(\frac{-a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3 > 0$.

[45]See [Cardano 1968, page 98] or "*Scipio Ferreus Bononiensis iam annis ab hinc triginta ferme capitulum hoc invenit, tradidit vero Anthonio Mariæ Florido Veneto, qui cum in certamen cum NicolaoTartales Brixellense aliquando venisset, occasionem dedit, ut Nicolaus invenerit et ipse, qui cum nobis rogantibus tradidisset, suppressa demonstratione, freti hoc auxilio, demonstrationem quæsivimus, eamque in modos, quod difficillimum fuit, redactam sic subijciemus*", see [Cardano 1545, Chapter XI, page 29v].

AM XI (1545) — *Proof.* Let the equation [$x^3 + 6x = 20$ and $\overline{GH} = x$ be a solution, that is] $\overline{GH}^3 + 6\overline{GH} = 20$. Cardano[46] assumes that $\overline{BC} = \overline{CK}$ and
$$\begin{cases} 20 = \overline{AC}^3 - \overline{BC}^3 \\ \frac{6}{3} = \overline{AC}\,\overline{BC} \end{cases}.$$

He wants to show that $\overline{AB} = \overline{GH}$, that is, that the equation $\overline{AB}^3 + 6\overline{AB} = 20$ holds. [This means that $\overline{AB} = x$ is also a solution of $x^3 + 6x = 20$.]

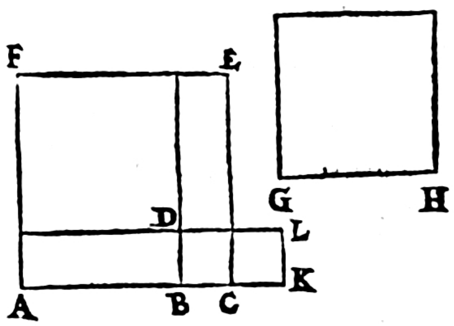


FIGURE 2.9 – *Ars magna* XI.

Cardano observes that the cubes $(DF)^3$, $(DC)^3$ respectively measure $\overline{AB}^3$, $\overline{BC}^3$ and that the parallelepipeds (AD), (DE) respectively measure $3\overline{AB}^2\,\overline{BC}$, $3\overline{AB}\,\overline{BC}^2$.

[Take $\overline{AB}$ to be a solution. We write $\overline{AB} = x$.] Since $\overline{AC}\,\overline{BC} = \frac{6}{3} = 2$, $3\overline{AC}\,\overline{BC} = 6$ follows, and then

$$(3\overline{AC}\,\overline{BC})\overline{AB} = 6x.$$

[Since $\overline{AC} = \overline{AB} + \overline{BC}$, I observe that we get $3\overline{AB}\,\overline{BC}^2 + 3\overline{AB}^2\,\overline{BC} = 6x$.] By $\overline{AC}^3 - \overline{BC}^3 = 20$, it follows that

$$\overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 = 20,$$

applying (AM VI.6) to $\overline{AC} = \overline{AB} + \overline{BC}$.

[46] See [CARDANO 1545, Chapter XI, pages 29v-30r].

Now, Cardano assumes that $\overline{BC} < 0$. He verifies in an analogous way that the last one of the above relations holds. Then, by[47] *Elements* I.35 and by[48] *Elements* XI.31, Cardano gets $\overline{AB} = \overline{GH}$. □

We observe that this kind of demonstration is opaque, meaning that the process of the discovery is not clearly explained. In fact, no hints are given on how Cardano had the idea to assume the system

$$\begin{cases} 20 = \overline{AC}^3 - \overline{BC}^3 \\ \frac{6}{3} = \overline{AC}\,\overline{BC} \end{cases},$$

which is the core of the proof. But the system reminds of that part of Tartaglia's poem about $x^3 + a_1 x = a_0$, which says that

Quando che'l cubo con le cose appresso
Se agguaglia à qualche numero discreto
Trovan dui altri differenti in esso.
Da poi terrai questo per consueto
Che 'l lor produtto sempre sia eguale
Al terzo cubo delle cose neto
El residuo poi suo generale
Delli lor lati cubi ben sottratti
Varra la tua cosa principale.

Indeed, we observe that Tartaglia and Cardano assume the same system up to changing variables. In fact, Tartaglia suggests to solve the system

$$\begin{cases} y - z = a_0 \\ yz = (\frac{a_1}{3})^3 \end{cases}$$

[47] *Elements* I.35: "Parallelograms which are on the same base and in the same parallels are equal to one another", see [HEATH 1956a, page 326].
[48] *Elements* XI.31: "Parallelepipedal solids which are on equal bases and of the same height are equal to one another", see [HEATH 1956c, page 337].

and to draw the solution of the equation from the difference of the sides of two cubes $x = \sqrt[3]{y} - \sqrt[3]{z}$. Taking $y = Y^3$ and $z = Z^3$, we get

$$\begin{cases} Y^3 - Z^3 = a_0 \\ Y^3 Z^3 = (\frac{a_1}{3})^3 \end{cases},$$

which is Cardano's system

$$\begin{cases} Y^3 - Z^3 = a_0 \\ YZ = \frac{a_1}{3} \end{cases}.$$

Let us now glance at the link between the cubic formula and the proof, that is to say, why the above proof shows that the cubic formula given is a solution of the equation $x^3 + a_1 x = a_0$. The link is not explicitly given, even if it is easy to recover it. If we assume $\overline{AC} = \overline{AB} + \overline{BC}$ and the system

$$\begin{cases} a_0 = \overline{AC}^3 - \overline{BC}^3 \\ \frac{a_1}{3} = \overline{AC}\,\overline{BC} \end{cases},$$

then we can draw $\overline{AC}, \overline{BC}$ depending on the given coefficients a_1, a_0. In order to solve the system, we have to solve the equation $\overline{BC}^6 + a_0 \overline{BC}^3 - \frac{a_1^3}{27} = 0$, or equivalently $y^2 + a_0 y - \frac{a_1^3}{27} = 0$ taking $y = \overline{BC}^3$. It is easy to verify that the solutions of the system are

$$\begin{cases} \overline{BC} = \sqrt[3]{-\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} \\ \overline{AC} = \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} \end{cases} \quad \text{and} \quad \begin{cases} \overline{BC} = \sqrt[3]{-\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} \\ \overline{AC} = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} \end{cases}.$$

We choose $x = \overline{AB}$ such that $\overline{AC} - \overline{BC} > 0$ and we get

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} - \sqrt[3]{-\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}}.$$

We remark – as we previously did in the preceding section – that the proof is based on fixing the reference of the given quantities by the means of the diagram within the geometrical environment. This enables Cardano to deal with the given quantities as if they were the measures of some geometrical objects, and thus to apply to them results concerning geometrical objects. In particular, note that Cardano explicitly employs (AM VI.6), the proof of which makes in turn reference to *Elements* II.4. Then, assessing whether Cardano's proof makes an essential use of geometry or not (and in which sense geometry is possibly used) comes back to what interpretation one gives of *Elements* II.4 (concerning either geometrical objects or numbers, which are the measures of these geometrical objects). The same is also true concerning *Elements* I.35 and XI.31, which never generated such a great debate as *Elements*, Book II. Moreover, there is another issue related to this topic. One can asks whether the proof contains the recipe for drawing the solution segment AB, or only the instructions to calculate its measure. In my opinion, we should place credit on the second option. In fact, one does not really need to pick up the rule and the compass to draw AB. It is enough to settle the problem in a language in which we have certain propositions that act as inference rules and let the machinery works. Moreover, it seems clear from the above explanation of the relations between the cubic formula and the proof that the way of thinking is arithmetically tackled. Since very similar arguments also hold in the following, I will omit from now on these remarks.

In 1570 and 1663 editions, this proof varies considerably. I will not present it here, as it is probably corrupted.[49] In the same editions, Cardano also explains how, taking two numbers to be a solution of $x^3 + a_1x = a_0$ and one of its coefficients, one can draw the equation, that is the remaining coefficient, using the fact that $a_0 = a_1x + x^3$ and $a_1 = \frac{a_0 - x^3}{x}$. Then, he starts from a depressed equation (like $x^3 + a_0 = a_1x$) of which he knows one solution, he takes this solution and one of its coefficients to be a solution and a coefficient of $x^3 + a_1x = a_0$, and draws the remaining coefficient. Finally, he says that "there may be a solution common to all types of cases [*æquatio hæc communis esse potest omnibus capitulis*]".

2.2.2. "On the cube equal to some things and a number". In **Chapter XII** Cardano shows that

AM XII. *Given the equation $x^3 = a_1x + a_0$, a solution is*

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}}.$$

[49]See Witmer's comment in [Cardano 1968, footnote 1, page 97].

AM XII — *Proof***.** Let the equation $x^3 = a_1x + a_0$. Cardano[50] assumes that

$$\begin{cases} a_0 = \overline{AB}^3 + \overline{BC}^3 \\ \frac{a_1}{3} = \overline{AB}\,\overline{BC} \end{cases}.$$

Then, he wants to show that $\overline{AC} = \overline{AB} + \overline{BC} = x$.

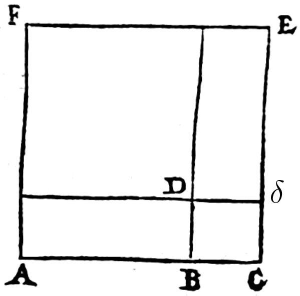


FIGURE 2.10 – *Ars magna* XII.

[Take $\overline{AC}$ to be a solution.] By $\overline{AB}\,\overline{BC} = \frac{a_1}{3}$, it follows that $3\overline{AB}\,\overline{BC} = a_1$, then

$$(3\overline{AB}\,\overline{BC})\overline{AC} = a_1\overline{AC}.$$

Cardano affirms that $(3\overline{AB}\,\overline{BC})\overline{AC} = 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2$. Note that he delays the proof of this statement. Then,

$$3\overline{AB}\,\overline{BC}^2 + 3\overline{AB}^2\,\overline{BC} = a_1\overline{AC}.$$

Therefore, by (AM VI.6),

$$3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{AB}^3 + \overline{BC}^3 = \overline{AC}^3$$

and we know by hypothesis that $a_0 = \overline{AB}^3 + \overline{BC}^3$. Then, the equation $\overline{AC}^3 = a_1\overline{AC} + a_0$ holds.

It remains to show that $(3\overline{AB}\,\overline{BC})\overline{AC} = 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2$. Cardano rather shows that $(\overline{AB}\,\overline{BC})\overline{AC} = \overline{AB}^2\,\overline{BC} + \overline{AB}\,\overline{BC}^2$.

[The multiplication between measures of segments – that is, between numbers – being associative, then] $(\overline{AB}\,\overline{BC})\overline{AC} = \overline{AB}(\overline{BC}\,\overline{AC})$. Since "all sides are equal to all sides" (namely, $\overline{AC} = \overline{CE}$, $\overline{BC}$ unchanged), it holds that

$$\overline{AB}(\overline{BC}\,\overline{AC}) = \overline{AB}(\overline{BC}\,\overline{CE}),$$

[50]See [CARDANO 1545, Chapter XII, page 31r].

that is

$$\overline{AB}(\overline{BC}\,\overline{AC}) = \overline{AB}(\overline{BC}(\overline{C}\,\overline{\delta} + \overline{\delta}\,\overline{E})) = \overline{AB}(\overline{BC}\,\overline{C}\,\overline{\delta} + \overline{BC}\,\overline{\delta}\,\overline{E}).$$

[The multiplication distributes over the sum] and again, since "all sides are equal to all sides" (namely, $\overline{C}\,\overline{\delta} = \overline{BC}$, $\overline{\delta}\,\overline{E} = \overline{AB}$), Cardano gets

$$\begin{aligned}\overline{AB}(\overline{BC}\,\overline{C}\,\overline{\delta} + \overline{BC}\,\overline{\delta}\,\overline{E}) &= \overline{AB}(\overline{BC}\,\overline{C}\,\overline{\delta}) + \overline{AB}(\overline{BC}\,\overline{\delta}\,\overline{E}) \\ &= \overline{AB}\,\overline{BC}^2 + \overline{AB}^2\,\overline{BC}.\end{aligned}$$

□

We observe that this proof follows the very same structure of the proof of (AM XI), with slightly variations. This is what we were expecting to, since we recall that in Chapter VI (see above, at page 70) Cardano said that the solution of $x^3 = a_1x + a_0$ is derived from the solution of $x^3 + a_1x = a_0$ in Chapter XI.

Anyway, there is a difference.In fact, Cardano additionally shows, as a sort of lemma, that

$$(3\overline{AB}\,\overline{BC})\overline{AC} = 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2,$$

whereas in the proof of (AM XI), where he also uses a similar consideration, he left this remark unsaid.

As before, no hints are given on how Cardano discovered the system in the hypotheses, which plays a central role. Again, it is the same as Tartaglia's one up to changing variables. In fact, Tartaglia suggests to solve the system

$$\begin{cases} y + z = a_0 \\ yz = (\frac{a_1}{3})^3 \end{cases}$$

and to draw the solution of the equation from the sum of two cube's sides $x = \sqrt[3]{y} + \sqrt[3]{z}$. Taking $y = Y^3$ and $z = Z^3$, we get

$$\begin{cases} Y^3 + Z^3 = a_0 \\ Y^3Z^3 = (\frac{a_1}{3})^3 \end{cases},$$

which implies Cardano's system

$$\begin{cases} Y^3 + Z^3 = a_0 \\ YZ = \frac{a_1}{3} \end{cases}.$$

As before, let us glance at the link between the cubic formula and proof. If we assume

$$\begin{cases} a_0 = \overline{AB}^3 + \overline{BC}^3 \\ \frac{a_1}{3} = \overline{AB}\,\overline{BC} \end{cases},$$

then we can draw $\overline{AB}, \overline{BC}$, and then $\overline{AC}$, depending on the given coefficients a_1, a_0. It is easy to verify that the solutions of the system are

$$\begin{cases} \overline{BC} = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} \\ \overline{AB} = \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} \end{cases} \quad \text{and} \quad \begin{cases} \overline{BC} = \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} \\ \overline{AB} = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} \end{cases}.$$

The resulting expression $\overline{AC} = \overline{AB} + \overline{BC}$ is symmetric, thus we get

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}}.$$

At the end of this chapter, Cardano makes a fundamental remark. He observes that, in order to apply this rule, the following condition has to be satisfied

> [w]hen the cube of one-third the coefficient of x is greater than the square of one-half the constant of the equation, which happens whenever the constant is less than three-fourths of this cube or when two-thirds of the coefficient of x multiplied by the square root of one-third the same number is greater than the constant of the equation,

that is

$$\left(\frac{a_0}{2}\right)^2 \geq \left(\frac{a_1}{3}\right)^3,$$

which[51] means that $\Delta_3 \geq 0$. If this condition does not hold (that is, if $\Delta_3 < 0$), we are in the *casus irreducibilis*, where a cubic equation has three distinct real

[51]See here 1.5.3, page 40.

solutions, but the cubic formula yields imaginary numbers.[52] This is why Cardano leaves this case out.

Then – he says[53] in 1545 edition – one should better refer to the "Aliza problem [*quæstio Aliza*]" or to Chapter XXV of the *Ars magna.* We remind that Chapter XXV deals with "particular" rules and solutions. There, Cardano explicitly states that the foregoing cases are "general" and in particular this is also referred to Chapter XII. But the fact that Chapter XII is "general" conflicts with Cardano's characterisation of "general" rules given in Chapter XXV, since indeed not every given problem (namely, the ones with $\Delta_3 < 0$) can be solved – for Cardano and his contemporaries – by the method in Chapter XII. Apart from this, what is more important is that Cardano's strategy[54] in Chapter XXV fails to provide a method safe enough to enable one to avoid imaginary numbers at all turns. At best, he manages to avoid to use the cubic formula, but this only happens in some lucky cases.

Later on, in the 1570 (and 1663) editions of the *Ars magna*, Cardano makes reference[55] to the *De regula aliza* alone, which was firstly published together with the *Ars magna* in 1570. I have already extensively dealt with this quotation from the *Aliza* in Section 1.3, at page 26. I only recall that this quotation is a key point for my interpretation, since it explicitly links the *Aliza* with the *casus irreducibilis.*

2.2.3. "On the cube and a number equal to some things". In **Chapter XIII** Cardano deals with the equation $x^3 + a_0 = a_1x$. First of all, he says that its solving method "derives from the preceding ones [*ex præcedenti trahitur*]". We

[52]See here Section 1.5.4, page 43.

[53]"*At ut cubis tertiæ partis numeris rerum, excedat quadratum dimidij numeri, æquationis, quod accedit quandocunque numerus æquationis est minor $\frac{3}{4}$ cubi illius, vel ubi ex $\frac{2}{3}$ numerum rerum, producitur in R $\frac{1}{2}$ eiusdem numeri maior numerus numero æquationis, tunc hoc dissolvitur per quæstionem Alizam, de qua in libro de quæstionibus Geometricis dictum est, sed si libet tantam effugere difficultatem, plerumque capitulu 25m huius tibi satisfaciet*", see [Cardano 1545, Chapter XII, page 31v].

[54]See here Section 2.1.4, page 96.

[55]"*At ut cubis tertiæ partis numeris rerum, excedat quadratum dimidij numeri, æquationis, quod accedit quandocunque numerus æquationis est minor $\frac{3}{4}$ cubi illius, vel ubi ex $\frac{2}{3}$ numerum rerum, producitur in R $\frac{1}{2}$ eiusdem numeri maior numerus numero æquationis, tunc consules librum Alizæ hîc adjectum*", see [Cardano 1663b, Chapter XII, page 251].

remind that in (AM I.5-6) Cardano explained that $x^3+a_0=a_1x$ and $x^3=a_1x+a_0$ have the same solutions, but that they are opposite in sign.

Then, Cardano states that

AM XIII. *Given*[56] *the equation* $x^3+a_0=a_1x$*, a solution is*

$$x=\frac{y}{2}\pm\sqrt{a_1-3\left(\frac{y}{2}\right)^2},$$

where y *is a solution of* $y^3=a_1y+a_0$*.*

Nothing guarantees that the above formula gives a real solution, except if we ask that $a_1-3\left(\frac{y}{2}\right)^2>0$. Moreover, despite the one example that Cardano gives (which is $x^3+3=8x$, the solutions of which are $-3, \frac{3}{2}-\frac{\sqrt{5}}{2}, \frac{3}{2}+\frac{\sqrt{5}}{2}$), nothing either guarantees that both the solutions in the formula are positive. And the last carelessness is that the equation has two (distinct) solutions only when $\Delta_3<0$, but it is not always the case.

This makes me think that the idea of the above formula is originated in a completely different manner of (AM XI) or (AM XIII). As always, no hint is given on how Cardano discovered this formula. But the striking similarity with the formula in (AM XXV.1) at page 85 (or in (AM VIII.2 i) at page 77) stands out, even though these formulae concern the family $y^3=a_1y+a_0$. Indeed, in the *Ars magna*, Cardano has available a path from the formula in (AM XXV.1) to the formula in (AM XIII), even though we have no evidence that he was aware of it. We assume that we are in the case $\Delta_3<0$ for $x^3+a_0=a_1x$. We consider $y^3=a_1y+a_0$, which accordingly also has $\Delta_3<0$. Suppose that one manages to find f,g positive such as in (AM XXV.1). Then, $y=\sqrt{f+\frac{g}{4}}+\frac{\sqrt{g}}{2}$, or $y=\sqrt{a_1-\frac{3}{4}g}+\frac{\sqrt{g}}{2}$ taking $f=a_1-g$, is a (positive) solution of $y^3=a_1y+a_0$. By (AM I.5-6), one now knows that $x=-y=-\sqrt{a_1-\frac{3}{4}g}-\frac{\sqrt{g}}{2}$ is a (negative) solution of $x^3+a_0=a_1x$. For the moment we keep it aside. Again, having found such f and g, $x=\sqrt{g}$ is a (positive) solution of $x^3+a_0=a_1x$ by (AM VIII.2). Then, $y=-x=-\sqrt{g}$ is a (negative) solution of $x^3+a_0=a_1x$. Substituting this

[56] Expressed as before, the cubic formula is

$$x=\sqrt[3]{-\frac{a_0}{2}+\sqrt{\left(\frac{a_0}{2}\right)^2-\left(\frac{a_1}{3}\right)^3}}+\sqrt[3]{-\frac{a_0}{2}-\sqrt{\left(\frac{a_0}{2}\right)^2-\left(\frac{a_1}{3}\right)^3}}.$$

Cardano does not provide the formula under this shape.

value in $x = -\sqrt{a_1 - \frac{3}{4}g} - \frac{\sqrt{g}}{2}$, we have one of the solutions in (AM XIII). Finally, we also know by (AM I.5-6) how to draw the third solution depending on the first two for each $x^3 + a_0 = a_1 x$ and $y^3 = a_1 y + a_0$, from where by a similar reasoning we get to the other solution in (AM XIII). Clearly, this is only my supposition. But we observe that, even though we need to assume that $\Delta_3 < 0$, in order to have the three real solutions, it is automatically gained that $a_1 - 3\left(\frac{y}{2}\right)^2 > 0$, since we took f, g positive such that $a_1 = f + g$ and $y = -\sqrt{g}$.

Anyway, even if my hypothesis hold, no mark is left in Cardano's proof.

AM XIII — *Proof***.** Let the equation $\overline{HK}^3 + F = \overline{A\beta}^2\,\overline{HK}$, where F is a [real] number.

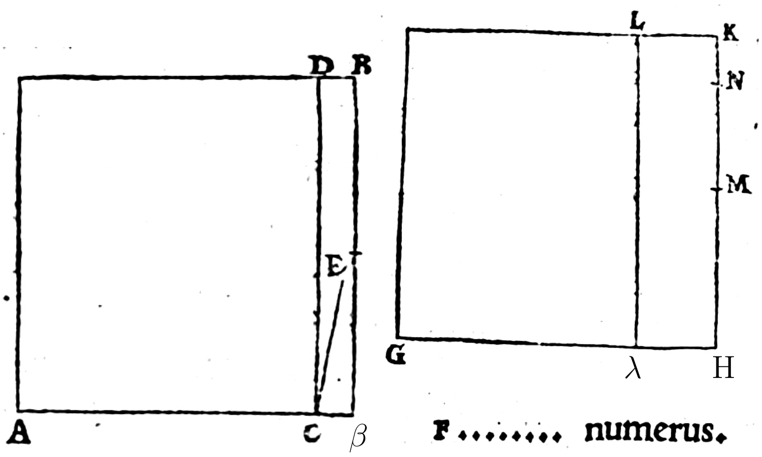


FIGURE 2.11 – *Ars magna* XIII.

Cardano[57] assumes that

$$\overline{HK}^2 = 4\overline{LK}\,\overline{HK}, \quad \overline{HK} = 2\overline{MK},$$

$$\overline{HK}^2 - (\overline{LK}\,\overline{HK}) = \overline{G\lambda}\,\overline{\lambda L} = \overline{AC}\,\overline{CD},$$

$$\overline{MN} = \overline{CE}, \quad \sqrt{\overline{C\beta}\,\overline{\beta B}} = \overline{CE}.$$

Then, he wants to show that either $\overline{HN}$ is a solution of $x^3+F = \overline{A\beta}^2 x$, that is that the equation $\overline{HN}^3 + F = \overline{A\beta}^2\,\overline{HN}$ holds, or $\overline{NK}$ is a solution of $x^3 + F = \overline{A\beta}^2 x$,

[57]See [CARDANO 1545, Chapter XIII, pages 31v-32r].

that is that the equation $\overline{NK}^3 + F = \overline{A\beta}^2\,\overline{NK}$ holds. [We will write in the first case $\overline{HN} = x$ and in the second case $\overline{NK} = x$.]

Cardano is looking for a writing of $\overline{A\beta}^2$ depending on the segments in the square on the right in the diagram

$$\begin{aligned}\overline{A\beta}^2\,[&= (\overline{AC}\,\overline{CD}) + (\overline{C\beta}\,\overline{\beta B})\\ &= \overline{HK}^2 - (\overline{LK}\,\overline{HK}) + \overline{CE}^2\\ &= \frac{3}{4}\overline{HK}^2 + \overline{MN}^2]\\ &= 3\overline{MK}^2 + \overline{MN}^2.\end{aligned}$$

In the first case with respect to $\overline{HN}$, he forces $\overline{HN}$ to appear in the expression for $\overline{A\beta}^2$

$$\begin{aligned}\overline{A\beta}^2 - \overline{HN}^2\,[&= 3\overline{MK}^2 + \overline{MN}^2 - \overline{HN}^2\\ &= 2\overline{MK}(\overline{MK} - \overline{MN})\\ &= 2\overline{MK}\,\overline{NK}]\\ &= \overline{HK}\,\overline{NK}.\end{aligned}$$

In this way, he is able to rewrite $\overline{HN}^3 = \overline{HN}\,\overline{HN}^2 = \overline{HN}(\overline{A\beta}^2 - \overline{HK}\,\overline{NK})$. Moreover, $\overline{A\beta}^2 = \overline{HN}^2 + \overline{HK}\,\overline{NK}$. Since the equation $\overline{HK}^3 + F = \overline{A\beta}^2\,\overline{HK}$ holds by assumption, Cardano looks for a writing of F depending on the segments in the second diagram, in particular depending also on $\overline{HN}$

$$\begin{aligned}F &= \overline{HK}(\overline{HK}^2 - \overline{A\beta}^2)\\ &= \overline{HK}((\overline{G\lambda}\,\overline{\lambda L} + \overline{LK}\,\overline{HK}) - (\overline{AC}\,\overline{CD} + \overline{C\beta}\,\overline{\beta B}))\\ &= \overline{HK}(\overline{LK}\,\overline{HK} - \overline{C\beta}\,\overline{\beta B})\\ &= \overline{HK}(\overline{MK}^2 - \overline{MN}^2)\\ &= \overline{HK}(2\overline{MN}\,\overline{NK} + \overline{NK}^2\\ &= \overline{HK}\,\overline{NK}\,\overline{HN}.\end{aligned}$$

Then,

$$\begin{aligned}\overline{HN}^3 + F &= \overline{HN}(\overline{A\beta}^2 - \overline{HK}\,\overline{NK}) + \overline{HK}\,\overline{NK}\,\overline{HN}\\ &= \overline{A\beta}^2\,\overline{HN},\end{aligned}$$

that is the equation $\overline{HN}^3 + F = \overline{A\beta}^2\,\overline{HN}$ holds.

In the second case with respect to $\overline{NK}$, Cardano shows in a very analogous way that, under his hypothesis, the equation $\overline{NK}^3 + F = \overline{A\beta}^2\,\overline{NK}$ holds. □

Let us glance at the link between the solution formula and the proof, that is to say why the above proof shows that the formula $x = \frac{y}{2} \pm \sqrt{a_1 - 3\left(\frac{y}{2}\right)^2}$, with y solution of $y^3 = a_1 y + a_0$, is a solution of the equation $x^3 + a_0 = a_1 x$. Cardano affirmed that this rule derives from (AM XII). In fact, he needs to know how to solve $y^3 = a_1 y + a_0$ in order to solve $x^3 + a_0 = a_1 x$. By (AM I.5-6), Cardano knows that, if there are two positive solutions x_1 and x_2 of $x^3 + a_0 = a_1 x$ (namely, in the case $\Delta_3 < 0$), then $y_3 = x_1 + x_2$ is a solution of $y^3 = a_1 y + a_0$. Cardano himself makes this remark.[58] We take $y = \overline{HK}$. Then, the solutions that Cardano is looking for are $x_1 = \overline{HN} = \overline{HM} + \overline{MN} = \overline{MK} + \overline{MN}$ and $x_2 = \overline{NK} = \overline{MK} - \overline{MN}$, that is

$$x_1 = \frac{1}{2}\overline{HK} + \overline{MN} \quad \text{and} \quad x_2 = \frac{1}{2}\overline{HK} - \overline{MN},$$

where $\overline{MN} = \sqrt{\overline{A\beta}^2 + 3\overline{MK}^2} = \sqrt{\overline{A\beta}^2 + 3\left(\frac{\overline{HK}^2}{2}\right)^2}$. Since $\overline{A\beta}^2 = a_1$ and $\overline{HK} = y$, we find the formula we were looking for.

Note nevertheless that, in order to use the above formula for $x^3 + a_0 = a_1 x$, one should be able to find a solution of $y^3 = a_1 y + a_0$, which is not always the case because of the *casus irreducibilis.*

Afterwards, Cardano adds another proposition.

[58]"From this demonstration it is evident that the solution for the cube equal to the first power and number is equal to the sum of the solutions for the cube plus the same number equal to the first power with the same coefficient", see [CARDANO 1968, page 106] or "*[e]x hac demonstratione patet, quod æquatio cubi æqualis rebus et numero, æqualis est ambabus æquationis cubi et eiusdem numeri æqualium totidem rebus simul iunctis*", see [CARDANO 1545, Chapter XIII, page 32r]. Note that in 1570 and 1663 editions this remark is shifted at the end of the chapter.

AM XIII bis. *Let us consider the equation $x^3 + a_0 = a_1 x$ and assume x_1 to be a solution. Then,*

$$x_2 = \sqrt{a_1 - 3\left(\frac{x_1}{2}\right)^2} - \frac{x_1}{2}$$

is another solution.

We remark that again one needs to ask that $\Delta_3 < 0$ to guarantee the existence of the second real solution. Once that one agrees with my reasoning in (AM XIII) at page 113 and, as a consequence, he acknowledges that Cardano knows that, under certain conditions, two of the solutions of $x^3 + a_0 = a_1 x$ can be written as $x_1 = \sqrt{g}$ and $x_3 = -\sqrt{a_1 - \frac{3}{4}g} - \frac{\sqrt{g}}{2}$, then the above formula is gathered by the opposite of the sum of these two solutions according to (AM I.5-6).

I will skip the proof in details, just sketching without calculations the idea behind it. At first sight, the proof is not so easy-reading, but it comes true that deep down it is very similar to (AM XVI bis).

AM XIII bis (idea) — *Proof***.** Cardano[59] takes $x_1 = \overline{AD}$, $a_1 - \overline{AD}^2 = \overline{AB}^2$, and $x_2 = \overline{FH}[= \sqrt{g}]$.

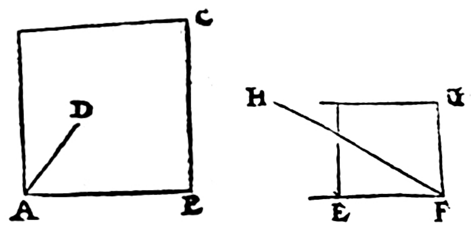


FIGURE 2.12 – *Ars magna* XIII bis.

He wants to show that such a $\overline{FH}$ is a solution of $x^3 + a_0 = a_1 x$.

He knows that it exists $\overline{EF}$ such that $\overline{EF}^2 = a_1 - \overline{FH}[= f]$. Now, it remains to show that $\overline{EF}^2\,\overline{FH} = a_0$, in order to infer by (AM VIII.2) that $\overline{FH}$ is a solution. For that, Cardano considers $\overline{AD}$ which is a solution of $x^3 + a_0 = a_1 x$ by hypothesis. Then, $a_0 = a_1\overline{AD} - \overline{AD}^3 = (a_1 - \overline{AD}^2)\overline{AD} = \overline{AB}^2\,\overline{AD}$. He finally shows that $\overline{EF}^2\,\overline{FH} = \overline{AB}^2\,\overline{AD}$. This ends the proof. □

[59]See [CARDANO 1545, Chapter XIII, page 32v].

We observe that this proposition provides a formula to find a second solution of $x^3 + a_0 = a_1 x$, which bears no conditions to be satisfied. Anyway, one still has to be able to guess no matter how a first solution of the same equation. This could help in bypassing the *casus irreducibilis* at least for the second solution, but does not help at all in finding the first solution.

In 1570 and 1663 editions, three corollaries are added. In the first, Cardano recalls that the (positive) solution of $x^3 = a_1 x + a_0$ is given by the sum of the (positive) solutions of $x^3 + a_0 = a_1$, which is a way to restate (AM I.5-6). The second corollary simply reminds (AM XIII bis). Finally, the third corollary is about the comparison of the shapes of the (positive) solutions of $x^3 = a_1 x + a_0$ and $x^3 + a_0 = a_1$. As usual trough an example, Cardano shows that, if the (positive) solution of $x^3 = a_1 x + a_0$ is a *binomium* (of the 2^{nd} or 5^{th} kind), then (one of the positive) solutions of $x^3 + a_0 = a_1$ is its *recisum*.

2.2.4. Summing up. Cardano gives in (AM XI) and (AM XII) two cubic formulae, respectively for $x^3 + a_1 x = a_0$ and $x^3 = a_1 x + a_0$. The proofs are independent but strictly related. They are essentially what one expects. In fact, Cardano simply verifies that a certain segment, when substituted in a certain equation, gives a true equality. In order to do that, he has to write down an expression for the segment in term of the coefficients of the equation. Thanks to his interpretation of the hypotheses in a geometrical environment, it is easy for him to do that. The trick behind these proofs is to write $x = y \pm z$. Then, Cardano basically uses the binomial formula in the case of the third power and the properties of the operations on the measures of segments. As previously argued, my opinion is that Cardano's proofs do not deliver the instruction to draw the solution segment, but rather to calculate its measure.

We soon run into the major inconvenient of Cardano's formulae, the so-called *casus irreducibilis*. Except for the cubic formula in (AM XI), Cardano has to check not to have negative quantities under the square roots and this is made by forcing a condition on the coefficients. This means that, under certain circumstances, Cardano is not able to draw a solution from the formula for that family of cubic equations. Since the formula in (AM XIII) refers back to the one in (AM XII), it also has to meet the same condition and then it bears the same problem.

Concerning the equation $x^3 + a_0 = a_1 x$, Cardano also finds a way to get another solution, thus bypassing the cubic formula in (AM XII). It is interesting to remark how this alternative formula could have been originated by "particular" arguments such as the ones in Chapter VII and XXV and by the transformations in Chapter I.

2.3. Solving cubic equation lacking in the first degree term

2.3.1. "On the cube equal to some squares and a number". In **Chapter XIV** Cardano deals with the equation $x^3 = a_2 x^2 + a_0$. He says that

> [i]f the cube is equal to the square and constant, the equation can be changed into one of the cube equal to the first power and constant by the first method of conversion, which is from the whole to the part. (The second method is from the part to the whole, the third by the difference between parts, and the fourth by proportion).[60]

We recall[61] that in Chapter VI Cardano spoke of a method of "similitude" to solve new cases, the third part of which was "the conversion of equations into [others] equivalent in nature". He now better specifies what this "method of conversion" is, saying that it is divided in parts.

AM XIV. *Given the equation* $x^3 = a_2 x^2 + a_0$, *a solution is*

$$x = \sqrt[3]{\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2} + \sqrt{\left(\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \sqrt[3]{\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2} - \sqrt{\left(\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \frac{a_2}{3}.$$

We observe that, in this case, we always have $\Delta_3 > 0$. In fact, taking in the formula (1.5.10) at page 42 $\alpha_3 = 1$, $\alpha_2 = -a_2 < 0$, $\alpha_1 = 0$, and $\alpha_0 = -a_0 < 0$,

[60] See [CARDANO 1968, page 110] or "*[q]uod, si cubus æqualis sit quadratis et numero, convertetur capitulum in cubum æqualem rebus et numero, primo conversionis modo, qui est a toto ad partem, nam secundus est a parte ad totum, tertius a differentia partium, quartus a proportione*", see [CARDANO 1545, Chapter XIV, page 33r].

[61] See here, page 70.

we get $\Delta_3 = \frac{-a_0}{108}(-27a_0 - 4a_2^3) > 0$. This fact also appears in the above formula, since $\left(\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6 = \frac{a_0^2}{4} + a_0\frac{a_2^3}{27} > 0$.

AM XIV — *Proof***.** Let the equation $\overline{AC}^3 = 6\overline{AC}^2 + 100$. [From now on, the diagrams appear no more in the proofs. This is because Cardano[62] makes reference to the diagram in Chapter XII, see the figure 2.10 at page 109.]

[Take $\overline{BC} = \frac{6}{3} = 2$. Then, Cardano shows that the equation $\overline{AC}^3 = 6\overline{AC}^2 + 100$ can be changed into $\overline{AB}^3 = 12\overline{AB} + 116$ *via* the substitution $\overline{AC} = \overline{AB} + 2$.]

[By *Elements* II.4,] Cardano knows that $\overline{AC}^2 = \overline{AB}^2 + 2\overline{AB}\,\overline{BC} + \overline{BC}^2 = \overline{AB}^2 + 4\overline{AB} + 4$. Then, by the assumed equation,

$$\begin{aligned}\overline{AC}^3 &= 6\overline{AC}^2 + 100\\ &= 6\overline{AB}^2 + 24\overline{AB} + 24 + 100.\end{aligned}$$

On the other way, by (AM VI.6) Cardano gets

$$\begin{aligned}\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 &= \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3\\ &= \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.\end{aligned}$$

Then, by comparison,

$$\begin{aligned}6\overline{AB}^2 + 24\overline{AB} + 24 + 100 &= \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8,\\ \overline{AB}^3 &= 12\overline{AB} + 116.\end{aligned}$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XII). The relation $\overline{AC} = \overline{AB} + \overline{BC}$ yields the solution of the equation in $\overline{AC}$. $\square$

Let us glance at the link between the cubic formula and the proof. Call $\overline{AC} = x$ and $\overline{AB} = y$. We observe that, substituting $x = y + \frac{a_2}{3}$ in $x^3 = a_2x^2 + a_0$, we get the following transformation

$$x^3 = a_2x^2 + a_0 \xrightarrow{x=y+\frac{a_2}{3}} y^3 = \frac{a_2^2}{3}y + (\frac{2}{27}a_2^3 + a_0),$$

which is basically what Cardano performs in the proof. Then, if we apply the cubic formula in (AM XII) to the transformed equation $y^3 = \frac{a_2^2}{3}y + \frac{2}{27}a_2^3 + a_0$, we

[62]See [CARDANO 1545, Chapter XIV, pages 33r-33v].

obtain the cubic formula of this chapter. We remark that this transformation is new compared to the ones in Chapters VI and VII for $x^3 = a_2x^2 + a_0$.

If we already knew how the following chapters deal with the remaining families of equations, we would realise that the common strategy is not to provide the formulae, but rather the transformations to get the formulae. Then, we can easily guess the transformation used here and restate the proposition as follows.

AM XIV restated. *Given the equation $x^3 = a_2x^2 + a_0$, a solution is*

$$x = y + \frac{a_2}{3},$$

where y is a solution of $y^3 = \frac{a_2^2}{3}y + \frac{2}{27}a_2^3 + a_0$.

Recall that in Proposition (AM IV.2) (see above, at page 83) Cardano displays the above substitution, even if it is not directly referred to this equation. Then, my guess is not that imaginative (moreover, see below, Proposition (AMA XXI.9) at page 185 and the figure 3.7 at page 222).

Note that, despite the fact that the equation in x always has $\Delta_3 > 0$, this is not *a priori* the case for the transformed equation in y. Anyway, if we go and check the discriminant of the transformed equation, we get

$$\Delta_3 = \frac{a_0^2}{4} + \frac{1}{27}a_0a_2^3,$$

which is always positive, since we took a_0, a_2 positive.

2.3.2. "On the cube and some squares equal to a number". In **Chapter XV** Cardano deals with the equation $x^3 + a_2x^2 = a_0$.

AM XV. *Given*[63] *the equation $x^3 + a_2x^2 = a_0$, a solution is*

$$x = y - \frac{a_2}{3},$$

[63] Expressed as before, the cubic formula is

$$x = \sqrt[3]{\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2} + \sqrt{\left(\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \sqrt[3]{\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2} - \sqrt{\left(\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} - \frac{a_2}{3}.$$

Cardano does not provide the formula under this shape.

where y satisfies the following conditions

i. *if $a_0' > 0$, then $y^3 + a_0' = a_1' y$,*

ii. *if $a_0' < 0$, then $y^3 = a_1' y + (-a_0)$,*

iii. *if $a_0' = 0$, then $y^3 = a_1' y$,*

with $a_1' = 3\left(\frac{a_2}{3}\right)^2 [= \frac{a_2^2}{3}]$ and $a_0' = 2\left(\frac{a_2}{3}\right)^3 - a_0$.

We remark that the number of cases increases, since Cardano always wants positive coefficients in the transformed equations. Let us consider for instance this case. If we perform the transformation $x = y - \frac{a_2}{3}$, we get to the equation

$$y^3 - \frac{a_2^2}{3}y + \frac{2}{27}a_2^3 - a_0 = 0,$$

where the sign of $\frac{2}{27}a_2^3 - a_0$ varies depending on the coefficients of the equation in x. Then, Cardano needs to distinguish the cases in order to be enabled to write explicitly the transformed equations.

Note moreover that there is only a loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). In fact, we have $p = -a_1' \leq 0$ and $q = a_0'$, which implies $q^2 \geq 0$. Anyway, this is not enough to completely determine the sign of Δ_3. We can only say that $\Delta_3 > 0$ in the first case $a_0' > 0$ and that $\Delta_3 \geq 0$ in the third case $a_0' = 0$.

Cardano firstly remarks that "[i]n this case we convert by the second method. The difference is that the first method showed the addition of one-third the coefficient of x^2, while the second shows its subtraction".[64] The "second method" of which Cardano is speaking is "from the part to the whole", accordingly to the corresponding remark in Chapter XIV.

Cardano gives a proof only for the case $2\left(\frac{a_2}{3}\right)^3 < a_0$, that is $a_0' < 0$.

AM XV — *Proof***.** Let the equation $\overline{AB}^3 + 6\overline{AB}^2 = 100$. [Cardano[65] considers the same diagram as in Chapter XII, see above, the figure 2.10 at page 109.] He takes $\overline{BC} = \frac{6}{3} = 2$.

[64]See [Cardano 1968, page 113] or "*[h]oc capitulum convertitur secundo modo, differentia autem est, quod primus modus ostendit addendam tertiam partem numeri quadratorum, et secundus minuendam*", see [Cardano 1545, Chapter XV, page 33v].

[65]See [Cardano 1545, Chapter XV, pages 33v-33r].

[Then, Cardano shows that the equation $\overline{AB}^3 + 6\overline{AB}^2 = 100$ can be changed into $\overline{AC}^3 = 12\overline{AC} + 84$ *via* the substitution $\overline{AB} = \overline{AC} - 2$.]

By (AM VI.6) and by $\overline{AB}^3 + 6\overline{AB}^2 = 100$, Cardano gets

$$\begin{aligned}\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 &= \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3 \\ &= \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + \overline{BC}^3 \\ &= 100 + 12\overline{AB} + \overline{BC}^3 \\ [&= 100 + 12(\overline{AC} - \overline{BC}) + \overline{BC}^3].\end{aligned}$$

Since $\overline{BC} = 2$, we get $\overline{AC}^3 = 12\overline{AC} + 84$.

Now, Cardano knows how to solve the transformed equation in $\overline{AC}$ by (AM XII). The relation $\overline{AB} = \overline{AC} - \overline{BC}$ yields the solution of the equation in AB. □

In 1545 edition, Cardano adds a corollary at the end of the proof. Since it is about the solving method in Chapter XVI, we will comment on it in the next chapter.

Afterwards, Cardano gives another solution method for this equation. He says that it is due to Ludovico Ferrari.

AM XV bis. *Given the equation $x^3 + a_2x^2 = a_0$, a solution is*

$$x = y^2 - a_2,$$

with y solution of $y^3 = a_2y + \sqrt{a_0}$.

We remark that this is the same substitution in (AM VII.6-7).

We observe that the main advantage of this method is that Cardano and Ferrari do no more need to explicitly distinguish the cases in the cubic formula. In fact, performing the substitution, we get

$$y^6 - 2a_2y^4 + a_2y^2 = a_0,$$

that is

$$(y^3 - a_2y)^2 = a_0,$$

where the coefficients are as usual all positive. Then, solving $x^3 + a_2x^2 = a_0$ depends only on solving $y^3 = a_2y + \sqrt{a_0}$ in (AM XII). Note that, considered the cases in (AM XV), it sounds very reasonable that the equation $x^3 + a_2x^2 = a_0$ is brought back only to one case. In fact, the transformed equation in y in the second case in (AM XV) has $\Delta_3 \lesseqgtr 0$, so that no restriction is put on the discriminant of the equation in x.

We moreover observe that this is no alternative method to (AM XV). In fact, one can show with calculations that both (AM XV) and (AM XV bis) lead to the same cubic formula (see footnote, 63 at page 121).

AM XV bis — *Proof.* Cardano and Ferrari[66] consider the following equation $\overline{AB}^3 + \overline{BD}\,\overline{AB}^2 = 100$, where $\overline{BD} = 6$.

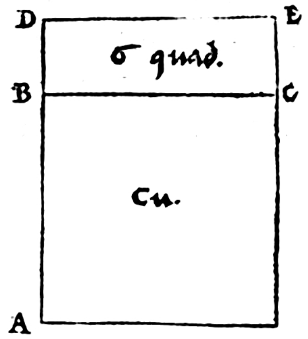


FIGURE 2.13 – *Ars magna* XV [complete the diagram drawing the parallelogram of height $\overline{AB}$ over the basis formed by the parallelepiped on AD, AB].

[They want to show that this equation can be changed into $\overline{AD}^3 = 6\overline{AD} + 10$, where $\overline{AD}$ is a square quantity. Then $\overline{AB} = \overline{AD} - \overline{BD}$.]

They assume $\overline{AD} = y^2$.

Then, $\overline{AB} = \overline{AD} - \overline{DB} = y^2 - 6$ and $\overline{AB}^2 = y^4 - 12y^2 + 36$.

Therefore, $\overline{AD}\,\overline{AB}^2 = y^6 - 12y^4 + 36y^2$. But $\overline{AD}\,\overline{AB}^2 [= (\overline{AB} + \overline{BD})\overline{AB}^2] = \overline{AB}^3 + \overline{BD}\,\overline{AB}^2 = 100$. Then,

$$y^6 - 12y^4 + 36y^2 = 100.$$

Taking the square root from both sides, they get

$$y^3 = 6y + 10.$$

[66] See [CARDANO 1545, Chapter XV, page 34v].

Now, they know how to solve the transformed equation in y by (AM XII). The relation $\overline{AB} = \overline{AD} - \overline{BC}$ yields then a solution of $\overline{AB}^3 + \overline{BD}\,\overline{AB}^2 = 100$. $\square$

Cardano says that the proof is "similar to our general [demonstration] in Chapter VII". In fact, the second transformation in (AM VII.6-7) implies as a particular case the following transformation

$$x^3 = a_1 x + a_0 \xrightarrow{y=x^2-a_1} y^3 + a_1 y^2 = a_0^2,$$

which is the same of the one in this chapter (note that x and y were exchanged in Chapter VII).

2.3.3. "On the cube and a number equal to some squares". In **Chapter XVI** Cardano deals with the equation $x^3 + a_0 = a_2 x^2$.

AM XVI. *Given*[67] *the equation* $x^3 + a_0 = a_2 x^2$*, a solution is*

$$x = \frac{\left(\sqrt[3]{a_0}\right)^2}{y},$$

with y *solution of* $y^3 + a_0 = a_2 \sqrt[3]{a_0}\, y$.

Cardano gives no proof of this rule. He just affirms that "this rule is clearly evident from the demonstration of the seventh chapter [*[h]oc capitulum per se patet, ex demonstratione 7i capituli*]". In fact, the first transformation in (AM VII.2-3) implies as a particular case the following one

$$x^3 + a_0 = a_2 x^2 \xrightarrow{y=\frac{(\sqrt[3]{a_0})^2}{x}} y^3 + a_0 = \left(\sqrt{a_0}a_2\right) y,$$

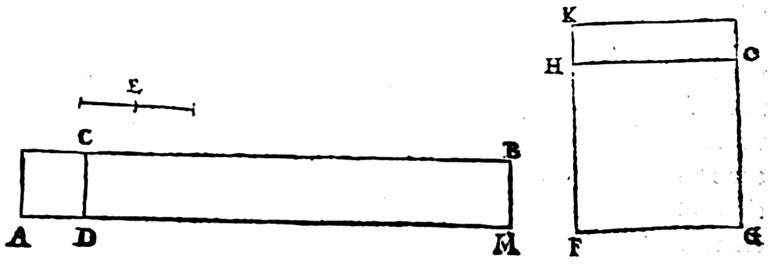
which is the one that we find in this chapter. The proof that Cardano gave in (AM VII.5) concerned just this particular case. We remind that in (AM VII.5)

[67]Expressed as before, the cubic formula is

$$x = \sqrt[3]{\left(\frac{a_2}{3}\right)^3 - \frac{a_0}{2} + \sqrt{\left(\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \sqrt[3]{\left(\frac{a_2}{3}\right)^3 - \frac{a_0}{2} - \sqrt{\left(\left(-\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \frac{a_2}{3}.$$

Cardano does not provide the formula under this shape.
We observe that it can be showed with calculations that this is the same cubic formula that one obtains using the substitution $x = y + \frac{a_2}{3}$.

Cardano drawed a geometrical model for the equation $x^3 + a_0 = a_2x^2$ taking $x = \overline{AD}$, $a_0 = \overline{AD}^2\,\overline{DM}$.



Ars magna VII.5.

Then, $a_2 = \overline{AM}$. Cardano showed that, assuming in addition that $a_1 = \sqrt[3]{a_0}\,a_2$ and $y = \frac{(\sqrt[3]{a_0})^2}{x}$, he managed to draw another geometrical model for the equation $y^3 + a_0 = a_1y$.

I cannot figure out why Cardano does not use here the substitution $x = y + \frac{a_2}{3}$ in (AM XIV) to make disappear the term of second degree. I can only say that if he did it so, he would bring back the equation $x^3 + a_0 = a_2x^2$ to the same three cases as in (AM XIV), whereas here he has just one case. Another possibility could have been to use one of the substitutions in Chapter VII, which bring back the equation $x^3 + a_0 = a_2x^2$ to (AM XIII).

In 1545 edition in Chapter XV, Cardano observes that

> [f]rom this is clear why the case of the cube and the constant equal to the square has not been demonstrated from the case of the cube and square equal to the constant. How, then, can the case of the cube and the number equal to the first power be demonstrated from the case of the cube equal to the first power and number? Since this case leads to the case of the cube and the number equal to the first power, it is better to proceed from the case of the cube and the number equal to the square directly to that of the cube and the number equal to the first power than to the same by the way of the case of the cube and

> square equal to the number. This avoids a far longer and more confusing demonstration.[68]

This quotation is quite tricky. We have seen that Cardano shows that the solving methods in (AM XIII) for $x^3 + a_0 = a_1 x$, in (AM XIV) for $x^3 = a_2 x^2 + a_0$, and partially in (AM XV) for $x^3 + a_2 x^2 = a_0$ depend directly on the solving method in (AM XII) for $x^3 = a_1 x + a_0$ (see below, at page 152). Also the solving method in (AM XV) depends for another part and directly on the solving method in (AM XIII), so that it depends indirectly on the solving method in (AM XII). Cardano explains why he does not want the solving method in (AM XVI) for $x^3 + a_0 = a_2 x^2$ to depend on the solving method in (AM XV) for $x^3 + a_2 x^2 = a_0$. In fact, since (at least partially) the solving method in (AM XV) depends on the solving method in (AM XIII), which depends in turn on the solving method in (AM XII), he can skip a passage making the solving method in (AM XVI) depends directly on the solving method in (AM XIII).

But, in 1570 and 1663 editions, the corollary disappears. I have no hypothesis on why. But I observe that the above quotation is not completely clear. Moreover, thanks to the alternative proof of (AM XV bis) in Chapter XV due to Ludovico Ferrari, the whole solving method in Chapter XV can be brought back to the solving method in (AM XII). In this way, one of Cardano's arguments in the above quotation seems to lose in strength.

Afterwards, Cardano gives a rule to find, given a solution of $x^3 + a_0 = a_2 x^2$, the other (positive) solution.[69] Again, this would enable Cardano to avoid to use the cubic formula (AM XII) in Chapter XII (from which the cubic formula (AM XIII) in Chapter XIII depends, and in turn the solving method in Chapter XVI), provided that one can find a solution of $x^3 + a_0 = a_2 x^2$.

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[68]See [Cardano 1968, page 115] or "*[e]x hoc est manifestum, cur capitulum, cubi et numeri æqualium quadratis, non demonstratur ex capitulo cubi et quadratorum æqualium numero. Quemadmodum capitulum cubi et numeri, æqualium rebus, demonstratum est ex capitulo cubi æqualis rebus et numero. Nam cum capitulum hoc perveniat aliquando ad capitulum cubi et numeri æqualium rebus, melius est igitur ducere capitulum cubi et numeri æqualium quadratis, immediate ad capitulum cubi et numeri æqualium rebus, quam ad idem capitulum, medio capituli cubi et quadratorum æqualium numero, nam et operatio longior, et demonstratio magis confusa evaderet*", see [Cardano 1545, Chapter XV, pages 34r,34v].

[69]Cardano knows by (AM I.8 ii) that in the case $\Delta_3 < 0$ the equation $x^3 + a_0 = a_2 x^2$ has two positive solutions.

AM XVI bis. *Let us consider the equation $x^3 + a_0 = a_2x^2$ and assume x_1 to be a solution. Then,*

$$x_2 = \sqrt{(a_2 - x_1)\left(x_1 + \frac{a_2 - x_1}{4}\right)} + \frac{a_2 - x_1}{2}$$

is another solution.

We observe that, as in (AM XIII bis), one needs to ask that $\Delta_3 < 0$ to guarantee the existence of the second real solution. Again, with a reasoning similar to the one at page 113, we could infer that Cardano get to this formula using (AM I.5-6), (AM VIII.2), and (AM XXV.14) (instead of (AM XXV.1)).

I will skip the proof in details, just sketching without calculations the idea behind it.

AM XVI bis (idea) — *Proof***.** Cardano[70] takes $x_1 = \overline{AB}$, $a_2 - \overline{AB} = \overline{AD}$ and $x_2 = \overline{EF}[= g]$.

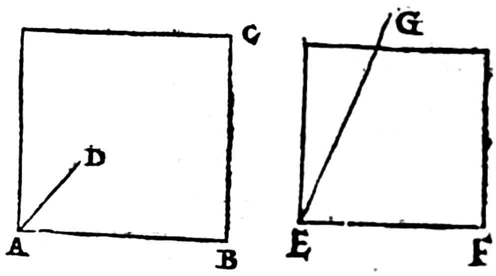

FIGURE 2.14 – *Ars magna* XVI bis.

He wants to show that such a $\overline{EF}$ is a solution of $x^3 + a_0 = a_2x^2$.

He knows that it exists $\overline{EG} = a_2 - \overline{EF}[= f]$. Now, it remains only to show that $\overline{EG}\,\overline{EF}^2 = a_0$ in order to infer by (AM VIII.2) that $\overline{EF}$ is a solution. For that, Cardano considers $\overline{AB}$ which is a solution of $x^3 + a_0 = a_2x^2$ by hypothesis. Then, $a_0 = a_2\overline{AB}^2 - \overline{AB}^3 = (a_2 - \overline{AB})\overline{AB}^2 = \overline{AD}\,\overline{AB}^2$. Then, he finally shows that $\overline{EG}\,\overline{EF}^2 = \overline{AD}\,\overline{AB}^2$. This ends the proof. □

We remark that the proof is very similar to (AM XIII bis (idea)).

Again, this proposition does provide a formula to find a second solution for $x^3 + a_0 = a_2x^2$ which bears no conditions to be satisfied. If this can help a bit in

[70]See [CARDANO 1545, Chapter XVI, pages 35r-35v].

bypassing the *casus irreducibilis* for the second solution, it does not help at all for the first solution.

2.3.4. Summing up. Transformation of equations is the way of proceeding in these chapters. In Chapter VII we already came across some of the transformations used here – and, in fact, one of the proofs in Chapter XVI is skipped, explicitly referring to those in Chapter VII. The other proofs plainly consist in verifying in the usual way that the given substitution leads to the desired equation, once that the hypotheses are appropriately interpreted. The proofs employ (AM VI.6) and they make reference to the proofs in Chapters XI and XII.

Cardano's treatment is not homogeneous, even if one substitution (up to changing signs) could have fitted for all these equations, namely $y = x \pm \frac{a_2}{3}$ (and, by the way, Cardano knows this fact by (AM IV.2), see above, at page 83). He employs different kinds of substitutions, maybe for the sake of not increasing the number of sub-cases. In fact, he uses $x = y + \frac{a_2}{3}$ in (AM XIV), $x = y - \frac{a_2}{3}$ or alternatively $x = y^2 - a_2$ in (AM XV), and $x = \frac{(\sqrt[3]{a_0})^2}{y}$ in (AM XVI). In the following table are resumed all the substitutions that Cardano knows concerning cubic equations lacking in first degree term.

Equation	Substitution	Leads to	In chapter
$x^3 = a_2x^2 + a_0$	$x = y + \frac{a_2}{3}$	$y^3 = a_1y + \frac{2}{27}a_2^3 + a_0$	XIV
	$x = \frac{(\sqrt[3]{a_0})^2}{y}$	$y^3 + \left(\sqrt[3]{a_0}\, a_2\right) y = a_0$	VII
	$x = y^2 + a_2$	$y^3 + a_2y = \sqrt{a_0}$	VII
$x^3 + a_2x^2 = a_0$	$x = y - \frac{a_2}{3}$	$\left\{\begin{array}{l} y^3 + 2\left(\frac{a_2}{3}\right)^3 - a_o = 3\left(\frac{a_2}{3}\right)^2 y \\ y^3 = a_0 - 2\left(\frac{a_2}{3}\right)^3 + 3\left(\frac{a_2}{3}\right)^2 y \\ y^3 = 3\left(\frac{a_2}{3}\right)^2 y \end{array}\right\}$	XV
	$x = y^2 - a_2$	$y^3 = a_2y + \sqrt{a_0}$	VII, XV
	$x = \frac{\sqrt[3]{a_0}}{y}$	$y^3 = \left(\sqrt[3]{a_0}\right)^2 \frac{1}{a_2}y + a_0$	VII
$x^3 + a_0 = a_2x^2$	$x = \frac{(\sqrt[3]{a_0})^2}{y}$	$y^3 + a_0 = \left(a_2\sqrt[3]{a_0}\right) y$	VII, XVI
	$x = \pm(a_2 - y)$	$y^3 + \sqrt{a_0} = a_2y$	VII

TABLE 2.2 – Substitutions in cubic equations lacking in second degree term in *Ars magna*, Chapters VII, XIV, XV, and XVI.

Note that, in order to apply the methods in (AM XV) and (AM XVI) to respectively solve $x^3 + a_2x^2 = a_0$ and $x^3 + a_0 = a_2x^2$, one should at least be able to find a solution of the family of equations in (AM XII), which is not always the case because of the *casus irreducibilis*. Then, in (AM XVI), as in (AM XIII), Cardano tries, given a solution of the considered equation, to provide a formula to find another solution. Both these methods could stand, on the one hand, as signs of Cardano's (unfulfilled) wish to find a way to skip the use of the cubic formula. On the other hand, they could have been originated by parallel researches on transformations and "particular" solutions.

2.4. Solving complete equations

2.4.1. "On the cube, some squares, and some things equal to a number". In **Chapter XVII** Cardano deals with the equation $x^3+a_2x^2+a_1x = a_0$.

AM XVII. *Given*[71] *the equation* $x^3 + a_2x^2 + a_1x = a_0$, *a solution is*

$$x = y - \frac{a_2}{3},$$

where y *satisfies the following conditions*

- ***i.*** *if* $a'_1 = 0$, *then* $y^3 = -a'_0$,
- ***ii.*** *if* $a'_1 > 0$, *then* $y^3 + a'_1y = -a'_0$,
- ***iii.*** *if* $a'_1 < 0$, *then*
  - ***a.*** *if* $a'_0 = 0$, *then* $y^3 = (-a'_1)y$,
  - ***b.*** *if* $a'_0 < 0$, *then* $y^3 = (-a'_1)y + (-a'_0)$,
  - ***c.*** *if* $a'_0 > 0$, *then* $y^3 + a'_0 = (-a'_1)y$,

with $a'_1 = -\frac{a_2^2}{3} + a_1$ *and* $a'_0 = -a'_1\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 - a_0[= -\frac{1}{3}a_1a_2 + \frac{2}{27}a_2^3 - a_0]$.

Note that Cardano uses '*Tpq̃d*' (which means '*Tertia pars numeri quadratorum*') to refer to $\frac{a_2}{3}$.

[71]From now on, I will not explicitly write down the cubic formula since it is a variation in signs of (1.5.9), see here page 42.

We remark that the number of cases increases more and more. In fact, since Cardano always wants positive coefficients in the transformed equations, he has now to check the sign of a'_1 and a'_0 that can both vary. In particular, we observe that, in the case $a'_1 = 0$, we get $a'_0 = -\left(\frac{a_2}{3}\right)^3 - a_0 < 0$. The same $a'_0 = -a'_1\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 - a_0 < 0$ also holds in the case $a'_1 > 0$. But, if $a'_1 < 0$, then the sign of a'_0 can vary further on. Similar observations can be made for the other chapters concerning the complete equations, and I will mostly omit them. Note moreover that there is only a loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). In fact, we have[72] $p = a'_1$ and $q = a'_0$. Anyway, this is not enough to completely determine the sign of Δ_3. We can *a priori* only say that $\Delta_3 \geq 0$ in the first case $a'_1 = 0$ and that $\Delta_3 > 0$ in the second case $a'_1 > 0$.

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[72]It is not surprising that Cardano adds (in 1545 edition) the following corollary. He remarks that the equation $x^3 + a_2x^2 + a_1x = a_0$ is brought back to $x^3 + a_1x = a_0$, $x^3 = a_0$, $x^3 = a_1x$, $x^3 = a_1x + a_0$, $x^3 + a_0 = a_1x$. Then, Cardano gives a clear insight of the method for complete equations.

> In all four-term cases, however, it is a common [characteristic] that one-third the coefficient of x^2 is always added to or subtracted from the value arrived at after they have been resolved into two or three terms, just as in this rule it is always subtracted, and it is also common in every case that the coefficient of y and the [second] constant are built up in the same way, just as in this instance the coefficient of y is the difference between the coefficient of x given in the four-term equation and the product of the coefficient of x^2 and one-third itself; and the constant of the equation into which it [i.e., the original equation] resolves itself is the difference between the product of the coefficient of y and one-third the coefficient of x^2 and the sum of the cube of one-third the coefficient of x^2 and the constant of the first equation

or

> *in omnibus autem capitulis quatuor denominationum, c omune est, cum fuerint resoluta in capitulum trium vel duarum denominationum, ut æstimationi inventæ addatur aut minuatur Tpq̃d: ut in hoc capitulo semper minuitur, et commune est etiam omni capitulo, ut rerum numerus et numerus ipse constituantur eodem modo, velut hic numerus rerum, est differentia numeri rerum assumptarum in capitulo quatuor denominationum, et producti ex numero quadratorum in sui tertiam partem, et numeru capituli in quod resolvitur, est differentia producti ex numero rerum iam inventarum, in Tpq̃d: et aggregati ex cubo Tpq̃d: et numero æquationis primo,*

see [Cardano 1968, page 124] or [Cardano 1545, Chapter XVII, Corollary, page 36v].

Finally, we remark that the transformed equation in y can have more than one solution. In this case, the above rule preserves the number of (real) solution (but, obviously, not necessarily their sign).

Cardano gives a proof for each one of the cases $a_1' \lesseqgtr 0$. Concerning the third case, he only proofs the sub-case $a_0' < 0$.

AM XVII — *Proof***.** Let the equation $\overline{AB}^3 + 6\overline{AB}^2 + 20\overline{AB} = 100$. [We then have $a_1' > 0$.]

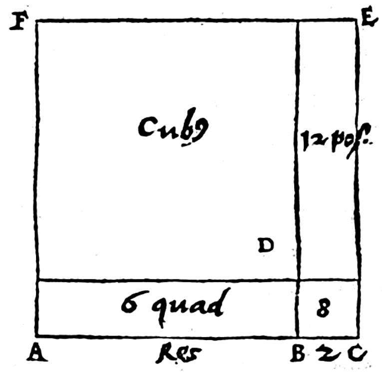


FIGURE 2.15 – *Ars magna* XVII.

Cardano[73] takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6), Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, Cardano has to add $8\overline{AB}$ in order to recover the relation given by the equation he wants to solve

$$\begin{aligned}\overline{AC}^3 + 8\overline{AB} &= (\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8) + 8\overline{AB}\\ &= (\overline{AB}^3 + 6\overline{AB}^2 + 20\overline{AB}) + 8\\ &= 108.\end{aligned}$$

[73]See [CARDANO 1545, Chapter XVII, page 36r].

But Cardano wants a transformed equation where $\overline{AC}$ is the unknown. He knows that $8\overline{AC} = 8(\overline{AB} + \overline{BC}) = 8\overline{AB} + 8\overline{BC}$, so he just has to add $8BC = 16$ to both sides to get

$$\overline{AC^3} + 8\overline{AB} + 8\overline{BC} = 108 + 16,$$

that is

$$\overline{AC^3} + 8\overline{AC} = 124.$$

Now, he knows how to solve the transformed equation in $\overline{AC}$ by (AM XI). The relation $\overline{AB} = \overline{AC} - \overline{BC}$ yields then a solution of $\overline{AB}^3 + 6\overline{AB}^2 + 20\overline{AB} = 100$.

Again, always referring to the same diagram, let the equation $\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} = 100$. [We then have $a_1' = 0$.] In this case, Cardano has to add $\overline{BC}^3 = 8$ to the left side of the equation he wants to solve in order to recover $\overline{AC}^3$

$$(\overline{AB}^3 + 6\overline{AB^2} + 12\overline{AB}) + 8 = 108,$$

that is

$$\overline{AC}^3 = 108.$$

Then, $\overline{AC} = \sqrt[3]{108}$ and $\overline{AB} = \overline{AC} - \overline{BC} = \sqrt[3]{108} - 2$.

Again, always referring to the same diagram, let the equation $\overline{AB}^3 + 6\overline{AB}^2 + 2\overline{AB} = 100$. [We then have $a_1' < 0$.] In this case, Cardano has to add $10\overline{AB} + \overline{BC}^3 = 10\overline{AB} + 8$ to the left side of the equation he wants to solve in order to recover $\overline{AC}^3$

$$(\overline{AB}^3 + 6\overline{AB}^2 + 2\overline{AB}) + 10\overline{AB} + \overline{BC}^3 = 10\overline{AB} + 108,$$

that is

$$\overline{AC}^3 = 10\overline{AB} + 108.$$

But Cardano wants a transformed equation where $\overline{AC}$ is the unknown, so he adds again $10\overline{BC} = 20$ to both sides and he gets

$$\begin{aligned}\overline{AC}^3 + 20 &= 10\overline{AB} + 10\overline{BC} + 108\\ &= 10\overline{AC} + 108,\end{aligned}$$

that is

$$\overline{AC}^3 = 10\overline{AC} + 88.$$

Now, he knows how to solve the transformed equation in $\overline{AC}$ by (AM XII). The relation $\overline{AB} = \overline{AC} - \overline{BC}$ yields then a solution of $\overline{AB}^3 + 6\overline{AB}^2 + 20\overline{AB} = 100$. $\square$

We observe that, in the three cases, the proof consists in the construction of the transformed equation completely relying on the substitution $x = y - \frac{a_2}{3}$, on the formula for the expansion of the cube of a binomial given in (AM VI.6), and obviously on the relation given by the original equation. Then, Cardano has only to adjust the coefficients in order to recover the original equation and he has to take care to obtain a transformed equation in $\overline{AC}$ with positive coefficients. In this way, he is able to bring back the complete equation $x^3 + a_2x^2 + a_1x = a_0$ to an equation that he knows how to solve thanks to (AM XI), (AM XII), or taking the cubic root of a number. We remark that the rule given above is a simple translation into an algorithmic language of the steps taken in the proof. Here again, similar observations can be made for all proofs in the following chapters, and I will mostly omit them.

2.4.2. "On the cube and some things equal to some squares and a number". In **Chapter XVIII** Cardano deals with the equation $x^3 + a_1x = a_2x^2 + a_0$.

AM XVIII. *Given the equation $x^3 + a_1x = a_2x^2 + a_0$, a solution is*[74]

$$x = y + \frac{a_2}{3},$$

where y satisfies the following conditions

- ***i.*** *if $a'_1 = 0$, then $y^3 = -a'_0$,*
- ***ii.*** *if $a'_1 > 0$, then $y^3 + a'_1y = -a'_0$,*
- ***iii.*** *if $a'_1 < 0$, then*
  - ***a.*** *if $a'_0 = 0$, then $y^3 = (-a'_1)y$,*

[74]Cardano adds that "having derived the solution [for y], add or subtract one-third the coefficient of x^2 as you will learn from the examples." In fact, in the first example, a solution of the transformed equation is $y = \sqrt[3]{17}$, which yields the solution of the original equation $x = \sqrt[3]{17} + 2$, whereas in the second example a solution of the transformed equation is $y = -\sqrt[3]{16}$, which yields the solution of the original equation $x = -\sqrt[3]{16} + 2$.
Cardano makes this remark only in this chapter, even if it holds also for other chapters. In fact, as said, from a negative solution of the transformed equation we can have a positive solution of the original equation, adding $\frac{a_2}{3}$. He mentions quickly this fact again in Chapter XXI, see below, page 145, and at the end of Chapter XXIII, see below, page 149.

b. *if* $a_0' < 0$, *then* $y^3 = (-a_1')y + (-a_0')$,

c. *if* $a_0' > 0$, *then* $y^3 + a_0' = (-a_1')y$,

with $a_1' = -\frac{a_2^2}{3} + a_1$ *and*[75] $a_0' = a_1 \frac{a_2}{3} - 2\left(\frac{a_2}{3}\right)^3 - a_0$.

Note the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a_1'$ and $q = a_0'$. Then, we can *a priori* only say that $\Delta_3 \geq 0$ in the first case $a_1' = 0$ and that $\Delta_3 > 0$ in the second case $a_1' > 0$.

Cardano gives a proof only for the case $a_1' < 0$.

AM XVIII — *Proof*. Let the equation $\overline{AC}^3 + 33\overline{AC} = 6\overline{AC}^2 + 100$. [From now on, the diagrams appear no more in the proofs. This is because Cardano[76] makes always reference to the diagram in Chapter XVII, see above, the figure 2.15 at page 132.]

Cardano takes $\overline{BC} = \frac{6}{3} = 2$. [By (AM VI.6), he knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,]$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, he has to add $33\overline{AC} = 33\overline{AB} + 33\overline{BC} = 33\overline{AB} + 66$ to $\overline{AC}^3$ in order to recover the relation given by the equation that he wants to solve

$$\begin{aligned}\overline{AC}^3 + 33\overline{AC} &= (\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8) + (33\overline{AB} + 66)\\ &= \overline{AB}^3 + 6\overline{AB}^2 + 45\overline{AB} + 74\\ &= 6\overline{AC}^2 + 100.\end{aligned}$$

But Cardano wants a transformed equation where $\overline{AB}$ is the unknown. So, using $6\overline{AC}^2 = 6(\overline{AB} + \overline{BC})^2 = 6\overline{AB}^2 + 12\overline{AB}\,\overline{BC} + 6\overline{BC}^2 = 6\overline{AB}^2 + 24\overline{AB} + 24$, by[77]

[75] We observe that $a_0' = a_1' \frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 - a_0$.

[76] See [Cardano 1545, Chapter XVIII, page 37v].

[77] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangles contained by the segments", see [Heath 1956a, page 379].

Elements II,4 he gets

$$\overline{AB}^3 + 6\overline{AB}^2 + 45\overline{AB} + 74 = (6\overline{AB}^2 + 24\overline{AB} + 24) + 100,$$

that is

$$\overline{AB}^3 + 21\overline{AB} = 50.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XI). The relation $\overline{AC} = \overline{AB} + \overline{BC}$ yields then a solution of $\overline{AC}^3 + 33\overline{AC} = 6\overline{AC}^2 + 100$. □

At the end of this proof, Cardano adds a quite opaque passage.[78] Fortunately, thanks to the explicit setting out of the rule, one is able to complete and interpret it as describing the other cases which Cardano does not consider in the proof. He says that their proofs are similar to the ones in Chapter XVII.

Then, Cardano makes some examples. As usual, passing from the transformed equation to the original one, the rule preserves the number of (real) solutions. We observe however that, when $y = 0$, we have a sort of limit case. At the end of his fourth example $x^3 + 15x = 6x^2 + 10$, Cardano himself needs to explain better this fact.

> Hence it is evident in this case, where the cube and first power are equal to the number, that if the difference between the numbers were nothing – as if we put 14 in the place of 10 – the solution would be one-third the coefficient of x^2, or 2, since the

[78]"It also clear that if the [number of] x's which were on the same side as the cube [in the original equation] had been equal to the product [of the coefficient of x^2 and one-third the same], we would have had a cube equal to a number alone [in the reduced equation]; and if [the number of] x's which were on the same side as the cube had been less [than that product], we would have had a first power on one side and a cube on the other. In this case, if the constant had been equal to a [certain] other [number], the first power would equal the cube, but if it [i.e., the new first power] were smaller [than the new cube] we would have had the first power and number equal to the cube and, if greater, the first power equal to the cube and a number, according to the same demonstration as in the preceding chapter", see [Cardano 1968, pages 128-9], or "*[m]anifestum est autem, quod ubi positione, quæ cum cubo erant, essent, æquales productis, haberemus cubum æqualem numero tantum, et ubi positiones quæ cum cubo erant, essent pauciores, haberemus rex ex una parte, et cubum ex alia, et tunc si numerus qui est cum cubo, foret æqualis alteri, essent positiones æquales cubo, et si maior, haberemus res æquales cubo et numero, ex eadem demonstratione, velut in præcedente capitulo*", see [Cardano 1545, Chapter XVIII, Proof, page 38r].

> derived equation would give us nothing to add or subtract, for $y^3 + 3y = 0$.[79]

The equation $x^3 + 15x = 6x^2 + 14$ falls within the case $a'_1 > 0$, $a'_0 = 0$ (which is not completely detailed in the above rule). We get then the transformed equation $y^3 + 3y = 0$, a solution of which is 0. So, one of the solutions of the original equation is $x = y + \frac{a_2}{3} = 0 + 2 = 2$. Again, in his fifth example $x^3 + 10x = 6x^2 + 4$, the transformed equation is $y^3 = 2y$, the solutions of which are $\sqrt{2}$ [and $-\sqrt{2}$ and 0, both not mentioned by Cardano]. These yield the solutions of the original equation $2 + \sqrt{2}$, $2 - \sqrt{2}$, and 2, which are nevertheless are all mentioned by Cardano. He will again hint at this fact in Chapter XXII, referring to both Chapters XXI and XXII, see here, page 146.

Afterwards, Cardano observes that

> [f]rom this it is evident that the coefficient of x^2, in the three examples in which there are three solutions for x, is always the sum of the three solutions. [...] Hence knowing two such solutions, the third always emerges. The reason for this appears at the beginning of this book.[80]

Cardano explicitly affirms that, in the case $x^3 + a_1x = a_2x^2 + a_0$ [when $\Delta_3 < 0$], he knows that $a_2 = x_1 + x_2 + x_3$, where x_1, x_2, x_3 are the three (real) solutions. To justify his statements, Cardano makes reference to Chapter I, where he spoke of "true" and "fictitious" solutions. Unluckily, this reference is empty. In fact, in Chapter I the equation $x^3 + a_1x = a_2x^2 + a_0$ is hardly mentioned. Anyway, in that same chapter, we find two statements (not justified) of similar rules for depressed cubic equations (see (AM I.5-6) with $a_2 = 0$) and for cubic equations lacking in the first degree term (see (AM I.8 ii)). Then, we can suppose that Cardano

[79]See [Cardano 1968, page 132] or "*Ideo patet quod in hoc casu, ubi cubus et res, æquantur numero, si differentia numerorum nulla foret, vel si loco 10 posuissemus 14, æstimatio rei esset Tpq̃d, scilicet 2, quia in æquatione inventa, nihil haberemus addendum vel minuendum, quia cubus et 3 res, æquerentur nihil*", see [Cardano 1545, Chapter XVIII, fourth example, page 39r].

[80]See [Cardano 1968, page 134] or "*[e]x hoc patet, qod numerus quadratorum, in his tribus exemplis, in quibus æstimatio rei triplicatur, semper componitur ex tribus æstimationibus iunctis simul [...], ideo duabus cognitis, tertia semper emergit, et causa est cognita in initio huius libri*", see [Cardano 1545, Chapter XVIII, page 39v].

knows in general the relation between the coefficient of the term of second degree and the roots.

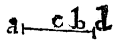
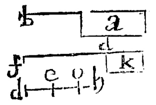
Again, Cardano observes that

> [i]t is also evident that whenever we arrive [at a case in which] the *y*s are separated from the cube, whether the number is joined to the y's or to y^3, three solutions always appear. The reason for this is likewise given in the same above place where we spoke of true and fictitious solutions. And it is also evident that all these methods can always be carried back to addition, for a minus that is added acts the same way as a plus that is subtracted.[81]

Cardano knows that, when $a_1' > 0$, the equation $x^3 + a_1x = a_2x^2 + a_0$ always has three solutions. Here again, Cardano refers back vainly to some justification in Chapter I. The only place where we can find it is an obscure addition to 1570 and 1663 editions.[82]

[81]See [CARDANO 1968, page 134] or "*[e]t manifestum est, quod cum pervenimus ad res, quæ a cubo separantur, seu numero rebus, seu cubo iungatur, semper emergunt tres æstimationes, et causa dicta est superius ibidem, ubi de vera et ficta æstimatione locuti sumus. Et patet etiam, quod omnes modi hi, ad additionem semper possunt referri, quamvis minus cum additur, vicem gerat plus cum detrahitur*", see [CARDANO 1545, Chapter XVIII, page 39v].

[82]Here follows the quotation.

> When a constant and the highest power are equal to one or more middle powers, there are two solutions for x since, whatever their coefficients, the middle terms can exceed the extreme term, as $100x^2$ can be greater than x^3. Now let $\overline{AB}$ be the value of x.
>
> 
>
>
> The cube can then be made to equal $100x^2$ either by decreasing the estimated value of x and allowing the given number to remain the same (in which case [the value of x becomes] $\overline{AC}$) or by decreasing the cube and thus increasing the value of x (in which case it becomes $\overline{AD}$). Therefore $100x^2 = x^3 + 1$ has two solutions.
>
> Similarly, if there are many middle terms, even a hundred of them, changes all the middle powers likewise increase or decrease. But if the extreme powers and the middle powers, taken alternately, are equated to each other (as if the cube and first power are equated to the square and a constant), I say that there may be three solutions.
>
> 
>

Finally, Cardano adds another rule discovered by Ferrari.

AM XVIII bis. *Given the equation* $x^3 + a_1 x = a_2 x^2 + a_0$*, a solution is*

- *if* $a'_1 = 0$*, then*
  - $*$ *if* $a'_0 < 0$*, then* $x = \frac{a_2}{3} + \sqrt[3]{a'_0}$*,*
  - $*$ *if* $a'_0 > 0$*, then* $x = \frac{a_2}{3} - \sqrt[3]{a'_0}$*,*
- *if* $a'_1 > 0$*, then*
  - $*$ *if* $a'_0 = 0$*, then* $x = \frac{a_2}{3}$*,*
  - $*$ *if* $a'_0 > 0$*, then* $x = \frac{a_2}{3} - y$*, where* y *is a solution of the equation* $y^3 + a'_1 y = a'_0$
  - $*$ *if* $a'_0 < 0$*, then* $x = \frac{a_2}{3} + y$*, where* y *is a solution of the equation* $y^3 + a'_1 y = -a'_0$*,*
- *if* $a'_1 < 0$*, then* $x = y + \frac{a_2}{3}$*, where* y *satisfies the following conditions*
  - $*$ *if* $a'_0 = 0$*, then* $y^3 = a'_1 y$*,*
  - $*$ *if* $a'_0 < 0$*, then* $y^3 = a'_1 y + (-a'_0)$*,*
  - $*$ *if* $a'_0 > 0$*, then* $y^3 + a'_0 = a'_1 y$*,*

with $a'_1 = -\frac{a_2^2}{3} + a_1$ *and* $a'_0 = a'_1 \frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 - a_0$.

We observe that this is substantially the same rule as before. In fact, the transformed coefficients a'_1 and a'_0 are the same (see above, footnote 75 at page 135) and this rule details all the cases depending on the sign of the solution y of

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> Thus, let A, the constant, and B, the number of squares, equal K, the cube, and F, the number of x's, and let x be $\overline{DE}$. If F is assumed to be big and $\overline{DE}$ small, an equation can be made, since the squares and the cube become smaller because of the smallness of $\overline{DE}$. But if the squares exceed the cube and if the x's are placed beside the latter, as said, there will be two solutions – either one that is increased because of the magnitude of x^3 or one that is diminished by the increase in the x's. Therefore, there will be three solutions [in all],case

see [Cardano 1968, pages 21-2].
Pietro Cossali explains the first part of the quotation as follows. Consider $x^3 + a_0 = a_2 x^2$ and suppose that it exists a real N such that $N^3 + a_0 = a_2 N^2$. Then, one can bring the inequality back to an equality in two ways, either increasing $a_2 N^2$ or decreasing N^3. This means decreasing N up to a certain N_1 which works, or increasing it up to N_2. Then, N_1 and N_2 are two solutions. Then, Cossali explains why this holds also for an equation of this kind with more than three terms, using the mathematics of his own time. After that, he continues dealing with the kind of equation in the second part of the quotation, but continuing in modern terms, see [Cossali 1799a, Volume II, Chapter VII, item III, number 12, pages 354-7].

the transformed equation, which were only sketched in the preceding rule (see above, footnote 74 at page 134).

We firstly find a proof for the case $a'_1 < 0$, $a'_0 > 0$.

AM XVIII bis i — *Proof.* Let the equation $\overline{AB}^3 + 100\overline{AB} = 6\overline{AB}^2 + 10$. [We then have $a'_1 < 0$, $a'_0 > 0$.]

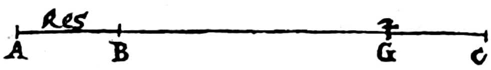

FIGURE 2.16 – *Ars magna* XVIII bis i.

Cardano and Ferrari[83] take $\overline{BC} = \frac{6}{3} = 2$ and mark G on $\overline{AC}$ such that $\overline{AG} = \overline{BC}$. Then, [$\overline{AB} = \overline{GC}$ and] $\overline{BG} = \overline{BC} - \overline{AB}$. By (AM VI.6), they know that

$$\begin{aligned}\overline{BG}^3 = (\overline{BC} - \overline{AB})^3 &= \overline{BC}^3 - 3\overline{BC}^2\,\overline{AB} + 3\overline{BC}\,\overline{AB}^2 - \overline{AB}^3 \\ &= \overline{BC}^3 + 3\overline{BC}\,\overline{AB}^2 - (3\overline{BC}^2\,\overline{AB} + \overline{AB}^3) \\ &= 8 + 6\overline{AB}^2 - (12\overline{AB} + \overline{AB}^3).\end{aligned}$$

They are looking for a transformed equation in $\overline{BG}$. By the given equation, they know that $6\overline{AB}^2 + 8 = (\overline{AB}^3 + 100\overline{AB}) - 2 = (\overline{AB}^3 + 12\overline{AB}) + 88\overline{AB} - 2$. Then, they get

$$\overline{BG}^3 = 88\overline{AB} - 2.$$

Since $\overline{BG} = \overline{BC} - \overline{AB} = 2 - \overline{AB}$, then $\overline{AB} = 2 - \overline{BG}$ and $\overline{BG}^3 + 2 = 88(2 - \overline{BG})$, that is

$$\overline{BG}^3 + 88\overline{BG} = 174.$$

Now, Cardano and Ferrari know how to solve the transformed equation in $\overline{BG}$ by (AM XI). The relation $\overline{AB} = 2 - \overline{BG}$ yields then a solution of $\overline{AB}^3 + 100x = 6x^2 + 10$. □

Afterwards, we find a proof for the case $a'_1 > 0$, $a'_0 < 0$.

AM XVIII bis ii — *Proof.* Again, let the equation $\overline{EF}^3 + 5\overline{EF} = 6\overline{EF}^2 + 10$. [We then have $a'_1 > 0$, $a'_0 < 0$.]

[83]See [CARDANO 1545, Chapter XVIII, pages 39v-40r].

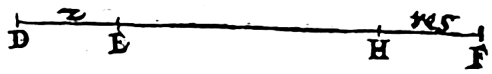

FIGURE 2.17 – *Ars magna* XVIII bis ii.

Cardano and Ferrari[84] take $\overline{DE} = \frac{6}{3} = 2$ [and mark H on $\overline{DF}$ such that $\overline{DH} = \overline{EF}$. Then $\overline{DE} = \overline{HF}$ and] $\overline{EH} = \overline{EF} - \overline{DE}$. Then, "from a demonstration similar to the foregoing" they get

$$\overline{EH}^3 = 7\overline{EH} + 16.$$

Now, Cardano and Ferrari know how to solve the transformed equation in $\overline{EH}$ by (AM XII). The relation $\overline{EF} = \overline{EH} + \overline{DE}$ yields then a solution of $\overline{AB}^3 + 100x = 6x^2 + 10$. □

No proof is given for the other cases.

2.4.3. "On the cube and some squares equal to some things and a number". In **Chapter XIX** Cardano deals with the equation $x^3 + a_2x^2 = a_1x + a_0$.

AM XIX. *Given the equation $x^3 + a_2x^2 = a_1x + a_0$, a solution is*

$$x = y - \frac{a_2}{3},$$

where y satisfies the following conditions

i. *if $a_0' = 0$, then $y^3 = (-a_1')y$,*
ii. *if $a_0' < 0$, then $y^3 = (-a_1')y + (-a_0')$,*
iii. *if $a_0' > 0$, then $y^3 + a_0' = (-a_1')y$,*

with $a_1' = -\frac{a_2^2}{3} - a_1$ and $a_0' = -a_1'\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 - a_0 [= \frac{1}{3}a_1a_2 + \left(\frac{a_2}{3}\right)^3 - a_0]$.

We remark that it is always $a_1' < 0$.

Moreover, note the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a_1'$ and $q = a_0'$. Then, we can *a priori* only say that $\Delta_3 < 0$ in the first case $a_0' = 0$.

Cardano gives only a proof for the case $a_0' < 0$.

[84] See [CARDANO 1545, Chapter XVIII, pages 39v-40r].

AM XIX — *Proof*. Let the equation $\overline{AB}^3 + 6\overline{AB}^2 = 20\overline{AB} + 200$. [Cardano[85] considers the same diagram as in Chapter XVII, see above, the figure 2.15 at page 132.]

He takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6) Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, using the relation given by the equation, Cardano gets

$$\begin{aligned}\overline{AC}^3 &= (20\overline{AB} + 200) + 12\overline{AB} + 8\overline{AB}\\ &= 32\overline{AB} + 208.\end{aligned}$$

But Cardano wants a transformed equation where $\overline{AC}$ is the unknown. He knows that $32\overline{AC} = 32\overline{AB} + 32\overline{BC}$, so he just has to add again $32\overline{BC} = 64$ to both sides to get

$$\overline{AC}^3 + 64 = 32\overline{AC} + 208,$$

that is

$$\overline{AC}^3 = 32\overline{AC} + 144.$$

Now, he knows how to solve the transformed equation in $\overline{AC}$ by (AM XII). The relation $\overline{AB} = \overline{AC} - \overline{BC}$ yields then a solution of $\overline{AB}^3 + 6\overline{AB}^2 = 20\overline{AB} + 200$. □

2.4.4. "On the cube equal to some squares, some things, and a number".

In **Chapter XX** Cardano deals with the equation $x^3 = a_2x^2 + a_1x + a_0$.

AM XX. *Given the equation* $x^3 = a_2x^2 + a_1x + a_0$*, a solution is*

$$x = y + \frac{a_2}{3},$$

[85] See [Cardano 1545, Chapter XIX, page 41r].

where y is a solution of $y^3 = (-a_1')y + (-a_0')$, with $a_1' = -\frac{a_2^2}{3} - a_1$ and[86] $a_0' = a_1'\frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 - a_0 [= -\frac{1}{3}a_1a_2 - \frac{2}{27}a_2^3 - a_0]$.

We remark that it is always $a_1' < 0$. Moreover, we also always have $a_0' < 0$. In fact,

$$\begin{aligned} a_0' &= (-\frac{a_2^2}{3} - a_1)\frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 - a_0 \\ &= -2\left(\frac{a_2}{3}\right)^3 - a_1\frac{a_2}{3} - a_0 < 0. \end{aligned}$$

Finally, note the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a_1'$ and $q = a_0'$. Then, we can *a priori* only say that $\Delta_3 \neq 0$.

AM XX — *Proof***.** Let the equation $\overline{AC}^3 = 6\overline{AC}^2 + 5\overline{AC} + 88$. [Cardano[87] considers the same diagram as in Chapter XVII, see above, the figure 2.15 at page 132.]

He takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6) Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, using the relation given by the equation, Cardano gets

$$\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8 = 6\overline{AC}^2 + 5\overline{AC} + 88,$$

[86]In Cardano's formulation of the rule there is a mistake concerning a_0'. In fact, he says that "To find the number, moreover, multiply the coefficient of y by one-third the coefficient of x^2 and add the product to the constant of the equation. From this subtract the cube of one-third the coefficient of x^2 and the remainder is the number", see [Cardano 1968, page 144], or "*pro numero autem, duc numerum rerum secundum in Tpq̃d et productum adde numero æquationis a quo minue cubum Tpq̃d. Residuum est numerus*", see [Cardano 1545, Chapter XX, Rule, page 42r], which means $a_0' = a_1'\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 - a_0$.
But, in the equation $x^3 = 6x^2 + 5x + 88$ considered in the demonstration and in the only example $x^3 = 6x^2 + 72x + 729$ of this chapter, Cardano makes the right calculations (with the '+') to obtain the transformed equations $y^3 = 17y + 114$ and $y^3 = 84y + 889$.
[87]See [Cardano 1545, Chapter XX, pages 41v-42r].

that is

$$\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} = 6\overline{AC}^2 + 5\overline{AC} + 80,$$

that is

$$\begin{aligned}\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} &= 6(\overline{AB}^2 + 2\overline{AB}\,\overline{BC} + \overline{BC}^3) + 5(\overline{AB} + \overline{BC}) + 80 \\ &= 6(\overline{AB}^2 + 4\overline{AB} + 4) + 5(\overline{AB} + 2) + 80 \\ &= 6\overline{AB}^2 + 29\overline{AB} + 114,\end{aligned}$$

that is

$$\overline{AB}^3 = 17\overline{AB} + 114.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XII). The relation $\overline{AC} = \overline{AB} + \overline{BC}$ yields then a solution of $\overline{AC}^3 = 6\overline{AC}^2 + 5\overline{AC} + 88$. □

2.4.5. "On the cube and a number equal to some squares and some things". In **Chapter XXI** Cardano deals with the equation $x^3 + a_0 = a_2x^2 + a_1x$.

AM XXI. *Given the equation* $x^3 + a_0 = a_2x^2 + a_1x$, *a solution is*

$$x = y + \frac{a_2}{3},$$

where y *satisfies the following conditions*

i. *if* $a_0' = 0$, *then* $y^3 = (-a_1')y$,
ii. *if* $a_0' < 0$, *then* $y^3 = (-a_1')y + (-a_0')$,
iii. *if* $a_0' > 0$, *then* $y^3 + a_0' = (-a_1')y$,

with $a_1' = -\frac{a_2^2}{3} - a_1$ *and* $a_0' = a_1'\frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 + a_0$.

We remark that it is always $a_1' < 0$.

Moreover, note the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a_1'$ and $q = a_0'$. Then, we can *a priori* only say that $\Delta_3 < 0$ in the first case $a_0' = 0$.

Cardano gives only a proof for the case $a_0' > 0$.

AM XXI — *Proof*. Let the equation $\overline{AC}^3 + 100 = 6\overline{AC}^2 + 24\overline{AC}$. [Cardano[88] considers the same diagram as in Chapter XVII, see above, the figure 2.15 at page 132.]

He takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6) Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, using the relation given by the equation, Cardano gets

$$(\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8) + 100 = 6\overline{AC}^2 + 24\overline{AC},$$

that is

$$\begin{aligned}\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 108 &= 6(\overline{AB}^2 + 2\overline{AB}\,\overline{BC} + \overline{BC}^2) + 24(\overline{AB} + \overline{BC})\\ &= 6(\overline{AB}^2 + 4\overline{AB} + 4) + 24(\overline{AB} + 2)\\ &= 6\overline{AB}^2 + 48\overline{AB} + 72,\end{aligned}$$

that is

$$\overline{AB}^3 + 36 = 36\overline{AB}.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XIII). The relation $\overline{AC} = \overline{AB} + \overline{BC}$ yields then a solution of $\overline{AC}^3 + 100 = 6\overline{AC}^2 + 24\overline{AC}$. □

At the end of the proof, Cardano adds that, in the case $a_0' < 0$, the transformed equation $y^3 = (-a_1')y + (-a_0')$ has three solutions, one positive and two negative. Since then we add $\frac{a_2}{3}$ to these solutions, $x^3 + a_0 = a_2x^2 + a_1x$ has two positive solutions and a negative one.[89]

[88] See [Cardano 1545, Chapter XXI, page 42v].

[89] "Remember, moreover, that when the original equation turns into an equation of the cube equal to the first power and a number, the true solution of the latter will be added to one-third the coefficient of x^2 and the smaller of the fictitious solutions will [likewise] be added negatively. Thus you will have both [true] solutions for the equation of the cube and the number equal to the first power and square, although that of the cube equal to the first power and constant has only one true solution", see [Cardano 1968, page 147], or "*[m]emineris tamen, quod quando capitulum hoc pervenerit ad capitulum cubi æquali rebus et numero, addenda erit vera æstimatio eius, et ex his quæ fictæ sunt minor per m: Tpq̃d ut habeas utramque æstimationem capituli*

Moreover, at the end of the next chapter while speaking of $x^3 + a_1x + a_0 = a_2x^2$, Cardano makes a remark that – he says – also holds for $x^3 + a_0 = a_2x^2 + a_1x$. In the case $a'_0 = 0$, the transformed equation is $y^3 = (-a'_1)y$ and "the fictitious solution does not differ from the true one in number. Therefore, for a second solution [for the principal equation], inasmuch as nothing is added to or subtracted from one-third the coefficient of x^2, this same one-third the coefficient of x^2 will be a true solution."[90] In fact, if $a'_0 = 0$, we get $y_1 = \sqrt{-a'_1}$, $y_2 = \sqrt{-a'_1}$, and $y_3 = 0$. For the last solution, the corresponding solution of the equation in x is then $x_3 = \frac{a_2}{3}$.

2.4.6. "On the cube, some things, and a number equal to some squares". In **Chapter XXII** Cardano deals with the equation $x^3 + a_1x + a_0 = a_2x^2$.

AM XXII. *Given the equation $x^3 + a_1x + a_0 = a_2x^2$, a solution is*

$$x = y + \frac{a_2}{3},$$

where y satisfies the following conditions

i. *if $a'_0 = 0$, then $y^3 = (-a'_1)y$,*
ii. *if $a'_0 < 0$, then $y^3 = (-a'_1)y + (-a'_0)$,*
iii. *if $a'_0 > 0$, then $y^3 + a'_0 = (-a'_1)y$,*

with $a'_1 = -\frac{a_2^2}{3} + a_1$ and[91] $a'_0 = a_1\frac{a_2}{3} - 2\left(\frac{a_2}{3}\right)^3 + a_0$.

Cardano remarks that, if $a'_1 > 0$, then "the case is impossible of a true solution", that is[92] $x < 0$. So, we assume $a'_1 < 0$.

cubi et numeri æqualis rebus et quadratis, cum capitulum cubi æqualis rebus et numero, unam tantum veram æstimationem habeat", see [Cardano 1545, Chapter XXI, Rule, page 42v].

[90]See [Cardano 1968, pages 151-2].

[91]We observe that $a'_0 = a'_1\frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 + a_0$.

[92]If $a'_1 > 0$, then $a'_0 = a'_1\frac{a_2}{3} + \left(\frac{a_2}{3}\right)^3 + a_0 > 0$. We would then have the transformed equation $y^3 + a'_1y + a'_0 = 0$, the real solution of which is negative.
In fact, by the cubic formula

$$y < 0 \Longleftrightarrow \sqrt[3]{-\frac{a_0}{2} + \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}} + \sqrt[3]{-\frac{a_0}{2} - \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}} \Longleftrightarrow -\frac{a_0}{2} < \frac{a_0}{2},$$

which is always true since we took $a_0 > 0$.
Anyway, Cardano gives no hints on how he discovered this results, whether by a similar calculation or generalising experience.

Note then the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a'_1$ and $q = a'_0$. Then, we can *a priori* only say that $\Delta_3 < 0$ in the first case $a'_0 = 0$.

We remind[93] that, in the case $a'_0 = 0$, $\frac{a_2}{3}$ is also a solution of $x^3 + a_1 x + a_0 = a_2 x^2$.

Cardano gives a proof for both cases $a'_0 > 0$ and $a'_0 < 0$.

AM XXII — ***Proof*****.** Let the equation $\overline{AC}^3 + 4\overline{AC} + 16 = 6\overline{AC}^2$. [We then have $a'_0 > 0$.] [Cardano[94] considers the same diagram as in Chapter XVII, see above, the figure 2.15 at page 132.]

He takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6) Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB^2} + 12\overline{AB} + 8.$$

Then, using the relation given by the equation, Cardano gets

$$(\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8) + 4\overline{AC} + 16 = 6\overline{AC}^2,$$

that is

$$\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8 + 4(\overline{AB} + \overline{BC}) + 16 = 6(\overline{AB}^2 + 2\overline{AB}\,\overline{BC} + \overline{BC}^2),$$

that is

$$\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8 + 4(\overline{AB} + 2) + 16 = 6(\overline{AB}^2 + 4\overline{AB} + 4),$$

that is

$$\overline{AB}^3 + 8 = 8\overline{AB}.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XIII). The relation $\overline{AC} = \overline{AB} + \overline{BC}$ yields then a solution of $\overline{AC}^3 + 4\overline{AC} + 16 = 6\overline{AC}^2$.

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[93] See above, at page 146.

[94] See [Cardano 1545, Chapter XXII, page 43r].

Again, always referring to the same diagram, let the equation $\overline{AC}^3+4\overline{AC}+1 = 6\overline{AC}^2$. [We then have $a_0' < 0$.] Cardano takes $\overline{BC} = \frac{6}{3} = 2$. As before, using the relation given by the equation, Cardano gets

$$(\overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8) + 4\overline{AC} + 1 = 6\overline{AC}^2,$$

that is

$$\overline{AB}^3 = 8\overline{AB} + 7.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AB}$ by (AM XII). The relation $\overline{AC} = \overline{AB}+\overline{BC}$ yields then a solution of $\overline{AC}^3+4\overline{AC}+16 = 6\overline{AC}^2$. $\square$

The same remark at page 146 also holds here. Then, if $a_0' = 0$, we get to $y_1 = \sqrt{-a_1'}$, $y_2 = \sqrt{-a_1'}$, and $y_3 = 0$. For the last solution, the corresponding solution of the equation in x is then $x_3 = \frac{a_2}{3}$.

2.4.7. "On the cube, some squares, and a number equal to some things". In **Chapter XXIII** Cardano deals with the equation $x^3 + a_2x^2 + a_0 = a_1x$.

AM XXIII. *Given the equation $x^3 + a_2x^2 + a_0 = a_1x$, a solution is*

$$x = y - \frac{a_2}{3},$$

where y is a solution of $y^3 + a_0' = (-a_1')y$, with $a_1' = -\frac{a_2^2}{3} - a_1$ and $a_0' = -a_1'\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 + a_0[= \frac{1}{3}a_1a_2 + \frac{2}{27}a_2^3 + a_0]$.

We remark that $a_1' < 0$. Moreover, we have $a_0' > 0$. In fact,

$$\begin{aligned} a_0' &= (\frac{a_2^2}{3} + a_1)\frac{a_2}{3} - \left(\frac{a_2}{3}\right)^3 + a_0 \\ &= 2\left(\frac{a_2}{3}\right)^3 + a_1\frac{a_2}{3} + a_0 > 0. \end{aligned}$$

Finally, note the loose correspondence between these cases and the sign of the discriminant $\Delta_3 = \frac{q^2}{4} + \frac{p^3}{27}$ (see above, the formula (1.5.10) at page 42). We have $p = a_1'$ and $q = a_0'$. Then, we can *a priori* only say that $\Delta_3 \neq 0$.

AM XXIII — *Proof*. Let the equation $\overline{AB}^3 + 6\overline{AB}^2 + 4 = 41\overline{AB}$. [Cardano[95] considers the same diagram as in Chapter XVII, see above, the figure 2.15 at page 132.]

He takes $\overline{BC} = \frac{6}{3} = 2$. By (AM VI.6) Cardano knows that

$$\overline{AC}^3 = (\overline{AB} + \overline{BC})^3 = \overline{AB}^3 + 3\overline{AB}^2\,\overline{BC} + 3\overline{AB}\,\overline{BC}^2 + \overline{BC}^3,$$

that is, in this case,

$$\overline{AC}^3 = \overline{AB}^3 + 6\overline{AB}^2 + 12\overline{AB} + 8.$$

Then, using the relation given by the equation, Cardano gets

$$\begin{aligned}\overline{AC}^3 &= (\overline{AB}^3 + 6\overline{AB}^2 + 4) + 12\overline{AB} + 4\\ &= 41\overline{AB} + 12\overline{AB} + 4\\ &= 53\overline{AB} + 4.\end{aligned}$$

But Cardano wants a transformed equation where $\overline{AC}$ is the unknown. He knows that $53\overline{AC} = 53\overline{AB} + 53\overline{BC}$, that is $53\overline{AB} = 53\overline{AC} - 53\overline{BC} = 53\overline{AC} - 106$, so he gets

$$\overline{AC}^3 = (53\overline{AC} - 106) + 4,$$

that is

$$\overline{AC}^3 + 102 = 53\overline{AC}.$$

Now, Cardano knows how to solve the transformed equation in $\overline{AC}$ by (AM XIII). The relation $\overline{AB} = \overline{AC} - \overline{BC}$ yields then a solution of $\overline{AB}^3 + 6\overline{AB}^2 + 4 = 41\overline{AB}$. $\square$

At the end of the chapter, Cardano adds that

> [y]ou must remember, however, that all the solutions for these cases are obtained always by adding from the true and the fictitious solutions of the equations into which they are resolved one-third of the coefficient of x^2. Then, provided that a number remains, even if that which is added is a pure negative, this

[95]See [Cardano 1545, Chapter XXIII, page 44r].

> remainder is a true solution for x. They can also be resolved into other cases of four powers when this is convenient.[96]

2.4.8. Summing up. We observe that Chapters XVII-XXIII have a similar structure as the preceding ones. They all contain a rule, a complete or partial proof of the rule, and some examples.

Moreover, the rules and the proofs themselves have all a similar structure. The general strategy is to bring back a complete cubic equation either to an equation lacking in second degree term (which Cardano already knows how to solve) or to take the cubic root of a number. As said about Chapter XVII, once that the given hypotheses are interpreted in a geometrical environment, the proofs consist in deriving the transformed equation completely relying on the substitution $x = y \pm \frac{a_2}{3}$, on the formula for the expansion of the cube of a binomial given in (AM VI.6), and obviously on the relation given by the given complete equation. The link between the rules and the corresponding proof is translating from an algorithmic language to a geometric language, and backwards.

All the proofs of (AM XVII)–(AM XXIII) follow a very similar pattern. As a consequence of the repetition of so many similar proofs, the diagrams disappear, or better the proofs of (AM XVIII)–(AM XXIII) make reference to the same diagram in (AM XVII) (even if only (AM XVIII) explicitly mentions it), which is nevertheless no more reproduced in the text. These repetitions give the impression that the proceeding is somehow going to become automatic and that the possible geometrical arguments that could justify (AM VI.6) get buried deeper and deeper.

[96] "*Memineris igitur quod homnes horum capitulorum æstimationes, habentur, addendo semper veras et fictas æstimationes capitulorum in quo resolvuntur Tpq̃d, et dummodo numerus relinquatur, etiam id quod additur sit m: purum, illus relictum est rei vera æstimatio. Possunt etiam resolvi in capitula alia quatuor denominationum, ut liquet*", see [Cardano 1545, Chapter XXIII, page 44r]. The translation is mine. Witmer changes the text, substituting "subtracting" and "subtracted" in the place of "adding" and "added", as he judge that this comment refers only to this chapter, see [Cardano 1968, page 154]. But in my opinion, this comment refers to all (AM XVII)-(AM XXIII). In this way, it seems to me more sounding that, adding a pure negative number to another number, Cardano has to take care that it makes no zero. Moreover, Witmer's translation of "*capitulorum*" as "cases" seems to refer to the sub-cases in the immediately preceding examples (all concerning the equation $x^3 + a_2x^2 + a_0 = a_1x$ as they are in Chapter XXIII), whereas "*capitula*" always refers to the whole cubic equation.

2.5. Inter-dependencies between the cubic equations in *Ars Magna*, Chapters XI-XXIII and the *casus irreducibilis*

Now, we would like to summarise the links that bind the cubic equations in Chapters XI-XXIII.

Note in particular that, even if the rules and the proofs in Chapters XI and XII follow a similar pattern, they never make reference to each other. Anyway, it exists a link between them, which is indeed twisted. In fact, in Chapter VII, Cardano provides some transformations of equations (see above, at pages 72 and 74). In particular, he is enabled to pass from $x^3 + a_1x = a_0$ in Chapter XI to $x^3 = a_2x^2 + a_0$ in Chapter XIV and *vice versa*, which means that, if he knows a solution of $x^3 + a_1x = a_0$, then he would also know a solution of $x^3 = a_2x^2 + a_0$, and *vice versa*. In turn, the solving method for the equation in Chapter XIV (allegedly) refers back to Chapter XII. In the end, these remarks link the solving methods in Chapters XI and XII.

Concerning the solving methods in all the other chapters, there are one-way connections between them, as showed by the respective rules and proofs in Chapters XIII-XXIII. A further (two-way) connection is sometimes given by the transformations in Chapter VII. So, when the given equation is connected to another one and when it happens that this last one falls into the *casus irreducibilis*, then also the given equation has to fall into it.

We sum up all the binds between the equations in Chapters XI-XXIII in the figure below. I write '(AM $\langle$*chapter number*$\rangle$.$\langle$*case*$\rangle$.$\langle$*sub-case*$\rangle$)' to make reference to the case '$\langle$*case*$\rangle$' (or sub-case of a case '$\langle$*case*$\rangle$.$\langle$*sub-case*$\rangle$') of the solving method for the equation with which the proposition '$\langle$*Chapter number*$\rangle$' deals. I use three kind of arrows. I write '(AM y) $\longrightarrow$ (AM x)' to mean that (AM y) gives rise to (AM x). This one-way connection is usually found in the solving method of the equation (AM x). When there moreover exists a transformation which enables to pass between the equation in Chapter y and the equation in Chapter x, and *vice versa*, I will write a double arrow. A dashed arrow means that the transformation is described neither in Chapter y nor in Chapter x, but it is anyway present in Cardano's text (namely, in Chapter VII). Note that when a dashed arrow and solid arrow overlap, only the solid arrow is visible. Actually, this does not happen but in two cases, namely between Chapters XI and XIV and between Chapters XVI and XVIII. Finally, when the arrow is dotted, the connection is not explicitly described in Cardano's text. This only happens once, namely between Chapter XIV and XII. There, regarding to Cardano's overall flowing of argumentation, I made a guess,[97] which is in my opinion a very reasonable one, since it is consistent with the rest of Cardano's text.

Note that I do not represent the trivial sub-cases where the transformed equation can be solved by taking the third or second root. Note moreover that I pointed out the sign of the discriminant whenever it is always $\Delta_3 > 0$.

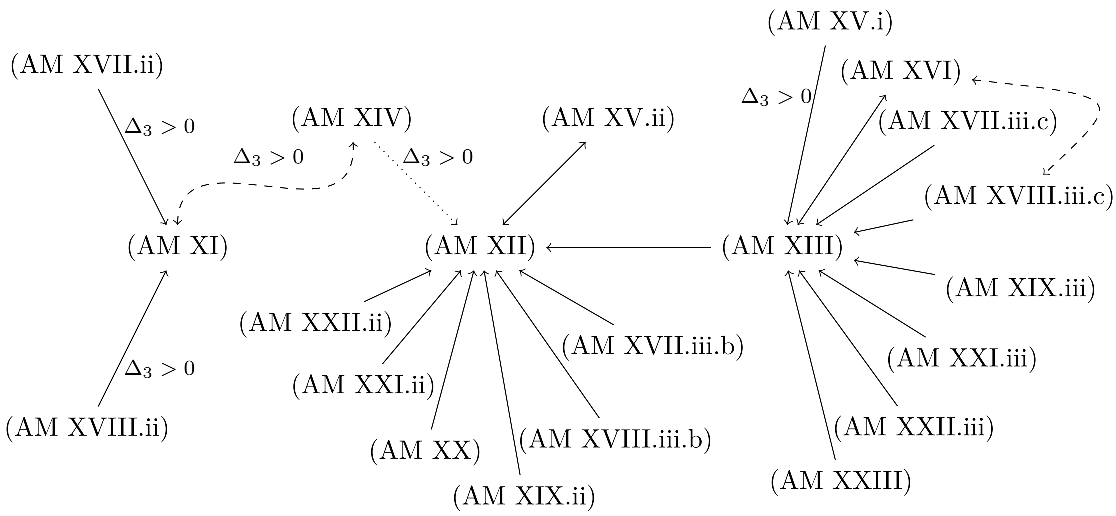

FIGURE 2.18 – Inter-dependencies between the solving methods of the cubic equations in the *Ars magna*.

Let us take a closer look at the diagram. We observe that (directly or not) there is a link between the solution of the equation $x^3 = a_1x + a_0$ in Chapter XII and all the others. Since that equation sometimes runs into the *casus irreducibilis*, it can *a priori* affects all the other equations, the solving method of which directly refers to $x^3 = a_1x + a_0$ (or to $x^3 + a_0 = a_1x$). The left side of the diagram, where (AM XI), (AM XIV), (AM XV.i), (AM XVII.ii), and (AM XVIII.ii) are, and (AM XV.i) are the exceptions, since they always have $\Delta_3 > 0$.

We can then appreciate at best the extent of the problem set by the *casus irreducibilis*. Firstly, it undermines the generality of the cubic formula for $x^3 = a_1x + a_0$ in Chapter XII, since Cardano has to add a condition on the

[97]See above, at page 121.

coefficients, which is equivalent to ask $\Delta_3 \geq 0$. But also all the other families of equations that can have (at least, a sub-case with) a negative discriminant (that is, all the other families of equations, except the ones in (AM XI) and (AM XIV)) are affected by this major inconvenient. If one wants to strongly ground the solving methods for cubic equations, the problem set by the *casus irreducibilis* must then be cleared. In the *Ars magna*, we find only one passage where Cardano has an active try in doing it. It is in Chapter XXV. We have already remarked[98] that Cardano's propositions in that chapter could help to guess a solution of a cubic equation, so that one can lower the degree by polynomial division. But his strategy does not settle at all the problem, even if in some cases it enables one to at least avoid to use the cubic formula. Obviously, Cardano is looking for a solution that works at all turns. In fact, in Chapter XII of the 1570 and 1663 editions of the *Ars magna* the reference to Chapter XXV is given up for the benefit of the *De regula aliza*. Therefore, we expect to find there a method that settle once and for all (and possibly in an algorithmic way) the *casus irreducibilis*.

2.6. Solving quartic equations

The next-to-last chapter, Chapter XXXIX, is entitled "On the rule by which we find an unknown quantity in several stages". It comes as the tail piece after Chapters XXVI-XXXVIII, where Cardano solves some miscellaneous problems thanks to a mix of his cubic formulae and traditional algebraic rules, such as the Golden rule or the Rule of equal position. All the problems considered there lead to quadratic or cubic equations, or to "derivative cases".

2.6.1. Cardano's list of the "most general" quartic equations. Before starting, a brief quotation from **Chapter I** follows.

> Although a long series of rules might be added and a long discourse given about them, we conclude our detailed consideration with the cubic, others being merely mentioned, even if generally, in passing. For, as *positio* [the first power] refers to a line, *quadratum* [the square] to a surface, and *cubum* [the cube] to a solid body, it would be very foolish for us to go beyond this

[98]See above, Section 2.1.4, at page 96.

> point. Nature does not permit it. Thus, it will be seen, all those matters up to and including the cubic are fully demonstrated, but the others which we will add, either by necessity or out of curiosity, we do not go beyond barely setting out.[99]

Despite these words, Cardano not only devotes Chapter XXXIX to solve quartic equations, but also gives some proofs.

In **Chapter XXXIX** Cardano tells how, thanks to a method discovered by Ferrari, they found "all the solutions for equations of the fourth power, square, first power, and number, or of the fourth power, cube, square, and number".[100] Let us try to see in which sense one has to interpret this statement. Cardano makes the following list of 20 quartic equations.

[99]See [Cardano 1968, page 9] or "*[e]t quanque longus sermo de his haberi posset, ac innumera capitulorum series subiungi, finem tunc exquisitæ considerationi in cubo faciemus, cætera, etiam si generaliter, quasi tamen per transennam tractantes. Nanque cum positio lineam, quadrato superficiem, cubus corpus solidum referat, nec utique stultum fuerit, nos ultra progredi, quo naturæ non licet. Itaque satis perfecte docuisse videbitur, qui omnia quæ usque ad cubum sunt, tradiderit, relique, quæ adijcimus, quasi coacti aut incitati, non ultra tradimus*", see [Cardano 1545, Chapter I, paragraph 1, pages 3r-v].

[100]See [Cardano 1968, page 237] or "*per eam habemus omnes æstimationes ferme capitulorum q̃d quadrati et quadrati rerum, et numeri, vel q̃d quadrati cubi, quadrati et numeri*", see [Cardano 1545, Chapter XXXIX, paragraph 2, page 72v].

1 q̄d'q̄d. æquale q̄d. rebus & numero
2 q̄d'q̄d. æquale q̄d. cubis & numero
3 q̄d'q̄d. æquale cubis & numero
4 q̄d'q̄d. æquale rebus & numero
5 q̄d'q̄d. cum cubis æqualia q̄d. & numero
6 q̄d'q̄d. cum rebus æqualia q̄d. & numero
7 q̄d'q̄d. cum cubis æqualia numero
8 q̄d'q̄d. cum rebus æqualia numero
9 q̄d'q̄d. cum q̄d. æqualia cub. & numero
10 q̄d'q̄d. cum q̄d. æqualia rebus & numero
11 q̄d'q̄d. cum q̄d. & rebus æqualia numero
12 q̄d'q̄d. cum q̄d. & cubis æqualia numero
13 q̄d'q̄d. cum q̄d. & numero æqualia cubis
14 q̄d'q̄d. cum q̄d. & numero æqualia rebus
15 q̄d'q̄d. cum numero æqualia cubis & q̄d.
16 q̄d'q̄d. cum numero æqualia cubis
17 q̄d'q̄d. cum numero æqualia rebus & q̄d.
18 q̄d'q̄d. cum numero æqualia rebus
19 q̄d'q̄d. cum cubis & numero æqualia q̄d.
20 q̄d'q̄d. cum rebus & numero æqualia q̄d.

FIGURE 2.19 – List of the quartic equations considered in *Ars magna*, Chapter XXXIX.

We can reasonably expect to find a list of the quartic equations that one can solve by Ferrari's method. We can also reasonably expect not to find the quartic equations that lower in degree, that is, that can be solved using the methods for degrees up to the third in an easy way (for instance, biquadratic equations). But we remark that two (actually, three) kinds of quartic equations which are truly quartic equations do not appear in the classification. They are the complete quartic equations, the three-terms quartic equations where only the second degree term is lacking (that is, the equations where the terms x^4, a_3x^3, a_1x, a_0 appear), and the two-terms quartic equation $x^4 = a_0$ (but one only needs to take the fourth root to solve it, so it is sensible that it does not appear in Cardano's classification). To a closest view, we remark that in this list there are only quartic equations where x^3 and x are never coupled.

Moreover, Cardano observes that

> [i]n all these cases, therefore, which are indeed [only] the most general as there are 67 others, it is convenient to reduce those [equations] involving the cube to equations in which x is present,[101]

such as – he goes on – reducing $x^4 + a_3x^3 = a_0$ to $x^4 = a_1x + a_0$ or $x^4 = a_2x^2 + a_3x^3 + a_0$ to $x^4 = a_2x^2 + a_1x + a_0$. I cannot figure out how Cardano counts the others 67 equations, to the amount of 87 equations. Maybe he includes some of the derivative cases, as he already did for cubic equations.[102]

These introductory remarks by Cardano are not as clear as one might expect. The following questions can be raised. Why are x^3 and x never coupled in the quartic equations in Cardano's list? Why does Cardano call 'general' his list of 20 equations? Why does Cardano judge convenient to reduce x^3 to x? We will try to answer to these questions in the following.

2.6.2. Solving "the square square, some squares, and a number equal to some things". I will not provide a general quartic formula, since it

[101]See [CARDANO 1968, pages 237-8] or "*[i]n his igitur omnibus capitulis, quæ quidem sunt generalissima, ut reliqua omnia sexaginta septem superiora, oportet reducere capitula, in quibus ingreditur cubus, ad capitula, in quibus ingreditur res*", see [CARDANO 1545, Chapter XXXIX, second rule].

[102]See [CARDANO 1968, pages 24-6] or [CARDANO 1545, Chapter II, paragraph 2, pages 7r-8r]. If one wants nevertheless to have a complete list of the (monic) quartic equations with the same features of the cubic equations considered in Chapters XI-XXIII, namely that do not lower in degree, with at least one real root (which means, since all the coefficients are taken positive, never to consider an equation where all the terms are on one side of the equal and 0 on the other side), and that cannot be solved by simply taking the fourth root (that is, $x^4 = a_0$), one finds

- 15 cases for complete equations, where the terms $x^4, a_3x^3, a_2x^2, a_1x, a_0$ appear,
- 7 cases for each of the following 3 families of four-terms equations, the one lacking in the third degree term (where the terms x^4, a_2x^2, a_1x, a_0 appear), in the second degree term (where the terms x^4, a_3x^3, a_1x, a_0 appear), and in the first degree term (where the terms x^4, a_3x^3, a_2x^2, a_0 appear),
- 3 cases for each of the following 2 families of three-terms equations, the ones where the terms x^4, a_3x^3, a_0 or x^4, a_1x, a_0 appear.

That makes 42 equations, out of which Cardano only lists 20 equations. If one adds the equations that lower in degree (7 cases for the family of four-terms equations where $x^4, a_3x^3, a_2x^2, a_1x$ appear, 3 cases for each of the following 4 families of three-terms equations where the terms x^4, a_3x^3, a_2x^2, x^4, a_3x^3, a_1x, x^4, a_2x^2, a_1x appear, and 1 case for each of the following 3 families of two-terms equations where the terms x^4, a_3x^3, x^4, a_2x^2, and x^4, a_1x appear), the 3 biquadratic equations where the terms x^4, a_2x^2, a_0 appear, and the one that can be solved by taking the fourth root one finds 68 equations. They are in no way 87. Anyway, it is more likely that Cardano includes some arbitrarily chosen derivative cases such as biquadratic equations than the other equations which lower in degree or which can be solved by taking the fourth root.

is so complicated to become pointless in practice. In any case, neither Cardano does. He rather gives a step-by-step, explicit procedure.

Firstly, Cardano[103] shows the formula for the square of a trinomial, as he already did in (AM VI.6) for the binomial. He does not explicitly formulate the statement, so I put it into square brackets.

AM XXXIX.3. *[The formula for the square of a trinomial is*

$$(AB + BC + CG)^2 = AB^2 + BC^2 + CG^2 + 2AB\,BC + 2BC\,CG + 2CG\,AB,$$

or

$$\left(\overline{AB} + \overline{BC} + \overline{CG}\right)^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2 + 2\overline{AB}\,\overline{BC} + 2\overline{BC}\,\overline{CG} + 2\overline{CG}\,\overline{AB}]$$

I will not transcribe the proof, only putting the diagram that Cardano uses as a hint. In fact, there are no substantial differences comparing to the proof in (AM VI.6).

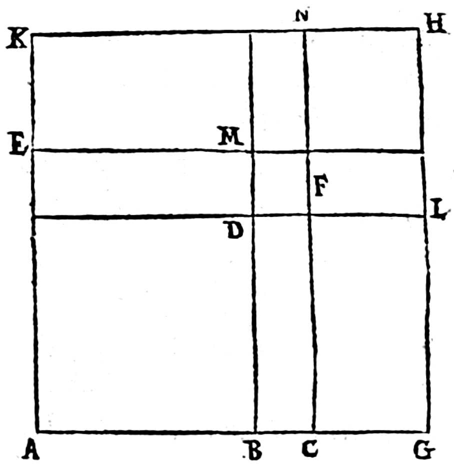

FIGURE 2.20 – *Ars magna* XXXIX.3.

We remark that Cardano will need to apply this formula to quartic equations taking $\overline{AB} = x^2$ and $\overline{BC}, \overline{CG}$ depending on the coefficients.

In the next step, Cardano[104] considers the equation $x^4 + a_2x^2 + a_0 = a_1x$. He remarks that, if $a_2 = 2\sqrt{a_0}$ holds, then it is easy to reduce the left side

[103]See [CARDANO 1968, page 238] or [CARDANO 1545, Chapter XXXIX, paragraph 3, pages 73r-73v].
[104]See [CARDANO 1968, pages 238-9] or [CARDANO 1545, Chapter XXXIX, paragraph 4, page 73v].

$x^4 + a_2x^2 + a_0$ of the equation to the perfect square $(x^2 + \sqrt{a_0})^2$. This is achieved by adding $(2\sqrt{a_0} - a_2)x^2$ to both sides of the equation.

The last step[105] will be to add "as many squares and such a number" to both sides of the equation in order to have the square of a trinomial on the left side and the square of a binomial on the right side in accordance to (AM XXXIX.3). Then, taking the square root, the solution will be easily derived by (AM V.4-6).

In the end, we get a very loose explanation of a general course of actions, especially concerning the last step. Cardano needs to specify it better with the example to "divide 10 into three proportional parts, the product of the first and the second of which is 6".[106] This is one of the problems with which Zuanne de Tonini da Coi used to challenge his adversaries, believing that it could not be solved. It yields to the following quartic equation.

AM XXXIX Problem V. *To solve the equation* $x^4 + 6x^2 + 36 = 60x$.

AM XXXIX Problem V — *Solution.* Cardano and Ferrari[107] add $6x^2$ to both sides of the equation in order to get the square of a binomial on the left side

$$(x^2 + 6)^2 = 6x^2 + 60x.$$

In order to obtain a square also on the right side, they want to add to both sides "enough squares and a number [*tot quadrati et numerus*]", say $b_2, b_0 > 0$,

$$x^4 + 12x^2 + 36 + (b_2x^2 + b_0) = 6x^2 + 60x + (b_2x^2 + b_0)$$

[such that the equation becomes

$$(x^2 + 6 + \gamma)^2 = (\delta x + \epsilon)^2,$$

where the real γ, δ, ϵ are to be further specified depending on the coefficients.] [Let $\overline{CG}$ be a real number (Cardano calls it in such a way to make reference to the diagram for (AM XXXIX.3)).] If they take $b_2 = 2\overline{CG}$ and $b_0 = \overline{CG}^2 + 12\overline{CG}$,

[105]See [Cardano 1968, page 239] or [Cardano 1545, Chapter XXXIX, paragraph 5, page 73v].

[106]See [Cardano 1968, page 239] or "*[f]ac ex 10 tres partes proportionales, ex quarum ductu primæ in secundam producantur 6*", see [Cardano 1545, Chapter XXXIX, problem V, page 73v].

[107]See [Cardano 1545, Chapter XXXIX, pages 73v-74r].

then they know by (AM XXXIX.3) that it holds that

$$x^4 + (12 + 2\overline{CG})x^2 + (36 + \overline{CG}^2 + 12\overline{CG}) = (6 + 2\overline{CG})x^2 + 60x + (\overline{CG}^2 + 12\overline{CG}),$$

that is

$$x^4 + (12 + 2\overline{CG})x^2 + (6 + \overline{CG})^2 = (6 + 2\overline{CG})x^2 + 60x + (\overline{CG}^2 + 12\overline{CG})$$

where the left side is the square $\left(x^2 + (6 + \overline{CG})\right)^2$.

[This means that $\left(x^2 + (6 + \overline{CG})\right)^2$, considered as a polynomial in x, is a square for each $\overline{CG}$. We look for a $\overline{CG}$ such that $(6+2\overline{CG})x^2+60x+(\overline{CG}^2+12\overline{CG})$ is also a square, considered as a polynomial in x. Take α a real such that $\alpha^2 = 6 + 2\overline{CG}$ and take β a real such that $\beta^2 = \overline{CG}^2 + 12\overline{CG}$. Then $\alpha^2 x^2 + 60x + \beta^2 = (\alpha x + \beta)^2$ if and only if $60x = 2\alpha x\beta$, that is, if and only if $(60x)^2 = (2\alpha x\beta)^2$, that is if and only if $\alpha^2\beta^2 = 900$. We get $(6 + 2\overline{CG})(\overline{CG}^2 + 12\overline{CG}) = 900$.] Then,

$$\overline{CG^3} + 15\overline{CG}^2 + 36\overline{CG} = 450.$$

By the cubic formula in (AM XVII), Cardano and Ferrari find

$$\overline{CG} = \sqrt[3]{190 + \sqrt{33903}} + \sqrt[3]{190 - \sqrt{33903}} - 5.$$

This means that Cardano and Ferrari are able to transform the given quartic equation in another quartic where there are two squares on both sides of the equal, provided that they manage to solve a certain cubic equation. This can be done by the cubic formulae in (AM XI)-(AM XXIII). Then – as said – they take the square root on both sides of the equal and solve the quadratic equation that comes out by (AM V.4-6). In this way, both the quadratic and cubic formulae have to be used in order to solve quartic equations.

We moreover remark that Cardano and Ferrari need as usual a formula for the square of the sum of certain quantities – a trinomial, in this case. Once hidden the possibly geometrical argument in the proof of (AM XXXIX.3), no other geometrical argument appears in the solution of (AM XXXIX Problem V). Besides, any geometrical argument cannot at all appear. In fact, Cardano and Ferrari consider $\overline{CG}^2 + 12\overline{CG}$. For homogeneity reasons, this forces the interpretation that all the quantities there have to be considered as numbers, and then the diagram becomes pointless.

Cardano also infers the following formula, to directly draw the cubic equation

$$\overline{CG}^3 + (1 + \frac{1}{4})2\sqrt{a_0}\overline{CG}^2 + a_0\overline{CG} = \frac{1}{2}\left(\frac{a_1}{2}\right)^2.$$

2.6.3. What about the other quartic equations? Using the above method, Cardano is able to solve the equations of the family $x^4 + a_2x^2 + a_0 = a_1x$. Then, some examples follow. We will check them in order to try to find out how Cardano solves the other 19 families of quartic equations that he listed at the beginning. Except (AM XXXIX Problem V), there are eight examples more – called "problems [*quæstiones*]". Problem VIII on $x^4 + 32x^2 + 16 = 48x$ uses the same Ferrari's methods as in (AM XXXIX Problem V).

Problem X on $x^4 = x^2 + 1$ do not need Ferrari's method to be solved, since it is a derivative case.[108]

Problem VI on $x^4 = x + 2$ is a simplified version[109] of Ferrari's method, since x^4 on the left side of the equation is already a perfect square. Then, one step can be skipped and the remaining part of Ferrari's method is unchanged. Namely, Cardano takes $b_2 = 2n$ and $b_0 = n^2$ where n is a (real) number. Then, $\alpha^2 = 2n$ and $\beta^2 = 2 + n^2$, and he has to solve the cubic equation $n^3 + 2n = \frac{1}{8}$.[110]

Problem IX on $x^4 + 4x + 8 = 10x^2$ and Problem XII on $x^4 + 3 = 12x$ have to be a little handled before applying Ferrari's method. In fact, in order to find on the left side of equation a perfect square where x^2 appears, Cardano[111] needs basically to arrange the second and first degree terms on the left side of the equation and the constant term on the right side. In this way, one can solve all the quartic equations where the third degree term does not appear.

[108]See [Cardano 1968, pages 250-1] or [Cardano 1545, Chapter XXXIX, problem X, pages 77v-78r].

[109]See [Cardano 1968, pages 243-6] or [Cardano 1545, Chapter XXXIX, problem VI, pages 75r-75v].

[110]Concerning again Problem VI, Cardano provides another way to solve the equation. He manages to lower the degree of the equation dividing both sides by an appropriate polynomial. Anyway, this is a particular method, since the possibility to find a polynomial which divides the equation depends on one's ability to guess a solution of the equation.

[111]See [Cardano 1968, pages 246-8] or [Cardano 1545, Chapter XXXIX, problem VIII, pages 76r-76v], see [Cardano 1968, pages 248-50] or [Cardano 1545, Chapter XXXIX, problem IX, pages 76v-77v], and see [Cardano 1968, page 252] or [Cardano 1545, Chapter XXXIX, problem XII, page 78v].

Then, it remains to show how the equations where the third degree term appears – which are the half of the 20 equations that Cardano listed above – can be solved by Ferrari's method. Problem VII is on $x^4 + 6x^3 = 64$. Cardano[112] uses (AM VII.6-7) (but also (AM VII.2-3) works) to transform $x^4+6x^3 = 64$ into $y^4 = 6y+4$. Then, he can apply Ferrari's method in his simplified version, since the left side of the equation x^4 is already a perfect square. With a similar justification, one can bring back also the equations $x^4 = a_3x^3 + a_0$ and $x^4 + a_0 = a_3x^3$ respectively to $y^4 + b_1y = b_0$ and $y^4 + b_0 = b_1y$. Moreover, we remark that (AM VII.12) enables to pass from $x^4 + a_2x + a_0 = a_3x^3$ to $y^4 + a_2y^2 + a_0 = a_1y$.

Also Problem XI on $x^4 - 3x^3 = 64$ could be solved in this way – as Cardano says, but he rather prefers to deal with it without transformations. In fact, using the transformations in Chapter VII, one can only deal with the few above cases. Cardano states then a slightly variation of Ferrari's method (the main idea to get two squares on both sides of the equal remaining unchanged) to solve all the quartic equations that he listed, where the third degree term appears.

> You ought to know two things: First, just as x should always remain on the side on which the number and the second power are and should not be on the side of x^4, so the cube, whether positive or negative, should remain with x^4. Secondly, just as the coefficient of x should not change, so also the coefficient of x^3 should not vary. To these we can add a third, namely that [just as] where there are x's we will arrive at the fourth power plus the square plus the number equal to the second power plus or minus the x's plus a number, so in this case we will arrive at the fourth and second powers plus the number equal to the fourth power plus or minus the cube plus the square.[113]

Let us now check how Cardano applies these prescriptions to Problem XI.

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[112] See [Cardano 1968, page 76r] or [Cardano 1545, Chapter XXXIX, problem VII, page 76r].

[113] See [Cardano 1968, page 251] or "*debes igitur scire duo. Primum, quod ut res debent semper manere ab alia parte, a qua est numerus cum quadratis, et non a arte q̃dq̃drati, sic cubi, seu p: seu m: debent manere cum q̃dq̃drato. Secundum, quod ut numerus rerum nunque debet variari, sic nec numerus cuborum. Et possumus addere tertius his, scilicet, quod ubi sunt res, pervenimus ad 1 q̀dq̀dratu p: quad.p: numero, æqualia quad.rebus p: vel m: et numero p: sic hic pervenimus ad q̃dq̃dratu p: quad.p: numero, æqualia q̃dq̃d. cubis p: vel m: et quad.p:*", see [Cardano 1545, Chapter XXXIX, Problem XI, page].

AM XXXIX Problem XI. *To solve the equation* $x^4 - 3x^3 = 64$.

AM XXXIX Problem XI — *Solution.* [We remark that 64 in the right side of the equation is already a perfect square.] Cardano[114] adds $\frac{n^2}{64}x^4 + 2nx^2$ (where n is a real number) to both sides in order to find, on the right side of the equation, a perfect square in which x^2 appears. Then,

$$x^4 - 3x^3 + \left(\frac{n^2}{64}x^4 + 2nx^2\right) = 64 + \left(\frac{n^2}{64}x^4 + 2nx^2\right),$$

that is

$$\left(\frac{n^2}{64} + 1\right)x^4 - 3x^3 + 2nx^2 = \left(8 + \frac{n}{8}x^2\right)^2.$$

[As before, this means that $\left(\frac{n^2}{64} + 1\right)x^4 - 3x^3 + 2nx^2$ is also a square, considered as a polynomial in x. We look for such a n. Take a real α such that $\alpha^2 = \frac{n^2}{64} + 1$ and a real β such that $\beta^2 = 2n$. Then $\alpha^2 x^4 - 3x^3 + \beta^2 x^2 = (\alpha x^2 + \beta x)^2$ if and only if $\alpha^2\beta^2 = \frac{9}{4}$. We get $(\frac{n^2}{64} + 1)2n = \frac{9}{4}$.] Then,

$$n^3 + 64n = 72.$$

[By (AM XI), Cardano finds a value for n. Then, since on both sides of the equal two squares appear, he gets a quadratic equation by taking the square root. Thanks to (AM V.4-6), Cardano knows how to solve it.]

And finally, what about the cases where a_3x^3 and a_1x appear together in the same equation? There are 22 (monic) quartic equations more that do not lower in degree, with at least one real solution, and that cannot be solved by simply taking the fourth root (that is, equations that should have had the right to appear in Cardano's list, but that he does not mention), where both a_3x^3 and a_1x appear. They are the 15 complete equations (which are not in Cardano's list) and the 7 cases for the family of equations where the terms x^4, a_3x^3, a_1x, a_0 appear (which instead are there).

[114] See [CARDANO 1968, pages 251-2] or [CARDANO 1545, Chapter XXXIX, problem XI, pages 78r-78v].

Cardano solves an equation of this last family. Problem XIII is indeed on $x^4 + 2x^3 = x + 1$. Unluckily, Cardano[115] deals very shortly and confusingly with this example. Anyway, at the end, he gives the right (positive, real) solution and affirms that "by this you know the methods for these rules if you have paid careful attention to the examples and the operations".[116] In fact, we can easily check that the methods in (AM XXXIX Problem V) – or, better, in (AM XXXIX Problem XI) – can be adapted to this case.[117]

2.6.4. Summing up. Summing up, Cardano has Ferrari's method available to solve equations like $x^4 + a_2x^2 + a_0 = a_1x$. Thanks to this method, one performs a double reduction to two squares (*a priori* of a trinomial and of a binomial) on both sides of the considered equation. In order to do that, a formula analogous

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[115] See [Cardano 1968, pages 252-3] or [Cardano 1545, Chapter XXXIX, problem XIII, pages 78v-79r].

[116] See [Cardano 1968, page 253] or "*per hæc intelligis modos harum regularum, si exempla hæc diligenter cum suis operationibus animadvertas*", see [Cardano 1545, Chapter XXXIX, problem XIII, page 79r].

[117] Following Cardano's suggestion, if we add x^2 to both sides of the equation $x^4 + 2x^3 = x + 1$, its left side becomes a perfect square $(x^2 + x)^2$. We want instead to add $b_2x^2 + b_1x + b_0$ to both sides

$$x^4 + 2x^3 + (b_2x^2 + b_1x + b_0) = x + 1 + (b_2x^2 + b_1x + b_0)$$

such that we get a perfect square of a trinomial, where the first two terms are x^2, x on the left side and another perfect square (*a priori* of a binomial) on the right side. Take the real γ to be the third term of the trinomial. Applying the formula for the square of a trinomial to $x^2 + x + \gamma$ and by comparison of the coefficients, we get $b_2 = 1 + 2\gamma$, $b_1 = 2\gamma$, and $b_0 = \gamma^2$. Then,

$$(x^2 + x + \gamma)^2 = (1 + 2\gamma)x^2 + (1 + 2\gamma)x + 1 + \gamma^2.$$

This means that $(1 + 2\gamma)x^2 + (1 + 2\gamma)x + 1 + \gamma^2$ is also a square considered as a polynomial in x. We look the for one of such γ. Take a real α such that $\alpha^2 = 1 + 2\gamma$ and take a real β such that $\beta^2 = 1 + \gamma^2$. Then,

$$(1 + 2\gamma)(1 + \gamma^2)x^2 = \left(\frac{1 + 2\gamma}{2}x\right)^2,$$

that is

$$8\gamma^2 + 4\gamma + 3 = 0.$$

The real solution being $\gamma = -\frac{1}{2}$, we get $b_2 = 0$, $b_1 = -1$, and $b_0 = \frac{1}{4}$. The quartic equation becomes

$$(x^2 + x - \frac{1}{2})^2 = \frac{5}{4},$$

from which it follows that

$$x^2 + x - \frac{1}{2} - \frac{5}{4} = 0$$

whose solutions are $x = \frac{1}{2}\left(-1 - \sqrt{3 + 2\sqrt{5}}\right)$ and $x = \frac{1}{2}\left(\sqrt{3 + 2\sqrt{5}} - 1\right)$. The positive solution is the one mentioned by Cardano.

to (AM VI.6) for the square of a trinomial is needed. It is in Proposition (AM XXXIX.3). Moreover, Cardano needs to introduce a certain quantity to be further specified thanks to a technical condition imposed on one of the squares. The condition leads to a cubic equation, which he solves by (AM XI)–(AM XXIII). Then, taking the square root and using (AM V.4-6), Cardano finds (at least) a (real) solution of the quartic equations.

This method can be plainly adapted to solve all the quartic equations that Cardano is likely to consider (that is, monic quartic equations that do not lower in degree, with at least one real root, and that cannot be solved by taking the fourth root). If the quartic equation contains only x^4, a_2x^2, a_1x, a_0 or x^4, a_1x, a_0 (that is, equations number 1, 4, 6, 8, 10, 11, 14, 17, 18, and 20 in Cardano's list), then one needs to apply Ferrari's method in (AM XXXIX Problem V), or the simplified version in Problem VI, or again the slight modification consisting in rearranging the terms as in Problems IX and XII. If the quartic equation contains x^4, a_3x^3, a_0 (that is, equations number 3, 7, and 16 in Cardano's list), one first of all transform the given equation in one that contains only x^4, a_1x, a_0 thanks to (AM VII.2-3) or (AM VII.6-7) as in Problem VII, and then goes on as above. Or one can also adapt Ferrari's method as in Problem XI. Finally, if the quartic equation contains x^4, a_3x^3, a_1x, a_0 (that is, equations number 2, 5, 9, 12, 13, 15, and 19 in Cardano's list), it is also possible to adapt Ferrari's method (see Problem XIII). In this case, one has to add a more composite number on both sides of the equation in order to get two squares on both sides, so that in the end we have a bit more complicated version.

For the remaining 15 families of quartic equations where a_3x^3 and a_1x appear together, no hints are given by Cardano himself. We can maybe suppose that Cardano knows the transformation to make the term of third degree vanish, so that he gets an equation in which only x^4, a_2x^2, a_1x, a_0 appear. Then, it is possible that Cardano calls 'general' the quartic equations in his list excluding the complete ones because their solving method is the one that can be widely applied to the others, once that the other families of equations are transformed to fit it.

We finally recall that, in this context, geometry possibly enters only in the proof for (AM XXXIX.3), as it was the case in (AM VI.6). But then, once anchored on a solid ground this formula, no more geometry is used in manipulating quartic equations.

CHAPTER 3

The prequel: Cardano got in touch with some general treatments of equations

In the *Ars magna*, we have seen what were the most advanced strategies and methods at Cardano's disposal to deal with equations. We have also remarked that his solving methods for cubic equations were supposedly strongly influenced by Tartaglia's poem, even if Cardano implemented it in a relevant way. Let us now take a step backward to go and check in the earlier treatises if we can find some foreshadows of these methods, or if we find different directions towards which Cardano previously headed. We will now focus on the *Practica arithmeticæ* and on the *Ars magna arithmeticæ*, since these are the only two other works – excepted the *Ars magna* and the *De regula aliza* – in which Cardano deals with cubic equations.

3.1. Scattered contacts with equations in the *Practica arithmeticæ*

Between 1536 and 1537, Cardano relates[1] that he gets back to one of his earlier mathematical works, called *Arithmetica rudimenta*, and revises it. In 1539 the outcome is published in Milan by Ioannis Antonins Castellioneus for Bernardini Calusci under the title *Hieronymi C. Cardani medici mediolanensis, Practica arithmetice, et mensurandi singularis, in quaque preter alias continentur, versa pagina demonstrabit* or *Practica arithmeticæ*. We also have a posthumous edition in the fourth volume of 1663 edition of Cardano's *Opera omnia*, printed in Lyon by Ioannis Antonii Huguetan and Marci Antonii Ravaud. No manuscript is known.[2]

[1] See [Cardano 1663g, page 41].
[2] See [www.cardano.unimi.it last checked January 22, 2014].

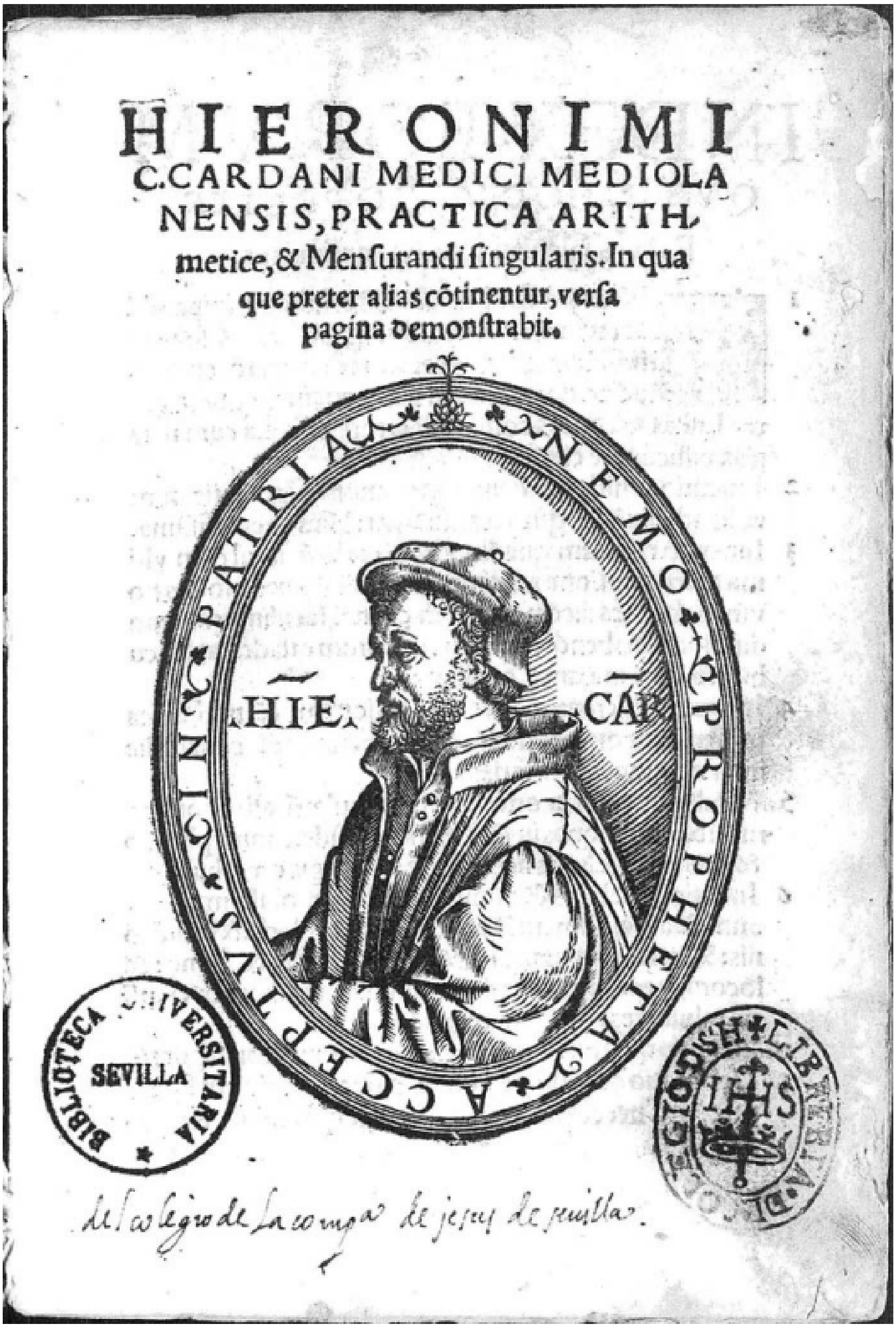
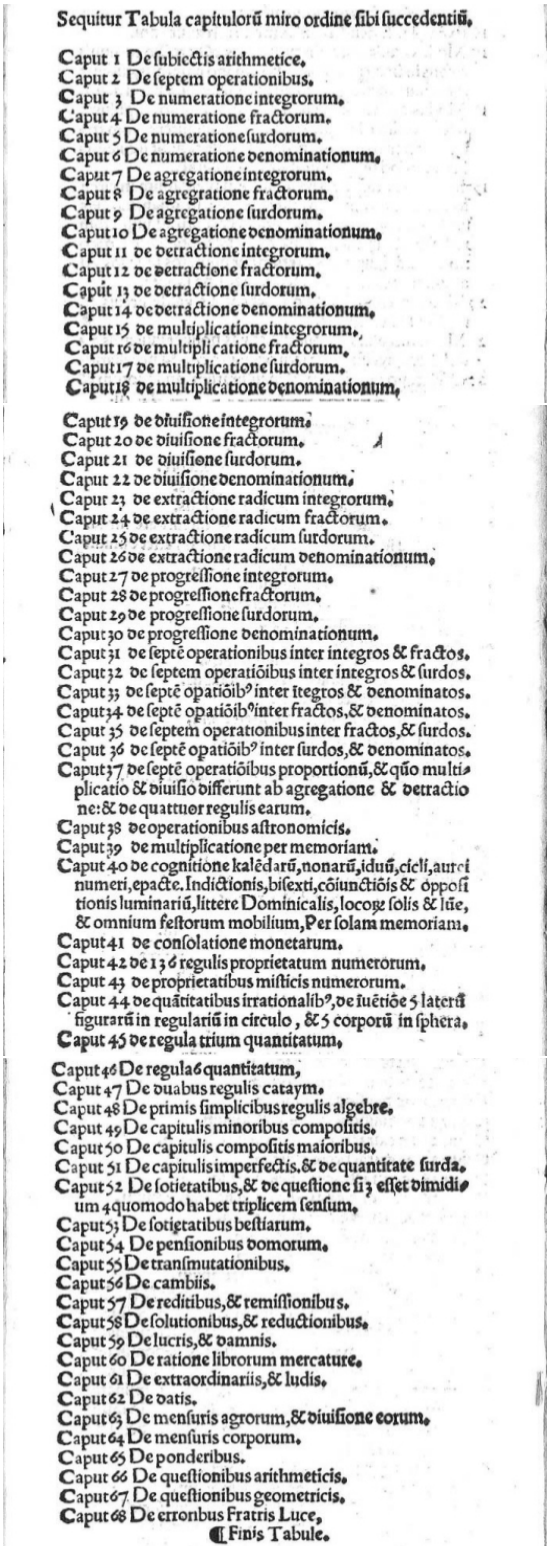


FIGURE 3.1 – Title page of the *Practica arithmeticæ* (1539)



Sequitur Tabula capitulorũ miro ordine ſibi ſuccedentiũ.

¶ Finis Tabule.

FIGURE 3.2 – Table of contents of the *Practica arithmeticæ* (1539).

The *Practica arithmeticæ* can be divided in two parts, namely Chapters I-LXV, devoted to the theory, and Chapters LXVI-LXVII, which gather some examples and problems. The most of the theoretical part concerns numbers and operations with numbers in a traditional style. Nevertheless, we find in Chapters XLV-LI a sort of short appendix dealing with equations.

In the next section, we will have a glance at the statements concerning equations. In particular, we will focuses on the statements in Chapter LI concerning cubic equations, which are the only ones to present innovations comparing to older *abaco* treatises.

3.1.1. The appendix devoted to equations. Let us see more in detail the contents of this appendix. Chapter XLVIII deals with two-terms equations from first to fourth degree where, in order to find a solution, one has simply to take the n^{th}-root, with $n = 1, 2, 3, 4$. Once and for all, Cardano observes that one has always to reduce to unity the coefficient of the higher degree term. Then, Chapter XLIX deals with the "minor composite chapters [*capitula minora composita*]",[3] that is, with quadratic equations and with equations that reduce to them (in the *Ars magna* they are called 'derivative'). Chapter L deals with the "greater composite chapters [*capitula composita maiora*]", that is, with biquadratic equations and with equations that reduce to them. Here, Cardano only the usual rules.

Finally, there is **Chapter LI** "On the imperfect cases [*De modis omnibus imperfectis*]", which is a miscellany. There, Cardano deals with surd numbers, continue proportions, taking n^{th}-roots, equations that can be reduced to quadratic

[3]The term 'chapter [*capitulum*]' is commonly used to refer to family of equations. Indeed, as Jeff Oaks argues in a private communication (thanks Jeff!), this term is very probably a mistranslation from the Arabic '*bāb*', which means 'door; gate; opening, gateway; entrance, chapter, section, column, rubric; group, class, category; field, domain' among other things. In most of the Arabic books '*bāb*' means 'chapter'. In the algebra books of Al-Khwārizmī, Abū Kāmil, and others it also takes the meaning of 'class, category, type, kind' when for instance they speak of the "six types" of equations. This is probably were the Latin translators as Robert of Chester or Gerardo of Cremona were confused. In fact, both of them translate Al-Khwārizmī's '*bāb*' as '*capitulum*' when they should have chosen a word which means 'kind' or 'type'. So they will often write the "*sex capitula*".
We moreover remark that the mistraslated term was part of the common mathematical terminology of the Italian algebraists of the time. In fact, we find for instance a "*Capitolo di potenze, e Tanti eguali à numero*" in Bombelli's *Algebra* and some "*Capitoli de Cosa, e Cubo equal à numero ed altri suoi aderenti*" in Tartaglia's *Quesiti et inventioni deverse*.

equations, the Rules *de modo*, *de duplici*, and *de medio*,[4] and with cubic equations. We will only consider a choice of interesting paragraphs from this chapter. A final remark before starting. Unlike in the *Ars magna*, all the results in the *Practica arithmeticæ* are stated – as it was usual for the time – by the means of numerical examples. In no way Cardano tries to resume the results in a general statement or formula. Nevertheless, I will do that, since it is more easily readable.

In the *Practica arithmeticæ*, there are mainly two strategies to deal with cubic equations, namely polynomial division and transformation of equations. Concerning polynomial division, Cardano reminds the following proposition.

PA LI.25.

$$\frac{x^3+n^3}{x+n} = x^2 - nx + n^2 \quad \textit{and} \quad \frac{x^3-n^3}{x-n} = x^2 + nx + n^2,$$

with n natural.

The two rules come from Chapter XXII, paragraphs 11, 12, and 16 where they are simply stated without any justification.

Then, Cardano shows how to lower some cubic equations to the second degree using these two formulae.

PA LI.26. *Consider the depressed cubic equations $x^3 = a_1x+a_0$ and $x^3+a_0 = a_1x$. If the following condition is fulfilled*

$$na_1 = n^3 + a_0,$$

with n natural, then one gets a quadratic equation by polynomial division.

In fact, consider firstly the equation $x^3 = a_1x + a_0$. If $na_1 = n^3 + a_0$, then $a_0 = na_1 - n^3$ and the equation becomes $x^3 + n^3 = a_1(x + n)$. Then, $x + n$ divides the left side of the equation by (PA LI.25), and the right side too. In this way, we obtain the quadratic equation $x^2 + n^2 = nx + a_1$. Similarly, the second equation becomes $x^3 - n^3 = a_1(x - n)$ and, dividing both sides by $x - n$, we get the quadratic equation $x^2 + nx + n^2 = a_1$.

PA LI.27 i. *Considers the complete cubic equations $x^3 + a_1x = a_2x^2 + a_0$, $x^3 + a_1x + a_0 = a_2x^2$, $x^3 + a_2x^2 = a_1x + a_0$, $x^3 + a_0 = a_2x^2 + a_1x$, and $x^3 =$*

[4]See [Gavagna 1999] and [Gavagna 2010].

$a_2x^2 + a_1x + a_0$. *If* $\pm n$*, with* n *natural, is a solution, then one gets a quadratic equation by polynomial division.*

This proposition works in a way similar to (PA LI.26). The two propositions are justified through some examples. Note that they partially correspond to (AM XXV.6) and (AM XXV.10), taking $n = f$.

We remark that, in all the examples by Cardano, n is natural, but we can with no extra difficulty suppose that n is real, since the only requirement imposed on it is to know how to take its third power n^3 or, conversely, its third root $\sqrt[3]{n}$. This was a common knowledge at that time, and Cardano taught in the preceding chapters how to perform this operation on real numbers.

At the very end of paragraph 27, Cardano remarks the following.

PA LI.27 ii. *In the equations where the "extreme denominations [*denominationes extremæ*]" (that is, the highest degree term and the constant term) are on the same side of the equal compared to the to the other terms, if there is a solution, there are always two.*

Again, his only justifications here are the examples where, in the case of cubic equations, given a solution, Cardano usually employs polynomial division to lower the degree of the equation. Since he obtains a quadratic equation, he knows how to solve it. The examples mentioned by Cardano are equations of the families $x^3 + a_0 = a_1x$ and $x^3 + a_0 = a_2x^2$. [5]

Concerning transformation of equations, in paragraph 28, Cardano makes one particularly remarkable statement (among various others) concerning continue proportions.

PA LI.28. *Consider* a, b, c, d *real such that* $a : b = b : c = c : d$. *Then*

$$(a + b)(a + d) = \left(\frac{a + d}{b} + \frac{a + d}{c}\right) b^2.$$

[5]We remind that Cardano makes the same consideration in the *Ars magna* at the end of Chapter I, see here note 82, page 139. Anyway, the here no further explanations are given.

As usual, Cardano do not justify it but by some examples.[6] Then, Cardano observes that this rule is useful to solve $x^3 + a_2x^2 = a_0$, $x^3 + a_0 = a_2x^2$, and $x^3 = a_2x^2 + a_0$. We remark that only $x^3 + a_0 = a_2x^2$ will be mentioned later on.

In paragraph 32, Cardano shows how to pass from $x^3+a_0 = a_2x^2$ to $y^3+a_0 = a_1y$. He considers $x^3 + 64 = 18x^2$ and takes $x = a$, $18 = a + d$, and $\sqrt[3]{64} = b$, where $a : b = b : c = c : d$. [We write $y = c$.] By the continue proportion, it follows that $d = \frac{c^2}{b} = \frac{y^2}{\sqrt[3]{64}}$. Then, $a = 18 - d = 18 - \frac{y^2}{\sqrt[3]{64}}$. By paragraph 28, Cardano knows that it holds

$$\left(18 - \frac{y^2}{\sqrt[3]{64}} + \sqrt[3]{64}\right) 18 = 18\sqrt[3]{64} + \frac{18}{y}\sqrt[3]{64},$$

that is $\sqrt[3]{\frac{729}{8}}x^3 + \sqrt[3]{23887872} = 324x$[, that is $x^3 + 64 = 72x$]. Moreover, Cardano gets that the first of the four continue proportionals a is solution of $x^3 + b^3 = (a + d)x^2$ and that the third c is solution of $x^3 + b^3 = b(a + d)x$. We can resume the rule as follows.

PA LI.32 (resumed). $x^3 + a_0 = a_2x^2 \xrightarrow{x=a_2-\frac{y^2}{\sqrt[3]{a_0}}} y^3 + a_0 = \left(a_2\sqrt[3]{a_0}\right) y.$

We remark that, by the continue proportion, it follows that $ac = b^2$, that is $xy = \left(\sqrt[3]{a_0}\right)^2$. Subsequently, we can restate the rule as follows.

PA LI.32 (restated). $x^3 + a_0 = a_2x^2 \xrightarrow{y=\frac{(\sqrt[3]{a_0})^2}{x}} y^3 + a_0 = \left(a_2\sqrt[3]{a_0}\right) y.$

We already met this transformation in the *Ars magna*. In fact,[7] this is one particular case of the rule (AM VII.2-3). We can now observe that Cardano achieved this rule through a property of continue proportions.

3.1.2. Summing up.

For the most part, the *Practica arithmeticæ* reminds of an *abaco* treatise, insomuch as Cardano himself never considered it as a

[6]Anyway, it is a straight calculation to check it. In fact,

$$(a + b)(a + d) = a^2 + ad + ba + bd.$$

By the continue proportions follows that $b^2 = ac$. Then

$$\left(\frac{a+d}{b} + \frac{a+d}{c}\right) b^2 = \frac{a+d}{b}b^2 + \frac{a+d}{c}ac = ab + db + a^2 + ad.$$

[7]See here, at page 72.

masterpiece.[8] Even if it does not contain much on cubic equations, it helps to enlighten all the same on the directions undertaken by Cardano before 1539, when he finally received Tartaglia's poem.

In this treatise, we find two techniques to deal with cubic equations, namely polynomial division and transformations of equations. Using polynomial division, provided to have a (real) solution of a cubic equation, Cardano is able to find the remaining solutions (if they are real). In certain cases, he is also able to discuss the number of solutions. Using transformation of equations, Cardano knows how to pass from $x^3 + a_0 = a_2x^2$ to $y^3 + a_0 = a_1y$, and *vice versa*. Both techniques come from surveys on continue proportions. Since these techniques do not ask for requirements on the coefficients of the equations, they are always effective. But neither of the two can provide by itself a way to get a solution of a cubic equation starting only from the coefficients, that is a formula. In fact, in order to fully apply these two techniques, one has to already have at hand a solution.

Therefore, in the *Practica arithmeticæ*, we come across Cardano's earlier investigations on cubic equations. He approaches at first the families of equations $x^3 = a_1x + a_0$, $x^3 + a_0 = a_1x$, and $x^3 + a_0 = a_2x^2$, but submits only non-systematical methods. At this first stage, each result on cubic equations comes after a corresponding result on continue proportion, so that in the *Practica arithmeticæ* the cubic equations play the part of side effects of proportions, meaning that they are not already a subject of interest by themselves. As mentioned above, traces of this approach will still be there six years after in the *Ars magna*.

3.2. A more systematic approach to cubic equations in the *Ars magna arithmeticæ*

As said, only recently the *Hieronymi Cardani Ars magna arithmeticæ, seu liber quadraginta capitulorum et quadraginta quæstionum*, or *Ars magna arithmeticæ*, has been restored to its fair chronological position. As such, it follows the *Practica arithmeticæ* and foreruns the *Ars magna*, and this is why we will deal with it now. The common belief that the *Ars magna arithmeticæ* was subsequent to the *Ars magna* had been raised by the fact that the only printed version that we

[8]See [Gavagna 2010, page 64].

have available is in the 1663 *Opera omnia*. There, where the publishers together with the editor Spon were supposed to follow a chronological sequence of the treatises, they inverted the order. We also have available the manuscript N 187 at the Biblioteca Trivulziana in Milan, which partially attests for the *Ars magna arithmeticæ*.

The contents of the *Ars magna arithmeticæ* are clearly three-folded. In Chapters I-XVII the so-called arithmetisation of *Elements*, Book X is developed. There, Cardano deals with rules concerning the irrational numbers called *binomia* and *recisa*.[9] In particular, in Chapters I-XII Cardano establishes the terminological correspondences between the different kinds of lines in Euclid's classification and the *binomia* and *recisa*. Moreover, he sets the rules to perform the basic operations on these numbers. In Chapters XIII, XIV, and XVII, he teaches how to handle with the powers of these quantities. Finally, Chapters XV and XVI are devoted to continue proportions.

On the other hand, Chapters XVIII-LX deal directly with equations – this is why we will mainly focus on them. In particular, in Chapters XVIII-XIX and XXII-XXVI Cardano states which types of *binomia* and *recisa* can or cannot be solutions of some cubic and quartic equations. Chapter XX is a lists of all the equations that Cardano considers. Chapter XXI is on transformations of equations. Chapter XXVII is on the ways to get in touch with solving equations. In Chapters XXVIII-XXX, XXXII, XXXIV, and XXXVI, Cardano solves depressed cubic equations and cubic equations lacking in the first degree term, whereas in Chapters XXXI, XXXIII, XXXV, and XXXVII he gives some "particular" rules on them. Chapter XXXIX contains a partial treatment of complete cubic equations. Finally, in Chapter LX, Cardano deals with the derivative cases and then presents, in the same style of Chapters XXIV-XL of the *Ars magna*, a miscellaneous of fringe problems, which are the "questions" recalled by the title of the *Ars magna arithmeticæ*. We remark that no chapter is specifically devoted to quadratic equations and that Cardano only glances over some families of quartic equations in Chapter XXXVIII (and, in any case, he never gives rules to solve them).

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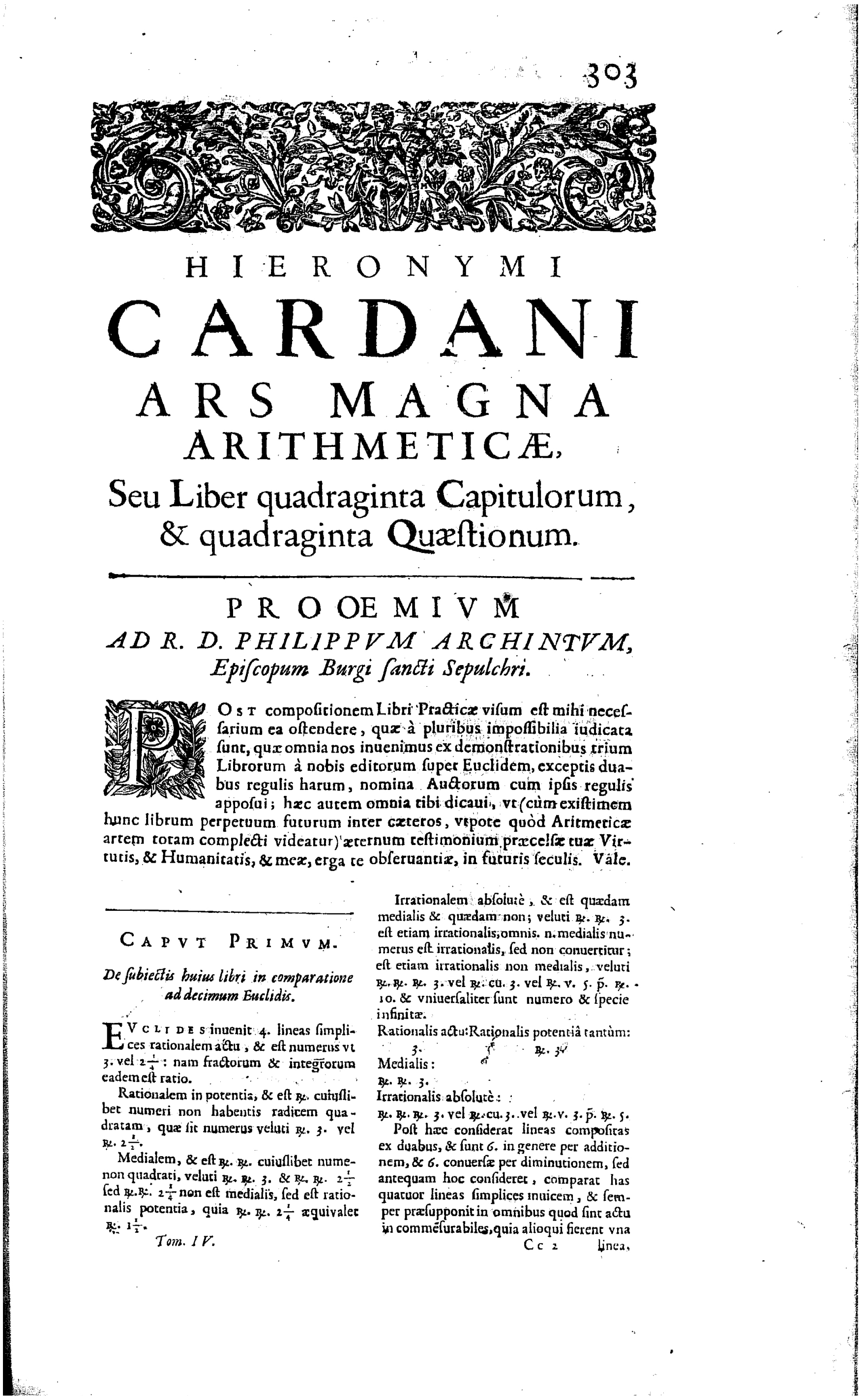
[9] See above, at page 82 or [HEATH 1956c, Book X, definitions II.1-6, pages 101-102].



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HIERONYMI

CARDANI

ARS MAGNA

ARITHMETICÆ,

Seu Liber quadraginta Capitulorum, & quadraginta Quæſtionum.

PROOEMIVM

AD R. D. PHILIPPVM ARCHINTVM, Epiſcopum Burgi ſancti Sepulchri.

POST compoſitionem Libri Practicæ viſum eſt mihi neceſſarium ea oſtendere, quæ à pluribus impoſſibilia iudicata ſunt, quæ omnia nos inuenimus ex demonſtrationibus trium Librorum à nobis editorum ſuper Euclidem, exceptis duabus regulis harum, nomina Auctorum cum ipſis regulis appoſui; hæc autem omnia tibi dicaui, vt (cùm exiſtimem hunc librum perpetuum futurum inter cæteros, vtpote quòd Aritmeticæ artem totam complecti videatur) æternum teſtimonium præcelſæ tuæ Virtutis, & Humanitatis, & meæ, erga te obſeruantiæ, in futuris ſeculis. Vale.

CAPVT PRIMVM.

De ſubiectis huius libri in comparatione ad decimum Euclidis.

EVCLIDES inuenit 4. lineas ſimplices rationalem actu, & eſt numerus vt 3. vel $2\frac{1}{4}$: nam fractorum & integrorum eadem eſt ratio.

Rationalem in potentia, & eſt ℞. cuiuſlibet numeri non habentis radicem quadratam, quæ ſit numerus veluti ℞. 3. vel ℞. $2\frac{1}{2}$.

Medialem, & eſt ℞. ℞. cuiuſlibet numenon quadrati, veluti ℞. ℞. 3. & ℞. ℞. $2\frac{1}{2}$ ſed ℞.℞. $2\frac{1}{4}$ non eſt medialis, ſed eſt rationalis potentia, quia ℞. ℞. $2\frac{1}{4}$ æquivalet ℞. $1\frac{1}{2}$.

Irrationalem abſolutè, & eſt quædam medialis & quædam non; veluti ℞. ℞. 3. eſt etiam irrationalis; omnis. n. medialis numerus eſt irrationalis, ſed non conuertitur; eſt etiam irrationalis non medialis, veluti ℞. ℞. ℞. 3. vel ℞. cu. 3. vel ℞. v. 5. p̄. ℞. 10. & vniuerſaliter ſunt numero & ſpecie infinitæ.

Rationalis actu: Rationalis potentiâ tantùm:
3. ℞. 3.
Medialis:
℞. ℞. 3.
Irrationalis abſolutè:
℞. ℞. ℞. 3. vel ℞. cu. 3. vel ℞. v. 3. p̄. ℞. 5.

Poſt hæc conſiderat lineas compoſitas ex duabus, & ſunt 6. in genere per additionem, & 6. conuerſæ per diminutionem, ſed antequam hoc conſideret, comparat has quatuor lineas ſimplices inuicem, & ſemper præſupponit in omnibus quod ſint actu incommēſurabiles, quia alioqui fierent vna

Tom. IV. Cc 2 linea,

FIGURE 3.3 – Title page of the *Ars magna arithmeticæ* (1663).

V.

Ars magna Arithmeticæ.

Index Capitum.

FIGURE 3.4 – Table of contents of the *Ars magna arithmeticæ* (1663).

Finally, after Chapter XL we find a miscellaneous of solved problems allocated over 40 questions.

In the following, we will consider as usual a choice of some interesting chapters, namely the ones that will turn out to be useful to interpret the *De regula aliza*. First of all and very briefly, I will show in Section 3.2.1 that some considerations on the square roots of negative quantities very similar to the ones in the *Ars magna* are also echoed in the *Ars magna arithmeticæ*. Going on to recall the topics already highlighted while analysing the *Ars magna*, I will report in Section 3.2.2 on Cardano's heuristics and in Section 3.2.3 on some techniques to transform equations. Obviously, I will deal with the cubic formulae. It will be done in Section 3.2.5. On the contrary, the study of "general" shapes for irrational solutions is the peculiarity of the *Ars magna arithmeticæ*. I will deal with this subject in Section 3.2.4. Together with Section 3.2.6 on "particular" solving methods, it will also be useful to clarify the opposition between "particular" and "general".

3.2.1. Shortly again on the square roots of negative numbers. To firstly evacuate a topic that pops up its full significance only in Chapter 4, I will very briefly deal with the square roots of negative numbers. We find them in the **Question 38** at the end of the *Ars magna arithmeticæ*.

There, among other problems, Cardano solves the systems

$$\begin{cases} 6 = x + y \\ 16 = xy \end{cases}$$

and

$$\begin{cases} 6 = x + y \\ 20 = xy \end{cases},$$

the solutions of which respectively are $x = 3 + \sqrt{-7}$ and $y = 3 - \sqrt{-7}$, and $x = 3 + \sqrt{-13}$ and $y = 3 - \sqrt{-11}$. Cardano simply gets to them applying the quadratic formula. Then, he verifies that those are the demanded values. For instance, in the first case

$$(3 - \sqrt{-7})(3 - \sqrt{-7}) = 9 - (-7) = 16,$$

since ‘minus minus 7 makes plus 7’.[10]

Finally, Cardano remarks that, since $\sqrt{9} = \pm 3$, then $\sqrt{-9}$ can be neither 3 nor -3, but rather it must be of a “third hidden nature”.[11]

We observe that we find here all the ingredients of *Ars magna*, Chapters I and XXXVII (and, by the way, the example provided are very similar when not identical).[12] Basically, there are the usual sign rules to multiply negative quantities and the idea that the square roots of negative numbers are to be kept separated from the other numbers. Again, we implicitly find that $(\sqrt{-a})^2 = -a$ must hold.

3.2.2. Ways to solve new cases. Cardano places **Chapter XXVII** “On the ways for finding new chapters [*De modis inveniendi capitula nova*]” immediately before the part strictly devoted to solve cubic equations. We will account for it in advance. As we expect from the title, it is analogous to Chapter VI in the *Ars magna*.

Cardano[13] suggests the following heuristic methods

AMA XXVII.1 by geometric demonstrations (“the nicest one [*pulcherrima*]”),

AMA XXVII.2 by undertaking a known issue in another way (“very nice [*valde pulcher*]”),

AMA XXVII.3 by assuming the solution,

AMA XXVII.4 by derivation,

AMA XXVII.5 by similitude:

- **i** increasing the solutions,
- **ii** transforming equations,
- **iii** of the solutions,
- **iv** of the equations.

In Cardano’s opinion, solving new cases “by a geometrical demonstration [*per demonstrationem Geometricam*]” (AMA XXVII.1) is the best way. It is “universal [*universalis*]” and “general [*generalis*]”. As an example of cases solved by

[10]“*Quod etiam m: m: 7 sit p: 7*”, see [Cardano 1663c, Question 38, page 372].

[11]“*Et nota quod R p: 9 est 3 p: vel 3m: nam p: [in p:] et m: in m: faciunt p:. Igitur R m: 9 non est 3 p: nec m: sed quædam tertia natura abscondita*”, see [Cardano 1663c, Question 38, page 373].

[12]See above, Section 2.1.1 at page 61.

[13]See [Cardano 1663c, Chapter XXVII, pages 338-41].

geometrical demonstrations, Cardano mentions the quadratic equations[14] and $x^3 + a_1x = a_0$. We have already founded a similar argument in Chapter VI, paragraph 5 in the *Ars magna*, see above at page 99.

Then, comes the method of solving new cases "by undertaking a known issue in another way [*ut accipias quætionem notam per aliquam viam*]" (AMA XXVII.2), which is also a good way in Cardano's opinion. The example given by Cardano is to solve

$$\begin{cases} x + y = y^2 \\ xy = 6 \end{cases}.$$

Depending on the substitution that one chooses, one obtains a depressed cubic equation or a cubic equation lacking in the second degree term. Then, being able to solve one out of the two equations, he will also know the solution of the other. We have already founded exactly the same argument (and almost the same example, with 8 instead of 6) in Chapter VI, paragraph 4 in the *Ars magna*, see above at page 70. There, this method was called "by transmutation".

Then, (AMA XXVII.3) consists in assuming a solution and draw a cubic equation – that is, the coefficients, given a certain family to which the cubic equation belongs – which has that same solution. (AMA XXVII.4) is the usual way to derive biquadratic equations – or, in general, higher degree equations – from the quadratic ones. These two methods also appear in the *Ars magna* (and with no major variations), even if we did not stop and pay attention to them, see [Cardano 1545, Chapter VI, paragraph 2 and 1, pages 14v-15v].

Finally, Cardano proposes the four-folded method of "similitude [*similitudo*]". It is directly comparable to the quotation from *Ars magna*, Chapter VI, paragraph 3 (see above at page 70). The method of "similitude by increasing the solutions [*similitudo augmentorum in æquationibus*]" (AMA XXVII.5 i) is not clearly

[14]Cardano speaks about "the first three composite chapters of the algebra [*prima tria capitula Algebræ composita*]". I interpret these as being quadratic equations, since they are the equations which appear in Chapters XLIX "On the smaller composite chapters [*De capitulis minoribus compositis*]" and L "On the greater composite chapters [*De capitulis compositis maioribus*]" of the *Practica arithmeticæ* and Chapter V "On finding the solutions for equations composed of minors [*Ostendit æstimationem capitulorum compositorum minorum, quæ sunt quadratorum, numeri, et rerum*]" of the *Ars magna*. In this interpretation, since quadratic equations are the topic which comes usually after having solved simple equations like $x^n = a_0$ with n natural, "composite" means 'to have more than one term'.

explained: it seems to be a mixing of (AMA XXVII.3) and (AMA XXVII.4). In fact, Cardano refers back to Chapter XXII, where he showed how to construct some quartic equations that lower to second degree starting from a given solution. It corresponds to *Ars magna*, Chapter VI, paragraph 3, point [2]. The method of "similitude by transformations of equations [*similitudo conversionis æquationis*]" (AMA XXVII.5 ii) corresponds to *Ars magna*, Chapter VI, paragraph 3, point [3] and to *Ars magna*, Chapter XIV (see above, at page 119). We will provide a detailed account on transformations of equations in the next section. The method (AMA XXVII.5 iii) is by "similitude of the solutions [*similitudo æquationum*]". This means that, knowing, for instance, that $\sqrt[3]{4} - \sqrt[3]{2}$ is a solution of $x^3 + 6x = 2$, one should pay attention to the behaviour of the cubic roots appearing in x^3 and $6x$, once that the solution is substituted and the calculation developed. In fact, since they are equal but opposite in sign, they vanish. Then, one should look for a similar behaviour when he forces $\sqrt[3]{4} - \sqrt[3]{2}$ to be a solution of a certain equation. It turns out that that the equation $x^3 = 6x + 6$ fits. This method corresponds to *Ars magna*, Chapter VI, paragraph 3, point [4], or at least to a part of it. The method (AMA XXVII.5 iv) is by similitude of the equations. Cardano remarks that, since the quadratic formulae for $x^2 = a_1x + a_0$ and $x^2 + a_1x = a_0$ are composed by the same quantities up to changing signs, the same should[15] holds for $x^3 = a_1x + a_0$ and $x^3 + a_1x = a_0$. If one stops to the mere quotation, this method could correspond to *Ars magna*, Chapter VI, paragraph 3, point [1]. But we have seen that, in the context of the *Ars magna*, this quotation had a deeper meaning.[16]

We remark that, in this chapter, almost the whole content (with the exception of the formula for the cube of a binomial) of *Ars magna*, Chapter VI is covered a in dispersed order. The heuristic conceptions of the *Ars magna*, then, stay very close to the ones in the *Ars magna arithmeticæ*. Even more noticeable is the fact that Cardano's heuristics gets more and more grounded in the *Ars magna arithmeticæ* conceptions, abandoning the starting point of the properties

[15]We observe nevertheless that the parallelism in not complete, since in the cubic case the quantities under the square root vary in sign.

[16]Namely, in the *Ars magna* there is much more than a simple relation between the shapes of the cubic formulae for $x^3 = a_1x + a_0$ and $x^3 + a_1x = a_0$. In fact, also their proofs are strictly related, see above, at pages 108 and 151.

of continue proportions as it was in the *Practica arithmeticæ*. In fact, even if Cardano still mentions his results on continue proportions in Chapters XV and XVI, they have no more remainders to the part on equations.

3.2.3. Transformations of equations. As an introduction to **Chapter XXI** "On the reciprocal exchange of chapters [*De permutatione capitulorum invicem*]", Cardano[17] begins stating that, among cubic equations, there are two principal cases, $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$, whereas the others four, namely $x^3 + a_0 = a_1x$, $x^3 + a_2x^2 = a_0$, $x^3 = a_2x^2 + a_0$, and $x^3 + a_0 = a_2x^2$, derive from those. Complete cubic equations are hardly considered in the above list. Anyway, in the transformations that follow we will find one family of complete cubic equations (and also some quartic equations).

AMA XXI.1.

$$x^3 = a_2x^2 + a_0 \xrightarrow[y=\frac{a_0}{a_2 x}]{} y^3 + \frac{a_0}{a_2}y = \frac{a_0^2}{a_2^3}.$$

Cardano says that this rule is taken "from the nineteenth question below [*ex decima nona quæstione inferius*]". Since he does not specify in which chapter the question should be, it is not easy to recover it. One could argues that he refers to the forty questions at the end of the last chapter of the *Ars magna arithmeticæ*, but the system solved there does not agree with the one that Cardano adds immediately after here. It is the following

$$\begin{cases} a : b = b : c \\ a + b = \frac{a_0}{a_2^2} \\ bc = \frac{a_0}{a_2} \end{cases}.$$

Cardano then suggests that (AMA XXI.1) comes from the resolution of the above system. In fact, he details through an example the two ways to solve the system. Taking $b = \frac{a_0}{a_2c}$, developing calculations, and then taking $x = c$, he gets the equation on the left side in (AMA XXI.1). On the other hand, taking $c = \frac{a_0}{a_2b}$, developing calculations, and then taking $y = b$, he gets the equation on the right side in (AMA XXI.1). Developing calculations, the relations between the

[17]See [CARDANO 1663c, Chapter XXI, pages 326-30].

coefficients of the two equations appear. We remark that this is the same method as in *Ars magna*, Chapter VI, paragraph 3 (see above, at page 70).

AMA XXI.2. *If* $a_0' > 0$*, then*

$$x^3 + a_2x^2 = a_0 \xrightarrow[y=\frac{a_2}{3}-x]{} y^3 = \frac{a_2^2}{3}y + \left(a_0 - \frac{2}{27}a_2^3\right).$$

If $a_0' < 0$*, then*

$$x^3 + a_2x^2 = a_0 \xrightarrow[y=\frac{a_2}{3}+x]{} y^3 + \left(a_0 - \frac{2}{27}a_2^3\right) = \frac{a_2^2}{3}y.$$

Moreover,

$$x^3 + a_2x^2 = a_0 \xrightarrow[y=\frac{\sqrt{a_0}}{x}]{} y^3 = a_2y + \sqrt{a_0}.$$

We remark that the first two folds of the above rule are the same as in (AM XV). Here, the limit case $a_0' = 0$ is lacking. On the contrary, the third fold is original. No further justifications are given for this rule.

AMA XXI.3.

$$x^3 + a_0 = a_1x \xrightarrow[x=\frac{y}{2}\pm\sqrt{a_1-3\left(\frac{y}{2}\right)^2}]{} y^3 = a_1y + a_0.$$

We remark that the above rule is the same as in (AM XIII). No further justifications are given for this rule.

AMA XXI.4.

$$x^3 + a_0 = a_2x^2 \xrightarrow[y=\frac{(\sqrt[3]{a_0})^2}{x}]{} y^3 + a_0 = \left(a_2\sqrt[3]{a_0}\right)y.$$

Cardano says that this rule is taken "from the thirty-second rule of chapter fifty-one [*ex trigesima secunda regula quinquagesimi primi capituli*]". The *Ars magna arithmeticæ* only contains forty chapters, but it exists a "chapter fifty-one" in the *Practica arithmeticæ*. If we go and check, we find that, in the paragraph 32, there is exactly the same rule (see above, at page 173). We also remind that this rule is a particular case of (AM VII.2-3) (see above, at page 72).

AMA XXI.5. *If* $x^4 = a_1x + a_0$ *and* $x = \sqrt{a} \pm b$*, then* $y^4 + a_1y = a_0$ *and* $y = \sqrt{a} \mp b$*.*

No further justifications are given for this rule.

AMA XXI.6.

$$x^4 + a_0 = a_3x^3 \xrightarrow[y=\frac{\sqrt{a_0}}{x}]{} y^4 + a_0 = (a_3\sqrt{a_0})\, y,$$

$$x^4 = a_3x^3 + a_0 \longrightarrow y^4 + (a_3\sqrt{a_0})\, y = a_0,$$

$$x^4 + a_3x^3 = a_0 \xrightarrow{y=\frac{\sqrt{a_0}}{x}} y^4 = (a_3\sqrt{a_0})\, y + a_0.$$

Cardano says that this rule is taken "from three questions [*ex tribus quæestionibus*]", without specifying where they can be founded. They are

$$\begin{cases} a : b = b : c \\ a + c = a_3 \\ ab = \sqrt{a_0} \end{cases} \quad \text{or} \quad \begin{cases} a : b = b : c \\ a = c + a_3 \\ ab = \sqrt{a_0} \end{cases} \quad \text{or} \quad \begin{cases} a : b = b : c \\ a + a_3 = c \\ ab = \sqrt{a_0} \end{cases}.$$

With the same method as in *Ars magna*, Chapter VI, paragraph 3 or (AMA XXI.1), it is easily checked that they respectively correspond to the three folds of the above rule. We moreover remark that the first and the third fold of the rule are particular cases of (AM VII.2-3) (see above, at page 72).

AMA XXI.7.

$$x^4 + a_2x^2 + a_0 = a_3x^3 \xrightarrow[y=\frac{\sqrt{a_0}}{x}]{} y^4 + a_2y^2 + a_0 = (a_3\sqrt{a_0})\, y.$$

Cardano says that this rule follows from

$$\begin{cases} a : b = b : c \\ a + b + c = a_3 \\ ab = a_2 \end{cases}$$

(by the same method of double resolution as above). We remark that this is the same as (AM VII.12) (see above, at page 75).

AMA XXI.8.

$$x^3 + a_0 = a_2x^2 \longrightarrow x^3 + \left(\frac{a_2}{2}\right)^2 x = a_2x^2 + \frac{a_0}{8}.$$

We remark that this is the same as (AM VII.13) (see above, at page 75).

AMA XXI.9. *Whenever the term* a_2x^2 *appears, two of the four substitutions* $x = y + \frac{a_2}{3}$ *and* $x = y + \frac{2}{3}a_2$*, or* $x = y - \frac{a_2}{3}$ *and* $x = y - \frac{2}{3}a_2$ *lead to a depressed cubic equation.*

Whenever the term a_1x *appears, both the two substitutions* $x = y + \sqrt{\frac{a_1}{3}}$ *and* $x = y - \sqrt{\frac{a_1}{3}}$ *lead to a cubic equation lacking in the first degree term.*

We remark that part of the first fold is the same as (AM IV.2) (see here, at page 83). No further justifications are given for this rule.

Summing up, the transformations of equations presented in the *Ars magna arithmeticæ* are very close to the ones in the *Ars magna.* The most of them (except (AMA XXI.1), the third fold in (AMA XXI.2), (AMA XXI.5), the second fold in (AMA XXI.6), and the second fold in (AMA XXI.9)) are exactly the same or particular cases of the transformations in the *Ars magna.* But, whereas in the *Ars magna* we had only the little hint given by Chapter VI on how these transformations could have been discovered, here there are still traces of their origins. In fact, in three cases Cardano details the system the solution of which leads to the transformations and in one case he also mentions the *Practica arithmeticæ.*

As Cardano explicitly says at the beginning of Chapter XXI, he plans to use these transformations to derive the solutions of the families of cubic equations lacking in first degree term and of $x^3 + a_0 = a_1x$ from the two principal depressed equations. But, at this stage, the connection[18] between (AM XIV) and (AM XI) is still missing. It will be later given as a generalisation of (AMA XXI.4) and partially of (AMA XXI.6) in Chapter VII of the *Ars magna.*

Complete cubic equations do not appear in Cardano's general planning. Anyway, he knows the transformations in (AMA XXI.9) and, as a matter of fact, we have seen that using in particular the first transformation reproduces the *Ars magna*'s way to reduce complete cubic equations to others that Cardano already knows how to solve.

3.2.4. "General" shapes for irrational solutions. In **Chapter XVIII** "On some necessary premises [*De aliquibus præmittendis necessariis*]", Cardano argues that, if one wants to discover the "universal [*universalis*]" rules of algebra,

[18]See above, the figure 2.18 concerning the *Ars magna* at page 152.

one should better investigate the *binomia* and *recisa* than the (rational) numbers, since a (rational) number can be written as a *binomium* or *recisum* but not the contrary.[19] Then, Cardano lists[20] all the types of *binomia* and *recisa* that – in his opinion – can be solution of a cubic equation.[21] They are

- the "*binomia* and *recisa* of the 1st or 4th, 2nd or 5th types":

$$a \pm \sqrt{b},\ \sqrt{a} \pm b;$$

---

[19]"*[C]um volueris invenire regulas aliquas capitulorum algebræ universales, aut condere ipsa capitula debes facere hoc per viam binomiorum et non simplicium numerorum, dato quod velis etiam experiri in numeris rationalibus*", see [CARDANO 1663c, Chapter XVIII, paragraph 3, page 322].

[20]"*[N]otandum, subiecta capitulorum sunt per se usque ad cubum duodecim, primum est binomium primum et quartum, ut 7 p: R 2. Secundum est recisum primum, et quartum, ut 7 m: R 2. Tertium est binomium secundum et quintum, ut R 7 p: 2. Quartum est recisum secundum et quintum, ut R 7 m: 2. Quintum est binomium cubicum simplex, ut* Rcu*: 32 p:* Rcu*: 2 vel universale ut* Rcu*: RV: 5 p: R 17 p:* Rcu*: RV: 5 m: R 17 et in hoc animadverte, quod semper componuntur ex binomio primo vel quarto cum suo reciso. Sextum est recisum cubicum simplex, ut* Rcu*: 9 m:* Rcu*: 3 vel est recisum universale cubi, ut RV: cu: R 33 p: 5 m: RV: cu: R 33 m: 5. Et in hoc animadverte quod semper componuntur ex binomio secundo vel quinto cum suo reciso. Septimum est trinomium ex binomium duarum RV: cubarum primi vel quarti binomij cum suo reciso, et tertia pars est R numeri producibilis ex multiplicatione dictarum partium invicem detracta ab illo binomio veluti RV: cu:* $9\frac{1}{2}$ *p: R* $89\frac{1}{4}$ *m: RV:* $9\frac{1}{2}$ *m: R* $89\frac{1}{4}$ *m: l 1. Octavum est trinomium quod contat ex duabus RV: cubicis vel simplicibus, et tertia parte quæ est R numeri qui producitur ex una parte R cubicæ, in aliam veluti* Rcu*:* $65\frac{67}{125}$ *p:* Rcu*:* $\frac{32}{125}$ *p:* $1\frac{3}{5}$*. Nunum est recisum secundum mixtu, cuius una pars, est* Rcu*: alia numerus, ut* Rcu*: 30 m: 2. Decimum est binomium secundum mixtum, cuius una pars est* Rcu*: alia numerus, ut* Rcu*: 30 p: 2. Undecimum est binomium primum mixtum, cuius una pars est numerus, alia* Rcu*: ut 3 p:* Rcu*: 2. Duodecimum est recisum primum mixtum cuius una pars est numerus, alia* Rcu*: veluti 3 m:* Rcu*: 2 ut in figura vides*"

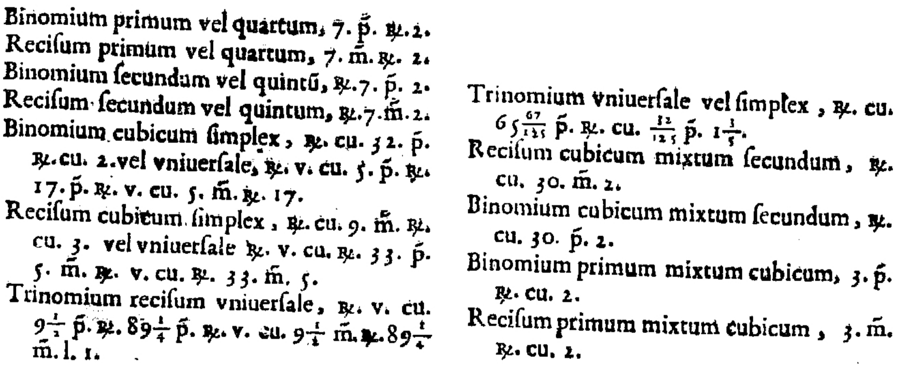
Binomium primum vel quartum, 7. p̃. ℞. 2.
Recisum primum vel quartum, 7. m̃. ℞. 2.
Binomium secundum vel quintũ, ℞. 7. p̃. 2.
Recisum secundum vel quintum, ℞. 7. m̃. 2.
Binomium cubicum simplex, ℞. cu. 32. p̃. ℞. cu. 2. vel vniuersale, ℞. v. cu. 5. p̃. ℞. 17. p̃. ℞. v. cu. 5. m̃. ℞. 17.
Recisum cubicum simplex, ℞. cu. 9. m̃. ℞. cu. 3. vel vniuersale ℞. v. cu. ℞. 33. p̃. 5. m̃. ℞. v. cu. ℞. 33. m̃. 5.
Trinomium recisum vniuersale, ℞. v. cu. $9\frac{1}{2}$ p̃. ℞. $89\frac{1}{4}$ p̃. ℞. v. cu. $9\frac{1}{2}$ m̃. ℞. $89\frac{1}{4}$ m̃. l. 1.
Trinomium vniuersale vel simplex, ℞. cu. $65\frac{67}{125}$ p̃. ℞. cu. $\frac{32}{125}$ p̃. $1\frac{3}{5}$.
Recisum cubicum mixtum secundum, ℞. cu. 30. m̃. 2.
Binomium cubicum mixtum secundum, ℞. cu. 30. p̃. 2.
Binomium primum mixtum cubicum, 3. p̃. ℞. cu. 2.
Recisum primum mixtum cubicum, 3. m̃. ℞. cu. 2.

FIGURE 3.5 – List of the types of *binomia* and *recisa* that can be solution of a cubic equation in *Ars magna arithmeticæ*, Chapter XVIII.

See [CARDANO 1663c, Chapter XVIII, paragraph 5, page 322].

[21]For the 1st type, see above, at page 82.

- the "simple cubic *binomia* and *recisa*":

$$\sqrt[3]{a} \pm \sqrt[3]{b},$$

(where $\sqrt[3]{ab}$ is rational);

- the "universal cubic *binomia*":

$$\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}},$$

where $a \pm \sqrt{b}$ are a *binomium* and *recisum* of the 1st or 4th type, and the "universal cubic *recisa*":

$$\sqrt[3]{\sqrt{a}+b} - \sqrt[3]{\sqrt{a}-b},$$

where $\sqrt{a} \pm b$ are a *binomium* and *recisum* of the 2nd or 5th type;

- the "universal *recisa trinomia*" such that

$$\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}} - \sqrt{c},$$

where $a \pm \sqrt{b}$ are a *binomium* and *recisum* of the 1st or 4th type and $c = (a+\sqrt{b})(a-\sqrt{b})$;

- the "simple cubic *trinomia*" such that

$$\sqrt[3]{a} + \sqrt[3]{b} \pm \sqrt{c},$$

where $c = \sqrt[3]{ab}$, and the "universal cubic *trinomia*" such that

$$\sqrt[3]{A} + \sqrt[3]{B} \pm \sqrt{c},$$

where A, B are a *binomium* and *recisum* (not specified by Cardano) and $c = \sqrt[3]{AB}$;

- the "mixed (cubic) *binomia* and *recisa* of the 1st, 2nd types" (later called "of the 1st or 4th, 2nd or 5th types"):

$$a \pm \sqrt[3]{b},\ \sqrt[3]{a} \pm b,$$

taking $a > b$ rational, positive, non zero[22] such that the square and cubic roots do not respectively vanish (and paying attention to combine the signs one with

[22] The assumption $a > b$ must be required, since it comes from the classification of the *binomia* and *recisa*, see above, at page 82

the other in order not to fall down on negative quantities). Cardano's list is for now not completely clear in all its details.

In **Chapter XIX** "On the properties of 3rd and 6th *binomium* and *recisum*, and on their inability to [fit] the chapters [*De proprietatibus tertij et sexti binomij et recisi, et inhabilitate ad capitula*]", Cardano states the following proposition.

AMA XIX. *The* binomia *and* recisa *of the 3rd or 6th types* $\sqrt{a} \pm \sqrt{b}$ *cannot be solution of whatever degree equations [with rational coefficients].*[23]

First of all, he observes that, if $\sqrt{a}, \sqrt{b}$ are irrational, then $\sqrt{a} \pm \sqrt{b}$ is irrational.[24] Then, he justifies the main statement looking at some particular cases of equations of second, third, fourth, and fifth degrees. Concerning cubic equations, Cardano says that $\sqrt{a} \pm \sqrt{b}$ can be solution neither of $x^3 = a_1x + a_0$ nor of $x^3 = a_2x^2 + a_0$. He justifies the first part of the statement saying that, if one replaces the value $\sqrt{a} \pm \sqrt{b}$ in $x^3 = a_1x + a_0$, "all [the terms from x^3] are square roots, then one cannot makes the number neither by addition by the first hypothesis nor by subtraction".[25] In fact, by substitution one gets

$$(\sqrt{a} \pm \sqrt{b})^3 = a_1(\sqrt{a} \pm \sqrt{b}) + a_0,$$

that is

$$(a + 3b)\sqrt{a} \pm (b + 3a)\sqrt{b} = a_1\sqrt{a} \pm a_1\sqrt{b} + a_0,$$

[23]"*[D]ico quod in tertio et sexto binomio et reciso non verificatur aliquod capitulum algebræ, nec de inventis nec de non inventis usque ad infinitum*", see [Cardano 1663c, Chapter XIX, paragraph 1, page 323].

[24]Cardano says that "*oportet scire quod ex duabus R simplicibus iunctis, numquam provenit numerus, cum ex duabus superficiebus simpliciter medialibus sive communicantes sint sive non, non possit fieri superficies rationalis*", see [Cardano 1663c, Chapter XIX, page 323].
We remark that Cardano's justification is a paraphrase of what he wants to justify, since he replaces "*R*" by "*superficies medialis*". Cardano statement is nevertheless true. Suppose by contradiction to have two rational, positive, non zero a, b such that $\sqrt{a}, \sqrt{b}$ is irrational and $\sqrt{a} \pm \sqrt{b}$ is rational. Then, $(\sqrt{a} \pm \sqrt{b})^2 = a \pm 2\sqrt{ab} + b$ is rational, which means that also $\sqrt{ab}$ is rational. This implies that also that $\sqrt{\frac{a}{b}}$ is rational, since $\sqrt{\frac{a}{b}} = \sqrt{\frac{a}{b}\frac{a}{a}} = \sqrt{\frac{a^2}{ab}} = \frac{a}{\sqrt{ab}}$. Then, $\sqrt{a} \pm \sqrt{b} = \sqrt{b}\left(\sqrt{\frac{a}{b}} \pm 1\right)$ implies that $\sqrt{b}$ is rational, which is absurd. Then $\sqrt{a} \pm \sqrt{b}$ is irrational.

[25]"*[U]t vides omnes sunt R igitur non potest per additionem fieri numerus ex primo supposito, nec per diminutionem*", see [Cardano 1663c, Chapter XIX, page 324].

where, by comparison, he remarks that there is no rational term on the left side of the equation that can correspond to the rational term of the right side. Similar arguments based on the comparison of rational and irrational terms after substitution also holds for whatever cubic equation (matching Cardano's requirements on the coefficients), even though Cardano does not make it explicit. Then, Cardano ends up saying that $\sqrt{a} \pm \sqrt{b}$ cannot be a solution of any equation (without justification it).[26] Nevertheless, we remark that Cardano's statement is not true in full generality. Despite his words, $\sqrt{a} \pm \sqrt{b}$ can be solution of a cubic equation with rational coefficients, namely[27] of $x^3 - \lambda x^2 - (a \pm 2\sqrt{ab} + b)x + \lambda(a \pm 2\sqrt{ab} + b) = 0$, where λ and $\sqrt{ab}$ are rational. Remark that we always have either a complete equation (for $\lambda \neq 0$) or an equation the degree of which easily lowers to two (for $\lambda = 0$).

It is clear that Cardano implicitly assumes that the coefficients are rational. In fact, at the end of the chapter, Cardano also says that it can happen that $\sqrt{a} \pm \sqrt{b}$ is a solution of some equations, if one admits that the coefficients are real.[28] Retrospectively, in order to give consistency to Cardano's words, we should then assume that, in Chapter XVIII and wherever else Cardano speaks of the shape of the solutions of cubic equations in term of *binomia* and *recisa*, he is considering equations with rational coefficients. Note that in the vast majority of the examples Cardano's equations really have rational coefficients.

From Chapters XXII to XXVI, we find a clarification of Cardano's statements in Chapter XVIII. In **Chapter XXII** "On the examination of the first, second, fourth, and fifth *binomium* with its *recisum* [*De examine primi, secundi, quarti et quinti binomij cum suis recisis*]" Cardano firstly affirms that

[26]"*Relinquitur igitur quod tam tertium quam sextum binomium et recisum sunt inutilia ad capitula omnia algebræ usque ad infinitum*", see [Cardano 1663c, Chapter XIX, page 324].

[27]In fact, $\sqrt{a} \pm \sqrt{b}$ is a root of a (unique) monic quadratic polynomial with rational coefficients if and only if $\sqrt{ab}$ is rational. In this case, the polynomial is $x^2 - (a \pm 2\sqrt{ab} + b)$. In particular, $\sqrt{a} \pm \sqrt{b}$ is also a root of the cubic polynomials $(x - \lambda)(x^2 - (a \pm 2\sqrt{ab} + b))$ for any rational λ.

[28]"*Dico tamen quod per accidens fiunt æquatione in tertio et sexto binomio et suis recisi, et etiam in quantitatibus universalibus ligatis, et est quod cum in aliquo capitulo fuerit numerus rerum, aut cuborum in R alicuius numeri*", see [Cardano 1663c, Chapter XIX, page 324]. The examples that Cardano makes are $x^2 = \sqrt{8}x + 10$ and $x^2 + \sqrt{8}x = 10$.

AMA XXII.i. *The* binomia *and* recisa *of the 1st or 4th types $a \pm \sqrt{b}$ and the* recisa *of the 2nd or 5th types $\sqrt{a} - b$ can be solutions of $x^3 + a_0 = a_1 x$.*

AMA XXII.i — *Proof***.** Cardano[29] consider at first a *binomium* or *recisum* of the 1st or 4th types $a \pm \sqrt{b}$. [He wants to show that it can be solution of a certain equation of the family $x^3 + a_0 = a_1 x$, that is, he wants to find the coefficients a_1, a_0 rational, positive, non zero depending on a, b such that $a \pm \sqrt{b}$ verifies the equation.]

By Chapter XVII, paragraphs 1 and 2, Cardano[30] knows that, writing $(a \pm \sqrt{b})^3 = c \pm \sqrt{d}$, it holds that $\frac{\sqrt{d}}{\sqrt{b}} > \frac{c}{a}$. Then, $\frac{\sqrt{d}}{\sqrt{b}}(a \pm \sqrt{b}) - (c \pm \sqrt{d})$ is rational. Cardano only needs to take $a_1 = \frac{\sqrt{d}}{\sqrt{b}}$ and $a_0 = \frac{\sqrt{d}}{\sqrt{b}}(a \pm \sqrt{b}) - (c \pm \sqrt{d})$. In this way, $x = a \pm \sqrt{b}$ is a solution of $x^3 + a_0 = a_1 x$.

Cardano proves in an analogous way the statements on the *recisa* of the 1st or 4th types $a - \sqrt{b}$ and of the 2nd or 5th types $\sqrt{a} - b$. □

For instance, take $a = 3$ and $b = 2$. Then, $a + \sqrt{b} = 3 + \sqrt{2}$ and $(a + \sqrt{b})^3 = 45 + 29\sqrt{2}$. Then, $\frac{29\sqrt{2}}{\sqrt{2}} > \frac{45}{3}$ and it is enough to take $a_1 = 29$ and $a_0 = 42$.

We observe that, if we replace x by $a \pm \sqrt{b}$ in $x^3 + a_0 = a_1 x$ and develop calculations, we will act as a matter of fact in the same way as Cardano does. In all the following statements in Chapters XXIII-XXVI, Cardano uses similar ways of arguing (sometimes in shorter forms, sometimes more articulated), so that I will no more detail them.

Cardano[31] shows the following statements.

AMA XXII.ii. *The* binomia *of the 2nd or 5th types $\sqrt{a} + b$ can be solution of $x^3 = a_1 x + a_0$.*

He observes that $x^3 + a_1 x = a_0$ cannot be solved by a *binomium* or *recisum* of the above types. His argument is that no more possible *binomia* or *recisa* are

[29]See [Cardano 1663c, Chapter XXII, pages 331-332].

[30]In fact, we remark that $c = a^3 \pm 3ab = (a^2 \pm 3b)a$ and $\sqrt{d} = (3a^2 \pm b)\sqrt{b}$. Then, $\frac{\sqrt{d}}{\sqrt{b}} = 3a^2 \pm b$ and $\frac{c}{a} = a^2 \pm 3b$. Moreover, $3a^2 \pm b > a^2 \pm 3b$, since $a^2 > b$ by the definition of *binomium* and *recisum*, see above, at page 82.

[31]See [Cardano 1663c, Chapter XXII, pages 331-333].

left, since $\sqrt{a} \pm \sqrt{b}$ cannot be solutions and $a \pm \sqrt{b}$ and $\sqrt{a} \pm b$ have already been assigned.[32]

He also observes that, since a solution of $x^3 + a_0 = a_1 x$ can go under three shapes, namely $a \pm \sqrt{b}$ and $\sqrt{a} - b$ where[33] in the first case $a > \sqrt{b}$ and in the second case $\sqrt{a} > b$, one cannot assign to them one "title [*titulus*]" nor find a "general chapter [*capitulum generale*]".[34] It seems then that Cardano affirms here that no *binomium* and *recisum* of the 1st, 2nd, 4th, or 5th types is suited to the role of a more comprehensive shape to encompass the remaining ones.

AMA XXII.iii. *The* binomia *and* recisa *of the 1st or 4th types $a \pm \sqrt{b}$ and the* binomia *of the 2nd or 5th types $\sqrt{a} + b$ can be solutions of $x^3 + a_0 = a_2 x^2$.*

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[32]"*[N]am cum tertio et sextum binomium cum suis recisis non sint utilia ad capitula per hanc regulam, primum vero et quartum binomium cum suis recisis et recisum secundum et quintum inserviunt tantum capitulo de rebus æqualibus cubo et numero, binomium vero secundum et quintum inserviunt tantum capitulo de cubo æquali rebus et numero relinquitur a sufficienti divisione quod cubus et res non possunt æquari numero in aliquo genere binomiorum aut recisorum*", see [Cardano 1663c, Chapter XXII, page 332].
Let us try to see why a *binomium* or *recisum* of the 1st-6th types cannot be solution of $x^3 + a_1 x = a_0$. Suppose $x = a \pm \sqrt{b}$ *binomium* or *recisum* of the 1st or 4th types. Then, we want to determine the rational coefficients a_1, a_0. We have

$$(a \pm \sqrt{b})^3 + a_1(a \pm \sqrt{b}) = a_0,$$

that is

$$(a^2 + 3b + a_1)a + (3a^2 + b + a_1)\sqrt{b} = a_0.$$

Then,

$$\begin{cases} a_1 = -3a^2 - b \\ a_0 = a^2 + 3b + a_1 \end{cases}.$$

This means that $a_1, a_0 < 0$. So, $x = a \pm \sqrt{b}$ satisfies $x^3 + a_0 = a_1 x$.
Similar observations hold supposing $x = \sqrt{a} \pm b$ *binomium* or *recisum* of the 2nd or 5th types. On the contrary, supposing $x = \sqrt{a} \pm \sqrt{b}$ *binomium* or *recisum* of the 3rd or 6th types, we get to the contradiction

$$\begin{cases} a_1 = -3a - b \\ a_1 = -a - 3b \\ a_0 = 0 \end{cases}.$$

[33]See above, at page 82.

[34]"*Notabile est quod cum res æqualis cubo et numero verificetur et de binomio primo et de suo reciso et de reciso secundo, quorum tria natura est diversa nec potest uno titulo assignari, quia in binomio primo numerus est maior R et uterque p: in reciso primo R est m: in binomio secundo numerus est minor R igitur nullum poterit assignari capitulum generale, quia æquatio cadit super tres omnino diversas [...] manet firmum sed in capitulo de rebus æqualibus cubo et numero inveniuntur tres æquationes diversæ nihil continentes firmu ita quod illud quod est in una æquatione sit in duabus reliquis igitur sequitur quod tale capitulum non potest habere regulam generalem*", see [Cardano 1663c, Chapter XXII, page 332].

AMA XXII.iv. *The* recisa *of the 2nd or 5th types* $\sqrt{a} - b$ *can be solution of* $x^3 + a_2x^2 = a_0$.

As before and by similar arguments, Cardano observes that $x^3 = a_2x^2 + a_0$ cannot be solved by a *binomium* or *recisum* of the 1st-6th types. He also observes that there cannot be a more comprehensive shape to encompass the *binomia* and *recisa* that are solutions of $x^3 + a_0 = a_2x^2$.

Chapter XXIII "On the examination of the cubic *binomium* and its *recisum* [*De examine binomij cubici et sui recisi*]" contains the following statements.

AMA XXIII.i. *The cubic* binomia $\sqrt[3]{a} + \sqrt[3]{b}$ *[such that* $\sqrt[3]{ab}$ *is rational] can be solution of* $x^3 = a_1x + a_0$, *where* $a_1 = 3\sqrt[3]{ab}$ *and* $a_0 = a + b$.

AMA XXIII.ii. *The cubic* recisa $\sqrt[3]{a} - \sqrt[3]{b}$ *[such that* $\sqrt[3]{ab}$ *is rational] can be solution of* $x^3 + a_1x = a_0$, *where* $a_1 = 3\sqrt[3]{ab}$ *and*[35] $a_0 = a - b\,[> 0]$.

Note that this rationality conditions put some structure on the *binomia*. Let us call the rational $\sqrt[3]{ab} = k$. Then, $\sqrt[3]{b} = \frac{k}{\sqrt[3]{a}} = \frac{k}{a}\sqrt[3]{a^2}$, and the *binomium* becomes

$$\sqrt[3]{a} \pm \frac{k}{a}\sqrt[3]{a^2}.$$

These statements[36] are proved in the same way as (AMA XXII.i). As a matter of fact, they are equivalent to replace the considered *binomium* or *recisum* in the equation and drawing the sign of the coefficients by developing calculations.

Concerning the first statement, we remark that, while detailing the example on $\sqrt[3]{6} + \sqrt[3]{2}$, Cardano comes across the equation $x^3 = \sqrt[3]{324}x + 8$, where the coefficient of the first degree term is not rational. This could contradict my previous guess on the fact that, in the context of discussing the shape of the solutions of the cubic equations in terms of *binomia* or *recisa*, we should consider (positive, non zero) rational coefficients. Anyway, immediately after the example, Cardano adds the condition that "to have the things in a number it is always

[35] We remark that here Cardano necessarily needs to assume that $a > b$ in order to avoid to get to the equation $x^3 + a_1x + a_0 = 0$.

[36] We remark that one can prove by Galois theory that $\sqrt[3]{a} \pm \sqrt[3]{b}$ is a root of a cubic polynomial if and only if either $\sqrt[3]{ab}$ or $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ is rational. Note that in the second case, assuming $\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = k$, we can write the root (the *binomium*) as $(k \pm 1)\sqrt[3]{b}$. Then the equation must have $a_1 = 0$ and $a_0 = (k \pm 1)^3 b$. This is why Cardano does not consider this second condition.

necessary that the roots, one multiplied by the other, produce a number".[37] This is why I included this condition in square brackets in the Propositions (AMA XXIII.i) and (AMA XXIII.ii).

Cardano finally observes (without justification) that the cubic *binomium* or *recisum* $\sqrt[3]{a} \pm \sqrt[3]{b}$ cannot be solution of a cubic equation lacking in the first degree term (otherwise – I add – either a or b must be zero, since $a_1 = 3\sqrt[3]{ab}$).

In **Chapter XXIV** "On the examination of the mixed, cubic *recisa* and their *binomium* [*De examine recisorum cubicurum mixtorum, et suorum binomiorum*]", Cardano states the following propositions.[38]

AMA XXIV.i. *The mixed, cubic* binomia *of the 1st or 4th types $a + \sqrt[3]{b}$ and the cubic mixed* binomia *and* recisa *of the 2nd or 5th types $\sqrt[3]{a} \pm b$ can be solutions of the equation $x^3 + a_1x = a_2x^2 + a_0$.*

AMA XXIV.ii. *The mixed, cubic* recisum *of the 1st or 4th types $a - \sqrt[3]{b}$ can be solution of the equation $x^3 + a_2x^2 + a_1x = a_0$.*

Finally, Cardano observes (without justification) that the mixed, cubic *binomia* or *recisa* cannot be solution of a non-complete cubic equation.

37"*[U]t autem habeas res in numero semper oportet ut partes R invicem multiplicatæ producant numerum*", see [Cardano 1663c, Chapter XXIII, page 335].

38Cardano never specifies the classification of cubic *binomia* and *recisa*. Anyway, analogously to the square (see above, at page 82) we could say that a cubic *binomium* is an expression that can be written in the form $\alpha + \beta$, where $\alpha^3 > \beta^3$. Moreover, when $\sqrt[3]{\alpha^3 - \beta^3}$ is commensurable with α, we say that it is of the

1st type if $\alpha = a$ is rational and $\beta = \sqrt[3]{b}$ is irrational, that is $\alpha + \beta = a + \sqrt[3]{b}$,

2nd type if $\alpha = \sqrt[3]{a}$ is irrational and $\beta = b$ is rational, that is $\alpha + \beta = \sqrt[3]{a} + b$,

3rd type if $\alpha = \sqrt[3]{a}$ is irrational and $\beta = \sqrt[3]{b}$ is irrational, that is $\alpha + \beta = \sqrt[3]{a} + \sqrt[3]{b}$,

and, when $\sqrt[3]{\alpha^3 - \beta^3}$ is incommensurable with α, we say that it is of the

4th type if $\alpha = a$ is rational and $\beta = \sqrt[3]{b}$ is irrational, that is $\alpha + \beta = a + \sqrt[3]{b}$,

5th type if $\alpha = \sqrt[3]{a}$ is irrational and $\beta = b$ is rational, that is $\alpha + \beta = \sqrt[3]{a} + b$,

6th type if $\alpha = \sqrt[3]{a}$ is irrational and $\beta = \sqrt[3]{b}$ is irrational, that is $\alpha + \beta = \sqrt[3]{a} + \sqrt[3]{b}$.

Likewise for a *recisum*, which is written in the form $\alpha - \beta$. Moreover, Cardano calls 'mixed, cubic *binomia* and *recisa*' the one of the 1st, 2nd, 4th, and 5th types. We could then soundly call 'of the 3rd and 6th types' the cubic *binomia* and *recisa* in Chapter XXIII.

In **Chapter XXV** "On the examination of useful *trinomia* and their *recisa* [*De examinatione trinomiorum utilium cum suis recisis*]", Cardano lists six types of *trinomia* and *recisa*, where two out of three terms are cubic roots.

$$\sqrt[3]{a}+\sqrt[3]{b}+c,$$
$$\sqrt[3]{a}+\sqrt[3]{b}-c,$$
$$\sqrt[3]{a}-\sqrt[3]{b}+c,$$
$$\sqrt[3]{a}-\sqrt[3]{b}-c,$$
$$-\sqrt[3]{a}+\sqrt[3]{b}+c,$$
$$-\sqrt[3]{a}-\sqrt[3]{b}+c,$$

with $\sqrt[3]{a}$, $\sqrt[3]{b}$ irrational, positive, non zero and c rational, positive, non zero. Note that the list is not complete, since $-\sqrt[3]{a}-\sqrt[3]{b}-c$ and $-\sqrt[3]{a}+\sqrt[3]{b}-c$ are missing. If we moreover assume that $a>b$ (as Cardano implicitly did in his list in Chapter XVIII, see at page 186), then the two above *trinomia* are negative for each rational, positive, non zero c, which justifies their absence.

Then, Cardano states the following propositions.

AMA XXV.i. *The* trinomia $\sqrt[3]{a}+\sqrt[3]{b}+c$ *such that* $c^2=\sqrt[3]{ab}$ *can be solution of the equation* $x^3=a_2x^2+a_0$.

AMA XXV.ii. *The* recisa $\sqrt[3]{a}+\sqrt[3]{b}-c$ *such that* $c^2=\sqrt[3]{ab}$ *can be solution of the equation* $x^3+a_2x^2=a_0$.

The *trinomia* of the above type are called 'proportioned [*proportionata*]', since the continue proportion $\sqrt[3]{a}:c=c:\sqrt[3]{b}$ holds. Note that this proportion puts some structure on the *trinomium*. Let us call the ratio $k=\frac{\sqrt[3]{a}}{c}=\frac{c}{\sqrt[3]{b}}$. Then, we remark that $c=k\sqrt[3]{b}$ and $\sqrt[3]{a}=kc=k^2\sqrt[3]{b}$ and that the *trinomium* is $k^2\sqrt[3]{b}+k\sqrt[3]{b}+\sqrt[3]{b}$. Moreover, since c is rational, also $k\sqrt[3]{b}$ must be rational, then $k=\sqrt[3]{b^2}$. Finally, the *trinomium* becomes

$$b\sqrt[3]{b^2}+b+\sqrt[3]{b},$$

and it holds that $k^2\sqrt[3]{b^2}=k^3\sqrt[3]{b}\sqrt[3]{b}$, which was the original condition written by Cardano.

Cardano observes at first that all the listed *trinomia* and *recisa* cannot be solution of a depressed cubic equation. Then, he specifies (without justification) that the remaining ones $\sqrt[3]{a}-\sqrt[3]{b}+c$, $\sqrt[3]{a}-\sqrt[3]{b}-c$, $\sqrt[3]{a}+\sqrt[3]{b}+c$, $-\sqrt[3]{a}+\sqrt[3]{b}+c$, $-\sqrt[3]{a}-\sqrt[3]{b}+c$ cannot be solution of any cubic equation.

In **Chapter XXVI** "On the remaining types of *binomia*, *trinomia*, and *quadrinomia* not useful to the chapters of algebra, and on their *recisa* [*De reliquis speciebus binomiorum, et quadrinomiorum, inutilium ad capitula algebra, et de recisis eorum*]", Cardano says that the following quantities "cannot make a chapter general and not even be part of a general value".[39] They are

- the "simple roots"[40]:
$$\sqrt{a};$$
- the *binomia* or *recisa* of the 3rd or 6th types:
$$\sqrt{a}\pm\sqrt{b};$$
- the cubic roots:
$$\sqrt[3]{a};$$
- the "median [*mediales*]" roots:
$$\sqrt[4]{a};$$
- the *binomia* such that
$$\sqrt{a}\pm\sqrt[3]{b},\ \sqrt[3]{a}\pm\sqrt{b};$$
- the "non-proportioned" *trinomia*:
$$\sqrt[3]{a}+\sqrt[3]{b}\pm\sqrt{c},$$
where $c\neq\sqrt[3]{ab}$;
- the "proportioned or not *trinomia* from square and universal roots";

[39]"*[Q]uantitates autem quæ sunt R simplices et binomia vel recisa ex tertio et sexto genere et R cubicæ et mediales, id est RR et binomiales ex R et R cubica et trinomiales non proportionatæ et RV: quadratæ triplicatæ sive proportionatæ sive non, et trinomia ex radicibus quadratis, et trinomia cubica in quibus prima vel tertia pars est m: vel est numerus, nullo modo faciunt capitulum generale, nec possunt esse pars æquationis generalis ita quod totum genus illius quantitatis inserviat uni capitulo*", see [Cardano 1663c, Chapter XXVI, page 338].

[40]Note that here "simple" is opposed to "cubic", "median", ..., but not to "universal" as before.

- the "*trinomia* from square roots":

$$\pm\sqrt{a} \pm \sqrt{b} \pm c;$$

- the cubic *trinomia* such that

$$-\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c},\ \sqrt[3]{a} + \sqrt[3]{b} - \sqrt[3]{c},\ a + \sqrt[3]{b} - \sqrt[3]{c},\ \sqrt[3]{a} + \sqrt[3]{b} \pm c,$$

taking $a > b$ rational, positive, non zero such that the square and cubic roots do not respectively vanish (and paying attention to combine the signs one with the other in order not to fall down on negative quantities). This last classification is quite unclear. Obviously, some of the above listed numbers can be solution of a somewhat equation. Then, Cardano's statement needs to be detailed. Cardano goes on arguing that,[41] even if it is really possible that $\sqrt{a}$, or a *trinomium* composed by a universal root, a square root, and a number, or a *binomium* composed by two universal roots, or a *binomium* composed by a universal root and a square root, or a *trinomium* composed by square proportioned roots could be solutions of a family of quadratic equations ($x^2 = a_1 x + a_0$, for instance), they would not be at the same time solutions of the corresponding families of higher degree equations (which is $x^3 = a'_1 x + a'_0$), and so on. Cardano ends this chapter with a quite obscure remark, by which he dismisses any other irrational quantity to be the shape of a solution of a cubic equation.[42]

Summing up, Cardano inquires the shapes under which an irrational solution of a cubic equation can go, assuming that the equation has rational coefficients. The following table resumes Cardano's results in Chapters XXVIII-XIX and XXII-XXVI.

[41]"*Quia bene possibile est (ut dictum est in capitulo) quod æuatio census æqualis rebus et numero perveniat ad æquationem unius R tantum, et similiter possibile est invenire trinomia ex RV: et R et numero, vel binomia ex duabus RV: vel ex RV: et R simplici quæ æquivalent numero vel alteri R in quam cadat æquatio capituli, sed hoc non est per se generale. Quod autem ita sit patet, nam si maxime hoc esset, posset esse in trinomio quadrato proportionaliter, cuius pars media sit numerus; veluti R 8 p: 2 p: R 2. Quod autem hoc non ita sit patet, nam in cubo proveniunt plus quam tres partem non communicantes, quare non potest æquari rebus et numero, et ut videas quod eadem ratione*", see [Cardano 1663c, Chapter XXVI, page 338].

[42]"*Si quis igitur dicat ad quid sunt aliæ quantitates irrationales non bimediales quæ possunt producere æquationem. Dico quod etsi non producunt æquationem per se, producunt per accidens, ut in capitulo decimo nono*", see [Cardano 1663c, Chapter XXVI, page 338].

Equation	Shape of an irrational solution			
	binomia and/or *recisa*			*trinomia*
	(normal)	cubic	mixed cubic	
$x^3 + a_1x = a_0$		$\sqrt[3]{a} - \frac{k}{a}\sqrt[3]{a^2}$		
$x^3 = a_1x + a_0$	$\sqrt{a} + b$	$\sqrt[3]{a} + \frac{k}{a}\sqrt[3]{a^2}$		
$x^3 + a_0 = a_1x$	$a \pm \sqrt{b}$, $\sqrt{a} - b$			
$x^3 = a_2x^2 + a_0$				$a\sqrt[3]{a^2} + \sqrt[3]{a} + a$
$x^3 + a_2x^2 = a_0$	$\sqrt{a} - b$			$a\sqrt[3]{a^2} + \sqrt[3]{a} - a$
$x^3 + a_0 = a_2x^2$	$a \pm \sqrt{b}$, $\sqrt{a} + b$			
$x^3 + a_2x^2 + a_1x = a_0$			$a - \sqrt[3]{b}$	
$x^3 + a_1x = a_2x^2 + a_0$			$a + \sqrt[3]{b}$, $\sqrt[3]{a} \pm b$	
Condition		$k \in \mathbb{Q}$		

TABLE 3.1 – Shapes of irrational solutions to cubic equations (with rational coefficients) in *Ars magna arithmeticæ*, Chapters XXII-XXV.

The usual way in which Cardano proves his statement is to replace the assumed value of the solution and then to develop calculations in order to find out the expected sign of the coefficients – if they exist rational.

Among the kind of irrational quantities that Cardano lists at the beginning as possible shapes,[43] there are $\sqrt[3]{a + \sqrt{b}} + \sqrt[3]{a - \sqrt{b}}$ and $\sqrt[3]{\sqrt{a} + b} - \sqrt[3]{\sqrt{a} - b}$ with some appropriate *binomia* and *recisa* under the cubic roots. They are no more mentioned after Chapter XVIII, but they can be studied in a way[44] similar to $\sqrt[3]{a} \pm \sqrt[3]{b}$, as Cardano probably also does with the "universal cubic *trinomia*" mentioned in Chapter XVIII. In the list also appears $\sqrt[3]{a + \sqrt{b}} + \sqrt[3]{a - \sqrt{b}} - \sqrt{c}$, where $a \pm \sqrt{b}$ is a *binomium* or *recisum* of the 1st or 4th types and $c = (a + \sqrt{b})(a - \sqrt{b})$. Except for the subsequent example $x^3 + 3x^2 = 21$, a solution of which is

[43]See above, at page 186.

[44]Let us rewrite the *binomia* and *recisa* at issue as $\sqrt[3]{A} \pm \sqrt[3]{B}$, where A, B are a *binomium* and *recisum* respectively of the 1st or 4th, 2nd or 5th types. In the list from Chapter XVIII, I wrote in a similar way the "universal cubic *trinomia*" $\sqrt[3]{A} + \sqrt[3]{B} \pm \sqrt{c}$, where $c = \sqrt[3]{AB}$.

$\sqrt[3]{\frac{19}{2}+\sqrt{\frac{357}{4}}}+\sqrt[3]{\frac{19}{2}-\sqrt{\frac{357}{4}}}-1$, Cardano does no more mention this shape in the following, so that at the end he does not justify this statement.[45]

In Chapters XXII and XXVI the term ‘general’ (referred to solutions) occurs. Similarly to the *Ars magna*, Chapter IV, a “general solution” is a solution the shape of which can encompass other solutions as “particular” – that is, more specific – cases. At the end of Chapter XXVI, it even seems that a “general” shape for a solution should not be restrained to classes of equations depending on their degrees. We have now more elements to try to specify a weaker version of our interpretation of “general solutions” and, as a consequence, of “particular” solutions at page 84. I will say that a “general solution” is a solution that goes under certain shapes that are usually expounded thanks to the *binomia* and *recisa*. We remind that Cardano establishes at the beginning of the *Ars magna arithmeticæ* a correspondence between the terms ‘*binomium*’ and ‘*recisum*’ and the traditional classification of the lines in *Elements*, Book X. In this context, ‘universal’ is used by Cardano as a synonym of ‘general’. By opposition, a “particular” solution is instead a solution that is not “general”. Then, a “general” solution has a shape which can encompass the one of a “particular” solution, but not in the other way round.

We will see in the next section that in the *Ars magna arithmeticæ* there is a good number of cubic formulae, or at least they are *in nuce* in the procedure that Cardano describes for solving the equation. In the abstract, it is through-and-through possible that Cardano got inspired by the method by which he solves certain equations to discuss the possible shapes for irrational solutions to certain

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[45]From a modern viewpoint, understanding for which a, b such a number $t=\sqrt[3]{a+\sqrt{b}}+\sqrt[3]{a-\sqrt{b}}-\sqrt{c}$, with $c=(a+\sqrt{b})(a-\sqrt{b})$, is a root of a polynomial of degree 3 is far from being a trivial problem.
In fact, t is always a root of a polynomial of degree 18. For random a, b this is the lowest possible degree. A sufficient condition for the degree to be 3 is that $a^2-b=u^6$ for some rational u. In this case, the polynomial is

$$x^3+3u^3x^2+(3u^6-3u^2)x+u^9-3u^5-2a=0.$$

Note that for such coefficients $a=-\frac{q}{2}$ and $b=\frac{q^2}{4}+\frac{p^3}{27}=\Delta_3$ where p, q are defined in (1.5.5), see above at page 40. Cardano’s example $x^3+3x^2=21$ is a particular case where $a=\frac{19}{2}$ and $b=\frac{357}{4}$, and then $u=1$.
But the converse is false. Indeed, if $a=1$ and $b=\frac{80}{81}$, then t is a root of $x^3+\frac{1}{3}x^2+\frac{1}{27}x-\frac{2186}{729}$. For such a polynomial we have $-\frac{q}{2}=\frac{3}{2}$ and $\Delta_3=\frac{9}{4}$.

cubic equations with real coefficients. But then one can only explain the presence of the chapters of which we spoke in this section by assuming Cardano's will to proceed in the arithmetisation of *Elements*, Book X that takes place in the first part of the *Ars magna arithmeticæ*. In practice, it seems to me by far more likely that he acted in the other way round: the inquiries on the possible shapes preexist to the solving methods. It is for sure easier to try to solve, or to guess a solution of, a cubic equation with rational coefficients rather than of an equation with real ones. Then, I interpret the study of *binomia* and *recisa* in the *Ars magna arithmeticæ* as one of Cardano's access points to the study of the structure of the solutions of an equation. But then, in the *Ars magna* the situation radically changes. We find some similar statements on the shape of solutions in Chapter IV, namely (AM IV.5) which corresponds both to (AMA XXIII.i) and (AMA XXIII.ii), and (AM IV.6) which corresponds both to (AMA XXV.i) and (AMA XXV.ii). I am likely to interpret these as some late remainders rather than early results still to be developed. In fact, they simply cannot point at the way to discover a formula. Moreover, in the *Ars magna* the considerations about the shape of the solutions really become marginal, leaving full room to the cubic formulae, which come to play the main role. We may notice in particular that in the *Ars magna* all the statements concerning the impossibility for a *binomium* or *recisum* to be solution disappear. This is maybe due to the fact that the results of this type can only be drawn for equations with (positive, non zero) rational coefficients, whereas in the *Ars magna* they are completely arbitrary (that is, real, positive, non zero coefficients).

3.2.5. The cubic formulae, or what gets closer to them. In **Chapter XX** "On the number of the chapters to be enquired [*De numero capitulorum quærendorum*]" Cardano lists some equations from second to tenth degrees. He puts a small cross next to the ones which are "known in a universal [that is, general] way [*univeraliter nota*]" and a star next to the ones which are "known in a particular way [*particulariter nota*]". Otherwise, no symbol is added.[46]

[46]"*Et nota quod ubi apponitur † talia capitula sunt universaliter nota, ubi ⋆ vero ponitur sunt particulariter nota tantum, ubi nihil, sunt ignota*", see [Cardano 1663c, Chapter XX, page 326].

Nouem Capitula triplicia neceſſaria.

† Cen. & res æqualia numero.
† Cen. & numerus æqualia rebus.
† Res & numerus æqualia cenſibus.
† Cubus & res æqualia numero.
† Cubus & numerus æqualia rebus.
† Res & numerus æqualia cubis.
† Cubus & cen. æqualia numero.
† Cubus & numerus æqualia cenſibus.
† Cen. & numerus æqualia cubis.

Duodecim Capitula triplicia neceſſari deriuatiua.

† Cenſus cenſus & cenſus æqualia numerc
† Cenſus cenſus & numerus æqualia cen ſibus.
† Cenſus & numerus æqualia cenſibus cen ſus.
† Cubus cenſus & cenſus æqualia numerc
† Cubus cenſus & numerus æqualia cenſi bus.
† Cenſus & numerus æqualia cubo cenſus
† Cubus cenſus & cenſus cenſus æquali numero.
† Cubus cenſus & numerus æqualia cenſi bus cenſus.
† Cenſus cenſus & numerus æqualia cub cenſus.
† Cen. cen. cen. & cen. cen. æqualia nu mero.
† Cen. cen. cen. & numerus æqualia cenſi bus cenſus.
† Cen. cen. & numerus æqualia cen. cen cen.

Sex Triplicia non neceſſaria.

* Cenſus cenſus & res æqualia numero.
* Cenſus cenſus & numerus æqualia rebus
* Res & numerus æqualia cenſibus cen ſuum.
* Cenſus cenſus & cubus æqualia numero
* Cenſus cenſus & numerus æqualia cubo
* Cubus & numerus æqualia cenſui cen ſus.

Sex Triplicia non neceſſaria deriuatiua.

* Cen. cen. cen. & cenſus æqualia numero.
* Cen. cen. cen. & numerus æqualia cenſibus.
* Cenſus & numerus æqualia cen. cen. cen.
* Cen. cen. cen. & cen. cubi æqualia numero.
* Cen. cen. cen. & numerus æqualia cen. cubi.
* Cen. cubi & numerus æqualia cen. cen. cen.

Septem Quadruplicia neceſſaria.

* Cubus & cenſus & res æqualia numero.
Cubus & cenſus & numerus æqualia rebus.
Cubus & res & numerus æqualia cenſibus.
Cenſus & res & numerus æqualia cubis.
Cubus & cenſus æqualia rebus & numeris.
* Cubus & res æqualia cenſibus & numero.
Cubus & numerus æqualia cenſibus & rebus.

Septem Quadruplicia neceſſaria deriuatiua.

* Cubus cenſus, & cenſus cenſus & cenſus æqualia numero.
Cubus cenſus, & cenſus cenſus & numerus æqualia cenſibus.
Cubus cenſus, & cenſus & numerus æqualia cenſibus cenſus.
Cenſus cenſus, & cenſus & numerus æqualia cubo cenſus.
Cubus cenſus, & cenſus cenſus æqualia cenſibus & numero.
* Cubus cenſus & cenſus æqualia cenſibus cenſus & numero.
Cubus cenſus & numerus æqualia cenſibus cenſus & cenſibus.

21. *Quadruplicia prætermiſſa.*

Cen. cen. & cen. & res æqualia numero.
Cen. cen. & cen. & numerus æqualia rebus.
Cen. cen. & res & numerus æqualia cenſibus.
Cen. & res & numerus æqualia cen. cen.
Cen. cen. & cenſus æqualia rebus & numero.
Cen. cen. & res æqualia cenſibus & numero.
Cen. cen. & numerus æqualia cenſibus & rebus.
Cen. cen. & cubi & res æqualia numero.
Cen. cen. & cubi & numerus æqualia rebus.
Cen. cen. & res & numerus æqualia cubis.
Cubus & res & numerus æqualia cenſibus cenſus.
Cen. cen. & cubi æqualia rebus & numero.
Cen. cen. & res æqualia cubis & numero.
Cen. cen. & numerus æqualia cubis & rebus.
Cen. cen. & cubi & cenſus æqualia numero.
Cen. cen. & cubi & numerus æqualia cenſibus.
Cen. cen. & cenſus & numerus æqualia cubis.
Cubus & cenſus & numerus æqualia cen. cen.
Cen. cen. & cubi æqualia cenſibus & numero.
Cen. cen. & cenſus æqualia cubo & numero.
Cen. cen. & numerus æqualia cubo & cenſibus.

21. Capitula quadruplicia prætermiſſa deriuatiua non poſui, quia ex præcedentibus faciliter cognoſcuntur, & difficile eſt peruenire ad æquationem, & rariſſimè cadunt in vſum.

Quinque Capitula quincuplicia prætermiſſa.

Cen. cen. & cubi & cenſus & res æqualia numero.
Cen. cen. & cubi & cenſus & numerus æqualia rebus.
Cen. cen. & cubi & res & numerus æqualia cenſibus.
Cen. cen. & cenſus & res & numerus æqualia cubis.
Cubus & cenſus & res & numerus æqualia cenſui cenſus.

Reliqua etiam ſua correſpondentia deriuatiua quinque capitula non poſui propter rationes ſuperius adductas.

Meminiſſe autem oportet, quòd cognito quocumque capitulo ſuum deriuatiuum ei correſpondens ſubitò cognoſcitur, ſiue ſit ex neceſſariis ſiue non, ſiue ex prætermiſſis ſiue ex triplicibus, ſiue ex quadruplicibus, ſiue quincuplicibus, nam æquatio deriuatiui ſemper eſt ℞. v. æquationis principalis, cuius exempla videbis in capitulo penultimo.

Eſt etiam aliud genus capitulorum quod vocatur aſſimilatorum & ſunt 9. & ſunt vniuerſalia, & aſſimilantur primis 9. & cognoſcuntur per illa ſine difficultate & ſunt hæc:

Nouem Capitula uniuerſalia aſſimilata.

† Cubus cenſus & cubus æqualia numero.
† Cubus cenſus & numerus æqualia cubis.
† Cubus & numerus æqualia cubo cenſus.
† Cubus cubi & cubus æqualia numero.
† Cubus cubi & numerus æqualia cubis.
† Cubus & numerus æqualia cubo cubi.
† Cenſus relati primi & ℞^{m} p^{m} æqualia numero.
† Cenſus relati primi & numerus æqualia ℞^{o} p^{o}.
† R^{m} p^{m} & numerus æqualia cenſui ℞^{i} primi.

FIGURE 3.6 – List of the equations considered in *Ars magna arithmeticæ*, Chapter XX.

Looking at the list, we observe that Cardano declares to know how to solve "in a universal way" the depressed cubic equations and the cubic equations lacking in the first degree term. He also knows how to solve "in a particular way" two families of complete cubic equations, but no hints are given on the remaining ones. We find Cardano's "universal" solving methods for these equations in Chapters XXVIII-XXX, XXXII, XXXIV, XXXVI, and XXXIX. Moreover, we remark that Cardano has no hints at all concerning the solution of the quartic equations.

In **Chapter XXVIII** "On the general chapter of the cube and things equal to a number, by *Magister* Niccolò Tartaglia from Brescia [*De capitulo generali cubi, et rerum æqualium numero, Magistri Nicolai, Tartabliæ, Brixiensis*]" Cardano deals with the equation $x^3 + a_1 x = a_0$. He also mentions his "second book on Euclid".[47]

Cardano says that the solution goes under the shape $\sqrt[3]{\sqrt{b}+a} \pm \sqrt[3]{\sqrt{b}-a}$, where $\sqrt{b} \pm a$ is a *binomium* or *recisum* of the 2nd or 5th types. Sometimes – he continues – the solution occurs as cubic roots of numbers or even just as a number. Therefore, the above shape is "general".

Then, Cardano provides the cubic formula.

AMA XXVIII. *Given the equation $x^3 + a_1 x = a_0$, a solution is*

$$x = \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} + \frac{a_0}{2}} - \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} - \frac{a_0}{2}}.$$

Only a few examples follow. Since the formula is abruptly stated with no justifications at all (not even a proof as in (AM XI)) and since Tartaglia is expressly recalled in the title of the chapter, it is very likely that Cardano received the formula by Tartaglia and did not have too much time to work on it.

In **Chapter XXIX** "On the cube equal to the squares and a number in general [*De cubo æquali censibus, et numero generali*]" Cardano deals with the equation $x^3 = a_2 x^2 + a_0$. He suggests two ways to get a solution. A few examples follow each way.

The first way is by transformations of equations. Cardano transforms $x^3 = a_2 x^2 + a_0$ into $y^3 + a_1 y = a_0'$, changing the coefficients according to (AMA XXI.1).

[47]See here, footnote 31, page 95.

Then, he draws a solution of $y^3 + a_1 y = a_0'$ thanks to the formula in the preceding chapter and transforms back the solution of the equation in y into a solution of the equation in x again by (AMA XXI.1). See the diagram below for a short version.

AMA XXIX.i. *Given the equation $x^3 = a_2 x^2 + a_0$, one draws a solution in the following way.*

$$\begin{tikzcd}
x^3 = a_2 x^2 + a_0 \arrow[rr, "\textit{AMA XXI.1}"] & & y^3 + \frac{a_0}{a_2} y = \frac{a_0^2}{a_2^3} \arrow[d, "\textit{AMA XXVIII}"] \\
x = \frac{a_0}{a_2\, y} & & y \arrow[ll, "\textit{AMA XXI.1}"]
\end{tikzcd}$$

The second way is – in Cardano's opinion – "by far easier".

AMA XXIX.ii. *Given the equation $x^3 = a_2 x^2 + a_0$, if f, g real exist such that*

$$\begin{cases} 2\left(\frac{a_2}{3}\right)^3 + a_0 = f + g \\ \left(\frac{a_2}{3}\right)^6 = fg \end{cases},$$

then $x = \sqrt[3]{f} + \sqrt[3]{g} + \frac{a_2}{3}$.

No hints are given on how Cardano discovered the propositions. In particular, we remark that (AMA XXIX.i) does not help in understanding why Cardano sets $2\left(\frac{a_2}{3}\right)^3 + a_0$ and $\left(\frac{a_2}{3}\right)^6$ apart. But, if we perform the substitution $x = y + \frac{a_2}{3}$ in (AMA XXI.9) (or in (AM IV.2)), we get to $y^3 = \frac{a_2^2}{3} y + (\frac{2}{27} a_2^3 + a_0)$, the coefficients of which appear in (AMA XXIX.ii).

We moreover remark that, in order to find such f, g, one only has to solve a quadratic equation. More precisely, it happens that

$$\begin{cases} q = f + g \\ \left(-\frac{p}{3}\right)^3 = fg \end{cases},$$

where p, q are defined in (1.5.5) (see above, at page 40), so that the quadratic equation $t^2 - qx + \left(-\frac{p}{3}\right)^3 = 0$ is the Lagrange's resolvent (up to the sign of the coefficients, see equation (1.5.8), at page 41). Then, (AMA XXIX.ii) is equivalent to the cubic formula[48] in (AMA XIV), even if Cardano does not explicitly state it.

Cardano ends saying that "this solution has no obstacle, therefore the chapter is the most general".[49] We remark that, in fact, in this case we always have $\Delta_3 > 0$.

In **Chapter XXX** "On the general case which has only one exception when the cube is equal to some things and a number [*De caitulo generali habente tantum unam exceptionem, quando cubus æquatur rebus, et numero*]" Cardano deals with the equation $x^3 = a_1x + a_0$. He says that this case is the converse of (AMA XXVIII).

In a very similar fashion, Cardano provides the cubic formula as in (AM XII). No justification is given, except one example to show that the formula works.

AMA XXX. *Given the equation $x^3 = a_1x + a_0$, a solution is*

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}}.$$

Cardano observes that, if a condition equivalent to $\Delta_3 \geq 0$ does not hold, then "no solution can follow [*nulla potest sequi æquatio*]". By the way, we remark that, in the example $x^3 = 6x + 10$ provided by Cardano, it happens that $\Delta_3 > 0$.

[48]See above (1.5.9), at page 42 (only the first line for real solutions) or the specific version for $x^3 = a_2x^2 + a_0$ in (AMA XIV), at page 119.
In fact, assuming the hypothesis in the system in (AMA XXIX.ii) and the cubic formula in (AMA XIV), the formula boils down to

$$x = \sqrt[3]{\frac{f+g}{2} + \sqrt{\left(\frac{f+g}{2}\right)^2 - fg}} + \sqrt[3]{\frac{f+g}{2} - \sqrt{\left(\frac{f+g}{2}\right)^2 - fg}} + \frac{a_2}{3},$$

that is $x = \sqrt[3]{f} + \sqrt[3]{g} + \frac{a_2}{3}$.
Conversely, assuming (AMA XXIX.ii), we get

$$\begin{cases} f = 2\left(\frac{a_2}{3}\right)^3 + a_0 - g \\ \left(\frac{a_2}{3}\right)^6 = \left(2\left(\frac{a_2}{3}\right)^3 + a_0 - g\right) g \end{cases},$$

that is $g^2 - \left(2\left(\frac{a_2}{3}\right)^3 + a_0\right) g + \left(\frac{a_2}{3}\right)^6 = 0$, which is the resolvent. Its solutions are the ones expected (and the same for f) in order to get the cubic formula.
[49]"*[T]alis equatio non habet impedimentum, ideo capitulum est generalissimum*", see [Cardano 1663c, Chapter XXIX, page 343].

Cardano ends saying that he found “this rule also by the rules of Chapter XXVII, and all the remaining [rules] except the ones of which [he] mentioned the authors in their places”.[50]

In **Chapter XXXII** “On the chapter of the cube and squares equal to a number which is general with only one exception [*De capitulo cubi et census æqualium numero, et est generale, habens unam exceptionem tantum*]” Cardano deals with the equation $x^3 + a_2x^2 = a_0$. As in Chapter XXIX, he suggests three ways to get a solution.

The first way corresponds to (AMA XXIX.ii).

AMA XXXII.i. *Given the equation* $x^3 + a_2x^2 = a_0$*, if* f, g *real exist such that*

$$\begin{cases} a_0 - 2\left(\frac{a_2}{3}\right)^3 = f + g \\ \left(\frac{a_2}{3}\right)^6 = fg \end{cases},$$

then $x = \sqrt[3]{f} + \sqrt[3]{g} - \frac{a_2}{3}$.

Also in this case, the corresponding remarks hold. In particular, (AMA XXXII.i) is equivalent to the cubic formula (1.5.9) (only the first line for real solutions, see above, at page 42) or the specific version for $x^3 + a_2x^2 = a_0$ in the footnote 63 (see above, at page 121).

The second way is by transformations of equations and corresponds to (AMA XXIX.i).

AMA XXXII.ii. *Given the equation* $x^3 + a_2x^2 = a_0$*, one draws a solution in the following way.*

$$\begin{tikzcd} x^3 + a_2x^2 = a_0 \arrow[r, "\textit{AMA XXI.2}"] & y^3 = \frac{a_2^2}{3}y + \left(a_0 - \frac{2}{27}a_2^3\right) \arrow[d, "*"] \\ x = y - \frac{a_2}{3} & y \arrow[l, "\textit{AMA XXI.2}"'] \end{tikzcd}$$

[50]“*[H]oc etiam ergo inveni ex regulis capituli 27i sicut et omnia reliqua præter illa duo quibus assignavi auctores suis locis*”, see [Cardano 1663c, Chapter XXX, page 343].

We expect $*$ to be (AMA XXX) in order to pursue the parallelism with (AMA XXIX.i). Instead, Cardano says that $*$ is

$$\begin{cases} a_0 - 2\left(\frac{a_2}{3}\right)^3 = f + g \\ \left(\frac{a_2}{3}\right)^6 = fg \end{cases}.$$

This means that, once applied the transformation in (AMA XXI.2), we fall again on (AMA XXXII.i), which was equivalent to the cubic formula for $x^3 + a_2x^2 = a_0$. In the end, $*$ is (AMA XXX), and $*$ together with the transformation in (AMA XXI.2) gives (AMA XXXII.i). In this case, we find in the text an insight on why Cardano sets $a_0 - 2\left(\frac{a_2}{3}\right)^3$ and $\left(\frac{a_2}{3}\right)^6$ apart, since the transformation in (AMA XXI.2) (or in (AMA XXI.9), or in (AM IV.2)) is explicitely recalled. We remark that this is basically the same idea as in (AM XV), even if less refined. In fact, here Cardano considers only the case $a_0 - \frac{2}{27}a_2^3 > 0$, whereas there also the cases < 0 and $= 0$.

The third way is, in short:

AMA XXXII.iii. *Given the equation $x^3 + a_2x^2 = a_0$, one draws a solution in the following way.*

$$\begin{tikzcd} x^3 + a_2x^2 = a_0 \arrow[rr, "\textit{AMA XXI.2}"] & & y^3 = a_1y + \sqrt{a_0} \arrow[d, "*"] \\ x = \frac{\sqrt{a_0}}{y} & & y \arrow[ll, "\textit{AMA XXI.2}"'] \end{tikzcd}$$

I recall that (AMA XXI.2) was composed by three folds. In this proposition, Cardano uses the third one.[51] Concerning $*$, Cardano only suggests to "find the solution [*invenias æquationem*]" without further specifying (neither in the short example which follows). So, I will suppose that here $*$ is again (AMA XXX).

[51]Cardano speaks of "the transformation of the cube and squares equal to a number in the cube equal to things and a number by the second way of this transformation in Chapter XXI [*conversionem cubi et census æqualium numero in cubum æqualem rebus et numero per secundum modum talis conversionis 21. capituli*]", see [Cardano 1663c, Chapter XXXII, page 346]. By the short example which follows it is clear that he is referring to the third fold of (AMA XXI.2).

In this chapter, Cardano never comes back on the "exception" mentioned in the title. Anyway, we remark that some "particular" rules are given for the equation $x^3 + a_2x^2 = a_0$ in Chapter XXXIII.

In **Chapter XXXIV** "On the general rule of the cube and number equal to the things [*De regula generali cubi et numeri æqualis rebus*]" Cardano deals with the equation $x^3 + a_0 = a_1x$. He suggests a transformation to get a solution.

AMA XXXIV.i. *Given the equation $x^3 + a_0 = a_1x$, one draws a solution in the following way.*

$$\begin{tikzcd}
x^3 + a_0 = a_1x \arrow[rr, "\textit{AMA XXI.3}"] & & y^3 = a_1y + a_0 \arrow[d, "\textit{AMA XXX}"] \\
x = \frac{y}{2} \pm \sqrt{a_1 - 3\left(\frac{y}{2}\right)^2} & & y \arrow[ll, "\textit{AMA XXI.3}"']
\end{tikzcd}$$

The formula is the one in (AM XIII).

Then, Cardano provides the following proposition.

AMA XXXIV.ii. *The solutions of $x^3 + a_1x = a_0$ and $x^3 = a_2x^2 + a_0$ can be* "binomia *or* trinomia *by one or more* universalia *cubic roots".*[52]

The solutions of $x^3 = a_1x + a_0$ and $x^3 + a_2x^2 = a_0$ can be part "binomia *or* trinomia *by cubic root", part* "binomia *or* trinomia *by one or more square roots".*[53]

The solutions of $x^3 + a_0 = a_1x$ and $x^3 + a_0 = a_2x^2$ can be "binomia *or* trinomia *by one or more square roots".*[54]

We remark that one finds a *binomium* in the case of depressed equations and a *trinomium* in the case of cubic equations lacking in the first degree term. This proposition is simply obtained by observing the shape of the solutions in the

[52] "*[Æ]quationes cubi et rerum æqualium numero, item cubi æqualis censibus et numero sunt RV: cubicæ bunimiales vel trinomiales*", see [Cardano 1663c, Chapter XXXIV, page 348].
[53] "*[Æ]quationes cubi æqualis rebus et numero et cubi et censuum æqualium numero sunt partim binomiales aut trinomiales R cubicæ partim quadratæ*", see [Cardano 1663c, Chapter XXXIV, page 348].
[54] "*[Æ]quationes cubi numeri æqualium censibus aut rebus sunt R binomiales aut trinomiales quadratæ*", see [Cardano 1663c, Chapter XXXIV, page 348].

previously mentioned chapters (and in Chapter XXXVI). Note that, now that Cardano has (explicitly or implicitly described by the solving method) some of the cubic formulae available, the qualitative description of the shape for the solutions radically changes. This could confirm my interpretation at page 198.

Then, Cardano ends saying that, since $x^3 + a_0 = a_1 x$ always has two (real, positive, non zero) solutions, then the following proposition holds.

AMA XXXIV.iii. *Given the equation $x^3 + a_0 = a_1 x$, let x_1 be a solution. Then*

$$x_2 = \sqrt{a_1 - x_1^2 + \frac{x_1^2}{4}} - \frac{x_1}{2}$$

is another solution.

This is (AM XIII bis). Cardano's words[55] remind of (AM VIII.2), but I did not find a similar statement in the *Ars magna*.

In **Chapter XXXVI** "On the universal chapter of the cube and number equal to the squares [*De capitulo universali cubi et numeri æqualium censibus*]" Cardano deals with the equation $x^3 + a_0 = a_2 x^2$. Similarly to Chapter XXXIV, he suggests two ways to get a solution by transformation and a way to get another (real, positive, non zero) solution provided that one has a (real, positive, non zero) solution.

AMA XXXVI.i. *Given the equation $x^3 + a_0 = a_2 x^2$, , one draws a solution in the following way.*

$$\begin{tikzcd}
x^3 + a_0 = a_2 x^2 \arrow[rr, "\textit{AMA XXI.4}"] & & y^3 + a_0 = a_2 \sqrt[3]{a_2}\, y \arrow[d, "\textit{AMA XXXIV}"] \\
x = \frac{(\sqrt[3]{a_0})^2}{y} & & y \arrow[ll, "\textit{AMA XXI.4}"]
\end{tikzcd}$$

The formula is the one in (AM XVI).

[55]"*Notum quod cum volueris invento uno valore æquationis cubi et numeri æqualium rebus habere reliquum valorem tu scis quod unus semper est R unius partis numeri radicum quæ multiplicata in reliquam partem producit numerum æquationis reliquus valor est etiam eiusmodi videlicet R unius partis quæ ducta in reliquam producit numerum æquationis*", see [Cardano 1663c, Chapter XXXIV, page 348].

AMA XXXVI.ii. *Given the equation $x^3 + a_0 = a_2x^2$, , one draws a solution in the following way.*

$$\begin{tikzcd}
x^3 + a_0 = a_2x^2 \arrow[r, "?"] & y^3 + a_2y^2 = a_0 \arrow[d, "AMA\ XXXII"] \\
x & y \arrow[l, "?"']
\end{tikzcd}$$

I have no hints on which transformation Cardano uses in *?*, since he never mentions it elsewhere.

Then, Cardano says that "the general solution of this chapter is the mixed solution".[56] This could have been seen from (AMA XXII.iii) or from the following proposition.

AMA XXXVI.iii. *Given the equation $x^3 + a_0 = a_2x^2$, let x_1 be a solution. Then*

$$x_2 = \sqrt{(a_2 - x_1)\left(x_1 + \frac{a_2 - x_1}{4}\right)} + \frac{a_2 - x_1}{2}$$

is another solution.

Again, this is (AM XVI bis). And again, Cardano's words[57] remind of (AM VIII.2).

In **Chapter XXXIX** "On the solutions of the chapters of *quadrinomia* and *quinomia* [*De æquationibus capitulorum quadrinomiorum et quinomiorum*]" Cardano deals with the complete cubic equations and some derivative cases. We will only follow him on complete cubic equations.

Cardano begins with two rules, which, according to the list in Chapter XX (see above, at page 200), should be the only "universal" (that is, "general") ones in this chapter.

AMA XXXIX.1. *Consider $x^3 + a_2x^2 + a_1x = a_0$ with $a_1 = \frac{a_2^2}{3}$.*

[56]"*[Æ]quatio generalis huius capituli est æquatio mixta*", see [Cardano 1663c, Chapter XXXVI, page 350].

[57]"*Cum volueris, in cubo et numero æqualibus censibus per unum valorem habere reliquum scias quod uterque valor est pars numeri censuum quæ quadrata et ducta in residuum producit numerum æquationis*", see [Cardano 1663c, Chapter XXXVI, page 350].

Then $x = \sqrt[3]{\left(\frac{a_2}{3}\right)^3 + a_0} - \frac{a_2}{3}$.

Cardano mentions Ferrari, who should have found the "geometrical demonstration" of this rule.[58] Actually, no proof follows.[59] We observe that Cardano made a similar, even if less detailed, remark in (AMA XXIV.ii). According to that, the above solution is a mixed, cubic *recisum* of the 1st or 4th types

AMA XXXIX.2. *Consider* $x^3 + a_1x = a_2x^2 + a_0$ *with* $a_1 = \frac{a_2^2}{3}$.
Then $x = \frac{a_2}{3} - \sqrt[3]{\left(\frac{a_2}{3}\right)^3 - a_0}$ *or* $x = \frac{a_2}{3} + \sqrt[3]{a_0 - \left(\frac{a_2}{3}\right)^3}$.

Again, no justification follows.[60] Cardano adds that "this solution falls on three kinds". If $\frac{a_2}{3} > \sqrt[3]{\left(\frac{a_2}{3}\right)^3 - a_0}$, then both solutions [are positive, non zero and they] go under the shape of the [cubic] *binomium* and *recisum* of the 1st [or 4th] types. If $\frac{a_2}{3} < \sqrt[3]{\left(\frac{a_2}{3}\right)^3 - a_0}$, then [only the second solution is positive, non zero and] it goes under the shape of the [cubic] *binomium* of the 2nd [or 5th] types.[61] We observe that Cardano made a similar, even if less detailed, remark in (AMA XXIV.i).

One should expect these rules to be "general", according to Cardano's classification in Chapter XX (see above, at page 200). According to my weaker interpretation at page 198, we are allowed to call these two rules "general" since they lead to "general solutions". In fact, in (AMA XXXIX.1) and in (AMA XXXIX.2) the solutions go under the standard form of the cubic *binomia* and *recisa*.

Then, Cardano goes on with some other rules.

AMA XXXIX.3. *Consider* $x^3 + a_2x^2 + a_1x = a_0$.

[58] "*Prima regua Ludovici de Ferrariis quam invenit geometrica demonstratione*", see [Cardano 1663c, Chapter XXXIX, paragraph 1, page 352].

[59] In any case, we can easily recover it from the equality

$$\left(x + \frac{a_2}{3}\right)^3 = \left(\frac{a_2}{3}\right)^3 + a_0.$$

[60] In any case, we can easily recover it from the equality

$$\left(x - \frac{a_2}{3}\right)^3 = a_0 - \left(\frac{a_2}{3}\right)^3.$$

[61] See above, at page 193.

Then, x^3 *and* a_1x *are "of the same nature" [that is, speaking of* binomia *and* recisa*, of the same type]. Moreover, the* recisum *of the 2nd [and 5th] types can be a solution.*

We remark that this equation has already been considered in (AMA XXXIX.1). Cardano justifies this proposition in a way similar to (AMA XXII.i). A few examples follow.

AMA XXXIX.4. *The* binomium *and* recisum *of the 1st and 4th types and the* recisum *of the 2nd and 5th types can be solutions of the equation* $x^3 + a_2x^2 + a_0 = a_1x$.

AMA XXXIX.5. *The* binomium *and* recisum *of the 1st and 4th types and the* binomium *of the 2nd and 5th types can be solutions of the equation* $x^3 + a_1x + a_0 = a_2x^2$.

AMA XXXIX.6. *The* binomium *of the 2nd and 5th types can be solutions of the equation* $x^3 = a_2x^2 + a_1x + a_0$.

For each one of the above three propositions, only a few examples follow after the statement.

By the classification in Chapter XX, one should expect these rules to be "particular", but in fact they are not – even according to our weaker interpretation. Indeed, Cardano says that "it is clear that all this chapters are very general except the last one, that is the cube equal to squares, things and a number. In fact, this case, although general, is nevertheless very involute".[62] We remark that, consistently with our weaker interpretation at page 198, there is no reason why (AMA XXXIX.3)-(AMA XXXIX.6) should not be considered as general as (AMA XXXIX.1) or (AMA XXXIX.2). Then, we could suppose an omission or a misprint in Chapter XX. But this does not explain why (AMA XXXIX.6) should have a different status.

[62]"*Manifestum est igitur quod hæc capitula valde generalia sunt excepto ultimo, videlicet quando cubus æquatur censibus, rebus et numero, nam tale caputilum, etsi si generale, attamen est valde involutum*", see [Cardano 1663c, Chapter XXXIX, paragraph 6, page 355].

Cardano then affirms that "the chapters with four terms depend on [the chapters with] three [terms]".[63] More precisely,

AMA XXXIX.iii. *The equation* $x^3 + a_2x^2 + a_1x = a_0$ *depends on*

- $x^3 + a'_1x = a'_0$ *(no solution in terms of* binomia *and* recisa*),*[64]
- $x^3 + a''_2x^2 = a''_0$*; they both have the* recisum *of the 2nd and 5th types as solution.*[65]

AMA XXXIX.iv. *The equation* $x^3 + a_2x^2 + a_0 = a_1x$ *depends on*

- $x^2 + a'_0 = a'_1x$*; they both have the* binomium *and* recisum *of the 1st and 4th types as solution,*[66]
- $x^3 + a''_0 = a''_1x$*; they both have the* binomium *and* recisum *of the 1st and 4th types and the* recisum *of the 2nd and 5th types as solution.*[67]

AMA XXXIX.v. *The equation* $x^3 + a_1x + a_0 = a_2x^2$ *depends on*

- $x^2 = a'_1x + a'_0$*; they both have the* binomium *of the 2nd and 5th types as solution,*[68]
- $x^3 + a''_0 = a''_2x^2$*; they both have the* binomium *and* recisum *of the 1st and 4th types and the* binomium *of the 2nd and 5th types as solution.*[69]

AMA XXXIX.vi. *The equation* $x^3 = a_2x^2 + a_1x + a_0$ *depends on*

- $x^3 = a'_1x + a'_0$*; they both have the* binomium *of the 2nd and 5th types as solution,*[70]
- $x^3 = a''_2x^2 + a''_0$ *(no solution in terms of* binomia *and* recisa*).*[71]

These propositions seem to be a more elaborated version of the corresponding (AMA XXXIX.3)-(AMA XXXIX.6).

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63 "*Pendent autem capitula quaternaria ex ternariis*", see [Cardano 1663c, Chapter XXXIX, paragraph 6, page 355].
64 See above, after (AMA XXII.ii).
65 See above (AMA XXII.iv).
66 At the beginning of Chapter XXII Cardano deals with the shape of the solutions of quadratic equations.
67 See above (AMA XXII.i).
68 See above, footnote 66.
69 See above (AMA XXII.iii).
70 See above (AMA XXII.ii).
71 See above, after (AMA XXII.iv).

AMA XXXIX.7. *The equation* $x^3 + a_0 = a_2x^2 + a_1x$ *depends on*

- $x^3 + a_0' = a_1'x$*; they both have the* binomium *and* recisum *of the 1st and 4th types and the* recisum *of the 2nd and 5th types as solution,*[72]
- $x^3 + a_0'' = a_2''x^2$*; they both have the* binomium *and* recisum *of the 1st and 4th types and the* binomium *of the 2nd and 5th types as solution.*[73]

AMA XXXIX.8. *The equation* $x^3 + a_1x = a_2x^2 + a_0$ *depends on*

- $x^2 + a_0' = a_1'x$*; they both have the* binomium *and* recisum *of the 1st and 4th types as solution,*[74]
- $x^3 = a_2''x^2 + a_0''$ *(no solution in terms of* binomia *and* recisa*).*[75]

We remark that this equation had already been considered in (AMA XXXIX.2).

AMA XXXIX.9. *The equation* $x^3 + a_2x^2 = a_1x + a_0$ *depends on*

- $x^2 = a_1'x + a_0'$*; they both have the* binomium *of the 2nd and 5th types as solution,*[76]
- $x^3 = a_1''x + a_0''$*; they both have the* binomium *of the 2nd and 5th types as solution.*[77]

We could then complete the table 3.1 at page 197 as follows.

[72] See above (AMA XXII.i).
[73] See above (AMA XXII.iii).
[74] See above, footnote 66.
[75] See above, after (AMA XXII.iv).
[76] See above, footnote 66.
[77] See above (AMA XXII.ii).

Equation	Shape of the irrational solution			
	binomia and/or *recisa*			*trinomia*
	(normal)	cubic	mixed cubic	
$x^3 + a_1x = a_0$		$\sqrt[3]{a} - \frac{k}{a}\sqrt[3]{a^2}$		
$x^3 = a_1x + a_0$	$\sqrt{a} + b$	$\sqrt[3]{a} + \frac{k}{a}\sqrt[3]{a^2}$		
$x^3 + a_0 = a_1x$	$a \pm \sqrt{b}$, $\sqrt{a} - b$			
$x^3 = a_2x^2 + a_0$				$a\sqrt[3]{a^2} + \sqrt[3]{a} + a$
$x^3 + a_2x^2 = a_0$	$\sqrt{a} - b$			$a\sqrt[3]{a^2} + \sqrt[3]{a} - a$
$x^3 + a_0 = a_2x^2$	$a \pm \sqrt{b}$, $\sqrt{a} + b$			
$x^3 + a_2x^2 + a_1x = a_0$	$\sqrt{a} - b$		$a - \sqrt[3]{b}$	
$x^3 + a_1x = a_2x^2 + a_0$	$a \pm \sqrt{b}$		$a + \sqrt[3]{b}$, $\sqrt[3]{a} \pm b$	
$x^3 + a_2x^2 = a_1x + a_0$	$\sqrt{a} + b$			
$x^3 = a_2x^2 + a_1x + a_0$	$\sqrt{a} + b$			
$x^3 + a_0 = a_2x^2 + a_1x$	$a \pm \sqrt{b}$, $\sqrt{a} \pm b$			
$x^3 + a_1x + a_0 = a_2x^2$	$a \pm \sqrt{b}$, $\sqrt{a} + b$			
$x^3 + a_2x^2 + a_0 = a_1x$	$a \pm \sqrt{b}$, $\sqrt{a} - b$			
Condition		$k \in \mathbb{Q}$		

TABLE 3.2 – Shapes of irrational solutions to cubic equations in *Ars magna arithmeticæ*, Chapter XXXIX.

Referring to the above propositions, Cardano uses “to depend [*pendeo*]”, “to be equivalent [*æquivaleo*]”, and “to compose [*compono*]” as synonyms, with a preference for the last term. But he does not explain in which sense a complete equation “depend” on, or “is equivalent” to, or “is composed” by two other equations. I cannot figure out a sensible meaning for that.[78] Moreover and in any case, no explanation on the different status of (AMA XXXIX.6) is given. This last part is quite confused. Since there are no definitively valid arguments for that, I prefer to trust Chapters XX and XXIV and consider that only $x^3+a_2x^2+a_1x = a_0$ and $x^3 + a_1x = a_2x^2 + a_0$ have a "general" rule.[79]

Summing up, we remark again that the hypothesis that the *Ars magna* comes from a further development of the *Ars magna arithmeticæ* is not at all unlikely. Especially regarding to the cubic equations, the *Ars magna arithmeticæ* is at an earlier stage, since the treatment of these equations is less systematic. Concerning depressed equations and equations lacking in the first degree term, we basically find (less refined) all the results of the *Ars magna*. The cubic formulae are not always explicitly written, but for the most of the non-complete equations they are already there *in nuce* (except maybe for $x^3 + a_0 = a_2x^2$ in Chapter XXXVI, of which we anyway know a *binomium* shape for one of its solutions, see (AMA XXXVI.iii)). The major difference with the *Ars magna* is that here there are no proofs. Moreover, the ordering of the results is different so that the dependence of each equation lacking in the first degree term from a depressed one is clearer.

As in the *Ars magna*, Tartaglia is mentioned only in relation with the equation $x^3 + a_1x = a_0$.

Concerning complete cubic equations, there is no more correspondence between the *Ars magna arithmeticæ* and the *Ars magna*. In fact, even if Cardano knows the substitution $x = y \pm \frac{a_2}{3}$ by (AMA XXI.9), he does not use it, so that he never gets to an equivalent of the cubic formula as in the *Ars magna*.

We have then the following correspondences.

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[78]We cannot always pass from a complete equation to three-terms equations by transformations, since also quadratic equations are involved. Or otherwise, if we consider the system formed by the three-terms equations, it is not true that it leads to the corresponding complete equation. Or finally, we could remark that the three-terms equations and the corresponding complete equation have a similar outfit so that, if we sum the three-terms equations, we get the corresponding complete equation. Then, if a number is a solution of one of the three-term equations, it will also be a solution of the complete equation. But in this way we are assuming a relationship between the solutions of the three-terms equations, which is not the case.

[79]We anyway remark that [Cossali 1799a] includes Chapter XXXIX among the "particular" rules. Nevertheless, Pietro Cossali also affirms that, at Cardano's time, there was probably no agreement on considering these rules "general" or "particular", see [Cossali 1799a, volume II, chapter II, Soluzioni speciali, page 183] and above, footnote 2.1.4 at page 99.

(AM XI)	$x^3 + a_1 x = a_0$	(AMA XXVIII)
(AM XII)	$x^3 = a_1 x + a_0$	(AMA XXX)
(AM XIII), (AM XIII bis)	$x^3 + a_0 = a_1 x$	(AMA XXXIV.i), (AMA XXXIV.iii)
(AM XIV)	$x^3 = a_2 x^2 + a_0$	(AMA XXIX.ii)
(AM XV)	$x^3 + a_2 x^2 = a_0$	(AMA XXXII.ii)
(AM XVI), (AM XVI bis)	$x^3 + a_0 = a_2 x^2$	(AMA XXXVI.i), (AMA XXXVI.ii)

TABLE 3.3 – Links between "general" propositions in the *Ars magna arithmeticæ* and *Ars magna*.

Finally, we remark that the *Ars magna arithmeticæ* is again linked to the opposition between "general" and "particular", as it appears from the title of all the considered chapters. Since Cardano does not state here whether "particular" is referred to the methods or to solutions, I will assume the most natural choice, that is, that "general" is referred to the methods employed. Anyway, this interpretation clashes in two points. Firstly, it seems that a "general" method can bear "exceptions", as in the titles of Chapters XXX and XXXII. Secondly, while speaking of complete cubic equations, I could not account for Cardano's use of "general" but interpreting "general" as referred to solutions.

3.2.6. "Particular" solving methods. In this section and when needed, we will refer to the functions of the coefficients in *Ars magna*, Chapter XXV using f', g', since my aim here is to explain the propositions concerning "particular" rules in the *Ars magna arithmeticæ* through the corresponding ones in the *Ars magna*.

We find Cardano's "particular" solving methods for cubic equations in Chapters XXXI, XXXIII, XXXV, and XXXVII.

In **Chapter XXXI** "On the chapter and particular rules of the cube equal to the things and a number [*De capitulo et regulis particularibus cubi æqualis rebus et numero*]", Cardano states the following propositions.

AMA XXXI.i. *Consider* $x^3 = a_1x + a_0$.
If $a_1\sqrt[3]{f^3} = f^3 + a_0$, *then* $x + f$ *divides* $x^3 + f^3$.
If $a_1\sqrt[3]{f^3} = f^3 - a_0$ *[with* $f^3 > a_0$*], then* $x - f$ *divides* $x^3 - f^3$.

No further explanation is given, except the example $x^3 = 16x + 21$ (with $\Delta_3 < 0$). Cardano explicitly remarks that this example cannot "be solved by the preceding chapter [*non potest solui per præcedens capitulum*]". In fact, we remind[80] that Chapter XXX deals with the "general" case of $x^3 = a_1x + a_0$ which nevertheless has "one exception". Then, as in (AM XII), it seems that a "general" rule or method can bear "exceptions".

We remark that this rule is (AM XXV.6) (with the same example) and (PA LI.26).

AMA XXXI.ii. *Consider* $x^3 = a_1x + a_0$.

If $\begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g} \end{cases}$, *then* $x = \sqrt{f + \frac{g}{4}} + \frac{\sqrt{g}}{2}$.

We remark that this is (AM XXV.1) and also one of the examples which follow $x^3 = 20x + 32$ (with $\Delta_3 < 0$) is the same.

AMA XXXI.iii. *Consider* $x^3 = a_1x + a_0$.

If $\begin{cases} f = \sqrt{a_1 + g} \\ a_0 = fg \end{cases}$, *then* $x = \sqrt{a_1 + g}$.

We remark that this is (AM XXV.2) and also one of the examples which follow $x^3 = 32x + 24$ (with $\Delta_3 < 0$) is the same.

In **Chapter XXXIII** "On the particular rules of the cube and squares equal to a number [*De regulis particularibus cubi et censuum æqualium numero*]", Cardano states the following propositions.

AMA XXXIII.i. *Consider* $x^3 + a_2x^2 = a_0$.

If $\begin{cases} a_2 = f + g \\ a_0 = fg^2 \end{cases}$, *then* $x = \sqrt{f\left(g + \frac{f}{4}\right)} - \frac{f}{2}$.

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80 See above, Section 3.2.5 at page 203.

We remark that this is (AM XXV.14) and also one of the examples which follow $x^3 + 20x^2 = 72$ (with $\Delta_3 < 0$) is the same.

Then, Cardano states in a less detailed way the proposition (AMA XXXII.ii), and this time explicitly specifying that $*$ is (AMA XXX).

Finally, he remarks that the solutions of $x^3 + a_2x^2 = a_0$ are opposite in sign to the ones of $x^3 + a_0 = a_2x^2$.[81]

In **Chapter XXXV** "On the particular rules of the cube and number equal to the things [*De regulis particularibus cubi et numeri æqualium rebus*]", Cardano states the following propositions.

AMA XXXV.i. *Consider*[82] $x^3 + a_0 = a_1x$.

$$\text{If } \begin{cases} \frac{a_1}{4} < g < \frac{a_1}{3} \\ g \text{ divides } a_0 \\ a_1 + \frac{a_0}{2\sqrt{g}} = 4g \end{cases}, \text{ then } x = \sqrt{g} \pm \sqrt{a_1 - 3g}.$$

Note that the condition $g < \frac{a_1}{3}$ grants that $a_1 - 3g > 0$, that is, that all the solutions are real. The condition $\frac{a_1}{4} < g$ grants that $\sqrt{g} - \sqrt{a_1 - 3g} > 0$. We remark[83] that this is (AM XXV.9) taking $f' = 4g - a_1$, and then $2\sqrt{g} = \sqrt{a_1 + f'}$.

AMA XXXV.ii. *Consider* $x^3 + a_0 = a_1x$.

$$\text{If } \begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g} \end{cases}, \text{ then } x = \sqrt{f + \frac{g}{4}} - \frac{\sqrt{g}}{2}.$$

Cardano remarks that this proposition corresponds to (AMA XXXI.ii). In fact, if we write a_1'', a_0'' for the coefficients and f'', g'' for the functions of the coefficients in (AMA XXXI.ii), then it is enough to take $f = f''$ and $\sqrt{g} = -\sqrt{g''}$, in order to verify that $a_1 = a_1''$ and $a_0 = -a_0''$.

We remark that this is (AM XXV.9), taking $f = -f'$ and $\sqrt{g} = -\sqrt{a_1 + f'}$.

AMA XXXV.iii. *Consider* $x^3 + a_0 = a_1x$.

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[81]See (AM I.8 ii).

[82]At the end of the chapter (see [Cardano 1663c, Chapter XXXV, pages 349-50]), Cardano adds a few words to explain why $\sqrt{a_1 - 3g}$ is not rational. Since they are really obscure, I do not comment them.

[83]Or, equivalently, this is (AM VIII.2 i), taking $G = -2\sqrt{g}$, $F = a_1 - 4g$, $\alpha_1 = -a_1$, and $\alpha_0 = a_0$.

If $a_0 - a_1 f = f^3$*, then* $x - f$ *divides* $x^3 + a_0 - a_1 f$*.*

Cardano remarks that this proposition corresponds to (AMA XXXI.i).

We remark that this is (AM XXV.9), taking $f = -f'$ and $\sqrt{g} = -\sqrt{a_1 + f'}$, and also (AM XXV.10) (with the same example) and (PA LI.26).

In **Chapter XXXVII** "On the particular rules of the cube and number equal to the squares [*De regulis particularibus cubi et numeri æqualium censibus*]", Cardano reminds that the *binomium* or *recisum* of the 1st or 4th types $a \pm \sqrt{b}$ and the *binomium* of the 2nd or 5th types $\sqrt{a}+b$ can be solutions of $x^3+a_0 = a_2x^2$ (see (AMA XXII.iii)). More precisely, he says that, when a solution is the *binomium* of the 1st or 4th types, then also its *recisum* is a solution and that, when a solution is an integer, then also the *binomium* of the 2nd or 5th types is a solution. Then, he wants to show that "the type of the *binomium* and *recisum* is not enough to meet each solution of this chapter".[84] But unluckily his justification seems to be totally unclear.

AMA XXXVII.i. *Consider* $x^3 + a_0 = a_2x^2$ *and let* x_1 *be a solution.*

Then

$$x_2 = \sqrt{fx_1 + \left(\frac{f}{2}\right)^2} + \frac{f}{2}$$

is another solution.

We remark that this is (AM XVI bis) and also (AMA XXXVII.iii), taking $f = a_2 - x_1$.

AMA XXXVII.ii. *Consider* $x^3 + a_0 = a_2x^2$*.*

$$\text{If } \begin{cases} \frac{a_2}{4} \leq g \leq \frac{a_2}{3} \\ \sqrt{\frac{a_0}{g}} \in \mathbb{Q} \\ a_2 + \frac{1}{2}\sqrt{\frac{a_0}{g}} = 4g \end{cases}, \text{ then } x = 2g \pm \sqrt{4g(a_2 - 3g)}.$$

Cardano remarks that this proposition corresponds to (AMA XXXV.i).

We remark that this proposition corresponds to (AMA XXXIII.i). In fact, if we write a_1'', a_0'' for the coefficients and f'', g'' for the functions of the coefficients

[84]"*[O]portet ostendere quod genus binomij et recisi non est sufficiens satisfacere omni æquationi huius capituli*", see [Cardano 1663c, Chapter XXXVII, page 350].

in (AMA XXXIII.i), then it is enough to take $f'' = -4g$ and $g'' = 4g - a_2$, in order to verify that $a_2 = -a_2''$ and $a_0 = -a_0''$.

We moreover remark that this is (AM XXV.15) taking $g = f'$ and also one of the example is the same. Or, equivalently, this is (AM VIII.2 ii), taking with $G = -\frac{1}{2}\sqrt{\frac{a_0}{g}}$, $F = 4g$, $\alpha_2 = -a_2$, and $\alpha_0 = a_0$.

Finally, Cardano states in a less detailed way the proposition (AMA XXI.4).

Summing up, we remark that all the above "particular" propositions can also be found in the *Ars magna*, mainly in Chapter XXV.

$x^3 = a_1x + a_0$		$x^3 + a_0 = a_1x$	$x^3 + a_2x^2 = a_0$		$x^3 + a_0 = a_2x^2$
(AM XXV.1) (AMA XXXI.ii)	$\longleftrightarrow$	(AM XXV.9) (AMA XXXV.i,ii)	(AM XXV.14) (AMA XXXIII.i)	$\longleftrightarrow$	(AM XXV.15) (AMA XXXVII.ii)
(AM XXV.2) (AMA XXXI.iii)	$\longleftrightarrow$	(AM XXV.17)	(AM XXV.18)		
(AM XXV.3)	$\longleftrightarrow$	(AM XXV.13)			
(AM XXV.4)	$\longleftrightarrow$	(AM XXV.11)			
(AM XXV.5)		(AM XXV.12)			
(AM XXV.6) (AMA XXXI.i) (PA LI.26)	$\longleftrightarrow$	(AM XXV.10) (AMA XXXV.iii) (PA LI.26)			

TABLE 3.4 – Links between "particular" propositions in *Ars magna*, Chapter XXV and in the *Ars magna arithmeticæ*.

Out of the above numbered propositions, only one is not also in Chapter XXV. It is (AMA XXXVII.i), which is the same as (AM XVI bis) and as (AMA XXXVI.ii). We remark by the way that there is no equivalent of (AM XIII bis).

Cardano says that these rules are "known in a particular way".[85] Comparing to the use of this term in the *Ars magna*, Cardano does not state here whether

[85] See [CARDANO 1663c, Chapter XX, page 326] or above, at page 199.

"particular" is referred to the methods or to solutions. Since Cardano provides no further specifications, I will assume that, as in the *Ars magna*, "particular" is here referred to the methods employed. In fact, only the families of equations that can have $\Delta_3 < 0$ are considered, and moreover all the twenty-six provided examples have as a matter of fact $\Delta_3 < 0$. Then, I interpret the above propositions to be "particular" in the sense that in concrete terms they only work if one can find such f, g, that is, a solution of the considered equation. In this sense, the term "particular" applies also to (AMA XXXVII.i), which teaches how to find a second solution of $x^3 + a_0 = a_2x^2$, provided that one already found a first one.

3.2.7. Summing up. All along this section, we have remarked that there is a path linking the *Ars magna*, the *Practica arithmeticæ*, and the *Ars magna arithmeticæ* and that the direction is from the *Practica arithmeticæ* to the *Ars magna*, passing trough the *Ars magna arithmeticæ*. Comparing the *Practica arithmeticæ* to the *Ars magna arithmeticæ*, we find in the latter more and by far more developed topics on cubic equations. It is passing from the *Practica arithmeticæ* to the *Ars magna arithmeticæ* that the role of equations radically changes from being side effects of proportions to (at least, starting to be) a subject of enquiry by itself.[86]

Comparing instead the *Ars magna arithmeticæ* to the *Ars magna*, we observe that a remarkable part of the *Ars magna*'s arguments for equations are already present in the *Ars magna arithmeticæ*, and this is especially true concerning Cardano's heuristic conceptions and transformations of equations. In both works, Cardano deals with them virtually in the same way, whereas other topics undergo to different treatments. Namely, it is the case of the study of the shape of irrational solutions (of equations with rational coefficients), which is still characterised by the use of the Euclidean terminology for irrational numbers and which almost disappear in the *Ars magna*. We moreover remark that at this stage, Cardano has only a partial treatment for cubic equations available, as the following table shows.

[86]See [Gavagna 2012].

(AM XI)	$x^3 + a_1x = a_0$	(AMA XXVIII)
(AM XII)	$x^3 = a_1x + a_0$	(AMA XXX), (AMA XXXI) "particular"
(AM XIII)	$x^3 + a_0 = a_1x$	(AMA XXXIV), (AMA XXXV) "particular"
(AM XIV)	$x^3 = a_2x^2 + a_0$	(AMA XXIX)
(AM XV)	$x^3 + a_2x^2 = a_0$	(AMA XXXII), (AMA XXXIII) "particular"
(AM XVI)	$x^3 + a_0 = a_2x^2$	(AMA XXXVI), (AMA XXXVII) "particular"
(AM XVII)	$x^3 + a_2x^2 + a_1x = a_0$	(AMA XXXIX) "particular"
(AM XVIII)	$x^3 + a_1x = a_2x^2 + a_0$	(AMA XXXIX) "particular"

TABLE 3.5 – Correspondences in the treatment of cubic equations between the *Ars magna* and the *Ars magna arithmeticæ*.

For the most of the non-complete equations (with maybe the exception of $x^3+a_0 = a_2x^2$), Cardano has a cubic formula available, or a procedure that eventually can lead to it. No proof is given. For some non-complete equations (the ones that can have $\Delta_3 < 0$), Cardano also provides some "particular" rules. His treatment of complete cubic equations, instead, is sometimes quite upsetting and as a matter of fact he does not have any formula. He only provides two "particular" rules. As seen, Cardano's treatment of cubic equations is tightly linked to the opposition between "general" and "particular", which anyway does not seem to be part of a completely stable terminology. The only significant rest of this opposition will be found in *Ars magna*, Chapter XXV while speaking of the case $\Delta_3 < 0$. Moreover, the order in which the chapters are displayed in the *Ars magna arithmeticæ* highlights a dependency of the equations lacking in the first degree term from the depressed equations, which in the *Ars magna* gets lost.

No treatment for quartic equations is available at all.

We would like now to draw for the *Ars magna arithmeticæ* a diagram to summarise the links that bind the cubic equations, similar to the one for the *Ars magna* at page 152. I keep the same notations. Note that now the dotted lines correspond to some propositions in Chapter XXI. In particular, I have dashed the link between $x^3 = a_1x + a_0$ in (AMA XXX)) and $x^3 = a_2x^2 + a_0$ in (AMA XXIX.i)), since the substitution is not only stated, but also referred to those equations (among others) in (AMA XXI.9).

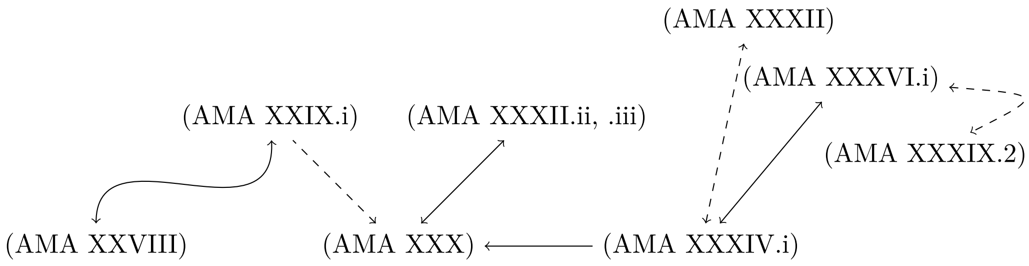


FIGURE 3.7 – Inter-dependencies between the cubic equations in the *Ars magna arithmeticæ*.

We observe that the *Ars magna*'s core of the bindings is already there. Anyway, we do not have the full system of the cubic formulae and – as said – the complete cubic equations are missing.

We remind that the only printed edition that we have is in the posthumous 1663 edition of the *Opera omnia*. These remarks on the links between the *Ars magna arithmeticæ* and the *Ars magna* could also shed light on why Cardano chase not to publish the *Ars magna arithmeticæ*. In fact, in [GAVAGNA 2012] it is argued that two of the factors that contributed are the problem posed by the *casus irreducibilis* and Cardano's willing to compose the *Opus arithmeticæ perfectum*. Gavagna affirms that the editorial plan of the *Opus arithmeticæ perfectum* raised the need to reallocate the contents of the *Ars magna arithmeticæ* in two blocks. The first of them should have been on irrational quantities, while the second should have been included in the algebra volume and make up the core of the subsequent *Ars magna*. This last part are the selected chapters of the *Ars magna arithmeticæ* that we have dealt with in detail in this section.

CHAPTER 4

Strategies to possibly overcome the *casus irriducibilis*: the *De Regula Aliza*

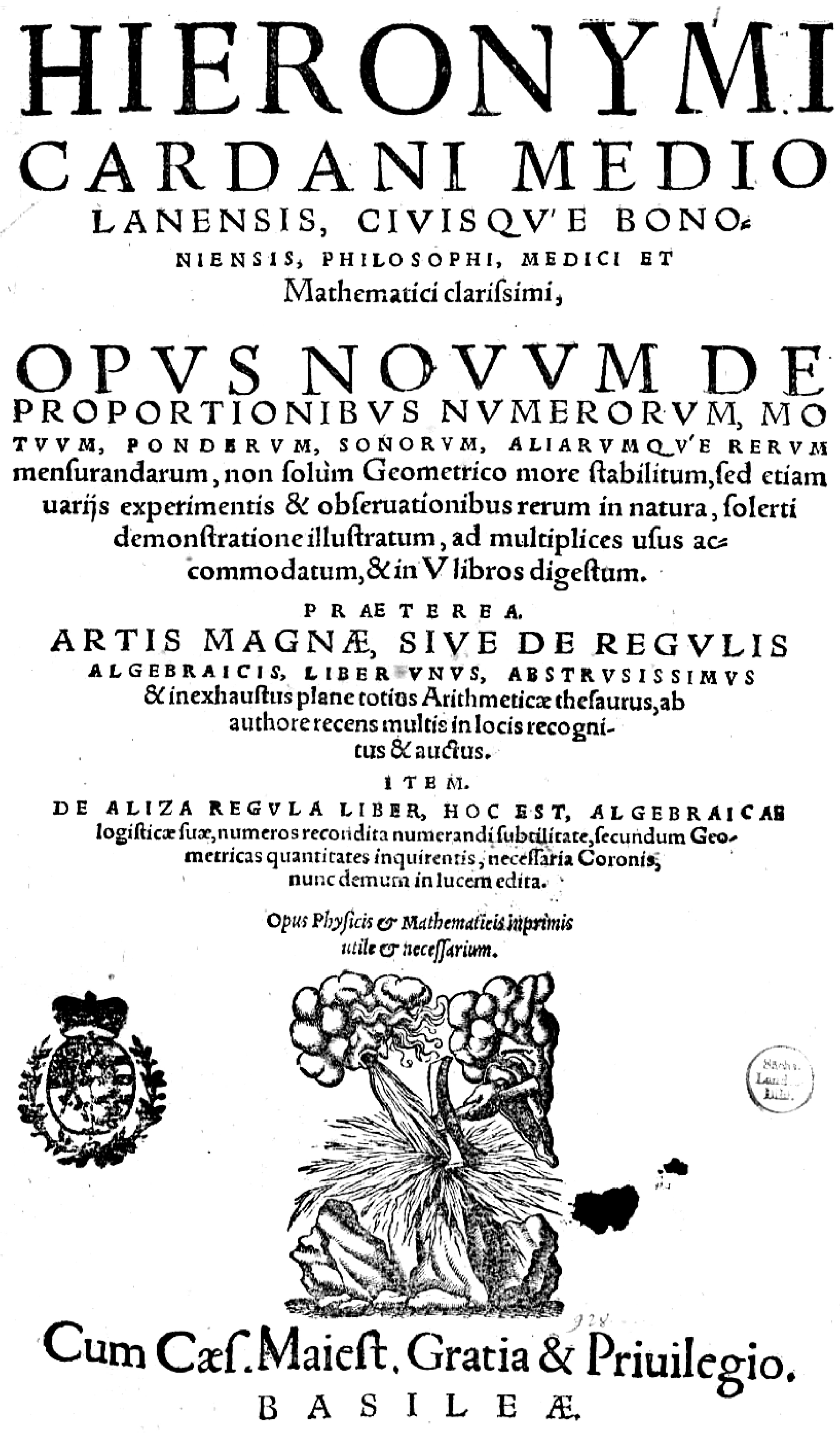
We are now ready to consider the *De regula aliza*.

4.1. Getting acquainted with the *De Regula Aliza*

There are two editions of this work. The first one is the *Hieronymi Cardani mediolanensis, civisque bononiensis, medici ac mathematici præclarissimi, de aliza regula, libellus, hoc est operi perfecti sui sive algebraicæ logisticæ, numeros recondita numerandi subtilitate, secundum geometricas quantitate inquirendis, necessaria coronis, nunc demum in lucem editæ.* It dates back to 1570 and was printed *in folio* in Basel by Officina Henricpetrina together with the *De proportionibus*[1] and with the second edition of the *Ars magna*. The second edition, which is shortly entitled *De regula aliza libellus*, is posthumous and is placed in the fourth volume of the 1663 edition of Cardano's *Opera omnia*, printed in Lyon by Ioannis Antonii Huguetan and Marci Antonii Ravaud. Some miscalculations and typos are corrected and sometimes the punctuation is improved, but as a whole it contains no major change compared to the first edition.

Several copies of the 1570 edition are available in Europe and America (Denmark, France, Germany, Italy, the Netherlands, Russia, Spain, Sweden, Switzerland, UK, but also Canada and USA). In Italy, there is a copy of the 1570 edition in some university libraries (Naples, Pisa, Rome), in national libraries (Florence, Naples, Rome, Turin), in public libraries (Mondovì, Lugo, Piacenza, Perugia, Ravenna), and in the diocesan library in Todi. In France, there is a copy of the 1570 edition at the National Library in Paris.

[1]The *De proportionibus* is quite a big work, composed by 233 propositions over 271 *folia*. As the title suggests, the proportions are the common thread of the book. Apart from the general rules of calculation with proportions, we do not only find proportions applied to mathematics (geometry and arithmetic) or physics (statics and dynamics), but also (even though to a lesser extent) to navigation, medicine, geography, mechanics, optics, gnomonics, and astronomy.



HIERONYMI
CARDANI MEDIO
LANENSIS, CIVISQV'E BONO-
NIENSIS, PHILOSOPHI, MEDICI ET
Mathematici clariſsimi,

OPVS NOVVM DE
PROPORTIONIBVS NVMERORVM, MO
TVVM, PONDERVM, SONORVM, ALIARVMQV'E RERVM
menſurandarum, non ſolùm Geometrico more ſtabilitum, ſed etiam
uarijs experimentis & obſeruationibus rerum in natura, ſolerti
demonſtratione illuſtratum, ad multiplices uſus ac-
commodatum, & in V libros digeſtum.

PRAETEREA.

ARTIS MAGNÆ, SIVE DE REGVLIS
ALGEBRAICIS, LIBER VNVS, ABSTRVSISSIMVS
& inexhauſtus plane totius Arithmeticæ theſaurus, ab
authore recens multis in locis recogni-
tus & auctus.

ITEM.

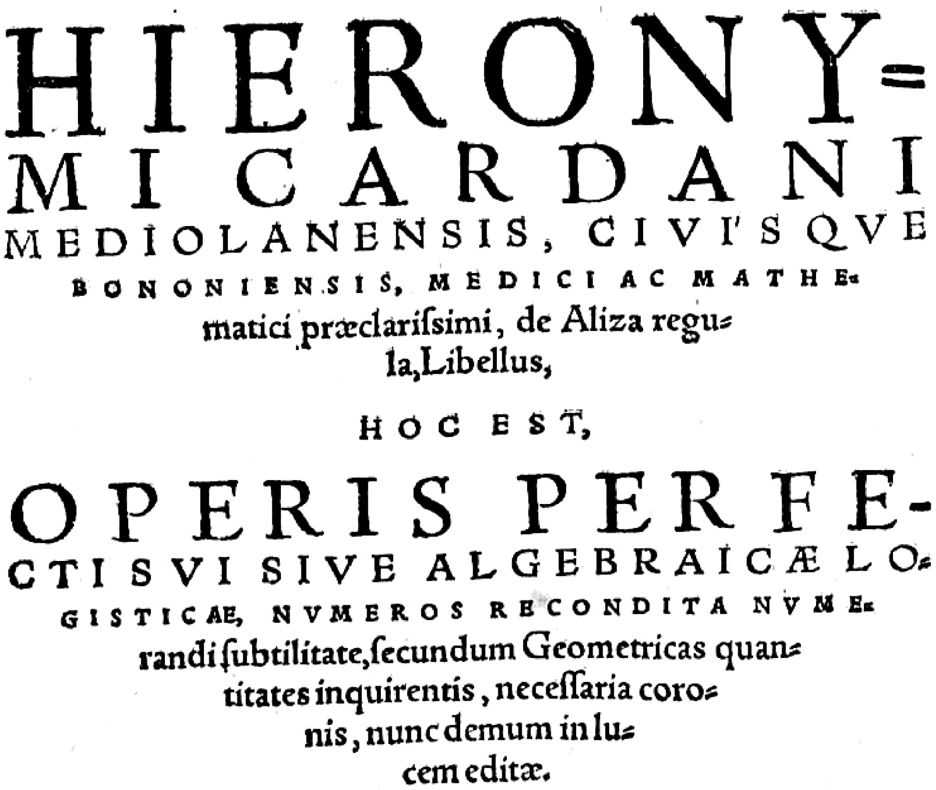
DE ALIZA REGVLA LIBER, HOC EST, ALGEBRAICAE
logiſticæ ſuæ, numeros recondita numerandi ſubtilitate, ſecundum Geo-
metricas quantitates inquirentis, neceſſaria Coronis,
nunc demum in lucem edita.

Opus Phyſicis & Mathematicis inprimis
utile & neceſſarium.

Cum Cæſ. Maieſt. Gratia & Priuilegio.

BASILEÆ.

FIGURE 4.1 – General title page of the joined edition of the *De proportionibus*, *Ars magna*, and *De regula aliza* (1570).



HIERONY=
MI CARDANI
MEDIOLANENSIS, CIVIS QVE
BONONIENSIS, MEDICI AC MATHE=
matici præclariſsimi, de Aliza regu=
la, Libellus,

HOC EST,

OPERIS PERFE-
CTI SVI SIVE ALGEBRAICÆ LO=
GISTICAE, NVMEROS RECONDITA NVME=
randi ſubtilitate, ſecundum Geometricas quan=
titates inquirentis, neceſſaria coro=
nis, nunc demum in lu=
cem editæ.

FIGURE 4.2 – Inner title page of the *De regula aliza* (1570).

As far as I know,[2] we do not have of any manuscript of the *Aliza*.

Even though Cardano claims that none of his works had been rewritten less than three times,[3] the *Aliza* seems not to enjoy this benefit. Probably, Cardano barely proofread the text or proofread it very quickly. In fact, apart from the widespread recurring typos, it happens that sometimes the figures are completely missing (as in Chapter II, see page 284) or that we find some oversights, like the unlikely multiplication of two areas (as in Chapter LX, see footnote 94 at page 295). As said, Cardano's Latin sentences suffer from a high complexity in their structure, where hypotaxis rules. But even more important, the structure of the book – as said in Section 1.2 at page 22 – is quite hard to detect, better not to say that there is no global structure at all. Therefore, I will not be able to give a picture of the *Aliza* as if it was one whole, uniform book. I will then regard it as a miscellany and search for the chapters, or for their parts, that show

[2] See [www.cardano.unimi.it last checked January 22, 2014].
[3] "*[N]ullus liber minus quam ter scriptum est*", see [CARDANO 1557, page 78].

a certain coherence. In this way – as we will see – we manage to identify one general leading idea, which concerns the algebraic shapes of the cubic equations and from which we can try to go back up to the *casus irreducibilis*.

The *Aliza*'s miscellany is composed by 60 highly patchy chapters, each one with its own title. They fill up 111 *folia* in the 1570 edition. Their length is very variable, from half a *folio* up to six or seven *folia*. We sometimes find an inner organisation, like numbered paragraphs, corollaries, *scholia*, or *paradoxa*. Each chapter virtually contains at least a diagram or a table where a calculation is displayed; there are 89 diagrams and 49 tables in all. In the margins and in the text, we find a good number of references, in the most of the cases referring to the *Aliza* itself or to the *Ars magna* and *De proportionibus*, which were published – as said – in the same joined edition. There are also references to the *Elements*.

The *Aliza* is written using the abbreviations and accents common to the Latin of the Renaissance (see at page 417). We should also consider the abbreviations that characterise Cardano's mathematics. I only recall that, acting as a man of his time, Cardano used to state the propositions in words, so that only a loose kind of formalism – or rather, some stenographic expressions – are involved. I refer to the preliminary Chapter Notations and more at page 5 for all the details. Without a fully developed formalism, it is not surprising that Cardano follows the habit of his time (which will last until the 17^{th} century and partially the 18^{th} century) of using a particular example – like an equation with numerical coefficients – to make a general discourse.[4] Limit cases as well are not usually considered.

As said, Cardano does not write a plain Latin. In general the flowing of the sentences is burdensome. To this, we must add the fact that speaking about mathematics in full (or abbreviated) words turns out to be to a certain amount and in the vast majority of the cases equivocal. In the main, it is much more easy to understand Cardano's arguments when at least a numerical example goes with them. Moreover, he usually creates his own mathematical terms, when needed. For instance, Cardano uses his own – as far as I know – terminology concerning the relations between the solids that compose the cube constructed on a divided

[4] As Karine Chemla says concerning another case, a statement can be general without being the most abstract, see [Chemla 2003].

segment. Let us consider a segment x divided in two parts such that $y + z = x$. Cardano calls y^2z and yz^2 (which are the orange and red parallelepipeds in the first cube of the below diagram) 'mutual [*mutua*]' parallelepipeds. Each one of the parallelepipeds yz^2 (in orange in the second cube of the below diagram) is "opposite [*adversum*, *alternum*, *altrinsecum*, *remotum*]" to the cube y^3 (in red in the second cube of the below diagram), while each one of the parallelepipeds y^2z (in orange in the third cube of the below diagram) is "connected [*coherens*, *proximum*]" to the cube y^3 (in red in the third cube in the diagram below).

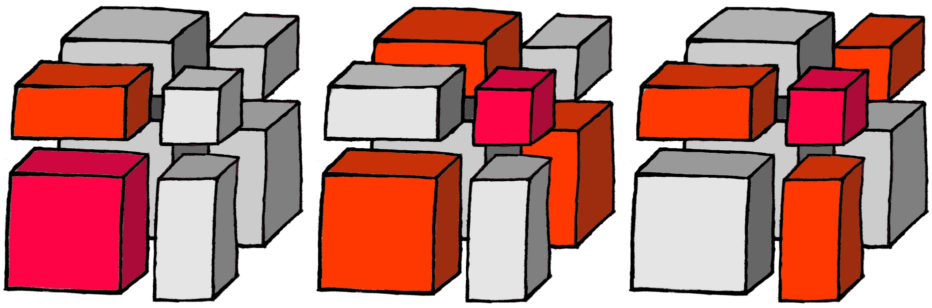


FIGURE 4.3 – "Mutual" parallelepipeds, parallelepipeds "opposite" to a cube, and parallelepipeds "connected" to a cube according to Cardano in the *Aliza*.

Furthermore, Cardano talks about the 'solid root [*radix solida*]',[5] the 'reduplicated ratio [*proportio reduplicata*]',[6] 'wild quantities [*quantites sylvestres*]',[7] or about the 'tetragonical side'.[8]

Trying to classify the *Aliza*'s chapters according to their contents is an extremely thorny issue. First of all – as said in Section 1.3 at page 26 – these chapters have probably been conceived over a period of about fifteen years, and it does

[5]Cardano never defines what a "solid root" is. Anyway, from the examples in Chapter L, we can infer that the "solid root" in the ratio of f to g is $f^3 + 2f^2g + fg^2$, see [CARDANO 1570a, Chapter L, paragraphs 2-3, page 93].

[6]Cardano says that "I call reduplicate ratio when the ratio of the parts is the square [of the ratio] of the remainders" or "*[v]oco proportionem reduplicatam cum fuerit proportio partium ut residuorum duplicata*", see [CARDANO 1570a, Chapter XXXII, pages 62-63].

[7]Cardano says that "a wild quantity, namely [a quantity] that is not in any kind of roots, not even composed by those, nor left by subtraction, as the quantity the square root of which multiplied by the remainder from 12 produces 2" or "*in quãtitate sylvestri, scilicet quæ non sit in aliquo genere radicum, nec composita ex illis, nec per detractionem relicta, velut quantitas cuius R ducta in residuum ad 12 producat 2*", see [CARDANO 1570a, Chapter X, page 20].

[8]Again, Cardano never explicitly defines what the "tetragonical side" is. Anyway, from the examples in Chapter IV, we can infer that the "tetragonical side" of a number is its fourth root, see [CARDANO 1570a, Chapter IV, page 8].

not seem that Cardano significantly proofread and uniformed them before the publication, in order to give some sort of global coherence to his treatise. There are chapters that are fully understandable, others that can be reconstructed to a greater or a lesser extent, and others that seem to be completely unintelligible. What immediately leaps out is that cubic equations are pervasive in the *Aliza*. We do not only have chapters that deal with $x^3 = a_1x + a_0$ (even if the number of pages devoted to this equation is strikingly huge compared to the others), but also the equations $x^3 + a_1x = a_0$ and $x^3 + a_0 = a_1x$ are taken into account as a proper subject of inquiry, and less frequently the equations $x^3 = a_2x^2 + a_0$ and $x^3 + a_0 = a_2x^2$. This does not anyway mean that we have any hope to organise the *Aliza* around these equations as it was the case for the *Ars magna arithmeticæ* or the *Ars magna*, since the concerned chapters are scattered all along the book. Generally speaking, we can attempt to identify some recurring patterns. As a stopgap measure, we can also use the titles of the chapters to get oriented – but sometimes they are simply too general or, worst, deceptive.

In this way, we can build a first – though rough – picture of the *Aliza*, which is resumed in the following table. Recall anyway that this is only a sketch of the contents of a book that is much more complex.

Pattern	Chapter(s)
A divided quantity and its powers	I, II, III, VII, VIII, IX, XV, XVIII, XIX, XX, XXI, XXV, XXX, XL, XLV, XLVI, XLVIII, L, LI, LIII, LIV, LV
The equation $x^3 = a_1x + a_0$	I, II, X, XI, XIII, XIV, XVI, XXIV, XXV, XL, XLIX, LIII, LVII, LVIII, LIX, LX
(Ir)rationality and (in)commensurability	I, IV, V, VIII, X, XI, XIII, XIV, XVI, XVII, XVIII, XLVII, LII, LVIII, LIX
The equation $x^3 + a_0 = a_2x^2$	III, XXIII, XXIV, XXVI, XXVIII
The equation $x^3 + a_0 = a_1x$	V, XIV, XXIII, XXVIII, XXIX, XLII
Sign rule	VI, XXII
Ratios	XV, XX, XXI, XXIV, XXXII, XXXIII, XXXIV, XXXV, XXXVI, XXXVII, XXXVIII, XXXIX, XLIV, LV, LVI
Solid roots	XVIII, XXIX, L
The equation $x^3 = a_2x^2 + a_0$	XXIII, XXIV, XXVI
Relations between the coefficients and/or the unknown	XXIII, XXVII, XLIII, XLVI, LX
Transformations	XXIV, XLIX
General and particular	XXVIII
The equation $x^3 + a_1x = a_0$	XXVIII, XXIX
Quartic equations	XXIV, XLI
The equation $x^3 + a_2x^2 = a_0$	XLIV

TABLE 4.1 – Recurring patterns in the *Aliza*.

Among the recurring patterns that I have managed to identify, the most common is what I have called 'a divided quantity and its powers'. This is a wide ranging label, under which I include a bunch of operations that one can perform after having divided a (known or unknown) quantity and raised it to the square or to the cube. For instance, Cardano explains how to compare the terms (or parts of them) of the cube or of the square of a binomial, teaches how to impose some conditions on these terms, or again couples and associate them to the coefficients of a certain equation (what I will call the 'splittings', obtained through the substitution $x = y + z$, see below Section 4.2 at page 235). It sometimes happens that the pattern 'a divided quantity and its powers' overlaps to the pattern 'the equation $x^3 = a_1x + a_0$'. This is for instance the case of the splittings. But it also happens that Cardano studies the equation $x^3 = a_1x + a_0$ without considering x as a divided quantity (or *vice versa* that the divided quantity x is the unknown of another kind of equation). In any case, most of the chapters that can be classified under these two labels will turn out to be very important in decoding the *Aliza*.

Another very recurring pattern is the one that I have classified under the label '(ir)rationality and (in)commensurability'. The study of particular kinds of irrational numbers, according to the terminology derived from Book X of the *Elements*, is of the utmost importance for Cardano's analysis of cubic equations with rational coefficients. In fact, when the equation falls into the *casus irreducibilis*, substituting $x = y + z$ (as in the splittings) leads to a system with some particular kind of irrational numbers. Eventually, one gets a cubic formula with imaginary numbers. Then, it is somehow natural to ask whether one can overcome the problem using another substitution, like $x = f(y, z)$. Thanks to Galois theory, we know that this is impossible, since the splitting field cannot be contained in $\mathbb{R}$. Anyway, Cardano discusses the (ir)rationality of the numbers into which he bumps (or their (in)commensurability with other numbers). He moreover suggests some calculation rules, especially concerning fractions with irrational denominators.

A third pattern, which however will not give rise to a thread significant in order to interpret the *Aliza*, concerns ratios. As we have already seen in the case of the *Practica arithmeticæ*, ratios were a common mathematical object at the time and equalities of ratios were used to express equations. Both can be occasionally find in the *Aliza*. Moreover, in the middle of the book, there is an (at least spatially) compact block of eight chapters that deal with the so-called 'reduplicated ratio'.

Then, there are a few chapters that stand quite aside from the others, like the ones on the sign rule or the ones that contain some hints on Cardano's conception of geometry. Both the sets of chapters that go under the labels 'sing rule' and 'geometry' will point at useful threads in the *Aliza*.

Finally, we can identify some patterns that glimpse here and there in the book. They are not recurring enough to set some real threads, but it is worthwhile to mention them in order to give a flavour of the variety of contents in the *Aliza*. They concern for instance the remaining non-complete cubic equations and two quartic equations, deal with the so-called 'solid roots', recall the discussions in the *Ars magna* on the comparison between "general" and "particular" and on the transformations of equations, or again discuss the relations between the coefficients and/or the unknown.

As said, some of these recurring patterns lead to identify some common threads o the *Aliza*. These will constitute the core of my analysis of the book. More precisely, I will complain with the following plan. In Section 4.2, I will present the main thread that can be identified in the *Aliza*: the splittings for the equation $x^3 = a_1x + a_0$. Cardano deals with them in the very first chapter, which is a real cornerstone for us. My hypothesis is that, through the splittings, Cardano is searching for some different cubic formulae that do not fall into the *casus irreducibilis*. In fact, it seems that he hopes that there exists a correspondence between the *casus irreducibilis* and the structure of the concerned equation, and thus he would like to discover a handling that leads to cubic formulae that only involve real numbers. But he did not managed to – and, as we know, he could not.[9] Still, Cardano wanted to publish the book and this could have also partially given rise to the obscurity of the *Aliza*.

I will thus organise a considerable number of chapters around these splittings and under different viewpoints. Basically, I will try to unravel where this idea come from and how one could employ the splittings in an unconventional, alleged

[9]See footnote 26 to Chapter 1 at page 24.

INDEX EORVM
QVÆ IN HOC LIBRO CONTINENTVR.

TABLE 4.2 – Table of contents of the *De regula aliza* (1570).

way. In Section 4.3.1, I will try to retrieve the origins of the substitution $x = y + z$ (which is fundamental in the splittings and in deriving the solving method for $x^3 = a_1x + a_0$) as a side-effect of the study of the shape of some irrational numbers that are solutions of cubic equations with rational coefficients. It turns out, in fact, that there are only two possible irrational shapes, and both of them are binomials. In Section 4.3.2, I will make a second hypothesis on the possible origin of the splittings. This time I will highlight the correlation with the *Ars magna*, and, in particular, with some rules from Chapter XXV that are intended to rephrase an equation in such a way that it is easier to guess one of its solutions. We recall that the link between the *Aliza* and *Ars magna*, Chapter XXV had already been suggested in *Ars magna*, Chapter XII. In Section 4.3.3, I will present another hypothesis on the origin of the splittings, based on my interpretation of the last chapter of the *Aliza*. It will appear that Cardano's researches could have been inspired by a certain kind of loosely interpreted geometry. In Section 4.3.4, I will deal with a last topic concerning the splittings, this time from a different viewpoint. In fact, the splittings can also be considered as a subject of inquiry by themselves. Cardano establishes some calculation rules on the splittings. These, in turn, could possibly also be useful in solving equations that fall into the *casus irreducibilis*, if one is willing to relax his requirements on the cubic formula.

Then, in Section 4.4, I will briefly complete the picture by further providing a few technical results (concerning irrational numbers and the echoes of (AM VIII.2)) that are exploited in the previously analysed chapters.

In Section 4.5, I will deal with a single chapter of the *Aliza*, where a different kind of geometry is exceptionally employed to deal with a problem that in the end involves a cubic equation. A comparison with Eutocius' and al-Khayyām's solutions to the very same problem will be useful in order to identify Cardano's position in the negative.

Finally, in Section 4.6, I will present another of the *Aliza*'s threads, which at that period is only a rough outline. As we will see, it could consist in preserving the cubic formula (together with the square roots of negative numbers that appear in the *casus irreducibilis*) and trying to develop an arithmetic of these nasty numbers. In this case, one needs first of all to deal with the sign of those numbers. Cardano makes a (failed) attempt in one controversial chapter of the *Aliza*.

As said, such an analysis cannot obviously account for the *Aliza* as a whole. It only concerns twenty-four chapters out of sixty (which are Chapters I, II, III, IV, V, VI, VII, X, XI, XII, XIII, XIV, XVI, XVII, XXII, XXIV, XXV, XXX, XL, LIII, LVII, LVIII, LIX, and LX), that is a little more than one third of the whole treatise. It happens that most of these chapters are placed at the two extremities of the book. Given the fact that the *Aliza* is a messy miscellany rather than a uniform work, I mistrust that it will be possible to finally settle each and single chapter in the same coherent logical structure.

The chapters that I will not consider in the following can be divided in two families. Those of one of them are negligible. Those of the other obscure. The chapters of the first family are those that explain some technical points, either directly concerning equations or not. More precisely, they are Chapters VIII, IX, XV, XXVII, and XLI. The second family gathers together the 31 remaining chapters. In turn, we can further split this family into the chapters that are completely obscure (like Chapters XVIII, XIX, XXI, XXVI, XXIX, XXXI, XXXII, XXXIV, XXXVI, XXXVII, XXXVIII, XXXIX, XLII, LIV, LVI, and the first part of Chapter XL) and the chapters in which we understand the particular calculations, but not their general aim (like Chapters XX, XXIII, XXVIII, XXXIII, XXXV, XLIII, XLIV, XLV, XLVI, XLVII, XLVIII, XLIX, L, LI, LII, and LV). Unluckily, among the chapters that are completely obscure, we count for instance Chapter XXXI "On the general value of the cube equal to some things and a number, which is called solid, and on its operations [*De æstimatione generali cubi æqualis rebus et numero solida vocata, et operationibus eius*]". As the title itself shows, it sounds very interesting, but I could not work it out.

Finally, I would like to shortly come back on the issue of the *Aliza*'s date. As it appears from Section 1.3 at page 26, it is unlikely to manage to precisely say when the book had been composed. At best, and considering it as a whole, I have given an overall estimate that ranges around 1550. But we shall not forget that this book is a miscellany. Then, one may want to try to establish an inner relative chronology of its chapters, starting from the hints that can be found in the text. Again, this is an extremely unlikely issue, but a few points on this topic deserve to be clarified.

The *Aliza* contains many references to the *Ars magna* and *De proportionibus*, which sounds very reasonable since the three books were published together. Unluckily, these references are completely ineffective in establishing a relative chronology of the *Aliza*'s chapters. In fact, even thought the *Aliza* had probably been not carefully proofread, their labels could have been in any case added at a second stage in the passages where Cardano employs the corresponding results.

In a few cases, I believe nonetheless that it is possible to give an estimate. First, the opening paragraphs of Chapter I convey the strong feeling that they could have been a late (maybe the latest) addition. In fact, they really seem to have the role of a sort of short introduction to (a considerable part of) the book, as we will see in Section 4.2.

Second, there is a handful of chapters, namely Chapters V, X, XIII, XVI, and especially Chapter XI, where Cardano studies the shapes for the irrational solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational. As we will see in Section 4.3.1 at page 256, the fact that he counts out some shapes that actually are possible shapes makes us suppose that he did not still have the cubic formula available. Third, as we will see at page 274, it seems that in Chapter XXIV Cardano does not have available Ferrari's method for quartic equations. Then, this chapter should have been composed before 1540 (see Appendix A.1 at page 352). The same also holds for Chapter XLI, that I will not mention in the following.

4.2. The method of the splittings in *Aliza*, Chapter I

As we have said, the equation $x^3 = a_1x + a_0$ is a central subject matter, and for sure the most recursive one, in the *Aliza*. This equation appears from the very beginning of the book, since its **Chapter I** "On the supposed [things] and on the ways [*De suppositis ac modis*]" is entirely devoted to it. This chapter, and especially its start, stands out, since it is the only place where we can find a few general, introductory words in a programmatic spirit – a sort of foreword. Moreover, this chapter gives rise to a very common thread, which will be recurring in the rest of the book, so that I will organise a sizeable number of other chapters around what we will call the 'splittings' in Chapter I. Note that all these chapters, which share a same core, could maybe be the remainders of the original "*aliza* problem", of which Cardano spoke in the *Ars magna*, Chapter XII (see above, at page 27).

It is then fully worthwhile to have an accurate look at the very first paragraph of Chapter I.

> Since I have already demonstrated in the *Ars magna* that all the chapters are transformed, provided that the two principal [chapters] had been discovered [to be] general and not by transformation, it is plain that, having discovered another general chapter in addition to the chapter of the cube and some things equal to a number and [the chapter] of the cube equal to some squares and a number, which is deduced by transformation from the preceding [one], even if [the other general chapter] was general, all the chapters either of three or of four terms would not only be known in general, but also demonstrated, provided that this very same [chapter] is discovered by a demonstration.[10]

As we have said, this is the only place in the whole *Aliza* where a sort of declaration of intent is made. Here, Cardano is explaining an overall strategy to deal with the *casus irreducibilis.* As a matter of fact, however, it is not shining as bright as one may hope for. What does Cardano want to mean here? He is firstly telling us a story that we know very well, also kindly giving the reference. As we have seen in Section 2.5 at page 151, the *Ars magna* shows that the solving methods of all the cubic equations are linked one to the other, and that all of them stand, in particular, in a special relation either with the solving method of the equation $x^3 + a_1x = a_0$ or with the one of the equation $x^3 = a_1x + a_0$. More precisely, all the proofs (except those for $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$) employ some kind of substitution to drive each equation back to the solving methods either of $x^3 + a_1x = a_0$ or of $x^3 = a_1x + a_0$. We have also remarked in Sections 2.2.1 and 2.2.2 that both the proofs for $x^3 + a_1x = a_0$ and for $x^3 = a_1x + a_0$ follow

[10]"*Cum iam in Arte magna demonstraverimus omnia capitula converti, modo duo principalia nec iam ex conversione inventa generalia fuerint, manifestum est, invento alio capitulo generali, præter capitula cubi et rerum æqualium numero et cubi æqualis quadratis et numero, quod ex priore per conversionem deducitur, et si generale sit, omnia capita seu ex tribus seu ex quatuor nominibus generaliter non solum cognita esse, sed et demonstrata, modo hoc ipsum demonstratione inventum sit*", see [Cardano 1570a, Chapter I, page 1].

a very similar pattern, though never making reference one to the other. Then, it is not surprising that here Cardano maintains that all the "chapters" or the family of equations[11] were transformed (by substitution) into two principal ones (which are $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$), which in turn were generally solved without substitutions. But we have also seen that, since the solving methods of all the equations are linked one to the other in such a way, if $x^3 = a_1x + a_0$ falls into the *casus irreducibilis*, then also all the equations the solving methods of which appeal to $x^3 = a_1x + a_0$ fall into it. In this way, the problem entailed by the *casus irreducibilis* spreads.

Cardano's plan to face the question is the following. He would like to find a certain family of equations such that it is at the same time "general" (but different from the family of equations of $x^3 + a_1x = a_0$ and $x^3 = a_2x^2 + a_0$) and solved by a method proved by a demonstration. Then, all the families of equations will also be solved. To get a full understanding of this plan, it is now essential to give an interpretation of the adjective "general". It is one of Cardano's beloved expressions – as we have already had the opportunity to verify (see for instance Sections 2.1.4, 3.2.4, and 3.2.6). We saw that Cardano correlates this term either to the shape of a solution of a cubic equation, or to the rule that gives a solution, or finally to the solving method of an equation. In the opening passage of the *Aliza* "general" is referred to a family of equations, and I am likely to make the comparison with the meaning of "general" referred to methods (at page 85). In fact, a family of equation is "general" (or can be generally solved) if its solving method is general, that is, if we can solve every given equation that belongs to that family by it.[12] In our particular case, this means that a family of equations is "general" if the concerned cubic formula (or the procedure from which the formula is derived) has no conditions. Under this viewpoint, the families of $x^3 + a_1x = a_0$ and of $x^3 = a_2x^2 + a_0$ are "general", since any arbitrary numerical equation belonging to those families can be solved through their cubic formulae (since

[11] See above, footnote 3 at page 170.

[12] Anyway, we have seen as well that Cardano's terminology is not always completely stable. In fact, in the *Ars magna*, the Chapter XII on $x^3 = a_1x + a_0$ is called "general", but it actually undergoes to a condition – this is the nitty-gritty of the issue! – and, in the *Ars magna arithmeticæ*, the same happens for Chapters XXX and XXXII respectively on $x^3 = a_1x + a_0$ and $x^3 + a_2x^2 = a_0$ (see here, at pages 108 and 199).

$\Delta_3 \geq 0$ and one can always get a real solution without running into imaginary numbers). By the way, note that there exist also some substitutions (see here, Propositions (AM VII.2-3) at page 72, (AM VII.6-7) at page 74, and (AMA XXI.1) at page 182) that link $x^3 = a_2x^2 + a_0$ to $x^3 + a_1x = a_0$ and *vice versa*, so that Cardano can say that the solving method for one equation is deduced from the other, even though both equations are "general". But not all the solving methods of the other cubic equations can be reduced to one of those two without loss of generality. In fact, there are some solving methods, more precisely those that resort to $x^3 = a_1x + a_0$, that require to previously know how to solve this equation. But the solving method that gives the cubic formula of the former equation stands independently from all the others, which means that it cannot be derived neither from that of $x^3 + a_1x = a_0$ nor from that of $x^3 = a_2x^2 + a_0$. These one-way connections can be grasped looking at the diagram at page 152. I claim that the "another chapter" that Cardano is referring to in the quoted passage is the one of $x^3 = a_1x + a_0$. Cardano's plan is then to find a family of equations such that it is "general", that is, to find a solving method for $x^3 = a_1x + a_0$ that does not have any condition. This means that he has to face the fact that sometimes square roots of negative numbers appear in the cubic formulae. To give consistency to my claim we need to address the rest of Chapter I.

It is indeed very reasonable to maintain that Cardano wants to focus on $x^3 = a_1x + a_0$. One the one hand, the problem of the *casus irreducibilis* arises at first in this equation so that, if one manages to impose no conditions on its formula, the problem will automatically be fixed also for the other equations. On the other hand, this equation is one of the most studied in the *Aliza*, and especially in Chapter I. In the remaining part of the opening passage, Cardano says that the solving methods for the cubic equations with three terms are easier to be discovered than the ones for the complete cubic equations. Moreover, among the solving methods for the cubic equations with three terms, that for $x^3 = a_1x + a_0$ is the easiest one – Cardano argues – for three reasons. First, the majority of this solving method is had by the rule in (AM XI). Second, it is discovered by the very same reasoning. Third, certain other solving methods are obtained starting from it by transformation.[13] But then, why the solving method for $x^3 = a_1x + a_0$ should be easier than the one for $x^3 + a_1x = a_0$? In fact, the above three reasons do not especially characterise $x^3 = a_1x + a_0$, since they also apply to $x^3 + a_1x = a_0$. In my opinion, Cardano is rather saying that the solving method for $x^3 = a_1x + a_0$ is the easiest one among those for the cubic equations with three terms that can have $\Delta_3 < 0$. In fact, only the equations that can have $\Delta_3 < 0$ can be reduced by transformation to the solving method for $x^3 = a_1x + a_0$.

In the second paragraph, Cardano shows what substitution he is going to use, namely $x = y + z$, and why he needs a substitution of this kind. I intend to deal with this topic in the next section, since there is a handful of chapters that can be grouped around it.

Cardano introduces then what Pietro Cossali calls '*spezzamenti*', and that we are going to call 'splittings' according to his semantic clue. Cardano briefly recalls the formula for the cube of a binomial, saying that he had already dealt with it (though he gives no precise reference; possibly he implicitly refers to (AM VI.6), see above at page 100). Even if it he does not make it explicit, it is clear that Cardano applies the substitution $x = y + z$ (where y, z are unknown to be further determined) to $x^3 = a_1x + a_0$, and uses the formula for the cube of a binomial to develop the calculations. In this way, he gets

$$y^3 + 3y^2z + 3y\,z^2 + z^3 = a_1(y + z) + a_0.$$

Afterwards, his strategy is to arrange the left side $y^3 + 3y^2z + 3y\,z^2 + z^3$ in the sum of two parts, both of which are going to be matched at first to $a_1(y + z)$ or to a_0 in the right side, and then (if possible) directly to one of the given coefficients a_1, a_0. Cardano states that this can be done in "in seven easiest ways [*septem modis facilioribus*]" – what we call the 'splittings', since the left side of the equality is split in two parts.

[13]"But among these [chapters] that consist of three terms the easiest one is the chapter of the cube equal to some things and a number, the greatest part of which is already obtained by the first rule [(AM XI)], and which stands according to the reasoning of the chapter already discovered [again (AM XI)], and also which we will show [to be] from that to the others not contrary to transformation" or "*[h]orum autem quæ tribus nominibus constant facillimum est capitulum cubi æqualis rebus et numero, cuius pars iam maxima ex prima regula habetur, et quod secundum rationem capituli iam inventi se habet; et etiam quod ex illo in alia non contra conversionem ostenderimus*", see [Cardano 1570a, Chapter I, page 1]. I interpret 'the first rule' as referring to the rule in the Chapter XI of the *Ars magna*.

A I.1. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = y^3 + z^3 \\ a_1x = 3y^2z + 3yz^2 \end{cases}, \quad \textit{that is} \quad \begin{cases} a_0 = y^3 + z^3 \\ a_1 = 3yz \end{cases},$$

holds, then the condition

$$\begin{cases} a_0 \geq \frac{1}{4}x^3 \\ [a_1x \leq \frac{3}{4}x^3] \end{cases}$$

follows.

Let us call $AB = y$ and $BC = z$. The first line in the condition is stated as follows.

> But since the cubes of AB and BC can never be smaller than one-fourth of the whole cube AD, and indeed this does not happen except when AC will be divided in equal [parts] by B, therefore being the number smaller than one-fourth of the whole cube AD, it will not be able to be equal to the cubes of AB, BC.[14]

Cardano says that, if $x = y + z$, then $y^3 + z^3 \geq \frac{1}{4}x^3$, and justifies it by "the 9th [proposition] of the second [book] of Euclid and by some Dialectic rules [*[p]er 9 secundi Elementorum et regula Dialec.*]". Note that both 1570 and 1663 editions have "*regula*". Since the preposition "*per*" takes the accusative case, we would have expected to find either '*regulam*' or '*regulas*', which is not the case. In these uncertain cases I prefer to leave the interpretation as loose as possible and I have then chosen to translate the term in plural.[15] This mention of the "Dialectic

[14] "*Sed quoniam cubi ab et bc nunque possunt esse minores quarta parte totius cubi ad, et hoc etiam non contingit nisi cum fuerit ac divisa per æqualia in b. Cum igitur numerus fuerit minor quadrante cubi totius ad, non poterit æquari cubis ab, bc*", see [Cardano 1570a, Chapter I, page 2].

[15] Note that Pietro Cossali interprets instead the term '*regula*' to be a singular. He affirms then that the "Dialectic rule" reminds of adding in an equality the same quantity on both sides of the equal, leaving thus the equality unchanged, but no real evidence supports his view. Cossali argues as follows: "*[q]uanto alla Regola Dialettica, si tratta della nota proprietà delle equazioni secondo la quale è possibile aggiungere o sottrarre ad ambo i membri dell'equazione una stessa espressione, proprietà che consente di spostare un termine da un membro all'altro dell'equazione*", see [Cossali 1966, Chapter I, paragraph 3, footnote 20, page 30]).

rules" seems to be a blurry reference to some classical logical rules. In fact, Cardano also wrote a *Dialectica*, printed in 1566 in Basel (in the second volume of [Cardano 1566]), which indeed contains some logical rules (and by the way also some mathematical examples to expound them). Anyway, since *Elements* II.9 implies[16] that $y^2 + z^2 \geq \frac{1}{2}x^2$, we are searching for one or more rules that enable to pass from the inequality with the square to the inequality with the cube, but I could retrieve none in the *Dialectica*. It is nevertheless not hard to derive[17] the searched inequality from *Elements* II.9. Note that, in order to verify this inequality, we must assume both y and z (real) positive.[18]

Then, once obtained $y^3 + z^3 \geq \frac{1}{4}(y+z)^3$, or $a_0 \geq \frac{1}{4}x^3$ because of the splitting (A I.1), it is immediate to get by the formula for the cube of a binomial that $3y^2z + 3yz^2 \leq \frac{3}{4}(y+z)^3$, or $a_1x \leq \frac{3}{4}x^3$.

Cardano says that, since this splitting has a condition, the family of equations is not "general" – and this agrees with the above interpretation of "general". Moreover (but not surprisingly), it turns out that this condition implies a condition

[16]*Elements* II.9: "If a straight line be cut into equal and unequal segments, the square on the unequal segments of the whole are double of the square on the half and on the square on the straight line between the points of section", see [Heath 1956a, page 392]. In our case, this means that $y^2 + z^2 = 2\left(\left(\frac{y+z}{2}\right)^2 + \left(\frac{y+z}{2} - z\right)^2\right)$.

[17]Let us assume that $y^2 + z^2 \geq \frac{1}{2}(y+z)^2$. We write

$$\begin{aligned}\frac{1}{4}(y+z)^3 = \frac{1}{2}(y+z)^2\,\frac{1}{2}(y+z) &\leq (y^2+z^2)\,\frac{1}{2}(y+z)\\ &= \frac{1}{2}(y^3 + y^2z + yz^2 + z^3)\\ &\leq y^3 + z^3.\end{aligned}$$

Cossali also gives two proofs of this, the second of which is an anachronistic attempt to reconstruct Cardano's reasoning (see [Cossali 1966, Chapter I, paragraph 3, pages 28-31] and [Cossali 1799a, Chapter VIII, paragraph 5, page 439]).

In modern terms, the searched inequality is a generalised mean inequality. We take an integer $n \geq 0$, and some real $p > 0$ and $x_1, \ldots, x_n \geq 0$. Consider the following generalisation of the arithmetic, geometric, and harmonic means

$$M_p(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}},$$

called 'generalised mean' or 'Hölder mean'. Then, for $p > q$, we have the generalised mean inequality $M_p \leq M_q$. Taking $n = 2$, $p = 1$, $q = 3$, we have that $\frac{1}{2}x \leq \sqrt[3]{\frac{1}{2}(y^3+z^3)}$, hence the inequality that Cardano wants to prove.

[18]Let us suppose by the absurd that $y \geq 0$ and $z \leq 0$, for example $y = 1$ and $z = -2$. Then, $-7 < -\frac{1}{4}$, which contradicts $y^3 + z^3 \geq \frac{1}{4}x^3$.

on the sign of the discriminant. In fact, the system

$$\begin{cases} a_0 \geq \frac{1}{4}x^3 \\ a_1 x \leq \frac{3}{4}x^3 \end{cases}$$

entails that

$$\begin{cases} a_0 \geq \frac{1}{4}x^3 \\ a_1 \leq \frac{3}{4}x^2 \end{cases}$$

and in turn that

$$\begin{cases} \frac{1}{4}a_0^2 \geq \frac{1}{4}\left(\frac{1}{4}x^3\right)^2 \\ a_1^3 \leq \left(\frac{3}{4}x^2\right)^3 \end{cases},$$

that is $\frac{1}{27}a_1^3 \leq \frac{1}{4^3}x^6 \leq \frac{1}{4}a_0^2$. Then, $\frac{1}{4}a_0^2 - \frac{1}{27}a_1^3 \geq 0$, or $\Delta_3 \geq 0$.

We finally observe that, if one knows the splitting (A I.1) for $x^3 = a_1 x + a_0$, he should also be able to derive its cubic formula, since there are no major conceptual gaps to be filled: a calculation is enough. We write the splitting in the following way

$$\begin{cases} a_0 = y^3 + z^3 \\ \frac{a_1}{3} = yz \end{cases}.$$

We take $y^3 = Y$ and $z^3 = Z$. Then, considering the real roots, we have

$$\begin{cases} a_0 = Y + Z \\ \frac{a_1}{3} = \sqrt[3]{Y}\sqrt[3]{Z} \end{cases},$$

that is

$$\begin{cases} a_0 = Y + Z \\ \frac{a_1^3}{27} = YZ \end{cases}.$$

Solving this system is equivalent to solve the quadratic equation $Y^2 + \frac{a_1^3}{27} = a_0 Y$. Its solutions are $Y_1, Y_2 = \frac{a_0}{2} \pm \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}$ and, because of the symmetry of the system, we have that $Y_1 = Z_2$ and $Y_2 = Z_1$. We recover then the cubic formula

$$x = \sqrt[3]{Y_1} + \sqrt[3]{Y_2} = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}}$$

(where, as said, I have only considered the real roots of $\sqrt[3]{Y_1}$, $\sqrt[3]{Y_2}$). We recall that this is the same method used by Cardano in *Ars magna*, Chapter XII (see here Section 2.2.2, page 111) to derive the cubic formula. It seems extremely unlikely to me that Cardano was not aware of this link with the cubic formula. Cossali maintains, very reasonably, that this splitting directly comes from Tartaglia's poem (see Appendix A.2 at page 356) and that, once discovered it, Cardano looked for other similar splittings.[19]

Given this first splitting, the one that is related to it in the simplest way is the following.

A I.2. *Consider* $x^3 = a_1x + a_0$. *Write* $x = y + z$ *with* y, z *two [real, positive] numbers.*
If y, z *can be chosen such that the splitting*

$$\begin{cases} a_0 = 3y^2z + 3yz^2 \\ a_1x = y^3 + z^3 \end{cases}, \quad \textit{that is} \begin{cases} a_0 = 3y^2z + 3y\,z^2 \\ a_1 = y^2 - yz + z^2 \end{cases},$$

holds, then the condition

$$\begin{cases} [a_0 \leq \frac{3}{4}x^3] \\ a_1x \geq \frac{1}{4}x^3 \end{cases}$$

follows.

This splitting is completely parallel to (A I.1), since the arrangement of the terms coming from the substitution $x = y + z$ in the equation $x^3 = a_1x + a_0$ is the same. The only difference is that the way in which the coefficients are assigned to the two arrangements of terms is swapped: $3y^2z + 3y\,z^2$ is now given to a_0 and $y^3 + z^3$ to a_1x. Also the condition is similarly stated,[20] except that it

[19]"*Giusta il metodo di Tartaglia questa equazione [*$y^3 + 3y^2z + 3y\,z^2 + z^3 - p(y + z) - q = 0$*] si spezza nelle due* $y^3 + z^3 = q$, $3y^2z + 3y\,z^2 = p(y + z)$*. È egli di natura sua generale questo spezzamento? È egli l'unico? E qui stendendo Cadano lo sguardo su le varie combinazioni de' termini, moltissimi gli si presentarono alla mente i supposti possibili a farsi*", see [COSSALI 1966, Chapter I, paragraph 3, page 27-28].

[20]"And because the parallelepipeds cannot be bigger than three-fourths of the whole cube, because the cubes taken together cannot be smaller than one-fourth, and therefore this chapter cannot be general, since, when the number will be bigger than three-fourths of the whole cube, it cannot be assigned to the parallelepipeds" or "*[e]t quia parallelipeda non possunt esse maiora dodrante totius cubi, quia cubi non possunt pariter accepti esse minores quadrante, ideo nec hoc capitulum potest esse generale, quoniam cum numerus fuerit maior dodrante totius cubi, non poterit tribui parallelipedis*", see [CARDANO 1570a, Chapter I, page 2].

cannot be translated in a condition on the discriminant (since the discriminant is defined in connection with the cubic formula that derives from (A I.1)). Note that Cardano uses *De proportionibus*, Proposition 146,[21] to state[22] the proportion $y^3 + z^3 : 3y^2z + 3yz^2 = y^2 + z^2 - yz : 3yz$, which in fact allows[23] to rewrite $a_1x = y^3 + z^3$ as $a_1 = y^2 - yz + z^2$.

Cardano then observes that neither (A I.1) nor (A I.2) are "general" because of their conditions, but that, if they are considered together, they are so, since their conditions, taken together, cover all the possible values of a_0, a_1 (see below, figures 4.4 and 4.5).

A I.3. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = 2y^2z + 2yz^2 \\ a_1x = y^3 + y^2z + yz^2 + z^3 \end{cases}, \quad \textit{that is} \quad \begin{cases} a_0 = 2y^2z + 2yz^2 \\ a_1 = y^2 + z^2 \end{cases},$$

holds, then the condition

$$\left[\begin{cases} a_0 \leq \frac{1}{2}x^3 \\ a_1x \geq \frac{1}{2}x^3 \end{cases}\right]$$

follows.

Note that Cardano only states the proportion $y^3 + z^3 + y^2z + yz^2 : 2y^2z + 2yz^2 = y^2 + z^2 : 2yz$, from which it follows[24] that $a_1x = y^3 + y^2z + yz^2 + z^3$ can be

[21] *De proportionibus*, Proposition 146: "The body that is made by a divided line times the surface equal to the squares of both parts having subtracted the surface of one part times the other, is equal to the aggregate of the cubes of both parts" or "*[c]orpus quod fit ex linea divisa in superficiem æqualem quadratis ambaru partium detracta superficie unius partis in alteram, est æquale agggregato cuborum ambarum partium*", see [Cardano 1570c, page 140r]. This means that $y^3 + z^3 = (y + z)(y^2 + z^2 - yz)$.

[22] "Or one may see by [the things] demonstrated in the book on Proportions that the proportion of the aggregate of the cubes to the aggregate of the six parallelepipeds is as [the proportion] of the aggregate of the squares AB and BC of the parts, subtracted the product of AB times BC from three times the product or the surface of AB times BC, or three times the surface AE" or "*[l]icet autem videre ex demonstratis in Libro de proportionibus quod proportio aggregati cuborum ad aggregatum sex parallelipedum est, veluti aggregati quadratorum ab et bc partium detracto producto ab in bc ad triplum producti seu superficiei ab in bc, seu triplum superficiei ae*", [Cardano 1570a, Chapter I, page 2].

[23] In fact, the proportion says that $3yz(y^3 + z^3) = 3yz(y + z)(y^2 + z^2 - yz)$.

[24] In fact, the proportion says that $2yz(y^3 + z^3 + y^2z + yz^2) = 2yz(y + z)(y^2 + z^2)$.

rewritten as $a_1 = y^2 + z^2$. This time he does not state any condition (though he does not even say that the splitting is "general", as he will do whenever it is so). Nevertheless, this splitting has the above condition, and Cossali identifies it in the following way.[25] Recall that Cossali gives two proofs of the condition for (A I.1) (see above, footnote 17 at page 241). Unlike the second one, the first proof does not attempt for reconstructing Cardano's reasoning: still, it is even more general. There, in an intermediate step, Cossali shows that the maximum value of $y^2z + yz^2$ is $\frac{1}{4}x^3$ (and is attained when $y = z$).[26] This let us easily gather that $2y^2z + 2yz^2 \leq \frac{2}{4}x^3$, or $a_0 \leq \frac{1}{2}x^3$ because of the splitting (A I.3), and then by the formula for the cube of a binomial that $y^3 + z^3 + y^2z + yz^2 \geq \frac{2}{4}x^3$, or $a_1x \geq \frac{1}{2}x^3$.

A I.4. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = y^2z + yz^2 \\ a_1x = y^3 + 2y^2z + 2yz^2 + z^3 \end{cases}, \quad \left[\textit{that is } \begin{cases} a_0 = y^2z + yz^2 \\ a_1 = y^2 + yz + z^2 \end{cases}\right],$$

holds, then the condition

$$\begin{cases} a_0 \leq \frac{1}{4}x^3 \\ [a_1x \geq \frac{3}{4}x^3] \end{cases}$$

follows.

The condition is immediately justified by the above considerations.

Let us resume through a diagram the conditions given by these first four splittings.

---

[25]See [Cossali 1966, Chapter I, paragraph 4, pages 32-33] and [Cossali 1799a, Chapter IX, paragraph 2, pages 445-446].

[26]Since $x = y + z$, Cossali writes $y = \frac{1}{2}x + t$ and $z = \frac{1}{2}x - t$, with $t \geq 0$, and then he calculates $y^2z + yz^2 = yz(y + z)$. He gets to $y^2z + yz^2 = \frac{1}{4}x^3 - xt^2$, which means that $\frac{1}{4}x^3$ is the maximum value (attained at $t = 0$), since all the numbers involved are positive. See [Cossali 1966, Chapter I, paragraph 3, pages 28-29] and [Cossali 1799a, Chapter VIII, paragraph 5, page 339].
Otherwise, one can simply verify that $y^2z + yz^2$ (taking $y = x - z$) attains a global maximum by deriving (twice) with respect to z and then calculates it.
We remark that Cardano do refer to the maximum value of $y^2z + yz^2$ elsewhere. In fact, In *Aliza*, Chapter XXV we find to this end a reference to *De proportionibus*, Proposition 209. For more details, see below, footnote 90 at page 289.

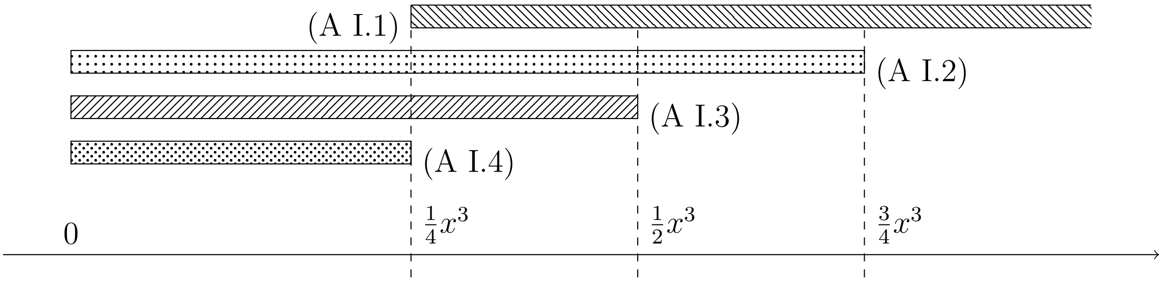


FIGURE 4.4 – Conditions on a_0 in the splittings (A I.1)-(A I.4).

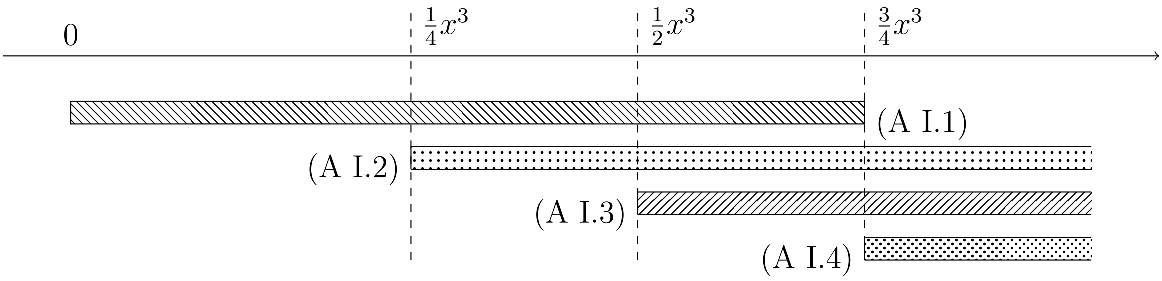


FIGURE 4.5 – Conditions on a_1x in the splittings (A I.1)-(A I.4).

Note that the condition in (A I.1) together with each one of the conditions in (A I.2)-(A I.4) cover all the possible values of a_0 and a_1. More precisely, the conditions in (A I.1) and (A I.4) are complementary, while the others overlap.

We moreover recall that there exist the connections between (A I.2) and (A XXV.5), between (A I.3) and (A XXV.3), and between (A I.4) and (A XXV.4), which we will analyse while dealing with Chapter II (see below, at page 284).

Let us now come back to the remaining splittings.

A I.5. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = y^3 \\ a_1x = 3y^2z + 3yz^2 + z^3 \end{cases}$$

*holds, then no [non trivial] condition follows. This splitting is "the worst of all [*deterius omnibus*]".*

Cardano's statement is in this case less clear. Obviously it is not true that – strictly speaking – no condition on the coefficients is entailed, since there is at least the trivial condition given by the splitting itself. It can be rewritten[27] in the following way

$$\begin{cases} a_0 \leq x^3 \\ a_1 x \leq x^3 \end{cases},$$

which is quite a useless condition, since it always holds when x is a solution of the equation. In fact, we have that $x^3 = a_1 x + a_0$ and both the coefficients a_1, a_0 and the solution x are positive, so that it is clear that a_0 and $a_1 x$ are smaller than, or equal to, x^3. Therefore, we account for Cardano's "no condition" as "no [non-trivial] condition".

Cardano then says that this splitting is the "the worst of all" since "it falls back on a chapter of four terms, thence from that to the first [*redit ad capitulum quatuor nominum inde ex eo ad primum*]". In fact, if we try to solve the system in y, z as we did to derive the cubic formula in (A I.1), we bump into the complete equation

$$z^3 + \sqrt[3]{a_0} z^2 + (\sqrt[3]{a_0^2} - a_1) z - a_1 \sqrt[3]{a_0} = 0$$

(which nevertheless can have $\Delta_3 < 0$). Deleting the term of degree two thanks to the substitution $z = w - \sqrt[3]{a_0}$ (which Cardano knows, see (AM IV.2) at page 83 and (AMA XXI.9) at page 185), we recover again $w^3 = a_1 w + a_0$, which was the equation from which we started.

A I.6. *Consider $x^3 = a_1 x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = y^3 + 3yz^2 \\ a_1 x = 3y^2 z + z^3 \end{cases}.$$

*holds, then no [non trivial] condition follows. This splitting is "misshaped [*difforme*]".*

[27]In fact, the first line of the condition is obtained by $y^3 \leq (y+z)^3 = x^3$, since all y, z are positive. The second line is obtained by studying $3y^2 z + 3yz^2 + z^3$, once substituted $y = x - z$, on the compact $[0, x]$ with respect to the variable z. Its maximum is x^3 when $y = 0$ and $z = x$.

As before, the splitting entails[28] the same trivial condition

$$\begin{cases} a_0 \leq x^3 \\ a_1 x \leq x^3 \end{cases}.$$

Cardano remarks that $(3y^2z + z^3) - (y^3 + 3yz^2) = (z - y)^3$ and provides an example that eventually leads to both equations $x^3 = \frac{158}{7}x + 185$ and $x^3 = \frac{185}{7}x + 158$, since the arrangement of the terms coming from the substitution $x = y + z$ in the equation $x^3 = a_1x + a_0$ is symmetric. It is exactly because of this symmetry that Cardano calls the splitting "misshaped [*difforme*]". In fact, he observes[29] that "similar [*simila*]" aggregates, like $y^3 + 3yz^2$ and $3y^2z + z^3$, are made equal to "dissimilar [*dissimila*]" ones, like a_0 and a_1x. This in particular means that a_0 and a_1x are to be considered "dissimilar" under a certain viewpoint. I believe that this viewpoint should be geometrical or, better, should concern a dimensional argument. More precisely, one would only say that a_0 and a_1x are "dissimilar" if x is interpreted as a segment and not as a number (being both a_0 and a_1 numbers).[30] On the other hand, Cardano calls the aggregates $y^3 + 3yz^2$ and $3y^2z + z^3$ "similar", and indeed they both represent some three-dimensional solids, namely a cube with the three "opposite" (see above at page 227 and below, Appendix B at page 359) parallelepipeds. We observe nevertheless that such a dimensional argument also holds for the other splittings. The difference with this

[28]In fact, the first line of the condition is obtained by studying $y^3 + 3yz^2$, once substituted $y = x - z$, on the compact $[0, x]$ with respect to the variable z. Its only critical point is the inflection point in $z = \frac{1}{2}x$. Then, it is trivially bounded by x^3. The same holds for the second line, since the splitting is symmetric in y, z.

[29]"Since what is made equal is the similar, namely the cubes with the three opposite parallelepipeds, to which the dissimilar is compared, in fact one aggregate is made equal to the number, the other [is made equal] to the things" or "*quoniam quod æquatur est simile scilicet cubi cum parallelipedis tribus adversis, cui æquatur dissimile, nam unum aggregatum æquatuor numero, aliud rebus*", see [Cardano 1570a, Chapter I, page 3].

[30]One could also argue that a_0 and a_1x are dissimilar, still a_0, a_1, x being all numbers. Indeed, this really happens in the *Aliza*, and more precisely in the second paragraph of the same Chapter I (see below, Section 4.3.1 at page 256). There, Cardano says that a_0 and a_1x are of a "different kind", since one is rational and the other is irrational.
Anyway, I do not believe that it is the case here. In fact, that second paragraph is also – in my opinion – a later addition that make reference to an issue on cubic equations with rational coefficients, which is expounded in a bunch of subsequent chapters. Then, I am likely to make a separation between the introductory opening of the chapter from the following part on the splittings.

case is that here the two aggregates y^3+3yz^2 and $3y^2z+z^3$ are symmetric. In my opinion, this remark by Cardano could testify to a trace of an early geometrical (dimensional) inspiration.

A I.7. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers.*
If y, z can be chosen such that the splitting

$$\begin{cases} a_0 = y^3 + 2y^2z + yz^2 \\ a_1x = y^2z + 2yz^2 + z^3 \end{cases}, \quad \left[\textit{that is } \begin{cases} a_0 = y^3 + 2y^2z + yz^2 \\ a_1 = yz + z^2 \end{cases} \right],$$

*holds, then no [non trivial] condition follows. This splitting is "misshaped [*difforme*]".*

As before, the splitting entails[31] the condition

$$\begin{cases} a_0 \leq x^3 \\ a_1x \leq x^3 \end{cases}.$$

Cardano remarks that it holds that $y^3 + 2y^2z + yz^2 : y^2z + 2yz^2 + z^3 = y : z$ and gives an example, which eventually leads to the equations $x^3 = 35x + 98$ and $x^3 = 14x + 245$. As before, due to the symmetry of the splitting, Cardano finds it "misshaped [*difforme*]".

According to Cardano's criteria (that is, always considering positive coefficients and writing no equality with zero), there are 62 possible splittings.[32] To count them all, we remark that splitting $y^3+3y^2z+3yz^3+z^3$ in two parts and assigning one to a_0 completely determines the choice for the other part (assigned to a_1x). We consider then all the possible combinations of the terms in $y^3+3y^2z+3yz^3+z^3$. We have two choices for each of y^3, z^3 (since we can have one or none of them) and four choices for each of y^2z, yz^2 (since we can have three, two, one, or none of them and we cannot distinguish them three by three). Moreover we have to count out the two limit cases, where there are all the terms or no term. This makes 62 possibilities.

[31] In fact, the first line of the condition is obtained by studying $y^3+2y^2z+yz^2$, once substituted $y = x - z$, on the compact $[0, x]$ with respect to the variable z. Then, it is trivially bounded by x^3. The same holds for the second line, since the splitting is symmetric in y, z.

[32] See also Cossali's account [Cossali 1966, Chapter I, paragraph 7, pages 47-49] and [Cossali 1799a, Chapter IX, paragraph 5, pages 452-454].

Cardano says that the remaining splittings are either "compositions [*compositiones*]", or "anomalous [*anomali*]", or "useless [*inutiles*]".

> The remaining [ways] are either compositions or anomalous, as if we had given the number to one parallelepiped, to three [parallelepipeds], to five [parallelepipeds], to two [parallelepipeds] not mutual, to four [parallelepipeds] two of which are not mutual, or to a cube and a parallelepiped, to two or to three [parallelepipeds] not of the same kind. The other [ways] are useless, as if we had given the number to the aggregate from both cubes and two or four parallelepipeds in whatever way. In fact, if it is not enough for the aggregate of the cubes, since the number is small, in what way is it enough for the same if the parallelepipeds are added?.[33]

He then provides some generic examples. Among the "compositions [*compositiones*]", Cardano lists

[33]"*Reliquæ autem compositiones, aut sunt anomale, velut si daremus numerum uni parallelipedo, vel tribus vel quinque vel duobus, non mutuis aut quatuor, ex quibus duo mutua non essent, aut uni cubo et uni parallelipedo, vel duobus vel tribus non eiusdem generis. Aliæ sunt inutiles, velut si daremus numerum aggregato ex ambobus cubis, et duobus parallelipedis aut quatuor quomodocumque, nam si numerus cum parvus sit, non sufficit aggregato cuborum, quomodo sufficit eidem si addantur parallelipeda*", see [Cardano 1570a, Chapter I, page 4].

$$\begin{cases} a_0 = y^2 z \\ a_1 x = y^3 + 2y^2 z + 3yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = 3y^2 z \\ a_1 x = y^3 + 3yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = 2y^2 z + yz^2 \\ a_1 x = y^3 + y^2 z + 2yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = 3y^2 z + 2yz^2 \\ a_1 x = y^3 + yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = 2y^2 z \\ a_1 x = y^3 + y^2 z + 3yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = 2y^2 z + 2yz^2 \\ a_1 x = y^3 + y^2 z + yz^2 + z^3 \end{cases},$$

$$\text{(A I.3)} \quad \begin{cases} a_0 = 3y^2 z + yz^2 \\ a_1 x = y^3 + 2yz^2 + z^3 \end{cases},$$

and the symmetrical splittings obtained by swapping y and z. Note that Cardano's account is not completely consistent, since he also classifies the splitting (A I.3) as a "composition". Among the "anomalous [*anomale*]" splittings, Cardano lists

$$\begin{cases} a_0 = y^3 + y^2 z \\ a_1 x = 2y^2 z + 3yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = y^3 + yz^2 \\ a_1 x = 3y^2 z + 2yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = y^3 + y^2 z + yz^2 \\ a_1 x = y^3 + 2y^2 z + 2yz^2 + z^3 \end{cases},$$

$$\begin{cases} a_0 = y^3 + 2y^2 z + yz^2 \\ a_1 x = y^2 z + 2yz^2 + z^3 \end{cases},$$

$$\text{(A I.7)} \begin{cases} a_0 = y^3 + y^2 z + 2yz^2 \\ a_1 x = 2y^2 z + yz^2 + z^3 \end{cases},$$

and their symmetrical splittings. Note again that Cardano's account is not consistent, since he classifies the splitting (A I.7) as "anomalous". Finally, among the "useless [*inutiles*]" splittings, Cardano lists

$$\text{(opposite of A I.2)} \begin{cases} a_0 = y^3 + y^2 z + yz^2 + z^3 \\ a_1 x = 2y^2 z + 2yz^2 \end{cases},$$

$$\begin{cases} a_0 = y^3 + 2y^2 z + z^3 \\ a_1 x = y^2 z + 3yz^2 \end{cases},$$

$$\text{(opposite of A I.7)} \begin{cases} a_0 = y^3 + 3y^2 z + yz^2 + z^3 \\ a_1 x = 2yz^2 \end{cases},$$

$$\text{(opposite of A I.4)} \begin{cases} a_0 = y^3 + 2y^2 z + 2yz^2 + z^3 \\ a_1 x = y^2 z + yz^2 \end{cases},$$

and the symmetrical splittings. We remark that the first and the fourth splittings are respectively the opposites[34] of (A I.2) and of (A I.4). Note moreover that the third of the above splittings is the opposite of (A I.7).

Counting the above splittings and their symmetrical ones, we have 28 splittings in all. In fact, we have 14 "compositions", 10 "anomalous" splittings, and 8 "useless" splittings, but the first and the second "useless" splitting are the symmetric respectively of the third and the seventh "composition", so that we have to count them out.

We recall that Cardano says that the splittings (A I.1)-(A I.7) are the "easiest" ones. I have not managed to find a pattern to justify this claim, not even in comparison with the above 28 splittings. I have neither detected the rules according to which Cardano lists the 28 above splittings. A reasonable idea could be to check if the classification depends on the conditions that those splittings entail, but it does not seem to be the case. In fact – as said, (A I.1)-(A I.4) have a non-trivial condition, while (A I.5)-(A I.7) have the trivial one. Considering the other splittings, all the "compositions" (except the last one, which is indeed (A I.3)), all the "anomalous" splittings, and the second and the third "useless" splittings have the trivial condition, while only the first and the fourth of the "useless" splittings have a non-trivial one (namely, the conditions opposite to the conditions in (A I.2) and (A I.4)). Moreover, I do not have any hint on the 27 remaining splittings on which Cardano keeps the lid.

I have claimed above that Cardano is looking for another cubic formula for the equation $x^3 = a_1x + a_0$, in which the square roots of negative numbers do not appear when all the solutions are real and distinct.[35] Let us consider again the seven "easiest" splittings. We have remarked that the splitting (A I.1) leads to the cubic formula that we know, since solving the system that characterises (A I.1) is equivalent to solve a cubic equation. We have also remarked that Cardano was almost for sure aware of this connection. It is indeed very reasonable that

[34] I say that two splittings are opposite if the arrangement of the terms coming from the substitution $x = y + z$ in the equation $x^3 = a_1x + a_0$ is the same and only their assignment to the coefficients is swapped. In this case, the lines in the condition are also swapped.

[35] But note that the splitting field that contains the imaginary unit $\imath$ is essentially unique by elementary Galois theory's reasons.

Cardano is looking for another cubic formula *via* one of the other splittings, since the device already worked in the first case. The method in (A I.1) has a straight connection with the solving method in *Ars magna*, Chapter XII, and therefore with Tartaglia's poem (see Appendix A.2 at page 356). The other splittings then can be seen as a sort of generalisation of it. Unluckily, Cardano's attempt was doomed to failure.[36]

We have already observed that, if we try to apply to (A I.5) the same method as in (A I.1), we fall back again on a cubic equation (namely, the same from which we started). And it is not better with the other splittings. In fact, let us consider the degree of the systems in the splittings, which is always 9. Sometimes, it happens that it can be lowered. This is the case for (A I.1)-(A I.4) and (A I.7), where the degree decreases to 6, so that also the equations associated to those systems have at most degree 6. It is not always an easy job to derive these equations, even when the degree lowers. In the context of the study of equations and of the deep structure of their coefficients during the 18$^{\text{th}}$ and 19$^{\text{th}}$ centuries, some elimination methods had been conceived. They enable one to derive the equation associated to a two-equations system (later extended to more than two) in one or more unknowns (usually two, but later extended to more than two) *via* the elimination of one term or of one of the unknowns.[37] The most influential characters were Newton, Euler, Cramer, Bézout, and Lagrange, but also Cossali got involved in the topic. What is interesting for us is that Cossali applies a previous study[38] of some elimination methods to the systems in the splittings in order to derive the associated equations. Cossali shows[39] that the systems in (A I.1)-(A I.4) are usually associated to an equation of degree 6. In particular, whereas the splitting (A I.4) leads to an equation lacking in the terms of degree five and three, and the splittings (A I.2) and (A I.3) lead to equations lacking in the term of degree five, the splitting (A I.1) leads to an equation in which only

[36]See footnote 26 to Chapter 1 at page 24.
[37]For a resume, see [STEDALL 2011, pages 131-145].
[38]See [COSSALI 1813].
[39]See [COSSALI 1966, Chapter I, paragraph 7, pages 49-57] and [COSSALI 1799a, Chapter IX, paragraph 5, pages 454-465].

the terms of degree six, three, and zero appear.[40] This particular feature of the associated equation is what opens the door to derive a cubic formula, since the equation can be solved as a quadratic one. The systems in the splittings (A I.5) and (A I.6) are instead associated to an equation of degree 9. Again, thanks to the particular structure of the equation associated to (A I.5) (only the terms of degree nine, six, three, and zero appear and the coefficients are appropriated), one is enabled to reduce the polynomial to the cube of a binomial, which is $(a_0 - y^3)^3$. Unluckily, this leads back to the starting point. Finally, the system in the splitting (A I.7) is associated to a complete cubic equation, which as such can have $\Delta_3 < 0$. Then, if Cardano's hope was to derive another cubic formula for $x^3 = a_1x + a_0$, the splittings (A I.2)-(A I.7) are useless (and indeed any other possible splitting will be useless as well[41]).

The splittings in the *Aliza* recall *Ars magna*, Chapter XXV (see here, Section 2.1.4 at page 85). In particular, we have a straight correspondence between (A I.2) and (AM XXV.5), between (A I.3) and (AM XXV.3), and between (A I.4) and (AM XXV.4) (see respectively the systems (AM XXV.5 bis) at page 88, (AM XXV.3 bis) at page 87, and (AM XXV.4 bis) at page 88). Considered the shift between 1545 and 1570 (and then 1663) editions in Cardano's remarks at the end of *Ars magna*, Chapter XII (see here, page 112), we expected this link. Indeed, we will see that Cardano explicitly mentions this link in the *Aliza* (see below, Section 4.3.2 at page 284). Nevertheless, the two methods are presented with different purposes in the two books. They both concern the problem entailed by the *casus irreducibilis*, but in the *Ars magna* – as we have already noticed at page 96 – the main gain was to help intuition to find a solution of the considered equation thanks to a rephrasing of the equation *via* a more friendly system. This enable one to lower the degree of the equation by polynomial division. In the

[40] As it often happens, Cossali's calculations contain little mistakes. He claims that the equation $-27y^6 + 27a_0y - a_1^3 = 0$ is associated to the system

$$\begin{cases} a_0 = y^3 + z^3 \\ a_1x = 3y^2z + 3yz^2 \end{cases}$$

in the splitting (A I.1). Actually, the equation is $-27y^6 + 27a_0y^3 - a_1^3 = 0$ (which can be solved as a quadratic equation). In fact, if we solve the quadratic equation in z in the second line of the system, we get $z_1 = -y$ and $z_2 = \frac{a_1}{3y}$. The first value leads to a contradiction, since from the substitution in the first line we have $-a_0 = 0$. The second value leads to the above equation.

[41] See here above, footnote 35

Aliza instead, Cardano wanted to fly higher.[42] The aim is no more to reach a case-by-case method, still subject to uncertainty, but an algorithmic one that hopefully works each time. This agrees with the fact that Cardano called "particular" the propositions in *Ars magna*, Chapter XXV, while in the *Aliza* he is looking for a "general chapter". There (see above, page 97), although the propositions made reference to an *a priori* "general" method, they could only be applied in some blessed cases, thus in practice one cannot solve every given problem by them. Here, Cardano is aspiring to a truly applicable "general" method, that is to an algorithm or a formula. It is hard to say whether the *Aliza* is a reworking of the *Ars magna* or *vice versa*. For sure, Cardano mentions the *Ars magna* in the *Aliza* and not the other way around – this is the only evidence that we have. I am likely to believe that Cardano firstly discovered the "particular" rules in *Ars magna*, Chapter XXV and then transformed some of them into the splittings. In fact, only a few rules of the *Ars magna* are compatible with the substitution $x = y + z$ in the *Aliza* and these could have been the core from which Cardano deduced the other splittings by analogy. Moreover, since no trace of similar propositions can be found either in the *Ars magna arithmeticæ* or in the *Practica arithmeticæ*, it makes sense to suppose that the particular rules in *Ars magna*, Chapter XXV were developed after the cubic formulae and the contact with Tartaglia.

4.3. Analysing the splittings

In the following sections I will deal with all the (understandable) chapters of the *Aliza* that can be grouped around the thread given by the splittings. In particular, I will use these connections in order to try to retrieve some possible origins for the splittings (including a geometrical one) and how they could have been unconventionally employed, if one is willing to relax the requirements on the cubic formula.

4.3.1. On the origin of the substitution $\mathbf{x = y + z}$: a hypothesis on the general shapes for irrational solutions. In the preceding section, I have suggested that the splittings can be seen as a sort of generalisation of the solving

[42] But note the little inconsistency in the *Aliza*, Chapter I when Cardano provides the following examples $x^3 = 29x + 140$, $x^3 = \frac{185}{7}x + 158$, $x^3 = \frac{158}{7}x + 185$, $x^3 = 35x + 98$, $x^3 = 14x + 245$ (which lead to equations with $\Delta_3 > 0$).

method in *Ars magna*, Chapter XII (or of Tartaglia's poem). The tricky part in deriving the splittings is to get the substitution $x = y + z$ and – as we will see – Cardano though hard and for a long time over it.

In Section 3.2.4 at page 185, we have expounded Cardano's researches in the *Ars magna arithmeticæ* about the algebraic shape that a "general" solution of a cubic equation with rational coefficients can have. A late cross reference to this approach moreover appears in *Ars magna*, Chapter IV, where nevertheless the coefficients are no more assumed to be rational (see above, Section 2.1.4 at page 81).

We remark that, in [Cossali 1966, Chapter II, paragraph 1, pages 59-67] and [Cossali 1799a, Chapter VII, paragraph 6, pages 399-405], Cossali strives for recapitulating all of Cardano's results concerning irrational solutions (in the *Ars magna arithmeticæ* and in the *Ars magna*, as well as in the *Aliza*). Unluckily, his account is far from being faithful to the spirit of Cardano's texts.

The topic of the algebraic shape of an irrational solution of an equation with rational coefficients surfaces in the *Aliza*. At the beginning of **Chapter I** (see above at page 239), even before mentioning the splittings, Cardano discusses the substitution $x = y + z$ for $x^3 = a_1x + a_0$. He shows that x consists at least of two incommensurable[43] parts, by an argument that appeals to "generality".

> Being therefore the cube equal to two quantities of a different kind (otherwise this would not be general, if it was extended to the numbers alone and to their parts), it is necessary both that it is unravelled in just two parts, one of which is a number and [is] equal to an assigned [one], the other [part] contains as many parts different in nature as there are in the things equal to these. Therefore it is necessary that the cube consists of at least two different parts, and therefore its side or the things, and indeed several quantities of different kinds cannot be made by a nature of a single kind through a multiplication however often repeated, as it is demonstrated by Euclid in the tenth Book (Proposition 20). Truly, if two quantities alien to the number

[43] For the definition of '(in)commensurable', see below, footnote 53 at page 265.

> were contained in the thing, it would be necessary that they were incommensurable one to the other, otherwise they would have been equivalent to one [quantity]. But in this way it is necessary that the cube, which contains three parts, a number and two irrationals, is made in order that it can be equal to some things and the number.[44]

Firstly, let us consider Cardano's remark in the parentheses. He states that, in the equation $x^3 = a_1x + a_0$, the terms a_1x and a_0 must be of a "different kind [*diversi generis*]", since he is aiming to a "general" method, that is, a method that work, for instance, also when x^3 is irrational. Right before, at the beginning of the second paragraph of Chapter I,[45] Cardano said that a_1 and a_0 are (rational) numbers. Then, his remark in the parentheses implies that x can be irrational, so that a_1x, and accordingly x^3, can be irrational too. In fact, let us suppose that a_1x and a_0 were of the same "kind". Then, a_1x will be rational, since a_0 is rational. But this cannot be, since $x^3 = a_1x + a_0$ and x^3 could be irrational. This way of reasoning recalls the study of the shape of irrational solutions that we have seen in the *Ars magna arithmeticæ* (see Section 3.2.4 at page 185). Having cleared this point, Cardano comes to the conclusion that x^3 must be composed (at least) of two parts – let us call them Y and Z, where

- Y must contain "as many parts different in nature [*totidem partes contineat natura varias*]" as there are in a_1x,

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[44]"*Cum ergo cubus æqualis sit duabus quantitatibus diversi generis (aliter non esset hoc generale, si ad solos numeros et eorum partes extenderetur), necesse est ut et ipse in duas tamtum partes resolvatur, quarum una numerus sit, et assignato æqualis: alia totidem partes contineat natura varias, quot in rebus continentur de eis æquales. Quo circa necesse est cubum saltem ex duabus partibus constare natura diversis, igitur et latus eius seu res, neque enim ab unius generis natura plures per multiplicationem quotiescunque repetitam plures quantitates diversorum generum fieri possunt, ut ab Euclide in decimo libro demonstratum est. Verum si in re contineantur duæ quantitates a numero alienæ, necesse est ut inter se sint incommensæ, aliter æquivalerent uni. At ex eiusmodi necesse est cubum fieri, qui tres partes contineat, numerum et duas rhete, ut rebus ac numero possit coæquari*", see [Cardano 1570a, page 1].

[45]"[I]n the chapter of the cube equal to some things and a number, two [numbers] are proposed in particular, the number of the equality and also the number of the things. But more generally let now the cube of a certain line or quantity to be equal to the proposed numbers, the simple [one] and [the one] of the things" or "*[i]n capitulo igitur cubi æqualis rebus et numero duo proponuntur speciatim, numerus æquationis, numerus etiam rerum. Generaliter autem, quod cubus nunc alicuius lineæ seu quantitatis, propositis numeris simplici et rerum æqualis est*", see [Cardano 1570a, Chapter I, page 1].

- Z must be rational and equal to the number a_0,
- ($Y + Z$ is not a rational number).

Thereby, Cardano says that also x must be composed of (at least) two parts – what we have called y and z. In fact, if x is composed only by one part, its cube x^3 is also composed only by one part, but this cannot be according to the above argument. Moreover, if x is the sum of two, or more than two, irrational (and no rational appears in the sum), then (at least two of) these irrational must be incommensurable between them. In fact, by the absurd, let us suppose for the sake of simplicity that $x = y + z$ with y, z irrational and commensurable. This means that $\frac{y}{z}$ is rational, that is $y = kz$ with k rational. Then $x = z(k + 1)$ with $k + 1$ rational, which cannot be, since we assumed that x cannot be composed only by one part. This explains Cardano's reasoning in the least detail.

Let us now consider two examples. We will make reference to the two "general" shapes (that is, those shapes that are expounded thanks to the *binomia* and *recisa* and that can encompass other more "particular" shapes for the solutions) mentioned in the *Ars magna arithmeticæ* as solutions of $x^3 = a_1x + a_0$ with a_1, a_0 rational. They are the *binomium* of the 2$^{\text{nd}}$ or 5$^{\text{th}}$ type $\sqrt{a} + b$ and the cubic *binomium* $\sqrt[3]{a} + \sqrt[3]{b}$ such that $\sqrt[3]{ab}$ is rational. We want to show that these two examples fit the above requirements on Y, Z. It is clear that both of them match the third one. They also match the first and the second ones. In fact, when $x = \sqrt{a} + b$ is a *binomium* of the 2$^{\text{nd}}$ or 5$^{\text{th}}$ type, its cube is

$$x^3 = 3ab + b^3 + 3b^2\sqrt{a} + \sqrt{a^3} = (3b^2 + a)\sqrt{a} + 3ab + b^3.$$

and we can take for instance $Z = b^3$, which is rational, and $Y = (3b^2 + a)\sqrt{a} + 3ab$. Then, Y contains two parts (a rational and an irrational one) as well as a_1x contains a rational and an irrational part (since $x = \sqrt{a} + b$). When instead $x = \sqrt[3]{a} + \sqrt[3]{b}$ is a cubic *binomium* such that $c = \sqrt[3]{ab}$ is rational, its cube is

$$x^3 = 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} + a + b = 3c(\sqrt[3]{a} + \sqrt[3]{b}) + a + b$$

and we can take $Z = a + b$, which is rational, and $Y = 3c(\sqrt[3]{a} + \sqrt[3]{b})$. Then, Y contains two (irrational) parts, which are $3c\sqrt[3]{a}$ and $3c\sqrt[3]{b}$ (and they are incommensurable, then they cannot be summed), as well as x.

We remark that in the above quotation Cardano pays particular attention[46] to the case in which neither y nor z are (rational) numbers, as in $x = \sqrt[3]{a} + \sqrt[3]{b}$. He specifies in fact that then y and z must be such that the equation $(y+z)^3 = a_1(y+z) + a_0$ can hold. In fact, the cube of a sum of two or more terms usually has more terms than the sum itself (except when some condition like '$\sqrt[3]{ab}$ is rational' is imposed) and Cardano wants to count this possibility out.

As said, Cardano's account on the substitution $x = y+z$ immediately foreruns the last part of Chapter I on the splittings. If his aim is – as I think – to justify the substitution, this account is not complete. In fact, Cardano does not explicitly show that x has at most two parts. This is the missing passage in order to show that x exactly has two parts, which is the result that Cardano wants to use. Anyway, we will see in the rest of this section that there is a sizeable number of *Aliza*'s chapters (as usual, haphazardly ordered) that recall and complete the argument. In my opinion, this paragraph is a later addition. In fact, on the one hand, it is the obvious continuation of the first paragraph, which I believe to be a later addition too. On the other hand, considering the framework of Chapter I, it is likely that Cardano needs to justify the substitution that he is going to use to derive the splittings. Then, it seems to me that he puts there just a few lines to give a semblance of uniformity, relying on the fact that more detailed arguments will follow elsewhere.

The study of the substitution $x = y + z$ is possibly originated by former studies on the shapes of the irrational solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational,

[46]"Truly, if two quantities alien to the number were contained in the thing, it would be necessary that they were incommensurable one to the other, otherwise they would have been equivalent to one [quantity]. But in this way it is necessary that the cube, which contains three parts, a number and two irrationals, is made in order that it can be equal to some things and the number. Having divided the thing in two parts, it is necessary that the cube produces three of this sort and that, if [the thing is divided] in three, it produces four and so on, and (as it is said) in order that in those the number [is] equal to the proposed number, [and] that in truth the remaining parts are from the nature of the parts of the sides and [are] equal to the aggregates of these" or "[v]erum si in re contineantur duæ quantitates a numero alienæ, necesse est ut inter se sint incommensæ, aliter æquivalerent uni. At ex eiusmodi necesse est cubum fieri, qui tres partes contineat, numerum et duas rhete, ut rebus ac numero possit coæquari. Cum ergo diviserimus rem in duas partes, oportet cubum tres eiusmodi progignere, et si in tres ut progignat quatuor atque ita deinceps, et (ut dictum est) ut in illis sit numerus numero proposito æqualis; reliquem partes vero ut sint ex natura partium lateris, et illarum aggregatis coæquales", see [Cardano 1570a, Chapter I, pages 1-2].

and, generally speaking, on irrational numbers. Concerning this last topic, I will provide some technical details in Section 4.4.1 at page 309. Concerning instead the first one, in the *Aliza* we find some chapters that recall or improve Cardano's results in the *Ars magna arithmeticæ* (see Section 3.2.4 at page 185).

More precisely, at the beginning of **Chapter V** Cardano states that

A V.i. *The* binomia *and* recisa *of the 3rd or 6th type* $\sqrt{a} \pm \sqrt{b}$ *cannot be solution [of a non-complete cubic equation with rational coefficients].*

Cardano does not specify of which equation $\sqrt{a} \pm \sqrt{b}$ cannot be a solution. Nevertheless, recall that we have already met a slightly more general version (for any degree) of this proposition in (AMA XIX). Since in the *Aliza* in general and in the rest of the chapter Cardano takes into account only cubic equations, I have considered them as the implicit object of this proposition too (with the correction in the footnote 27 at page 189). Moreover, Cardano's justification here corresponds to the missing part of the justification in the *Ars magna arithmeticæ* (see above at page 188), since he only justifies the proposition for $x^3 = a_2x^2 + a_0$.

> I say that the value by a *binomium* or *recisum* in which there is no number is not suitable in this case, because, [the value being] subtracted from the number, three non-composite quantities, a number and two roots, are left, and from those roots multiplied by themselves it is nothing but a number and a root of a number, then in the product they cannot erase each other.[47]

Let us take by the absurd $x = \sqrt{a} + \sqrt{b}$ to be a solution of $x^3 = a_2x^2 + a_0$, with a_2, a_0 rational. Then, we get the contradiction $x^2(x - a_2) \neq a_0$, since

$$(a + 2\sqrt{ab} + b)(\sqrt{a} + \sqrt{b} - a_2) = (a + 3b)\sqrt{a} + (3a + b)\sqrt{b} - 2a_2\sqrt{ab} - a_2(a + b)$$

cannot be rational. We remark that the justification sticks to the very same pattern (substituting by the absurd the (un)expected value and deriving a contradiction through calculations) as in the *Ars magna arithmeticæ*, though this time Cardano rewrites the equation as $x^2(x - a_2) = a_0$.

[47]"*Dico quod æstimatio in binomio vel reciso, in quo non est numerus, non est idonea in hoc casu, quia detracta a numero relinquit tres quantitates incompositas, numerum et duas radices, et ex radicibus illis in se ductis non sit nisi numerus, et una radix numeri, ergo in producto non poterunt se delere*", see [Cardano 1570a, Chapter V, page 13].

In **Chapter X** "In what way the parts with the proposed line are appropriate to the parallelepiped [*Quomodo conveniant partes cum linea proposita in parallelipedo*]" Cardano directly employs the splittings (A I.2)-(A I.4) to draw some more systematic conclusions on the shape of the solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational. Also the Proposition (A V.i) is rephrased.

A X.1. *Consider the equation $x^3 = a_1x + a_0$ and take y, z such that $y + z = x$ is a solution.*
If moreover x is rational, then either y, z are rational or y, z are a binomium *and its* recisum.

Cardano does not better specify the type of the *binomium* and *recisum.*

A X.1 — *Proof***.** Let a_1, a_0, and $x = y + z$ be given rational.

By (A I.2), Cardano knows that $a_0 = 3y^2z + 3yz^2$. Then, $yz = \frac{a_0}{3(y+z)}$ is rational. Therefore, y, z are either both rational or they are a *binomium* and its *recisum.*[48]

The same is done using (A I.3) or (A I.4) and replacing the factor 3 in the denominator with 2 or 1. □

We remark that Cossali[49] gives a statement similar to the above one, though his proof completely departs from Cardano's one.

The following propositions are proved in an analogous way. Note that, since they all contain a negative statement, Cardano employs the same method by the absurd to contradict the fact that the coefficients are rational. I will only give the proof of the following proposition as an example.

A X.2. *Consider the equation $x^3 = a_1x + a_0$. Then, no irrational $\sqrt{a}$ can be a solution of the equation.*

[48]Since y, z are the solutions of the associated quadratic equation. Nevertheless, Cardano (as usual) justifies this last step only through an example, namely $x^3 = 30x + 36$ (where $\Delta_3 < 0$).

[49]"La forma irrazionale $t \pm \sqrt{u}$ non soddisfa all'equazione $x^3 - px \mp q = 0$, che a condizione, che essa equazione abbia una radice razionale", see [Cossali 1966, Chapter II, paragraph 1, page 61] and [Cossali 1799a, Chapter VII, paragraph 6, pages 400-401].

A X.2 — *Proof.* Let a_1, a_0 be given rational and $x = y + z$ be given irrational. Suppose moreover that $x = \sqrt{a}$ and by the absurd let it be a solution of the equation.

As in the previous proof, Cardano gets to the system

$$\begin{cases} y + z = x \\ yz = \frac{a_0}{3(y+z)} \end{cases}.$$

In particular, $y, z = \frac{x}{2} \pm \sqrt{\left(\frac{x}{2}\right)^2 - \frac{a_0}{3(y+z)}}$ [being y, z the solutions of the associated quadratic equation]. Anyway, this cannot be, since a_1 is rational. In fact, using the splitting (A I.2) and (A XXX), Cardano infers from $a_1 x = y^3 + z^3$ that $a_1 = y^2 - yz + z^2$. [But, using the above values of y, z, one (obviously) finds that $y^2 - yz + z^2 = x^2 - \frac{a_0}{x}$, which is irrational since $\frac{a_0}{x}$ is so.]

The same can be done using (A I.3) or (A I.4) and replacing the factor 3 in the denominator with 2 or 1. □

We remark that Cardano could have replaced $x = \sqrt{a}$ in the equation to immediately derive a contradiction, but he does not so. He rather chooses to stick to the pattern of the previous proof by rephrasing $a_1 = x^2 - \frac{a_0}{x}$.

A X.3. *Consider the equation $x^3 = a_1 x + a_0$. Then, no irrational $\sqrt[3]{a}$ can be a solution of the equation.*

A X.4. *Consider the equation $x^3 = a_1 x + a_0$. Then, no irrational $\sqrt{a} \pm \sqrt{b}$ can be a solution of the equation.*

We remark that this proposition is a particular case of (A V.i). The proof given here, analogous to the one of (A X.2), fixes the uncertainties of the proof of (A X.3).

A X.5. *Consider the equation $x^3 = a_1 x + a_0$. Then, no irrational $\sqrt[4]{a}$ or $\sqrt[6]{a}$ can be a solution of the equation.*

A X.6. *Consider the equation $x^3 = a_1 x + a_0$. Then, no irrational $\sqrt[n]{a \pm \sqrt{b}}$ with $n \geq 2$ can be a solution of the equation.*

The conclusion of this chapter is puzzling.

> Therefore, it is necessary that the universal value of this sort is either under the *binomium* in which there is a number or not, or under the *trinomium* in which there is the number or not, or in more terms in which there is [the number] or not, or in a wild quantity, namely [a quantity] that is not in any kind of roots, not even composed by those, nor left by subtraction.[50]

At this stage, Cardano recaps the situation as follows. “Universal” (or “general”) values for the equation $x^3 = a_1x + a_0$, with a_1, a_0 rational, can go under the shapes of any *binomium* (and *recisum*), any *trinomium* (and *recisum*), or any “wild [*sylvester*]” quantity – whatever a “wild” quantity is, since Cardano subsequent explanation is completely unclear.[51] In this way, also the *binomia* (and the *recisa*) of the 3rd or 6th type $\sqrt{a} \pm \sqrt{b}$ are encompassed, though they had already been counted out on two occasions.

Concerning clearness, **Chapter XI** “How many and what parts of the cube, and on the necessity of those, and what incommensurable [parts are] [*Partes cubi quot et quæ, et de necessitate illarum, et quæ incommensæ*]” is by far worse. Its extreme obscurity is a good example of Cardano’s bad style in the *Aliza*. But the chapter instantly draws our attention, since it starts by rephrasing the paragraph from Chapter I that I have quoted at the beginning of this section:

> [t]herefore we repeat and we say that, regarding to the side of the cube the quantity of which is searched, if the cube must be equal to two [parts], some things and a number, it is necessary that the cube [is] divided in such a way at least in two, then its

[50]“*Necesse est igitur ut huiusmodi æstimatio universalis sit aut sub binomio, in quo sit numerus, aut in quo non sit, aut trinomio in quo sit numerus, aut in quo non sit, aut in pluribus nominibus in quo sit, aut in quo non sit, aut in quãtitate sylvestri, scilicet quæ non sit in aliquo genere radicum, nec composita ex illis, nec per detractionem relicta*”, see [Cardano 1570a, Chapter X, paragraph 6, page 20].

[51]Cardano adds a further explanation obscure as well. “as the quantity the square root of which multiplied by the *residuum* to 12 produces 2, whence the chapter cannot be discovered” or “*velut quantitas cuius R ducta in residuum ad 12 producat 2, ubi capitulum inventum non esset*”, see [Cardano 1570a, Chapter X, paragraph 6, page 20].
It seems then that a “wild quantity” s is such that $\sqrt{s}(12 - s) = 2$. The solutions of the equation $s^{\frac{3}{2}} + 2 = 12s^{\frac{1}{2}}$ with $s \geq 0$ correspond to the solutions of the equation $p^3 + 2 = 12p$ with $p \geq 0$ *via* the substitution $p = s^{\frac{1}{2}}$. Since this last equation has $\Delta_3 < 0$, the formula contains imaginary numbers. Then, a “wild quantity” could maybe be related to imaginary numbers. This is the only occurrence of the term in the whole *Aliza*.

> side. In fact, from one it comes nothing but one. Then, at least in two.[52]

First of all, Cardano reminds that the unknown must be divided at least in two parts. We consider then $x = y + z$. In the following, using a very Euclidean terminology for irrational numbers, Cardano takes again into account the remaining possibilities (since the case 'y commensurable in length to z' has just been counted out). The steps follow such an awfully haphazard and tiresome layout that I had to considerably reorganise them.

According to the terminology of *Elements*, Book X, two numbers y, z can have different kinds of relationships. They can be commensurable or incommensurable,[53] or commensurable or incommensurable to the square power.[54] Indeed, we can easily generalise the definition of '(in)commensurable to the square power' to '(in)commensurable to the 3$^{\text{rd}}$ power' (and in truth to '(in)commensurable to the n^{th} power' for any natural n), encompassing then the definition of 'commensurable' in 'commensurable to the 1$^{\text{st}}$ power'.[55] This is not only a more comprehensive way of speaking, but as a matter of fact suits one of Cardano's needs. In fact, when for the first time he says that two numbers are "commensurable to the power

[52]"*Repetamus igitur et dicamus quod latus cubi, cuius quantitas quæritur, si debet æquari cubus duobus rebus et numero, oportet ut cubus sic divisus in duo saltem, ergo latus eius, nam ex uno non provenit nisi unum. Ergo in duo saltem*", see [Cardano 1570a, Chapter XI, page 20].

[53]"Those magnitudes are said to be commensurable which are measured by the same measure and those incommensurable which cannot have any common measure", see [Heath 1956c, Book X, definition 1, page 10]. *Elements* X.5-6: "Commensurable magnitudes have to one another the ratio which a number has to a number" and *vice versa*, and *Elements* X.7-8: "Incommensurable magnitudes have not to one another the ratio which a number has to a number" and *vice versa*, see [Heath 1956c, pages 24-28]. Then, y, z are commensurable (respectively incommensurable) if and only if $\frac{y}{z}$ is rational (respectively irrational).

[54]"Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure", see [Heath 1956c, Book X, definition 2, page 10]. Then, y, z are commensurable (respectively incommensurable) in square if and only if $\left(\frac{y}{z}\right)^2$ is rational (respectively irrational). Note that Heath chooses to use only 'in square' instead of 'to the square power', see his commentary on definition 2 [Heath 1956c, pages 11].

[55]Note that, fixed one n, the relation 'being incommensurable to the n^{th} power' is the negation of the relation 'being commensurable to the n^{th} power'. This last relation is transitive.
Note moreover that 'being commensurable to the n^{st} power' implies 'being commensurable to the m^{th} power' for each natural m that is a multiple of n, or equivalently 'being incommensurable to the n^{st} power' implies 'being incommensurable to the m^{th} power' for any natural m that divides n.

[*potestate commensæ*]", he does not specify to which power, but indeed his examples concern the pairs a, $\sqrt[3]{b}$, and $\sqrt[3]{a}$, $\sqrt[3]{b}$ (where a, b are rational such that their cubic roots are irrational), which are indeed commensurable only to the 3rd power. Nevertheless, referring to those kind of numbers, Cardano never explicitly uses '(in)commensurable to the 3rd power', but rather a generic '(in)commensurable to the power', or a deceptive '(in)commensurable to the 2nd power'. For instance, he states that $\sqrt[6]{32}$ and $\sqrt{2}$ are commensurable to the 2nd power, but he actually shows that they are commensurable to the 3rd power. Then, I believe that we need the above extended terminology in order to clarify Cardano's one.

Cardano employs this pseudo-Euclidean terminology for two purposes. On the one hand, it gives a criterion on which one can build a classification for the irrational shapes of the solutions of a cubic equation with real coefficients. On the other hand, it conveys a distinction that Cardano will often use in his justifications. In fact, it eventually highlights whether one can sum or not the terms in the cube of the binomial $x^3 = (y+z)^3 = y^3 + 3y^2z + 3yz^2 + z^3$. As we have already hinted to at page 259, if two numbers v, w are commensurable, then $\frac{v}{w}$ is rational. This means, for instance, that $v = kw$ with k rational, and therefore $v + w = w(k+1)$, where $k+1$ is rational. In this way, one manages to write a certain sum of two terms as one monomial. The point at issue clearly arises if we consider that in the most of the cases Cardano uses numerical examples, where, to expound a generic argument, one automatically writes a sum such as the one above as a monomial. Then, when we take the cube of a certain numerical binomial, the (in)commensurability of the terms of the binomial says something about the number of terms of its cube. In the end, we recall that these surveys on the (in)commensurability relationships are linked to the method of the splittings. In fact, knowing how many terms there are in the cube of the binomial $(y+z)^3$ is one of Cardano's key arguments to set the substitution $x = y + z$.

As a preliminary step, Cardano remarks that, when y, z are incommensurable to the 1st power, then $\frac{yz^2}{y^3}, \frac{y^2z}{z^3}, \frac{y^2z}{yz^2}$ (which are all equal to $\frac{y}{z}$) are irrational (and thus we cannot lower the number of the terms in the cube of the binomial). When instead y, z are commensurable to the 3rd power (and not to the 1st), then $\frac{y^3}{z^3}$ is rational (thus $y^3 + z^3$ can be written as one term) and $\frac{y^3}{y^2z}, \frac{y^2z}{yz^2}, \frac{y}{z}$ are irrational (thus we cannot lower the number of the terms). As said, Cardano uses the fact that certain terms in the cube of the binomial $x = y + z$ can be summed (or not) to count how many terms (and of which kind) remains. This can entail a contradiction when the cube x^3 is equated to $a_1x + a_0$, excluding thus some of the possible shapes for a solution. This general plan is quite clear, but unluckily not its details. More or less following this overall ordering, but with a lot of repetitions, Cardano affirms that:

A XI.i when $y = \sqrt[3]{a}$, $z = \sqrt[3]{b}$, and y, z are incommensurable to the 1st power [and to the 2nd power[56]], then y^3, z^3 are rational, but not y^2z, yz^2.[57] Then, the sum $y + z$ cannot be a solution of $x^3 = a_1x + a_0$;[58]

A XI.ii when y, z are not cubic roots and are commensurable to the 3rd power [and incommensurable to the 1st power], then the ratio $\frac{y^2z}{yz^2}$ is irrational.[59] In particular, if $y = \sqrt[6]{32}$ and $z = \sqrt{2}$, then y^2z, yz^2 are irrational, and also y^3, z^3;[60]

[56]This last condition is automatically verified if y, z are incommensurable to the 1st power. In fact, consider the polynomial $P(u) = u^3 - \frac{a}{b} \in \mathbb{Q}[u]$ and call $t = \sqrt[3]{\frac{a}{b}}$. Then, t satisfies $P(u)$, which is its minimal polynomial, since $P(u)$ (with no rational roots) is irreducible. Let us now consider another polynomial $Q(u) \in \mathbb{Q}[u]$ such that $Q(t) = 0$. Then, $P(u)$ divides $Q(u)$ and the degree of $P(u)$ is greater than, or equal to, 3. Suppose by the absurd that t^2 is rational (that is, y, z are commensurable to the 2nd power). Then, the choice $Q(u) = u^2 - t^2$ is satisfied by t, but has degree 2, whence the contradiction.

[57]"In fact, if AC and CB were not commensurable, therefore neither g and f, nor d and e [would be commensurable], but d and g are numbers, because [they are] cubes of cubic roots, therefore e and f cannot be numbers" or "*nam si non sint commensæ ac et cb, igitur nec g et f, nec d et e, sed d et g sunt numeri, quia cubi Rcu: igitur e et f non possunt esse numeri*", see [Cardano 1570a, Chapter XI, page 21]. We have called $AC = y$, $CB = z$, $d = y^3$, $e = y^2z$, $f = yz^2$, and $g = z^3$.

[58]"Then AB cannot be composed by two incommensurable cubic roots, because the parallelepipeds were not numbers. [...] Then, in any case, if the number was not near, none of the two quantities cannot satisfy the parallelepipeds according to the number in order that the remaining [part] of the cube satisfy the things" or "*[n]on ergo potest esse ab composita ex duabus R cubicis incommensis, quia parallelipeda non essent numeri [...]. Nullæ ergo duæ quantitates aliquo modo si non adsit numerus possunt satisfacere parallelipedis pro numero, ut reliquum cubi satisfaciat rebus*", see [Cardano 1570a, Chapter XI, page 21].

[59]"In truth if they were commensurable to the second [third] power [...], but, if they were not cubic roots, e will be to f as AC to CB, but AC is not commensurable to CB, therefore not even e to f" or "*[s]i vero commensæ potentia secunda [...], sin autem no essent Rcu: erit e ad f ut ac ad cb, sed ac non est commensum cb, ergo nec e cum f*", see [Cardano 1570a, Chapter XI, page 22].

[60]"The said example is $\sqrt[6]{32} + \sqrt{2}$. According to the single parallelepipeds to avoid the effort, the The cube is $\sqrt{72} + \sqrt[6]{2048} + \sqrt[6]{8192}$. Here it is agreed that no number is made, therefore it cannot be appropriate. I only say that no parallelepipeds can be a number under these hypotheses" or "*[e]xemplum dictum est Rcu:quad. 32 p: R 2 cubus secũdum simplicia parallelipeda ad*

A XI.iii when $y, z = \sqrt[3]{a \pm \sqrt{b}}$ [that is, y, z are incommensurable to the 1st, 2nd, and 3rd powers], then their sum cannot be a solution of the considered equation.[61]

Cardano's arguments are sometimes unclear. Unluckily, he only provides the numerical example in (A XI.ii). Cardano's statements are quite deceptive, up to the point of simply being wrong by low accuracy. A blatant case is (A XI.i), where he does not mention the solution $\sqrt[3]{a} + \sqrt[3]{b}$, with $\sqrt[3]{ab}$ rational, which is a clear counter-example to the statement. Moreover, if we take a, b irrational and consider (A XI.iii) as a generalisation of (A XI.i), then the canonical shape for a solution given by the cubic formula is also not taken into account. In my opinion, the Euclidean terminology makes Cardano mislead. If this terminology enables Cardano to study the number of the terms in the cube of a binomial, it also prevents him from having a good understanding of the fact that the sum of two cubic roots can actually be a solution. In this last case, he merely did not choose the most befitting language.

Facing such an enigmatic chapter, I have made my way as follows. Consider $x = y + z$ and its cube $y^3 + 3y^2z + 3yz^2 + z^3$. We want to infer some information on the (in)commensurability to the n^{th} power (for $n = 1, 2, 3$) of the terms in the cube depending on the relations between y and z. Then, we will be able to count how many parts there are in the cube and to specify y, z such that x is a solution of $x^3 = a_1x + a_0$, with a_1, a_0 rational. More precisely, we want to study what happens when the ratios $\frac{y^3}{y^2z}, \frac{yz^2}{z^3}$ (both equal to $\frac{y}{z}$), $\frac{y^3}{yz^2}, \frac{y^2z}{z^3}$ (both equal to $\frac{y^2}{z^2}$), and $\frac{y^3}{z^3}$ are either rational, or not. There are five cases.

If $\frac{y}{z}$ is rational, then $\frac{y^2}{z^2}$ and $\frac{y^3}{z^3}$ are rational too. In this case, x has one (rational or irrational) part, and can be a solution of the equation. This is one of the cases that Cardano charges to be non-"general". If $\frac{y}{z}$ is irrational and $\frac{y^2}{z^2}$, $\frac{y^3}{z^3}$ are rational, then y^3, y^2z are incommensurable to the 1st power as well

laborem fugiendum est Rquad. 72 p: Rcu:quad. 2048 p: Rcu:quad. 8192. Hic constat nullum fieri numerum, ideò convenire non potest. Dico modo quòd nullum parallelipedum potest in his suppositis esse numerus", see [Cardano 1570a, Chapter XI, page 22].

[61]"Neither these two parts could be universal cubic roots. Since, [...] if [the parts are] incommensurable, there will be made four incommensurable parts in the cube, therefore one one will be superfluous" or "*Neque poterunt hæ duæ partes esse RV:cu.. Quoniam [...] si incommensæ fient quatuor partes in cubo incommensæ, ergo una erit superflua*", see [Cardano 1570a, Chapter XI, page 22].

as yz^2, z^3 and y^2z, yz^2. Moreover, y^3, yz^2 are commensurable to the 1st power as well as y^2z, z^3 and y^3, z^3. But this cannot be. In fact, y^3, y^2z, yz^2, z^3 will be all commensurable to the 1st power by transitivity. If $\frac{y}{z}$ is irrational, $\frac{y^2}{z^2}$ is rational, and $\frac{y^3}{z^3}$ is irrational, then y^3 and y^2z, yz^2 and z^3, y^2z and yz^2, y^3 and z^3 are incommensurable to the 1st power, while y^3 and yz^2, and y^2z and z^3 are commensurable to the 1st power. In this case, x^3 has two parts. This is for instance what happens when $x = \sqrt{a} + b$ is a *binomium* of the 2nd or 5th type. If $\frac{y}{z}$ and $\frac{y^2}{z^2}$ are irrational and $\frac{y^3}{z^3}$ is rational, then y^3 and y^2z, yz^2 and z^3, y^2z and yz^2, y^3 and yz^2, y^2z and z^3 are incommensurable to the 1st power, while y^3 and z^3 are commensurable to the 1st power. In this case, x^3 has three parts. This is for instance what happens when $x = \sqrt[3]{a} + \sqrt[3]{b}$ with $\sqrt[3]{ab}$ rational. Finally, if $\frac{y}{z}$, $\frac{y^2}{z^2}$, and $\frac{y^3}{z^3}$ are irrational, then all the terms in the cube are two by two incommensurable to the 1st power. In this case, x^3 has four parts. But it cannot be equated to $a_1x + a_0$, since it only has three parts. Note that the relations of 'being (in)commensurable to the n^{th} power' for $n > 3$ are not influential in this classification, since only rational numbers up to the third power appear in the cube.

Cardano's above statements fit this scheme: (A XI.i) (and partially (A XI.ii)) correspond to the fourth case, while (A XI.iii) corresponds to the fifth case. Moreover, in this chapter and in others, Cardano has already dealt with the first case.

At around the middle of the chapter, Cardano starts to recap:

A XI.iv the solution of the considered equation may not contain cubic roots at all (whether universal, or with numbers or not).[62] Take a, b, c rational such that their cubic roots are irrational. Then:

- when $y = a$ and $z = \sqrt[3]{b}$, then their sum cannot be solution of $x^3 = a_1x + a_0$ (by substitution and comparison);[63]

[62] "In the end it is deduced from this that that part of the chapter of the cube equal to some things and a number cannot consist of quantity composed by two simple or universal cubic roots or by the number and a cubic root" or "*[e]x quo tandem concluditur, partem illam capituli cubi æqualium rebus et numero non posse consistere in quantitate composita ex duabus R cubicis simplicibus aut universalibus, aut numero et R cubica*", see [Cardano 1570a, Chapter XI, page 23].

[63] "In fact in the number and cubic root it will be necessary to give the cubes to the number, because [the cubes] will be numbers, then they will not satisfy a small number. Besides the

- when $y = \sqrt[3]{a}$ and $z = \pm\sqrt[3]{b}$, then their sum cannot be a solution of $x^3 = a_1x + a_0$;[64]
- the *trinomium* $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ [where $\sqrt[3]{a}$, $\sqrt[3]{b}$, $\sqrt[3]{c}$ are two by two incommensurable to the 1st and 2nd powers and $\sqrt[3]{abc}$ is irrational],

parallelepipeds will be incommensurable and [will be] two cubic roots and in the thing there is nothing but one part that is a cubic root, therefore the number of the things will not be [given]'" or "*in numero et R cubica oportebit dare cubos numero, quia erunt numeri, ergo in numero parvo non satisfacient, præterea parallelipeda incommensa erunt et duæ Rcu: et in re non est nisi una pars quæ sit Rcu: igitur non erit numerus rerum*", see [Cardano 1570a, Chapter XI, page 23].

[64] "And not even if both parts are cubic roots, since if you gave the parallelepipeds to the number, firstly they will not necessarily fit the cubes if [the cubic roots] are not commensurable, then the cubic roots will not be a number. Besides the cubes will be numbers, then they could not contain the thing according to the number, since the thing consists of two cubic roots multiplied by the number, they would produce the number. And we cannot even give both cubes, when they are positive, to the number, where the number was not smaller than the fourth part of the whole cube, as we taught. But where it is bigger or equal, we give [both cubes to the number] and that part of the chapter of the cubes equal to some things and a number that is already known is made. Therefore the remaining part in that equality never takes place. And we cannot even give the difference of the cubes to the number, as in a positive and a negative cubic root like $\sqrt[3]{6} - \sqrt[3]{2}$, because the negative parallelepipeds will be bigger and the positive [parallelepipeds] [will be] smaller. Then, since in the thing the positive cubic root are necessarily bigger than the negative cubic root, the things could in no way be contained in the parallelepipeds according to the number, but, joining well the positive [part] of the things with the negative [one] and the negative [part] with the positive [one] with the cube, the chapter of the cube and some things equal to a number is perfectly made" or "*[n]eque si ambæ partes sint Rcu: , quoniam si dederis parallelipeda numero primum non convenient cubi necessario, si non sint commensa, sint Rcu: ergo non numerus. Præterea cubi erunt numeri, ergo non poterunt res continere per numerum, cum res constet ex duabus Rcu: ductæ in numerum, producerent numerum. Neque possumus dare utrunque cubum p: numero ubi numerus sit minor quarta parte totius cubi, ut docuimus, ubi autem est maior vel æqualis damus, et fit illa pars capituli cubi æqualis rebus et numero, quæ iam nota est, igitur reliqua pars in hac æquatione nullum habet locum. Neque possumus dare differentiam cuborum numero, ut in Rcu: p: et m:, velut Rcu: 6 m: Rcu: 2, quia parallelipeda m: erunt maiora et p: minora, ergo cum in re Rcu: p: sit maior Rcu: m: necessario, nullo modo res poterunt contineri per numerum in parallelipedis, sed bene iungendo p: cum m: et m: cum p: rerum cum cubo fiet ad unguem capitulum cubi et rerum æqualium numero*", see [Cardano 1570a, Chapter XI, page 23].

Cardano's argument is two-folded. First, take $x = \sqrt[3]{a} + \sqrt[3]{b}$ to be a solution of $x^3 = a_1x + a_0$ with a_1, a_0 rational. Consider $a + 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} + b = a_1\sqrt[3]{a} + a_1\sqrt[3]{b} + a_0$. It is evident that Cardano cannot take $3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} = a_0$ and $a + b = a_1\sqrt[3]{a} + a_1\sqrt[3]{b}$. But also the assignment $a + b = a_0$ and $3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} = a_1\sqrt[3]{a} + a_1\sqrt[3]{b}$ does not always work, namely – Cardano argues making implicitly reference to (A I.1) – when $a_0 < \frac{1}{4}x^3$ (see here, page 240). Cardano omits to mention the condition $\sqrt[3]{ab}$ rational, under which the assignment partially works.

Second, take $x = \sqrt[3]{a} - \sqrt[3]{b} > 0$ (with $\sqrt[3]{a} > \sqrt[3]{b}$) and consider $a - 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} - b = a_1\sqrt[3]{a} - a_1\sqrt[3]{b} + a_0$. Then, the assignment $a - b = a_0$ and $-3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} = a_1\sqrt[3]{a} - a_1\sqrt[3]{b}$ only works for $x^3 + a_1x = a_0$, since $-3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} < 0$.

such as $\sqrt[3]{6} + \sqrt[3]{5} + \sqrt[3]{2}$, cannot be a solution of $x^3 = a_1 x + a_0$ (by substitution and comparison);[65]

A XI.v when $y = \sqrt{a}$ and $z = b$, then their sum cannot be a general solution of $x^3 = a_1 x + a_0$ [referring to Chapter I];[66]

A XI.vi when $y = a$ and $z = \sqrt{b \pm \sqrt{c}}$, then their sum cannot be a solution of $x^3 = a_1 x + a_0$ (by substitution and comparison);[67]

A XI.vii when $y = a$ and $z = \sqrt[3]{b \pm \sqrt{c}}$, then their sum cannot be a solution of $x^3 = a_1 x + a_0$ (by substitution and comparison).[68]

As said, this chapter is quite messy. Due to the Euclidean terminology employed, it stands a bit aside from the other chapters considered in this section. It is composed by three parts, respectively containing the preliminary remarks, the statements (A XI.i)-(A XI.iii), and the statements (A XI.iv)-(A I.7). It happens that sometime one part contains a statement that is not consistent with a statement in another part. In my opinion, we can explain this contradiction saying that the three parts have been juxtaposed precisely because of the Euclidean terminology employed. The fact that Cardano employs a terminology with ancient roots and which is manifestly not up-to-date with his aims, together with the fact that he fails to recognise a shape for the irrational solutions of $x^3 = a_1 x + a_0$

65"Therefore let AB to be divided in three parts that all are incommensurable cubic roots and not in the same proportion and it is agreed that eight kinds of bodies are made [...] the cube will not be equal to some things under any number" or "*[i]am ergo ventum est necessario ad triarios, sit ergo ab divisa in tres partes, quæ omnes sint Rcu: incommensæ, nec in eadem proportione, et constat quod fient octo genera corporum [...] cubus non poterit æquari rebus sub aliquo numero*", see [Cardano 1570a, Chapter XI, pages 23-24].

66"But the value cannot consist of a number and a square root in order to be general, in fact this is shown above" or "*[s]ed neque æstimatio potest constare ex numero et R quadrata, ut sit generalis, hoc enim est demonstratum supra*", see [Cardano 1570a, Chapter XI, page 23].

67"[The value] cannot consist of a number and a universal square root, because two incommensurable parts will be in the cube beside the number (because the universal root is not a first power commensurable to the number, see Chapter III at the end) and only one [part] in the thing, then the number of the things does not correspond" or "*neque potest constare ex numero et RV: quadrata quia in cubo erunt duæ partes præter numerum incommensæ (quia RV: non est potentia prima commensa numero) et in re una tantum ergo non constabit numerus rerum*", see [Cardano 1570a, Chapter XI, page 23].

68"And not even by a number and a universal cubic root. It will be necessary to give the cubes to the number and the parallelepipeds will be two incommensurable, then as before, being only one universal cubic root, the thing will not be able to be contained in the cube under any number" or "*[n]eque ex numero et RV: cu. quoniam oportebit dare cubos numero et parallelipeda erunt duo incommensa, ergo ut prius cum sit tantum una RV: cu. non poterunt res numero aliquo contineri in cubo*", see [Cardano 1570a, Chapter XI, page 23].

(that he would have not missed if he had had the cubic formula available), makes me suppose that Chapter XI attests one of the earliest inquiries by Cardano on the topic.

Chapter XIII "On the discovery of the parts of the cubic *trinomium*, which produces the cube with two only cubic parts [*De inventione partium trinomii cubici, quod cubum producit cum duabus partibus tantum cubicis*]" is devoted to the study of the *trinomia* as solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational. Cardano maintains the following statements.

A XIII.i. *Consider the equation* $x^3 = a_1x + a_0$*. Then, no irrational* $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ *with* $\sqrt[3]{abc}$ *rational (or, equivalently,*[69] *such that* $\sqrt[3]{a} : \sqrt[3]{b} = \sqrt[3]{b} : \sqrt[3]{c}$*) can be a solution of the equation.*

A XIII.ii. *Consider the equation* $x^3 = a_1x + a_0$*. Then, no irrational* $\sqrt[3]{a} + \sqrt[3]{b} + c$ *[with* $\sqrt[3]{abc}$ *rational or, equivalently,] such that* $\sqrt[3]{a} : \sqrt[3]{b} = \sqrt[3]{b} : c$ *can be a solution of the equation.*

Cardano plainly justifies the statements by substitution and comparison, employing a very few Euclidean terminology from Chapter XI. Here the (in)commensurability relationships are no more the centrepiece: the classification of the shapes for the solution of the considered equation is no more based on them. They are used only in so far as they enable one to infer how many parts there are in the cube x^3, given a certain shape for the solution x.

For instance, in the case (A XIII.i), similarly to what was done in the third point of (A XI.iv), Cardano remarks[70] that, if by the absurd $x = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$,

[69]In fact, if $\sqrt[3]{abc}$ is rational, then we can write $\sqrt[3]{c} = \frac{k}{\sqrt[3]{ab}}$, with k rational, and the proportion holds. *Vice versa*, if the proportion $\sqrt[3]{a} : \sqrt[3]{b} = \sqrt[3]{b} : \sqrt[3]{c}$ holds, then $\sqrt[3]{b^2} = \sqrt[3]{ac}$, which implies that $\sqrt[3]{abc}$ is rational.

[70]"And I only say that, if the cubic *trinomium* the parts of which multiplied produced the number was assumed, only two parts that are cubic roots would be produced, but there are three incommensurable parts in the thing – as it is said, therefore the parts of the cube cannot contain the parts of the things according to the number" or "*dico modo quod si assumatur trinomium cubicum, ex cuius ductu partium producatur numerus, quod producentur duæ partes tantum, quæ R cubicæ, sed in re sunt tres partes incommensæ, ut dictum est, igitur partes cubi non possunt continere partes rerum secundum numerum*", see [Cardano 1570a, Chapter XIII, page 27].

with $\sqrt[3]{abc}$ rational, had been assumed, then its cube would have been

$$x^3 = (a+b+c+6\sqrt[3]{abc}) + 3(\sqrt[3]{ab^2} + \sqrt[3]{a^2c} + \sqrt[3]{bc^2}) + 3(\sqrt[3]{a^2b} + \sqrt[3]{b^2c} + \sqrt[3]{ac^2}),$$

where the parentheses help to identify the groups of terms commensurable to the 1st power one to the other. In fact, since $\sqrt[3]{abc}$ rational implies that $\sqrt[3]{c} = \sqrt[3]{\frac{b^2}{a}}$, we can rewrite the cube as follows

$$x^3 = (a+b+c+6\sqrt[3]{abc}) + 3(a+2b)\sqrt[3]{\frac{b}{a}} + 3(2a+b)\sqrt[3]{\frac{b^2}{a^2}}.$$

Equivalently, let us call the irrational $\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{b}}{\sqrt[3]{c}} = k$. Then, k^3 is rational, $\sqrt[3]{b} = k\sqrt[3]{c}$, and $\sqrt[3]{a} = k^2\sqrt[3]{c}$. The cube can thus be rephrased as

$$x^3 = (k^6c + k^3c + c + 6k^3c) + 3c(k^3+1)k + 3c(k^3+2)k^2.$$

Then, Cardano concludes that x^3 cannot be equated to $a_1x + a_0$ because the number and the kinds of cubic roots on the two sides of the equal are not the same.

In **Chapter XVI** "That a *quadrinomium* from cubic roots is reduced to three parts, two of which are only cubic roots, or by far many [*Quod quadrinomii ex radicibus cub. cubus ad tres partes quarum duæ sint tantum R cubæ reducitur, aut longe plures*]" Cardano finally takes into account the *quadrinomia* as solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational. This chapter is in the same style (justifications by substitution and comparison, with a very few Euclidean terminology) as Chapter XIII.

A XVI.i. *Consider the equation $x^3 = a_1x + a_0$. Then, no irrational $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}$ such that $\sqrt[3]{a} : \sqrt[3]{b} = \sqrt[3]{b} : \sqrt[3]{c} = \sqrt[3]{c} : \sqrt[3]{d}$ can be a solution of the equation.*

Note that the condition stated in the continued proportion implies that $\sqrt[3]{abc}$ is rational. More precisely, let us call the irrational $\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{b}}{\sqrt[3]{c}} = \frac{\sqrt[3]{c}}{\sqrt[3]{d}} = k$. Then, $\sqrt[3]{c} = k\sqrt[3]{d}$, $\sqrt[3]{b} = k^2\sqrt[3]{d}$, and $\sqrt[3]{a} = k^3\sqrt[3]{d}$, and therefore $\sqrt[3]{abc} = k^6d$ is rational, since k^3 is rational. We moreover remark that the *quadrinomium* is in truth a *trinomium*, since $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} = (k^3+1)\sqrt[3]{d} + k^2\sqrt[3]{d} + k\sqrt[3]{d}$.

Similar considerations also hold for the following two propositions.

A XVI.ii. *Consider the equation $x^3 = a_1x + a_0$. Then, no irrational $\sqrt[3]{a} + \sqrt[3]{b} + c + \sqrt[3]{d}$ such that $\sqrt[3]{a} : \sqrt[3]{b} = \sqrt[3]{b} : c = c : \sqrt[3]{d}$ can be a solution of the equation.*

A XVI.iii. *Consider the equation $x^3 = a_1x + a_0$. Then, no irrational $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}$, with[71] $\sqrt[3]{a^2bc}$ or $\sqrt[3]{bcd^2}$ rational, can be a solution of the equation.*

A XVI.iv. *Consider the equation $x^3 = a_1x + a_0$. Then, no irrational $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}$, with no relation between $\sqrt[3]{a}$, $\sqrt[3]{b}$, $\sqrt[3]{c}$, $\sqrt[3]{d}$, can be a solution of the equation.*

We remark that in this case there are much more cubic roots incommensurable to the 1st power contained in the cube than in the preceding propositions (namely, 16 cubic roots in the development of the cube are incommensurable one to the other). Then – Cardano ends – "no *quadrinomium* by cubic roots is suitable [*nullum quadrinomium ex* R *cubicis non est idoneum*]".

Chapters V, X, XI, XIII, and XVI are located in the first third of the *Aliza*. In the main, they are spaced out by chapters on different topics.[72] Then, a bit outdistanced, we find **Chapter XXIV** "Demonstration that shows that no chapter except the discovered [ones] can be known generally [*Demonstratio ostendens quod caput nullum præter inventa generale sciri potest*]". Unluckily, it is another of the *Aliza*'s chapters in which Cardano's bad Latin gets the better on clearness. The chapter is divided in two parts. In the first one, which corresponds to the first short paragraph, Cardano maintains that no equation with more than four different terms (regarding to their powers, that is, any equation of degree higher than four) that cannot be reduced to another with fewer terms

[71] Cardano only says to take the four cubic roots such that " two [cubic roots] times the square of another produce a number[*duæ in quadratum alterius producant numerum*]". Indeed, considering that $\sqrt[3]{c} = k\sqrt[3]{d}$, $\sqrt[3]{b} = k^2\sqrt[3]{d}$, and $\sqrt[3]{c} = k^3\sqrt[3]{d}$, there are two possibilities. In fact, either $\sqrt[3]{a^2bc} = k^9d\sqrt[3]{d}$ or $\sqrt[3]{bcd^2} = k^3d\sqrt[3]{d}$, both assuming that $\sqrt[3]{d}$ is rational, can only be rational (the other possibilities contain the powers of k that are not multiple of 3). Then, the *quadrinomium* is in truth a *trinomium* one term of which is the number.

[72] See for instance Chapter VI on the sign rule (see below, at page 332) or Chapter XII on the geometrical proof for the existence of a (positive, real) solution of $x^3 = a_1x + a_0$ (see below, at page 317). Note that, despite the title, Chapter XIV teaches how to find the coefficients of an equation of the families $x^3 + a_0 = a_1x$ and $x^3 = a_1x + a_0$, once the shape of one of its solutions has already been given. Also Chapter XV, in which Cardano harks back on the splittings, does not concern the study of the possible shape for the solutions of $x^3 = a_1x + a_0$.

(that is, either to an equation of lower degree that Cardano is able to solve, or to an equation that can be solved by substitution, like the biquadratic ones) can be solved in general. When, instead, there are four terms up to the cube (that is, a complete cubic equation), one should – according to Cardano – get an equation with three terms,[73] and, from there, the equation $x^3 = a_1x + a_0$.[74] We remark that, if my understanding is correct, Cardano is suggesting again the same strategy as in the first paragraph of Chapter I. Note moreover that at this stage the quartic equations seem not to be included among the equations that Cardano is able to solve.

Cardano goes on observing that, according to (AM XXV.1) and to the third corollary to (AM XIII) (see above, respectively at pages 85 and 118),

[73]This (and in particular reduce a complete cubic equation to a depressed one) was a common strategy, achieved through some well known substitutions (see, for instance, the substitutions in Section 2.4 at page 2.4). And indeed, in his poem Tartaglia mentions only depressed cubic equations.

[74]The paragraph is quite tricky, then here the quotation follows so that the reader can judge by himself: "[i]t remains to show what has been proposed at the beginning, because of which we wrote this, namely that there is no other general chapter that can be known beyond those [ones] that are bequeathed. Since, more than four different kinds [of terms] cannot be reduced to fewer, either by a division, or [by taking the] root, or by a change, or [by] a own rule, or [by] suppressing, or for the sake of the origin, or by a Geometrical demonstration, because there are great inequalities in the single [terms] that can hardly be understood in four quantities and, if the perfection cannot be found in these, the less [it can be found] in those. Then on this, if there are four quantities up to the cube, it is now wise to reduce to three quantities and [to reduce] all the chapters of three quantities to the cube equal to some things and a number. Therefore, if I will show that this cannot be general, the purpose is also clear in the unknown part" or "*[r]eliquum est ut ostendamus quod ab initio propositum est, cuius causa hæc scripsimus, scilicet non esse capitulum aliud generale, quod sciri possit, ultra ea quæ tradita sunt, quoniam ultra quatuor diversa genera nisi possit reduci ad pauciora, vel per divisionem, vel radicem, aut per mutationem, aut regulam propriam, vel deprimendo, aut ob originem, aut per demonstrationem geometricam, cum in singulis sint magnæ inæqualitates, quæ vix possunt intellegi in quatuor quantitatibus, nec in eis potuerit invenire perfectio quanto minus in illis. De his ergo, si sint quatuor usque ad cubum, iam doctus est reducere ad tres quantitates, et capita trium quantitatum omnia ad cubum æqualem rebus et numero. Si igitur ostendero hoc non posse esse generale, etiam in parte ignota liquet propositum*", see [Cardano 1570a, Chapter XXIV, page 49]. Note that my understanding is based on interpreting 'four different kinds [*quatuor diversa genera*]' as 'four different kinds of terms' in an equation, and precisely as 'four different kinds of terms according to their power'. This interpretation is supported by the second sentence (in the Latin version), where the "four different kinds" becomes "four up to the cube [*quatuor usque ad cubum*]", which can subsequently be reduced to "three quantities [*tres quantitates*]", and in turn to $x^3 = a_1x + a_0$.
The close of the above quotation is upsetting, especially because this chapter actually ends by stating the generality of a solution.

the two (positive, real) solutions of $x^3 = a_1x + a_0$ (when $\Delta_3 < 0$ and the coefficients are rational) are a number and a *binomium* of the (2^{nd} and) 5^{th} type(s). Therefore, "it is necessary that the general value can be shared by the number and by the fifth *binomium* [*oportet æstimationem generalem posse communicari numero et quinto binomio*]". Cardano then focuses on the radical part of the *binomium* at issue. He firstly remarks that the cube of any *binomium* is again a *binomium* where the same radical appears. For instance, we have that $(\sqrt{a}+b)^3 = (a+b^2)+2b\sqrt{a}$. Then, taking this *binomium* as a solution and equating $(\sqrt{a}+b)^3 = a_1(\sqrt{a}+b)+a_0$, he finds some conditions on the inner structure (or shape) of the root $\sqrt{a}$. He considers the cases in which a is a perfect square and $a = \sqrt{a'}+a''$, or $a = \sqrt{a'}+\sqrt{a''}+a'''$, or $a = \sqrt[4]{a'}+\sqrt{a''}+a'''$, and by substitution and confrontation comes to the conclusion that only $a = \sqrt{a'}+a''$ is possible.

In particular, Cardano considers the equation $x^3 = 29x+42$ (with $\Delta_3 < 0$), since he is able to write its coefficients as

$$\begin{cases} 29 = (18+\sqrt{72})+(11-\sqrt{72}) \\ 42 = (18+\sqrt{72})\sqrt{11-\sqrt{72}} \end{cases}.$$

Then, he applies (AM XXV.1), taking $f = 18+\sqrt{72}$ and $g = 11-\sqrt{72}$ (and $\sqrt{g} = \frac{3}{2}-\frac{\sqrt{2}}{2}$), to draw

$$x = \sqrt{\frac{83}{4}+\frac{18}{4}\sqrt{2}}+\frac{3}{2}-\frac{\sqrt{2}}{2}.$$

In this case, x has (almost) the shape of the considered *binomium* with another *binomium* under the square root. Then $x = 6$, since he knows that $\sqrt{\frac{83}{4}+\frac{18}{4}\sqrt{2}} = \frac{9}{2}+\frac{\sqrt{2}}{2}$. It is interesting that Cardano says that "this $[\sqrt{\frac{83}{4}+\frac{18}{4}\sqrt{2}}+\frac{3}{2}-\frac{\sqrt{2}}{2}]$ is the true value of the thing [*[h]æc igitur est vera æstimatio rei*]", and not 6! Again, "this kind of value is general, because it can be made equal to a number or not, and [it can be] made equal to the *binomium* [*hoc genus æstimationis est generale, quia potest æquari numero, et non æquari, et posset æquari binomio*]".[75] On the one hand, this strengthens my interpretation of "general" solutions as a solutions that go under certain codified shapes and that can encompass other shapes (see

[75]See [CARDANO 1570a, Chapter XXIV, pages 51-52].

here Sections 2.1.4 and 3.2.4). On the other hand, this is could be – in my opinion – one of the earliest (chronologically speaking) passages where Cardano observes that x can be written as the sum of two parts (above called y, z), which in this case are the two radicals $\sqrt{\frac{83}{4}+\frac{18}{4}\sqrt{2}}$ and $\sqrt{\frac{11}{2}-\frac{6\sqrt{2}}{2}}$. In fact, it is remarkable that Cardano exploits the "particular" rule (AM XXV.1) in order to study the shape of the solutions of $x^3 = a_1x + a_0$ instead of using the formula in (AM XII). It is true that, if one wants to amend the cubic formula, he may appeal to other results. Nevertheless, the fact that no mention of Ferrari's method for solving quartic equations is made makes me believe that this chapter has been written some time before the big discoveries of the *Ars magna*.

Finally, at the very end of the *Aliza*, Cardano harks back on the topic of the possible shape for the solutions of $x^3 = a_1x+a_0$ with real coefficients. In **Chapter LVII** "On the treatment of the general value of the cube equal to some things and a number [*De tractatione æstimationis generalis capituli cubi æqualis rebus et numero*]" Cardano states that

> the general value of the chapter of the cube equal to some things and a number is had neither by a general nor by a special rule, except by that [which is to] find a quantity that multiplied by a second [one] produces the number of the equality. That second quantity behaves like the gnomon, and is the first root or the side of the aggregate by the number of the things, and [is] discovered according to that quantity.[76]

The rule referred to is (AM XXV.2) (see above, page 86). In my opinion, the above quotation could also fall under the same pre-formula context as in Chapter XXIV.

In this chapter, Cardano massively appeals to the very end of **Chapter XL** "On three necessary [things] that is necessary to put before the discovery [*De tribus necessarijs quæ præmittere oportet ad inventionem*]". There, the last paragraph

[76]"*Iam docui te quod æstimatio generalis capituli cubi æqualis rebus et numero non est habita, neque per regulam generalem neque specialem, nisi per illam, ut invenias quantitatem quæ ducta in secundam, producat numerum æquationis, et illa secunda quantitas gerit vicem gnomonis, et sit prima radix seu latus aggregati ex numero rerum, et secunda illa quantitate inventa*", see [Cardano 1570a, Chapter LVII, page 104].

stands aside compared to the rest and deals with the equation $x^3 = a_1 x + a_0$. Cardano states the following proposition.

A XL. *Consider the equation* $x^3 = a_1 x + a_0$. *Then,*[77] $x = \sqrt{a_1 + \frac{a_0}{x}}$.

His justification trots out a so-called "orthogonal triangle [*triangulum orthogonius*]",[78] which is nowhere else mentioned in the *Aliza*.

A XL — *Proof***.** As usual, Cardano interprets the equation in a geometrical environment, taking $x = \overline{BC}$, $a_1 = (\overline{AD})^2$, $a_0 = g$.

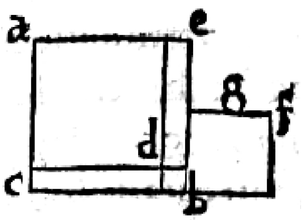


FIGURE 4.6 – *De regula aliza*, Chapter XL.

In this interpretation, where $a_1 x = (\overline{AD})^2 \overline{BC}$, it is natural to take $g = (\overline{C\,D\,E})\overline{BC}$ (and in turn Cardano makes it equal to $(\overline{BF})^2$ for the sake of brevity). Then $\overline{BC}^3 = \overline{BC}((\overline{AD})^2 + (\overline{BF})^2)$ and $\overline{BC}^2 = (\overline{AD})^2 + (\overline{BF})^2$, from where he gets the searched formula. $\square$

Actually, the proposition as it is stated is wrong. In fact, Cardano only considers the positive root (we have already remarked it at page 86, while dealing with (AM XXV.2), since in fact it is the same rule as there). By the way, this chapter sheds new light on how the "particular" rule in (AM XXV.2) was understood by Cardano. At the basis, the proposition describes a decomposition in two parts of

[77] We remark that this proposition could also provide an algorithm to calculate an approximate solution of the considered equation. For instance, let us consider $x^3 = 3x + 3$. We call $f(x) = \sqrt{3 + \frac{3}{x}}$ and we take $x_0 = 1$. Then, if we evaluate $f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$, and so on, up to the tenth iteration we obtain a very good approximation of the real value.
We also recall that, even though Cardano does not seem to use here the proposition in this way, there is a whole chapter of the *Ars magna* that is devoted to calculate approximate solutions through a modified version f the method of the double false position, see [CARDANO 1545, Chapter XXX, pages 52v-54r].

[78] "And its origin is from the orthogonal triangle" or "*est origo eius ex triangulo orthogonio*", see [CARDANO 1570a, Chapter XL, page 82].

the cube as a three-dimensional body. Anyway, and as before, no real positional property of the geometrical objects involved is singled out, since in fact it can be imitated by basic arithmetical operations.

Coming back to Chapter LVII, Cardano uses the above proposition[79] (or (AM XXV.2)) to test (again) the possible shape for an irrational solution of $x^3 = a_1 x + a_0$. He considers x respectively to be a *binomium* of the 2nd or 5th type $\sqrt{a} + b$, any *recisum*, and a *binomium* of the 1st or 4th type $a + \sqrt{b}$. He remarks that, substituting one of these values for x in $\sqrt{a_1 + \frac{a_0}{x}}$, he gets either a *binomium* or a *recisum*, or always a *binomium*, or a *recisum*. Then, only the shape $\sqrt{a} + b$ is admissible, since it is the only one that can lead to a true equality, which is ‘a *binomium* equal to a *binomum*’ (while the others respectively are ‘a *recisum* equal to a *binomum*’ and ‘a *binomium* equal to a *recisum*’). Cardano's conclusion is then the following:

> [i]t is also agreed that this value [$\sqrt{a_1 + \frac{a_0}{x}}$] is common to the cubic *binomium* discovered in a part of the chapter and to the surface *binomium* here indicated and the common quantity is the general value.[80]

No explanation of the term “surface [*superficiale*]” *binomium* is given there or elsewhere. I am likely to compare it with the previously mentioned cubic *binomium*.

The subsequent **Chapter LVIII** “On the common quantity from two incommensurable [quantities], in how many ways it is said [*De communi quantitate duabus incommensis quot modis dicatur*]” seems to start where Chapter LVII ended. The topic is again the “common quantity [*quantitas communis*]”. Cardano recalls once more that the possible shapes for the (irrational) solutions

[79]Note that in Chapter LVII Cardano abruptly employs some new stenographic expressions. There ‘*c. p.*’ means ‘with what is produced from [*cum producente*]’, ‘*d. n*’ means ‘n divided by (something, which in our case is x) [*diviso*]’, and ‘*f. n*’ means ‘fragment of n [*fragmento*]’. This kind of (to say the truth, not functional) formalism is employed only once elsewhere in the *Aliza* and with the very same examples, which are “20 *c. p.* 32”, “20 p: *d.* 32”, and “20 *f.* 32”. They all stands for $20 + \frac{32}{x}$. See [Cardano 1570a, Chapter LIII, page 98 and Chapter LVII, page 104].

[80]“*Constat etiam quod talis æstimatio est communis binomio cubico invento in parte capituli, et binomio superficiali hic declarato et communis quantitas est æstimatio generalis*”, see [Cardano 1570a, Chapter LVII, page 104].

of $x^3 = a_1 x + a_0$ (with a_1, a_0 rational) are the *binomium* of the 2nd or 5th type $\sqrt{a} + b$ and the cubic *binomium* $\sqrt[3]{a} + \sqrt[3]{b}$ (with $\sqrt[3]{ab}$ rational). His aim is to search for a "common quantity" or a "common value [*æstimatio communis*]" to them. Unluckily (again), Cardano's start is not very plain.[81] But then the things get better. Cardano explicitly states that

> it is necessary that the common quantity is neither of the kind of AB nor of BC. And this can be, in fact the animal is common to the human being, to the the donkey, to the ox, and to the horse, thus AB and BC are contained in a certain common quantity, which as long as it is common to all [AB and BC] has only this property, that, the simple number of the equality [being] divided by that same, it is the root of the things with what comes out.[82]

He takes $AB = \sqrt{a} + b$ a *binomium* of the 2nd or 5th type and $BC = \sqrt[3]{a} + \sqrt[3]{b}$ a cubic *binomium* (with $\sqrt[3]{ab}$ rational). We saw that in Cardano's opinion these are "general" solutions, meaning that they are written under certain codified shapes and can encompass other more "particular" shapes (like a rational number). The "common value" that Cardano is looking for may then be an even more encompassing shape, since it should also include those two irrational shapes. At first, Cardano hopes that the encompassing shape could be the sum of the two above irrational shapes. But then he realises that the only property of this

[81] "Then two values of the cube equal to some things and to a number are already known, one in the biggest part of the number and is [the value] of the cubic *binomium*, and the other in the smallest part of the number [and is the value] of the second or fifth *binomium* by square roots, and the common value that cannot be incommensurable, in fact they would have been commensurable between them, and the fourth [value is known], namely [the one] that is realised in the smallest part of the number. Therefore it is necessary that the common fails, as it is said, by conjunction" or "*Sunt ergo iam notæ duæ æstimationes cubi æqualis rebus et numero, una parte in maiore numeri, et est binomii cubici, alia in parte minoris numeri binomii ex R quadratis secundi vel quinti, et communis æstimatio quæ non potest esse incommensis, essent enim inter se commensæ, et quarta scilicet quæ intelligitur in parte minoris numeri, deficere igitur commune oportet, ut dicatur per coniunctionem*", see [Cardano 1570a, Chapter LVIII, page 106].

[82] "*Necesse est igitur ut sit quantitas communis genere non ab nec bc. Et hoc esse potest, nam animal est commune homini et asino et bovi et equo, ita ab et bc continentur sub communi aliqua quantitate, quæ donec communis est omnibus habet solam eam proprietatem, quod cum dividitur numerus simplex æquationis, per illam ipsam est R numeri rerum cum eo quo prodit*", see [Cardano 1570a, Chapter LVIII, page 106].

"common value" is to be a solution of $x^3 = a_1x + a_0$. This implies that the equality in Chapters XL and LVII $x = \sqrt{a_1 + \frac{a_0}{x}}$ holds, which is clearly enough (even though a part of the text is missing) checked in the two examples $\sqrt{8} + 2$ and $\sqrt[3]{4} + \sqrt[3]{2}$. Cardano ends[83] by remarking – as he already did in Chapter LVII – that the "nature [*natura*]" of (one of) the solution(s) must be the same of $\sqrt{a_1 + \frac{a_0}{x}}$.

Finally, I will very briefly deal with an application. In **Chapter XIV** "On the discovery of the kind of the value [*De inventione generis æstimationis*]" Cardano considers not only the equations of the family $x^3 = a_1x + a_0$ with rational coefficients, but also of the family $x^3 + a_0 = a_1x$. Here he works in the other way round and explains how to draw the numerical value of the coefficients, knowing that they must be rational and given a certain shape for a solution (namely, a *binomium* of the 2$^{\text{nd}}$ or 5$^{\text{th}}$ type for $x^3 = a_1x + a_0$ or a *binomium* of the 1$^{\text{st}}$ or 4$^{\text{th}}$ type and a *recisum* of the 2$^{\text{nd}}$ or 5$^{\text{th}}$ type for $x^3 + a_0 = a_1x$). Despite all the rambling discussions in the chapters of the *Aliza* that we have analysed above, here Cardano rather behaves in a very similar way to (AMA XXII.i) and (AMA XXII.ii), where he basically acted by substitution and comparison. My main point being only to shortly echo the substitution techniques from the *Ars magna arithmeticæ*, I will skip the details (which by the way are plainly understandable). It is interesting to remark that, as a sort of complement to the *Ars magna arithmeticæ*, Cardano also gives a proof of the formulae that he finds for the coefficients. As usual, he translate the hypotheses in a geometrical, non-positional environment. Moreover, in this case he only performs a sequence of steps completely parallel to the ones that he would have done if he had arithmetically drawn the coefficients.

[83]"Therefore, in both divisions, as you see, it is common that the *recisum* comes forth and, added the number of the things, it goes through a nature similar to the square of the thing. Therefore the number of the things changes its nature, which comes from the division of the number of the equality by the thing" or "*[c]ommune est ergo ut vides in utraque divisione prodire recisum, quod additum numero rerum, transeat in naturam similem quadrato rei. Numerus igitur rerum mutat naturam eius, quod provenit ex divisione numeri æquationis per rem*", see [Cardano 1570a, Chapter LVIII, page 107].

Summing up, in the context of Cardano's surveys on $x = y + z$ in the equation $x^3 = a_1x + a_0$, it turns out that he lingers for more than a few moments on this substitution, or rather on a topic strongly tied to it. As we expected – since we are acquainted with the parallel results in the *Ars magna arithmeticæ* – the idea of the substitution could have been a side effect of the study of the possible algebraic shapes for the irrational solutions of the equation $x^3 = a_1x + a_0$, with a_1, a_0 rational. We remark that appreciating whether these algebraic shapes can or cannot express a solution of an equation is one of Cardano's most recurring (and probably ancient) strategies in dealing with equations. In fact, concerning that equation, it comes to light that an irrational shape for one of its solutions must contain two parts (which in Section 4.2 we used to call y and z) that bear certain relationships between them. We recall that, in the end, these shapes are the *binomium* of the 2^{nd} or 5^{th} type $\sqrt{a} + b$ and the cubic *binomium* $\sqrt[3]{a} + \sqrt[3]{b}$ with $\sqrt[3]{ab}$ rational. Nowadays, it is clear that, for instance, if $\sqrt{a} + b$ is a solution of the equation $x^3 = a_1x + a_0$, the associated polynomial is reducible. Moreover, another solution is $-\sqrt{a} + b$ and the third solution is rational. Anyway, Cardano does not have the concept of irreducible polynomial available so that he must analyse the coefficients of the equation and their shapes. The same happens for $\sqrt{a} + \sqrt{b}$. We know that it cannot be a solution of $x^3 = a_1x + a_0$ (since the splitting field of the equation is of order 4, which does not divide 6), but Cardano needs to show that he gets an impossible system. The link between the substitution $x = y + z$ and the possible shapes for the irrational solutions is firstly made at the very beginning of Chapter I. Afterwards, we meticulously went and searched for the subsequent chapters in which the topic is developed. The messiness of the *Aliza*'s structure strongly appears.

The stages of Cardano's survey are the following. One stage, which explicitly recalls the substitution $x = y + z$ and is linked to Chapter I, is Chapter XI. There, Cardano's classification of y, z according to the Euclidean terminology for irrational numbers is employed in order to count how many parts there are in x^3. Afterwards, Cardano compares them to the rational and irrational parts in $a_1x + a_0$ and draws some information on how many parts there are in x. But Cardano's conclusions turn out to be sometimes inaccurate and the Euclidean terminology prevents him from having a clear view.

Another stage is Chapter X (which also recaps the result in Chapter V). There, by substitution and comparison, Cardano counts out many possible shapes for the irrational solutions of the considered equation.

An intermediary stage could be the surveys in Chapters XIII and XVI. We still have a few remainders of the Euclidean terminology (even if not as massively as in the classification in Chapter XI), but the more effective method by substitution and comparison is also employed to respectively count out all the *trinomia* and *quadrinomia* (and, accordingly, the *multinomia* with more than five terms). Chapter XIV attests that the techniques of the *Ars magna arithmeticæ* are still in use.

Furthermore, another stage is Chapter XXIV, where the guidelines are (AM XXV.1) and the third corollary to (AM XIII). There, Cardano wants to study the irrational part of a shape for an irrational solution.

Finally, but strictly related with the preceding one, there is the stage in which (AM XXV.2)) is employed in order to detect an even more encompassing shape that should account for both $\sqrt{a}+b$ and $\sqrt[3]{a}+\sqrt[3]{b}$. This is made in Chapters LVII and LVIII.

The study of the shapes for the irrational solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational, also helps in giving a (relative) estimate in time for the concerned chapters. In fact, it seems that these surveys come before the discovery of the cubic formulae. This is at least true for Chapter XI, where Cardano strikingly counts out some shapes that actually are possible shapes (and maybe he would have not missed them, if he had had the cubic formula available), and for Chapter XXIV, where Cardano does not still seem to be aware of Ferrari's method for solving quartic equations. But in my opinion also the results in Chapters V, X, XIII, and XVI are prior, or at worst contemporary, to the cubic formulae. In fact, they are fully comparable (when not akin) to some of the results in the *Ars magna arithmeticæ* that I have analysed in Section 3.2.4. We recall the interpretation of the corresponding part of the *Ars magna arithmeticæ* at page 198, where I have argued that the study of the shapes of the irrational solutions is an access point to the structure of the solutions themselves, and then to the formula. These (possibly previous) surveys are recovered in the *Aliza* with an additional aim: justify the fact that a solution x of $x^3 = a_1x + a_0$ is composed by two parts $x = y + z$. This, in turn, leads to the method of the splittings, which are intended to amend the cubic formula.

4.3.2. On the origin of the splittings: a hypothesis in relation with *Ars Magna*, Chapter XXV. As we have seen in Section 2.1.4, the "particular" rules in *Ars magna*, Chapter XXV are very tightly associated with (AM VIII.2). We will analyse the chapters of the *Aliza* that refer to (AM VIII.2) in Section 4.4.2 at page 311. Nonetheless, I choose to deal here with the chapters of the *Aliza* that concern *Ars magna*, Chapter XXV because they are in a stronger connection with *Aliza*, Chapter I. We moreover recall that this connection (or, at least, the connection between *Ars magna*, Chapter XXV and the "*aliza* problem") already existed in *Ars magna*, Chapter XII (see above, at page 112).

In **Chapter II** "On the special rules of Chapter XXV of the *Ars magna* of the cube equal to some things and a number [*De Regulis specialibus Capituli XXV Artis magnæ cubi æqualis rebus et numero*]" Cardano deals again with all the "particular" rules in *Ars magna*, Chapter XXV that concern $x^3 = a_1x + a_0$ (see here, Section 2.1.4 at page 84). The chapter's start is upsetting. The first paragraph looks like a proof, but I have not been able at all to recover what Cardano wants to prove. In fact, on the one hand, he never says what his aim is. Worst, on the other hand, the proof itself is completely unclear. It has quite a lot of references to a certain diagram supposedly filled up with many letters (from A to O). Unluckily, the diagram is displayed neither there (in both editions), nor elsewhere (namely, in *Ars magna*, Chapter XXV or in *Aliza*, Chapter I, which are both quoted in the rest of the chapter, or in *Ars magna*, Chapter VIII, which could have also been a reasonable option), and I have not been able to retrieve it from the text. Without the diagram I could not understand the paragraph, which is massively based on it. This could be an evidence of the fact that the *Aliza* was published in a hurry, since maybe the publisher had not realised that he also needed to reproduce the diagram.

Anyway, at the end of this paragraph, Cardano gets the equation $x^3 = a_1x+a_0$, and then a sequence of propositions follow.

A II.1. *The*[84] *system in the proposition (AM XXV.1) does not come from any splitting in* Aliza*, Chapter I.*

We remind in fact that, while commenting (AM XXV.1) at page 85, we did not bring it back to the substitution $x = y + z$.

Then, Cardano says that, from the demonstration at the beginning of Chapter II, it follows that $x^3 + a_0 = a_1 x$ is "simpler [*simplicius*]" than $x^3 = a_1 x + a_0$ and that one can pass from the knowledge of the first equation to the knowledge of the second.[85] He implicitly makes reference to the system common to (AM VIII.2) (or (A V.ii) for the particular case of $x^3 + a_0 = a_1 x$, see below at page 314) and to (AM XXV.1). The explanation is not very well detailed, but one can observe that the solution given in the first case, namely $\sqrt{g}$, is regarding to its shape "simpler" than the one given in the second case, namely $\sqrt{f + \frac{g}{4}} + \frac{\sqrt{g}}{2}$. Cardano's statement is in any case an evidence of the fact that he was aware of the connection between (AM VIII.2) and (AM XXV.1), as I have supposed at page 112.

Finally, Cardano remarks that "this rule is not general by itself to the whole chapter of the cube equal to some things and a number [...]. But it is general to the chapter of the cube and a number equal to some things [*Hæc etiam regula non est generalis per se toti capitulo cubi æquali rebus et numero, quia ubi numerus esset maior non satisfaceret, sed est generalis capitulo cubi et numeri æqualium rebus*]". I cannot account for this last remark, except assuming that the meaning of 'general' is not always well fixed. We recall, for instance, that we have already found a likely incongruity in the *Ars magna*, where Cardano calls Chapter XII "general" (and this disagrees with his characterisation of the term in the rest of the book, see above at page 112). Maybe something similar is happening here in the *Aliza*, but I cannot provide any reasonable interpretation of "general" under which the above quotation makes sense.

[84] "This demonstration shows that this chapter [$x^3 = a_1 x + a_0$] does not arise from those seven ways [in Chapter I]" or '*[H]demonstratio ostendit quod hoc capitulum non oritur ex illis septem modis*", see [Cardano 1570a, Chapter II, page 5].

[85] "From this it also follows that the chapter of the cube and a number equal to some things is simpler and from it [it is] more easy to [get to] the knowledge in the chapter of the cube equal to some things and a number" or "*[e]x hoc etiam sequitur quod capitulum cubi et numeri æqualium rebus est simplicius, et ex se magis obvium cognitioni capitulo cubi æqualium rebus et numero*", see [Cardano 1570a, Chapter II, page 5].

A II.2. *The system in the Proposition (AM XXV.2) does not come from any splitting in* Aliza*, Chapter I.*

Again, when commenting on (AM XXV.2) at page 86, we did not bring it back to the substitution $x = y + z$. Rather, Cardano explained that the idea behind (AM XXV.2) is to write $g^3 = (a_1 + f)g$. The same explanation is also recalled here in the *Aliza*.

A II.3. *The system in the Proposition (AM XXV.3) comes from the splitting (A I.3).*

A II.4. *The system in the Proposition (AM XXV.4) comes from the splitting (A I.4).*

A II.5. *The system in the Proposition (AM XXV.5) comes from the splitting (A I.2).*

At page 86, we have already displayed a way to rephrase the three propositions of the *Ars magna* that highlights the substitution $x = y + z$. Then, the three above propositions are clear, if one compares the way in which they have been rephrased with the propositions at issue in *Aliza*, Chapter I (respectively at pages 244, 245, and 246).

A II.6. *The system in the Proposition (AM XXV.6) does not come from any splitting in* Aliza*, Chapter I.*

Cardano calls the Proposition (AM XXV.6) "special [*specialis*]". In fact – as we have already remarked in Section 2.1.4 at page 89, that proposition is mainly meant to perform polynomial division and, as such, stands aside from the others, since it does not depend on (AM VIII.2).

A II.7. *The system in the Proposition (AM XXV.7) comes from the system (AM XXV.5)) [or from the splitting (A I.2)].*

Unluckily, in Cardano's further commentary on the above proposition, his description[86] does not fit with (AM XXV.5), though he quotes the very same example

[86] "The seventh [rule] arises from the the fifth [one]. But it is seen that [it is] different from that, because in that [fifth rule] the whole root, namely $\sqrt{28 - 3x^2}$, is supposed, in this [seventh rule] the half [of the root], [namely] $\sqrt{7 - \frac{3}{4}x^2}$, [is supposed]. And because in one [fifth rule]

$x^3 = 7x + 90$. Actually, the system in (AM XXV.7) seems rather to come from the system in (AM XXV.4), or in (A I.4), where the shape of the term of degree zero of the equation is $a_0 = y^2z + yz^2$.

A II.8. *The system in the Proposition (AM XXV.8) comes from the splitting (AM XXV.3) [or (A I.3)].*

Cardano does not provide any detailed argument, but in this case the correspondence holds, since the term of degree zero of the equation has the shape $a_0 = 2y^2z + 2yz^2$.

Quite far from Chapter II, another whole chapter of the *Aliza* is devoted to *Ars magna*, Chapter XXV. It is **Chapter XXV** "On the examination of the third rule of Chapter XXV of the *Ars magna* [*De examine tertiæ regulæ Capituli XXV Artis magnæ*]". More precisely, it is devoted to the comparison of the "special,[87] non-general rule [*regula specialis non generalis*]" (AM XXV.3) with the "general, non-special rule [*regula generalis non specialis*]" (AM XII). We remind that both rules are about $x^3 = a_1x + x_0$ and that the second one gives the cubic formula.

Note that here Cardano does not explicitly mention (AM XII) here. Anyway, in the examples, it is clear that he is using the cubic formula. In fact, instead of immediately providing the integer solution (note that in this chapter all the equations, except the last one, have only one real solution, which is 6), Cardano writes it at first as the sum of two cubic radicals. This means that he should not have guessed the solution, but he is possibly using the cubic formula. Or rather, when Cardano details the calculations, we notice that he is not exactly using (AM XII), but instead a proposition that is directly implied by it. Maybe for the comparison's sake, this proposition is stated in the same style as (AM XXV.3). Here it follows.

the parts are reciprocally multiplied by the squares, in the other [seventh rule] the aggregate [is multiplied] by the product of the parts. Nevertheless they are the same as it is demonstrated in the book on Proportions" or "*[s]eptima oritur ex quinta, sed videtur ab ea diversa, quia in illa supponitur R tota scilicet RV: 28 m: 3 quad., in hac dimidium R 7 m: $\frac{3}{4}$ quad.. Et quia in una ducuntur partes mutuo in quadrata, in alia aggregatum in productum partium; sunt tamen idem ut demonstratum est in Libro de proportionibus*", [Cardano 1570a, Chapter II, page]. I could not find the exact reference to the *De proportionibus*.

[87]See the remarks at pages 97 and 112 concerning the terms "particular" and "general" referred to these rules

A XXV.i. *Consider* $x^3 = a_1 x + a_0$.
If

$$\begin{cases} a_0 = f + g \\ \left(\frac{a_1}{3}\right)^3 = fg \end{cases},$$

then $x = \sqrt[3]{f} + \sqrt[3]{g}$.

If we take $f = \frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}$ and $g = \frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}$, the conditions in the system are fulfilled. Then, (A XXV.i) is implied by (AM XII).

Let us come back to Cardano's comparison between (AM XXV.3) and (AM XII) (or rather (A XXV.i)). He says that some equations of the family $x^3 = a_1x + a_0$ undergo to (AM XXV.3), others undergo to (AM XII), and finally there are certain equations that undergo to both.[88] He explains that in the first case $a_1 > a_0$, whereas in the second case $a_1 < a_0$. Cardano does not say that the third case is when $a_1 = a_0$, but we are led to suppose it. In fact, the numerical equation that he mentions, namely $x^3 = 6x + 6$, pertains to this case, since 6 is not a solution of it, despite $a_1 = a_0$. Anyway, even dismissing the alleged condition on the third case, the remaining conditions are inaccurate.

In fact, Cardano further specifies that (AM XII) (or rather (A XXV.i)) is not useful for $x^3 = 22x + 84$, since the equation has $\Delta_3 < 0$. Unluckily, the example is very badly chosen, because the equation actually has $\Delta_3 > 0$. Since Cardano's explanation is not ambiguous,[89] I consider that the condition in (AM XII) is the usual one on the discriminant, namely $\Delta_3 \geq 0$ or $\left(\frac{a_0}{2}\right)^2 \geq \left(\frac{a_1}{3}\right)^3$. On the other hand, Cardano says that (AM XXV.3) is not useful for $x^3 = 17x + 114$ (and for $x^3 = 15x + 126$) referring to a certain boundary given by a maximum problem.

[88] "Therefore there are three parts in this chapter, the first that serve the special, non-general rule when the number of the things is bigger in comparison to the number of the equality. The second that serve the general, non-special rule when the number of the equality is bigger in comparison the the number of the things. The third that serve both" or "*tres sunt partes in hoc capitulo, prima quæ servit regulæ speciali non generali, cum numerus rerum est magnus in comparatione numeri æquationis. Secunda quæ servit regulæ generali non speciali cum numerus æquationis est magnus comparatione numeri rerum. Tertia quæ servit utrique*", see [CARDANO 1570a, Chapter XXV, page 52].

[89] "[T]he general rule cannot reach $x^3 = 22x + 84$, because 21, the fourth part of 84, makes a square neither bigger nor equal to the cube of the third part of the things $\frac{22}{3}$" or "*non potest regula generalis attingere ad 1 cub. æqualem 22 rebus p: 84, quia 21 quarta pars 84 non facit quadratum, neque maius neque æquale cubo* $7\frac{1}{3}$ *tertiæ partis rerum*", see [CARDANO 1570a, Chapter XXV, page 52].

He knows that, if $f+g=a_1$, the maximum value of $\sqrt{f}g+f\sqrt{g}$ is attained when $f=g=\frac{a_1}{2}$ (and is $a_1\sqrt{\frac{a_1}{2}}$).[90] Finally, we remark that, even though there is a connection between (AM XXV.3) and (A I.3) (highlighted for instance in Chapter II), the condition in (AM XXV.3) is not comparable with the one stated in (A I.3), since it does not involve a_1x. In the two above examples, this maximum is strictly smaller than $\frac{a_0}{2}$, so that one cannot find those (positive) f,g. Then, we conclude that (AM XXV.3) has the condition $a_1\sqrt{\frac{a_1}{2}}\geq\frac{a_0}{2}$, that is $\frac{a_1^3}{2}\geq\left(\frac{a_0}{2}\right)^2$. Let us resume in the following diagram the conditions stated in (AM XXV.3) and (AM XII) (or rather (A XXV.i)).

[90]Cardano justifies his assertion on the maximum value through a cross reference to *De proportionibus*, Proposition 209: "If it appears [that] a rectangular surface [is] divided in two equal parts, which are both squared, likewise in two unequal [parts], the parallelepiped from the side of the middle part times the whole surface will be bigger than the aggregate of the parallelepipeds from the unequal parts times the sides of the other part by what is reciprocally made from the difference of the side of the smaller part from the side of the middle [part] times twice the difference of the bigger part of the surface from the middle [part] of the surface and from the difference both of the unequal sides joined to both sides and of the equal [sides] joined times the smaller part of the surface" or "*[s]i superficies rectangula in duas partes æquales divisa intelligatur, quæ ambę quadratæ sint, itemque in duas inæquales, erit parallelepipedum ex latere mediæ partis in totum [totam] superficiem maius aggregato parallelipedorum ex partibus inæqualibus, in latera alterius partis mutuo in eo, quod fit ex differentia lateris minoris partis a mediæ latere in differentiam maioris partis superficiei a media superficie bis, et ex differentia amborum laterum inæqualium iunctorum ad ambo latera, æqualia iuncta in minorem partem superficiei*", see [Cardano 1570c, pages 241-242]. It says that, if we take three real numbers u,v,w such that $2u=v+w$, then

$$2u\sqrt{u}=v\sqrt{w}+w\sqrt{v}+\big(2(\sqrt{u}-\sqrt{v})(w-u)+(2\sqrt{u}-\sqrt{v}-\sqrt{w})v\big)\,.$$

This implies that the maximum value of $\sqrt{f}g+f\sqrt{g}$ is $2u\sqrt{u}$, and is attained when $v=w=u$. A few words on the proof. As usual, it employs the language of geometry, and mainly consist in drawing an effective diagram thanks to which the quantity at issue are identified. Then, readjusting the parts in the diagram, the proposition follows easily. The point is to make a good assignment of the quantities to the geometrical objects. More precisely,

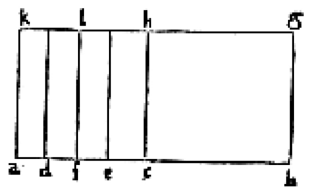


Figure 4.7 – *De proportionibus*, Proposition 209.

Cardano takes $(\overline{AH})^2=(\overline{HB})^2=u$, $\overline{AC}=\overline{CB}=\sqrt{u}$, $(\overline{KD})=v$, $(\overline{DG})=w$, $\overline{AF}=\sqrt{v}$, and $\overline{EB}=\sqrt{w}$. Then, he only needs to draw $\overline{FC}=\sqrt{u}-\sqrt{v}$, $\overline{DH}=w-u$, and $\overline{FE}=2\sqrt{u}-\sqrt{v}-\sqrt{w}$, and this can be done by looking at the diagram.

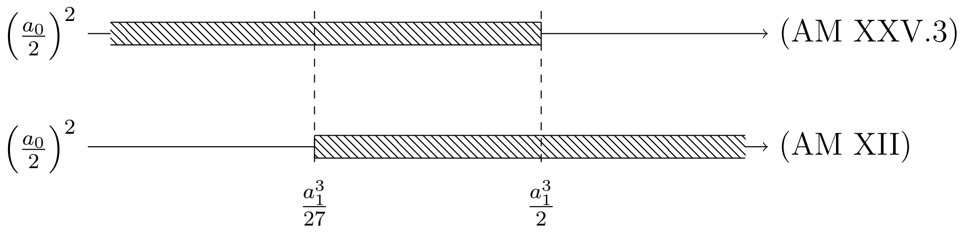


FIGURE 4.8 – Comparison between the conditions on (AM XXV.3) and (AM XII) (or rather (A XXV.i)) in *Aliza*, Chapter XXV.

The final aim of the comparison concerns the equations that fall under the third case, for which Cardano gives the following proposition.

A XXV.ii. *Consider* $x^3 = a_1 x + a_0$.

When both (AM XXV.3) and (AM XII) [or rather (A XXV.i)] apply, then f, g *in (AM XXV.3) can be obtained in the following way: find a solution* α *of the equation and take*

$$\sqrt{f} = \frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \frac{a_0}{2\alpha}} \quad \textit{and} \quad \sqrt{g} = \frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \frac{a_0}{2\alpha}}.$$

Note that this proposition is in truth (AM XXV.8), since $a_1 = \alpha^2 - \frac{a_0}{\alpha}$ holds. By the way, the link between the Propositions (AM XXV.3) and (AM XXV.8) has already been highlighted in (A II.8).

We observe that the point of the proposition cannot obviously be to draw a solution of the equation (note that all the examples that Cardano provides have $\Delta_3 > 0$, so he perfectly knows how to draw a solution). Rather, his aim here is to study the shape of f, g in (AM XXV.3). If one wants to find a solution to the problem entailed by the *casus irreducibilis*, this could be of a certain importance, since, when (AM XII) does not apply (that is, when $\Delta_3 < 0$), (AM XXV.3) still applies.

Finally, In **Chapter LIX** "On the order and on the examples in the second and fifth *binomia* [*De ordine et exemplis in binomiis secundo et quinto*]" Cardano provides two methods to solve two particular classes of equations of the family $x^3 = a_1 x + a_0$. He does not explicitly state the methods as such, but he provides

69 numerical equations with integer coefficients, divided in five lists, from which we gather two rules.

Firstly, from the first four lists we draw the following proposition.

A LIX.i. *Set* $i = 1, 2, 3$ *or* 4 *and consider the family of equations*

$$x^3 = (n + i^2)x + ni$$

for any integer $n \geq 0$. *Then* $x = \sqrt{\frac{4n+i^2}{4}} + \frac{i}{2}$.

In the examples, Cardano limits himself to consider $0 \leq n \leq 16$ when $i = 1, 2, 3$ and $0 \leq n \leq 17$ when $i = 4$. Note that the proposition can be trivially extended to any natural i. Moreover, we remark that all the equations $x^3 = (n + i^2)x + ni$ have the same (negative) rational solution, which is $-i$, and that thereby they have three distinct real solutions (otherwise, note that the term under the square root is $4n + i^2 \geq 0$). All the equations have then $\Delta_3 < 0$. Therefore, this proposition enables Cardano to devise a considerable number of equations that he is able to solve with $\Delta_3 < 0$.

In my opinion, it is very likely that Cardano comes to this proposition using (AM XXV.1). On the one hand, we can easily verify that one can use (AM XXV.1) to derive (A LIX.i), taking $n = f$ and $g = i^2$. On the other hand, Cardano himself recalls the algorithm of (AM XXV.1), even if he does not explicitly mentions the proposition (he only says "in the previous [*in priore*]"). It is in the fourth corollary to this proposition, where he compares, through an example, the formulae given in (A LIX.i) and in (AM XXV.1), and remarks that they lead to the same result.[91]

[91] Concerning the equation $x^3 = 19x + 30$, Cardano says that "[i]n the following, we will take 1, the fourth part of 4, and we will add [it] to the difference 15, 16 is made, the root of which is 4 added. It constitute the value 5. In the previous, we will add $\frac{9}{4}$, the fourth part of 9, to the difference 10, $\frac{49}{4}$ is made, the root of which is $\frac{7}{2}$, added $\frac{3}{2}$, the half of 3, [which is] $\sqrt{9}$, the value of the thing 5 is made as before" or "*[i]n posteriore accipiemus 1, quartam partem 4, et addemus ad 15, differentiam fit 16, cuius R, quæ est 4, addito constituit ęstimationem 5. In priore addemus 2*$\frac{1}{4}$*, quartam partem 9, ad 10 differentiam, fit 12*$\frac{1}{4}$*, cuius R, quæ est 3*$\frac{1}{2}$*, addito 1*$\frac{1}{2}$*, dimidio 3, R 9, fit 5, ut prius rei æstimatio*", see [CARDANO 1570a, Chapter LIX, corollary 4, page 109].

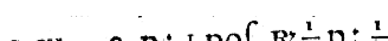
Out of the other three corollaries, two of them describe the formula in (A LIX.i). The first is a commentary on the term '$\frac{i}{2}$' in the formula and the third[92] points out that all the equations have $\Delta_3 < 0$. In the second corollary, Cardano states that "the chapter cannot be general, because the first number is necessarily a square [*capitulum non potest esse generale, quia primus numerus necessario est quadratus*]". It happens in fact that, when $n = 0$, then $a_1 = i^2$ (as Cardano has already remarked at the very beginning of the chapter). He spends some words explaining why a_1 is an (integer) square, whereas its connection with the non-"generality" of the chapter seems to be obvious for him. Then, we can guess that (A LIX.i) is not "general", since it concerns the class of equations $x^3 = (n + i^2)x + ni$, which is not "general", because the coefficients are not so.

From the fifth and last list of equations we gather the following proposition.

A LIX.ii. *A solution of the family of equations* $x^3 = nx + (\alpha^3 - \alpha n)$ *is* α *for any integer* $n \geq 0$.

In the examples, Cardano only considers $0 \leq n \leq 16$ and $\alpha = 6$ (but the proposition can be trivially adjusted in order to return any natural value for the solution α instead of 6). No corollary concerns this proposition. It likely comes from the rephrasing $a_0 = \alpha^3 - a_1\alpha$.

We remark that this chapter is the only passage (in all the texts that we have analysed) where Cardano writes that a coefficient can be '0'. Namely, it happens in the first line of each list. For instance, we find in the first list:


1 cu. 0 p: 1 pos. R 1/4 p: 1/2

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[92] Unluckily, by a non completely clear argument: "the number of the equality can never be increased to such a point that the square of the half of that is bigger than the cube of the third part of the number of the things. In fact, then by the first rule the value would be the cubic *binomium* and by that rule [(A LIX.i)] the squared *binomium*, and thus one would be equal to the other. Which can be allowed [...]. Nevertheless it cannot be extended and the value is loosened in an integer number" or "*nunquam numerus æquationis potest adeo augeri, ut quadratum dimidii eius sit maius cubo tertiæ partis numeri rerum; nam tunc per primam regulam fieret æstimatio binomium cubicum. Et per hanc regulam binomium quadratum, et ita unum æquale esset alteri. Quod licet esse possit, ut in hoc exemplo RV: cu.20 p: R 329 [392] p: RV: cu. 20 m: R 392, et est 2 p: R 2 et 2 m: R 2, quod est 4, non potest tamen continuari, et æstimatio resolvitur in numerum integrum*, see [Cardano 1570a, Chapter LIX, corollary 4, page 109]. The "first rule" maybe refers to (AM XII), as it is stated in the *Ars magna*, or to (A XXV.i).

FIGURE 4.9 – First line of the first list of equations in *De regula aliza*, Chapter LIX.

In these lists, the equations of the family $x^3 = a_1x + a_0$ are usually written in a shortened form. In the first column there is x^3, while in the second and third columns there is $a_0 + a_1x$. In each line of the lists, between the first and the second columns, we should add '=' in order to correctly read the equation. Note then that the '0' in the second column of the above diagram means '$a_0 = 0$'.

Summing up, Chapters II, XXV, and LIX of the *Aliza* are characterised by the comparison with *Ars magna*, Chapter XXV. Chapter II associates *Ars magna*, Chapter XXV with *Aliza*, Chapter I, while Chapter XXV is on the shape of f, g in (AM XXV.3) (not asking this time for rational coefficients), and Chapter LIX is mainly meant to provide a list of equations with rational coefficients and $\Delta_3 < 0$ that Cardano is able to solve.

These chapters (together with the chapters considered in Section 4.4.2) have in common the interest for those "particular" rules that enable one to solve an equation by finding two numbers f, g that fulfil certain requirements. Sometimes the *Ars magna* is explicitly mentioned, sometimes not. Whether all these chapters of the *Aliza* had been written before or after the *Ars magna* is – as said – a matter up to discussion.

It is likely that Cardano conceived the propositions in *Ars magna*, Chapter XXV starting from the study of the irrational solutions of equations with rational coefficients. In fact, on the one hand, for the most of the time Cardano's equations only sporadically have irrational coefficients. On the other hand, the propositions in *Ars magna*, Chapter XXV (and the chapters of the *Aliza* that link them with the splittings, or that analyse some of them in details) have been refined – in my opinion – after that Cardano realised that the cubic formulae had a problem. Note that indeed there is nothing similar in the *Practica arithmeticæ* or in the *Ars magna arithmeticæ*, and that in *Ars magna*, Chapter XII Cardano himself says that the propositions of Chapter XXV are a temporary relief to the *casus irreducibilis*, while the *Aliza* should be the conclusive treatment. Then, I guess that some of the splittings, namely (A I.2)-(A I.4), could have been originated by rethinking some propositions in *Ars magna*, Chapter XXV, namely (AM XXV.3)-(AM XXV.5).

4.3.3. Cardano's last say on the origin of the splittings. Up to now, I have tried to gather from the *Aliza* as much information as possible concerning the origins of the substitution $x = y + z$ and of the splittings. All my hypotheses had a mainly arithmetical source. Nevertheless, there could also have been a loose geometrical inspiration at the origin of Cardano's conjectures.

We can track it down in **Chapter LX** "General demonstration of the chapter of the cube equal to some things and a number [*Demonstratio generalis capituli cubi æqualis rebus et numero*]". Just by reading the title, we have great expectations from this chapter. Unluckily, it belongs to that crowded bunch of not-so-clear chapters of the *Aliza*. Its abstract speeches are quite murky, even though it is not as badly written as other chapters are. Luckily, there is also a part that contains some calculations, and this turns out to be of a certain importance. Cardano's start seems to be very promising.

> And, since there is this special rule that concerns the value, therefore it is also not remarkable if it is also special in the way of the discovery, supposing a squared number. Therefore, in order to be generally considered, we propose [...].[93]

It is not clear to which "special" (or "particular") rule Cardano is referring to. As we have seen at page 290, the foregoing Chapter LIX implicitly refers to (AM XXV.1) and the rules in *Ars magna*, Chapter XXV are considered to be "particular". But this is a poor connection. I rather believe that a more suitable option could be a generic reference to the rules in *Aliza*, Chapter I (to which the term "particular" can also be applied, and some of which are indeed equivalent to some of the rules in *Ars magna*, Chapter XXV, as Cardano himself remarked in the *Aliza*, see Section 4.3.2). In fact, in the following – as we will see – Cardano assumes that $x = y + z$, which is closer to the hypotheses in *Aliza*, Chapter I, rather than to those in *Ars magna*, Chapter XXV. Apart from identifying the exact reference, the rough idea that the very first lines of Chapter LX convey

[93]"*Et cum sit regula hæc quòd ad æstimationem attinet specialis, ideò etiam non mirum est si sit etiam specialis in modo inveniendi, cum supponat numerum quadratum. Ergo ut generaliter consideretur proponamus [...]*", see [Cardano 1570a, Chapter LX, page 109].

is that Cardano wants to explain the "general" way through which he got the relevant "rule" – whatever it may be. And – as we are now too much aware of – the passages in which Cardano speaks about what he is going to do are a real gold dust.

Cardano considers the equation $x^3 = a_1x + a_0$ and rewrites it as $x^2 = a_1 + \frac{a_0}{x}$. We have seen that it is one of his usual techniques. Then, he distinguishes three cases, depending on $a_0 = x^2$, $a_0 > x^2$, or $a_0 < x^2$. No commentary is given in the first case. In the second case Cardano provides a geometrical interpretation, where $\overline{AB} = x$ and $\overline{AB}\,\overline{AD} = a_0$.

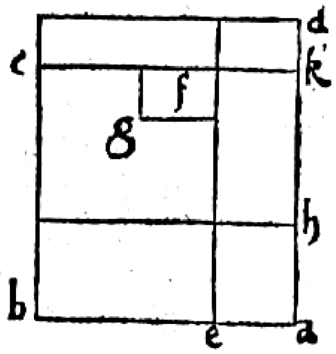

Figure 4.10 – *Aliza*, Chapter LX.

Unfortunately, the description of this case is imprecise and sometimes simply borders on being inexplicable. For instance, Cardano takes $(\overline{BD}) > (\overline{AE})^2\,(\overline{DE})$, that is $\overline{AB}\,\overline{AD} > \overline{AE}^3\overline{AD}$, which is quite a bizarre statement regarding to dimensional coherence, since it is a comparison between a rectangle and a four-dimensional body.[94] According to the diagram and to what he makes in the third case, he should have taken instead $\overline{AB}\,\overline{AD} > \overline{AC}^2$. Then, we drop this case, since this is only the first of a very long list of incongruities. Let us turn to the third case, which is much more sensible. In fact, we even happily manage to retrace

[94]It is very likely that the text has been corrupted, since we find 'AE' instead of the expected 'AC' (and exchanging 'E' with 'C' is a common – maybe the most common – misprint in the *Aliza*). Anyway, it must have been corrupted in a massive way, since it is not only a matter of misprinting a type, but there is also the multiplication by $(\overline{DE})$ that makes no sense.
We remark that in the 1663 edition the letters in the description of the second (and of the third) cases are capitalised, as if someone revised the passage. This is a unique case in the 1663 edition of the *Aliza*. There, the above misprint has been emended. In fact, we find $(\overline{BD}) > (\overline{AC})^2(\overline{DE})$ instead of $(\overline{BD}) > (\overline{AE})^2(\overline{DE})$. Nevertheless, the dimensional problem persists.
This could be another sign of the fact that the publisher did not proofread the draft of the *Aliza*, or that he proofread it very quickly.

the description of the second case by analogy with the description of this third case. Since the above diagram is indeed two diagrams in one (for the second and third cases), I will only copy the part that concerns the third case.

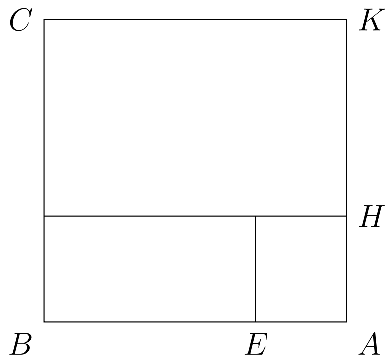

FIGURE 4.11 – *Aliza*, Chapter LX (partial).

Cardano takes $\overline{AB} = x$ and $\overline{AB}\,\overline{AH} = a_0$ (so that morally $\overline{AH} = \frac{a_0}{x}$). Anyway, since at the beginning he stated that he wanted $\frac{a_0}{x}$ to be a surface, he assumes $\overline{AE} = 1$ and $\overline{AH}\,\overline{AE} = \frac{a_0}{x}$. Since $x^2 = a_1 + \frac{a_0}{x}$, Cardano knows that $\overline{AB}^2 = a_1 + \overline{AH}\,\overline{AE}$, hence by the diagram $a_1 = (H\,C\,E)$. Then,

$$(H\,C\,E) - \overline{AB}\,\overline{AH} = (\overline{HK}\,\overline{AB}) - (\overline{AH}\,\overline{AE}).$$

Let us write $y = \overline{AH} = \frac{a_0}{x}$ and $z = \overline{HK}$ such that $x = y + z$. The last equality can be rewritten as

$$a_1 - a_0 = xz - y.$$

Moreover, we have that $xy = x\frac{a_0}{x} = a_0$. To show that it is correct, Cardano repeats the whole procedure through an example in which he assumes $x = \sqrt[3]{12}+2$ and $a_1 - a_0 = 3$, drawing the values of y, z and verifying that he falls back on the good values for $a_1 - a_0$ and a_0.

Afterwards, Cardano converts the above procedure into the following statement.[95]

A LX. *Consider the equation* $x^3 = a_1x + a_0$.

[95]"[F]ind the quantity that can be divided in two parts so that the product of the wholeductum totum by one produces, for example, 3 and [the product of the whole] by the remaining [part] added the preceding [one] produces, for example, 8" or "*[i]nvenias quantitatem quę possit dividi in duas partes ut ductum totum in unam producat 3. Gratia exempli, et in reliquam partẽ addito priore producat 8 pro exemplo*", see [CARDANO 1570a, Chapter LX, page 111].

If [there are two positive, real numbers y, z such that]

$$\begin{cases} xy = a_0 \\ xz - y = a_1 - a_0 \end{cases},$$

then $x = y + z$.

Cardano says that this problem can be settled "by the Rule *de modo* and [*de*] *positione* [*ex regula de modo et positione*]".[96] Then, the chapter (and the book) end with some cryptic remarks on huge fractions.

It is extremely interesting to have a closer look at the system in (A LX). The proposition is similar to the ones in *Aliza*, Chapter I, but it cannot be retrieved with the same method. It happens nevertheless that, once that one makes the substitution $a_0 = (y+z)y = y^2 + yz$ in the second line of the system, one gets $a_1 x = y^3 + 3y^2 z + 3yz^2 + z^3 - y^2 - yz$. Obviously, the sum $a_1 x + a_0$ gives $(y+z)^3$, but one can hardly argue that Cardano gets the system *via* the formula for the cube of a binomial, as it happened in Chapter I. In my opinion, things should have gone the other way round, that is to say, that Cardano gets the idea of splitting $(y+z)^3$ from jiggering with the system in (A LX) (or with a similar one).

In this way the problem becomes to know where the Proposition (A LX) comes from, and in this chapter Cardano himself gives an answer. The idea behind (A LX) – as we have said – is to rewrite the equation as $x^2 = a_1 + \frac{a_0}{x}$ and to interpret in the usual geometrical environment $\frac{a_0}{x}$ as a surface. Since Cardano knows very well how to sum two surfaces (but not how to sum a surface and a line), it is quite a natural choice to take $\frac{a_0}{x}$ as a surface. Since it is very easy to divide a square the side of which is x in two rectangles, the most obvious choice would have been to take a_1 one rectangle and $\frac{a_0}{x}$ the other. But it leads nowhere. Cardano has then the idea to consider the unity segment AE and in this way he manages to interpret $\frac{a_0}{x}$ as the surface $\overline{AHAE}$. Finally, an appropriate division of the square leads to the division of its side in $AH = y$ and $HK = z$.

[96]The Rule *de modo* is a method to solve systems usually with two equations in two unknowns, see [Gavagna 2010, pages 71-74]. The rule *de positione* is simply to assign the unknown to a certain quantity in a problem; this can be done in many ways, see for instance [Cardano 1545, Chapter XXXII].

This last chapter of the *Aliza* contains a hint on where the splittings at the beginning of the book could have been derived from. It is reasonable to expect that Cardano does not pull such a method out of the bag (for instance, considering a symbolic writing, which he did not have available). Instead, this chapter shows that also a kind of figurative (or geometrical) thinking plays a role. Still, the role of geometry in Chapter LX is very restricted. Saying it better, the geometrical argument here involves reflections on the dimension of the geometrical objects at issue. It leads to assume that $\frac{a_0}{x}$ is a surface. Then, it is a very specific kind of geometry, since it does not essentially depends on the position of its objects.

4.3.4. Studying the splittings by themselves. Finally, I will deal with a few chapters of the *Aliza* in which the splittings are considered from a different viewpoint, becoming a subject of inquiry by themselves.

In **Chapter VII** "On the examination of the values assumed by the second and third rules of the first chapter [*De examine æstimationum sumptarum ex regula secunda et tertia primi capituli*]" Cardano analyses one of the splittings of Chapter I.

First of all, a remark on the title of the chapter. In 1570 and 1663 editions it originally is "On the examination of the values assumed by the second and third rules of the second chapter [*De examine æstimationum sumptarum ex regula secunda et tertia secundi capituli*]". The title would supposedly make reference to certain "values assumed" in (A II.2) and (A II.3). Actually, there are no values assumed in those propositions. I rather think that the reference is to *Aliza*, Chapter I, since Cardano is indeed using (A I.2) in the rest of Chapter VII. Moreover, in all the splittings considered in Chapter I Cardano really assumes a value (even if not numerical) for a solution of the equation, namely he assumes that a solution is written as $x = y + z$. Nevertheless, (A I.2) is the only splitting explicitly mentioned in Chapter VII, so that the reference to the "third rule" is empty. Note that Cossali[97] not only replaces the "second chapter" in the title by a reference to *Aliza*, Chapter I (as we did), but he also changes the numbers of the propositions (then, according to Cossali, Cardano would have quoted (A I.1)

[97]"*[N]el Capo VII De Regula Aliza erroneamente intitolato:* De examine æstimationum sumptarum ex regula secunda et tertia secundi capituli *in cambio di:* ex regula prima et secunda capituli primi", see [Cossali 1966, Chapter I, paragraph 6, page 42].

and (A I.2)). But, as a matter of fact, no true reference to (A I.1) can be found in the text.

Let us now address the rest of the chapter.

A VII. *Let the equation* $x^3 = a_1x + a_0$ *be given as well as one of its solutions* α. *Consider* Y, Z *such as in (A I.2).*
Then, $YZ = \frac{a_0}{3\alpha}$,

$$Y = \frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - YZ}, \quad \textit{and} \quad Z = \frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - YZ}.$$

No proof is given, but Cardano shows as usual the procedure in detail on some examples. The only justification that he quotes is *Elements* II.5. In the examples we remark that, given an equation $x^3 = a_1x + a_0$ and one of its solutions α, Cardano takes $\alpha = Y + Z$ as in (A I.2). The product YZ can be immediately found from the first line $a_0 = 3Y^2Z + 3YZ^2$ of the system in the splitting. Knowing the product and the sum of Y, Z, *Elements* II.5 provides an equality[98] that enables Cardano to immediately draw the values of Y and Z.

This chapter is short and overall clear. Then, let us look for its real place in the *Aliza*'s framework. Since only numerical equations with $\Delta_3 > 0$ are considered, I guess that here Cardano is not considering the splittings as a method to possibly overwhelm the problem entailed by the *casus irreducibilis*. Rather, he considers the splittings by themselves. More precisely, Cardano seems to be interested in a kind of algebra of the splittings (meant as a collection of calculation rules), which could possibly turn out to be useful when he wants to solve an equation, but which, as such, is independent from solving equations.

Cossali goes one step further and guesses that Cardano is explaining how to pass from one splitting to another, and *vice versa*. Cossali takes firstly into

[98] *Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [Heath 1956a, page 382]. This means that $yz + \left(\frac{y+z}{2} - y\right)^2 = \left(\frac{y+z}{2}\right)^2$.
Obviously, we remark that giving $Y + Z = \alpha$ and $YZ = \frac{a_0}{3\alpha}$ is equivalent to give the equation $x^2 - \alpha x + \frac{a_0}{3\alpha} = 0$. Then, if one allows a loose interpretation of *Elements*, Book II, it would give the quadratic formula. Anyway, we do not need to go that far to account for Cardano's calculations.

account the splittings (A I.1) and (A I.2), and then also (A I.3) and (A I.4),[99] as he finds a compact way to write them all in one.[100] Cossali takes the solution of the equation $x^3 = a_1x + a_0$ to be $\alpha = y + z$, where y, z are such as in (A I.1). Then, by (A VII) he assumes that $\alpha = Y + Z$, infers the product YZ, and eventually Y, Z depending on $y+z = Y+Z$. The remark is non trivial at all and his interpretation is very appealing. However, it is borderline on overinterpreting. In fact, let us consider Cardano's first example $x^3 = 18x + 30$, the (real) solution of which is $\sqrt[3]{18} + \sqrt[3]{12}$. It is true that $y = \sqrt[3]{18}$ and $z = \sqrt[3]{12}$ verify the condition in the splitting (A I.1). Cardano's justification of how he found the solution is the following:

> the value of the thing $\sqrt[3]{18} + \sqrt[3]{12}$ is discovered according to the part of the chapter, and it is extended to infinite increasing beyond the number[101]

(and no justification at all is given for the subsequent examples). Then, the validity of Cossali's interpretation entirely depends on how one is likely to interpret 'the part of the chapter', namely if it is (very implicitly) referred to (A I.1) or rather to (AM XII).

Anyway, Cossali's interpretation is not completely unrelated to Cardano's intents. Cossali could have had the idea through another chapter of the *Aliza*, which is

[99]"*[N]el Capo VII De Regula Aliza erroneamente intitolato:* De examine æstimationum sumptarum ex regula secunda et tertia secundi capituli *in cambio di:* ex regula prima et secunda capituli primi *Cardano espone la via di uno spezzamento all'altro. Io la estenderò agli spezzamenti 3^o e 4^o*", see [Cossali 1966, Chapter I, paragraph 6, page 42].
[100]The following proposition recaps Cossali's statements.

A I.2-4 (Cossali). *Consider $x^3 = a_1x + a_0$. Write $x = Y + Z$ with Y, Z two [real, positive] numbers.*
If Y, Z can be chosen such that the splitting

$$\begin{cases} Y^3 + Z^3 + bY^2Z + bYZ^2 - a_1(Y+Z) = 0 \\ (3-b)Y^2Z + (3-b)YZ^2 - a_0 = 0 \end{cases},$$

holds with $b = 0, 1, 2$, then the following condition is entailed

$$\begin{cases} a_0 \leq \frac{3-b}{4}(Y+Z)^3 \\ [a_1(Y+Z) \geq \frac{1+b}{4}(Y+Z)^3] \end{cases}.$$

We have (A I.2) for $b = 0$, (A I.3) for $b = 1$, and (A I.4) for $b = 2$.
[101]"*[R]ei æstimatio iuxta partem capituli inventam sit Rcu: 18 p: Rcu: 12 et supra augendo numerum extenditur in infinitum*", see [Cardano 1570a, Chapter VII, page 17].

tightly linked to Chapter VII, namely **Chapter LIII** "On the accurate consideration of [the things] above said in Chapter VII [*De diligenti consideratione quorundam superius dictorum Capitulo 7*]". There, Cardano explicitly mentions (A I.1) and moreover he compares two splittings.

At first, Cardano considers the equation $x^3 = 12x + 20$, the (real) solution of which is $x = \sqrt[3]{16} + \sqrt[3]{4}$. He says that the solution

> can be assigned giving similarly the number 20 to the cubes, and the number can also be given to both cubes and to two mutual [parallelepipeds], and [the number can] also [be given] to both cubes and to four mutual parallelepipeds, and thus in three ways.[102]

He is indeed describing the first line of the splitting (A I.1) and of the splittings opposite to (A I.3) and (A I.4) (see footnote 34 at page 253). Then, Cardano evaluates $2\left(\frac{x}{2}\right)^3 = \frac{1}{4}x^3$ when $x = \sqrt[3]{16} + \sqrt[3]{4}$ (which is smaller than 20, as it should be in the condition of (A I.1)). He remarks that, when a_0 becomes smaller, the splitting can no more be applied (since the condition is not fulfilled).[103]

Afterwards, Cardano considers the equation $x^3 = 12x + 34$, the (real) solution of which is $x = \sqrt[3]{32} + \sqrt[3]{2}$, and the splitting opposite[104] to (A I.3). It happens that the coefficients of this equation match the condition of the splitting opposite to (A I.3), but not the one of (A I.1) (as well as the coefficients in the preceding

[102]"*Et iam dicamus quod cubus æqualis sit 12 rebus p: 20, et rei æstimatio est R cub.16 p: Rcu: 4, et hæc potest tribui dando 20 numerum cubis similiter, et potest idem numerus dari ambobus cubis et duobus mutuis, et etiam ambobus cubis et quatuor mutuis parallelipedis, et ita trifariam*", see [Cardano 1570a, Chapter LIII, page 96].

[103]"Therefore it is clear that that root cannot be divided because of the smallness of the number" or "*liquet igitur non posse dividi sic hanc R propter numeri parvitatem*", see [Cardano 1570a, Chapter LIII, page 96].

[104]It is the following.

A I.3 opposite. *Consider $x^3 = a_1x + a_0$. Write $x = y + z$ with y, z two [real, positive] numbers. If y, z can be chosen such that the splitting*

$$\begin{cases} a_0 = y^3 + y^2z + yz^2 + z^3 \\ a_1x = 2y^2z + 2yz^2 \end{cases}, \quad \textit{that is} \quad \begin{cases} a_0 = (y+z)(y^2+z^2) \\ a_1 = 2yz \end{cases},$$

holds, then the condition

$$\begin{cases} a_0 \geq \frac{1}{2}x^3 \\ a_1x \leq \frac{1}{2}x^3 \end{cases}$$

follows.

example $x^3 = 12x + 20$ match the condition of (A I.1), but not the one of the splitting opposite to (A I.3)). Then, as in (A VII), Cardano draws the values of Y, Z in the splitting opposite to (A I.3), the sums $Y + Z = x$ and $Y^2 + Z^2$ being given.

A LIII.i. *Let the equation* $x^3 = a_1x + a_0$ *be given as well as one of its solutions* α. *Consider* Y, Z *such as in the splitting opposite to (A I.3).*
Then, $Y^2 + Z^2 = \frac{a_0}{\alpha}$, *and*

$$Y = \frac{\alpha}{2} + \sqrt{\frac{Y^2+Z^2}{2} - \left(\frac{\alpha}{2}\right)^2}, \quad \textit{and} \quad Z = \frac{\alpha}{2} - \sqrt{\frac{Y^2+Z^2}{2} - \left(\frac{\alpha}{2}\right)^2}.$$

As before, once that one assumes that $\alpha = Y + Z$, the value of $Y^2 + Z^2$ immediately comes from the splitting. Eventually, one infers Y, Z, this time using *Elements* II.9 instead of *Elements* II.5 (none of the two propositions is however mentioned here).

But then Cardano oversteps the ending point of Chapter VII and seems to suggest a comparison between some of the splittings, such as the one proposed by Cossali[105] for Chapter VII. Cardano devises a geometrical environment in which he interprets the splitting (A I.1) and that opposite to (A I.3), both applied to the example $x^3 = 6x + 40$.

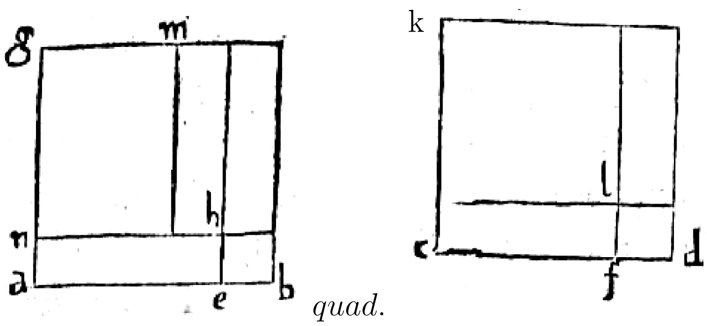

FIGURE 4.12 – *Aliza*, Chapter LIII.

Namely, he takes $\overline{AB} = \overline{CD} = 4$, which is the (real) solution of the considered equation. Moreover, in the first square, he takes $\overline{AE} = y$ and $\overline{EB} = z$ as in (A I.1) and, in the second square, he takes $\overline{CF} = Y$ and $\overline{FD} = Z$ as in the splitting

[105]By the way, Cossali's interpretation of Chapter LIII is centred on the passage from (A I.1) to one of the splittings opposite to (A I.3) and (A I.4), and *vice versa*, see [COSSALI 1966, Chapter I, paragraph 6, pages 44-46].

opposite to (A I.3).[106] Then, he deduces y, z solving the system in (A I.1) (or using the formula, which is the same). He could have also deduced Y, Z using (A LIII.i), but – here the problems begin – he says that

> that rule [the splitting opposite to (A I.3)] does not help to that equation so understood. Therefore it is necessary to discover another [equation] typical of that [rule].[107]

He suggests then an appropriate example, which is $x^3 = 12x + 34$, that – as we have already seen – matches the condition in (A I.3). I can only suppose that the equation $x^3 = 6x + 40$ was bad chosen as an example. Then, the things get worst, since we expect that Cardano applies, as above, the same splittings to $x^3 = 12x+34$, but he rather considers the ones opposite to them, which are (A I.2) and (A I.3) (so that our nice talk on the conditions of the splittings is completely disrupted). However, Cardano says that one cannot find such a Y, Z by the means of one splitting, but only the y, z of the other splitting.[108] The splitting opposite to (A I.4) is no more mentioned. Then, despite all the incongruities that there may be in the text, I believe that the idea of the comparison between the splittings is really there, thus legitimising Cossali's interpretation.

Cardano ends this chapter with a note, in which he reconsiders his opinion on the fact that the propositions in *Ars magna*, Chapter XXV were "particular". In

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[106]Cardano explicitly describes the splitting (A I.1) and that opposite to (A I.3), without explicitly mentioning them. In the comparison between the two, he shows that $(y+z)(y^2-yz+z^2) = 40 = (Y+Z)(Y^2+Z^2)$. We remark that, in order to prove the first part in the above equality, he needs to identify in the first square the geometrical object the measure of which is $y^2 - yz$ (he already knows that $z^2 = \overline{EB}^2$), since he assumed that $y^3 + z^3 = 40$ because of (A I.1) and he knows that $y^3 + z^3 = (y+z)(y^2 - yz + z^2)$. Passing from the (imaginary) three-dimensional diagram (with height AB) to the two-dimensional diagram as it is printed in the text, he clears out the term $(y+z)$. Then, he only needs to build the rectangle (MH), the measure of which is equal to $(\overline{AH}) = yz$, and to take it away from GH, the measure of which is y^2.
We remark that here Cardano is accurately imitating in a geometrical environment the calculations that he makes with those y, z, or $\overline{AE}, \overline{EB}$.

[107]"*[N]ec regula illa servit huic æquationi sic intellectæ, ergo oporteret invenire aliam ei propriam*", see [Cardano 1570a, Chapter LIII, page 97].

[108]"And with the equality $x^3 = 12x+34$ one may not discover CD so that it is composed by CF and FD, but by the other rule. But we will find AB so that it is divided in the parts AE, EB" or "*nec licebit cum æquatione 1 cu. æqualis 12 rebus p: 34, invenire cd ut composita est ex cf et fd, sed ex alia regula. Sed inveniemus ab ut est divisa in partes ae, eb*", see [Cardano 1570a, Chapter LIII, page 97].

fact, he says[109] that they are actually "general", as we have already observed (see above, Section 2.1.4 at page 84).

We recall (see above at page 9) that in this chapter Cardano introduces a new kind of notation to talk about a solution of an equation depending on the conditions given in some propositions in *Ars magna*, Chapter XXV. For instance, referring to $x^3 = 20x + 32$,

'20 d. p. ℞ p. 32'

means '20 divided in the part and the root that produce 32 [*divisum in partem et radicem producentes 32*]' and refers in a very convoluted way to the solution of the considered equation depending on the f, g in (AM XXV.1). I could not retrieve for sure the proposition to which

"32 p. 20 cum p. 32",

or "what produces 20 with what produces 32 [*producentis 20 cum producente 32*]", refers, since it is a very generic description. On the contrary,

'Ag. ℞ p: 20 p: n: 16'

means 'the aggregate of the roots of the parts of 20 that reciprocally multiplied produce 16 [*aggregatum radicum partium 20, quæ mutuo ductæ producunt 16*]' (or the half of 32), and makes reference to the f, g in (AM XXV.3) (or (A I.3)). This kind of stenography is also used in **Chapter LVII**, where Cardano explicitly mentions Chapter LIII and the very same example $x^3 = 20x + 32$. We recall (see here at page 277) that at the beginning of this chapter the "particular" rule (AM XXV.2) is mentioned as an introduction to the study of the shape of a solution. We also remind that another access point to the topic was the writing $x = \sqrt{a_1 + \frac{a_0}{x}}$ in (A XL). The stenography used by Cardano is employed with the intent of describing this last writing. In fact, Cardano correlates the three

[109]"From [the things] seen here and above it appears clearly that all the rules of the chapter twenty-five of the Ars magna that they call special are general and they are said [to be] special for the only reason of the kind of the value" or "*[e]x visis hic et superius apparet liquido, quod omnes regulæ vigesimiquinti capituli Artis magnæ, quas vocant speciales, sunt generales, et dicuntur speciales solum ratione generis æstimationis*", see [CARDANO 1570a, Chapter LIII, note, page 97].

following stenographic writings:

‘32 p: 20 c. p. 32’,

which means ‘what produces 20 with what produces 32 [*producens 20 cum producente 32*]’,

“R 20 p: d. 32”,

which means ‘the root of 20 plus 32 divided by the same root [*R 20 p: diviso 32 per ipsam radicem*]’, and

“R 20 f. 32”,

which means ‘the root of 20 with a fragment of 32 [*R 20 cum fragmento 32*]’ (a “fragment [*fragmentum*]” is “what comes forth from a division [*quod ex divisione prodit*]”), as to have the same meaning. Despite the obscurities of the above writings, at least the last one points to $\sqrt{20+\frac{32}{x}}$, considering that Cardano says that “$\sqrt{20}$ f. 32 is the value of $x^3 = a_1x + a_0$ [*R 20 f. 32 est æstimatio cubi ęqualis 20 rebus p: 32 numero*]” (even though we expected a universal root instead of ‘R’).

Chapters VII and LIII deal with the study of the splittings by themselves. The fact that all the examples have $\Delta_3 > 0$ could indeed point at the little interest showed here for the *casus irreducibilis*. This study eventually leads to establish some calculation rules on the splittings, which could possibly in turn be useful when one wants to solve an equation, also (and especially) when it falls into the *casus irreducibilis*. In Chapter VII, given the coefficients of an equation of the family $x^3 = a_1x + a_0$ and one of its solutions, Cardano explains how to find Y, Z in the splitting (A I.2) (depending on the sum and on the product of Y, Z). There is a similar argument in Chapter LIII, where, under the same assumptions, Cardano explains how to find Y, Z in the splitting opposite to (A I.3) (depending this time on the sum of Y, Z and on the sum of their squares).

Nevertheless, in Chapter LIII Cardano goes further on – and this makes us review our interpretation of Chapter VII in the wake of Cossali’s guidance. Despite all the incongruities, in Cardano’s text there is indeed the idea of passing from one of the first four splitting of Chapter I to another, possibly also including the splittings opposite to (A I.3) and to (A I.4). In fact, if we write y, z according to the splitting of departure and Y, Z according to the splitting of arrival, and if we assume that $y + z = Y + Z$ (since they are both equal to a solution of the equation), the second condition on Y, Z will be given by the splitting of arrival itself. What goal could allegedly have this method? Regarding to my hypothesis on the final aim of the splittings, namely – in short – that Cardano studies (A I.2)-(A I.7) in order to find another cubic formula, the following observation strikes me. If one manages to find a way to pass from one splitting to another when the conditions of two splittings are satisfied, then he could pass from that same splitting to the other also when the conditions of one of the two splittings are not satisfied. This would be extremely useful, in particular, when the conditions on the splitting (A I.1) do not hold. Then, in this case, one would not need to embrace the drastic relief to the *casus irreducibilis* and completely discard the cubic formula for $x^3 = a_1x + a_0$. In the small, one could also be contented with keeping the usual cubic formula and using another one (non-"general", as the one in (A I.1) was not either) when the first one cannot be applied. This entails that one should relax his requirements on what a cubic formula is, namely it will no more be unique.

4.3.5. Summing up. There is a sizeable number of chapters in the *Aliza* that can be gathered around the thread given by the splittings in Chapter I for the pivotal equation $x^3 = a_1x + a_0$. In the above sections, I have analysed seventeen chapters out of sixty. All together, these could have maybe formed the original core, or part of the original core, of that "*aliza* problem" mentioned in the *Ars magna*. In short, I have identified two main topics linked to the splittings. One concerns their alleged origin, while the other is a possibly unconventional way in which the splittings could have been employed in order to face the problem entailed by the *casus irreducibilis*.

The topic on which I have found the most of evidences is the first one. Basically, I have conceived three different hypotheses on the alleged origin of the splittings. Firstly, in Section 4.3.1, I have questioned the substitution $x = y + z$. Whether he got the idea from Tartaglia's poem or not, the *Aliza* attests the deep interest that Cardano had in this shape for a solution of $x^3 = a_1x + a_0$. It is, by the way, a long standing interest, since similar surveys were also in the *Ars magna arithmeticæ*. Regarding to these early beginnings, I have made the hypothesis that the study of the shapes of the solutions started before the discovery of the cubic formulae. In Cardano's mathematical writings this is indeed a well-recognisable issue, which more precisely aims to describe the irrational solutions of cubic equations with rational coefficients depending on some codified shapes. In my opinion, Cardano's dexterity in calculations with irrational numbers (see, for instance, Section 4.4.1), opened the door to the development of his researches towards the shapes of the irrational solutions. Anyway, in the *Aliza*, unlike in the *Ars magna arithmeticæ*, these researches are not only scattered all along the book, but also very heterogeneous. There are in fact great dissimilarities among the methods employed in the different chapters (which either refer to Book X of the *Elements*, or proceed by substitution and comparison, or use propositions from the *Ars magna*). These could maybe testify to the different developing stages of Cardano's surveys. But, in the end, Cardano comes to the same conclusions as in the *Ars magna arithmeticæ*, namely that an irrational solution of $x^3 = a_1x + a_0$, with a_1, a_0 rational, must be a *binomium* of the 2nd or 5th type $\sqrt{a} + b$ or a cubic *binomium* $\sqrt[3]{a} + \sqrt[3]{b}$ such that $\sqrt[3]{ab}$ is rational. In this way, a very important remark surfaces: the shape of the solutions of the considered equation must contain two parts. This, in turn, possibly leads to write $x = y + z$. Hence, the idea of substituting $x = y + z$ in $x^3 = a_1x + a_0$ could have been a side effect of the study of the possible shapes for its irrational solutions.

Secondly, in Section 4.3.2, I have analysed the connection with *Ars magna*, Chapter XXV, which is quite an obvious link, as *Ars magna*, Chapter XII already suggested. Moreover, when analysing *Ars magna*, Chapter XXV, we have already remarked that three propositions could be written in a way that recalls three splittings – and now we clearly see it. But, even though both the three splittings and the corresponding propositions in *Ars magna*, Chapter XXV concern the problem entailed by the *casus irreducibilis*, they are intended to achieve different purposes. Respectively, the splittings are meant to provide a new formula, while the rules in *Ars magna*, Chapter XXV only to give a case-by-case method. I have suggested that the splittings could have also come from a further improvement of some of the propositions in that chapter of the *Ars magna*, and hence that they respond to the need of a conclusive relief to the problem entailed by the *casus irreducibilis*. Nonetheless, it happens that in the considered chapters of the *Aliza* Cardano limits himself to rational coefficients. This led me to conjecture that, if Cardano did not conceive the propositions in *Ars magna*, Chapter XXV (and thus some of the splittings) starting from the study of equations with rational coefficients, he at least planned to apply these propositions to equations with rational coefficients. This is indeed what happens in the great majority of the cases. Moreover, these rules must have been refined after that Cardano realised the problem that the cubic formulae entail, since no mention of such a method is made either in the *Practica arithmeticæ* or in the *Ars magna arithmeticæ*.

Thirdly, in Section 4.3.3 I have accounted for Cardano's last say on the topic of the splittings. As it was indeed very reasonable to expect, a geometrical inspiration is also at the origin of the mainly arithmetical methods that Cardano devised. Anyway, it is a very specific kind of geometry, where the position of the geometrical objects is not essential and the main constraint is played by dimensional coherence.

Finally, a few words on the second topic, which concerns another possible application of the splittings. In Section 4.2 I have made the hypothesis that, since (A I.1) leads to the cubic formula, Cardano displays the other splittings hoping that they lead to an alternative cubic formula for $x^3 = a_1x + a_0$, which will eventually replace the usual one. We now know that this cannot be done, but it could have sounded as a very promising strategy. Nevertheless, it is also possible that Cardano envisaged to use more than one non-"general" cubic formula at a time (if only he had managed to retrieve more than one), thus relaxing his requirements on what a 'cubic formula' is. In fact, in the *Aliza* we find a few chapters that approach the splittings as a proper subject of inquiry. In particular, this leads to establish some calculation rules on them and, in turn, these could possibly be useful in order to complement the conditions of the usual formula originated by (A I.1).

4.4. Some technical complements

In the present section I will briefly deal with some chapters that, on the one hand, complete my analysis in the preceding sections by providing some technical details, even though they do not display a real uniformity respect to their contents. On the other hand, their variety helps in giving a flavour of the diversity of the contents in the *Aliza*.

4.4.1. Preliminary study of irrational numbers. Many chapters of the *Aliza* seems to provide the starting point for the subsequent deeper survey on the shapes of the irrational solutions of $x^3 = a_1x + a_0$ with rational coefficients (see Section 4.3.1 at page 256) and consequently on the substitution $x = y + z$ (see Section 4.2 at page 235). These chapters deal essentially with different kinds of irrational numbers.

Pretty soon we come across a classification of the six types of *binomia* and *recisa* by square roots,[110] and we find some calculation rules and properties. As such, these chapters could maybe be paired to the arithmetisation of Book X of the *Elements* in the first part of the *Ars magna arithmeticæ* (even though they do not constitute such a compact block as there). In fact, these chapters are scattered all along the *Aliza* and I could only gather them regarding to (the main consistent part of) their contents.

In **Chapter IV** "On the way to reduce all the quantities that are called 'first sides' by the tenth [Book] of Euclid in short [*De modo redigendi quantitates omnes, quæ dicuntur latera prima ex decimo Euclidis in compendium*]" Cardano suggests a classification for irrational numbers. The treatment is essentially analogous to the one in the *Ars magna arithmeticæ* that I have recalled in Section 2.1.4 at page 82. Note that Euclid is explicitly mentioned at the beginning of the chapter and his Book X is a steady reference. Cardano moreover quotes a "third book [*tertio libro*]" – very likely referring to the *Opus arithmeticæ perfectum* (as he often does when he does not specify the title of the referred writing), the third book of which should have been devoted to irrational numbers.[111] Then, he teaches case-by-case how to take the square root of all the six *binomia* and *recisa*, interspersing his

[110]According to the definitions II.1-6 in the second part of Book X, see [Heath 1956c, pages 101-102].
[111]See above, at page 22.

explanation by some rules.[112] These deal with the (in)commensurability and order relationships between the different types of *binomia* and *recisa*, teach how to multiply radicals, and show how to rationalise fractions where square roots appear in the denominator and a rational number in the numerator.

This last topic, together with the rationalisation of fractions where cubic roots appear, is the starting point of **Chapter XVII** "In how many ways the number can be produced by the non-number [*Quot modis numerus possit produci ex non numero*]". It helps to the final aim of the chapter, which is to show how a rational number can be obtained by performing operations on irrationals. In particular, taking a, b rational, Cardano considers the products of the form $(a+\sqrt{b})(a-\sqrt{b})$ or $(a+\sqrt{b})(ka-k\sqrt{b})$, with k rational, the differences of the form A^2-B^2, with $A=\frac{b-a^2}{2a}+a$ and $B=\frac{b-a^2}{2a}$, and the powers of the form $(\sqrt[n]{a})^n$, with n natural, which all give a rational result.

In **Chapter XXX** "What [is] the equality of the cubes of the parts of a divided line [*Qualis æqualitas cuborum partium lineæ divisæ*]" Cardano shows the following proposition.

A XXX. *If a line AB is divided in C, then*

$$\overline{AC}^3+\overline{CB}^3=(\overline{AC}+\overline{CB})(\overline{AC}^2-\overline{AC}\,\overline{CB}+\overline{CB}^2).$$

This proposition is useful in the context of the splittings. In fact, if we take $\overline{AC}=y$ and $\overline{CB}=z$ (as we did at page 240) and rewrite (A XXX) as $y^3+z^3=(y+z)(y^2-yz+z^2)$, we remark that the proposition justifies the passage from the system

$$\begin{cases} a_0 = 3y^2z+3yz^2 \\ a_1x = y^3+z^3 \end{cases}$$

to the system

$$\begin{cases} a_0 = 3y^2z+3y\,z^2 \\ a_1 = y^2-yz+z^2 \end{cases},$$

which was left implicit in the splitting (A I.2) (see above, at page 243).

[112] We remark that a similar topic had already been presented in *Ars magna arithmeticæ*, Chapters XIII, XIV, and XVII, see here at page 175.

Since this chapter is quite a detailed explanation of a simple identity, it is unusually easy to detect in the proof (that I will not reproduce[113]) the common pattern of (AM VI.6). As before, the geometrical environment in which the proposition is interpreted fixes the reference of the objects involved. Then, the relative position of the geometrical objects identified in this way is no more essential and the proof strikingly reduces to an arithmetical calculation.

Summing up, as said, these few chapters do not contain new, bright results on irrational numbers. They rather contain some technical propositions that complete the picture of the *Aliza* in the above sections. Collaterally, they also give an insight of the extreme dexterity and acquaintance with calculation methods that Cardano shared with the mathematicians of his time.

4.4.2. Again on Proposition (AM VIII.2). At the very beginning of the *Aliza*, there are two chapters that make both implicitly reference to the same proposition of the *Ars magna*, namely (AM VIII.2). Since in the framework of the *Ars magna* it was proved that this proposition had many connections, I will briefly deal with its echoes in the *Aliza*.

In **Chapter III** "On the way to discover the quantities that help to the chapters by the product of one part by the other and by the square of the difference of the parts [*De modo inveniendi quantitates quæ serviant capitulis per producta unius partis in aliam, et quadratum differentiæ partium*]" Cardano uses (without explicitly giving the reference) a particularised version of (AM VIII.2). We remind that in the end this proposition is nothing more than a help to the intuition to guess a solution of the considered equation. What is remarkable in the *Ars magna* is that Cardano considers equations of any degree, provided that they belong to the large class of $x^q + a_0 = a_p x^p$ with $0 < p < q$ (see here, at page 75). In the *Aliza*, instead, Cardano considers only a family of cubic equations (more precisely, the ones lacking in the first degree term), but we find a more detailed insight. This is not surprising at all in a book devoted to the *casus irreducibilis*. The final aim of the chapter is to try to better determine the two numbers f, g that enable him to find a solution of the considered equation. Then, Cardano takes into account the shape of f, g and searches for some constraints

[113]See [Cardano 1570a, Chapter XXX, pages 59-60].

on it. In particular, he forces the coefficients of the equation to be rational and exploits (AM VIII.2).

A III.i. *Let* $x^3 + a_0 = a_2x^2$ *be given [with* a_2, a_0 *rational] and assume that two positive, real numbers* f, g *exist such that*

$$\begin{cases} a_2 = f + g \\ a_0 = fg^2 \end{cases}.$$

Then [$x = g$ *and], if* f *is a* binomium *of any type,* g *is a* recisum *of the same type, and* vice versa*.*

Note that Cardano does not specify that the coefficients of the equation are rational. We can only observe that Cardano's subsequent calculations require that they are so (see below, footnote 117). Moreover, we recall that we have already run into similar considerations in the *Ars magna* and *Ars magna arithmeticæ*, and there the coefficients were rational.

We also remark that, if Cardano is fully aware that $x = g$ is a solution, he does not seem to have an understanding of the fact that the *binomia* and *recisa* of the 3rd or 6th type cannot be a solution of a (non-complete) cubic equation[114] (see for instance *Aliza*, Chapters V and X, in Section 4.3.1 at page 256). Still, in the examples that follow the proposition, he obviously only considers some *binomia* and *recisa* of the 1st or 4th and of the 2nd or 5th types.

We observe that we have already run into a similar (even if more detailed) proposition in (AM XXV.15). We remind that (AM XXV.15) is linked to (AM VIII.2) and gives, according to certain conditions, two solutions of $x^3 + a_0 = a_2x^2$ under the shape of a *binomium* and a *recisum* (see above, at page 94). In fact, we recall that $\frac{f}{2} \pm \sqrt{\frac{f^2}{4} + fg}$ are the two other solutions (see above, at page 79).

[114]"Then the first part [g] can be the first, the second, and the third *binomium* or *recisum* and can also be the fourth, the fifth, and the sixth [*binomium* or *recisum*], nevertheless not the *recisum*, because, being the first [part] a root, the second part is necessarily a *binomium*, because the first is a *recisum*, therefore in both there is a positive root, therefore that number that is divided at the beginning cannot be" or "*[p]rima ergo pars potest esse binomium vel recisum primum, secundum, et tertium et potest etiam esse binomium quartum, quintum, et sextum, non tamen recisum, quia cum prima sit R et secunda pars necessario sit binomium, quia prima est recisum, igitur in utraque esse R p:, igitur non potest esse numerus ille qui ab initio divisus est*", see [Cardano 1570a, Chapter III, page 6].

Cardano does not provide any justification for (A III.i). One may – and maybe Cardano did – consider that the examples that follow the next proposition display a clear enough explanation also for this proposition. For instance, Cardano considers the equation $x^3 + 8 = 7x^2$ and suggests the couples $f = 3 - \sqrt{8}$, $g = 4 + \sqrt{8}$ and $f = 3 + \sqrt{8}$, $g = 4 - \sqrt{8}$.

A III.ii. *Let the equation $x^3 + a_0 = a_2x^2$, where only a_2 is given positive, rational. We write $a_2 = F + G$ with F, G positive rational.*
Then, referring to the proposition (A III.i), we have

$$f = F \pm \sqrt{H} \quad \textit{and} \quad g = G \mp \sqrt{H},$$

where $H = 2FG - G^2$. Moreover,

$$a_0 = FG^2 + FH - 2GH.$$

We remark that in this case the rational solution is $F - G$.

Moreover, note that Cardano considers $g = G - \sqrt{H}$ only when $g > 0$, since g is also a solution, and obviously $H = 2FG - G^2$ only when $H > 0$. His abstract explanation of this proposition could seem confusing,[115] but his examples (all with $\Delta_3 < 0$) are indeed very clear.[116] As it frequently happens, Cardano justifies his statement only through these examples, showing that the algorithms that he

[115]"Then I say that, divided the number of the squares in two parts, that we call principal, and we call principal [the part] that is multiplied by itself, multiply the first by the double of the second to find the other two parts. And you will deduce the square of the first from the product and the root of the *residuum* is the part to add or to subtract to the principal [parts] with the agreed conditions. And similarly, to produce the positive number, multiply the difference of the principal [parts] between them and the product by the double of the first principal, and what is produced is the searched number" or "*[d]ico ergo quod diviso numero quadratorum in duas partes, quas vocabimus principales, et eam quæin se ducitur, vocabimus principales, et pro alijs duabus partibus inveniendis, duc primam in duplum secundæ et a producto deducito quadratum primæ, et R residui est pars addenda principalibus, aut detrahenda cum conditionibus condictis. Et similiter pro numero producendo duc differentiam principalium in se, et productum in duplum primæ principalis, et quod producitur, est quæsitus numerus*", see [Cardano 1570a, Chapter III, page 6].

[116]See [Cardano 1570a, Chapter III, pages 6-7]. The examples concern the equations $x^3 + 8 = 7x^2$, $x^3 + 6 = 7x^2$, $x^3 + 48 = 7x^2$, $x^3 + 24 = 8x^2$, $x^3 + 40 = 8x^2$, $x^3 + 45 = 8x^2$, and $x^3 + 75 = 8x^2$.

gave work. It is nevertheless not difficult to recover the formulae of the above proposition by a straight calculation.[117]

On the one hand, as already observed, the *Ars magna* gives a more comprehensive result (for more equations with any coefficients). This result is nothing but a rephrasing of the original equation, thanks to which one switches from searching for a solution of the original equation to searching for the f, g in the rephrasing. On the other hand, (A III.ii) particularises this result to a certain family of equation with rational coefficients[118] (but note that this does not make easier to find F, G).

In **Chapter V** "On the consideration of the *binomia* and *recisa* that contain a rational figure, whence on the value of the chapters [*De consideratione binomiorum et recisorum continentium figuram rheten, ubi de æstimatione capitulorum*]" Cardano uses again a particular version of (AM VIII.2), which now concerns $x^3 + a_0 = a_1x$ (again without any explicit reference). Consider f, g as in (AM VIII.2) and assume that the solution x is a certain type of *binomium* or *recisum*. One of the aims of this chapter is then to specify which type of *binomia* and *recisa* f, g are.

A V.ii. *Let*[119] $x^3 + a_0 = a_1x$ *[with a_0, a_1 positive, rational] and one of its solutions x be given.*
[We consider (AM VIII.2). In particular, if two positive, real numbers f, g exist such that

$$\begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g} \end{cases},$$

[117]In fact, we take two positive, rational numbers F, G such that $a_2 = F + G$ and a positive, rational number H. We write $f = F \pm \sqrt{H}$ and $g = G \mp \sqrt{H}$. We want to draw a value for H and a_0 depending on F, G.
Then $g^2 = G^2 + H \mp 2G\sqrt{H}$. We have that

$$fg^2 = FG^2 + FH - 2GH + \sqrt{H}(\mp 2FG \pm G^2 \pm H).$$

By (A III.i) we know that $a_0 = fg^2$ must be rational. Therefore, we need that $\sqrt{H}(\mp 2FG \pm G^2 \pm H) = 0$. Then, either $H = 0$ (but Cardano does not consider this case), or $\mp 2FG \pm G^2 \pm H = 0$, from where we infer the value of H. Moreover, $a_0 = FG^2 + FH - 2GH$.

[118]Note that – as we have already remarked, in the vast majority of Cardano's examples the equations have rational coefficients.

[119]My commentary on the Proposition (A V.i) is at page 261. I use this numbering since it reproduces the order of the results in Cardano's text.

then $x_1 = \sqrt{g}$, $x_2 = -\frac{\sqrt{g}}{2} + \sqrt{\frac{g}{4} + f}$, *and* $x_3 = -\frac{\sqrt{g}}{2} - \sqrt{\frac{g}{4} + f}$ *(see above at page 77).]*

Let x *be a solution of the equation. Then, the following relations between* x, f, *and* g *hold:*

- *if* x *is* binomium *of the 1st type, then* g *is a* binomium *of the 1st type and* f *is a* recisum *of the 1st type;*
- *if* x *is* recisum *of the 1st type, then* g *is a* recisum *of the 1st type and* f *is a* binomium *of the 1st type;*
- *if* x *is* binomium *of the 4th type, then* g *is a* binomium *of the 1st type and* f *is a* recisum *of the 4th type;*
- *if* x *is* recisum *of the 4th type, then* g *is a* recisum *of the 1st type and* f *is a* binomium *of the 4th type;*
- *if* x *is* recisum *of the 2nd type, then* g *is a* recisum *of the 1st type and* f *is a* binomium *of the 2nd type;*
- *if* x *is* recisum *of the 5th type, then* g *is a* recisum *of the 1st type and* f *is a* binomium *of the 5th type.*

The proposition is expounded by some examples (again, all with $\Delta_3 < 0$). We recall that Cardano is well acquainted with the operations of rising to the power and taking the root of these kind of numbers (for instance in *Aliza*, Chapter IV, see above at page 309, and in the *Ars magna arithmeticæ*, see above at page 175), on which all the calculations that are at the bottom of this statement depend.[120] Note that – as Cardano himself remarks – the *binomia* of the 2nd or 5th type are

[120]Note that the relation between x and $g = x^2$ is always true, namely that the square of a *binomium* of the 1st or 4th type is a *binomium* of the 1st type and that the square of a *recisum* of the 1st, 2nd, 4th, or 5th type is a *recisum* of the 1st type. In fact, let us firstly consider a *binomium* or *recisum* of the 1st or 4th type $a \pm \sqrt{b}$ (that is, with $a^2 - b > 0$). Then, $(a \pm \sqrt{b})^2 = (a^2 + b) \pm 2a\sqrt{b}$. This is a *binomium* or *recisum* of the 1st type, since $(a^2 + b)^2 - 4a^2b > 0$ and $\frac{\sqrt{(a^2+b)^2-4a^2b}}{a^2+b} = \frac{a^2-b}{a^2+b}$ is always rational (in particular, it does not depend on the type of $a \pm \sqrt{b}$, that is on fact that $\frac{\sqrt{a^2-b}}{a}$ is rational or not). Secondly, let us consider a *recisum* of the 2nd or 5th type $\sqrt{a} - b$ (that is, with $a - b^2 > 0$). Then, $(\sqrt{a} - b)^2 = (a + b^2) - 2b\sqrt{a}$. This is a *recisum* of the 1st type, since $(a + b^2)^2 - 4ab^2 > 0$ and $\frac{\sqrt{(a+b^2)^2-4ab^2}}{a+b^2} = \frac{a-b^2}{a+b^2}$ is always rational (in particular, it does not depend on the type of $\sqrt{a} - b$, that is on fact that $\frac{\sqrt{a-b^2}}{\sqrt{a}}$ is rational or not).
On the other hand, the conclusions on f depend on the intrinsic nature of the coefficients of the equation.

not listed among the possible solutions. Indeed, when a *recisum* of the 2nd or 5th type $\sqrt{a} - b$ is one of the irrational solutions (and the equation has more than one real solution, that is $\Delta_3 < 0$), then the other irrational solution is $-(\sqrt{a} + b) < 0$.

Finally, Cardano makes some remarks that describe the proposition and recalls the statement in (AM XIII bis). This abrupt reference to (AM XIII bis) stroke me. He simply states the formula for the second solution of $x^3 + a_0 = a_1x$. But the fact that he chooses to recall it right in the same chapter in which he is speaking of the connection between the shapes of the *binomia* and *recisa* and a solution of the considered equation suggests – in my opinion – that the idea of the formula in (AM XIII bis) (and also the one in (AM XIII)) could have been somehow originated in relation to the environment of the *binomia* and *recisa*. Unfortunately, I cannot provide any evidence, except maybe the fact that in this way we can account for some of the carelessness in those propositions of the *Ars magna*, as I have already remarked at page 113.

Summing up, Chapters III and V of the *Aliza* (even though they do not display any uniformity) both recall the same result from the *Ars magna*, which is no more proposed as such in the *Aliza*. On the one hand, one may believe that there is no need to reprint it, since in the 1570 edition the *Aliza* is appended to the *Ars magna*. On the other hand, it is useless to recall a too general result when one is interested only in cubic equations (or rather – I would say – in cubic equations with $\Delta_3 < 0$, since all the examples are of this kind). Cardano considers the two families of equations $x^3 + a_0 = a_1x$ and $x^3 + a_0 = a_1x$, and takes f, g as in (AM VIII.2). His main aim in the *Aliza* is to try to better specify f, g, possibly in order to get a solution of the equations. In the *Ars magna*, the whole difficulty has been displaced from directly solving the considered equation to find such a f, g – and we recall that finding those f, g is not trivial at all. There, fixing f, g depended on intuition. Now, Cardano assumes to have rational coefficients and suggests another rephrasing of (AM VIII.2), which is maybe supposed to make more easier to find those f, g. As we have seen in Section 4.3.2 while dealing with the chapters of the *Aliza* that refer to *Ars magna*, Chapter XXV (which is tightly linked to (AM VIII.2)), limiting to rational coefficients is quite a common strategy in the *Aliza*, also considering that the most of Cardano's examples have only sporadically irrational coefficients.

4.5. How far goes geometry in the *Aliza*?

In Section 1.3 at page 26, we have seen that the book that we call nowadays '*Aliza*' was possibly originated by an "*aliza* problem" that Cardano planned to include in a certain "book of the geometrical problems", very likely referring to the last book of the (never written) *Opus arithmeticæ perfectum.* Therefore, we have been led to expect a certain amount of geometry in this treatise. Considered that – as we have seen – the *Ars magna* contains many proofs in a loose geometrical fashion, the interesting question is 'is there any geometry in the *Aliza* and, if yes, which kind of geometry is it?'

Since we are now more acquainted with this book, we are able to remark that in the *Aliza* there is not as much geometry as we were expecting to be. We only came across a few examples, for instance while dealing with Chapter XXX (at page 310), Chapter XL (at page 277), or Chapter LX (at page 295). In all these cases we found that, in the geometrical environment in which Cardano translates his hypotheses, the relative positions of the involved geometrical objects do not play an essential role. We recall that this is exactly what happened in the *Ars magna.* Moreover, we have seen in Section 4.3.3 at page 294 that the splittings themselves could have had a loose geometrical inspiration.

Given these premises, it is time to have a look at another chapter of the *Aliza* in which geometry – and an essentially different kind of geometry – is massively employed.

4.5.1. Cardano's proof of the existence of a solution for $\mathbf{x^3 + a_0 = a_2x^2}$.

In the *Aliza*, **Chapter XII** "On the way to show geometrically the value of a cube and a number equal to some squares [*De modo demonstrandi geometrice æstimationem cubi et numeri æqualium quadratis*]" we find for the very first (and unique) time a different geometrical approach.

Cardano is mainly interested in the equation $x^3 + a_0 = a_2x^2$ when it falls into the *casus irreducibilis.* In fact, he says that, if a condition equivalent to $\Delta_3 > 0$

holds, "the purpose is false [*propositum esset falsum*]."[121] If, instead, a condition equivalent to $\Delta_3 = 0$ holds, Cardano is able to solve the equation (since in the cubic formula he does not come across square roots of negative numbers) and spends a very few words to explain how to find a solution.[122] We resume his reasoning in the following proposition.

A XII.i. *Consider* $x^3 + a_0 = a_2x^2$. *If* $a_0 = \frac{4}{27}a_2^3$, *then* $x = \frac{2}{3}a_2$.

Let us then assume $x^3 + a_0 = a_2x^2$ with $\Delta_3 < 0$. In this case, the cubic formula contains some square roots of negative numbers and Cardano needs to face this problem to get a (real) solution. Yet, there is a particular case in which Cardano is able to find a real solution through the formula. This happens when a condition equivalent to $q = 0$ (see above, Section 1.5.3 at page 40) holds. Then, Cardano is able to draw a solution using the fact that $\sqrt[3]{-y} = -\sqrt[3]{y}$.[123]

A XII.ii. *Consider* $x^3 + a_0 = a_2x^2$. *If* $a_0 = \frac{2}{27}a_2^3$, *then* $x = \frac{a_2}{3}$.

Once that the limit cases have been cleared out, we can focus on Cardano's main issue in this chapter: the equation $x^3 + a_0 = a_2x^2$, when square roots of negative numbers appear in the cubic formula and cannot be deleted (that is, when $\Delta_3 < 0$ and $q \neq 0$). In particular, this is the case of $x^3 + 192 = 12x^2$. Through this example Cardano shows that this family of equations (u the above conditions) always has a (real) solution. As he reminds, he will use the very same method that Eutocius of Ascalon employed in his commentary on the Proposition II.4 of *On the sphere and cylinder* by Archimedes.[124]

[121]Note that Cardano makes reference to "the things said above [*ex supradictis*]" (more specifically, to a certain "Proposition 27") to justify the fact that, if $a_2 = 12$ and $a_0 > 256$, then "the purpose is false". We remark that the condition on the coefficients implies that $\Delta_3 > 0$. Unluckily I did not find such a "Proposition 27" neither in the preceding chapters of the *Aliza*, nor in the *Ars magna*, or in the *De proportionibus*.

[122]In fact, taking $\Delta_3 = 0$, that is $a_0 = \frac{4}{27}a_2^3$, we have $q = \frac{2}{27}a_2^3$ (see above, Section 1.5.3 at page 40). Then, $x = \sqrt[3]{q} + \frac{a_2}{3} = \frac{2}{3}a_2 > 0$ is the (positive) double root. Moreover, $x = -2\sqrt[3]{q} + \frac{a_2}{3} = -\frac{1}{3}a_2 < 0$ is the (negative) remaining root.

[123]In fact, taking $q = 0$, we have $\Delta_3 < 0$ and $x = \sqrt[3]{\sqrt{\Delta_3}} + \sqrt[3]{-\sqrt{\Delta_3}} + \frac{a_2}{3}$. Since the cubic root is an odd function, $x = \sqrt[6]{\Delta_3} - \sqrt[6]{\Delta_3} + \frac{a_2}{3} = \frac{a_2}{3}$ gives a real, positive solution.
Remark that in this case we need to use the properties of imaginary numbers to obtain the other solutions. In fact, $x = \pm\frac{i\sqrt{3}}{3}\sqrt[6]{-a_2^6}$.

[124]Cardano says that Eutocius gave two solutions, but that he will quote only the first. He also says that he will not quote instead the propositions by Euclid, see [CARDANO 1570a, Chapter XII, page 25].

A XII.iii. *Consider the equation $x^3 + 192 = 12x^2$. Then, it has a [real] solution.*

A XII.iii — *Proof*. Cardano takes $\overline{AB} = 12$ and E on AB such that $\overline{EB} = 2\overline{AE}$. He considers $\overline{DQ}^2\overline{DZ} = 192$ [with $\overline{DZ} = \overline{AE}$].[125] He takes u such that $\overline{EB} : \overline{DQ} = \overline{DQ} : \overline{u}$ and AC perpendicular to AB such that $\overline{DZ} : \overline{AC} = \overline{EB} : \overline{u}$.

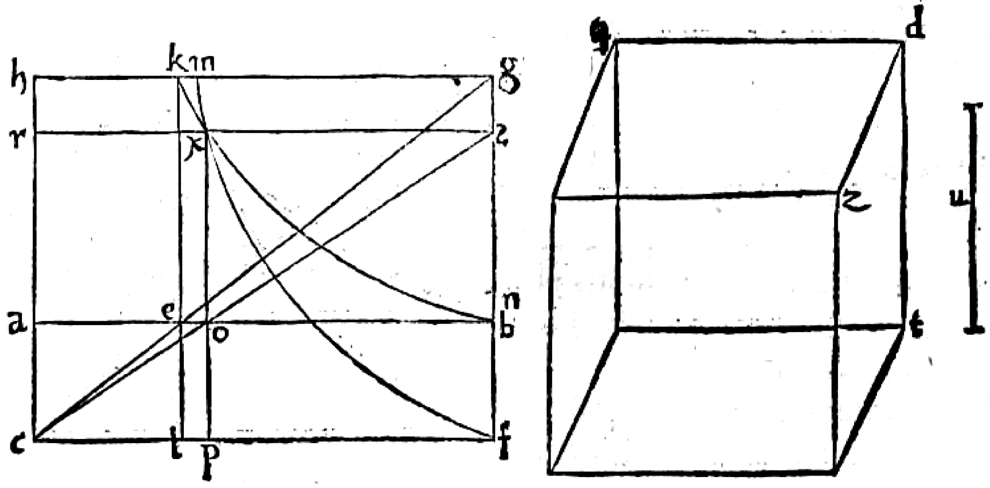

FIGURE 4.13 – *Aliza*, Chapter XII.

From these hypotheses it follows that $\overline{DZ} : \overline{AC} = \overline{EB}^2 : \overline{DQ}^2$, that is $\overline{AC}\,\overline{EB}^2 = \overline{DQ}^2\overline{DZ}$. Cardano wants to find a point O on AB such that $\overline{AO}\,\overline{OB}^2 = \overline{AC}\,\overline{EB}^2$.

He performs the following construction. He takes the line passing by C and E in order to draw the rectangle of vertexes C, F, G, H. Then, he takes M on GH such that $\overline{GM} = \overline{DQ}$ and considers $\overline{FN}$ such that $\overline{AB} : \overline{EB} = \overline{EB} : \overline{FN}$. Referring to Book II of Apollonius' *Conics*,[126] he draws the parabola with axis FG, parameter $\overline{FN}$, passing through F and the hyperbola with asymptotes HC, CF, passing through B.

In his commentary, Eutocius reports in details on three solutions to the same problem: one found in an old book, one by Dionysodorus, and one by Diocles. As we will see, Cardano follows Eutocius's first solution almost step by step. He does not mention the other two solutions.

[125] Cardano does not explicitly assume that $\overline{DZ} = \overline{AE}$, but it is clear that he must have done so. In fact, immediately after the construction, he states that $\overline{AE} : \overline{AC} = \overline{EB}^2 : \overline{GM}^2$, having drawn $\overline{GM} = \overline{DQ}$.

[126] In particular, *Conics* II.4 says that "[g]iven the asymptotes and a point P on a hyperbola, to find the curve", see [HEATH 1896, page 56].

[Then, Cardano shows by a continuity argument on the sly that the parabola and the hyperbola intersect.] On the one hand, since $\overline{DZ} : \overline{AC} = \overline{EB}^2 : \overline{DQ}^2$, $[\overline{AE} = \overline{DZ}]$, $[\overline{EB}^2 = \overline{AB}\,\overline{FN}$ by the hypothesis on $\overline{FN}]$, $[\overline{AB} = \overline{CF}]$, and $\overline{GM} = \overline{DQ}$, it holds that $\overline{AE} : \overline{AC} = \overline{CF}\,\overline{FN} : \overline{GM}^2$. [By similar triangles,] $\overline{AE} : \overline{AC} = \overline{CF} : \overline{FG}$ [and then $\overline{AE} : \overline{AC} = \overline{CF}^2 : \overline{FG}\,\overline{CF}$]. Therefore, $\overline{CF}\,\overline{FN} : \overline{GM}^2 = \overline{CF}^2 : \overline{FG}\,\overline{CF}$. By *Elements* VIII.9,[127] Cardano gets $\overline{FG} : \overline{GM} = \overline{GM} : \overline{FM}$, [that is $\overline{GM}^2 = \overline{FN}\,\overline{FG}$,] which means by Apollonius' *Conics* I[.11][128] that M is on the parabola. On the other hand, since $\overline{(HE)} = \overline{(EF)}$, adding $\overline{(AL)}$ to both, it holds that $\overline{FB} : \overline{HK} = \overline{FG} : \overline{GH}$. This means that K is on the hyperbola. [Since the parabola passes through F and M, the hyperbola passes through B and K, and both are continuously traced, they must intersect.] Cardano calls X the point of intersection.

Then, he performs the last construction. He draws the parallel and perpendicular lines to AB passing through X. The point O is where the perpendicular cuts AB.

Cardano needs now to show that the intersection point S between FG and the parallel to AB is aligned with C, O, so that he can exploit the properties of similar triangles. By the hyperbola definition, $\overline{(XP)} = \overline{(AF)}$ and, taking away the common $\overline{(AP)}$, he gets $\overline{(RO)} = \overline{(OF)}$. [By the converse of *Elements* I.43,[129]] they are the complements of two parallelograms around the diameter that pass through C, O, S. Then, $\overline{CF} : \overline{FS} = \overline{AO} : \overline{AC}$. Since $\overline{EB}^2 = \overline{CF}\,\overline{FN}$, Cardano gets $\overline{FS}\,\overline{FN} : \overline{EB}^2 = \overline{AO} : \overline{AC}$. By the definition of the parabola and since $\overline{SX} = \overline{OB}$, he finds that $\overline{OB}^2 : \overline{EB}^2 = \overline{AO} : \overline{AC}$, that is $\overline{AO}\,\overline{OB}^2 = \overline{AC}\,\overline{EB}^2$ by *Elements* XI.34.[130] □

[127]"If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion", see [Heath 1956b, page 358].
[128]"First let the diameter PM of the section be parallel to one of the sides of the axial triangle as AC, and let QV be any ordinate to the diameter PM. Then, if a straight line PL (supposed to be drawn perpendicular to PM in the plane of the section) be taken of such a length that $PL : PA = BC^2 : BA\,AC$, it is to be proved that $QV^2 = PL\,PV$", see [Heath 1896, page 8].
[129]"In any parallelogram, the complements of the parallelograms about the diameter are equal to one another", see [Heath 1956a, page 340].
[130]"In equal parallelepipedal solids the basis are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal", see [Heath 1956c, page 345].

In short, Cardano provides as usual the good assumptions under which the concerned equations can be translated in an appropriate geometrical environment. In this case, this means that the coefficients of the equation $x^3 + 192 = 12x^2$ are represented by the measures of some geometrical quantities – the segment AB and the parallelepiped DQ^2DZ with a squared basis. Note that Cardano introduces the point E, the segments AC and u, and later the point M in order to interpret DQ^2DZ in the same diagram as AB. Cardano wants then to show that it exists O on AB such that $\overline{AO}\,\overline{OB}^2 = \overline{DQ}^2\overline{DZ}$ holds. This means in turn that it exists $\overline{OB} = x$ that is a solution of the equation $x^3 + 192 = 12x^2$, since $\overline{AO} = 12 - x$ and $\overline{DQ}^2\overline{DZ} = 192$. Cardano does what is needed in order to draw the parabola and the hyperbola, shows (partially implicitly by a continuity argument) that they intersect, and finally verifies that the intersection point identifies the segment searched for. We remark indeed that the proof does not provide any actual instructions on how to calculate the measure of the segment OB. The segment exists, but one can explicitly find its measure only if he knows how to calculate the abscissa of the intersection point of the parabola and the hyperbola, which is indeed equivalent to solve the cubic equation.[131] But clearly enough – as Cardano recalls in the close of this chapter[132] – this is far beyond his skills and the aim of the proposition. We cannot say for sure whether Cardano wrote this chapter before or after having found the cubic formulae. The close could fit both alternatives, since "the arithmetical operation that does not satisfy" could refer either to the fact that Cardano has not yet found the cubic formula or that he has found a cubic formula that does not always work. In any case, the geometrical environment in which the proposition at issue is translated is not suitable to draw the numerical value of the solution or an algorithm that gives it.

[131]In fact, if we take C to be the origin of a Cartesian coordinate system, FC on the x-axis (right to left oriented, such that $\overline{OB} = x > 0$), and FG on the y-axis, the equations of the parabola and hyperbola respectively are $x^2 = \frac{4}{9}a_2 y$ (or $x^2 = \frac{16}{3}y$) and $(a_2 - x)y = \frac{9}{4}\frac{a_0}{a_2}$ (or $(12 - x)y = 36$). Then, their intersection points are given by the solutions of $x^3 + 192 = 12x^2$. We moreover remark that the parabola and hyperbola have three intersection points (since the equation has three real solutions). They are one on the left and one on the right of the intersection point showed by Cardano in the diagram.

[132]"And the easy geometrical operation is the most difficult in arithmetic and it does not even satisfy" or "*Et ideo facilis operatio Geometrica difficillima est arithmetice, nec etiam satisfacit*", see [Cardano 1570a, Chapter XII, page 27].

At the end of the proof, Cardano adds a few words as an explanation. He firstly observes how "the reasoning of the construction [*ratio constructionis*]" relies on the definitions of the parabola (for instance, $\overline{GM}^2 = \overline{FN}\,\overline{FG}$) and of the hyperbola (for instance, $\overline{(XP)} = \overline{(AF)}$). Then, he remarks that the parameter of the parabola depends only on the coefficient of the term of degree two, while the hyperbola depends on both the coefficients of the considered equation. Finally, Cardano seems to suggest that one can draw $\overline{FS}$, and consequently $\overline{OB}$, thanks to the parabola.[133] Unluckily, this last part is quite corrupted,[134] and in any case contains a serious miscalculation.[135]

We remark that, in this proof, some geometry is undeniably employed and is of a different kind than usual, beginning with the diagram itself, which is not the one that Cardano usually draws. This, for sure, also depends on the fact that the aim of the proof is not to calculate a solution, but only to show its existence.

In order to seize more clearly the implications of Cardano's proof, let us have a look at where the Proposition (A XII.iii) comes from.

4.5.2. Comparing Cardano's proof with his forerunners' one. Cardano explicitly quotes Eutocius' commentary on the Proposition II.4 of Archimedes' *On the sphere and cylinder* to be the source of his proof. Therefore, let us go and briefly check this forerunner.

Being a basic treatise with possibly a good deal of applications, *On the sphere and cylinder* had a relatively lucky history compared to other Archimedes' books. Anyway, it experienced, as it was common, some alterations. For instance, already during the 2nd century BC, Diocles and Dionysodorus had no more available the proof that Archimedes maintained to have written at the end of the Proposition II.4. During the 4th century AD, Eutocius confirms that this proof is missing from all the codex that he has consulted. Nevertheless, in a passage from an old book, he bumped into a theorem that he ascribes to Archimedes and that he

[133]This would correspond to draw y in the system given by the equations of the parabola and hyperbola, but the calculations do not correspond to Cardano's ones.

[134]See for example "*Posita ergo fs quad. c*", [Cardano 1570a, Chapter XII, page 27].

[135]While "reducing to one the cube [*reduxerimus ad unum cubum*]", Cardano finds 8 for the coefficient of the term of degree two, whereas the correct equation – according to all his hints up to now – should have been $3\sqrt{3}y^2 - y^3 = 9\sqrt{3}$. Moreover, not even the formulae provided by Cardano to calculate these coefficients are consistent with his previous procedures.

believes to be the missing proof. It must be said that nowadays Eutocius' opinion is no more trusted, since his arguments (amounting to the use of Doric dialect and of some ancient terminology for the conic sections) are not decisive.

Proposition II.4 is about cutting a sphere in two parts such that they have a given ratio. This result relies on a so-called auxiliary problem,[136] the proof of which was promised by Archimedes and got lost.

Archimedes' auxiliary problem to *On the sphere and cylinder*, II.4. *Let*[137] *the segment AB and a point E on AB such that $\overline{EB} = 2\overline{AB}$, and let $\overline{AC} < \overline{AB}$.*
To find a point O on AB such that $\overline{AO} : \overline{AC} = \overline{EB}^2 : \overline{OB}^2$.

For comparison's sake, I have used the same lettering as Cardano's. Then, we can make reference to the diagram 4.13 at page 319. We remark that this is exactly what Cardano needs to prove in (A XII.iii) in order to show that the family of equations $x^3 + a_0 = a_2x^2$, with $\Delta_3 < 0$ and $q \neq 0$, has a (real) solution.

Eutocius proves a slightly more general version in his commentary.

Eutocius' auxiliary problem to *On the sphere and cylinder*, II.4. *Let*[138] *AB, AC be given segments and Δ be a given surface.*
To find a point O on AB such that $\overline{AO} : \overline{AC} = \overline{\Delta} : \overline{OB}^2$.

This version requires a condition of existence ("*diorismos*") on O. In fact, $\overline{AO} : \overline{AC} = \overline{\Delta} : \overline{OB}^2$ implies that $\overline{AO}\,\overline{OB}^2 = \overline{AC}\,\overline{\Delta}$, where $\overline{AC}\,\overline{\Delta}$ is given by hypothesis. But, if the segment AB is cut in any point E, the maximum value of $\overline{AE}\,\overline{EB}^2$ is attained when E is such that $\overline{EB} = 2\overline{AE}$ (as Eutocius shows later on[139]). Then, it must be $\overline{AC}\,\overline{\Delta} \leq \overline{AE}\,\overline{EB}^2$. Note that the condition is automatically matched by the particular $\Delta = EB^2$ chosen by Cardano. In truth, the condition holds when the problem is associated (in the same way as Cardano does) to any equation in which[140] $\Delta_3 \leq 0$.

[136] See [Heath 1897, pages 62-64] or [Ver Eecke 1921, pages 101-105].
[137] See [Heath 1897, page 64] or [Ver Eecke 1921, pages 103 and 635].
[138] See [Heath 1897, page 66] or [Ver Eecke 1921, page 636].
[139] I will skip the proof, since it will lead us too far from our interests.
[140] In fact, it is enough to show that $\overline{AC} \leq \overline{AE}$ and this is true because $\overline{AC} = \frac{9}{4}\frac{a_0}{a_2^2}$, $\overline{AE} = \frac{a_2}{3}$, and $a_0 \leq \frac{4}{27}a_2^3$ since $\Delta_3 \leq 0$.

Eutocius gives an analysis and a synthesis for this problem. I will only sketch the idea behind the analysis,[141] since it was the part that, according to the common practice, Cardano did not make explicit.

Eutocius' auxiliary problem to *On the sphere and cylinder*, II.4 — *Analysis.* [Assume that O is given.]

Eutocius considers the lines that passes by C and O in order to draw the rectangle of vertexes C, F, S, R and the line perpendicular to AB that passes through O. He takes $\overline{FN}$ such that $\overline{\Delta} = \overline{AB}\,\overline{FN}$.

[By hypothesis, by some trivial identities, and by the properties of similar triangles,] it follows that $\overline{XS}^2 = \overline{FN}\,\overline{FS}$, that is, that the parabola of axis FS, parameter $\overline{FN}$, that passes through F also passes through X. Similarly, it follows that $\overline{AB}\,\overline{FB} = \overline{RX}\,\overline{XL}$, that is, that the hyperbola with asymptotes RC, CF, that passes through B also passes through X. Since the parabola and the hyperbola are given, the intersection point X is also given, and then the projection point O is given.

Note that this analysis explains how the intersection point X of the parabola and hyperbola is linked to the point O, which identifies the segment sought for, but gives no hint on how the parabola and hyperbola were discovered.

The synthesis[142] is identically retraced by Cardano's proof of (A XII.iii), so I will omit it. From now on I will speak of 'Cardano's proof of the auxiliary problem' to highlight the peculiar rephrasing of the statement that links the auxiliary problem to a cubic equation *via* certain assumptions – but keep in mind that it is a particular case of Eutocius' synthesis. As Cossali remarks,[143] Cardano's idea is to apply the solution of the auxiliary problem to the case in which its statement hides a cubic equation that falls into the *casus irreducibilis*.

We have pointed out that the Proposition (A XII.iii) and its proof follow Eutocius' general version of the auxiliary problem and his synthesis. We know for sure that Cardano was overall interested in Archimedes' works. In fact, in the

[141] See [Heath 1897, pages 67-68] or [Ver Eecke 1921, pages 636-639].
[142] See [Heath 1897, pages 68-69] or [Ver Eecke 1921, pages 639-641].
[143] "*Di Cardano fu dunque propriamente solo l'avere a tale equazione condotta la geometrica ricerca della radice dell'equazione $x^3 + q = nx^2$ nel caso, che nella formola di algebraica risoluzione presentasi da immaginario implicamento svisata*", see [Cossali 1799a, page 434].

1544 edition of his autobiography,[144] Cardano explains that he was planning to achieve a revision of the *Elements*, which should have also collected the whole geometrical knowledge from Euclid on and some original propositions by Cardano himself. Cardano reveals that he left aside Archimedes' works, but that he had already agreed with Filippo Archinto (to which the *Ars magna arithmeticæ* was dedicated and who helped Cardano to get a position at the Scuole Piattine in 1534) to add them.

Then, it is natural to wonder what version of *On the sphere and cylinder* Cardano could have at hand.[145] When Charles of Anjou, called by the pope Clement IV, took possession of the kingdom of Frederick II, some Greek manuscripts went from Naples to the Papal Court. Among them, there was almost the whole of Archimedes' works (including *On the sphere and cylinder*) and Eutocius' commentaries. In 1269 the literal Latin translation by Willem van Moerbeke, which was based on these manuscripts, was achieved. Anyway, this was not enough to boost the study of Archimedes' texts. In fact, its mathematics lingered fairly unknown during the Middle Ages, since, on the one hand, it did not matched the prevailing interest for calculation of that time. On the other hand, it was a difficult mathematics that required a very good knowledge of the Greek *corpus*, such as the proportion theory, the conic sections, and Book XII of the *Elements*, and all of them were missing in the Middle Ages. In 1455 *On the sphere and cylinder* was translated again in Latin by Iacopo da Cremona, but his translation was even less popular than Moerbeke's one.

Later on, around the first half of the 16th century, Archimedes' works became better reknown. Johannes Müller von Königberg, also known as Regiomontanus, reexamined Iacopo's translation for the first time from a completely different viewpoint. He was a real scientist and an expert mathematician whose aim was to give a new birth to the Greek science. But Regiomontanus' ambitious project was prevented by his death. His papers were partially used by Thomas Geschauff, also known as Venatorius, to publish in 1544 in Basel the Greek *editio princeps*

[144]"*Euclides omnia collegit, quæ in Geometria ab antiquis scripta essent, nos quæ invenimus, suis autoribus adscriptis, nostris inventis adiunximus, uno Archimede relicto, quem cum Archinto illi adiungere decrevimus*", see [Cardano 1663d, page 57]. See also [Gavagna 2003, page 127].
[145]For the following, I will mainly refer to [Napolitani 2001].

of Archimedes' works (containing also Eutocius' commentaries). This is the first printed (and consequently truly widespread) version.

Note that I have only mentioned Archimedes' editions and manuscripts that are previous to 1570 and in which *On the sphere and cylinder* appears. By the way, also Tartaglia made an edition of Archimedes' works in 1543, but it does not contain the book in which we are interested. We do not know which edition of Archimedes (and of Eutocius) Cardano could have read, but it is extremely unlikely that he had available one of the handwritten translations by Moerbeke or Iacopo.[146] Then, only one plausible possibility remains, that is to say, that Cardano had the *editio princeps* available with Eutocius', Dionysodorus', and Diocles' solutions to the auxiliary problem.

At this point we ought to spend a few words on the solution to the very same problem[147] given by 'Umar al-Khayyām. Once and for all, it must be clearly said that we are not privy to any relationships with Cardano's proof. Nevertheless, I will briefly deal with it, since Khayyām's strikingly similar use of the intersection of a parabola and a hyperbola to get the searched point will help to shed a new light on Cardano's usage.

Both Archimedes' *On the sphere and cylinder* and Eutocius' commentary were translated into Arabic, but, in most of the cases, separately. Archimedes' translation was made around the 9$^{\text{th}}$ century and was mostly unabridged, which was not the case for Eutocius' commentary. Not later than in the 10$^{\text{th}}$ century, the commentary was fragmentary translated and very often did not go together with Archimedes' book.[148] Then, it is not surprising that many Arabic mathematicians, probably unaware of Eutocius' efforts, tried to complete Archimedes' text.

We find Khayyām's solution in the *Treatise of algebra and al-muqābala* (probably written in the first part of the 12$^{\text{th}}$ century). Apart from the fact that Khayyām explicitly mentions the Proposition II.4 of *On the sphere and cylinder*, we know virtually nothing on his relationship with Eutocius' text. As it is well

[146] Nevertheless, one of the manuscripts by Iacopo circulated in the same Milanese milieu frequented by Cardano some years before his birth. In fact, Giorgio Valla attests that his mathematics teacher Giovanni Marliani, medicine professor in Pavia and member of the Collegio dei fisici in Milan, had a copy of one of Iacopo's translations. Marliani died in 1483.

[147] For a detailed history of the auxiliary problem, see [Netz 2004].

[148] See [Netz 2004, pages 139-131].

known,[149] Khayyām's aim is not merely to solve the auxiliary problem. Rather, the problem is embedded in a family of equations, and its solution comes as a drawback of the proof of the existence of (at least) a (real) solution of this family of equations. Khayyām starts with a complete classification for equations up to degree three – following some criteria (namely, to have positive coefficients and positive solutions, never to equate something to zero, and to group the equations according to the number of terms) that will be reminded by Cardano's ones. Whenever Khayyām is able to, he solves the equations under an arithmetical interpretation to provide the value of the solution(s). This is obviously not the case for the equation $x^3 + a_0 = a_2 x^2$ at issue.

This equation fills up the fifth position in the group of six equations in three terms. Modern editors used to number the equations one after the other, so that this equation became 'Equation 17'. Khayyām's treatment is highly articulated. On the one hand, assuming that the solutions must be positive entails a condition on the coefficients (namely, that $\sqrt[3]{a_0} < a_2$). Khayyām argues by the absurd in order to count out the other cases. On the other hand, the geometrical environment in which he interprets his hypotheses leads him to further discuss three cases. Since we are only interested in the comparison with Cardano's text, I will directly make the good assumptions in order to be in the same case as $x^3 + 192 = 12x^2$.

Khayyām's Equation 17. *Consider*[150] *the equation* $x^3 + a_0 = a_2 x^2$*. Assume*[151] *that* $\sqrt[3]{a_0} < \frac{a_2}{2}$*.*
Then, it has a [real] solution.

Khayyām's Equation 17 — *Proof***.** Khayyām takes $\overline{AC} = a_2$. We consider a_0 as a parallelepiped with a squared basis. By a previous lemma,[152] Khayyām

[149]See for example [Rashed and Vahabzadeh 1999] and for an interpretation [Panza 2007, pages 125-142].
[150]See [Woepke 1851, pages 40-44] and [Rashed and Vahabzadeh 1999, pages 46-52, 174-180].
[151]Since $\frac{1}{8} < \frac{4}{27}$, the assumed condition implies that $a_0 < \frac{4}{27} a_2^3$, that is $\Delta_3 < 0$.
[152]"*Étant donnés le carré* $ABCD$*, base du parlallélépipède rectangle* $ABCDE$*, et le carré* MH*, construire sur la base* MH *un parallélépipède rectangle égal au solide donné* $ABCDE$", see [Woepke 1851, pages 30-31], or also [Rashed and Vahabzadeh 1999, pages 156-158].

knows that it exists a segment H such that $\overline{H}^3 = a_0$. He takes B on AC such that $\overline{BC} = \overline{H}$.

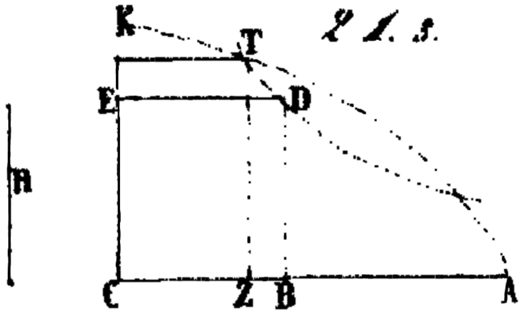

FIGURE 4.14 – Woepke's diagram in Khayyām's *Treatise of algebra* for the proof concerning Equation 17.

By hypothesis it follows that $\overline{BC} < \overline{AB}$.

Then, Khayyām performs the following construction. He draws the square of BC, the vertexes of which are C, B, D, E. He draws the parabola of axis AC, parameter $\overline{BC}$, passing through A and the hyperbola with asymptotes AC, CE, passing through D. [Since $\overline{BC} < \overline{AB}$, then $\overline{BC}^2 < \overline{AB}\,\overline{BC}$ and] the point D is inside the parabola. Therefore, the parabola and the hyperbola intersect in two points. Khayyām calls T one of them and Z its projection on AC. He wants to prove that ZC is a solution of the equation.

[Since T, D belong to the hyperbola,] $(\overline{TC}) = \overline{BC}^2$, and then $\overline{ZC} : \overline{BC} = \overline{BC} : \overline{TZ}$. Since T belongs to the parabola, $\overline{TZ}^2 = \overline{BC}\,\overline{AZ}$, and then $\overline{BC} : \overline{TZ} = \overline{TZ} : \overline{AZ}$. Therefore, $\overline{ZC} : \overline{BC} = \overline{BC} : \overline{TZ} = \overline{TZ} : \overline{AZ}$, then $\overline{ZC}^2 : \overline{BC}^2 = \overline{BC} : \overline{AZ}$, and then $\overline{BC}^3 = \overline{ZC}^2\overline{AZ}$. Finally, $\overline{ZC}^3 + a_0 = \overline{ZC}^3 + \overline{ZC}^2\overline{AZ} = \overline{ZC}^2(\overline{ZC} + \overline{AZ}) = \overline{ZC}^2\overline{AC}$, and then $\overline{ZC}^3 + a_0 = a_2\overline{ZC}^2$. $\square$

Note that the structure of Khayyām's proof does not substantially differ from Eutocius and Cardano's ones. In fact, once chosen the appropriate conic sections, the core of the proof is to use their properties to show that (one of) their intersection point(s) defines the searched segment.

Then, we shall search elsewhere for the difference. The choice of the conic sections[153] is a good option to inquire into. We are able to guess a sensible analysis

[153]Note that Khayyām's choice of the conic sections differs from Cardano's one. Let us suppose $a_0 = 192$ and $a_2 = 12$. If we take C to be the origin of a Cartesian coordinate system, AC on the x-axis (right to left oriented), and EC on the y-axis, the equation of the parabola and

for Khayyām's proof[154] and, contrary to Eutocius' one, it let us understand how Khayyām chose the conic sections. In fact, it is enough to retrace backwards the proof of Khayyām's Equation 17 at page 327.

Khayyām's Equation 17 — *Analysis.* Consider $x^3 + a_0 = a_2x^2$. It follows that [$a_0 = a_2x^2 - x^3$ and] $a_0 = x^2(a_2 - x)$, [that is, the coefficient of the term of degree zero can be written under the shape of a parallelepiped with squared basis.[155]] Thanks to a lemma,[156] we know that it exists a (real) number h such that $h^3 = x^2(a_2 - x)$. Then, $x^2 : h^2 = h : a_2 - x$. It follows that it exists a (real) number y such that $x : h = h : y = y : a_2 - x$. [This gives the equations of the conics sections, if we take $h = \sqrt[3]{a_0}$. The parabola is $y^2 = h(a_2 - x)$ and the hyperbola is $xy = h^2$.[157]]

If we try to gather some clues on how Cardano chase his conic sections simply by retracing backwards the last steps of the proof of (A XII.iii) (that is, the analysis of (Eutocius)), we do not get so straight the conic sections. Rather, we do not get them at all, since indeed we need to already know how they are defined. Nevertheless, it is not a major problem to imitate Khayyām and wildly derive Cardano's (or Eutocius') supposed conics directly from the equation.[158] The point is that in Cardano's text you do not find anything similar. This suggests that Cardano draws the conic sections in a way that does not directly depend on the equation. I do not have any guess on that, but we remark that Cardano's conic sections are tightly related to $\frac{4}{9}a_2$, which is the parameter FN of the parabola

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hyperbola respectively are $y^2 = 4\sqrt[3]{3}(x + 12)$ (or $y^2 = \sqrt[3]{a_0}(x + a_2)$) and $xy = -16\sqrt[3]{9}$ (or $xy = -\sqrt[3]{a_0^2}$). Then, their intersection points are given by the solutions of $x^3 + 192 = 12x^2$. We recall that Cardano's conics are $x^2 = \frac{4}{9}a_2y$ and $(a_2 - x)y = \frac{9}{4}\frac{a_0}{a_2}$, see above, page 321.

[154] I was inspired by [Panza 2007, pages 136-137] and I thank Marco Panza for his notes.

[155] We remark that Khayyām employs the common language for geometrical objects and numbers. See [Panza 2007, pages 127-128].

[156] See above footnote 152

[157] Note that, up to the sign, these are the same conic sections of the footnote 321, at page 321. This sign depends on the orientation of the x-axis that I chose in order to have the same sign for the solutions compared to Cardano's one.

[158] Consider $x^3 + a_0 = a_2x^2$. Then, it follows that $x^2(a_2 - x) = a_0$. We write $x^2(a_2 - x) = a_0\left(\frac{9}{4a_2^2}\right)\left(\frac{4}{9}a_2^2\right)$, or better $x^2(a_2 - x) = \left(\frac{9}{4}\frac{a_0}{a_2^2}\right)\left(\frac{4}{9}a_2^2\right)$.
Up to now, this is simply Cardano's retraced. But then he uses the definition of the parabola, so that we cannot hope to derive it. Let us leave Cardano's way. We write $x^2 = \left(\frac{4}{9}a_2\right)\left(\frac{9}{4a_2}\frac{a_0}{(a_2 - x)}\right)$. Then, the parabola is $x^2 = \frac{4}{9}a_2y$ and the hyperbola is $y = \frac{9}{4a_2}\frac{a_0}{(a_2 - x)}$.

and which appears in the hyperbola together with a_0. In turn – as Cardano says at the end of the proof – $\frac{4}{9}a_2$ depends only on $AB = a_2$ and on E, which is taken at two-thirds of AB. The point E is needed when the problem is shaped under the terms of the auxiliary problem, since one needs to have the point in which the maximum value of $\overline{AE}\,\overline{EB}^2$ is attained in order to state the *diorismos*. Then, Cardano's conic sections are tightly related to the point E, which arises because of the geometrical (positional) properties of the diagram. But this does not happen in Khayyām's text, where the conic sections are chosen starting from the equation itself. As Netz says, "[s]ince Khayyam's study of cases is logically prior to his study of geometrical properties, he is not interested in the geometrical properties of the points that define cases".[159]

4.5.3. Summing up. Summing up, while analysing Chapter XII, we have talked out all the geometrical arguments that are in the *Aliza*. As said, Cardano usually employs a kind of geometry with no positional features, which, as such,

[159]See [Netz 2004, page 168].
His point is that

> [t]he author faced raw geometrical reality and transformed it into a statement in words, and the words still have impressed on them this fresh stamp of reality. The same is no longer true for Khayyam. While we do not know exactly which works Khayyam was aware of, we know – from his own words [while dealing with $x^3 + a_0 = a_2x^2$, Khayyām quotes Abū al-Jūd and al-Māhānī] – that he was acquainted with several treatments, some successful, some not, of the very same problem. Thus, when Khayyam sets out to produce his new version, he faces not geometrical reality in the raw, but geometry already transformed int verbal forms. He did not transform reality into words, but words into words,

see [Netz 2004, page 160]. Netz provides some linguistic arguments to support his view, of which I am not fully convinced. Anyway, I choose to pick up the ones that can also be applied to Cardano so that the reader can appreciate them.
First, there is the line called '*H*'. Since it does not intersect with other geometrical objects, it cannot be distinguished by any of his points. Then, "[t]his heterogeneous way of naming lines makes it somewhat less natural to see the expressions '*H*', '*AC*' as mere symbols. As mere symbols, they are homogeneous; their heterogeneity is a function of the geometrical configuration". Still, there is the "permutability of names", as AZ and ZA, which destroys the identity as symbols of two-lettered objects.
Finally, Netz ends: "[i]n short, we see that Khayyam opens up the possibility of considering his objects symbolically, as elements manipulated by the rules of calculation; yet essentially conceives them as components in a geometrical configuration". For more details, see [Netz 2004, page 163].

imitates arithmetic. This chapter of the *Aliza* turns out to be the exception that proves the rules.

It is indeed true that the comparison with Khayyām's choice of the conic sections highlights that, on the contrary, in Cardano's case, this choice is tightly related with a certain point that depends in turn on the positional properties of the diagram. But we have also seen that here Cardano is substantially reproducing Eutocius' commentary on the Proposition II.4 of *On the sphere and cylinder*. No surprise then that he is using his language of proportions together with his conception of geometry. But Cardano does not add any significant feature to Eutocius' proof. In my opinion, Chapter XII is on of those scattered notes that has been collected and included in the *Aliza* because it is focused on a cubic equation. The chapter stands, in fact, aside form the others and, in particular, I do not see any true connection with the splittings or with some method linked to them.

Therefore, if one wants to search for the genuine geometrical inspiration in Cardano's reasoning – which is a completely reasonable aim – I believe that it is better to refer, for instance, to *Aliza*, Chapter LX rather than to Chapter XII.

4.6. Trying for an arithmetic of nasty numbers

In Sections 4.2 and 4.3, we have focused on Cardano's main strategy to overcome the problem of the *casus irreducibilis*. In short, the idea was to try to replace the cubic formula for $x^3 = a_1x + a_0$ with another one that had no conditions on the coefficients. This should have been achieved through the splittings. Then, I gathered as many chapters as possible around this topic. But, if one wants to ride out the problem entailed by the *casus irreducibilis*, there is at least another possibility to consider. One could simply accept the cubic formula as it is and try to live with the square roots of negative numbers that appear there. These ways are not mutually exclusive, and indeed we also find evidences for the second one in the *Aliza*, namely in Chapter XXII.

If one chooses to preserve the cubic formula, he must show how to deal with these nasty numbers. First of all, he needs to look for their arithmetic. Eventually, he hopes that in the formula these numbers are going somehow to vanish. In fact, it is possible that one has experienced some cases – and Cardano for sure had – in which a nice (that is, natural) solution of a cubic equation was known independently from the formula, but some nasty numbers were returned when he tried to calculate this solution through the formula. Apart from speculations, Cardano must have foreseen this direction. In fact, if we believe to Tartaglia's account,[160] Cardano asked him how to find the rule to derive the equalities $\sqrt[3]{\sqrt{108}+10} = \sqrt{3}+1$ and $\sqrt[3]{\sqrt{108}-10} = \sqrt{3}-1$. Note that those cubic roots are exactly the ones that the formula returns[161] for $x^3+6x = 20$. It is not unlikely that he could search for a similar rule when there are negative numbers under the square roots.

If then one undertakes this second way, he would like to define how to perform operations with square roots of negative numbers. One of the very first natural questions that arises is which sign these numbers should have. As we will see, Cardano tries for the easy way: to make the square roots of negative numbers disappear by imitating what happens with the cubic roots, where $\sqrt[3]{-a} = -\sqrt[3]{a}$. For that, he suggests an alternative sign rule, which, as such, is wrong. But the point is that Cardano's attempt displays (again) his will to overpass the problem entailed by the *casus irreducibilis*.

In the *Aliza*, **Chapter VI** "On the operations plus and minus according to the common usage [*De operationibus p: et m: secundum communem usum*]" Cardano recalls the usual sign rules, in particular that "minus multiplied by minus [...] produces always plus".[162] More precisely, he firstly does that for (rational) numbers and then for the *binomia* and *recisa*.

He moreover says that, according to the common usage, the square root of a positive number is positive, but the square root of a negative number does not exists.[163] We observe that this partially contradicts Cardano's statement in the

[160] See [Tartaglia 1959, Libro IX, Quesito XL, pages 125-6].

[161] We remark that this equation is one of the examples in *Ars magna*, Chapter XI. There, it is clear that Cardano does not know that $\sqrt{108}+10$ and $\sqrt{108}-10$ are two cubes, since he leaves the (real) solution written with the two cubic radicals.

[162] "*In multiplicatione et divisione p: fit semper ex similibus, m: ex contrariis, unde p: ductum in p: et divisum per p:, et m: ductum in m: et divisum per m: producunt semper p:. Et ita p: in m: vel m: in p: vel p: divisum per m: vel m: per p: producit m:*", see [Cardano 1570a, Chapter VI, paragraph 1, page 15].

[163] "*R p: est p: R m: quadrata nulla est iuxta usum communem, sed de hoc inferius agemus*", see [Cardano 1570a, Chapter VI, paragraph 3, page 15].

Ars magna, Chapter I and in the *Ars magna arithmeticæ*, Question 3, both on the fact that $\sqrt{9} = \pm 3$.

But in **Chapter XXII** "On the contemplation of plus and minus, and that minus times minus makes minus, and on the causes of these according to the truth [*De contemplatione p: et m: et quod m: in m: facit m: et da causis horum iuxta veritatem*]", Cardano provides a radically different account. I will not try to dictate an interpretation over this text, since it is one of those chapters where unluckily there is very little consistency. I will content myself with highlighting the relevance of Cardano's doubts about the usual sign rule, namely concerning the part that states that a negative number times a negative number makes a positive number. Eventually, we will interpret these doubts as another attempt to overpass the problem entailed by the *casus irreducibilis*.

Cardano softly starts by quite an orthodox beginning. When one sums two numbers – he says – it happens that he can write them as a unique term (for instance, $6+2 = 8$) or not (for instance, $6+\sqrt{2}$ or $6+x$). He incidentally adds – and we keep it in mind for later – that "a multiple quantity is nevertheless equivalent to a simple [one], and this mostly happens with universal and abstruse roots, as it is declared in the *Ars magna* (Chapter 11) that $\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$ is the same as 2".[164] In any case, when one wants to calculate the square of the sum of simple or "composite" numbers, like 8 or $6+\sqrt{2}$ and $6+x$, Cardano suggests to apply *Elements* II, 4.

Afterwards, he addresses the square of the difference of two numbers. Here we find his argument to maintain that the usual sign rule is wrong and that it should be replaced by the following one.

A XII. *A negative number times a negative number makes a negative number.*

I reveal in advance that Cardano's argument is wrong – it must be. Let us have a look at it in detail.

164"*[A]liquando tamen quantitas multiplex æquivalet simplici, et hoc maxime accidit in RV: et abstrusis, velut declaratum est a nobis in Arte magna, quod RV: cu.R 10, 8 p: 10 m: RV: cu.R 10, 8 m: 18, idem est quod 2*", see [Cardano 1570a, Chapter XXII, page 43].

A XXII — *Proof***.** Cardano[165] takes $\overline{AB} = 10$ and $\overline{CB} = -2$.

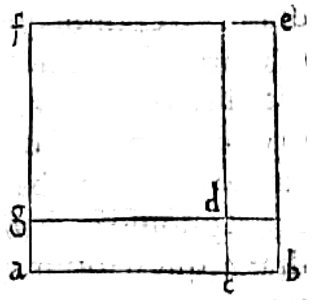

FIGURE 4.15 – *Aliza*, Chapter XXII.

Then, $[\overline{AB}^2 = 100,]$ $\overline{AC} = 8$, and $\overline{AC}^2 = 64$. He considers now the gnomon (GCE). On the one hand, by the same *Elements* II, 4, $\overline{(GCE)} = 2\overline{AC}\,\overline{CB} + \overline{CB}^2 = \overline{(AD)} + \overline{(DE)} + \overline{(BD)}^2$. On the other hand, $\overline{(GCE)} = [\overline{AB}^2 - \overline{AC}^2 = 100 - 64 =]36$. But, since $\overline{(AD)}, \overline{(DE)} < 0$ [as they both are the product of the positive number $\overline{AC}$ and of the negative number $\overline{CB}$, and] $\overline{(AD)} + \overline{(DE)} = -32$, then it must be that $\overline{(BD^2)} = \overline{CB}^2 = -4$, otherwise one would have had that $\overline{(GCE)} = [-32 + 4 =]28$ and $\overline{AC}^2 = [\overline{AB}^2 - \overline{(GCE)} = 100 - 28 =]72$. This means that "the square BC is minus, and is made by a minus multiplied by itself, then a minus multiplied by itself is a minus". □

[165]The full quotation of this controversial passage follows. "Then it is clear that AC is truly 8 and its square DF will be 64, but the gnomon is the whole remainder, as I have similarly said, and if BC is of someone else, therefore the whole gnomon also [belongs] to him, as I will show. And it is agreed that that gnomon is made by AC times CB twice, and they are the rectangles AD, DE, with the square of BC by the same proposition (fourth [Proposition] of the second [Book] of the *Elements*). But that whole gnomon is 36, because the square of AB is 100 and FD [is] 64, therefore the remaining gnomon GCE is 36, and AD and DE are minus and are 32, and the gnomon is -36. Therefore the square of BC, which is 4, is also minus. In fact, if it would have been plus, the gnomon was not minus, if not 28, and DF 72, and AC $\sqrt{72}$ and not $\sqrt{64}$, which is 8. Therefore the square of BC is minus and is made by minus multiplied by itself. Therefore minus multiplied by itself produces minus" or "*liquet ergo quod ac vere est 8, et eius quadratum df erit 64, sed totus residuus gnomo est, ut dixi perinde, ac si bc esset alterius, ideoq́ue totus gnomo etiam illius, ut ostendam, et constat quod ille gnomo per eandem propositionem fiet ex ac in cb bis, et sunt rectangula ad de cum quadrato bc iste autem gnomo totus est 36, quia ab quadratum est 100 et fd 64, igitur gce gnomo residuus est 36, et ad et de sunt m: et sunt 32, et gnomo est 36 m: igitur quadratum bc, quod est 4, est etiam m:, nam si esset p: non esset gnomo m: nisi 28 et df 72 et ac R 72, et non R 64, quod est 8. Igitur quadratum bc est m: et fit ex m: in se ducto, igitur m: in se ductum, producit m:*", see [CARDANO 1570a, Chapter XXII, page 44].

As said, Cardano is mistaken. First of all, we remark that he assumes that the segment CB has a negative measure – which he never did before and will never do later, since he does not define what a segment with a negative measure is. The trick of his argument flows from this. In fact, Cardano repeatedly shifts between considering $\overline{CB} = -2$ and $\overline{CB} = 2$. Namely, he takes at first $\overline{CB} = -2$, but then, deriving $\overline{AC} = 8$, he acts as if $\overline{CB} = 2$. This makes him infer a value for the measure of the gnomon $(G\,C\,E)$. After that, he comes back to his first assumption, that is $\overline{CB} = -2$, to deduce that the measures of AD, DE (both equal to $AC\,CB$) are negative, and therefore that also the value of the measure of the gnomon must be negative. This argument is followed by a second one, which is completely likely, except that the diagram is a rectangle instead of a square.

Hereafter, Cardano explains why it is instead commonly believed (as he has said in Chapter VI) that the product of two negative numbers is positive. Referring to the above diagram [and[166] by *Elements* II, 7], he gets that $\overline{AB}^2 + \overline{CB}^2 = 2\overline{AB\,CB} + \overline{AC}^2$, that is $\overline{AB}^2 + \overline{CB}^2 - 2\overline{AB\,CB} = \overline{AC}^2$. Since $2\overline{AB\,CB} = \overline{(G\,C\,E)} + \overline{CB}^2$, then

$$\overline{AB}^2 + \overline{CB}^2 - \overline{(G\,C\,E)} - \overline{CB}^2 = \overline{AC}^2.$$

Once cleared the ground, Cardano argues that

> only the gnomon is really a minus. Then we subtract as much as the square BC plus what we had [to subtract] from the square AB as if [it were] plus. Therefore, in order to rebuild that minus that we subtracted beyond measure, it is necessary to add as much as the square BC plus. Therefore, as BC is minus, we say that -2 is the inverted square in plus, then that minus times minus produces plus. But it is not true. We added as much as

[166] In the very same paragraph Cardano states that one must employ *Elements* II, 4 for the square of a *binomium* and *Elements* II, 7 for the square of a *recisum*: "as well as with the *binomia* we perform operations by the fourth proposition and according to the substance of the composite quantity, so also with the *recisa* we perform operations by the same [to know] the substance, but by the seventh of the same to know the terms" or "*in binomiis operamur per quartam propositionem, et secundum substantiam quantitatis compositae , ita etiam in recisis quo ad substantiam et vere operamur cum eadem. Sed ad nominum cognitionem operamur in virtute septimæ eiusdem*", see [Cardano 1570a, Chapter XXII, page 45]. This partially contradicts his foregoing practice (when he employed *Elements* II, 4 to calculate the square of a difference), but inconsistencies are the overall course in the *Aliza*.

> the square BC is plus. [It is] not that the square BC is plus, but that another arbitrarily assumed quantity equal to BC is added and made plus.[167]

Finally, Cardano needs to justify the remaining sign rules. At this point, he abruptly starts to call the minus 'alien [*alienum*]'. Tautologically, he argues that "nothing is able beyond its powers, then plus is able of as much as itself"[168] and that the product of two positive numbers is positive. The same argument holds for the product of a positive number and a negative one, which makes a negative number.

Starting from these rules for the multiplication, Cardano says that one can infer the corresponding rules for the division and for taking the square root. He explicitly suggests only the rules for the division, and here come the drawbacks of his brand-new sign rules. In fact, in this way it happens that the division of a positive number by a negative one is no more well-defined. Moreover, a negative number divided by another negative number gives both a positive and a negative number (though Cardano does not mention it here, but rather in the *Sermo de plus et minus*, see below at page 338). Cardano does not go on detailing the consequences for taking the square root. If he had done so, he would have found that the square root of a negative quantity is negative, hence contradicting again his earlier claims in Chapter VI and both the ones in the *Ars magna* and *Ars magna arithmeticæ*. Let us schematically resume Cardano's new rules and their consequences. I will write for short '+' for a positive number and '−' for a negative one.

[167]"*[S]solus gnomo vere sit m:, quia ergo detrahimus quantum est quadratum bc, plusquam deberemus a quadrato ab, tamquam p:, ideo ad restitutionem illius m: quod detrahimus præter rationem oporteret addere, quantum est quadratum bc p:, et ideo cum bc sit m:, dicemus quod 2 m: quadratum conversum est in p: ideo quod m: in m: produxit p:. Sed non est verum; sed nos addidimus quantum est quadratum bc p: non quod quadratum bc sit p:, sed alia assumpta quantitas pro arbitrio nostro æqualis bc addita est, et facta est p:*", see [Cardano 1570a, Chapter XXII, page 44].

[168]"*[N]ihil potest ultra vires suas, ergo p: potest quantum est ipsum*", see [Cardano 1570a, Chapter XXII, page 45].

Multiplication	Division	Square root
$+ \cdot + = +$	$\frac{+}{+} = +$	$\sqrt{+} = +$
$+ \cdot - = -$	$\frac{-}{+} = -$ and $\frac{-}{-} = +$	
$- \cdot + = -$	$\frac{-}{-} = +$ and $\frac{-}{+} = -$	
$- \cdot - = -$	$\frac{-}{-} = -$	$\sqrt{-} = -$

TABLE 4.3 – New sign rules for the multiplication and division in *Aliza*, Chapter XXII, from which I derive the consequences concerning the square root.

All along this chapter we followed Cardano attempting to an alternative sign rules for numbers – and maybe also for nasty numbers, that is for square roots of negative numbers. He employs an overstretched argument (which also involves *Elements* II, 4) to justify that the product of two negative numbers is negative. Anyway, I believe that we should not focuses on the potential (and wrong) result, but rather on the reason that pushes Cardano towards rethinking such a topic. I suggest to frame the whole problem in connection with the remark on simple and "composite" numbers at the very beginning of the chapter. This also has the advantage to provide the chapter with a little more internal cohesion. In particular, I consider the example $\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} = 2$, which was quoted from the *Ars magna*. My point is that, if it had been true that the product of two negative numbers is negative, then $\sqrt{-a} = -\sqrt{a}$ for any number a. Thus, Cardano would have been enabled to deal with all the expressions, such as the ones coming from a cubic formula, that contains square roots of negative numbers.

At this point, a mention of the *Sermo de plus et minus* is due. This is a very short pamphlet, never published during Cardano's lifespan (and never mentioned in his autobiography), but included in the fourth volume of his *Opera omnia*. It is a reaction[169] to Rafael Bombelli's *Algebra*, which was published in 1572: this means that the *Sermo* is a very late work. It is far from my purpose to enter all the details of this text, since if something unclearer than the *Aliza* exists, here it is. I will just briefly taking into account its beginning,

[169] See [TANNER 1980a].

The *Sermo* starts where the *Aliza*, Chapter XXII stopped. There is even an explicit cross-reference to it.[170] Here, the survey on the consequences of the sign rules for division is fully achieved, since Cardano lists the cases $\frac{+}{+} = +$, $\frac{-}{+} = -$, $\frac{-}{-} = +$, $\frac{-}{-} = -$, and "nothing [*nihil*]" for $\frac{+}{-}$. The second paragraph starts in a very rambling way, but it is here that the Proposition II, 7 of the *Elements*, Rafael Bombelli, and the equation $x^3 = a_1x + a_0$ are mentioned all together. In the middle, we read (in Tanner's translation) that

> [o]n the other hand, Raphael Bombellus of Bologna confined this to $\sqrt[3]{}$ of a binomial and its conjugate, because this minus appeared useful exclusively for the complete treatment of cube equal to unknown and number. But there the $\sqrt[3]{}\, l.$ is two-fold, namely that of the binomial and that of the conjugate of the binomial. So he was right to reserve this minus solely for these two cases, namely the $\sqrt[3]{}\, l.$ of a binomial and of its conjugate.[171]

Cardano carries on by remarking that what Bombelli called '*p: m:*' and '*m: m:*' are to be intended as '*p: di m:*' and '*m: di m:*' (which mean $+\sqrt{-}$ and $-\sqrt{-}$). This yields to Tanner's "first impression that Bombelli's notation was the result of an intentional misprint, whether condoned by him or not, and misguidedly perpetuated in its modern reprinting".[172] Then, Cardano recalls the sign rules given by Bombelli for these numbers ("*p: di m:*" times "*m: di m:*" and "*m: di m:*" times "*p: di m:*" make "*p:*", "*p: di m:*" times "*p: di m:*" makes "*m:*", and "*m: di m:*" times "*m: di m:*" makes "*m:*") and says that Bombelli has found (a) cubic root of $2 \pm \sqrt{-121}$. This enables Bombelli to draw a real solution of the equation $x^3 = 15x + 4$, which is

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4,$$

[170]Tanner remarks the misprint '*Li. Aliza cap. 2*' instead of '22', see [Tanner 1980a, page 168].

[171]See [Tanner 1980a, pages 169-170]. "*Raphael autem Bombellus Bononiensis contraxit hanc ad R cub. Binomij et recisi, quia non videbatur m: hoc utile nisi pro perfectione cubi æqualis rebus et numero. Sed ibi est R cub. l. duplex binomii scilicet et sui recisi. Ideo recte contraxit hoc m: ad illas duas conditiones R cub. scilicet l. et binomii cum suo reciso*", see [Cardano 1663j, page 435].

[172]See [Tanner 1980a, page 171].

by the cubic formula. I have not been able to retrieve other useful information from the very obscure remaining three pages. Therefore, I am dropping here the *Sermo*.

I would only suggest a last, short remark on the *Sermo*. It is true that there is no explicit relation between Cardano's sign rule on the product of negative numbers and Bombelli's one on "*m: di m:*" times "*m: di m:*". But nevertheless it remains the fact that Cardano himself found a link between the two, since he started the *Sermo*, which is devoted to Bombelli's "*p: di m:*" and "*p: di m:*", by recalling the *Aliza*, Chapter XXII. This strengthens my hypothesis that the grounding reason of Cardano's rethinking of the sign rules lies in his willing to handle the square roots of negative numbers.

Summing up, we have good reasons to believe that Cardano has seriously been questioning about the square roots of negative numbers, and in particular about their sign. Cardano tries out various possibilities: they exist but behave differently, they do not exist, they are negative. We recall that in both the *Ars magna arithmeticæ* and the *Ars magna* Cardano keeps the safest road, since he only suggests that the square roots of negative numbers do not possibly behave like the others. In fact, he says – they cannot be neither a positive or a negative number, since we have, for instance, that $\sqrt{9} = \pm 3$. He assumes that $(\sqrt{-a})^2 = -a$ and that, once that the square root has vanished, they behave as usual. Thus, the common sign rules hold – everything is fine. This low-profile handling of the square roots of negative numbers is in the same style as *Aliza*, Chapter VI, even if it is not completely consistent with it (in the *Aliza* Cardano seems rather to believe that the square roots of negative numbers do not exist).

Then, we crush into Chapter XXII's rift, which is carried on in the *Sermo de plus et minus*. We do not have to be too much surprised by them. Inconsistency being the general feature of the *Aliza*, we only have to pay attention not to throw the baby out with the bathwater. And the baby here is the aim that Cardano wanted to attain, no matter if he failed. Rethinking the common sign rules, in fact, has an intrinsic value by itself, since it is a symptom of another attempt to overpass the problems caused by the presence of these nasty numbers in the cubic formula.

Conclusions

The treatise on which my dissertation has been focused, the *De regula aliza*, does not have the features of a masterpiece – but it definitely is a work by one of the most brilliant mathematicians in the 16^{th} century. Reading this book is a constant fight against the messiness of its structure. Apart from some extrinsic causes, like misprints and calculation mistakes, there is a profound reason behind: Cardano's attempt – or, at least, Cardano's attempt in the chapters that we have taken into account – was indeed unattainable. Nevertheless, in the framework of Cardano's solving methods for cubic equations, the missed purpose of these chapters is still very talkative.

My dissertation is not thought to be an exhaustive work on the *Aliza* – on the chance that that such a kind of work may ever be written – but rather one step in decoding it. We recall that an early step in this direction, however controversial, has already been made by the mathematician Pietro Cossali at the end of the 18^{th} century.

Even though the *Aliza* is published twenty-five years after the latest of Cardano's works on equations, it easily finds its own place among his other mathematical treatises.

In the *Practica arithmeticæ*, *Ars magna arithmeticæ*, and *Ars magna*, we have followed the development of Cardano's treatment of equations. They pass from being a side-effect of proportions in the *Practica arithmeticæ*, which is still an *abaco*-oriented treatise, to become more and more a proper subject of inquiry. The mark of this passage is in the *Ars magna arithmeticæ*, where a first part on irrational numbers, expounded with the Euclidean terminology of Book X, is juxtaposed to the second part where the study of some cubic equations is the central subject matter. Then, Cardano's masterpiece, the *Ars magna*, comes. All the cubic equations are solved: algorithmic rules or sometimes even out-and-out formulae are given together with their geometrical proofs. Unluckily, Cardano cannot provide a completely general method, since it happens that under certain conditions his formulae are nonsensical. In fact, when a cubic equation has three real distinct solutions, the formula (necessarily) involves imaginary numbers – this is the *casus irreducibilis*, as it has been lately called.

In this flowing, the *Aliza*, or at least a part of it, however inconsistent and obscure, is meant to overcome the problem. A very few pages has been written on this writing and – as said – only Pietro Cossali's work is a detailed account. Nevertheless, we should have again a look at the *Aliza*, mainly aiming to integrate its mathematics with the general framework of Cardano's solving methods for cubic equations.

There are two main strategies to overcome the *casus irreducibilis*. In short, one is to reject it, the other is to try to live with it. As usual, the mathematical discovery is rarely made by cleanly moving from one step to the next, but by traversing many routes, sometimes simultaneously, and sometimes retracing one's steps over old ground.

We find both strategies in the *Aliza*, even if Cardano makes a much greater effort in the first direction. Rejecting the *casus irreducibilis* basically means to reject the cubic formulae of the *Ars magna*. In particular, since the problem entailed by the *casus irreducibilis* is originated in the cubic formula for $x^3 = a_1x + a_0$, one would reject the cubic formula for that family of equations, and try to search for another one that have no conditions. This is the main idea that leads to the splittings. Since Cardano has managed to derive the cubic formula from the first splitting for $x^3 = a_1x + a_0$, why not to try to derive from some other splitting another formula to replace this first one? Unluckily, this is as much unattainable as avoiding the square roots of negative numbers in the formula.

From our viewpoint, nevertheless, the splittings have a fundamental role, since they are the main thread that I have managed to trace in order to give coherence to the *Aliza*. In fact, there is a sizeable number of chapters of the *Aliza* that can be grouped around them. This has enabled me to advance three hypotheses on their origins. In short, the first one links the choice of the substitution $x = y + z$ with the study of the algebraic shapes for the irrational solutions of $x^3 = a_1x + a_0$, with a_1, a_0 rational. Thus, the splittings are also connected to the very early surveys on irrational numbers, like the ones in the *Ars magna arithmeticæ*, which possibly comes before the discovery of the cubic formulae. We recall that these surveys, scattered all along the *Aliza*, are very heterogeneous, and could attest the development of Cardano's enquiries. We remark that appreciating whether these algebraic shapes can or cannot express a solution of an equation is one of Cardano's most recurring strategies in dealing with equations. The fact that, in the end, it turns out that there are only two possible irrational shapes, both binomials, could then have been at the origin of Cardano's substitution $x = y + z$.

The second hypothesis suggests that the splittings come from a further development of some rules in *Ars magna*, Chapter XXV, which are intended to rewrite the equation in a more comfortable way so that it is easier to guess a solution. We recall that the link between the *Aliza* and *Ars magna*, Chapter XXV had already been suggested in *Ars magna*, Chapter XII. Nevertheless, even though the splittings in the *Aliza* and the rules in *Ars magna*, Chapter XXV respond to the same need, which is to overcome the problem entailed by the *casus irreducibilis*, they are intended to achieve different purposes. In fact, the splittings are meant to provide a new formula, while the rules in *Ars magna*, Chapter XXV only to give a case-by-case method. The third hypothesis concerns a loose geometrical inspiration. It involves a very specific kind of geometry, where the focus is not on the relative position of the geometrical objects, but rather on their dimension. Hence, Cardano's treatment of the splittings in the *Aliza* shows how he possibly used to deal with equations. The solving methods in the *Ars magna* are a sort of black box that makes a clean sweep of all the previous inquiries, which are probably seen from that point on as worthless. It is therefore very reasonable to believe that, when Cardano wanted to overcome the problems of these formulae, he came back to his previous researches.

Otherwise, there is a strategy that is a branching of the main one: one rejects the *casus irreducibilis*, at the same time trying to relax the constraints on the cubic formulae. For instance, one could accept to have two (or more) formulae, or solving methods, to deal with the same family of equations, each one with a condition, provided that these conditions are complementary. Then, when the condition of a formula are not matched, one can resort to the other, and *vice* *versa*. In the *Aliza*, the splittings also becomes a subject of inquiry by themselves, when Cardano establishes some calculation rules on them. In turn, these could be allegedly used to complement the conditions on the usual cubic formula for $x^3 = a_1x + a_0$.

The second strategy is, instead, to live with the *casus irreducibilis*. In truth, this will be the way to follow in order to progress in the study of equations, and it will eventually end up with the birth of the field of complex numbers. But it is by far too early for Cardano's time. In the *Aliza*, he tries to give an alternative sign rule, which could possibly be of help in transforming the square roots of negative numbers in square roots of positive numbers, but his account is not satisfactory at all.

During our survey we have also gathered some hints on the kind of geometry that Cardano employs. It would be very astonishing if Cardano had not been driven by a geometrical way of thinking. But it is a very special kind of geometrical intuition. In the vast majority of the cases (and, when it is not so, it is because he is rephrasing some ancient sources, like Eutocius) Cardano does not need any positional features, which means that the relative position of the geometrical objects involved is unessential. As such, this kind of geometry is the most suitable to imitate arithmetic. More precisely, the geometrical feature to which Cardano aims is the dimensional coherence, since he makes operations only with geometrical objects of the same dimension and that he should not overshoot the third dimension. Nevertheless, when equations are involved, this is not as plain as it could seem, and Cardano sometimes has to find tricks to respect the dimensional constraint.

In our surveys, we have also taken into account in details the results of the *Practica arithmeticæ*, *Ars magna arithmeticæ*, and *Ars magna* concerning equations. This is a fundamental step that could not be skipped. In fact – as said – the threads that I have highlighted above are so well hidden in the *Aliza* that one has to try to make his way as plain as possible by appealing to all the other sources.

At this point, there is at least another question that I cannot fully answer to. Why did Cardano choose to publish such an inconclusive book? He was a man of his time, when sharing knowledge was not a priority, and in my opinion it is not likely that he published the *Aliza* to let someone else progress after him on the topic. Anyway, it is also true that in those days publishing books was a way to become notorious, and thus to earn one's living. Then, publishing a book with a lots of techniques – though without the great expected result – could have been better than nothing. Moreover, we should also consider that Cardano chooses to publish the *Aliza* together with the *De proportionibus* and the *Ars magna*. While the link with the *Ars magna* is explicit, the same does not hold for the *De proportionibus*. This could be a ground to investigate further.

I would like to end with a quotation by Reviel Netz. It is about Archimedes and the auxiliary problem to *On the sphere and cylinder* II.4, but it also applies very well to Cardano's *Aliza* and to the *casus irreducibilis*. In fact, we could say that the *De regula aliza*, or a part of it,

> is dedicated to what may be the most striking problem studied by Archimedes [to be meant: Cardano] – so striking, difficult, and rich in possibilities, that it could serve, on its own, as an engine for historical change. Time and again, it had attracted mathematicians; time and again, it had challenged the established forms of mathematics. Quite simply, this is a very beautiful problem.[173]

[173]See [Netz 2004, page 10].

APPENDIX A

A short history of Cardano's life

Cardano's life is known fairly in details, thanks to several editions of his autobiography, which he wrote when he was a very old man.[1] In the following I will try to resume the most outstanding episodes of his life, with particular focus on the mathematical side.

Girolamo Cardano was born in Pavia, Italy, in 1501, probably of illegitimate birth. His father Fazio was a lawyer in Milan, but he also had wide interests in mathematics, medicine, and occult sciences. Cardano himself inherited this typical attitude of his time. Aside from the lawyer practice, his father lectured on geometry at the University of Pavia and at the Scuole Piattine in Milan. His proficiency was so deep that Leonardo da Vinci himself asked him an advice on some geometrical problems.

Cardano got his first education at home. His father thought him writing and reading in Vernacular and Latin, some basic arithmetic, a smattering of occult sciences, astrology. Since Cardano was twelve years old, he began to study the first six books of Euclid's *Elements*. He became his father's flunky more than his assistant, so he soon aspire to get an education at the university. Cardano studied medicine in Pavia and in Padua, where both his capacities of brilliant student and his strong character were noticed. In Pavia, he distinguished himself in such a way to be called to lecture on Euclid's geometry when he was only twenty-one years old. Nevertheless, he had to leave the city a few years later, because of the war between France and Spain to conquer Milan. Then, he moved to Padua, where he graduated in 1524. But his repeated applications for the Collegio dei fisici in Milan were all refused because of his illegitimate birth. His stubborn

[1] See [Cardano 1544], [Cardano 1998], [Cardano 1557],[Cardano 1562], [Cardano 1663g], [Cardano 1663h]. See also [Cardano 1854] for an English translation, and [Ore 1953] for an accurate resume.

and unkind character does not earn him many friends, so that he had troubles in finding a work. Cardano never had great resources and gambling became for him not only a pastime, but a source of income. Eventually, he moved to a village near Padua where he managed to practise medicine and knew his wife – in his words, the happiest period of his life. They had three children.

Cardano obtained in 1534 a teaching position at the Scuole Piattine in Milan, thanks to a personal relationships with Filippo Archinto, a diplomat in Milan who knew his father's reputation. He should have lectured on arithmetic, geometry, and astrology (or astronomy) on non-working days, but, in order to attract a larger audience, he taught architecture instead of arithmetic and geography instead of geometry. Thanks to the reputation earned in this way, the German printer Petreius offered him to publish whatever manuscript he had to submit. It is in this period, and more precisely after failing to gain with his astrological works the patronage of Pope Paul III, that Cardano devoted himself more and more to mathematics.

In the meanwhile, he illegally practised medicine to supplement his exiguous salary, since he had not the Collegio's authorisation. He gradually acquired some reputation as physician. When he published his first book on bad practice in medicine, Cardano became insulting against the Collegio. In this way, he finally managed to draw the attention of the Collegio, which had to accept him as a member. Within a few years, he became the most prominent physician in Milan. He began travelling. This was the highest point in his career.

In 1546 Cardano's wife died. In 1560 one of his sons was sentenced to death for having poisoned his unfaithful wife, despite of all the support that Cardano gave to him in every possible way. After the loss of his son, Cardano retired from his public engagements.

In 1562 Cardano managed to get a position at the University of Bologna through the intervention of the future cardinal Francesco Alciati and of Federico Borromeo. He moved there with his youngest son, who at that time had already begun a criminal career. In Bologna, Cardano was recognised as the noteworthy scholar he was and he spent another good period of his life. He created his own network of friendships, which helped him in obtaining the freedom of the city and tax exemption. But he was the more and more on strained terms with his youngest son. In 1569 he stole a large amount of his father's properties and Cardano had to report the theft to the authorities, who banished his son from the city.

The last years of Cardano's life were marked by misfortune. In October 1570 he was arrested and put in jail without warning. The causes had never been completely clarified. Maybe his son revenged himself denouncing him, but the charges are not known. More likely, Cardano's arrest was related with the renewal impulse given by the Counter-Reformation. Among the varied works that he wrote, it was not difficult to find statements that could be interpreted as impious. The most quoted example is a horoscope of Jesus Christ, but even Emperor Nero's eulogy or the comparison between Christianity and other religions suited well. At that time Cardano was an old man. He was kept in jail for a few months, apparently without torture. Public opinion was favourable to him and he never directly opposed the church, so that the inquisitors were merciful. Cardano does not provide a more detailed account, probably because they imposed to him the secret. He had to abjure, but not publicly, was excluded from the university, was prevented from lecturing in public, and forbidden to publish any other work – he was deprived of all the worthwhile things in his life. Furthermore, he was obliged to pay a certain amount of money to the ecclesiastical authorities. They finally accorded to him a modest annual income.

In 1571 Cardano moved to Rome, where Rodolfo Silvestri, one of his good friends who had also been a former pupil of him, lived. When the new pope Gregory XIII succeeded Pius V, Cardano's modest annual income was commuted into an even more modest pension, which nevertheless carried greater prestige as a sign of papal favour. Cardano still continued writing books until the very end of his life, hoping to get the restrictions on publishing and teaching lifted. At last, in 1572 he received the licence to publish his existing medical works and in 1574 the right to publish again new works. But, at that time he already had destroyed 130 books of his treatises. In 1574 he was agreed at the Collegio dei fisici. In 1576 Cardano finally got the right to return teaching in Bologna.

Cardano died on September 20, 1576.

A.1. Chronicle of a controversy

During Cardano's lifespan, a typical way of testing and developing knowledge was controversies between scholars. Cardano himself was often involved to and likes to take part in this kind of debates.

Usually, scholars challenged each others on professional opinions, but it was not infrequent that a controversy degenerate into personal attacks. In mathematical contests, they used to send letters to the adversaries and to a notary, specifying a time interval within which certain problems had to be solved. An implicit rule was not to submit to the adversary problems that one did not know how to solve by himself. Traditionally, a controversy ended with a public debate. Depending on the standing of the adversaries, the debate could take place before the authorities of the university and some judges. The winner could be granted with a money price, but in any case he gained prestige.

One can fully grasp how important was to win controversies only if he contextualise this usage in the Italian educational system of the time. As said, since the 13^{th} century, the *abaco* schools spread from Italy throughout Europe. In the 15^{th} century, after an elementary level of instruction, a children could enter a so-called *scuola di grammatica*, and then a university, or an *abaco* school, which prepared for a profession or an apprenticeship. In particular, in the *abaco* schools, students learnt how to solve problems that came from mercantile life and practical mathematics were taught. It often happened that the handbooks coming from teaching, the so-called *trattati d'abaco*, were published. In the main, they were based on simplified revisions of the *Liber Abaci* by Leonardo Pisano. But the universities and the *abaco* schools were not always parallel paths. In a controversy, the high reputation obtained by the winner could even get him students who would have pay to be taught. Moreover, the system of controversies explain the leaning of the time to keep secret new results and discoveries.

In the history of the solution of cubic equations, controversies played a relevant role.

Cardano became renewed for having published for the first time in 1545 the cubic formulae in the *Ars magna*. But the formulae's paternity is highly disputed. Allegedly, Scipione del Ferro first discovered some cubic formulae to solve the depressed equations, or at least one of them. We know very little of del Ferro except that he was professor of arithmetic and geometry for a long time in Bologna. He kept the lid on his results, but it is said that just before his death in 1526 he revealed his discoveries to his pupil Antonio Maria Fiore. Furthermore, at Scipione's death his papers came into possession of his son-in-law Annibale della Nave.

In the meanwhile Niccolò Fontana, also known as Tartaglia, *maestro d'abaco* in Verona, applied himself to the problem of solving cubic equations. Because of his poor economic conditions, he was very used to enter controversies. In 1530 he was challenged by Zuanne de Tonini da Coi, who taught mathematics in Brescia. Zuanne proposed to him two problems, which were equivalent to solve the cubic equations $x^3 + 3x^2 = 5$ and $x^3 + 6x^2 + 8x = 1000$. Tartaglia, recalling the authority of Luca Pacioli who estimated impossible to solve cubic equations, doubted whether the weak mathematician that Zuanne was could have find by himself the solutions. So, he threatened Zuanne to report him to the authorities for boasting. In a subsequent letter to Zuanne, Tartaglia claimed to know how to solve the first equation, but not the second. Then, Antonio Maria Fiore challenged Tartaglia five years later in Venice. with thirty problems. Exploiting his teacher del Ferro's results, Fiore prepared thirty questions all equivalent to solve a depressed cubic equation of the family $x^3 + a_1x = a_0$ with rational coefficients. Tartaglia, on the contrary, submitted a variety of problems. During one sleepless night, shortly before the end of the time interval granted to solve the problems, Tartaglia found out the cubic formula he needed. The day after – it was February 13^{th} according to Tartaglia's account – he came to the solution of the depressed equations of the family $x^3 = a_1x + a_0$. Fiore, who was strong in calculation but weak in theory, did not manage to solve his problems and was declared the loser.

Zuanne was informed about the controversy. He deduced that Tartaglia found out how to deal with (at least one of) the problems in their past controversy and asked Tartaglia to explain to him his method. At that time, Tartaglia was not inclined to spread his secret. This caused the relationship between Zuanne and Tartaglia to be broken off. Anyway, Zuanne's role is not negligible, since he also acted as connection between Tartaglia and Cardano. In fact, Zuanne informed Cardano of the contest and reported him Tartaglia's denial to release the formulae. Cardano was greatly impressed, all the more so because he was preparing the oncoming *Practica arithmeticæ*. The years passed by, and even if Cardano attempted by his own to discover the formulae, he was unsuccessful.

Then, Tartaglia was approached in Venice by the bookseller Zuanantonio da Bassano in 1539. On Cardano's behalf, Zuanantonio asked Tartaglia for the cubic formulae. Neither flattery nor the promise to publish Tartaglia's results under his own name (or not to publish them at all) convinced Tartaglia to release the formulae. He continued to sidestep Cardano's requests until he realised that a connection with Cardano could turn out to be advantageous. Cardano had some good relationships in the court of Milan, especially with the Spanish governor of Lombardia, Alfonso d'Avalos, who was renown to be an unusually generous patron. Tartaglia hoped to be introduced by him to the court of Milan and maybe to be engaged as a technical expert. To cut a long story short, after Cardano's repeated solicitations, Tartaglia imparted in the end to Cardano the cubic formulae for the depressed equations in verse. I will explain the poem in details in the next section. Cardano, who was just about to pass for press the *Practica arithmeticæ*, had to promise to Tartaglia not to publish his results. Eventually, Tartaglia never entered the court in Milan and returned to Venice, from where he continued the correspondence with Cardano, clarifying his further questioning. But he was anxious to verify that Cardano kept his promise and asked him a copy of his *Practica arithmeticæ*. So, Cardano sent to him a copy, but an unbounded one.

Just after having received the poem, Cardano should have begun to work on the cubic equations lacking in the term of second degree and on the complete ones. Quite soon he should have come about the so-called *casus irreducibilis*, when a cubic equation has three real distinct solutions but the cubic formula involves imaginary numbers. So, he knew that in some cases the formulae fail (or seem to fail) to return the solutions. On August 4$^{\text{th}}$, 1539 Cardano wrote to Tartaglia, asking about the *casus irreducibilis*, but he got no answer.

Then, other results concerning equations popped up. Ludovico Ferrari, Cardano's talented pupil and his best friend, solved quartic equations in 1540. The two met four years before, when Ferrari was a young boy. He became soon

Cardano's secretary and was provided with an excellent education. But despite his popularity as mathematician, Ferrari choose a more remunerative position as tax assessor in Bologna. He always remained faithful to Cardano and was his occasional assistant. He was one of the supporters for Cardano's moving to the university of Bologna. So, Cardano and his pupil were step by step accumulating a large body of knowledge on third and fourth degree equations and Cardano's promise to Tartaglia remained an obstacle to make it public. It is still not clear why Tartaglia did not hurry to publish the results that he discovered.

But then something changed. In 1542 Cardano and Ferrari found Scipione del Ferro's papers in the house of Annibale della Nave. Thus, Cardano felt free from his promise, since it was clear that Tartaglia could be no more credited as the fist discoverer of the cubic formulae. He hurried to polish his result and only three years later the *Ars magna*, which contained all the cubic formulae and Ferrari's method for quartic equations, went into press. There, Cardano mentioned both

Scipione del Ferro and Tartaglia twice,[2] and Tartaglia alone once.[3] The *Ars magna*

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[2]In *Ars magna*, Chapter I Cardano says that

> [i]n emulation of him [Scipione del Ferro], my friend Niccolò Tartaglia of Brescia, wanting not to be outdone, solved the same case [$x^3 + a_1x = a_0$] when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my entreaties, gave it to me. For I had been deceived by the words of Luca Paccioli, who denied that any more general rule could be discovered than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this though and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil. Hereinafter those things which have been discovered by others have their names attached to them; those to which no name is attached are mine. The demonstrations, except for the three by Mahomet and the two by Ludovico, are all mine

or

> *Huius æmulatio Nicolaus Tartalea Brixellensis, amicus noster, cum in certamen cum illius discipulo Antonio Maria Florido venisset, capitulum idem, ne vinceretur, invenit, qui mihi ipsum multis precibus exoratus tradidit. Deceptus enim ego verbis Lucæ Paccioli, qui ultra sua capitula, generale ullum aliud esse posse negat. quanque tot iam antea rebus à me inventis, sub manibus esset, desperabam tamen invenire, quod quærere non audebam. Inde autem, illo habito, demonstrationẽ venatus, intellexi complura alia posse haberi. Ac eo studio, auctaque iam confidentia, per me partim, ac etiam aliqua per Ludovicum Ferrarium, olim alumnum nostrum, inveni. Porro quæ ab his inventa sunt, illorum nominibus decorabuntur, cætera, quæ nomine carent, nostra sunt. At etiam demonstrationes, præter tres Mahometis et duas Lodovici, omnes nostræ sunt*,

see [Cardano 1968, pages 8-9] or [Cardano 1545, Chapter I, paragraph 1, page 3r]. The second reference is in Chapter XI

> Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule [to solve $x^3 + a_1x = a_0$] and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though with-holding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult

or

> *Scipio Ferreus Bononiensis iam annis ab hic triginta ferme capitulum hoc invenit, tradidit vero Anthonio Mariæ Florido Veneto, qui cum in certamen cum Nicolao Tartalea Brixellense aliquando venisset, occasionem dedit, ut Nicolaus invenerit et ipse, qui cum nobis rogantibus tradidisset, suppressa demonstratione, freti hoc auxilio, demonstrationem quæsivimus, eamque in modos, quod difficillimum fuit*,

see [Cardano 1968, page 96] or [Cardano 1545, Chapter XI, page 29v].

[3]It is in Chapter VI

was immediately recognised as a masterpiece by the mathematicians of that time, and had a deep influence in the developing of 16$^{\text{th}}$ century mathematics.

And then, the controversy begun. One year later, the *Quesiti et inventioni diverse* by Tartaglia appeared. The last part is devoted to the exchange with Cardano, reconstructing the settling of his promise and its breaking.

Cardano never directly replied, but then Ferrari came to his master's defence. He firstly attacked Tartaglia with one *cartello* and the exchange lasted six *cartelli* over one year and a half, until when Tartaglia and Ferrari met in Milan. It was a gala occasion where the controversy took place. Ferrante Gonzaga himself, the governor of Milan, was named arbiter. Many distinguished personalities attended. The topics on which Tartaglia and Ferrari challenged were the problems contained in the *cartelli.* We have no records of the controversy and only a brief, one-sided account by Tartaglia remains. In the late afternoon of August 18, 1548 in the Church of the Garden of the Frati Zoccolanti, Ferrari turned out to be the winner and Tartaglia was ingloriously defeated.

I would like to end with Oystein Ore's words:

> Cardano and Ferrari represent by far the greater mathematical penetration and wealth of novel ideas. Tartaglia was also, doubtless, an excellent mathematician, but his great tragedy was the head-on collision with the only two opponents in the world who could be ranked above him.
>
> [...]
>
> But on one point a deep injustice has persisted. Some of the early Italian algebraists referred to the formula for the solution of the cube and the cosa equal to a number proposition as "del Ferro's formula", but the influence of the *Ars magna* was so great

---

> When, moreover, I understand that the rule that Niccolò Tartaglia handed to me had been discovered by him through a geometrical demonstration, I thought that this would be the royal road to pursue in all cases

or

> *Cum autem intellexissem capitulum quod Nicolaus Tartalea mihi tradiderat, ab eo fuisse demonstratione inventum geometrica, cogitavi eam viam esse regiam ad omnia capitula venanda*,

See [Cardano 1968, page 52] or [Cardano 1545, Chapter VI, paragraph 5, page 16v].

that it has forever since been known as "Cardano'a formula". Cardano never made any claim to this invention. In present-day mathematical texts it should in no way be too late to begin to pay homage again to the original discoverer, del Ferro, and thus give credit where credit is due.[4]

A.2. Tartaglia's poem

On March 25, 1539 Cardano receives from Tartaglia a poem containing the cubic formulae for the depressed equations. Here it follows.

Quando che'l cubo con le cose appresso
  Se agguaglia à qualche numero discreto
  Trovan dui altri differenti in esso.
Da poi terrai questo per consueto
  Che 'l lor produtto sempre sia eguale
  Al terzo cubo delle cose neto
El residuo poi suo generale
  Delli lor lati cubi ben sottratti
  Varra la tua cosa principale.

In el secondo de codesti atti
  Quandi che 'l cubo restasse lui solo
  Tu osservarai quest'altri contratti,
Del numero farai due tal part'àl volo
  Che l'una in l'altra si produca schietto
  El terzo cubo delle cose in stolo
Delle qual poi, per comun precetto
  Torrai li lati cubi insieme gionti
  E tal somma sara il tuo concetto.

El terzo poi de questi nostri conti
  Se solve col secondo se ben guardi
  Che per natura son quasi congionti.
Questo trovai, et non con passi tardi

[4] [Ore 1953, page 106].

Nel mille cinquecentè, quatro e trenta
Con fondamenti ben sald'è gagliardi
Nella città dal mar'intorno centa.[5]

Firstly, considering the equation $x^3 + a_1 x = a_0$, Tartaglia suggests to solve the linear system

$$\begin{cases} y - z = a_0 \\ yz = \left(\frac{a_1}{3}\right)^3 \end{cases}$$

in two unknowns. Solving this system is equivalent to solve the quadratic equation $t^2 + a_0 t - \left(\frac{a_1}{3}\right)^3 = 0$, which was an easy exercise by this time. Then, Tartaglia says that a solution of the equation can be found thanks to the difference of the sides of certain cubes, that is $x = \sqrt[3]{y} - \sqrt[3]{z}$. If we want to draw an explicit formula, we would obtain

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}} - \sqrt[3]{-\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3}}.$$

Analogously, considering the equation $x^3 = a_1 x + a_0$, Tartaglia suggests to solve the system

$$\begin{cases} y + z = a_0 \\ yz = (\frac{a_1}{3})^3 \end{cases},$$

which is equivalent to solve $t^2 + a_0 t + \left(\frac{a_1}{3}\right)^3 = 0$. Then, Tartaglia says that a solution of the equation is a certain $x = \sqrt[3]{y} + \sqrt[3]{z}$. If we want again to draw an explicit formula, we would obtain

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}}.$$

Finally, Tartaglia says that the solution of the equation $x^3 + a_0 = a_1 x$ can be easily reduced to the solution of the equation $x^3 = a_1 x + a_0$.

We remark that none of the above calculations and formulae (except the systems) are conveyed by Tartaglia's poem. But Tartaglia writes to Cardano the key idea which leads to the cubic formulae: that the sum or the difference of

[5]See [Tartaglia 1959, Quesito XXXIII fatto con una lettera dalla eccellentia de Messer Hieronimo Cardano l'Anno 1539 A di 19 Marzo, page 119].

the sides of two certain cubes has to be considered. Anyway, no hints on how Tartaglia had this idea and on how he got the systems are given.

APPENDIX B

A mathematical vocabulary from the *De Regula Aliza*

Here follows the list of the technical terms employed by Cardano in the *De regula aliza*, especially when they are employed in a mathematical sense.

adjicio, adjicere, adjeci, adjectus : to add

adversus, adversa adversum : opposite [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

æquatio, æquationis : equality

æquo, æquare, æquavi, æquatus : to make equal

æstimatio, æstimationis : value

alienus, aliena alienum : alien

alogus, aloga, alogum : *alogai*

alternum, alterni : alternate [meaning the conjugate of a *multinomium*]

altrinsecus, altrinseca, altrinsecum : on the other side, opposite [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

aufero, auferre, abstuli, ablatus : to remove

bes, bessis : two-thirds

bimedium, bimedii : bimedial [line]

binomium, binomii : *binomium* [see at page 82]

binomium superficiale : surface *binomium* [see at page 279]

capitulum, capituli : chapter [of a book], "chapter" [that is, everything that concerns in a loose way the solution of an equation, see at page 237]

coherens, coherentis : connected [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

converto, convertere, converti, conversus : to transform

commensurabilis, commensurabilis, commensurabile : commensurable

denominatio, denominationis : denomination

denominator, denominatoris : denominator

difformis, difformis, difforme : misshaped

dodrans, dodrantis : three-fourths

duco, ducere, duxi, ductus : to draw [geometrical objects], to multiply [numbers], to raise [to the power]

duplicatus, duplicata, duplicatum : duplicated [ratio]

figura, figuræ : figure

fractus, fracta, fractum : fractional [number]

fractio, fractionis : fraction

genus, generis : kind

incommensurabilis, incommensurabilis, incommensurabile : incommensurable

individuus, individua, individuum : indivisible

invenio, invenire, inveni, inventus : to discover

inventum, inventi : discovery

irrationalis, irrationalis, irrationale : irrational [line or number]

latitudo, latitudinis : width

latus secundum : second side [that is, fourth root]

(al)ligatus, (al)ligata, (al)ligatum : bounded [root]

medium, medii : medial, mean

media proportio : mean proportional

medietas, medietatis : half

monas, monadis : monad

multinomium, multinomii : *multinomium*

multiplex, multiplicis : multiple

multiplico, multiplicare, multiplicavi, multiplicatus : to multiply [geometrical objects]

mutuus, mutua, mutuum : mutual [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

natura, naturæ : nature

nomen, nominis : term

numerator, numeratoris : numerator

numerus æquationis : the coefficient a_0 of the term of degree zero

numerus rerum : the coefficient a_1 of the term of first degree

numerus quadratorum : the coefficient a_2 of the term of second degree

opero, operare, operavi, operatus : to perform operations

operatio, operationis : operation

ordo, ordinis : order

parallelipedus, parallelipedi : parallelepiped

probo, probare, probavi, probatus : to prove

proportio, proportionis : ratio [except '*proportio continua*', which I translate as 'continued proportion']

proximus, proxima, proximum : near, connected [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

quadrinomium, quadrinomii : *quadrinomium*

radix, radicis : root

radix relata : fifth root

radix solida : solid root [see at page 227]

rationalis, rationalis, rationale : rational [line or number]

recisum, recisi : *recisum* [see at page 82]

reduplicatus, reduplicata, reduplicatum : reduplicated [ratio]

remotus, remota, remotum : remote, opposite [for two parallelepipeds, regarding to their reciprocal position when composing a cube, see at page 227]

residuum, residui : remainder [note that it sometimes ambiguously means *recisum*]

rhetes, rhetes : *rhete*

sexquitertius, sexquitertia, sexquitertium : four-thirds

solidus, solidi : solid

species, speciei : type

subduplicatus, subduplicata, subduplicatum : subduplicated [ratio]

subtriplicatus, subtriplicata, subtriplicatum : subtriplicated [ratio]

superficies, superficiei : surface

sylvester, sylvestris : wild [quantity, see at page 227]

terminus, termini : boundary

(latus) tetragonicum : tetragonical side [that is, a quantity that is under a square root, see at page 227]

trinomium, trinomii : *trinomium*

APPENDIX C

List of internal and external references in and to the *De Regula Aliza*

Here follows the list of the internal and external references made by Cardano in and to the *De regula aliza*. When the reference is between square brackets, it is implicit in the text.

Internal references to the *Aliza* in the *Aliza*	
The Chapter	**refers to:**
A VII	[A I.2]
A XI	A [I], III, X
A XIII	A [XI], LI
A XIV	A V
A XV	[A I]
A XVI	A XIII
XVIII	A X, XV
A XXII	[A VI]
A XXIII	A X

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The Chapter	**refers to:**
A XXIV	A XIII Corollary 2
A XXVIII	A XXVI
A XXXIX	A XXVI, XXVII
A XLII	A XL
A LI	A XLVI
A LIII	A II, VII, XXVIII, XXXI, XL, LVII
A LVII	A IV, XL, LIII

External references in the *Aliza*		
The Chapter	**refers to:**	**is referred in:**
A I	DP 146 *Elements* II.9, X.20	
A II	AM XXV DP *Elements* II.4, 5	
A III	DP 148	
A IV	OAP, book III, XIX.2 (see AMA XIX.2) *Elements* X.6, 16, 17, 33, 39, 40, 41, 54, 66, 112, 113, 114	
A VII	*Elements* II.5	
A VIII	*Elements* X.6, 10, 11, 33, and Definition 6	
A IX	DP 143 *Elements* XI.24	
A X	*Elements* II.5	
A XI	*Elements* I. 43, II.4	
A XII	DP 27 *Elements* VIII.9, XI.34 Apollonius' *Conics* I[.11], II[.4] Archimedes' *On the sphere and cylinder* II[.4] and Eutocius' commentary	

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The Chapter	**refers to:**	**is referred in:**
A XIII	*Elements* VI.17, XI.32	
A XIV	AM XIII	
A XX	*Elements* XI.34	
A XXI	DP Proposition 2, 34 *Elements* XI.32	
A XXII	*Elements* II.4, 7	*Sermo de plus et minus*
A XXIII	*Elements* X.20, 21	
A XXIV	AM XIII Corollary 1, XXV.1 *Elements* II.4	
A XXV	AM XII, XXV.3 DP 209	
A XXXII	*Elements* V.8, 10, 19	
A XXXIII	*Elements* V.11, 19	
A XXXIV	*Elements* I.44, II.1, V.19, VI.12, 16	
A XXXVI	AM	
A XXXIX	AM XXXIX.2 DP Definition 18, Propositions 135, 143 *Elements* II.5, 7, VI.1, 10, 11	

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The Chapter	**refers to:**	**is referred in:**
A XL	AM XXV.2	
A XLI	AM [XXXIX Problem IX]	
A XLIII	*Elements* XI.34	
A XLVI	AM	
A XLVII	*Elements* XI	
A XLVIII	AM XIII	
A XLIX	AM XVIII [OAP] , Book III	
A LII	*Elements* VI.20	
A LIII	AM XXV	
A LIV	*Elements* II.1, 4	
A LVI	*Elements* I.4	
A LVII	PA LI AM XXV, XXIX *Elements* I.47, X.54	
A LIX	AM XXV.1 *Elements* X.115	

APPENDIX D

List of Cardano's numerical cubic equations

Here follows the list of the cubic equations in one unknown that Cardano suggests as examples in the selection of the Chapters of the *Practica arithmeticæ*, *Ars magna arithmeticæ*, *Ars magna* that I have considered. Concerning the *De regula aliza*, I have taken into account all the chapters. Note that sometimes Cardano does not explicitly state the equation as such, but only gives some hints from which it can be reconstructed. In these cases, I have also included the reconstructed equations.

Cubic equations in the *Practica arithmeticæ*		
PA LI.26	$3x^3 = 24x + 21$	
	$3x^3 = 15x + 21$	
	$3x^3 = 15x + 36$	
	$3x^3 + 21 = 24x$	
	$3x^3 + 6 = 15x$	
PA LI.27	$x^3 + 7x = 4x^2 + 4$	$2x^3 + 5x^2 = 10x + 16$
	$x^3 + x + 2 = 4x^2$	$x^3 + 3 = 4x^2 + 2x$
	$x^3 + 3x^2 = 7x - 3$	$x^3 = 4x^2 + 6x + 1$
	$x^3 + 4x = 4x^2 + 1$	$x^3 + 2x^2 = 2x + 3$
	$3x^3 + 3 = 7x^2 + 7x$	$x^3 + 2 = 5x$
PA LI.32	$x^3 + 64 = 18x^2$	

Cubic equations in the *Ars magna arithmeticæ*

AMA XXI	$x^3 = 3x^2 + 45$	$x^3 = 53x + 88$
	$x^3 + 15x = 75$	$x^3 + 12 = 34x$
	$x^3 = 3x^2 + 36$	$x^3 = 34x + 12$
	$x^3 + 12x = 48$	$x^3 + 216 = 27x^2$
	$x^3 = \frac{24}{5}x^2 + \frac{288}{5}$	$x^3 + 16 = 6x^2$
	$x^3 + 12x = 30$	$x^3 + 16 = 12\sqrt[3]{3}x$
	$x^3 + 3x^2 = 21$	$x^3 + 8 = 7x^2$
	$x^3 = 3x + 19$	$x^3 + \frac{49}{4}x = 7x^2 + 1$
	$x^3 + 6x^2 = 40$	$x^3 + 40 = 8x^2$
	$x^3 = 12x + 24$	$x^3 + 16x = 8x^2 + 5$
	$x^3 + 12x = 56$	$x^3 + 5 = 12x$
	$x^3 + 72 = 48x$	$x^3 + 6x^2 = 11$
	$x^3 + 4x^2 = 225$	$x^3 + 21 = 6x^2$
	$x^3 = 4x + 15$	$x^3 + 7 = 6x^2$
	$x^3 + 2x^2 = 441$	$x^3 = 12x + 9$
	$x^3 = 2x + 21$	$x^3 + 81 = 12x^2$
	$x^3 + 3x^2 = 20$	$x^3 + 12x^2 = 175$
	$x^3 + 88 = 53x$	
AMA XXII	$x^3 + 42 = 29x$	
	$x^3 + 4 = 17x$	
	$x^3 = 17x + 4$	
	$x^3 + \frac{49}{6} = \frac{45}{6}x^2$	
	$x^3 + \frac{1}{4} = \frac{17}{4}x^2$	
	$x^3 + \frac{17}{4}x^2 = \frac{1}{4}$	

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AMA XXIII	$x^3 = 3\sqrt{12}x + 8$
	$x^3 = 6x + 6$
	$x^3 + 6x = 2$
AMA XXV	$x^3 = 6x^2 + 18$
	$x^3 + 6x^2 = 50$
AMA XXVII	$x^3 + 6x = 36$
	$x^3 = x^2 + 6$
	$x^3 + 6x = 9$
	$x^3 = 6x + 9$
	$x^3 + 6x = 2$
	$x^3 = 6x + 6$
AMA XXVIII	$x^3 + 9x = 6$
	$x^3 + 9x = 26$
	$x^3 + 6x = 10$
	$x^3 + 3x = 10$
AMA XXIX	$x^3 = \frac{27}{2}x^2 + \frac{243}{2}$
	$x^3 = 6x^2 + 18$
	$x^3 = 6x^2 + 4$
AMA XXX	$x^3 = 6x + 10$

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AMA XXXI	$x^3 = 16x + 21$
	$x^3 = 20x + 32$
	$x^3 = 29x + 52$
	$x^3 = 65x + 8$
	$x^3 = 23x + 28$
	$x^3 = (24 + \sqrt{6})x + 12$
AMA XXXII	$x^3 + 6x^2 = 36$
	$x^3 = 6x + 6$
	$x^3 + 3x^2 = 21$
	$x^3 = 3x + 19$
AMA XXXIII	$x^3 + 20x^2 = 72$
	$x^3 + 11x^2 = 72$
	$x^3 + (4 + \sqrt{10})x^2 = 40$
	$x^3 + 8x^2 = 40$
	$x^3 + 7x^2 = 50$
AMA XXXIV	$x^3 + 88 = 48x$
	$x^3 = 48x + 88$
	$x^3 + 32 = 20x$
	$x^3 + 12 = 34x$

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AMA XXXV	$x^3 + 12 = 34x$	
	$x^3 + 88 = 53x$	
	$x^3 + 10 = 23x$	
	$x^3 + 4 = 17x$	
	$x^3 + 12 = 19x$	
	$x^3 + 16 = 17x$	
	$x^3 + 88 = 48x$	
AMA XXXVI	$x^3 + 12 = 5x^2$	
AMA XXXVII	$x^3 + 50 = 7x^2$	
	$x^3 + 32 = 10x^2$	
	$x^3 + 16 = 6x^2$	
	$x^3 + 40 = 8x^2$	
	$x^3 + 10 = 9x^2$	
	$x^3 + 48 = 10x^2$	
	$x^3 + 8 = 7x^2$	
AMA XXXIX	$x^3 + 6x^2 + 12x = 22$	$x^3 + x^2 + 3 = 13x$
	$x^3 + 12x = 6x^2 + 38$	$x^3 + 2x^2 + 2 = 9x$
	$x^3 + 12x = 6x^2 + 11$	$x^3 + x^2 + 11 = 47x$
	$x^3 + \frac{9}{2}x^2 + x = \frac{1}{2}$	$x^3 + x + 7 = 5x^2$
	$x^3 + 5x^2 + 3x = 1$	$x^3 + 3x + \frac{14}{3} = \frac{16}{3}x^2$
	$x^3 + 6x^2 + 4x = 16$	$x^3 + 8x = \frac{37}{6}x^2 + \frac{7}{6}$
	$x^3 + x^2 + 49 = 35x$	$x^3 + x + \frac{1}{2} = \frac{9}{2}x^2$
	$x^3 + 2x^2 + 56 = 41x$	$x^3 + 2x + \frac{3}{4} = \frac{19}{4}x^2$
	$x^3 + 3x^2 + 63 = 47x$	$x^3 + 3x + 1 = 5x^2$

Cubic equations in the *Ars magna*

AM I	$x^3 + 6x = 20$	$x^3 + 3x + 18 = 6x^2$
	$x^3 + 16 = 12x$	$x^3 + x^2 + 2x = 16$
	$x^3 = 12x + 16$	$x^3 + 2x + 16 = x^2$
	$x^3 + 9 = 12x$	$x^3 = 2x^2 + x + 16$
	$x^3 = 12x + 9$	$x^3 + 2x^2 + 16 = x$
	$x^3 + 21 = 2x$	$x^3 + 3x^2 + 6 = 20x$
	$x^3 = 2x + 21$	$x^3 = 3x^2 + 20x + 6$
	$x^3 = 3x^2 + 16$	$x^3 + 72 = 6x^2 + 3x$
	$x^3 + 3x^2 = 20$	$x^3 + 6x^2 = 3x + 72$
	$x^3 + 20 = 3x^2$	$x^3 + 4 = 3x^2 + 5x$
	$x^3 + 11x^2 = 72$	$x^3 + 3x^2 = 5x + 4$
	$x^3 + 72 = 11x^2$	$x^3 + 10 = 6x^2 + 8x$
	$x^3 + 6x^2 + 3x = 18$	$x^3 + 6x^2 = 8x + 10$
AM VI	$x^3 + 40 = 12x^2$	
	$x^3 + 3x^2 = 14x + 20$	
	$x^3 = x^2 + 8$	
	$x^3 + 8x = 64$	
AM VII	$x^3 + 80 = 18x$	$x^3 + 8 = 9x^2$
	$x^3 = 9x + 10$	$x^3 = 3x^2 + 10$
	$x^3 = 6x^2 + 16$	$x^3 + 3x = \sqrt{10}$
	$x^3 + 12\sqrt[3]{3}x = 16$	$x^3 + 16 = 14x^2$
	$x^3 + 18x = 8$	$x^3 + 49x = 14x^2 + 2$
	$x^3 = 9x^2 + 8$	$x^3 + 40 = 8x^2$
	$x^3 + 8 = 18x$	$x^3 + 16x = 8x^2 + 5$

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AM VIII	$x^3 + 3 = 10x$
AM XI	$x^3 + 6x = 20$
	$x^3 + 3x = 10$
	$x^3 + 6x = 2$
	$x^3 + 12 = 34x$
AM XII	$x^3 = 6x + 40$
	$x^3 = 6x + 6$
AM XIII	$x^3 + 12 = 34x$
	$x^3 + 12 = 10x$
	$x^3 + 3 = 8x$
	$x^3 + 60 = 46x$
	$x^3 + 21 = 16x$
AM XIV	$x^3 = 6x^2 + 100$
	$x^3 = 6x^2 + 20$
AM XV	$x^3 + 6x^2 = 100$
	$x^3 + 6x^2 = 25$
	$x^3 + 6x^2 = 16$
	$x^3 + 6x^2 = 7$
	$x^3 + 6x^2 = 40$
	$x^3 + 6x^2 = 36$

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AM XVI	$x^3 + 64 = 18x^2$	
	$x^3 + 24 = 8x^2$	
AM XVII	$x^3 + 6x^2 + 20x = 100$	
	$x^3 + 6x^2 + 12x = 22$	
	$x^3 + 3x^2 + 9x = 171$	
	$x^3 + 6x^2 + x = 14$	
	$x^3 + 12x^2 + 27x = 400$	
	$x^3 + 6x^2 + 2x = 3$	
AM XVIII	$x^3 + 33x = 6x^2 + 100$	$x^3 + 12x = 6x^2 + 8$
	$x^3 + 12x = 6x^2 + 25$	$x^3 + 12x = 6x^2 + 9$
	$x^3 + 48x = 12x^2 + 48$	$x^3 + 12x = 6x^2 + 7$
	$x^3 + 15x = 6x^2 + 24$	$x^3 + 20x = 6x^2 + 24$
	$x^3 + 15x = 6x^2 + 10$	$x^3 + 20x = 6x^2 + 15$
	$x^3 + 10x = 6x^2 + 4$	$x^3 + 20x = 6x^2 + 33$
	$x^3 + 26x = 12x^2 + 12$	$x^3 + 9x = 6x^2 + 2$
	$x^3 + 100x = 6x^2 + 10$	$x^3 + 9x = 6x^2 + 4$
	$x^3 + 5x = 6x^2 + 10$	$x^3 + 21x = 9x^2 + 5$
AM XIX	$x^3 + 6x^2 = 20x + 56$	
	$x^3 + 6x^2 = 20x + 112$	
	$x^3 + 6x^2 = 20x + 21$	
AM XX	$x^3 = 6x^2 + 5x + 88$	
	$x^3 = 6x^2 + 72x + 729$	

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AM XXI	$x^3 + 100 = 6x^2 + 24x$	
	$x^3 + 64 = 6x^2 + 24$	
	$x^3 + 128 = 6x^2 + 24x$	
	$x^3 + 9 = 6x^2 + 24x$	
AM XXII	$x^3 + 4x + 16 = 6x^2$	
	$x^3 + 4x + 8 = 6x^2$	
	$x^3 + 4x + 1 = 6x^2$	
AM XXIII	$x^3 + 6x^2 + 4 = 41x$	
	$x^3 + 6x^2 + 12 = 31x$	
AM XXV	$x^3 = 20x + 32$	$x^3 + 21 = 16x$
	$x^3 = 32x + 24$	$x^3 + 18 = 19x$
	$x^3 = 10x + 24$	$x^3 + 8 = 18x$
	$x^3 = 19x + 30$	$x^3 + 18 = 15x$
	$x^3 = 7x + 90$	$x^3 + 48 = 25x$
	$x^3 = 16x + 21$	$x^3 + 20x^2 = 72$
	$x^3 = 4x + 15$	$x^3 + \frac{45}{2}x^2 = 98$
	$x^3 = 14x + 8$	$x^3 + 48 = 10x^2$
	$x^3 + 12x = 34x$	

Cubic equations in the *De regula aliza*	
A I	$x^3 = 29x + 140$
	$x^3 = \frac{185}{7}x + 158$
	$x^3 = \frac{158}{7}x + 185$
	$x^3 = 35x + 98$
	$x^3 = 14x + 245$
A II	$x^3 = 7x + 90$
A III	$x^3 + 8 = 7x^2$
	$x^3 + 6 = 7x^2$
	$x^3 + 48 = 7x^2$
	$x^3 + 24 = 8x^2$
	$x^3 + 40 = 8x^2$
	$x^3 + 45 = 8x^2$
	$x^3 + 75 = 8x^2$
A V	$x^3 + 24 = 32x$
	$x^3 = 32x + 24$
	$x^3 + 12 = 34x$
	$x^3 = 34x + 12$
	$x^3 + 8 = 18x$
	$x^3 + 18x = 39$
A VII	$x^3 = 18x + 30$
	$x^3 = 18x + 58$
	$x^3 = 18x + 75$
	$x^3 = 18x + 33$
	$x^3 = 18x + 42$

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A X	$x^3 = 30x + 36$
	$x^3 = 38x + 2$
A XII	$x^3 + 256 = 12x^2$
	$x^3 + 128 = 12x^2$
	$x^3 + 192 = 12x^2$
A XIV	$x^3 + 12 = 19x$
A XV	$x^3 + 1 = 3x$
	$x^3 + 2x = \frac{8}{3}x^2 + \frac{4}{3}$
	$x^3 = \frac{10}{17}x + \frac{700}{729}$
	$x^3 + x = 2x^2 + 2$
A XVIII	$x^3 = 6x + 6$
A XIX	$x^3 = 6x + 6$
	$x^3 = 9x + 9$
A XX	$x^3 + 6x^2 = 24$
	$x^3 + 8x^2 = 24$
	$x^3 + x^2 = 4$
A XXIII	$x^3 + \frac{7}{64} = \frac{2}{3}x$
	$x^3 = 4x + 48$
	$x^3 = 4x + 47$
	$x^3 + 12 = 34x$
	$x^3 + \frac{59172}{4913} = 34x$
	$x^3 + 52 = 30x$
	$x^3 = 4x + 50$

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A XXIV	$x^3 = 20x + 32$	$x^3 = 29x + 60$
	$x^3 = 6x + 4$	$x^3 = 29x + 52$
	$x^3 = 29x + 28$	$x^3 = 29x + 20$
	$x^3 = 29x + 50$	$x^3 = 29x + 42$
A XXV	$x^3 = 18x + 108$	
	$x^3 = 21x + 90$	
	$x^3 = 15x + 126$	
	$x^3 = 22x + 84$	
	$x^3 = 17x + 114$	
	$x^3 = 6x + 6$	
A XXVI	$x^3 = 9x^2 + 8$	$x^3 + 408 = 200x$
	$x^3 + 8 = 9x^2$	$x^3 + 927 = 300x$
	$x^3 + \frac{8569}{1000} = 9x$	$x^3 + 8829 = 900x$
	$x^3 + 200 = 100x^2$	
A XXVII	$x^3 = 6x^2 + 400$	
	$x^3 + 32 = 6x^2$	
	$x^3 + 12 = 10x$	
A XXVIII	$x^3 + 6 = 8x,$	
	$x^3 + 8x = 6$	
A XXXI	$x^3 = 13x + 60$	
	$x^3 = 6x + 1$	
A XXXV	$x^3 = 4x + 8$	
	$x^3 = \sqrt{2}x^2 + \sqrt{8}$	
	$x^3 + 2x^2 = 8$	
	$x^3 + 2x = \sqrt{8}$	
	$x^3 = 2x + 4$	

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A XXXVII	$x^3 + 576 = 25x^2$	
	$x^3 = 22x^2 + 576$	
	$x^3 + 576 = 22x^2$	
	$x^3 + 36x = 252$	
A XXXIX	$x^3 + 16 = 9x^2$	
	$x^3 = 9x^2 + 16$	
	$x^3 + 24 = 8x^2$	
	$x^3 + x = 25$	
	$x^3 + 108 = 36x$	
	$x^3 + 64 = 36x$	
	$x^3 = 27x + 46$	
A XL	$x^3 + 24 = 32x$	$x^3 + 4 = 5x$
	$x^3 + 12 = 34x$	$x^3 + 6 = 7x$
	$x^3 + 8 = 18x$	$x^3 + 4 = 12x$
	$x^3 + 153 = 64x$	$x^3 + 8 = 8x$
	$x^3 + 4 = 36x$	
A XLII	$x^3 + 4 = 6x$	
	$x^3 + 10 = 9x$	
	$x^3 + 12x = 34$	
	$x^3 + 6 = 7x$	
	$x^3 + 8 = 8x$	
	$x^3 + 12 = 34x$	
	$x^3 + 20 = 15x$	
	$x^3 + 8 = 8x$	

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A XLIV	$x^3 + 2x^2 = 8$
A XLV	$x^3 + 252 = 78x$
	$x^3 = 26x + 60$
	$x^3 + 252 = 48x$
	$x^3 + 252 = 45x$
	$x^3 + 252 = 42x$
A XLVI	$x^3 = 24x + 5$
	$x^3 = 6x + 95$
	$x^3 = 4x + 105$
	$x^3 = 5x + 100$
	$x^3 = 3x + 120$
	$x^3 = 18x + 100$
A XLVIII	$x^3 = 13x + 60$
	$x^3 + 70 = 39x$
A XLIX	$x^3 = 6x + 40$
	$x^3 = 6x + 20$
	$x^3 + \frac{81}{4}x = \frac{81}{4}$
	$x^3 + 27x = \frac{9}{2}x^2 + 54$
A L	$x^3 = 36x + 36$
A LIII	$x^3 = 12x + 20$
	$x^3 = 12x + 34$
	$x^3 = 6x + 40$
	$x^3 + 20x = 32$
A LV	$x^3 = x^2 + x + 3$

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A LVI	$x^3 = 20x + 32$	
	$x^3 = 19x + 12$	
A LVII	$x^3 = 20x + 32$	
	$x^3 = 39x + 18$	
	$x^3 = 19x + 12$	
A LVIII	$x^3 = 20x + 16$	
	$x^3 = 6x + 6$	
A LIX	$x^3 = x$	$x^3 = 4x$
	$x^3 = 2x + 1$	$x^3 = 5x + 2$
	$x^3 = 3x + 2$	$x^3 = 6x + 4$
	$x^3 = 4x + 3$	$x^3 = 7x + 6$
	$x^3 = 5x + 4$	$x^3 = 8x + 8$
	$x^3 = 6x + 5$	$x^3 = 9x + 10$
	$x^3 = 7x + 6$	$x^3 = 10x + 12$
	$x^3 = 8x + 7$	$x^3 = 11x + 14$
	$x^3 = 9x + 8$	$x^3 = 12x + 16$
	$x^3 = 10x + 9$	$x^3 = 13x + 18$
	$x^3 = 11x + 10$	$x^3 = 14x + 20$
	$x^3 = 12x + 11$	$x^3 = 15x + 22$
	$x^3 = 13x + 12$	$x^3 = 16x + 24$
	$x^3 = 14x + 13$	$x^3 = 17x + 26$
	$x^3 = 15x + 14$	$x^3 = 18x + 28$
	$x^3 = 16x + 15$	$x^3 = 19x + 30$
	$x^3 = 17x + 16$	$x^3 = 20x + 32$
	$x^3 = 18x + 17$	$x^3 = 21x + 34$

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A LIX	$x^3 = 9x$	$x^3 = 25x + 36$
	$x^3 = 10x + 3$	$x^3 = 26x + 40$
	$x^3 = 11x + 6$	$x^3 = 27x + 44$
	$x^3 = 12x + 9$	$x^3 = 28x + 48$
	$x^3 = 13x + 12$	$x^3 = 29x + 52$
	$x^3 = 14x + 15$	$x^3 = 30x + 56$
	$x^3 = 15x + 18$	$x^3 = 31x + 60$
	$x^3 = 16x + 21$	$x^3 = 32x + 64$
	$x^3 = 17x + 24$	$x^3 = 216$
	$x^3 = 18x + 27$	$x^3 = x + 210$
	$x^3 = 19x + 30$	$x^3 = 2x + 204$
	$x^3 = 20x + 33$	$x^3 = 3x + 198$
	$x^3 = 21x + 36$	$x^3 = 4x + 192$
	$x^3 = 22x + 39$	$x^3 = 5x + 186$
	$x^3 = 23x + 42$	$x^3 = 6x + 180$
	$x^3 = 24x + 45$	$x^3 = 7x + 174$
	$x^3 = 25x + 48$	$x^3 = 8x + 168$
	$x^3 = 26x + 51$	$x^3 = 9x + 162$
	$x^3 = 16x$	$x^3 = 10x + 156$
	$x^3 = 17x + 4$	$x^3 = 11x + 150$
	$x^3 = 18x + 8$	$x^3 = 12x + 144$
	$x^3 = 19x + 12$	$x^3 = 13x + 138$
	$x^3 = 20x + 16$	$x^3 = 14x + 132$
	$x^3 = 21x + 20$	$x^3 = 15x + 126$
	$x^3 = 22x + 24$	$x^3 = 16x + 120$
	$x^3 = 23x + 28$	$x^3 = 6x + 40$
	$x^3 = 24x + 32$	

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A LX	$x^3 = 25 + 20x$
	$x^3 = 36 + 30x$
	$x^3 = 18x + 8$
	$x^3 = 11x + 3$

We remark that the equations that have $\Delta_3 < 0$ are 13 out of 16 in the *Practica arithmeticæ*, 68 out of 105 in the *Ars magna arithmeticæ*, 68 out of 119 in the *Ars magna*, and 133 out of 214 in the *De regula aliza*.

APPENDIX E

List of Cardano's results on cubic equations

Bibliography

Sources

ARCHIMEDES (1544). *Archimedis syracusani philosophi ac geometriae excellentissimi opera: quae quidem extant omnia, latinitate iam olim donata, nuncque primum in lucem edita*. Basel: Hervagius.

BOMBELLI, RAFAEL (1572). *L'algebra*. Bologna: Giovanni Rossi.

CARDANO, GIROLAMO (1539). *Heorinymi C. Cardani Medici mediolanensis, Practica arithmetice, et mensurandi singularis. In quaque preter alias continentur, versa pagina demonstrabit*. Milan: Io. Antonins Castellioneus for Bernardini Calusci.

CARDANO, GIROLAMO (1544). *De sapientia libri quinque. Eiusdem De consolatione libri tres, alias æditi, sed nunc ab eodem authore recogniti. Eiusdem De libris propriis, liber unus. Omnia locupleti indice decorata*, ch. Libellus de libris propriis, cuius titulus est ephemerus, pp. 419–431. Nuremberg: Johannes Petreius.

CARDANO, GIROLAMO (1545). *Hieronymi Cardani, præstantissimi mathematici, philosophi, ac medici, Artis magnae sive de regulis algebraicis, lib. unus. Qui et totius operis de arithmetica, quod Opus perfectum inscripsit, est in ordine decimus*. Nuremberg: Iohannes Petrius.

CARDANO, GIROLAMO (1554). *Hieronymi Cardani mediolanensis medici De subtilitate libri XXI nunc demum recogniti atque perfecti*. Basel: Ludovicum Licium.

CARDANO, GIROLAMO (1557). *Liber De libris propriis, eorumque ordine et usu, ac de mirabilibus operibus in arte medica per ipsum factis*. Lyon: Guillaume Rouillé.

CARDANO, GIROLAMO (1562). *Somniorum synestesiorum, omnis generis insomnia explicantes libri IIII. Quibus accedunt eiusdem hæc etiam: De libris* *propriis. De curationibus et prædictionibus admirandis. Neronis encomium. Geometriæ encomium. De uno. Actio in Thessalicum medicum. De secretis. De gemmis et coloribus. Dialogus de morte. [Dialogus] de humanis consiliis, tetim inscriptus. Item ad somniorum libros pertinentia: De minimis et propinquis. De summo bono*, ch. De libris propriis, eorumque usu, liber recognitus, pp. 1–116. Basel: Heinrich Petri.

Cardano, Girolamo (1566). *Heorinymi Cardani mediolanensis civisque bononiensis, Ars curandi parva, quae est absolutiss. medend. methodus, et alia, nunc primum aedita, opera, in duos tomos diuisa, quae versa pagina indicabit; omnia autem qualia sint autoris epistola vere praedicat.*. Basel: Officina Henricpetrina.

Cardano, Girolamo (1570a). *Hieronymi Cardani mediolanensis, civisque bononiensis, philosophi, medici et mathematici clarissimi, Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum mensurandarum, non solum geometrico more stabilitum, sed etiam varijs experimentis et observationibus rerum in natura, solerti demonstratione illustratum, ad multiplices usus accommodatum, et in V libros digestum. Prætera Artis magnæ, sive de regulis algebraicis, liber unus, abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus, ab authore recens multis in locis recognitus et auctus. Item De aliza regula liber, hoc est, algebraicæ logisticæ suæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirentis, necessaria coronis, nunc demum in lucem edita*, ch. De aliza regula libellus, hoc est Operis perfecti sui sive algebraicæ Logisticæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirenti, necessaria coronis, nunc demum in lucem editæ. Basel: Oficina Henricpetrina.

Cardano, Girolamo (1570b). *Hieronymi Cardani mediolanensis, civisque bononiensis, philosophi, medici et mathematici clarissimi, Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum mensurandarum, non solum geometrico more stabilitum, sed etiam varijs experimentis et observationibus rerum in natura, solerti demonstratione illustratum, ad multiplices usus accommodatum, et in V libros digestum. Prætera*

Artis magnæ, sive de regulis algebraicis, liber unus, abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus, ab authore recens multis in locis recognitus et auctus. Item De aliza regula liber, hoc est, algebraicæ logisticæ suæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirentis, necessaria coronis, nunc demum in lucem edita, ch. Artis magnae sive de regulis algebraicis, lib. unus. Qui et totius operis de arithmetica, quod Opus perfectum inscripsit, est in ordine decimus. Ars magna, quam volgo cossam vocant, sive regulas algebraicas, per D. Hieronymum Cardanum in quadraginta capitula redacta, et est liber decimus suæ Arithmeticæ. Basel: Oficina Henricpetrina.

Cardano, Girolamo (1570c). *Hieronymi Cardani mediolanensis, civisque bononiensis, philosophi, medici et mathematici clarissimi, Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum mensurandarum, non solum geometrico more stabilitum, sed etiam varijs experimentis et observationibus rerum in natura, solerti demonstratione illustratum, ad multiplices usus accommodatum, et in V libros digestum. Prætera Artis magnæ, sive de regulis algebraicis, liber unus, abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus, ab authore recens multis in locis recognitus et auctus. Item De aliza regula liber, hoc est, algebraicæ logisticæ suæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirentis, necessaria coronis, nunc demum in lucem edita*, ch. In Librum de Propotionibus Hieronymi Cardani Mediolanensis, civisqúe Bononiensis, Medici. Basel: Oficina Henricpetrina.

Cardano, Girolamo (1663a). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 4, ch. De regula aliza libellus. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663b). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 4, ch. Hieronymi Cardani, Artis magnae sive de regulis algebraicis, liber unus. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663c). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 4, ch. Ars magna arithmeticæ, seu liber quadraginta capitulorum et quadraginta quæstionum. Lyon: Ioannis Antonii

Huguetan and Marci Antonii Ravaudn. Edited by Charles Spon.

Cardano, Girolamo (1663d). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 1, ch. Libellus De libris propriis, cui titulus est ephemerus. Ad Hieronymum Cardanum medicum, affinem suum, pp. 55–59. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663e). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 1, ch. Liber De libris propriis, eorumque ordine et usu, ac de mirabilibus operibus in arte medica per ipsum factis, pp. 60–95. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663f). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 1, ch. Liber De libris propriis, eorumque ordine et usu, ac de mirabilibus operibus in arte medica per ipsum factis, pp. 96–150. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663g). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 1, ch. "*Liber a me conscripti, quo tempore, cur, quid acciderit*", Hieronymi Cardani De propria vita, liber (1576), pp. 1–54. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663h). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 1, ch. Vita Cardani ac de eodem iudicium. Par Gabrielem Naudæum. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. (without page numbering). Edited by Charles Spon.

Cardano, Girolamo (1663i). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 4, ch. Practica aritmeticæ generalis omnium copiosissima et utilissima. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1663j). *Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta*, vol. 4, ch. Sermo de plus et minus. Lyon: Ioannis Antonii Huguetan and Marci Antonii Ravaud. Edited by Charles Spon.

Cardano, Girolamo (1821). *Vita di Girolamo Cardano filosofo medico e* *letterato celebratissimo*: Sonzogno. Edited by Vincenzo Mantovani.

Cardano, Girolamo (1854). *Jerome Cardan. The life of Girolamo Cardano, of Milan, physician*: Chapman and Hall. Edited by Henry Morley.

Cardano, Girolamo (1968). *Ars magna or the rules of algebra*: Dover. Edited and translated by T. Richard Witmer.

Cardano, Girolamo (1982). *Della mia vita*: Serra e Riva Editori. Edited by Alfonso Ingegno.

Cardano, Girolamo (1998). *De libris propriis [MS 1550],* edited by Baldi, Marialuisa and Canziani, Guido. Rivista di storia della filosofia, 4, 767–98. With the foliation (1-12r) of the MS F II.38 Nr 1, Öffentliche Bibliothek der Universität Basel.

Cardano, Girolamo (2004). De libris propriis. *The editions of 1544, 1550, 1557, 1562, with supplementary material*: Franco Angeli. Edited by Ian MacLean.

Commandino, Federico (1572). *Euclidis Elementorum libri XV*. Pisa: Jacobus Chriegber.

Cossali, Pietro (1779). *Particularis methodi de cubicarum æquationum solutione a Cardano luci traditæ generalis posterioru analystarum usus ex cap. I De regula aliza ipsius Cardani vitio lucentissime evictus atque Mysterium casus irreducibilis post duo sæcula prorsus recta causa sublatum speciem analystarum primum Petri Cossal Cl. Rec. Th.*. Venetiis: Caroli Palesii.

Ferrari, Ludovico and Tartaglia, Niccolò (1974). *Cartelli di sfida matematica*: La Nuova cartografica. facsimile reproduction of the 1547-1548 originals by Arnaldo Masotti.

Gegenbauer, Leopold (1893). *Über den sogenannten* casus irreducibilis *der Cardanishen formel*. Monatshefte für Mathematik und Physik, 4, 155–158.

Heath, Thomas (ed) (1896). *Apollonius of Perga. Treatise on conic sections*: Cambridge University Press.

Heath, Thomas (ed) (1897). *The works of Archimedes*, vol. I: Cambridge University Press, 2nd ed.

Heath, Thomas (ed) (1956a). *The thirteen books of Euclid's Elements*, vol. I: Dover, 2nd ed.

Heath, Thomas (ed) (1956b). *The thirteen books of Euclid's Elements*, vol. II:

Dover, 2nd ed.

Heath, Thomas (ed) (1956c). *The thirteen books of Euclid's Elements*, vol. III: Dover, 2nd ed.

Hölder, Otto (1891). *Über den* casus irreducibilis *bei der Gleichung dritten Grades*. Matematische Annalen, 38, 307–312.

Kneser, Adolf (1892). *Bemerkungen über den Sogenannten* casus irreducibilis *bei cubischen Gleichungen*. Mathematische Annalen, 41, 344–348.

Lefèvre d'Étaples, Jacques (1516). *Euclidis megarensis geometricorum elementorum libri XV. Campani galli transalpini in eosdem commentariorum libri XV. Theonis alexandrini Bartholamaeo Zamberto veneto interprete, in tredecim priores, commentariorum libri XIII. Hypsiclis alexandrini in duos posteriores, eodem Bartholamaeo Zamberto veneto interprete, commentariorum libri II. Utcunque noster valuit labor conciliata sunt haec omnia, ad studiosorum non paruam (quam optamus) utilitatem: id Magnifico D. Francisco Briconneto postulante. Si haec beneuole suscipiantur, et fructum adferant quem cupidimus: alia eiusdem authoris opera prodibunt in lucem, successum praestante deo, et adiutoribus (ubiubi gentium sint) ad bonarum literarum institutionem probe affectis Gallis, Italis, Germanis, Hispanis, Anglis. Quibus omnibus prospera imprecamur: et puram pro dignitate veramque cognitionis lucem*. Paris: Henrici Stephani.

Lorgna, Anton Maria (1776). *De casu irreducibili tertii gradus et seriebus infinitis dissertatio analitica*. Veronæ: Marci Moroni.

Mollame, Vincenzo (1890). *Sul* casus irreductibilis *dell'equazione cubica*. Rendiconto dell'Accademia delle scienze fisiche e matematiche (sezione della società Reale di Napoli) II, 4, 167–171.

Mollame, Vincenzo (1892). *Sulle radici primitive dell'unità negativa*. Rendiconto dell' Accademia delle scienze fisiche e matematiche (sezione della società Reale di Napoli) II, 6, 179–183.

Neumann, Peter M. (2011). *The mathematical writings of Évariste Galois*: European Mathematical Society.

Roth, Peter (ed) (1608). *Roth, Petrus, Arithmetica philosophica, oder schöne künstliche Rechnung der Coss oder Algebrae. Im 1. Theil werden Herrn D. H. Cardani dreyzehn Reguln esetzt. Dessgleichen noch drey andere Reguln, zu* *den ersten drey cubiccosischen aequationibus gehörig*. Nuremberg: Johann Labenberger.

Ruffini, Paolo (1799). *Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*: Stamperia di S. Tommaso D'Acquino.

Ruffini, Paolo (1813). *Riflessioni intorno alla soluzione delle equazioni algebraiche generali*: Società Tipografica di Modena.

Schemmel, Matthias and Stedall, Jackie (last checked January 22, 2014). *Topics in the manuscripts of Thomas Harriot (1560-1621).* Online at `http://echo.mpiwg-berlin.mpg.de/content/scientific_revolution/harriot/harriot-bl/maps/2.4.14_cardano.pt`.

Stevin, Simon (1585). *L'arithmétique*. Leiden: Christophle Plantin.

Tartaglia, Niccolò (1959). *Quesiti et inventioni diverse*: Ateneo di Brescia. facsimile reproduction of the 1554 second edition by Arnaldo Masotti.

Ver Eecke, Paul (1921). *Les œuvres complètes d'Archimède, traduites du grec en français avec une introduction et des notes*: Desclèe, De Brouwer and co.

Wantzel, Pierre Laurent (1842). *Classification des nombres incommensurables d'origine algébrique*. Nouvelles annales de mathématiques 1ère série, 2, 117–127.

Woepke, Franz (ed) (1851). *L'Algèbre d'Omar Alkhayyami*: Duprat.

www.cardano.unimi.it (last checked January 22, 2014). *Sito Girolamo Cardano. Strumenti per storia del Rinascimento nell'Italia settentrionale*. Online at `http://www.cardano.unimi.it/`.

www.maurolico.unipi.it (last checked January 22, 2014). *Il progetto Maurolico*. Online at `http://www.maurolico.unipi.it/`.

Secondary literature

Alkhateeb, Haitham and Oaks, Jeffrey A. (2007). *Simplyfing equations in Arabic algebra*. Historia mathematica, 34, 45–61.

Avanzini, Giuseppe (1822). *Elogio di Pietro Cossali*. Memorie di matematica e fisica della Società italiana delle scienze (detta dei XL), pp. 413–458.

Baldi, Maria Luisa and Canziani, Guido (eds) (1999a). *Girolamo Cardano. Le opere, le fonti, la vita*: Franco Angeli.

Baldi, Maria Luisa and Canziani, Guido (1999b). *La circolazione dei manoscritti cardaniani. Alcuni documenti.* In Baldi, Marialuisa and Canziani, Guido (eds): *Girolamo Cardano. Le opere, le fonti, la vita*, pp. 477–498: Franco Angeli.

Baldi, Maria Luisa and Canziani, Guido (eds) (2003). *Cardano e la tradizione dei saperi*: Franco Angeli.

Baldi, Maria Luisa and Canziani, Guido (eds) (2009). *La pensée scientifique de Cardan*: Les Belles Lettres.

Baldini, Ugo (2003). *L'edizione dei documenti relativi a Cardano negli archivi del Sant'Ufficio e dell'Indice. Risultati e problemi.* In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi. Milano (23-25 maggio 2002)*, pp. 457–515: Franco Angeli. Edited by Marialuisa Baldi and Guido Canziani.

Bashmakova, Isabella and Smirnova, Galina (2000). *The beginnings and evolution of algebra*: The Mathematical Association of America.

Betti, Gian Luigi (2009). *Cardano a Bologna e la sua polemica con il Tartaglia nel ricordo di un contemporaneo.* Bruniana e campanelliana, 15, 159–169.

Bortolotti, Ettore (1926). *I contributi del Tartaglia, del Cardano, del Ferrari e della scuola matematica bolognese alla teoria algebrica delle equazioni cubiche.* Studi e memorie per la storia dell'Università di Bologna, 10, 55–108.

Bortolotti, Ettore (1947). *La storia della matematica nell'Univerisità di Bologna*: Zanichelli.

Bos, Henk J. M. (2001). *Redefining geometrical exactness. Descartes transformation of the Early Modern concept on construction*: Springer.

Boyer, Carl Benjamin (1991). *A history of mathematics*: Wiley. Second edition.

Cajori, Florian (1904, 1969). *An introduction to the theory of equations*: Dover.

Cajori, Florian (1917-1918). *Pierre Laurent Wantzel.* Bulletin of the American Mathematical Society, B24, 339–47.

Cantor, Moritz (1892). *Vorlesungen über die Geschichte der Mathematik*: Teubner. Volume 2.

Capelli, Alfredo (1902). *Istituzioni di analisi algebrica*: Pellerano.

CHEMLA, KARINE (1992). *De la synthèse comme moment dans l'histoire des mathématiques*. Diogène, 160, 97–114.

CHEMLA, KARINE (2003). *Generality above abstraction. The general expressed in terms of the paradigmatic in mathematics in ancient China*. Science in context, 16, 413–458.

CIFOLETTI, GIOVANNA (1994). *Cardano's algebra in the manual of Peletier and Gosselin*. In *Girolamo Cardano. Philosoph, Naturforscher, Artz*, pp. 243–64: Harrassowitz. Edited by Eckhard Keßler.

CLAGETT, MARSHALL (1964-1984). *Archimedes in the Middle Ages*: Reale Tipografia Parmense. Volume 3.

COSSALI, PIETRO (1782). *Sul quesito analitico proposto all'Accademia di Padova per il premio dell'anno 1781 di una assoluta dimostrazione della irriducibilità del binomio cubico*: per gli eredi di Marco Moroni.

COSSALI, PIETRO (1799a). *Origine, trasporto in Italia, primi progressi in essa dell'Algebra*: Reale Tipografia Parmense. Two volumes.

COSSALI, PIETRO (1799b). *Particularis methodi de cubicarum æquationum solutione a Cardano luci traditæ. Generalis posteriorum analystarum usus ex cap. I De Regula Aliza ipsius Cardani vitio luculentissime evictus. Atque mysterium casus irreducibilis post duo sæcula prorsus retecta causa sublatum specimen analyticum primum*: Caroli Palesii.

COSSALI, PIETRO (1813). *Disquisizione sui varj metodi di eliminazione con il componimento di uno nuovo*. Memorie di matematica e di fisica della Società italiana delle scienze, XVI, first part, 272–330.

COSSALI, PIETRO (1966). *Storia del caso irriducibile*: Istituto veneto di scienze, lettere ed arti. Edited by Romano Gatto.

FLAMENT, DOMINIQUE (2003). *Histoire des nombres complexes*: CNRS Éditions.

FRANCI, RAFFAELLA (1996). *Tra latino e volgare. La lingua della trattatistica matematica dal 1200 al 1600*. Archimede, 4, 163–70.

FRANCI, RAFFAELLA and TOTI RIGATELLI, LAURA (1985). *Towards a history of algebra from Leonardo of Pisa to Luca Pacioli*. Janus, 72, 17–82.

FREGUGLIA, PAOLO (1987). *Niccolò Tartaglia e il rinnovamento delle matematiche nel '500*. In *Cultura, scienze e tecniche nella Venezia del Cinquecento*, pp. 203–216: Istituto veneto di scienze, lettere ed arti.

Freguglia, Paolo (1994). *Sur la théorie des équations algébriques entre le XVIème et le XVIIème siècle*. Bollettino di storia delle scienze matematiche, 14, 259–98.

Freguglia, Paolo (1999). *La geometria fra tradizione e innovazione*: Bollati Boringhieri.

Galuzzi, Massimo, Maierù, Luigi, and Santoro, Nadia (2012). *La tradizione latina dell'algebra, Fibonacci, le scuole d'abaco, il cinquecento*: A.

Gamba, Enrico and Montebelli, Vico (1987). *La matematica abachistica tra recupero della tradizione e rinnovamento scientifico*. In *Cultura, scienze e tecniche nella Venezia del Cinquecento*, pp. 169–202: Istituto veneto di scienze, lettere ed arti.

Gavagna, Veronica (1999). *Alcune osservazioni sulla* Practica Arithmeticæ *di Cardano e la tradizione abachistica quattrocentesca*. In Baldi, Marialuisa and Canziani, Guido (eds): *Girolamo Cardano. Le opere, le fonti, la vita*, pp. 273–312: Franco Angeli.

Gavagna, Veronica (2003). *Cardano legge Euclide. I* Commentaria in Euclidis Elementa. In Baldi, Marialuisa and Canziani, Guido (eds): *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi. Milano (23-25 maggio 2002)*, pp. 125–144: Franco Angeli.

Gavagna, Veronica (2010). *Medieval heritage and new perspectives in Cardano's* Practica arithmeticæ. Bollettino di storia delle scienze matematiche, 1, 61–80.

Gavagna, Veronica (2012). *Dalla* Practica arithmeticæ *all'*Ars magna*. Lo sviluppo dell'algebra nel pensiero di Cardano*. In *Pluralité de l'algèbre à la Renaissance*, pp. 237–268: Champion. Edited by Maria-Rosa Massa-Esteve, Sabine Rommevaux, and Maryvonne Spiesser.

Gericke, Helmut (ed) (1990). *Mathematik im Abendland. Von den römischen Feldmessern bis zu Descartes*: Marlx Verlag.

Giusti, Enrico (1992). *Algebra and geometry in Bombelli and Viète*. Bollettino di storia delle scienze matematiche, 12, 303–28.

Grattan-Guinness, Ivor (ed) (1994). *Companion encyclopedia of the history and philosophy of the mathematical sciences*: John Hopkins University Press.

Gregori, Carlo (1991). *L'elogio del silenzio e la simulazione come modelli di* *comportamento in Gerolamo Cardano*. In *Sapere e/è potere. Dalle discipline ai ruoli sociali*, p. 76: Istituto per la Storia di Bologna. Edited by Angela De Benedictis.

HEEFFER, ALBRECHT (2010). *Philosophical aspects of symbolic reasoning in Early Modern mathematics*, ch. From the second unknown to the symbolic equation, pp. 57–122: College Publications. Edited by Albrecht Heeffer and Maarten Van Dyck.

HEEFFER, ALBRECHT (last checked January 22, 2014). *On the nature and origin of algebraic symbolism*. Online at `http://logica.ugent.be/albrecht/thesis/PMP2007Heeffer.pdf`.

HERSTEIN, ISRAEL N. (1975). *Topics in Algebra*: Wiley.

HØYRUP, JENS (2002a). *Lengths, widths, surfaces. A portrait of old Babylonian algebra and its kin*. In *Studies and sources in the history of mathematics and physical sciences*: Springer.

HØYRUP, JENS (2002b). *Pre-Modern "algebra". A concise survey of that which was shaped into the technique and discipline we know*. Quaderni di ricerca in didattica, 11, 84–97.

HØYRUP, JENS (2006). *Jacopo da Firenze and the beginning of Italian vernacular algebra*. Historia mathematica, 33, 4–42.

HØYRUP, JENS (2010). *Philosophical aspects of symbolic reasoning in Early Modern mathematics*, ch. Hesitating progress. The slow development toward algebraic symbolization in abbacus and related manuscripts, c. 1300 to c. 1550, pp. 3–56: College Publications. Edited by Albrecht Heeffer and Maarten Van Dyck.

HUTTON, CHARLES (1812). *Tracts on mathematical and philosophical subjects*, ch. Tract 33. History of algebra, pp. 219–224: F. C. and J. Rivington.

IRVING, RONALD S. (2004). *Integers, polynomials, and rings. A course in algebra*: Springer.

KESSLER, ECKHARD (ed) (1994). *Girolamo Cardano. Philosoph, Naturforscher, Arzt*: Harrassowitz.

KLEIN, JACOB (1968). *Greek mathematical thought and the origin of algebra*: MIT Press.

KLINE, MORRIS (1972). *Mathematical thought from Ancient to Modern times*:

Oxford University Press.

Knorr, Wilbur Richard (1986). *The ancient tradition of geometric problems*: Birkhäuser.

Knorr, Wilbur Richard (1989). *Textual studies in ancient and medieval geometry*: Birkhäuser.

Kouskoff, Georges (1980). *Quelques aspects du vocabulaire mathématique de Jerôme Cardan*. In *Acta conventus neo-latini Turonensis*, pp. 661–674: Vrin. Edited by Jean-Claude Margolin.

Lang, Serge (2002). *Algebra*: Springer-Verlag, third ed.

Liddel, Henry George and Scott, Robert (1996). *A Greek-English lexicon*: Oxford University Press, 9th ed.

Loria, Gino (1931, 1950). *Storia delle matematiche. Dall'alba della civiltà al secolo XIX*: Hoepli.

MacLean, Ian (1999). *Interpreting the* De libris propriis. In Baldi, Marialuisa and Canziani, Guido (eds): *Girolamo Cardano. Le opere, le fonti, la vita*, pp. 13–34: Franco Angeli.

MacLean, Ian (2007). *Girolamo Cardano. The last years of a polymath*. Renaissance Studies, 5(21), 587–607.

MacLean, Ian (2009). *Learning and the market place. Essays in the history of the early modern book*: Brill.

Maracchia, Silvio (2005). *Storia dell'algebra*: Liguori.

Napolitani, Pier Daniele (1998). *La tradizione archimedea*. In *Luca Pacioli e la matematica del Rinascimento*, pp. 77–102: Petruzzi. Edited by Enrico Giusti.

Napolitani, Pier Daniele (2001). *Archimede. Alle radici della scienza moderna*. Le Scienze. I grandi della scienza, 22, 1–109.

Netz, Reviel (1999). *The shaping of deduction in Greek mathematics*: Cambridge University Press.

Netz, Reviel (2004). *The transformation of mathematics in the Early Mediterranean world. From problems to equations*: Cambridge University Press.

Neugebauer, Otto (1952). *The exact sciences in antiquity*: Princeton University Press.

Oaks, Jeffrey A. (2005). Māl*, enunciations, and the prehistory of Arabic* *algebra*. Historia mathematica, 32, 400–425.

Oaks, Jeffrey A. (2007). *Arabic algebra as an artificial language*. Journal of Indian Philosophy, 35, 543–575.

Oaks, Jeffrey A. (2009). *Polynomials and equations in Arabic algebra*. Archive for history of exact sciences, 63, 169–203.

Oaks, Jeffrey A. (2010). *Polynomials and equations in medieval Italian algebra*. Bollettino di storia delle scienze matematiche, 30, 23–60.

Oaks, Jeffrey A. (2013a). *Algebraic notation in medieval Arabic*. Philosophica. To appear.

Oaks, Jeffrey A. (2013b). *Irrational coefficients in Renaissance algebra*. Draft.

Ore, Oystein (1953). *Cardano. The gambling scholar*: Princeton University Press.

Panza, Marco (1997). *Analysis and synthesis in mathematics. History and philosophy*, ch. Classical sources for the concepts of analysis and synthesis, pp. 365–414: Kluwer.

Panza, Marco (2005). *Newton et l'origine de l'analyse*: Blanchard.

Panza, Marco (2007). *What is new and what is old in Viète's* analysis restituta *and* algebra nova*, and where do they come from? Some reflections on the relations between algebra and analysis before Viète*. Revue d'histoire des mathématiques, 13, 85–153.

Panza, Marco (2010). *Philosophical aspects of symbolic reasoning in Early Modern mathematics*, ch. What more there is in early modern algebra that its literal formalism, pp. 193–228: College Publications. Edited by Albrecht Heeffer and Maarten Van Dyck.

Panza, Marco (2011). *Rethinking geometrical exactness*. Historia mathematica, 38, 42–95.

Procissi, Angiolo (1951). *Il caso irriducibile dell'equazione cubica da Cardano ai moderni algebristi*. Periodico di matematiche, 5, 263–80.

Rahman, Shahid, Street, Tony, and Tahiri, Hassam (eds) (2008). *The unity of science in the Arabic tradition*: Springer.

Rashed, Roshdi (ed) (1994). *The development of Arabic mathematics. Between arithmetic and algebra*: Kluwer.

Rashed, Roshdi and Vahabzadeh, Bijan (eds) (1999). *Al-Khayyām mathématicien*: Blanchard.

Rommevaux, Sabine (2003). *La réception des* Éléments *d'Euclide au Moyen Age et à la Renaissance*. Revue d'histoire des sciences, 56, 267–273.

Sabaino, Daniele (2003). *Gerolamo Cardano* musicus. In *Gerolamo Cardano nel suo tempo*, pp. 89–124: Edizioni Cardano.

Sansone, Giovanni and Conti, Roberto (1958). *Lezioni di analisi matematica*: CEDAM.

Smith, David Eugene (1908). *Rara arithmetica. A catalogue of the arithmetics written before the year 1601*: Ginn and Company Publishers.

Stedall, Jacqueline (2011). *From Cardano's great art to Lagrange's reflections. Filling a gap in the history of algebra*: EMS.

Stedall, Jacqueline and Robson, Eleanor (2009). *The Oxford handbook of the history of mathematics*: Oxford University Press.

Tamborini, Massimo (2003). *Per una storia dell'*Opus Arithmeticæ Perfectum. In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi. Milano (23-25 maggio 2002)*, pp. 157–190: Franco Angeli. Edited by Marialuisa Baldi and Guido Canziani.

Tanner, Rosalind C. H. (1980a). *The alien realm of the minus. Deviatory mathematics in Cardano's writings*. Annals of science, 37, 159–178.

Tanner, Rosalind C. H. (1980b). *The ordinary regiment of the minus sign. Off-beat mathematics in Harriot's manuscripts*. Annals of science, 37, 127–158.

Toscano, Fabio (2009). *La formula segreta. Tartaglia, Cardano e il duello matematico che infiammò l'Italia nel Rinascimento*: Sironi.

Unguru, Sabetai (1975). *On the need to rewrite history of Greek mathematics*. Archive for history of exact sciences, 15, 67–114.

Unguru, Sabetai (1979). *History of Ancient mathematics. Some reflections on the state of the art*. Isis, 70, 555–565.

Vacca, Giovanni (1937). *L'opera di Girolamo Cardano nel quarto centenario del suo insegnamento in Milano*. Rendiconti del seminario matematico e fisico di Milano, 9, 22–40.

Valente, Michela (2003). *"Correzioni d'autore" e censure dell'opera di Cardano*. In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale* *di studi. Milano (23-25 maggio 2002)*, pp. 437–456: Franco Angeli. Edited by Marialuisa Baldi and Guido Canziani.

Van Egmond, Warren (1976a). *The commercial revolution and the beginning of Western mathematics in Renaissance Florence, 1300-1500*. PhD thesis, Indiana University.

Van Egmond, Warren (1976b). *Practical mathematics in the Italian Renaissance. A catalog of Italian abbacus manuscript and printed books to 1600*: Istituto e museo di storia della scienza.

Vitrac, Bernard (2005). *Peut-on parler d'algèbre dans les mathématiques grecques anciennes?*. online at `http://hal.archives-ouvertes.fr/docs/00/17/49/33/PDF/Algebre_grecque_Ayene_Miras.pdf`.

Waerden, Bartel Leendert van der (1975). *Defence of a "shocking" point of view.* Archive for history of exact sciences, 15, 199–200.

Weil, André (1978). *Who betrayed Euclid? (Extract from a letter to the Editor).* Archive for history of exact sciences, 19, 91–93.

Zardin, D (2003). *La lettera a una giovane sua creatura di Cardano e la tradizione dell'institutio cinquecentesca.* In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi. Milano (23-25 maggio 2002)*, pp. 281–305: Franco Angeli. Edited by Marialuisa Baldi and Guido Canziani.

ADDENDUM

De Regula Aliza. A compared transcription of 1570 and 1663 editions with a partial English translation

In this part, follow a compared transcription of the two printed editions (1570 in Basel and 1663 in Lyon) of the *De regula aliza* and a partial English translation of the chapters that we have taken into account in the first part.

HIERONYIMI CARDANI
MEDIOLANENSIS,
CIVISQUE BONONIENSIS,

philosophi, medici et mathematici clarissimi,

OPUS NOVUM
DE PROPORTIONIBUS NUMERORUM,

MOTUUM, PONDERUM, SONORUM

aliarumque rerum mensurandarum

non solum geometrico more stabilitum,

sed etiam variis experimentis et observationibus rerum in natura,

solerti demonstratione illustratum,

ad multiplices usus accommodatum, et in V libros digestum.

PRÆTEREA

ARTIS MAGNÆ,

SIVE DE REGULIS ALGEBRAICIS, LIBER UNUS,

abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus,

ab authore recens multis in locis recognitus et auctus.

ITEM

DE REGULA ALIZA LIBER,

hoc est algebraicæ logisticæ suæ,

numeros recondita numerandi subtilitate,

secundum geometricas quantitates inquirentis, necessaria coronis,

nunc demum in lucem edita.

Opus physicis et mathematicis inprimis utile et necessarium

Cum Cæsaris maiestatis gratia et privilegio

BASILEÆ

HIERONYMI CARDANI
MEDIOLANENSIS,
CIVISQUE BONONIENSIS,

MEDICI AC MATHEMATICI PRÆCLARISSIMI

de aliza regula, libellus,

HOC EST,

OPERIS PERFECTI SUI
SIVE ALGEBRAICÆ LOGISTICÆ

NUMEROS RECONDITA NUMERANDI SUBTILITATE,

secundum geometricas quantitates inquirentis,

necessaria coronis,

nunc demum in lucem edita.[1]

---

[1]1570 has "*editæ*". Taking into account the general title page above reproduced, I presume that there is a misprint here (since "*editæ*" must be referring to "*coronis*").

Note. In the following transcription, the 1570 edition of the *De regula aliza* by Girolamo Cardano is reproduced. This edition has been compared with the 1663 version in Cardano's *Opera omnia*. Up to now, no manuscript is known.

In the few places where the two editions disagree, the 1663 version is recalled in the footnotes. The notes in the margin and the diagrams are placed according to their position in the 1570 version.

The main purpose of this transcription is to make Cardano's text more easily readable for a modern reader. For that, whenever it has been possible to emend the mistakes in the text, the correction has been substituted to the original text and an explanatory footnote has been added. My additions have been put into square brackets.

All along the transcription, 'u' and 'U' in the original text has been dissimilated in 'u', 'v' and 'U', 'V', while 'i', 'j' and 'I', 'J' in the original text has been assimilated in 'i' and 'I'. Note moreover that, during the 16th century, the letter 's' is usually written as 'ſ' , except at the end of a word and when it is double ('ſs'). In any case, it has been transcribed as 's'.

Concerning the accents and other special types that were common in the Renaissance typography, the following substitutions has been done: 'ã' becomes 'am' or 'an', 'ë' and 'ę' become 'æ', 'ẽ' becomes 'em' or 'en', '&' becomes 'et', 'õ' becomes 'om' or 'on', 'p̃' becomes 'præ', 'ꝑ' becomes 'pre', 'ꝓ' becomes 'pro', 'q̃' becomes 'qua', 'qȝ' becomes 'que', 'ꝗ' becomes 'qui', 'q̊' becomes 'quo', 't̃' becomes 'tur', and 'ũ' becomes 'um' or 'un'. Exceptionally, 'aũt' becomes '*autem*', 'em̃' becomes '*enim*', and 'tñ' becomes '*tunc*'. Moreover, the grave accents that according to the common usage were employed to mark the adverbs and prepositions has been removed. Concerning the abbreviations, all the words that were abbreviated in the original text using a point and that are employed as a technical jargon have been kept abbreviated. Instead, the abbreviated words that belong to the common language have been replaced by their full writing.

For the same sake of readability, the punctuation, the capital letters, and the italics have been adapted to the modern usage.

Index eorum quæ in hoc libro continentur

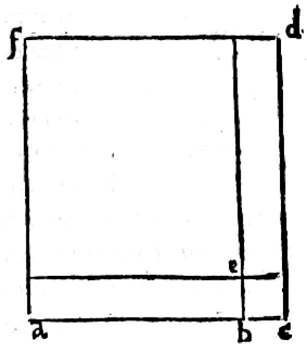
Caput I

De suppositis ac modis

Cum iam in Arte magna demonstraverimus omnia capitula converti, modo duo principalia nec iam ex conversione inventa generalia fuerint, manifestum est, invento alio capitulo generali, præter capitula cubi et rerum æqualium numero et cubi æqualis quadratis et numero, quod ex priore per conversionem deducitur, et si generale sit, omnia capita seu ex tribus seu ex quatuor nominibus generaliter non solum cognita esse, sed et demonstrata, modo hoc ipsum demonstratione inventum sit. At vero faciliora inventu sunt quæ ex tribus nominibus constant quam quæ ex quatuor, non solum quia hæc pluribus partibus constent, sed quoniam hæc per illa habeantur. Horum autem quæ tribus nominibus constant facillimum est capitulum cubi æqualis rebus et numero, cuius pars iam maxima ex prima regula habetur, et quod secundum rationem capituli iam inventi se habet; et etiam quod ex illo in alia non contra conversionem ostenderimus. Horum omnium causa de illo agemus.

In capitulo igitur cubi æqualis rebus et numero duo proponuntur speciatim, numerus æquationis, numerus etiam rerum. Generaliter autem, quod cubus nunc alicuius lineæ seu quantitatis, propositis numeris simplici et rerum æqualis est. Oportet autem ut generaliter hoc inveniamus et demonstrative et facillime. Cum ergo cubus æqualis sit duabus quantitatibus diversi generis (aliter non esset hoc generale, si ad solos numeros et eorum partes extenderetur), necesse est ut et ipse in duas tamtum partes resolvatur, quarum una numerus sit, et assignato æqualis: alia totidem partes contineat natura varias, quot in rebus continentur de eis æquales. Quo circa necesse est cubum saltem ex duabus partibus constare natura diversis, igitur et latus eius seu res, neque enim ab unius generis natura plures per multiplicationem quotiescunque repetitam plures quantitates diversorum generum fieri possunt, ut ab Euclide in decimo libro demonstratum est. Verum Propositio 20
si in re contineantur duæ quantitates a numero alienæ, necesse est ut inter se sint incommensæ, aliter æquivalerent uni. At ex eiusmodi necesse est cubum

fieri, qui tres partes contineat, numerum et duas rhete, ut rebus ac numero possit coæquari. Cum ergo diviserimus rem in duas partes, oportet cubum tres eiusmodi progignere, et si in tres ut progignat quatuor atque ita deinceps, et (ut dictum est) ut in illis sit numerus numero proposito æqualis; reliquem partes vero ut sint ex natura partium lateris, et illarum aggregatis coæquales.



Rursum ut repetamus quæ dicta sunt, sit cubus ad ex linea ac divisa tribus et constat quod in eo erunt quatuor partes diversæ principales: cubus ab, cubus bc, triplum ab in quadratum bc, et triplum bc in quadratum ab. Oportet igitur accommodare numerum, et potest fieri septem modis facilioribus, ut diximus, si non possumus invenire æstimationem in faciliori, quomodo in difficiliori inveniemus? Primus ergo modus est ut numerus tribuatur cubis. Atque hic modus est inventus, et est pars illa capituli quæ habetur in qua accipimus R cubicas partium numeri pro rei partibus, et ita cubi illarum sunt numeri qui iuncti æquantur aggregato cuborum cc [ce] et ef, et res ipsæ æquantur parallelipedis sex, quæ ex cubo ad residua sunt. Sed quoniam cubi ab et bc nunque possunt esse minores quarta parte totius cubi ad, et hoc etiam non contingit nisi cum fuerit ac divisa per æqualia in b. Cum igitur numerus fuerit minor quadrante cubi totius ad, non poterit æquari cubis ab, bc. Et ideo capitulum hac in parte non fuit generale.

Per 9 secundi Elementorum et regula [*sic*] Dialec.

Sequitur ergo secundus modus, et est ut parallelipeda omnia dentur numero et cubi rebus. Et quia parallelipeda non possunt esse maiora dodrante totius cubi, quia cubi non possunt pariter accepti esse minores quadrante, ideo nec hoc capitulum potest esse generale, quoniam cum numerus fuerit maior dodrante totius cubi, non poterit tribui parallelipedis. Cum ergo neutrum istorum capitulorum possit esse generale per se, ambo tamen iuncta constituit[1] capitulum generale.

[1] 1663 has "*constituunt*".

Etiam primum servit quando numerus non fuerit minor quadrante totius cubi, seu triente rerum, quod idem est; secundus cum numerus non fuerit maior dodrante totius cubi, seu maior triplo quantitatis rerum, quod ad idem pertinet. Ex quo liquet quod cum numerus fuerit a quadrante ad dodrantem totius cubi, et est magna latitudo scilicet semissis, tunc æstimatio potest haberi per utranque regulam, quia numerus potest tradi cubis et parallelipeda rebus, et converso modo numerus parallelipedis et res ipsis cubis. Æstimatio ergo erit eadem, et duobus modis inventa. Licet autem videre ex demonstratis in Libro de proportionibus Propositio
quod proportio aggregati cuborum ad aggregatum sex parallelipedum est, veluti 146
aggregati quadratorum ab et bc partium detracto producto ab in bc ad triplum producti seu superficiei ab in bc, seu triplum superficiei ae.

Tertius, quartus, quintusque modus non sunt adeo elegantes tametsi priores duo quippiam habeant præcipui. Tertius siquidem est cum quatuor parallelipeda numero dantur, reliqua duo cum cubis rebus. Est autem hoc inter corpora illa præcipuum, quod proportio ipsorum corporum est, ut quadratorum partium simul iunctorum ad ambo producta. Velut, capio rem 7, divisam in 5 et 2, quatuor parallelipeda sunt 140, cubi partium cum duobus parallelipedis sunt 203, proportio 203 ad 140 est velut 29 aggregati quadratorum 5 et 2 ad 20 duplum producti 5 in 2. Et similiter, in quarto modo numerus datur duobus tantum parallelipedis mutuis. Hoc tamen habet præcipui, quod extenditur ad quadrantem ad unguem numeri, unde videtur ad unguem perficere capitulum cum prima regula. Manifestum est ergo, quod in secundo modo oportet producere tertiam partem numeri ex mutuis parallelipedis, in tertio medietatem, in hoc autem totum numerum. Et semper aggregatum ex duobus mutuis parallelipedis æquale est ductui producti partium invicem in aggregatum earum, seu in rem. Sed in quinto modo damus numerum uni cubo, reliqua septem corpora rebus, ideo est valde difficilis, et redit ad capitulum quatuor nominum inde ex eo ad primum, ideo est deterius omnibus. Si tamen posset inveniri, esset generale ut duo sequentia.

Sextus modus est ut demus numerum uni cubo et tribus parallelipedis quæ fiunt ex latere cubi illius in quadrata lateris alterius cubi, et reliqua quatuor corpora, scilicet, cubum cum tribus parallelipedis rebus. Et ideo est difforme, quoniam quod æquatur est simile scilicet cubi cum parallelipedis tribus adversis, cui æquatur dissimile, nam unum aggregatum æquatuor numero, aliud rebus.

Præcipuum tamen est his corporibus, ut differentia aggregatorum sit æqualis cubo differentiæ laterum, velut in exemplo posito primum aggregatum est 185, secundum 158, differentia est 27, cubus 3 differentia 5 et 2. Et hoc capitulum si inveniretur, esset generale.

Septimus modus est cum numerum damus aggregato ex cubo et duobus cohærentibus parallelipedis cum uno adverso et res reliquis quatuor corporibus, velut in exemplo ad 125 cubum 5 addo 100 duplum parallelipedi 2 in 25, quadratum 5 et 20, parallelipedum 5 in 4, quadratum 2, et totum fit 245, et similiter reliquum erit 8 p: cubo, et 40 p: duplo 5 in 4, quadratum 2 et 50, parallelipedum 2 in quadratum, ut omnia sint 98. Præcipuum in hoc est quod utraque pars habet rationem quadrati, et R quadrata fit ex 75 in R unius partis, velut R 245 fit ex 7 in R 5, et 98 ex 7 in R 2. Et proportio talium corporum est velut partium rei, id est velut 5 ad 2. Patitur tamen et hoc difficultatem eandem cum priore, scilicet quod corpora similia generatione comparantur naturis diversis per se in genere, ut numero et rebus. Reliquæ autem compositiones, aut sunt anomale, velut si daremus numerum uni parallelipedo, vel tribus vel quinque vel duobus, non mutuis aut quatuor, ex quibus duo mutua non essent, aut uni cubo et uni parallelipedo, vel duobus vel tribus non eiusdem generis. Aliæ sunt inutiles, velut si daremus numerum aggregato ex ambobus cubis, et duobus parallelipedis aut quatuor quomodocumque, nam si numerus cum parvus sit, non sufficit aggregato cuborum, quomodo sufficit eidem si addantur parallelipeda.

Caput II

De Regulis specialibus Capituli XXV Artis magnæ cubi æqualis rebus et numero

Prima hic ad numerus rerum ea ratione constructa superficie rectangula, ut altitudo eius cd ducta in residuum, deducto quadrato bd, quod sit ae producat f numerum, liquet ergo quod cubus cd seu cb cum numero f æquatur numero rerum, id est parallelipedo ex bc in ad, sumatur ergo bl quarta pars bd, et totius al latus am, cui adiiciatur mn dimidium bc, quæ in bl, erit ergo posita an re corpus ex an in ad rursus res ipsæ dico quod cubus an æquatur totidem rebus an, id est secundum numerum ad et numero f. Nam quadratum an est æquale quadrato am et mn, quare superficiebus al et bl ex supposito et duplo am in mn, quod est æquale duplo quadrati mn (posita mo æquali mn) producti ex mn in mo et duplo producti mn in ao, duplum autem producti mn in mo, seu quadrati mn est ex supposito æquale duplo bl. Quadratum igitur an est æquale superficiei al et triplo bl et duplo ap in mn, al autem cum triplo bl est tota superficies ad. Quadratum igitur an est æquale ad superficiei ad et duplo mn in ao, est autem cd dupla mn igitur superficiei ex cd in ao, cubus ergo an qui fit ex an in quadratum an æqualis est parallelipedo ex an in ad et in cd in ao, sed ex an, cd, ao fit idem quod ex cd in superficiem ex an in ao quæ est æqualis superficiei ae, nam ex an in ao fit cum quadrato mn, quod est bl quadratum am, eo qui mn et mo sunt æquales, igitur detracta conmuni superficiei bl ex an in ao fit ac, et ex cd in ao ex supposito fit f numerus, igitur cubus an æqualis est parallelipedis ex an in ad, et ex cd in superficiem ae, quod est æquale f, numero igitur cubus an est æqualis parallelipedo ex an in ad cum numero f, at ex an in ad est numerus rerum positarum supponendo an rem, quia adfuit numerus rerum, igitur cubus est æqualis rebus et numero propositis. Hæc demonstratio ostendit quod hoc capitulum non oritur ex illis septem modis sed alia ratione. Ex hoc etiam sequitur quod capitulum cubi et numeri æqualium rebus est simplicius, et ex se magis

Per 4. 2. Elementorum

Per 5. 2. Elementorum

obvium cognitioni capitulo cubi æqualium rebus et numero. Nam in eo sufficit ut invenias partem in numero rerum, cuius radix ducta in reliquam partem producat numerum propositum. Hæc etiam regula non est generalis per se toti capitulo cubi æquali rebus et numero, quia ubi numerus esset maior non satisfaceret, sed est generalis capitulo cubi et numeri æqualium rebus. Secunda quoque regula nec ex his demonstratur, sed accepta quacunque parte cubi pro numero reliqua est æqualis rebus ex supposito, igitur superficies est æqualis numero rerum. Si ergo ea superficies cum eo quod provenit diviso numero per eandem quantitatem fuerit quadratum illius quantitatis, igitur quantitas illa est res. Sed neque hoc ad hoc propositum pertinet, cum ex illa divisione rei non pendeat. Et licet ad quadrati divisionem quod cubi basis est, res divisa intelligatur, attamen illa divisio magis pertinet ad capita cubi et numeri comparatorum quadratis quam rebus. Tertia regula oritur ex tertio modo præcedentis divisionis. Quarta regula similiter ex quarto modo demonstratur, nam duo cubi cum quatuor parallelipedis ad duo reliqua parallelipeda eam obtinent proportionem quam quadrata partium rei divisæ cum superficie unius partis in alteram ad alteram superficiem quæ quadratum complet. Cum ergo duo quadrata et superficies ex una parte in aliam sint tres quantitates continuæ proportionis[1] et radices extremarum, seu latera quadratorum ex supposito ducta in ipsa quadrata mutuo producant numerum æquationis et parallelipeda etiam, igitur parallelipeda sunt æqualia numero, reliqua autem corpora rebus. Ergo cum cubus sit æqualis illis octo corporibus, erit etiam æqualis rebus et numero. Quod vero sex illa corpora sint æqualia rebus, constat ex constitutione cubi et demonstratis Libro de proportionibus. Quinta regula ex secundo modo originem ducit, verum cum ibi demonstrata sit, non est ut eam repetam. Sexta regula huic proposito non congruit, nam specialis est. Septima oritur ex quinta, sed videtur ab ea diversa, quia in illa supponitur R tota scilicet RV: 28 m: 3 quad., in hac dimidium R 7 m: $\frac{3}{4}$ quad.. Et quia in una ducuntur partes mutuo in quadrata, in alia aggregatum in productum partium; sunt tamen idem ut demonstratum est in Libro de proportionibus. Pendet autem per regulam de modo cum 1 cu. æqualis sit 7 reb. p: 90, ex 7 m: quadrato differentiæ, quod est 1 quad. posita differentia 1 pos. (tanquem producto partium, et est semper æquale differentiæ aggregati quadratorum a producto unius partis

[1] 1570 has "*propositionis*".

in alteram), in RV: 28 m: 3 quadrata, inventum per regulam de modo, fit 30 tertia pars numeri æquationis. Sic octava pendet ex tertia eodem processu. Sed quia positæ sunt solum ad inventionem generis quantitatum quæ[2] multiplicatæ in quadrata vel radices producant numerum ideo omitto.

[2] 1570 has "$\tilde{q}$", which normally is "*qua*".

Caput III

De modo inveniendi quantitates quæ serviant capitulis per producta unius partis in aliam, et quadratum differentiæ partium

Cum dixerit quis 1 cu. p: 8 æqualia 7 quadratis, tunc divides 7 in duas partes, ex quarum una in alterius quadratum fit numerus, et semper oportet ut reliqua pars quæ non in se ducitur sit binomium vel recisum primum, quia quadratum alterius necessario est binomium vel recisum primum. Ex Euclide igitur si debet numerum efficere ductum in reliquam partem, oportet ut sit illa secunda pars binomium vel recisum primum. Prima ergo pars potest esse binomium vel recisum primum, secundum et tertium, et potest etiam esse binomium quartum, quintum et sextum, non tamen recisum, quia cum prima sit R et secunda pars necessario sit binomium, quia prima est recisum, igitur in utraque esse R p: igitur non potest esse numerus ille qui ab initio divisus est. Dico ergo quod diviso numero quadratorum in duas partes, quas vocabimus principales, et eam quæin se ducitur, vocabimus principalem, et pro aliis duabus partibus inveniendis, duc primam in duplum secundæ et a producto deducito quadratum primæ, et R residui est pars addenda principalibus aut detrahenda cum conditionibus condictis. Et similiter pro numero producendo duc differentiam principalium in se, et productum in duplum primæ principalis, et quod producitur est quæsitus numerus. Exemplum ergo in proposito, divido 7 in 4 et 3, duco 4 in duplum 3, fit 24, deduco cb quadratum 4, relinquitur 8, cuius R addita vel detracta et ita addita vel detracta a 3 converso modo constituit partes 4 p: R 8 et 3 m: R 8 vel 4 m: R 8 et 3 p: R 8. Et ideo notandum est quod sub æquatione eadem prima pars principalis et secunda idem faciunt per binomium et recisum. Utraque enim harum æstimationum scilicet 4 p: R 8 et 4 m: R 8 est æstimatio 1 cu. p: 8 æqualium 7 quadratis. Pro numero ergo habendo æstimationis, seu qui producitur, cape 1 differentiam 4 et 3 partium principalium, et duc in se, fit 1, duc in 8 duplum primæ principalis, fit 8 numerus quæsitus. Exemplum ergo aliud, divido 7 in 3 et 4, et sit 3 pars prima, duco in duplum 8 fit 24, aufero 9 quadratum primæ fit 15. Et erit pars ℞ 15 detrahenda a 4 et addenda 3 propter ea quæ dicta sunt. Pro numero sume differentiam quæ est 1, duc in se fit 1, duc in duplum 3 primæ principalis, fit 6 numerus quæsitus. Habebo igitur 1 cu. p: 6 æqualia 7 quadrata. Et similiter divido 7 in $1\frac{1}{2}$ et $5\frac{1}{2}$, et duco u [*sic*] duplum $5\frac{1}{2}$ in $1\frac{1}{2}$, fit $16\frac{1}{2}$, detraho[1] $2\frac{1}{4}$ quadratum $1\frac{1}{2}$, relinquitur $14\frac{1}{4}$, huius igitur radici adde $1\frac{1}{2}$ et detrahe a $5\frac{1}{2}$, et fierent partes ℞ $14\frac{1}{4}$ p: $1\frac{1}{2}$ et $5\frac{1}{2}$ m: ℞ $14\frac{1}{4}$. At productum fit ex differentia $5\frac{1}{2}$ et $1\frac{1}{2}$ in se et fit 16, et ducto 16 in 3 duplum $1\frac{1}{2}$ fit 48, simili modo fere ex 8 duas partes, ex quarum ductu unius in quadratum alterius fiat 9 posita una $4\frac{1}{2}$ alia $3\frac{1}{2}$, per secundam regula patet propositum, scilicet quod producemus 9 vel 7. Nam quadratum differentiæ est 1, et ductum in duplum $4\frac{1}{2}$ constituit 9, et in duplum $3\frac{1}{2}$ constituit 7, et ita si dividatur in 5 et 3, producetur 40 vel 24. In prima ergo divisione erunt partes $4\frac{1}{2}$ p: vel m: ℞ $11\frac{1}{4}$ et $3\frac{1}{2}$ p: vel m: eadem ℞ $11\frac{1}{4}$, et ita si diviseris in $2\frac{1}{2}$ et $5\frac{1}{2}$ habebis 45, et erunt partes ℞ $21\frac{1}{4}$ p: $2\frac{1}{2}$ et $5\frac{1}{2}$ m: ℞ $21\frac{1}{4}$. nam aliter esse non potest, ut ab initio diximus. Et si diviseris in $1\frac{1}{2}$ et $6\frac{1}{2}$ habebimus 75. Et ex hac operatione patet, quod præter integra dimidia non potest ulla divisio esse utilis. Sic enim duc divisum in a et b, et sit c differentia, et quia si dividatur per[2] $4\frac{1}{2}$ et $5\frac{1}{2}$, vel $6\frac{1}{2}$ et $3\frac{1}{2}$, et sic de singulis differentia est numerus integer, ergo cum a et b non sunt neque integra neque media, erunt maiora vel minora, ergo c est maius vel minus integro. Et quia numerus qui debet produci necessario fit ex quadrato c in duplum a, ubi a sit prima pars, et iam a non est nec integer numerus nec dimidium, igitur duplum a non est numerus, sed aliquis ex eadem denominatione cum c. At quia ducitur in se, et est ultra integrum, aliquid erit productum, factum genere denominationis quadratæ, ut si sit $2\frac{1}{2}$ erit $5\frac{4}{9}$, at ductum 5 in $\frac{4}{9}$ in $7\frac{2}{3}$ duplum minoris, quia denominatio composita est ad $5\frac{4}{9}$ producit numerum ex genere fractorum, quorum denominator est 27, ut pote $41\frac{20}{27}$ ut demonstratum est suo loco, igitur productum non potest esse numerus aliquis integer, sed 10 non potest dividi per integra et media nisi decem modis, ergo numerus æquationum non potest esse nisi decem. At 10 quadrata æqualia cubo et numero possunt æquari, ut demonstratum est in Libro de proportionibus usque

[1] 1570 and 1663 have '$1\frac{1}{4}$'.
[2] 1570 and 1663 have '$4\frac{1}{4}$'.

ad 148 numeris integris et singulis, ergo divisio binomiorum et recisorum, et per integra non satisfacit, sed desunt 138 numeri integri. Præter illos, in quibus sunt adiectæ partes ipsæ numerorum quibus eadem ratione hæ quantitates satisfacere non possunt. Sed pro nunc, sufficiat ostendisse de integris, et quia capitula omnia convertuntur, liquet quod idem defectus est in illis.

Considerandum præterea quod ex hac regula habetur proportio numeri cum additione producti ad numerum sine additione, velut, si 8 divisum in 3 et 5, producit 45, et additis partibus quæ sunt ex regula producitur 24. Ut superius dixi R ex regula secunda, dico quod proportio 45 ad 24, ut demonstratione patet se habet, ut 15 productum 5 in 3 ad 8 duplum quadrati differentiæ. Et ita dividendo 8 in 6 et 2, fit 24 primo modo, et per secundum regula fit 64, et proportio 24 ad 64, et velut 12 producti ex 6 in 2 ad 32 duplum quadrati 4 differentiæ.

Caput IV

De modo redigendi quantitates omnes, quæ dicuntur latera prima ex decimo Euclidis in compendium

Euclides constituit viginti quatuor lineas alogas, id est irrationales, et unam rhete seu rationalem vicem habentem numeri.

Rationalis seu rhete, ut sex vel septem. Aloga simpliciter quæ numero et rheti potentia tantum est commensa, ut R sex vel septem, id est latus tetragonicum superficiei rhete, sed non quadratæ. Media et est latus tetragonicum superficiei alogæ simpliciter, ut R R 6 et R R 7. Ex comparatione autem alogarum inter se vel cum rhetis consurgant sex genera binomiorum, de quibus dicemus.

Proprium primi binomii et similiter recisi est quod prima pars sit numerus, et secunda aloga, et quadrata harum differant numero quadrato, ut 3 p: R 5, quorum quadrata sunt 9 et 5, differentia est 4, latera autem 9 et 4, sunt 3 et 2, et ita 3 m: R 5 erit residuum primum. Et ita 4 p: R 12 conveniunt autem, ut dictum est in tertio Libro, tam binomii quam recisa quod primi et quarti prima pars est numerus, 2 R, 2 et 5 secunda pars est numerus 2, [1] R, 3 et[1] 6, ambæ partes sunt R , sed primus ordo, id est 1, 2 et 3, differunt a 2 ordine, id est 4, 5 et 6, quod in prima ordine pars maior, seu prima semper est potentior minore parte, seu secundo quadrato quantitate commensæ primæ parti, in secundo ordine incommensæ. Modus autem generalis omnibus binomiis et recisis habendi radicem est ut ducas secundam partem in se, ut R 12 per se, fit 12, cuius sume quartam partem semper, et est 3, fac ex prima parte 4 duas partes producentes 3, et erunt 3 et 1. Horum radices iunctæ faciunt 4 p: R 12, et in producendo quadrata efficiunt semper rhete, et aloga fit ex duplo unius partis in alteram, et est regula quam posuimus in tertio Libro Operis perfecti. Cum vero Euclides non poneret latus linearum, hoc invenit, ut acciperet superficiem ex rheti et binomio primo, cuius latus tetragonicum dixit esse aliquod binomium. Constat

Capitulum 19. Regula 2

[1] 1570 and 1663 have "*61*".

Liber 10 Propositio 54 autem talem superficiem etiam esse binomium primum, et est tertia regula quæ ut antedicta excipitur ad Euclide. Omnis enim quantitas commensa binomio Propositio 66 primo est binomium primum. Sunt autem commensæ cum fuerit proportio earum Propositio 6 ut[2] numeri ad numerum ex dictis ab illo in decimo Libro. Sed non sunt omnia binomia eiusdem speciei inter se commensa, et est quarta regula. Nam ut visum est 3 p: R 5 et 4 p: R 7 sunt binomia prima, et tamen inter se non sunt commensa Propositio 17 ex eodem Euclide. Idem in omni genere contingit alogarum, scilicet ut commensæ sint eiusdem speciei, non tamen quæ sunt eiusdem speciei invicem conmensæ sint. Inveni postmodum, quod idem est ducere R 5 in 3, et fit R 45, quæ si ducatur in R 5, fit R 225, et est 15; si duxerimus R 5 in se, fit[3] R 25, et 5 in 3, fit 15, vide quanto facilius sit, et hæc est quinta regula. Sexta autem continet quatuor propositiones quæ invicem convertuntur, et est quod cum fuerint duæ quantitates ab[4] maior et c minor, et divisa fuerit ab in d, ita ut inter bd et da cadat media, pars dimidia c, tunc si ab est potentior c, quadrato commensæ bd est commensa da etsi non, non. Et si bd est commensa da, tota ab est potentior c commensa ipsi ab, etsi non, non. Hoc autem pendet ex hoc quod abscisa de æquali da quadratum ab superat quadratum c, quod est æquale quadruplo bd in da per octavam secundi Elementorum in quadrato be. Usus autem harum linearum est, ut manifestum est, ad binomia et recisa invenienda; et ambas dictas bimedias. Septima regula sumitur ab Euclide, et est quod si superficies æqualis quadrato binomii ad rheten adiugatur, latus secundum est binomium primum. Et est eadem tertiæ regulæ licet videatur conversa. Cum enim ut in illa diximus latus binomii primi sit aliquod binomium, igitur quadrata binomiorum omnium sunt binomia prima, at latus illud rhete est tanquam proportio, et non variat speciem, igitur latus illud alterum necessario est[5] est binomium primum. Et omnia quæ hic dicuntur de binomiis, intelliguntur de suis residuis, et sunt generalia in omnibus quantitatibus comparando aggregatum ad recisum seu residuum, ideo per hanc octavam regulam tractabimus solum de sex generibus binomiorum, et quinque aliis, et ponemus nomina singillatim hic a latere.

[2] 1570 has "*et*".
[3] 1570 and 1663 have only "*25*".
[4] 1570 and 1663 have "*ab*".
[5] 1570 has "necio".

Rhete	Aloga	Media	
Binom 1	Binom	Reſ. 1	Reſid
Binom 2	Bimed 1	Reſ. 2	Reſ.med 1
Binom 3	Bimed 2	Reſ. 3	Reſ.med 2
Binom 4	linea ma.	Reſ. 4	linea mi.
Binom 5	Pot.in Rat & Med	Reſ. 5	cũ Rat & Med
Binom 6	Pot.in duo Med	Reſ. 6	cũ Med & Med

Modus quoque iungendi R has, ut apparet in eodem tertio Libro commodior est ut dividas maiorem radicem per minorem, et exeuntis accipe radicem cui adde, id est, et duc in se, et productum in quadratum minoris radicis, R productum est quæsiti. Decima regula est, quod huiusmodi divisiones sunt magis conspicuæ in figura quam in numeris, veluti posita R maximi aloga constat per 44. primi Elementorum posse super datam rhetem fieri superficiem æqualem illi. Eius ergo latus secundum erit, ut dixi in septima regula, naturæ eiusdem cuius est superficies, et cum eo latere potero dividere superficiem rationalem seu rheten, et confestim ex prima diffinitione secundi Elementorum habebo latus secundum quod vix in numeris haberi potest, et cum habetur fit magno labore, ibi vero statim est conspicuum, sed ars generalis nondum est inventa in numeris. Est autem iuxta undecimam regulam, ut invenias recisum usque ad quatuor quantitates, velut volo dividere[6] 10 per R 6 p: R 5 p: R 3 p: R 2, pones recisum ex æquis partibus contrariis, et habebis dividendum et divisorem:

10 R 6 p: R 5 p: R 3 p: R 2
R 6 p: R 5 m: R 3 m: R 2

6 p: R 120 m: R 24
R 600 p: R 500 m: R 300 m: R 200

6 p: R 120 p: R 24
133 p: R 17280

132 m: R 17280

qui est 6 p: R 120 m: R 24, cui appone trinomium quod ductum in recisum producit 132 p: R 17280. Duc etiam trinomium illud in quadrinomium, et habebis dividendum, quem tædii et brevitatis causa omitto. Rursus appono recisum et duco in binomium, et fit 144 pro divisione, ducto autem reciso in quantitatem quæ constat ex duodecim nominibus, fiet dividendo viginti quatuor nominum. Forsan poterunt reduci ad pauciora, quia radices illæ sint commensæ. Et si

[6] 1570 and 1663 have "*10 per R 6 p: R 15 p: R 5 p: R 2*".

transeant quatuor quantitates non commensas in duobus casibus adhuc poterit esse divisus. Vel quando habuerit radicem, ut 6 p: R 24 p: R 12 p: R 8, vel cum habuerit divisorem. Velut si quis dicat, divide 10 per R 24 p: R 15 p: R 15 p: R12 p: R 10 p: R 5 p: R 2 p: R 2. Hic quia producitur ex R 6 p: R 5 p: R 2 in R 3 p: R 2 p: 1, dividemus per regulam datam per alterum horum, inde quod exit per reliquum. Et ideo possumus reducere ad unum casum quando divisor dividi potest per multinomium, ut ita dicam, ita ut minuatur numerus nominum. Nam inventio lateris est quædam divisio. Consideranda est ultimo ratio proportionis superficiei ad lineam. Et dico quod superficies a ad lineam bc est ut super bc fiat superficies rectangula æqualis a, dico quod cd latus secundum est proportio, vel id quod adæquatur proportioni a ad bc, quia unus ex[7] proportione ducta in terminum fit, alter terminus, ut in numeris videmus. Et est ex diffinitione proportionis, et ex latere cd in bc fit a, quia sit bd æqualis a, igitur cd dicetur proportio vera superficiei a ad lineam bc. Et est magis conspicua quam superficiei ad superficiem et lineæ ad lineam. Euclides tamen (ut dixi) prætermisit, quoniam videbat lineas in latitudine esse individuas. Nos tamen dicimus quod componitur ex lineis, sicut ex fluxu puncti fit linea, et instantis tempus, et alia eodem modo, et hæc est duodecima regula.

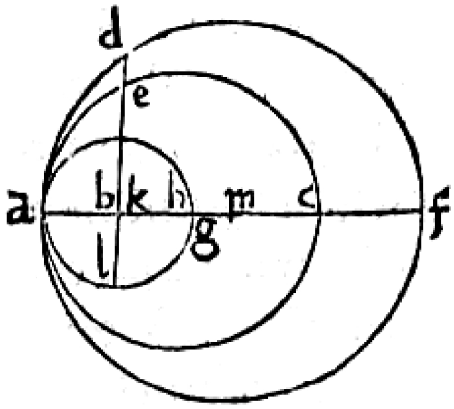
Binomii secundi latus est bimedia prima, ut docet Euclides suo modo, poterit igitur vel sub nomine RV: ostendi vel recta ratione. Capiamus ergo R 72 p: 8, et per secundam regulam ducamus 8 in se fit 64, cuius pars quarta est 16, fac ex R 72 duas partes ex quarum ductu invicem fiet 16. Accipio per quintam secundi Elementorum dimidium R 72, quod est R 18, et duco in se, fit 18, abiicio 16, remanet 2, cuius R addita et detracta a R 18, et facit R 32 et R 8 per nostram regulam. Igitur R harum partium, id est[8] R R 32 p: R R 8, constituunt bimediam primam, et subtractæ una ab altera residuum bimedii primi. Sunt enim[9] commensæ tantum potentia, nam quadratum R R 8 est dimidium quadrati R R 32, nam R 8 est dimidium R 32, et continent superficiem rheten, et maior est potentior breviore in quadrato R 8 cuius latus est R R 8, incommensa in

[7] 1570 and 1663 have "*quia n ex*".
[8] 1570 and 1663 have only "*i.*".
[9] 1570 has "n.".

longitudine R R 32, quod est propositum. Et iste modus est omnibus aliis longe facilior, et a nobis pro exemplo explicatur.

Binomii tertii sumatur radix ut R 32 p: R 24, et erit R R 18 p: R R 2, quæ potestate tantum commensæ sunt, et continent superficiem mediam 1 R R 36, seu R 6, et in hoc differt a priore, et maior est potentior minore in quadrato R R 8, quæ est R R 18 incommensa in longitudine, sicut R dimidii ad radicem totius; aut radix 3 ad R 2, et hæc vocatur bimedia secundam.

Binomii quarti ut 3 p: R 6 invenio radicem; capioque ad vitandas fractiones 6 p: R 24, cuius accipio R, quæ est RV: 3 p: R 3 p: RV: 3 m: R 3, ducito primum RV: 3 p: R 3 in se, fit 3 p: R 3, duc[10] RV: 3 m: R 3, fit 3 m: R 3, iunge, fiunt 6, duc RV: 3 p: R 3 in RV: 3 m: R 3, fit quadratum, ducendo unamquamque seorsum in se, et fiunt 3 p: R 3 et 3 m: R 3, inde invicem, et fiunt 6, inde accipiendo R quæ erit R 6, et hoc erit productum RV: 3 p: R 3 in RV: 3 m: R 3. Sed quia oportet bis multiplicare fient, duc R 6 quæ sunt R 24. Sed quia per quartam secundi Elementorum quadratum RV: 3 p: R 3 p: RV: 3 m: R 3 æquale est quadratis partium cum duplo unius in alteram. Igitur quadratum RV: 3 p: R 3 p: RV: 3 m: R 3 est 6 p: R 24, est quadratum RV: 3 p: R 3 p: RV: 3 m: R 3 [*sic*].



In figura autem capiemus ab, cui triplam faciemus bc in directum coniunctam, et sumemus medium ac, quod sit h, delineabimus super id circulum aec, et ducemus ex ab ad perpendiculum be, quæ erit latus seu R 3, hanc adiungemus ac in directum, et abscindemus ab ac, et fient bf 3 p: R 3 et bg 3 m: R 3 super ca, igitur et af divisis per æqualia in k et m ducemus circulos adf et al, et ex parte e producemus bed latus bf, et ex adverso bl latus bg. Itaque id erit æqualis RV: 3 p:

[10]1570 and 1663 have "*R 3 m: R 3*".

R 3 et RV: 3 m: R 3. Ex hoc patet quod geometrica ostensio est clarior arithmetica; et ut ita dicam evidentior, ea vero qua fit per numeros est fidelior, certior et securior, quia experimento probatur, ut supra feci. Et est tertia decima regula, sequitur etiam alia pulchra quartadecima, scilicet in ordine (licet non ad artem multum) et est quod sicut unum est principium in rebus naturalibus, ita etiam in transitu arithmeticorum ad geometricas figuras monas, quam quidam appellant unitatem, est principium necessarium inventionis, super qua fundatur tota ars.
39a decimi Dico modo quod RV: 3 p: R 3 p: RV: 3 m: R 3 conveniunt omnes proprietates
Per 16 Propositionem linæe maioris. Nam sunt duæ quantitates potentia incommensæ, omne enim binomium est incommensum suo reciso, etiam est vera in omnibus alogis, et facile demonstratur, et est regula quintadecima, et ambo quadrata pariter accepta sunt rhete, et productum unius in alteram mediam, nam quadrata iuncta faciunt 6,
Propositio 33 et productum unius in alteram est R 6. Notandum quod apud Euclidem addit una operatio, scilicet quod partes in se ducuntur, et additur quadratum mediæ partis minoris inde sumitur RV: 1. Et hæc operatio in numeris est superflua, quia possumus accipere radicem cuiuslibet quantitatis. Binomii quinti, et est ut R 24 p: 4, ducemus dimidium minoris in se pro secundam regulam, fiet 4, fac ex R 24 duas partes producentes 4 et erunt medietates R 6, quæ ductæ in se faciunt 6, adiice 4, relinquitur 2, cuius R addita ad R 6 facit R 6 p: R 2, et detracta R 6 m: R 2, et harum quantitatum RV: constituunt quantitatem quæ
Propositio 40 potest in rheten et mediam. Quadrata quidem harum sunt R 24 et productum unius in alteram est R 4, quod est 2, cuius duplum est 4 numerus binomii. Et sunt potentia incommensæ, quoniam sunt ut dixi binomium et recisum R 6 p: R 2 cum R 6 m: R 2. Omnes igitur hæ quantitates cum sint radices binomiorum in se ductæ producunt suum binomium. Et est regula sextadecima. Eadem ut dixi intelligenda sunt de recisis et residuis, qua sunt radices recisorum. Binomii sexti, et est ut R 24 p: R 12, ducemus R 12 in se, fit 12, cuius quarta pars est 3, faciemus ex R 24 duas partes quæ producant 3, et ducemus R 6, dimidium R 24, in se, fit 6, aufer 3, relinquitur 3, cuius R addita et detracta ex R 6 facit R 6 p: R 3 et R 6 m: R 3, quarum quantitatum radices universales constituunt iunctæ quantitatem quæ potest in duo media, nam sunt potentia primum incommensæ, quia quadrata illarum sunt binomium et recisum. Deinde quia compositum ex quadratis est R 24 medium, et productum unius in alterum est R 3, et R 3 est incommensa

R 24, est enim proportio unius ad alteram R 8, constat propositum ex Euclide. Præter hoc demonstrat quod dictæ alogæ quantitates, aliter dividi non possunt ut sint ex eodem genere in quo erant ante separationem. Ut pote 6 p: R 20 est divisum in 6 et R 20, et constituit binomium primum, aliter ut idem constituat dividi non potest. Cum divisa fuerint quantitates rhete per residuum aliquod, exibit binomium eiusdem ordinis commensum partibus suis illi residuo. Etsi per binomium exibit residuum eiusdem ordinis, similiter conmensum partibus, et erunt partes illæ binomiorum cum recisis, et etiam binomiorum et recisorum eadem proportione, et ex ductu residui in binomium semper producitur rhete. Et hæc demonstrantur ab Euclide in fine 10 Libro. Ex his constat quod binomio commenso binomio residui, aut residuo commenso residuo binomii rhete semper producitur. Item si latus secundum superficiei æqualis quadrato lineæ potentia tantum rationalis dividatur binomio vel residuo, exibit binomium vel recisum cum eadem proportione partium, quandoque eiusdem ordinis quandoque diversi. Velut dividendo R 24 p: R 3 p: R 2, exit R 72 m: R 48. At proportio R 72 ad R 48 est ut R 3 ad R 2 et sunt eiusdem ordinis. At si dividas eandem R 24 pro 2 p: R 2, exibit R 24 m: R 12, qua licet habeant partes in eadem proportione, eadem tunc sunt binomium cum reciso eiusdem ordinis, sed binomium est ordinis quarti et recisum sexti. Ex quo patet unum mirum, quod licet non possint esse quatuor quantitates in eadem proportione, quarum tres sint numeri et quarta sit potentia tantum rationalis, possunt tamen esse quatuor quantitates quarum tres erunt potentia tantum rationales et una erit numerus, et poterit esse quantitatis alogæ ad numerum proportio, velut alterius alogæ ad alogam, seu ut duarum alogarum. Sequitur etiam quod duæ quantitates incommensæ habebunt ambas partes conmensas, ut 2 p: R 3 et 5 p: R 12, nam cum sint binomia primi et quarti ordinis sunt incommensa et tamen 2 et 5 sunt commensa, et similiter R 12 et R 3 cum una sit dupla ad aliam.

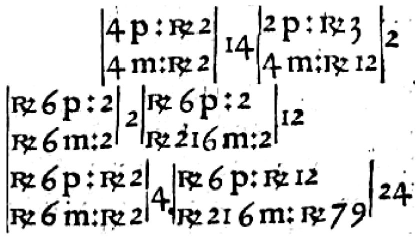
Propositio 41

Propositiones 112, 113 et 114

Caput V

De consideratione binomiorum et recisorum continentium figuram rheten, ubi de æstimatione capitulorum

Cum omne binomium et recisum possit esse latus superficiei numeratæ, ideo non distinguam nisi ratione partium, in quibusdam enim maior pars est numerus, in quibusdam minor, in quibusdam neutra. Proponam autem exemplum in omnibus.


4 p: ℞ 2 | 14 | 2 p: ℞ 3 | 2
4 m: ℞ 2 | | 4 m: ℞ 12 |
℞ 6 p: 2 | 2 | ℞ 6 p: 2 | 12
℞ 6 m: 2 | | ℞ 216 m: 2 |
℞ 6 p: ℞ 2 | 4 | ℞ 6 p: ℞ 12 | 24
℞ 6 m: ℞ 2 | | ℞ 216 m: ℞ 79 |

Dico quod æstimatio in binomio vel reciso, in quo non est numerus, non est idonea in hoc casu, quia detracta a numero relinquit tres quantitates incompositas, numerum et duas radices, et ex radicibus illis in se ductis non sit nisi numerus, et una radix numeri, ergo in producto non poterunt se delere. Examinemus ergo rem per singula capita, et dicamus quod si cubus et[1] 24 æquetur 32 rebus rei æstimatio cum sit duplex est 3 p: ℞ 5 et 3 m: ℞ 5, et ipsæ[2] conficiunt iunctæ 6, æstimationem cubi æqualis 32 rebus p: 24, et quia ex 32 oportet facere duas partes, ex quarum una in radicem alterius fiat 24, duco ergo 3 p: ℞ 5 in se, fit 14 p: ℞ 180, detraho ex 32, relinquuntur 18 m: ℞ 180. Et hic ductus in 3 p: ℞ 5 debet producere 24 numerum æstimationis.

[1] 1570 has "*est*", while 1663 has nothing.
[2] 1570 has "*ipse*".

18 m: R 180
2 p: R 5

24

14 p: R 180
3 p: R 5
18 m: R 180

Apparet ergo in hoc primo exemplo quod oportet divisionem fieri in binomio primo, nam 18 m: R 180 et 14 p: R 180 sunt binomia prima, quia habent radicem, et illa etiam oportet ut sit binomium primum, quia ducta in binomium primum producit numerum. Et si residuum non fuisset binomium primum, sed quartum, etiam radix binomii primi fuisset binomium quartum, aliter non potuisset producere numerum. Secundum exemplum igitur sit 1 cu. p: 12 æqualia 34 rebus, rei æstimationes sunt 3 p: R 7 et 3 m: R 7, quæ componunt 6, æstimationem 1 cu. æqualis 34 rebus p: 12, duco ergo 3 p: R 7 gratia exempli in se, fit 16 p: R 252, detraho ex 24, relinquuntur 18 m: R 252, ex quo et 3 p: R 7 producitur 12 ad unguem. Ista sunt plana. Tertium est 1 cu. p: 8 æquatur 18 rebus, et æstimatio est R 6 m: 2 (omitto nunc integram), quadratum R 6 m: 2 est 10 m: R 96, residuum R 96 p: 8. Causa est ergo quod binomium primum relinquitur residuum quin et converso. Quia ergo fuit radix residuum quintum res bene se habet.

℞ 96 p:8
℞ 6 m:2

8

1 cu.p:8 æqual.18 pos.
æstim.℞ 6 m:2

1 cu.p:48 æqual 25 rebus
æstim.℞ $3\frac{1}{4}$ m: $1\frac{1}{2}$

1 cu.p:21 æqual. 16 rebus
æstim.℞ $9\frac{1}{4}$ m: $1\frac{1}{2}$

1.cu.p:18 æqual.19 rebus
æstim. ℞ $17\frac{1}{4}$ m: $\frac{1}{2}$

1 cu.p:18 æqual.15 rebus
æstim.℞ $8\frac{1}{4}$ m: $1\frac{1}{2}$

1 cu.p:18 æqual.39 rebus
℞ 12 m:3

1 cu.p:12 æqual. 34 rebus
æstim.3 p:℞ 7,uel 3 m:℞ 7

1 cu.p:24 æqual 34 rebus
æstim.3 p:℞ 5,uel 3 m:℞ 5

Idem dico de binomio et residuo secundis. Et in hoc genere habet ferme plura exempla quam in primo velut vides. Et R 12 m: 3 est recisum secundum, et R 6 m: 2 est recisum quintum, et 3 m: R 5 recisum primum, et 3 m: R 7 recisum quartum. Habes igitur omnia exempla.

Primum igitur considerandum est quod in primo et quarto potest esse rei æstimatio binomium et recisum, ut vides in duobus ultimis exemplis, sed in secundo et quinto non potest esse nisi recisum. Probatur, nam si sit binomium primum, igitur residuum erit vel residuum primi vel quarti modi, ergo per præcedentem ductum in recisum secundi vel quinti generis non producit numerum. Sunt igitur sex æstimationum genera binomium, primum, quartum et recisum primi, quarti, itemque secundi et quinti modi. Secundum est quod, cum sint duæ æstimationes in hoc capitulo cubi et numeri æqualium rebus, vel æquales vel inæquales, et in reciso secundo et quinto non possit esse suum binomium, et secunda æstimatio habeatur per primam (ducto illius dimidio in se et triplicato producto et detracto a numero rerum) R residui deducto dimidio, æstimationis primæ est æstimatio secunda. Velut 1 cu. p: 18 æquatur 39 rebus, et rei æstimatio est R 12 m: 3, duco R 3 m: $1\frac{1}{2}$ in se, fit $5\frac{1}{4}$ m: R 27, triplica, fit 153 m: R 243, detrahe ex 39 numero rerum, relinquitur $33\frac{1}{4}$ p: R 243, a cuius radice universali detrahe R 3 m: $1\frac{1}{2}$, dimidium primæ æstimationis, erit secunda æstimatio RV: $23\frac{1}{4}$ p: R 243, ablata R 3 m: $1\frac{1}{2}$, et hoc totum constat esse æquale[3] 6, necesse est ut secunda æstimatio sit numerus, vel aliquid quod se habeat ad priorem æstimationem, ut RV: $23\frac{1}{4}$ p: R 243 p: l $1\frac{1}{2}$ m: R 3 ad recisum secundum. Et ita liceret per eandem regulam invenire secundam æstimationem. Tertium est quod cum in eodem numero, puta 18, inveniantur plures æstimationes, ut pote puta R $17\frac{1}{4}$ m: $\frac{1}{2}$ et R $8\frac{1}{4}$ m: $1\frac{1}{2}$ et R 12 m: 3. Ita oportet sub eodem numero rerum idem facere. Et hoc magis conveniret ad rei intelligentiam.

Quartum, quod videmus numerum rerum in numeros non solum integros sed etiam fractos, velut in quarto exemplo 19 dividitur in $17\frac{1}{2}$ et $1\frac{1}{2}$. In quinto autem 15 in $10\frac{1}{2}$ et $4\frac{1}{2}$, et in secundo 25 in $3\frac{1}{2}$ et $21\frac{1}{2}$, et in tertio 16 in $11\frac{1}{2}$ et $4\frac{1}{2}$. Considerare igitur oportet num in alias.

Quintum, quod videmus numerum æquationis si sit compositus, ut 18, 12, 24, facile habere æstimationem et plures etiam, si autem primus difficile est invenire unam solam.

[3] 1570 and 1663 have "*6*".

Caput VI

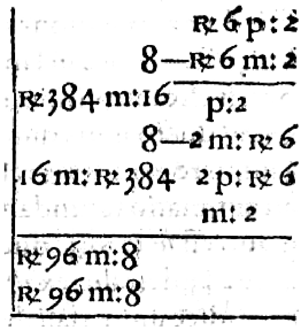
De operationibus p: et m: secundum communem usum

1. In multiplicatione et divisione p: fit semper ex similibus, m: ex contrariis, unde p: ductum in p: et divisum per p:, et m: ductum in m: et divisum per m: producunt semper p:. Et ita p: in m:, vel m: in p:, vel p: divisum per m:, vel m: per p: producit m:.

2. In additione omnia retinent suam naturam, in detractione commutant, ut p: additum fit p:, m: detractum p:.[1] Sin autem vincatur, relinquitur id a quo detrahitur, ut m: 4 a m: 6, relinquitur m: 2, quia m: a quo detrahitur p: maius.

3. R p: est p:,[2] R m: quadrata nulla est iuxta usum communem, sed de hoc inferius agemus,[3] de cubica dubium non est, nam R 1 cu. m: 8 est m: 2.

4. Si quis dicat divide 8 p: 2 p: R 6 vel R 6 p: 2, tum invenies ambo recisa R 6 m: 2 et 2 m: R 6, quod est vere m: ducas ergo recisa in 8 pro quantitate dividenda, fiunt R 384 m: 16 et 16 m: R 384, quod est m: hoc igitur cum primum sit dividendum per p: 2, exit manifeste R 96 m: 8. Secundum dividitur per m: 2, exit ex prima regula, idem scilicet R 96 m: 8.



Recisum autem quod componitur ex p: et m: potest habere radicem et illa constat ex p: et m:, ut 5 m: R 24, eius R est 3 m: R 2.

[1] 1570 and 1663 have "*m: m: detractum p: vicem gerit m: m: detractum p:*".
[2] 1663 omits "*p:*".
[3] 1570 has "*agemns*".

Caput VII

De examine æstimationum sumptarum ex regula secunda et tertia primi capituli

Proponamus[1] quod cubus æqualis sit 18 rebus p: 30, unde rei æstimatio iuxta partem capituli inventam sit Rcu: 18 p: Rcu: 12, et supra augendo numerum extenditur in infinitum. Et si dederimus parallelipeda omnia numero, oportebit ex hac æstimatione facere duas partes, ex quarum ductu in quadrata mutuo fiat 10, tertia pars numeri. Quare etiam ex ductu aggregati seu æstimationis in productum fiet idem. Dividam ergo 10 per Rcu: 18 p: Rcu: 12, exit Rcu: 12 m: 2 p: Rcu: $5\frac{1}{3}$ productum, dividam Rcu: 18 p: Rcu: 12 in duas partes, quæ ductæ invicem producant Rcu: 12 m: 2 p: Rcu: $5\frac{1}{3}$, et erunt partes

℞ cu $2\frac{1}{4}$ p: ℞ cu. $1\frac{1}{2}$ p: ℞ v: 5 m: ℞ cu. $\frac{3}{16}$ m: ℞ cu. $\frac{1}{12}$.
Et ℞ cu. $2\frac{1}{4}$ p: ℞ cu. $1\frac{1}{2}$ m: ℞ v: 5 m: ℞ cu. $\frac{3}{16}$ m: ℞ cu. $\frac{1}{12}$.

Dico ergo quod cum duo parallelipeda cum simili æstimatione possint æquari etiam 30, sex parallelipeda poterunt æquari 90 et multo amplius, veluti cubus æquatur 18 rebus p: 58, rei æstimatio est Rcu: 54 p: Rcu: 4. Et si cubus æquetur 18 rebus p: 75, rei æstimatio erit Rcu: 72 p: Rcu: 3. Et si quis dicat 1 cu. æquatur 18 rebus p: 33, rei æstimatio per eandem regulam erit Rcu: 24 p: Rcu: 9. Dividam ergo 11 per Rcu: 24 p: Rcu: 9, exit Rcu: $21\frac{1}{3}$ m: 2 p: Rcu: 3. Partes igitur erunt

Per quintam 2 Elementorum

℞ cu. 3 p: ℞ cu. $1\frac{1}{8}$ p: ℞ v: 5 m: ℞ cu. $\frac{1}{3}$ m: ℞ cu. $\frac{3}{64}$.
p: ℞ c. 3 p: ℞ cu. $1\frac{1}{8}$ m: ℞ v: 5 m: ℞ cu. $\frac{1}{3}$ m ℞ cu. $\frac{3}{64}$.

Similiter si cu. æqualis sit 18 rebus p: 42, erit æstimatio Rcu: 36 p: Rcu: 6. Et parallelipeda 14, divide 14 ergo per Rcu: 36 p: Rcu: 6, exit R 48 m: 2 p: Rcu: $1\frac{1}{3}$,

[1] In the title, 1570 and 1663 have "*secundi capituli*". See footnote 1, at page 669.

duc Rcu: $4\frac{1}{2}$ p: Rcu: $\frac{3}{4}$ in se, fit Rcu: $20\frac{1}{4}$ p: 3 p: Rcu: $\frac{9}{16}$, detrahe ex hoc quod produci vis, id est aggregatum, relinquitur[2] 5 m: Rcu: $\frac{3}{4}$ m: Rcu: $\frac{1}{48}$, partes erunt

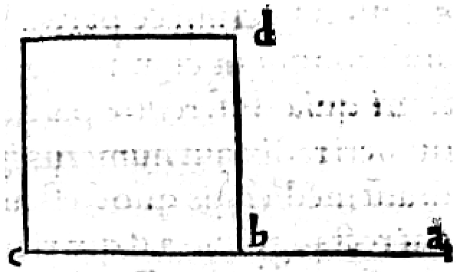
℞ cu. $4\frac{1}{2}$ p: ℞ cu. $\frac{3}{4}$ p: ℞ v: 5 m: ℞ cu. $6\frac{3}{4}$ m: ℞ cu. $\frac{1}{48}$ p:
℞ cu. $4\frac{1}{2}$ p: ℞ cu. $\frac{3}{4}$ m: ℞ v: 5 m: ℞ cu. $6\frac{3}{4}$ m: ℞ cu. $\frac{1}{48}$.

[2]1570 and 1663 have "*5 m: Rcu:* $\frac{3}{4}$ *m: Rcu:* $\frac{1}{48}$ *p: R*".

Caput VIII

De natura laterum parallelipedorum

Sit parallelipedum ex ab in cd quadratum æquale numero,



et dico primo quod si ab fuerit latus cubi et cubus bc numerus, erunt ab et bc commensæ. Nam proportio ab ad bc est ut numeri ad numerum, igitur sunt commensæ, quod si ab sit latus cubi et non commensum bc, clarus est quod cubus bc non potest esse numerus per præcedentem, neque bc ipsa. Tertio dico quod si cubus ba non sit numerus et parallelipedum sit numerus, nec bc est latus cubicum numeri. Aliter essent parallelipedi ad cubum, ut ab ad bc, et ideo ut numeri ad numerum, et ab commensa bc quod est contra Euclidem. Omnis enim commensa lateri cubi est latus cubi. Dico demum quod in hoc casu ab non est commensa bc. Nam cum cubus bc non sit numerus et parallelipedum sit numerus, ergo parallelipedum est incommensum cubo bc, sed ab ad bc ut parallelipedi ad cubum, igitur ab est incommensa bc, quod est quartum.

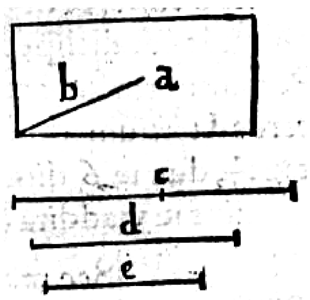
Per sextam diffinitionem 10 Elementorum

Per 33am 11i, et 10am et 6am Propositiones 10 Elementorum

Per 10am 10i Elementorum

Caput IX

Quomodo ex quacunque linea constituantur duo parallelipeda, non maiora quarta parte cubi lineæ propositæ



Sit parallelipedum a cuius altitudo b, proposita linea cuius duplum cubi medietatis non sit minus parallelipedo proposito, volo datam lineam sic dividere ut contentum sub c altitudine in superficiem partium sit æquale a parallelipedo. Inter c et b statuatur media proportione d, et fiat ut c ad d ita lateris tetragoni a ad e lineam, erit ergo superficiei a ad quadratum c velut c ad d duplicata, quare ut c ad b, quæ etiam est dupla proportioni c ad d parallelipeda, ergo ex c in quadratum e et ex ab in a erunt æqualia, quia ergo parallelipedum ex b in a non est maius parallelipedo ex c in quadratum medietatis eius, neque ergo parallelipedum ex c in quadratum e maius erit parallelipedo ex c in quadratum medietatis ipsius c, ergo e non est maius medietate c. Ex c igitur facio duas partes quarum rectangulum sit æquale quadrato c per ea quæ demonstravimus in geometria et habebimus partes lineæ propositas. Cum igitur parallelipedum ex c in superficiem ex suis partibus sit æquale parallelipedo ex b in a sequitur per demonstrata in Libro de proportionibus, quod duo mutua parallelipeda partium c lineæ propositæ sunt æqualia parallelipedo ex b in a, quod est propositum.

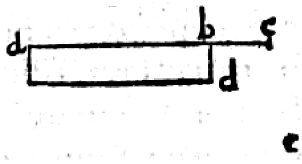
Per 24am 11i Elementorum

Per 143 Propositionem

Caput X

Quomodo conveniant partes cum linea proposita in parallelipedo

1. Sit proposita primum

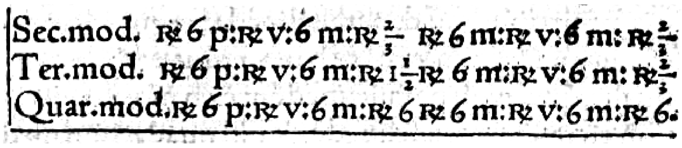


ac linea divisa in b ut parallelipedum ex tota ac in superficiem ad ex ab in bc sit æquale numero, et sit primo ac numerus, constat quod oportet ad esse numerum et partes ab, bc numeros aut binomium cum reciso, et potest demonstrari quia differentia partium necessario est numeri radix aut numerus ipse ad hoc ut quadratum medietatis quod est numerus, quia ac tota est numerus, excedat rectangulum ad qui est numerus, quoniam ex illo in ac numerum sit numerus e. Exempli causa, sit ac tota b, et e numerus 36, possum ex secundo modo tribuere numerum sex parallelipedis, et tunc duo erunt 12, divide 12 per ac, exit 2 superficies ad, igitur partes erunt 3 p: R 7 et 3 m: R 7. Et duo cubi erunt 90 p: R 8092. Et 90 m: R 8092, quod totum est 180, unde parallelipedis relinquuntur 36. Possum dare dimidium e iuxta tertium modum uni parallelipedo et erunt partes 3 p: R 6 et 3 m: R 6, possum iuxta quartum modum tribuere totum 36 duobus parallelipedis et partes erunt 3 p: R 3 et 3 m: R 3. Et in primo casu cubus æquabitur 30 rebus p: 36. In secundo cubus æquabitur 30 rebus etiam p: 36. Et in tertio rursus eodem modo, sed discrimen est quoniam in primo casu 30 res æquatuor cubis solum, in secundo cubis et duobus parallelipedis, in tertio duobus cubis et quatuor parallelipedis.

Ex quinta secundi Elementorum

2. Ponantur rursus e 12, ac R 24, dividam e totum, sed melius est reducere ad tertium modum dividendo dimidium, scilicet 6 per R 24, exit R $1\frac{1}{2}$, duc R

6 dimidium R 24 in se fit 6, abiice R $1\frac{1}{2}$, et fit 6 m: R $1\frac{1}{2}$, huius RV: addita et detracta a R 6 ostendit partes,[1]



Sec.mod. ℞ 6 p:℞ v:6 m:℞ $\frac{2}{3}$ ℞ 6 m:℞ v:6 m: ℞ $\frac{2}{3}$
Ter.mod. ℞ 6 p:℞ v:6 m:℞ $1\frac{1}{2}$ ℞ 6 m:℞ v:6 m: ℞ $\frac{2}{3}$
Quar.mod. ℞ 6 p:℞ v:6 m:℞ 6 ℞ 6 m:℞ v:6 m:℞ 6.

proponam autem iuxta singulos modos. Constat hanc æstimationem inutilem esse, nam habemus cubum per parallelipeda duo vel quatuor vel sex æqualia numero. At reliquum ergo est R aliqua, ut pote R 13824 m: 12 aut m: 6 vel m: 4, sed hoc diviso per R 24, quæ est res, nullus potest prodire numerus, igitur cubus non potest æquari rebus sub aliquo numero integro vel fracto.

3. Simili ratione sed alia tamen causa ostendo quod, si ac sit Rcu: numeri simplex, quod non potest satisfacere. Proponamus ergo quod ac sit Rcu: 40, et e sit 2, et accipio quartum modum in hoc casu ut faciliorem seu simpliciorem, divido 2 per Rcu: 40, exit Rcu: $\frac{1}{5}$ superficies ad, divido ac per æqualia, fit Rcu: 5, duco in se, fit Rcu: 25, detraho Rcu: $\frac{1}{5}$, relinquitur Rcu: $12\frac{4}{5}$, huius R addo et detraho ad Rcu: 5, partes erunt Rcu: 5 p: Rcu: R $12\frac{4}{5}$ et Rcu: 5 m: Rcu: R $12\frac{4}{5}$. Istarum igitur partium duo tantum parallelipeda faciunt 2, reliquum igitur a cubo Rcu: 40, manifestum est quod necesse sit esse numerum et est 38. 38 igitur divisum per rem quæ p: Rcu: 40 necessario producit Rcu:, igitur numerus rerum non potest esse numerus verus, sed Rcu:, ut si quis dicat cubus æquatur rebus Rcu: 100 p: 10, hoc autem non venit in usum. Quærimus enim nos cubum æqualem numero rerum, et numero seu integro seu fracto. Et dato quod incideremus in talem casum hoc esset raro, nec habemus regulam generalem, sed posset inveniri, velut in binomiis vel recisis pro ut nunc subiungemus.

4. Proponatur nunc (postquam priores tres modi parum utiles sunt, nam primus est notus etiam sine capitulis et est cuique obviam, nec est generalis, nec

[1]The correct values are:

$$\text{second way } \sqrt{6}+\sqrt{6-\sqrt{\tfrac{2}{3}}}\ \ \sqrt{6}-\sqrt{6-\sqrt{\tfrac{2}{3}}},$$
$$\text{third way } \sqrt{6}+\sqrt{6-\sqrt{\tfrac{3}{2}}}\ \ \sqrt{6}-\sqrt{6-\sqrt{\tfrac{3}{2}}},$$
$$\text{fourth way } \sqrt{6}+\sqrt{6-\sqrt{6}}\ \ \sqrt{6}-\sqrt{6-\sqrt{6}}.$$

ut in pluribus saltem, reliqui duo prorsus inutilis, nec ulla R alia simplex et R R vel R R p: vel R quadrata potest eadem ratione esse utilis, quia cubus eius necessario esset e genere primæ R et detracto e recisum, ergo divisum per rem non posset exire numerus ullus) quod ac sit duæ R quadratæ, dico quod et hic modus inutilis est, nam detracto numero e relinquetur cubus recisum, et ita non potest dividi per rem ut prodeat numerus, nam in cubo binomii vel recisi ubi ambæ partes sint radices non potest prodire numerus ut constat.

5. Et neque potest ac esse[2] R R vel Rcu: quadrata, quia tales perveniunt ad radices eiusdem generis, unde detracto numero fiunt recisæ, sed recisum non potest dividi per R ullam unius generis, ut prodeat numerus, igitur non poterit esse numerus rerum verus in æquatione. Sed neque ex R 2 et ex R R 18, nec ex R R R 8 et R R 2, nam quamvis tria parallelipeda in prima sint 18 et in secunda 6, relinquuntur tamen tres naturæ diversæ, ut in prima R 8 et R R 5832 et R R 23328, constat autem quod R R non potest magis esse commensa R simplici quam R simplex numero. Ergo R R 23328, nec R R 5832 possunt esse commensæ cum R 8, sed neque inter se, quia[3] R R 5832 fit ex R 18 in R R 18, et R R 23328 fit ex 6 in R R 18, at 6 et R 18 non sunt commensæ, licet divisæ 23328 per 5832 exeat 4, et ideo contingit, quia divisa R R 23328 per R R 5832 exit R R 2, igitur non sunt commensæ. Cum ergo in re non sint nisi duo genera quantitatum in dividendo et est residuum cubi tria, non poterit prodire numerus rerum. Et ita in omnibus similibus.

Corollarium 1. Ex quo patet quod hoc est generale, licet explicuerimus de parallelipedo, qualiscumque tribuatur pars cubi ipsi numero, reliquum erit plurium partium non commensarum quam sint in re, igitur non poterunt esse res sub numero aliquo.

Corollarium 2. Ex hoc etiam sequitur quod quo plures erunt eiusmodi partes incommensæ, eo fiet discrimen numeri partium cubi detracto numero a partibus radicis maius, ergo multo minus poteris residuum divisum per rem reddere numerum ut proponebatur.

[2] 1570 and 1663 have "*R R vel R R vel Rcu: quadrata*".
[3] 1570 and 1663 have "*R 5832*".

6. Nec potest esse una RV: quadrata, neque cuba, neque alterius generis. Nam si sit R quadrata, idem sequitur quod in secunda regula. Sin autem cubica dissolvetur, ergo non poterit continere rem, id est RV: cu., sub aliquo numero. Neque RV: quadrata iuncta alteri R simplici, nam ut dixi in corollario[4] secundo præcedentis, quo plures fuerint partes incommensæ in re, eo plures erunt in cubo in comparatione ad reliquas.

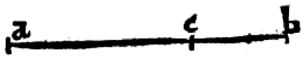
Necesse est igitur ut huiusmodi æstimatio universalis sit aut sub binomio, in quo sit numerus, aut in quo non sit, aut trinomio in quo sit numerus, aut in quo non sit, aut in pluribus nominibus in quo sit, aut in quo non sit, aut in quantitate sylvestri, scilicet quæ non sit in aliquo genere radicum, nec composita ex illis, nec per detractionem relicta, velut quantitas cuius R ducta in residuum ad 12 producat 2, ubi capitulum inventum non esset.

[4]1570 has "*corrolario*".

Caput XI

Partes cubi quot et quæ, et de necessitate illarum, et quæ incommensæ

Repetamus igitur et dicamus quod latus cubi, cuius quantitas quæritur, si debet æquari cubus duobus rebus et numero, oportet[1] ut cubus sic divisus in duo saltem, ergo latus eius, nam ex uno non provenit nisi unum. Ergo in duo saltem, cum ergo fuerint duæ partes, dum fit cubus, necesse est ut fiat quadratum totius, et hoc constat ex tribus partibus diversis natura, et si prima potentia ac et ab sint incommensæ omnibus invicem incommensis.



Nam proportio quadrati ac ad id quod fit ex ac in cb est velut AC ad CB, et similiter proportio eius quod fit ex ac in cb ad quadratum bc eodem modo, ergo quod fit ex ac in cb est incommensum utrique. Quadrata etiam ac et cb inter se incommensa, ergo per demonstrata ab Euclide erunt tres superficies in quadrato ab, quadratum ac, cb, duplum ac in cb, omnes incommensæ, cubus autem ab constat ex ac et cb in tres dictas superficies, et fiunt quatuor genera corporum: unum ex ac in quadratum ac, aliud ex bc in quadratum bc, tria ex ac in quadratum ab, et tria ex bc in quadratum ac. (Hæc autem non recito, quia revocare velim constructionem cubi in memoriam, sed cum alibi sint demonstrata, ut possim quæ opus est ostendere) Quare primum quæ fiunt[2] ex ac in quadrata bc sunt incommensa cubo bc, quia se habent ut ac ad cb, et eadem ratione quæ fiunt ex bc in quadrata ac, et similiter quæ fiunt ex ac in quadrata bc ad ea quæ fiunt ex bc in quadrata ac, sunt enim omnia ut dixi in proportione ac ad cb, ut a latere vides.

4a 2i Elementorum

d	e	f	g
8	12	18	27

[1]1663 has “oportebit”.
[2]1663 has “sunt”.

Sed posui numeros ut clarius videres proportionem, et ipsos ductus mutuos semel tantum repræsentantes parallelipeda ex ac in quadratum ab et cb in quadratum ac. Dico etiam quod per absurdum esset accipere partes commensas invicem ac et cb, quia sic esset perinde ac si essent una et eadem quantitas. Hoc stante habes proximas esse incommensas. At proportio f ad d est duplicata ei quæ est ac ad cb, ubi ergo ac et cb essent potestate commensæ d et f essent commensæ, et e[3] et g. Si vero secunda potestate, ut si una esset numerus alia Rcu:, vel ambæ Rcu: commensæ, tunc cubi invicem essent commensæ, sed ad parallelipeda utraque incommensi. Etiam ipsa parallelipeda incommensa sunt, ut liquet inter se, quoniam sunt in proportione ac ad cb. Et similiter R quad. 2 et Rcu: quad. 32, et R quad. 3 et Rcu: quad. 108 sunt commensæ potestate secunda. Primæ enim ductæ ad cubum producunt R quad. 8 et R quad. 32, secundæ R quadrata 27 et R quadrata 108[4] quæ sunt invicem duplæ. Unde notandum quod aliud est Rcu: esse secunda potestate commensas, nam omnes tales sunt, et earum cubi necessario sunt numeri aliud ipsas esse commensas, velut R cub. 16 p: Rcu: 2, vel m:. Ipsæ enim solæ sunt quarum parallelipeda sunt numeri, quod demonstratur. Nam si non sint commensæ ac et cb, igitur nec g et f, nec d et e, sed d et g sunt numeri, quia cubi Rcu:, igitur e et f non possunt[5] esse numeri. Non ergo potest esse ab composita ex duabus R cubicis incommensis, quia parallelipeda non essent numeri; neque commensis, quia esset una R cubum et cubus totus numerus. Nullæ ergo duæ quantitates aliquo modo si non adsit numerus possunt satisfacere parallelipedis pro numero, ut reliquum cubi satisfaciat rebus, nam si omnino sint incommensæ longitudine prima et secunda potentia, erunt quatuor producta incommensa. Ergo dato quód unum esset numerus, tria illa reliqua non possent continere duas quæ sunt in rebus numero, ergo non datur numerus rerum. Si autem essent commensæ longitudine, essent una quantitas, igitur non satisfaceret. Si vero commensæ potentia secunda et essent Rcu:, cubi essent numeri non parallelipeda, sin autem non essent Rcu: erit e ad f ut ac ad cb, sed ac non est commensum cb, ergo nec e cum f, igitur duo incommoda sequentur primum, quod si unum parallelipedum est numerus, alterum non erit, quare non

Ex Capitulo 10

[3]1570 and 1663 have "*c*".
[4]1570 and 1663 have "*180*".
[5]1663 has "non possunt esse non numeri".

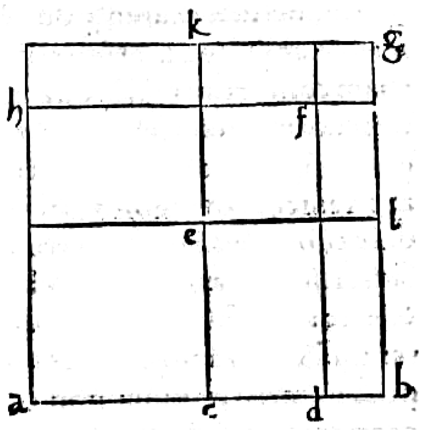
poterit fieri regula generalis. Secundum quod aggregatum cuborum, quod erit eiusdem naturæ, non poterit uni parti convenire secundum numerum, quia est in proportione commensa ad quamcunque partem cum quadrato unius earum, cum ergo sit quadratum non numerus nisi quantitas sit R, et si sit, tunc est contra dicta, constet quod non potest fieri æquatio. Exemplum dictum est Rcu: quad. 32 p: R 2, cubus secundum simplicia parallelipeda ad laborem fugiendum est R quad. 72 p: Rcu: quad. 2048 p: Rcu: quad. 8192. Hic constat nullum fieri numerum, ideo convenire non potest.

a—b—c
d—e—f

Dico modo quod nullum parallelipedum potest in his suppositis esse numerus. Aliter sint a et b non R cubicæ in secunda potentia commensæ, et producant parallelipedum c numerum si fieri potest, et quia sunt potentia secunda commensæ, capio duas Rcu: in eadem proportione d et e, quæ producant f parallelipedum, erit ergo ex dictis f Rcu: numeri, at c numerus est, ergo proportio c ad f ut numeri ad Rcu: talis est triplicata ex Euclide ei quæ est a ad d, at d Rcu: est alicuius numeri, igitur a est numerus vel Rcu: numeri, quod est contra suppositum. Sed neque possunt esse potentia prima commensæ partes, quia sic esset f ad g, et g ad e ut numeri ad numerum. Essent ergo hæ duæ quantitates, si igitur una est numerus reliqua non potest continere ac et cb, quæ sunt longitudine incommensæ. Si nulla ergo cubus non æquatur numero. Neque poterunt hæ duæ partes esse RV: cu.. Quoniam si commensæ erunt una; hoc autem demonstratum est esse non posse, si incommensæ fient quatuor partes in cubo incommensæ, ergo una erit superflua. Relinquitur tandem ut una sit numerus alia R, ut videbimus, vel ut sint plures quam duæ partes. Videamus ergo de tribus partibus primum cubicis omnibus et incommensis, ut sunt Rcu: 6 Rcu: 5 et Rcu: 2, cognosces autem esse incommensas longitudine, quando (ut dixi) numeri illarum ducti in quadratum, alterius non producunt numerum cubum, neque tunc Rcu: unius ducta in alterius quadratum producit numerum convertunt, ergo sic producere numerum et mutuo producere, et numeros producere eodem modo numerum cubum, et radices illas commensas esse. Et contraria horum etiam convertuntur. Ex quo tandem concluditur, partem illam capituli cubi æqualium rebus et numero non posse consistere in quantitate composita ex duabus R cubicis simplicibus aut universalibus, aut numero et R cubica. Nam in numero et R cubica oportebit dare cubos numero, quia erunt numeri, ergo in numero parvo non satisfacient, præterea parallelipeda incommensa erunt et duæ Rcu: et in re non est nisi una pars quæ sit Rcu: igitur non erit numerus rerum. Neque si ambæ partes sint Rcu:, quoniam si dederis parallelipeda numero primum non convenient cubi necessario,[6] si non sint commensa, sint Rcu: ergo non numerus. Præterea cubi erunt numeri, ergo non poterunt res continere per numerum, cum res constet ex duabus Rcu: ductæ in numerum, producerent numerum. Neque possumus dare utrunque cubum p: numero ubi numerus sit minor quarta parte totius cubi, ut docuimus, ubi autem est maior vel æqualis damus, et fit illa pars capituli cubi æqualis rebus et numero, quæ iam nota est, igitur reliqua pars in hac æquatione nullum habet locum. Neque possumus dare differentiam cuborum numero, ut in Rcu: p: et m:, velut Rcu: 6 m: Rcu: 2, quia parallelipeda m: erunt maiora et p: minora, ergo cum in re Rcu: p: sit maior Rcu: m: necessario, nullo modo res poterunt contineri per numerum in parallelipedis, sed bene iungendo p: cum m: et m: cum p: rerum cum cubo fiet ad unguem capitulum cubi et rerum æqualium numero. Sed neque æstimatio potest constare ex numero et R quadrata, ut sit generalis, hoc enim est demonstratum supra, neque potest constare ex numero et RV: quadrata, quia in cubo erunt duæ partes præter numerum incommensæ (quia RV: non est potentia prima commensa numero) et in re una tantum ergo non constabit numerus rerum. Neque ex numero et RV: cu., quoniam oportebit dare cubos numero et parallelipeda erunt duo incommensa, ergo ut prius cum sit tantum una RV: cu. non poterunt res numero aliquo contineri in cubo. Iam ergo ventum est necessario ad triarios,

Capitulum 3 in fine

[6]1570 has "*non convenient cu. necessario*", while 1663 has "*non convenient cum necessario*".



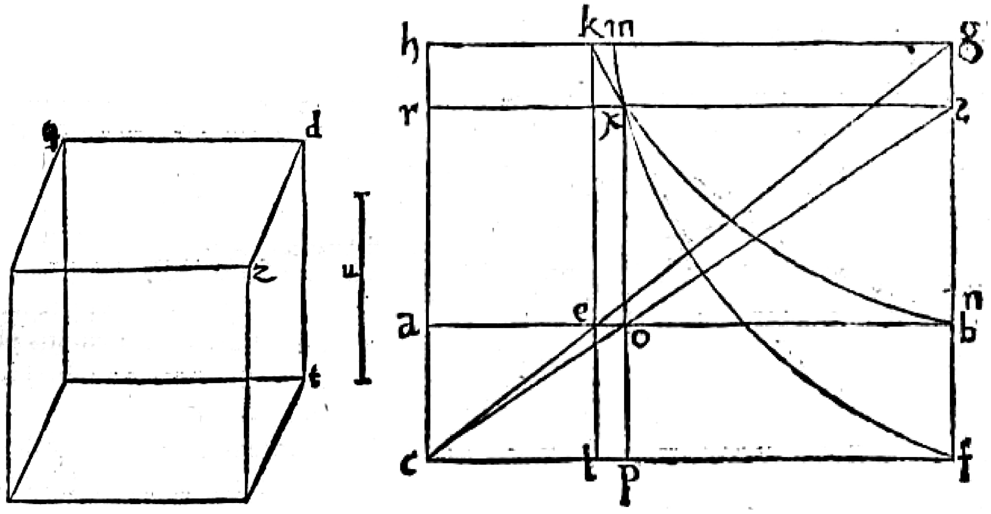
sit ergo ab divisa in tres partes, quæ omnes sint Rcu: incommensæ, nec in eadem proportione, et constat quod fient octo genera corporum, unum quod erit numerus qui constabit ex cubo singularum partium. Cum enim ac, cd, db sint Rcu: numerorum, erunt cubi earum numeri; quare et aggregatum eorum numerus. Secundum corpus constabit e sexcuplo corporis, cuius latera sunt omnes partes scilicet ac, cd, db, iam ergo habes novem corpora, reliqua decem octo cum sint tria et tria æqualia, erunt ergo sex, primum constabit ec, cd in triplum quadrati ac, secundum ex bd in triplum quadrati ac, tertium ex ac in triplum quadrati cd, quartum ex bd in triplum quadrati cd, quintum ex ac in triplum quadrati bd, sextum ex cd in triplum quadrati bd, cum ergo sint septem partes incommensæ in cubo et tres tantum in re, cubus non poterit æquari rebus sub aliquo numero. Ostendo modo quod ita sit. Nam in superficie ag sunt tria quadrata ae, ef, fg et sex superficies quarum binæ, et binæ sunt æquales de, eh et dl, hk et lf, fk. At ex ac, cd, db in sua quadrata fiunt tres cubi, ex ac vero in fl, fk idem fit quod ex cd in dl, hk, et ex bd in de, eh, igitur constat de novem iam corporibus in duo redactis. Dico modo quod ex una parte in quadratum alterius fiunt tria corpora, ut pote ex ac in ed, eh et ex cd in [*sic*], fiunt tria parallelipeda ex cd in quadratum ac, igitur cum binæ quantitatis residuæ multiplicentur, in quadratum tertiæ fient sex aggregata ex tribus parallelipedis, omnia igitur viginti septem reducta ad octo.

Per 43. 1i Elementorum

Caput XII

De modo demonstrandi Geometrice æstimationem cubi et numeri æqualium quadratis

Sint quadrata 12 æqualia cubo et 192 numero, gratia exempli, et constat
ex supradictis quod si numerus esset maior 256, quod propositum esse falsum Propositio 27
et si esset ipse numerus 256, quod latus cubi esset 8 seu bes eiusdem numeri quadratorum et ideo proposuimus numero illo minorem. Et ex eisdem constat quod si numerus esset dimidium maximi numeri, scilicet 128 quod res esset tertia pars numeri quadratorum propositorum, quia proportio quadrati bessis ad quadratum trientis est velut bessis ad id quod provenit diviso 128 solido proposito per quadratum bessis, quod est 64, exit enim 2 qui est quarta pars 8, ut 16 quadratum 4, trientis est quarta pars 64 quadrati 8 bessis numeri quadratorum propositi. Nos ergo sumpsimus alium numerum ab his ut dixi.



Proponatur ergo corpus solidum dqtz rectilineum et æquidistantium laterum ac superficierum, cuius ima superficies sit dqt quadrata et sit totum solidum 192, scilicet numerus propositus, et eius altitudo sit linea dz, et sit ab data 12 æqualis, scilicet numero quadratorum proposito, et divisa ita ut be sit dupla ad ea. Et duabus cb[1] et dq subtendatur linea quædam u, et sit dz ad ac ut eb ad u, erit ergo quadrati eb ad quadratum qt ut eb ad u, quare ut dz ad ac. Igitur solidum quod sub ac et quadrato eb æquale solido dqtz, propositum igitur est sic dividere ab, ut solidum ex una parte in quadratum alterius sit æquale solido ex ac in quadratum eb. Et hoc nos docet facere Eutocius Ascalonita in secundum de Sphæra et cylindro bifariam, sed sufficiat adduxisse primam illius demonstrationem. Non adducam autem propositiones ex Euclide tanquam notissimas, erigo ergo ac ad perpendiculum super ab, et compleo superficiem abcf, et duco ce usque occurrat fg in g, et compleo similiter superficiem æquidistantium laterum hgcf, et duco ex e æquidistantem ch, quæ sit lek, et resecetur gm, æqualis dq, et duabus lineis ab et eb subtendatur in continua proportione fn. Ducatur ergo super gf axe paraboles quæ transibit per m, ut ostendam, et similiter ex b, ducatur circa coincidentes hc et cf hyperboles quæ transibit per k, per ea rursus quæ demonstrata sunt ab Apollonio in secundo [conicorum[2] Elementorum. Ubi ergo se divident kb et mf in x, ducam rxs æquidistantem ab et xp æquidistantem rc, quæ secabit ab in o, quem punctum dico esse quæsitum. Ducam ergo co[3] quam ostendam pertingere ad s, quia ergo ut ea ad ac ita, quadratum be ad quadratum gm et ideo rectanguli ex cf in fn ad idem, at ut ea ad ac ita cf ad fg, et ut cf ad fg sic[4] quadratum cf ad id quod sub cf, fg, quare ut id quod sub cf, fn ad quadratum gm ita quadratum cf ad id quod sub cf, fg. Igitur quadratum cf ad id quod sub cf, fn ut id quod sub cf, fg ad quadratum gm. At ut quadratum cf ad contentum sub cf fn ita cf ad fn, et ut cf ad fn ita contenti sub cf, fg ad contentum sub fg et fn, igitur ut cf, fg ad quadratum gm ita contenti sub cf, fg ad contentum sub gf, fn igitur quadratum gm æquale ei quod fit ex gf in fn. Igitur gm est media proportione inter gf et fn. Ducta ergo paraboles ex primo Conicorum Apollonii per f, axe gf, cadet in m punctum,[5] quod est primum. Et quia he est æqualis ef, sunt enim supplementa, erunt hl et af æqualia, et coincidentes hc, cf. Ergo hyperbole ducta ex b resecabit proportione respondentem fb ipsi gf ex gh. Igitur cadet in k,

Per nonam octavi Elementorum

[1] 1570 and 1663 have "*eb*".
[2] 1570 has "*cronicorum*".
[3] 1570 and 1663 have "*eo*".
[4] 1570 has "*dic*".
[5] 1570 has "*[d]ucta ergo parabolę ex primo Conicorum Apollonij per f, ita ut ductæ possint ad fn axe gf cadet in m punctum*", while 1663 has "*[d]ucta ergo parabola ex primo Conicorum Apollonij per f, ita ut ductæ possint ad fn axe gf cadet in m punctum*".

quod est secumdum. Cum ergo hc et cf sint coincidentes, et rectangula rxp et abf tangant hyperbolem, igitur invicem sunt æqualia, detracta igitur communi aopc, erunt duæ superficies arxo et obpf[6] æquales. Et cum sint supplementa erunt circa eandem diametrum. Igitur co cadat in s, quod est tertium. Quoniam ergo cf ad fs ut cp ad po, et ideo ut ao ad ac, et ut cf ad fs ita contenti sub cf, fn, quod est quadratum eb ad contentum sub sf, fn, erit quadrati eb ad contentum sub sf, fn, velut oa ad ac, et contentum sub nf, fs æquale quadrato sx, propter parabolam assumptam, super nf igitur quadrati eb ad quadratum ob, quod est æquale quadrato xs, velut oa ad ac, igitur solidum ex ao in quadratum ob est æquale solido ex ac in quadratum cb, quod fuit demonstrandum. Et fuit quartum, liquet autem quod ratio constructionis huius problematis pendet ex his duobus, primum quod assumpto puncto n æqualiter distante a vertice paraboles, qui est f, ita ut paraboles resecet æqualem ex perpendiculari ducta ex n ad parabolem ipsi nf semper ducta ad perpendiculum ex illo axe gf, quantuscunque sit ad parabolem, media illa est inter nf, et lineam a vertice ad punctum, ex quo perpendicularem eduxisti. Alterum pendet ex constructione hyperbolis, nam cum ducta ad perpendiculum super axe, et axis inciderint in duas rectas ad perpendiculum illæ in quas incidunt, vocantur coincidentes, et semper faciunt superficies æquales extra contentas. Ut in exemplo sumpto puncto k cum vertice kc est æqualis cb, et sumpto puncto x cum vertice fit xc æqualis eidem cb. Ex quo sequitur manifeste quod xc est æqualis kc, ideo hoc evenit, quia semper kr[7] est æqualis xl, ubicunque punctus x statuatur. Apparet ergo quod proposita ab 12 semper fn erit eadem quia in proportione ad, ab et eb, et ideo $5\frac{1}{3}$. Et si dqtz ponatur 192, erit ac 3, et si 128, erit ac 2, et si 64, erit 1. Et in primo casu xc semper erit 36, in secundo 24, in tertio erit 12. Posita ergo fs quad. [fs quad.,[8] erit xs in omni casu res numero R $5\frac{1}{3}$, igitur rx erit 12 m: rebus[9] R $5\frac{1}{3}$, quia ergo xp est æqualis sf, erit superficies rp, atque ideo af 12 quad. m: cu. R $5\frac{1}{3}$, et hoc potest esse æquale 36 et 24 et 12, vel cuicunque numero. Cum ergo reduxerimus ad unum cubum, fient in omni casu[10] $\sqrt{27}$ quadrata, scilicet ducta

Per 34am undecimi Elementorum

[6]1570 and 1663 have “*abpf*”.
[7]1570 and 1663 have “*k*”.
[8]1570 and 1663 have “*c*”.
[9]1570 and 1663 have “ ° $5\frac{4}{3}$”.
[10]1570 and 1663 have “*8*”.

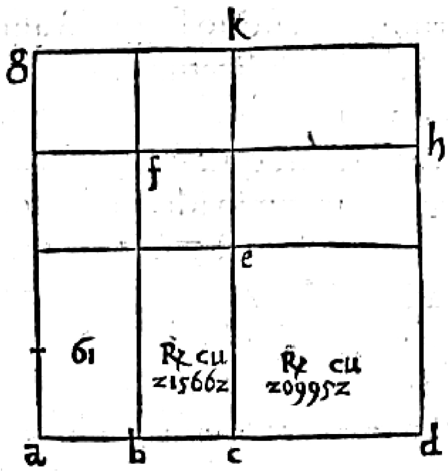
quantitate ab, quæ est 12 per fn, quæ est R $5\frac{1}{3}$, fit 64 tum est 8. Igitur supposita ab solum 12, quantumcunque sit solidum dqtz, erunt semper 8 quadrata æqualia cubo et numero, qui producitur ducta ac et ab, et producto in R $5\frac{1}{3}$, si ergo bc fit 36, erit numerus[11] R 6912, et si fuerit 24, erit R 3172, et si fuerit 12, erit R 168. Et æstimatio in se ducta producet sf, quæ ducta in nf, et eius sumpta radice proveniet sx pars quæsita, nam ipsa est æqualis ob. Ergo ducta in se, et detracta ab ab, et uno in alterum ducto proveniet solidum dqtz. Et ideo facilis operatio geometrica difficillima est arithmetice, nec etiam satisfacit.

[11] 1570 and 1663 have "*6912*".

Caput XIII

De inventione partium trinomii cubici, quod cubum producit cum duabus partibus tantum cubicis

Et dico modo quod si assumatur trinomium cubicum, ex cuius ductu partium producatur numerus, quod producentur duæ partes tantum, quæ sint R cubicæ, sed in re sunt tres partes inconmensæ, ut dictum est, igitur partes cubi non possunt continere partes rerum secundum numerum. Ex quo sequitur quod cum res fuerit ex tribus radicibus cubicis in continua proportione, quod idem sequetur,



nam Rcu: 12 p: Rcu: 6 p: Rcu: 3 producunt quantum media ducta ad cubum, igitur invicem ductæ producunt Rcu: 216, quæ est 6. Hoc igitur generaliter sic demonstratur. Supponatur trinomium cubicum abcd, solum cum hac conditione, quod corpus ex ab, bc, cd sit numerus, constat ergo quod sunt novem corpora, quæ sunt æqualia numero. Reliqua decem octo sunt tria, ut dictum est, ex ab in quadratum bc et ex cd in quadratum ab, quæ dico esse commensa, velut [*sic*]

Per 17am sexti Elementorum

R cu.3	R cu.4	R cu.18.	
1296.	216	972.	729
162.	27	288.	216
48	8	36	27

et ex bc in quadratum cd. Nam quod fit ex ab in quadratum bc ad id quod fit ex cd in quadratum ab se habet ut quadratum bc ad id quod fit ex ab in cd, at quadratum bc se habet ad id quod fit ex ab in cd ut numerus ad numerum, nam ex bc in quadratum suum fit cubus, qui est numerus, et ex bc in rectangulum ab in cd fit parallelipedum æquale numero, igitur proportio bc quadrati ad superficiem ab in cd est velut numeri ad numerum. Ea igitur ratione etiam quod ex bc in quadratum cd, igitur fient duo tantum Rcu: incommensæ, at in radice sunt tres, igitur non possunt res æquari cubis assumptæ per rerum. At si proposueris partem numerum, velut Rcu: 32 p: Rcu: 16 p: 2, proveniunt 152 p: Rcu: 131072 p: Rcu: 128000. Ideo cum sit longe minor proportio quam partium rei, non poterit cubus æquari certo numero rerum. Vide etiam infra.

Per 32am undecimi Elementorum

Capitulum 51

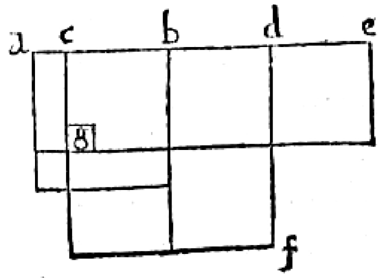
Caput XIV

De inventione generis æstimationis

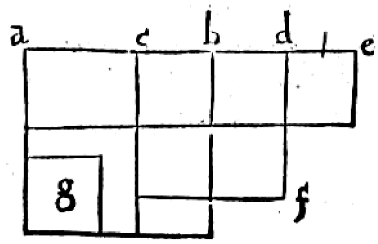
Cum sit constitutum quod æstimatio cubi et numeri sit duplex. aut binomium et suum recisum primum, aut recisum quintum, et ex utraque fiat æstimatio cubi æqualis rebus et numero, necesse est ut, cum ex binomio et suo reciso fiat numerus et ex reciso quinto et numero binomium quintum, ut capituli cubi æqualis rebus et numero sit tantum inventa æstimatio binomii quinti ultra numerum. Ideo primum quæramus habita æstimatione cubi et numeri, sed non dati æqualium rebus numerum rerum et æquationis. Proposito ergo binomio primo vel quarto, aut residuo eorum, seu residuo secundo aut quinto, ducatur pars quæ est numerus in se, et triplicetur, et ei addatur quadratum partis quæ est radix, et conflabitur numerus rerum. Deinde pro numero æquationis duplica partem quæ est numerus, et duc in se, et residuum a numero rerum ducatur in idem duplum numeri, et fiet numerus æquationis. Exemplum R 7 m: 2. Primum duc 2 in se, fit 4, triplica fit 12, adde quadratum R 7, fit 7, et totum 19 numerus rerum. Inde pro numero æquationis dupla 2, fit 4, duc in se, fit 16, differentia a 19 numero rerum est 3, duc in 4 duplum 2, fit 12 numerus æquationis. Igitur 1 cu. p: 12 æquatur 19 rebus.

Capitulum 13 Artis Magnæ

Vide Capitulum 5 supra



Huius causa est quod, posita ab re divisa quomodocunque in c ita quod bc sit numerus et ac alia quantitas, et iuxta quadratum bc addantur duo alia quadrata ei æqualia, et eadem sumantur cum quadrato ac pro numero rerum, illæ res erunt æquales cubo assumptæ lineæ ab cum eo quod fit ex duplo bc in differentiam numeri rerum a duplo quadrati bc, quod fit bd, nam si tria quadrata bc cum quadrato ac sunt numerus rerum, ergo res sunt æquales tribus cubis bc et triplo ac in quadratum bc et cubo ac et parallelipedo ex bc in quadratum ac, detraho igitur cubum ab ex illis corporibus, relinquetur differentia dupli cubi bc, duplo bc, in quadratum ac, at vero numerus ex supposito sit ex duplo bc in differentiam ef a numero rerum quæ est quadratum bc minus quadrato ac, nam quadratum cf continet quater quadratum bc, et numerus rerum continet quadratum bc, ter et insuper quadratum ac, igitur facto g æquali quadrato ac, duplum quadrati bc excedit numerum rerum in gnomone g, at demonstratum est quod res excedunt cubum ab in differentia dupli cubi ab ab eo quod fit ex bc in quadratum ac, quod est g, ideque in duplo bc in gnomonem, igitur numerus hic additus cubo æquatur rebus. Idem dices si numerus esset minor radice, sed ab residuum, eodem enim modo procedit demonstratio, sed oportet mutare figuras.



Idem vero contingit ubi cubus sit æqualis rebus et numero, et sit binomium secundum aut quintum, ut sit numerus rerum triplum quadrati bc cum quadrato ac eritque numerus, id quod fit ex duplo cb in differentiam quadrati ac[1] a quadrato dupli bc, quod sit cf. Erunt enim tres triplum cubo bc, triplum parallelipedi ac in quadratum bc, cubus ac, et parallelipedum ex bc in quadratum ac, detrahantur hæc octo corpora ex cubo ab, relinquetur differentia cubi ab ab octo corporibus duplum bc in quadratum ac a duplo cubi bc, at numerus fit ex supposito ex duplo bc in differentiam cf quadrati a tribus superficiebus quadratis bc cum quadrato ac, hæc autem est quantum differentia quadrati ac a quadrato bc cum triplum quadrati bc sit commune utrisque quantitatibus, fiat igitur g quadratum bc in quadrato ac, cum sit minus ergo numerus æquatur duplo bc in gnomonem g, cubus autem ab excedit res, ut demonstratum est in differentia dupli bc in quadratum ac a duplo cubi bc, sed duplum bc in gnomonem g est æquale duplo

[1] 1570 and 1663 have "*cf*".

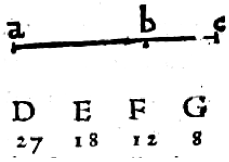
excessus bc in quadratum ac a duplo cubi bc, quoniam sunt eædem altitudines et superficies, ergo cubus æquatur rebus et numero assumptis.

Corollarium. Ex hoc patet quod hæc æquatio est inæqualis, ideo neque generaliter potest tradi regula, nam numerus datur duplo parallelipedi minoris partis in gnomonem, qui est differentia quadratorum partium, liquet etiam quod talis gnomo in omni casu est æqualis rebus solis, ubi partes suppositæ sint dimidium numeri plus una re et minus una re.

Caput XV

De inventione partium rei per partes cubi

Ubi propositæ sint partes cubi ac notæ,



ex quibus velis scire quantitatem lineæ ac. Quinque suppositis id ages cognitis. Primum ut scias proportionem partium singularum aut excessum, quæ continetur his regulis. Prima proportio cubi ad cubum est triplicata lateris ad latus. Secunda proportio cubi ad parallelipedum, quod fit ex quadrato lateris sui, est ut partium lineæ. Tertia proportio cubi ad parallelipedum alternum est ut partium lineæ duplicata. Quarta proportio parallelipedorum invicem est ut partium lineæ. Quinta proportio aggregati cuborum ad aggregatum duorum mutuorum parallelipedum est velut aggregati quadratorum partium lineæ, detracto parallelogrammo ipsarum ad illud parallelogramum. Sexta proportio cubi cum parallelipedo proximo ad parallelipedum alternum cum alio cubo est ut partium lineæ duplicata. Septima proportio aggregati ex cubo et parallelipedo alterno ad aggregatum ex parallelipedo proximo et alio cubo est velut partium lineæ. Octava differentia aggregati ex cubo et triplo parallelipedorum alternum ab aggregato parallelipedorum trium proximorum et alterius cubi est cubus differentiæ partium lineæ. Secundum suppositum debet reducere parallelipeda et cubos semper ad unum præterquam in hac octava regula, ut unum uni comparetur. Tertium suppositum proportiones partium reducuntur ad proportionem 1 cu., 1 quad., 1 pos., et 1 ipsius proportionis, ita ut 1 pos. sit ipsa proportio. Unde si quis dicat, fuit parallelipedum 4 et residuum cubi 104, dico tu scis constitutionem cubi, et pones 1 cu. p: 1 pro re p: 1, et habebis 1 cu. cu.[1] p: 3 cu. quad. p: 3 cu. p: 1, et hoc est in proportione ad 1 cu. ut 108, quod fit restituto 4 ad 104, ad 4 ut 27 ad 1 R,

[1] 1570 and 1663 have only "*cu.*".

autem cubica 1 cu. cu. p: 3 cu. quad. p: 3 cu. p: 1, radix cu. 1 cu. est 1 pos. Rcu: 27 est 3, igitur 1 cu. p: 1 æquatur 3 pos. Et in hoc supposito ingreditur scientia compositionis cubi ex cubis partium et sex parallelipedis, quorum tria sunt similia et æqualia et tria similiter inter se, et quod fiunt ex una parte in alterius quadratum et cognitio extrahendi Rcu: et quadratum et dividendi per communem divisorem, cum fuerint plures denominationes. Quartum suppositum est ut scias quod cum volueris iungere aliquas R eiusdem generis aut detrahere, divides unum per aliud, et accipe R illius generis proventus, et pro additione adde 1 et pro detractione auferto, et quod fit, ducito ad quadratum, si fuit R quadrata vel ad cubum si cuba, et productum multiplicabis per divisorem, et quo provenit est quæsitum. Quintum suppositum est ut adiuves te cum regulis generalibus algebraticis et de modo.

Si quis ergo dicat cubus et duo parallelipeda altrinsecus sunt 24 et cubus alter cum duobus parallelipedis 18, igitur per tertium suppositum 1 cu. p: 2 pos. est sexquitertium 2 quad. p: 1, et æquale $2\frac{2}{3}$ quad. p: $1\frac{1}{3}$, et habebis $\frac{10}{27}$ rerum p: $\frac{700}{729}$ æqualia cubo et rei æstimatio cum $\frac{8}{9}$ TPNQ[2] est rei æstimatio.

Et generaliter posito uno cubo, puta ab, cum quartis parte nota, erit reliquum notum. Quia ab nota. Ergo si cubus bc, igitur bc, igitur parallelipeda, vel si parallelipedum, diviso eo per ab, vel quadratum ab prodibit bc. Igitur tota ac.

Ex difficilioribus autem modis primus est cum cubi et tria parallelipeda proxima cognita sunt. Habebis tamen rei æstimationem R cubica totius, velut unum aggregatum sit 48, aliud 16, erit res 4 latus cubicum 64 totius. Secundum adhuc difficilius cum cubi et duo parallelipeda proxima nota fuerint, nam nec licebit assequi rem, ut in priore, subiacet tamen inventioni, et habet æqualitatem. Ultimum est cum est anomalum, ut aggregatum ex duobus parallelipedis unius generis et uno vel tribus ex alio genere, vel duo cubi cum uno parallelipedo, vel duobus ex uno genere, alio ex alio genere, vel cubus et tria parallelipeda unius generis cum parallelipedo alterius generis. E ita de aliis modis inæqualitatis.

Demus est compositio notior cubi cum duobus parallelipedis proximo et uno remotiore, nam primum talia aggregata sunt in proportionem partium lineæ. Et singula eorum habent radicem quadratum, velum unum sit 20, aliud 80, erit

[2]This abbreviation means 'tertia pars numeri quadratorum'. It is very often used in the *Ars magna*, especially starting from Chapter XVII, see here, page 130.

proportio partium lateris totius cubici quadrupla, igitur ponemus unam 1 pos., aliam 4, inde producemus 64 cu. p: 48 cu. p: 12 cu. p 1 cu. et hæc sunt æqualia 100 res, igitur est Rcu: $\frac{4}{5}$. Aliter ponemus proportionem 1 quad. erunt partes ut a latere: dissolve in duo aggregata proposita, habebis partes, ut vides.

1 cu. q̃n. p: 3 q̃d' q̃d. p: 3 q̃d. p: 1.
1 cu q̃d. p: 2 q̃d' q̃d. p: 1 q̃d. — 1 cu. p: 1 pos.
1 q̃d' q̃d. p: 2 q̃d. p: 1.
4 q̃d' q̃d. p: 8 q̃d. p: 4 — 2 q̃d. p: 2.
$\frac{1}{2}$ pos.

Igitur cum sit proportio illarum quadrupla, multiplicabis aggregatum minus per 4, et assumes R quadratum partium quam semper se habent, quia ab initio habuerunt et post duxisti per numerum, habes igitur 1 cu. p: 1 pos. æqualia 2 quad. p: 2, et ideo semper poteris dividere unum per aliud, habebit ergo $\frac{1}{2}$ pos. æqualia duplæ. Circa quod nota quod cum 1 cu. p: 1 pos. sit æquale 2 quad. p: 2 pos.. Igitur dividendo unum per aliud, provenit unum, et iam provenit $\frac{1}{2}$ pos.. Igitur $\frac{1}{2}$ pos. est 1 et 1 pos. est 2, et ego posui 1 quad. pro proportione, res ergo redit ad idem. Sed hoc volui ostendere ob reliqua.

Caput XVI

Quod quadrinomii ex radicibus cub. cubus ad tres partes quarum duæ sint tantum R cubæ reducitur, aut longe plures

Et si primo quadrinomium ex R cubicis in continua proportione, in quibus non sit numerus, ut Rcu: 3 p: Rcu: 6 p: Rcu: 12 p: Rcu: 24 est. Ergo Rcu: 24 ad Rcu: 3 triplicata proportio, at 3 ad 3 ut Rcu: 6 ad Rcu: 3 triplicata, igitur Rcu: 3 est dimidium Rcu: 24, igitur faciunt Rcu: quæ est R cu.[1] 81, sed Rcu: 81 ducta in Rcu: 72, productum ex Rcu: 6 in Rcu: 12, producit Rcu: 729, scilicet 9 duplicatam, igitur per supradicta quadrinomium illud reductum ad cubum non habet nisi duas R cub.. Ideo res non possunt æquari cubo. Ostendo modo quod ex producto ex cu. 72 in Rcu: 81 fiat numerus, quia per dicta productum ex Rcu: 3 in Rcu: 6, inde in Rcu: 12, sint in continua proportione.[2] Pariter ex Rcu: 6 in Rcu: 12, et exinde in Rcu: 24, fit numerus. Igitur ex producto Rcu: 6 in Rcu: 12 in Rcu: 3 et in Rcu: 24, fiunt numeri, ergo in aggregatum fit numerus, quod erat demonstrandum.

Si vero inter illas R cub. sit numerus, velut R cub. 2 p: Rcu: 4 p: Rcu: 8 p: Rcu: 16. Idem continget ut palam est, producuntur enim Rcu: 128 et 4, et reliquæ omnes illis commensæ sunt, ut facile demonstrari potest, idem fiet in trinomio solo ex Rcu: 16 p: Rcu: 4 p: Rcu: 2. Et eo magis ut de hoc non sit dubitatio, quia sumus in casu priore.

Si vero sint tales ut duæ in quadratum alterius producant numerum iam conmensæ sunt. Et ideo non sunt amplius quatuor, sed tres; si una R alterius nihil refert in hoc casu, capiamus ergo Rcu: 2 Rcu: 3 Rcu: 4 Rcu: 5. Et manifestum

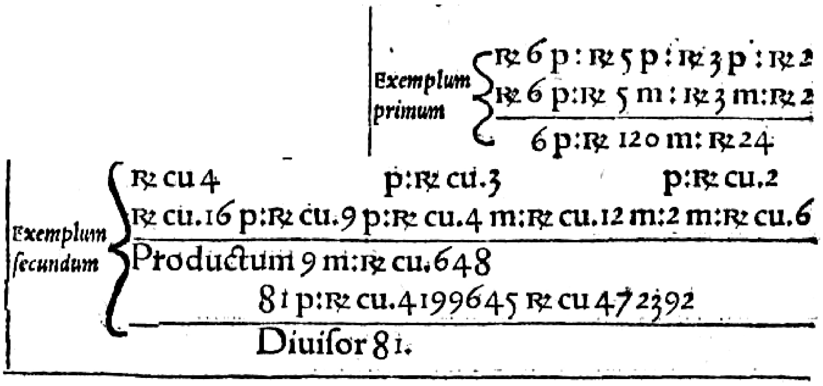
[1] 1570 has "*au.*".
[2] 1570 has "*ex producto ex cu. 72 in Rcu: 81 fiat numerus, quia per dicta productum ex Rcu: 3 in Rcu: 6, inde in Rcu: 12, cum sint in continua proportione*", while 1663 has "*ex producto ex cu. 72 in Rcu: 81 fiat numerus, quia per dicta productum ex Rcu: 3 in Rcu: 3 in Rcu: 6, inde in Rcu: 12, cum sint in continua proportione*".

est quod fiunt plures Rcu: non commensæ quam quatuor reducendo ad cubum. Ergo nullum quadrinomium ex R cubicis non est idoneum.

Caput XVII

Quot modis numerus possit produci ex non numero

Primum numerus quilibet producitur ex his numeris a quibus dividi potuit. Et ita si volo dividere 10, potest dividi per numerum alogon qui constet quatuor radicibus incommensis, non tamen ultra etiamsi sit una pars numerus. Et si sint R cub. non ultra tres. Et fit hoc per recisa, ut in tertio Libro Operis perfecti, velut si dividens sit R 6 p: R 5 p: R 3 p: R 2, invenies suum recisum, ut vides.



Et ita, si volo dividere per Rcu: 4 p: Rcu: 3 p: Rcu: 2 ut docuit Scipio Terrus Bononiensis. Et manifestum est quod assumuntur quadrata illarum p: et producta invicem pro m:. Productum autem est aggregatum cuborum partium[1] 4 et 3 et 2, quod est 9, m: R cubica tripla ei quæ fit ex prima in extremam, scilicet Rcu: 24, quæ triplicata producit Rcu: 648. Et eadem rationem invenies suum trinomium eodem modo, ut vides, ducendo partes in seipsas et inter se. Et ita conficies numerum divisorem. Sed hæc ut dixi alio pertinet.

Hoc ipsum est quod volebam docere, scilicet quod, ubi non possis dividere numerum propter multitudinem partium, sufficiet supponere, velut volo dividere 10 per R 6 p: R 5 p: R 3 p: R 2 p: sufficiet supponere divisorem dividendo, et habebis $\frac{10}{\text{R } 6 \text{ p: R } 5 \text{ p: R } 3 \text{ p: R } 2 \text{ p: R } 1}$. Et cum hoc potes operari multiplicando, dividendo, addendo et detrahendo ad unguem, sicut in fractis numeris fieri solet.

[1] 1570 has “*1 4 et 3 et 2*”.

Velut volo dividere 20 per hunc numerum, exibit R 24 p: R 20 p: R 12 p: R 8 p: R 2. Et ita habes quod 10 producitur ex quibusvis numeris dividentibus cum suo alterno.

At proprie fit primo ex quibusvis binomiis et recisis, quorum differentia partium est in se ductarum, est ille idem numerus, velut ex R 11 p: R 1 et R 11 m: R 1, et R 12 m: R 2 et R R 12 p: R 2, et ita de aliis, et ita potest produci ut 4 p: R 6 et 4 m: R 6, et 5 p: R 15 et 5 m: R 15.

Secundo potest produci ex binomio et reciso proportionem habentibus, ut 4 p: R 12 et 2 m: R 3, producunt enim 2, proposito ergo quovis binomio vel reciso invenias suum alternum, et duc invicem, et producto divide numerum propositum, et id quod exit duc per secundo inventum, et habebis quæsitum, velut volo invenire numerum qui ductus per 3 p: R 7 producat 10, invento 3 m: R 7, et ductis invicem fit 2, divide 10 per 2, exit 5, duc 5 in 3 m: R 7, fit 15 m: R 175, duc 15 m: R 175 in 3 p: R 7, fit 45 m: R 1225, cuius R est 35, detrahe a 45, relinquitur 10.

Tertio, fit ex fractis eodem modo, quo in primo R $17\frac{1}{5}$ p: $3\frac{1}{5}$ in R $17\frac{1}{3}$ m: R $7\frac{1}{3}$, producunt 10, et ita in numeris, velut $3\frac{1}{6}$ p: R $\frac{1}{36}$, et $3\frac{1}{6}$ m: R $\frac{1}{36}$, et ita in alogis, velut $3\frac{1}{3}$ p: R $\frac{6}{25}$ et $3\frac{1}{5}$ m: R $\frac{6}{25}$.

Quarto, si velis duos numeros quorum quadrata differant in 10, facile hoc est cum quovis numero, exemplum capio 2 et 10, duc 2 in se fit 4, detrahe a 10, remanent 6, divide per 2 exit 3, huius dimidium per se sumptum et additum ad 2 producit quadrata quorum differentia est 10, nam quadrata $3\frac{1}{2}$ et $1\frac{1}{2}$ differunt in 10. Similiter capio $2\frac{1}{5}$ duco in se, fit $4\frac{21}{25}$, detraho ex 10 relinquuntur $5\frac{4}{25}$, divide per $2\frac{1}{5}$, et est ac si dividas $\frac{129}{25}$ per $\frac{11}{5}$, quod est ac si divideres 1295 per 55, exeunt ergo $2\frac{19}{55}$, cuius dimidium est $1\frac{19}{110}$, adde ad $2\frac{1}{5}$, fiet $3\frac{14}{110}$ et $1\frac{19}{110}$, producunt quadrata quorum differentia est 10.

Producitur insuper a numeris simplicibus quibuslibet, qui sunt in ordine eodem, et producunt numerum iuxta naturam eius, ut a R 10 in se; et a R R quavis, quæ ducta in aliam R R producat R 100, et ab analogis ut R 20 in R 5 producunt 10, quia proportio R 20 ad R 10 est ut R 10 ad R 5. Et proportio R 50 ad R 10 ut R 10 ad R 2. Et quævis R cubicæ invicem ductæ producentes 1000, cubum 10, ut Rcu: 1000 in Rcu: 10, Rcu: 200 in Rcu: 5, et Rcu: 50 in Rcu: 20. Et ita 10 producitur ab omnibus R relatis producentibus 100000 R primum 10.

Caput XVIII

Quod ultima divisio cubi non satisfacit proposito

Capitulum 15

Porro ultima divisio cubi superius data, licet sit speciosa valde non satisfacit capitulo proposito, uti neque æstimatio binomiorum vel recisorum primi vel quarti ordinis, nec recisorum secundi vel quinti. Sed nec æstimatio differentiæ R cub. nec R quadrata aut cuba differt a numero per R quadratam aut cubam aliter recisum esset æquale radici simplici, quod esse non potest, sed est differentia recisum. Sed veniamus ad aggregata, quorum unum parti sit æquale numero, si capitulum debet esse generale et divisio utilis, igitur hoc modo triplex est radix quadrata, cubica, et corporea. Quadrata est per rationem compositionis ex duabus partibus, atque duabus velut diviso cubo 125 in 100, et 25 radix quadrata 100 est composita ex 8 et 2, et tota est 10, et quadratum 8 est 64, et quadratum 2 est 4, et duplum 8 in 2 est 32, qui omnes iuncti faciunt 100. Et ita est compositum ex quatuor corporibus, 64, 16, 16, et 4. Et ita reliquæ partis, quæ est 25 radix, est 4 et 1, et corpora 16, 4, 4, et 1, quæ sunt 25. Proportio igitur partium est ut partium cubi, nam 8 ad 2 et 4 ad 1 ut 100 ad 25. Et ita si 1 cu. æquetur 6 rebus p: 6. Rei æstimatio est Rcu: 4 p: Rcu: 2, et si velimus dare corpus unum ipsi numero, id est ipsi 6 reliquum rebus erit illud æquale necessario R 864 p: Rcu: 432. Igitur qualis proportio Rcu: 864 p: Rcu: 432 ad 6 talis partium R quadratæ 6 invicem. Quare ut Rcu: 864 p: Rcu: 432 p: 6 ad 6 ita R 6 ad partem radicis minorem per compositum proportionem igitur ducto 6 in R 6, et producto quod p: R 216, diviso per R cub. 864 p: R cub. 432 p: 6, exibit pars illa. Divido ergo R 216 per R cub. 864 p: Rcu: 432 p: 6, et assumam suum recisum, quod est R cub. 432 m: 6, et ducam cum priore, et fit 36 divisor, ergo dividendum esset R 216, ductum in R cub. 432 m: 6, quare dividam R 216 per 36, exit R $\frac{1}{6}$, id duco per Rcu: 432 m: 6, et producitur Rcu: 6 quad. 864 m: R 6 pars minor, quare maior est R 24 m: Rcu: quad. 864. Hæc nolui addere tanquam milia ad institutum, sed ob operationem.

Scholium

Ex hoc habetur quod cum duæ radices cubicæ fuerint in continua proportione cum numero, radices illæ invicem ductæ, producunt

3/ 27 . 8 /2
18 . 12
2/ 18 . 12 /3
12 . 18
75 . 50

numerum, sint a b c in continua proportione et sit a numerus et b c R cub.. Dico productum b in c esse numerum, subiungatum d in continua proportione, eritque d ad a triplicata ei quæ est b ad a, at b ad a est velut Rcu: ad numerum, igitur d ad a est ut numeri ad numerum, sed a est numerus, ergo d est numerus: igitur productum d in a est numerus, sed id æquale producto ex b in c, quia quantitates sunt in continua proportione, igitur productum b in c est numerus. Hoc dixi, ut ostenderem Rcu: 864 ductum in Rcu: 432 producere numerum, scilicet quantum producitur ex 6 in 12 quartam quantitatem, quæ est in continua proportione scilicet 72, a quo detracto 36 m:, relinquebatur 30, ut fuit assumptum in supposito.

Secundum genus R est cubicum, de quo toties actum est. Sed hic non est in proposito.

Tertium autem genus vocatur corporeæ radicis, et est velut dividendo 125 cub. 5 in 75 pro 15 rebus, et 50 pro numero, et radix est eadem utrique parti, et etiam toti scilicet 3 et 2, et fiunt quatuor corpora

a num.
b ℞ cub.
c ℞ cub.
d

utrinque cubus, et productum alterius partis in quadratum proprium 615, et productum eiusdem in quadratus alterius semel. Cum ergo una pars supponatur numerus, erit divisio illa ad hoc, ut cubus eius sit cubus partis eius radicis, aliter divisio esset inutilis. Quare pars illa est numerus necessario aut Rcu: numeri, cum ergo reliquæ partes ponantur quadrata eius Rcu:, ducta in aliam, erunt numerus rerum et quadratum æqualia numero, igitur erit secunda pars, aut numerus aut recisum, ergo æstimatio rei non potest esse generalis, quia (ut dixi) æstimatio est capituli, et totius et partis, ergo nihil profecimus. Si vero prima pars sit R cub. velut 1 cu. æqualis est 6 rebus p: 6, et ponatur prima pars Rcu: 3. Ponemus secundam 1 pos., igitur partes erunt 3, scilicet Rcu: 3 et reliquum quad. Rcu: 3 p: pos. Rcu: 72, nam ducendo 1 pos. secundam partem in quadratu primæ partis, quod est Rcu: 9, nam prima pars fuit R cub. 3, fiunt res R cub. 9, et quia assumimus duplum illius quadrati, erunt res numero R cub. 72, et hoc est æquale 3, residuo 6 detracto cubo primæ partis scilicet 3, igitur reducendo ad 1 quadrata, id est dividendo per Rcu: 3, fiet igitur 1 quad. p: rebus Rcu: 24 æqualia Rcu: cuius sequere capitulum, et erit rei æstimatio R R cub. 72 m: Rcu: 3. Esset igitur una pars, scilicet maior Rcu: 3, minor R Rcu: 72 m: Rcu: 3. Et res ipsa R Rcu: 72. At constat quod cubus huius non potest esse æqualis 6 rebus p: 6, sed neque ulli numero rerum, cum numerus non possit æquari ex Rcu: quare haec divisio licet speciosa non potest generaliter satisfacere, ita simpliciter sumpta. Quod autem necesse sit ad hoc genus quantitatis

| a c d
| b

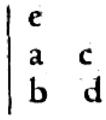
simplicis R R cub. devenire demonstro. Nam posita a prima parte et b secunda, assumitur numerus rerum in creatio corporis in duplo quadrati a, quod sit c, et numerus quadratorum est a, igitur ut numerus quadratorum deducatur ad unum oportet dividere per a, ergo etiam oportet dividere per a, quia ergo c est duplum quadrati a, divisum c per a, exibit duplum a, quod sit d, igitur numerus rerum, quæ cum quadrato æquantur numero cuius qui sit e, erit duplum a, at in capitulo inveniendæ æstimationis quadrati et rerum æqualium numero oportet ducere dimidium numeri rerum in se, numerus

| a c d
| b e

autem rerum fuit d duplum a, igitur oportebit ducere a dimidium d in se, et addere ipsi numero e, et totius excipere R a, qua detrahemus dimidium numeri rerum 1 ipsum a R, erit æstimatio secundæ partis semper R quad. R cub. aggregati ex quadrato a et e diviso, per a detracto a, sed prima pars est a, igitur tota æstimatio est R quadrata aggregati duarum Rcu:, scilicet a in se ducti, et e divisa per a. Ita ergo posito numero e 8 ut dixi et a Rcu: 2 sufficiet ducere Rcu: 2 in se, fit Rcu: 4, et detrahere 2 cub. Rcu: 2 ex 8, relinquitur 6, hoc divide per Rcu: 2, exit Rcu: 108, hoc autem est commensum Rcu: 4, ideo iunctæ faciunt Rcu: 256, igitur secunda quantitas b p: Rcu: 256 m: Rcu: 2, sed a fuit Rcu: 2 , igitur tota res est R Rcu: 256. Non est igitur idonea hæc divisio.

Per Capitulum 10

Scholium II



Dico igitur quod duæ illæ R cubicæ, scilicet quadratum a et residuum e, quod est numerus (quia e est numerus et cubus est numerus) divisum per a facit commensum. Quia enim diviso cubo a, qui est numerus, exit quadratum a, quod sit b, et diviso c numero qualicunque per a exit d, sit autem cubus ae, erit e ad c ut b ad d, quare cum c et e sint numeri ex supposito, erit commensum d, quod est propositum.

Constat enim ex hoc quod, diviso cubo æquali 10 rebus p: 8 in duas partes, ut in ultimo modo et supposita una parte rei 1 pos., erit reliqua pars R $\frac{8}{\text{pos.}}$ m: 1 pos., hac igitur deducta ad cubum et in quadratum alterius semel, et quadrati duplo etiam in eandem fiet totum æquale 10 rebus primis, at 10 res primæ sunt decuplum utriusque partis rei 1 R $\frac{800}{\text{pos.}}$. Et ideo hoc erit æquale illi parti cubi compositæ ex illis quatuor partibus dictis. Et quia proportio rerum ad numerum est sicut proportio partium rei invicem, et proportio rerum ad numerum est manifeste $1\frac{1}{4}$ ipsius rei. Ideo oporteret facere ex 8 duas partes eo modo, ita ut ducta minore in totum cum quarta parte produceret maiorem. Dico de radicibus cubicis.

De æstimatione autem binomii primi aut quarti vel recisorum ratio est, quia tria quadrata numeri et unum radicis necessario sunt maiora tribus quadratis radicis et uno numeri, quia in his numerus semper est maior radice, ergo cum volueris æquare radices, ut cadant, numerus in rebus superabit numerum in cubo, et ita cubus non poterit æquari rebus et numero, sed res potius cubo et numero. Et ita in reciso secundo et quinto, apparet ratio dum deduces, et formabis cubum ex partium.

De cubo radice posita 3 et Rcu: 2, differentia partium est 3 m: Rcu: 2, detracta enim bc ex ab, relinquitur ac. Ergo differentia ab, quæ est 3, et ac quæ est 3 m: R cub. 2, cum sit bc, manifeste erit Rcu: 2. Tanta vero est 3 p: Rcu: 2 a 3. Ergo nulla R differt a numero in radice. Ex his tandem patet quod non datur æstimatio generalis pro capitulo cubi æqualis rebus et numero in parte ea quæ nondum est inventa, sed dantur multæ æstimationes, quæ simul iunctæ satisfaciunt, ut si sciri possit, nondum cognita sit generaliter.

Caput XIX

Quod ubi æstimatio satisfacit quovis modo dividatur cubus satisfacit, si non, non

Propono ergo istud, quod si Rcu: non satisfacit gratia exempli aut[1] R quad. quad. cum R cuba, quomodo vis dividatur cubus, nunquam satisfaciet. Item dico quod si 1 cub. æqualis sit 6 rebus p: 6 et rei æstimatio dando numeros cubis sit R cub. 4 p: Rcu: 2. Eadem æstimatio satisfaciet diviso cubo in duas partes quomodo vis. Non tamen sequitur quod, si hæc quantitas, ut pote Rcu: 4 p: R cub. 2, sic divisa non satisfaciat cubi diviso in corpora similia, non ob id sit æstimatio satisfacit tamen alio modo, ut a latere vides.

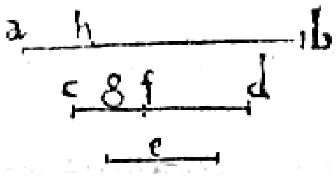
| R cu 64 R cu. 8
| R cu. 256 R cu. 128
| R cu 16 R cu. 32

Ergo si cubus æquatur 9 rebus p: 9, cum sit aliqua æstimatio, poterit satisfacere iuxta quamcunque divisionem, des non ut ipsa divisa est, et iuxta quamcunque divisionem, sed non ut cubus erit divisus.

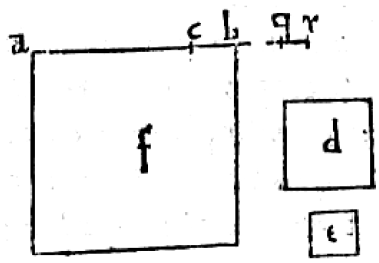
[1] 1570 has "*R qd̃ qd̃*", while 1663 has "*R qd. qd.*".

Caput XX

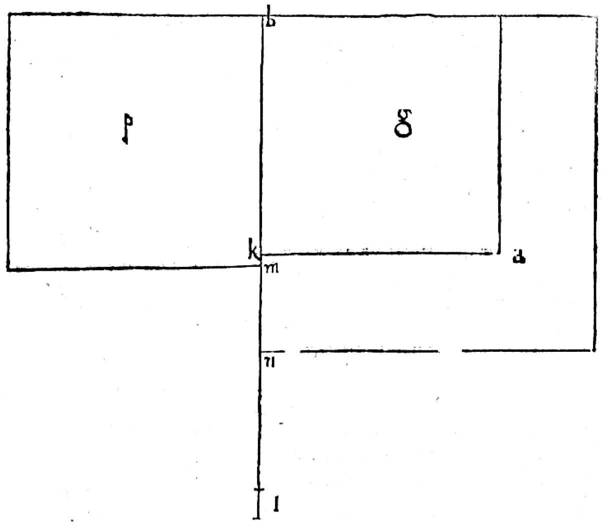
Data linea quomodo quadrifariam dividatur in duas partes, ut sit proportio unius ad productum totius in alteram data



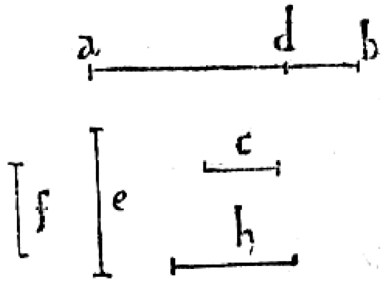
Sit data ab quam volo dividere ita in puncto h, ut sit proportio ab et ah ad bh ut cd ad e, abscindo df æqualem e, et divido ef per æqualia in g, et facio ut gd ad df ita ab ad bh, cum ego sit ita, erit gd ad gf ut ab ad ah, quare coniungendo ab ah ad ah, ut gd gf, quare ut cd ad gf, at rursus ab ad bh ut gd ad df. Igitur disiungendo ah ad hb ut gf ad fd, quare per eam quam vocant proportionem æquam cd ad df ut ab ah ad hb.



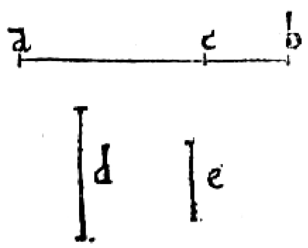
Secundo volo dividere ab in e ut sit quadrati ac ratio ad id quod fit ex ab in bc ut d ad e, facio g quadratum ad f ut d ad e. Et rursus facio per eandem hl ad hk ut hk ad ab, eritq̀ue hl ad ab ut d ad e, et sit hm dimidium hl, et eius quadratum p, cui æqualem gnomonem circumpono quadrato g, ita ut totum quadratum quod vocetur o sit æquale quadratis g et p, facio igitur ac æqualem mn, dico ab recte esse divisam in c.



Quadratum enim hn, cum sit æquale quadratis hk et hm, et quadratis hm, mn, et duplo hm in mn detracto communi quadrato hm, relinquetur quadratum g æquale quadrato mn et duplo mn in mh, quare quadrato mn, et ei quod fit ex mn in hl, siquidem hm est dimidium hl, cum igitur supposuerimus ac æqualem mn, erit quadratum ac cum eo quod fit ex ac in hl æquale quadrato g, igitur quadratum ac cum eo quod fit ex ac in hl habet proportionem ad quadratum f quam habet d ad e, nam talem habuit quadratum g ad ipsum quadratum f. Addo ad ab 69 ei æqualem, et 92 ad quam sit proportio ut ac ad 69. Cum ergo quadratum ab sit æquale quadratis ac, cb, et duplo ac in cb, at duplum ac in cb est æquale ei quod fit ex ac in cq, quia cq est dupla bc, et quadratum bc est æquale ei quod fit ex ac in 92, erit quod fit ex ac in ar æquale quadrato f, igitur quod fit ex ac in se et in hl se habet ad id quod fit ex ac in ar ut d ad e. Quare hl et ac ad ar ut d ad e. At quia ac, cb et 92 sunt in continua proportione ex supposito, erit coniungendo ab ad 62 ut ac ad cb. Igitur quod fit ex br in ac est æquale ei quod fit ex ab in cb. At proportio hl et ac ad ar est veluti ac ad br, quia hl ad ab fuit ut d ad e, et hb cum ac ad br ut d ad e, igitur hl cum ac ad ar ut hl ad ab, quare permutando hl cum ac ad hl ut ar ad ab, igitur disiungendo hl ad ac ut ab ad br, quare rursus permutando hl ad ab ut ac ad br, sed hl ad ab ut d ad e, et ac ad br ut ac quadrati ad id quod fit ex ab in bc, igitur ac quadrati ad id quod fit ex ab in bc ut d ad e.

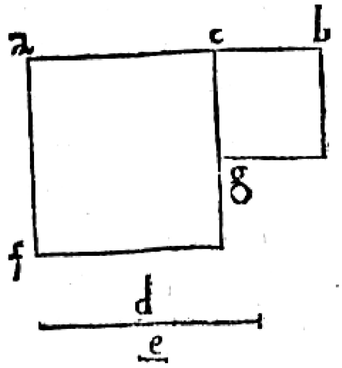


Tertio, ponatur eadem ab, et data ratio ad monadem c, volo dividere ab in d ut sit ratio rectanguli ab in bd ad lineam ad qualis e ad f. Potest, et generalius proferri, ut quod fit ex ab in bd ad id quod fit ex ad in c habeat proportionem e ad f. Quod est ut rectangulum ab in bd æquale sit rectangulo ex ad in h, quæ h se habeat ad c ut c ad f. Adeo ut reducatur ad hoc, et est generale, divisa ab in d ut sit proportio ab ad h ut ad ad bd. Hoc autem est quasi per se manifestum, nam coniunctis h et ab ut fiat abh ad h, ita ab ad bd erit disiungendo ad db ut ab ad h.



Quartum est ut dividamus ab, datam in c, ut sit ratio cubi ac ad id quod fit ex ab in quadratum bc, aut bc in quadratum ab ut d ad e. Et dico quod oportet ut in primo casu proportio ac ad cb habeat rationem duplicatam ab ad ac. Et in secundo, ut ratio ab ad ac sit duplicata ei quæ est ac ad cb. Et quia in prima quæstione reducitur res ad cubum cum rebus æqualia numero, et istud est cognitum, ideo declarabo solum secundam ut proponatur quod cubus ac sit nona plus producto ab in bc, et describam quadrata ac et bc, et quia si essent æqualia, essent basis afc et beg in proportione ab ad ac, quare ab ad ae duplicata ei quæ est ac ad cb, cum igitur sit d ad e nonupla, erit afc ad bcg nonupla eius quæ est ab ad ac.

Per 34 undecimi Elementorum



Nam si 216 est nonuplum ad 24, et 24 constat ex 24, et 216 e 36, et 6 proportio 36 ad 1 est nonupla eius quæ est 24 ad 6, ergo posuerimus ac unum quadratum et bc 4 m: quadrato uno, erit cub. ae cub. quad et bc quad 16 p: 1 quad. quad. m: 8 quad.. Si igitur proportio d ad e sit nonupla, erit 1 cu. quad. æqualis 576 p: 36 quad. quad. m: 288 quad., et si proportio d ad e sit sexdecupla, erit 1 cu. quad æqualis 5024 p: 64 quad quad. m: 52 quad . Igitur in primo casu accipiendo radices quadrata partium habebimus 1 cu. æqualem 24 m: 6 quad.. Et in secundo 1 cu. æqualem 3 m: 8 quad. et si essent æquales, esset 1 cub. æqualis 4 m: 1 quad.. Habes igitur æstimationes, ut vides quatuor æqualis quadruplæ nonuplæ sexdecuplæ.

1 cu.p:2 q̄d.æq̄l.8
1 cu.p:4 q̄d.æq̄l.16
1 cu.p:6 q̄d.æq̄l.24
1 cu.p:8 q̄d. æq̄l.32.

Cum ergo prima tria exempla solvi possint ex capitulo, ultimum non possit, et demonstratio geometrica sit universalis, pateat eam non esse generalem rationem capituli ad inveniendam æstimationem, sed esse longe meliorem.

Caput XXI

Demonstratio ostendens æquationis necessitatem

Et proponatur ab divisa in c et per præcedentem est una demonstratio proportionis cubi ac ad solidum ex ab in quadratum bc in parte cognita et incognita, ubi proportio est maior sed parum, aut minor semper nota, at hæc proportio composita ex duabus quarum una est nota altera data. Nota quidem est ex præcedenti proportio cubi ac ad solidum ex cb in quadratum ab, cum sit cubi et rerum æqualium numero generalis, alia est data solidi bc in quadratum ab ad solidum ex ab in quadratum bc, semper velut cb ad ba, fiunt enim illa ex rectangulo eodem ab in bc, alterum iuxta altitudinem ab, alterum iuxta altitudinem bc, cum ergo interposito solido ex bc in quadratum ab inter cubum ac et solidum ab in quadratum bc, componetur proportio cubi ac ad solidum ab in quadratum bc, quomodolibet constat propositum.

Per conversam 32 undecimi Elementorum

Per secundam Propositionem Libro de proportionibus

Paradoxum

Ex hoc patet quod, divisa linea inter puncta data in proportione data cubi partis unius ad solidum ex tota in alterius quadratum, ut sit proportio horum solidorum (quævis linea sit aut pars) data et cognita quantitate partium sub uno numero divisæ lineæ, non erit cognita quantitas earundem partium sub eandem divisione, sed mutato solum numero seu denominatione assumptæ lineæ. Et hoc contingit, quia ultima pars præcedentis propositionis non est perfecte nota, quia quantitates natura similes non possunt esse in proportione lineæ, velut linea ad lineam superficies ad superficiem et corpus ad corpus non possunt esse in proportione unius lineæ, sed lineæ ad lineam, ut visum est in Libro de proportionibus. Dico ergo quod si data est ab inter duo puncta data et proportio cubi ac ad solidum ab in quadratum cb secundum totam lineam ab, vel secundum quamlibet illius partem, veluti ad ut omnia hæc sint data et immota nihilo minus,

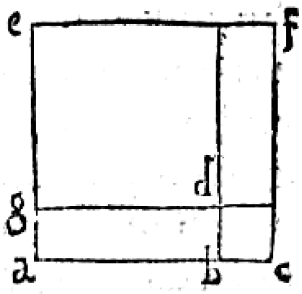
Propositio 34

si constituamus ab totam sub numero parvo, puta quatuor aut sex, perveniemus per ultimam partem præcedentis quantacumque supposita ad modo sit pars ab ad cognitionem ac, quia perveniemus ad 1 cu. p: quadratis, non pluribus quam quatuor æqualibus numero alicui, qui poterit convenire æquationi ima cognitæ. Et supposita ab centum exempli gratia, licet sit eadem linea quæ prius nec maior, et proportione sub eadem ad poterit esse ut perveniamus ad æquationem eiusdem capituli et non cognitam. Et hoc est (quia ut dixi) proportio talium solidorum, non potest esse vere linea ab, neque ad, sed vel ut linea ab vel ad ad aliquam aliam lineam, aut simpliciter denominationis ab vel ad, quæ sumitur in comparatione ad monadem. Unde si quis inveniat demonstrationem, ut dixi, veram proportionis cubi ac ad solidum ex ab in quadratum cb, secundum lineam ad, tunc inventa æstimatione sub ab denominata, ut decem inveniretur sub denominata, ut centum. Et ita sub duo bis inveniretur sub decem et est mirabile pulchrum et arduum.

Caput XXII

De contemplatione p: et m:, et quod m: in m: facit m:, et de causis horum iuxta veritatem

Cum dico 6 p: 2 clarum est, quod est 8 secundum rem: sed iuxta nomen est compositum ex 6 et 2, et similiter cum dico 10 m: 2, secundum rem est 8, iuxta nomen autem est 10 detracto. Et ideo in operatione quod ad finem attinet 6 p: 2 debet producere 64, quia 8 in se ductum producit 64, et ita 10 m: 2, quia est 8, debet producere idem 64. Sed quod ad modum operandi, quia 8 est divisum in 6 p: 2, seu in 10 m: 2, oportet operari per quartam secundi Euclidis. Et in 6 p: 2 est manifestum,



ut in figura ponatur ab 6, bc 2, fient ad 12, dc[1] 4, df 12, de 36. Totum igitur 64, et de hoc non est dubium, sed si ponatur ac 10 et bc 2 m: erit quadratum ac nihilominus 64, id est quadratum de, quia ab vere est 8. Est ergo ac, si quis diceret habes agrum decem pedum quadratum, cuius duo pedes sunt alterius, et quadratum partis tuæ est, tuum reliquum totum est alterius, igitur tu haberes de solum, quod est 64, et gnomo illæ gbf esset alterius, et esset 36, ut liquet.

Causa ergo divisionis p: vel m: est duplex, nam si essent eiusdem naturæ, ut 6 et 2, vel 10 et 2, vel 6 et $\frac{1}{2}$, aut 10 m: $3\frac{1}{3}$, stultum esset et superfluum dicere 6 p: 2, aut 6 p: $\frac{1}{2}$, aut 10 m: 2, aut 10 m: $3\frac{1}{2}$, sed deberemus dicere 8 m: [=] 6 p: 2, aut 10 m: 2, vel $6\frac{1}{2}$ in 6 p: $\frac{1}{2}$, vel 10 m: $3\frac{1}{2}$, et esset facilius pro multiplicatione et divisione. Et præcipue quod in divisione semper oportet reducere quantitatem

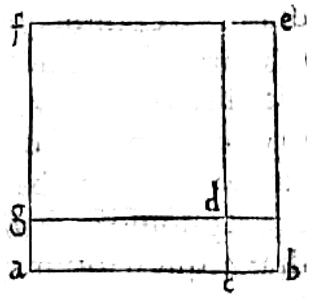
[1]1570 and 1663 have "*de*".

significatam per plura nomina, seu p: seu m:, ad unam simplicem quantitatem. Sed causa talium nominum p: seu m: est vel quia quantitas quæ additur vel detrahitur non est eiusdem naturæ cum prima, ut 6 p: R 2, aliter binomium esset rhete aut alogum, id est numerus aut radix numeri, quod demonstratum est ab Euclide esse non posse. Et ita 6 m: Rcu: 2, quia sunt diversarum naturarum nec possunt significari uno nomine, necesse fuit iungere illas quantitates per p: vel m: neque etiam possunt significari uno nomine per viam R, nam 6 p: R 2, et RV: 38 p: R 288, et licet videatur, simplex est tamen R unius compositi numeri seu quantitatis, id est 38 et R 288. Secunda causa est cum secunda quantitas aut tertia adiuncta vel detracta est ignota, velut si dicamus 6 p: 1 pos., licet enim poneremus quod positio esset 2, et ita totum hoc esset vere 8, quia tamen nescimus quanta sit positio, ideo cogimur dicere 6 p: 1 pos., 10 m: 1 pos., ex quo constat quod in primo casu nunquam nisi per fortunam multiplex potest reduci ad unam naturam, neque enim ut dictum 6 p: R 2 potest effici unus numerus, nec unius naturæ, sed in secundo casu aliquando potest, aliquando non. Ut si dicamus 10 m: 1 pos. et pos. sit 2, tunc æquivaleret 8. At si positio esset R 2 vel R cub.3 manifestum est quod nunquam posset reduci ad unam naturam, sed æquivaleret semper binomio vel reciso, vel aliæ quantitati alogæ, ut 6 m: R 2, 6 m: Rcu:3. Dixi in primo casu quod aliquando tamen quantitas multiplex æquivalet simplici, et hoc maxime accidit in RV: et abstrusis, velut declaratum est a nobis in Arte magna, quod RV: cu. R 108 p: 10 m: RV: cu. R 108 m: 18,[2] idem est quod 2. Et hoc etiam accidit in quadratis, ut RV: 6 p: R 9 est 3. Ergo ut dixi ob duas illas causas necesse fuit ponere p: et m:.

Capitulum 11

Propositio 4

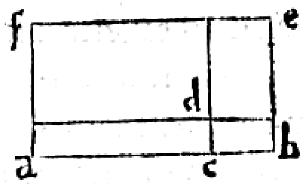
Hoc viso cum operatio p: sit clara et demonstratis ab Euclide in secundo Elementorum reliquum est, ut ostendam illud idem de m:,



[2]1570 has "*RV: cu. R 10, 8 p: 10 m: RV: cu. R 10, 8 m: 18*", while 1663 has "*RV: cu. R 108 p: 10 m: RV: cu. R 108 m: 18*".

et ponatur ab 10, ut prius et bc 2 m: liquet ergo quod ac vere est 8, et eius quadratum df erit 64, sed totus residuus gnomo est, ut dixi perinde, ac si bc esset alterius, ideoq́ue totus gnomo etiam illius, ut ostendam, et constat quod ille gnomo per eandem propositionem fiet ex ac in cb bis, et sunt rectangula ad de cum quadrato bc iste autem gnomo totus est 36, quia ab[3] quadratum est 100 et fd 64, igitur gce gnomo residuus est 36, et ad et de sunt m: et sunt 32, et gnomo est 36 m: igitur quadratum bc, quod est 4, est etiam m:, nam si esset p: non esset gnomo m: nisi 28 et df 72 et ac ℞ 72, et non ℞ 64, quod est 8. Igitur quadratum bc est m: et fit ex m: in se ducto, igitur m: in se ductum, producit m:,

4. secundi Elementorum



et similiter statuatur ab 10, et bc m: 2, erit ergo vere ac 8, et ponatur af 4 et ag 1 m: gratia exempli, erit igitur vere fg 3, quare fd 24, tota autem ae superficies est 40, igitur gnomo gc est 16 residuum, et fit ex ac in cd, ideoq́ue superficies adest 8, et ex bc in gf, superficies de,[4] et ex bc in cd, et est 2, quod totum est 16, sed hoc est m:, quia est differentia productorum 10 in 4, et 10 m: 2 in 4 m: 1, igitur tam m: in m:, id est bc in cd, quarum utraque est m: producit bd m:, quæ est 2, quam ac p: in cd m:, et bc m: in fg p:, quæ ex confesso apud illos producunt m:. Et ideo patet communis error dicentium, quod m: in m: producit p: neque enim magis m: in m: producit p: quam p: in p: producat m:. Et quia nos ubique diximus contrarium, ideo docebo causam huius, quare in operatione m: in m: videatur producere p: et quomodo debeat intellegi. Supponamus ergo in secunda figura quod ab sit[5] 10 ut prius et bc sit 1 pos. m:, manifestum est quod oporteret iuxta hanc operationem ducere ac in se, et bc in se, et ac in bc bis, sed cum ac sit ignota, est 10 m: 1 pos. accipimus ab, quæ est nota. Est enim 10, ut operamur cum ab et bc 9, et quia quadratum ab cum quadrato bc est æquale quadrato ac cum duplo ab in bc, ideo detrahimus, duplum ab in bc ex quadratis ab et bc, et quoniam duplum ab in bc superat gnomonem gce in quadrato bd, ut constat,

[3]1570 and 1663 have "*ac*".
[4]1570 and 1663 have "*deb*".
[5]1570 and 1663 have "*20*".

ideo detrahimus quantum est quadratum bc plusquam oporteret et ponimus m:, cum solus gnomo vere sit m:, quia ergo detrahimus quantum est quadratum bc, plusquam deberemus a quadrato ab, tamquam p:, ideo ad restitutionem illius m: quod detrahimus præter rationem oporteret addere, quantum est quadratum bc p:, et ideo cum bc sit m:, dicemus quod[6] 2 m: quadratum conversum est in p: ideo quod m: in m: produxit p:. Sed non est verum; sed nos addidimus quantum est quadratum bc p: non quod quadratum bc sit p:, sed alia assumpta quantitas pro arbitrio nostro æqualis bc addita est, et facta est p:. Idem dico in tertia figura, quia operamur per ab et af loco ac et fg. Ideo in operando videtur quod m: in m: producat p:. Et sit ergo, ut in secundo exemplo ab sit 10, bc 1 pos., et sic 2 vere erit ac 8, igitur df erit 64, et gnomo[7] gce 16 pos. p: 1 quad. 36, nam 16 pos. sunt 32 et 1 quadrat. 4, quod totum est 36. Et tunc debet dici ab iunctum et separatum non proprie m:, si vero operemur cum tota ab et bc,[8] habebimus 100 p: 1 quad. m: 20 pos., ecce quod in priore æquatione non habebas nisi 16 pos. m:, hic vero habes 20, et ideo cum in priore æquatione haberes 1 quad. m: et hic habeas 1 quad. p:, ideo oportuit addere numero pos. 4, 1 a 16 ad 20, seu quia addidisti illas 4 pos. m: plusquam oporteret, ideo subtraxisti 1 quad. m: et etiam loco eius addidisti 1 quad. p: et ideo ad hoc devenisti, ut diceres m: in m: producere p: quod tamen est falsum, non enim contingit ex operatione multiplicationis, set ut pervenires ad maiorem noticiam per illam septimam propositionem secundi Euclidis,

5 m: R 3
5 m: R 2

similiter dico, si multiplicas 3 m: R 2 in 5 m: R 3, vere oporteret ducere ac in fg, et haberes verum productum. Sed quia nec 3 m: R 2 nota est vere sub uno nomine, nec 5 m: R 3 est nota sub uno nomine, et omnis multiplicatio et divisio sit singillatim per simplices quantitates, ideo in recisis necesse est operari per septimam propositionem secundi Euclidis loco quartæ. Et ita quia in illa includitur additio illa quadrati m: in multiplicatione unius partis integræ, in partem detractam bis supra gnomonem, ideo oportet addere ad p: quantum est

[6]1570 and 1663 have "*62*".
[7]1570 and 1663 have "*gce 16 pos. p: 1 quad. 136*".
[8]1663 has "*abc*".

quadratum partis illius quæ est m:. Ideo ut in binomiis operamur per quartam propositionem, et secundum substantiam quantitatis compositæ, ita etiam in recisis quo ad substantiam et vere operamur cum eadem. Sed ad nominum cognitionem operamur in virtute septimæ eiusdem.

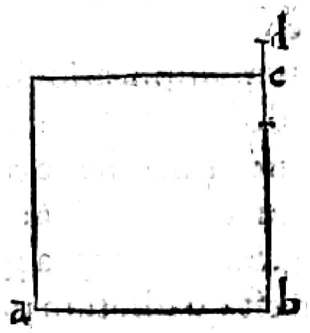
Quartum et ultimum est, quod erat considerandum, cur p: in p: solum faciat p: et m: in m: et in p: faciat m:. Et dico quod (ut dixi) m: oportet supponere tanquam non fit de ipso p: est enim alienum, ideo ad construendum oporteret assumere plura, ad destruendum sufficit unum. Ad hoc ergo ut p: constituatur, oportet ut p: in p: ducatur, nam cum ducitur p: in m: seu in alienum fit m:, quia nihil potest ultra vires suas, ergo p: potest quantum est ipsum, igitur cum ducitur extra ipsum, producit m:, aliter posset plus producere quam potestate esset. Sed cum ducitur in aliud p: non potest etiam nisi quantum potest in partes illius p:. Exemplum, 6 ducitur in 10, igitur in 6 et 4, sed ut [6] in 6 ex demonstratis non potest ultra 36, ut autem 4 ducitur in 6 non potest, nisi ut in 4 et 2, et ut in 4, nisi ut in seipsum, igitur non potest nisi usque ad 16, et ut residuum 2 in 4, nisi ut in 2, et 2 igitur non potest nisi 4 et 4 est, sed 36, 16, 4, et 4 producunt 60, igitur 6 in 10 non potest producere nisi 60, igitur m: in m: seu alienum in alienum, et m: in p: seu p: in m: seu quod est in alienum, seu alienum in id quod est, producunt m: solum, seu alienum quod erat demonstrandum.

Corollarium 1 supra Capitulum 6. Ex quo intelliges veram rationem ducendi m: et dividendi per m: et accipiendi R tam quadratam quam cubam (nam de cuba dubium non est, quod est m:) non antea cognita.

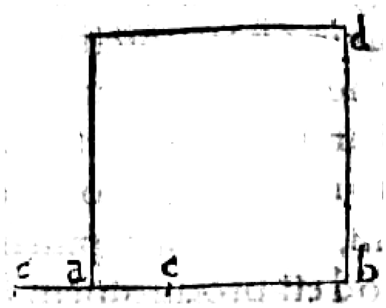
Corollarium 2. Ex hoc etiam patet quod diviso m: per p: exit m:, nam ducto m: in p: fit m:, ergo diviso m: per p: exit m:. Et diviso m: per m: exit m: et p:, quia ex m: in p: et m: fit m:, igitur diviso eo producto, quod est m: exit alterutrum, scilicet p: vel m:. Diviso autem p: per m: nihil exit, nam seu exiret p: seu m: ex m: in idem p: vel m: produceretur p: quod est contra demonstrata.

Caput XXIII

De examine capituli cubi et numeri æqualium rebus



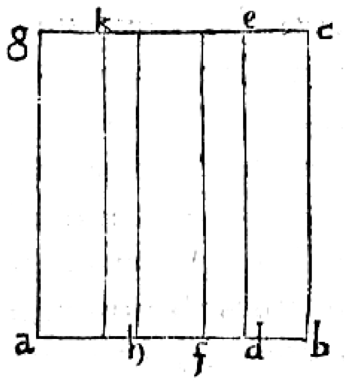
Proponatur primo ab res, et quadratum eius ac, et sit bd numerus quadratorum æqualium cu. 6 et numero, dico quod bd est maior ba, nam si minor esset cubus ab maior quadratis, igitur multo maior esset cubus ab cum numero, quadratis ipsis non ergo æqualis. Contraria ratione sequitur quod si cubus æquaretur quadratis et numero, necesse est ab rem esse maiorem numero quadratorum. Per idem si bd sit numerus rerum æqualium cubo et numero, necesse est bd esse maiorem ab, modo ab sit æqualis aut maior monade. Nam si ab esset maior bd, esset ac maius superficiei ab in bd, quare si ab est maior monade, cubus ab erit multo maior rebus. Ergo cubus ab cum numero multo maior rebus secundum numerum bd, non ergo possunt esse æquales, sed ubi ab esset minor monade, posset esse in hoc casu cubus cum numero æqualis[1] rebus, ut 1 cu. p: $\frac{7}{64}$ æqualis $\frac{2}{3}$, rei tunc ab est $\frac{3}{4}$, quod est maius $\frac{2}{3}$. Quod si cubus æquetur rebus et numero,



[1] 1570 has "*æquati*".

ut sit bd numerus rerum et quadratum eius bda, ut etiam ab sit numerus idem rerum et æqualis bd, tunc si quadratum bd numeri rerum additum numero æquationis sit æquale cubo numeri rerum, tunc æstimatio rei, id est bc erit numerus rerum, velut bd sit 4 numerus rerum et numerus æquationis 48, ex 48 et 16 quadrato 4 fit 46 cubus eiusdem 4, igitur 4 est æstimatio rei. Sed si quadratum bd cum numero rerum fuerit minus cubo bd, erit bc æstimatio rei, bc minor ba, ut si cubus æquetur 4 rebus p: 47, quia 16 et 47 faciunt 63, minus 64, cubo 4, numeri rerum, erit bc minor ba. Et si quadratum esset cum numero æquationis maius cubo esset æstimatio rei maior numero rerum. Veluti 1 cub. æquetur 4 rebus p: 50, tunc æstimatio rei erit bc, maior ba, qui est 4 numerus rerum.

Demonstratio



Quibus stantibus proponatur res, et bc numerus rerum, et parallelogrammum abc quantitas ipsarum rerum collectarum, et sint res sub numero bc, puta 34, æquales 1 cub. p: 12. Et erit per lemma præcedens bc maior ba, item oportet ex demonstratis in Libro de proportionibus, ut cubus tertiæ partis bc sit æqualis aut maior numero æquationis. Sic ergo numerus æquationis superficies dbce, eritq̀ue bd necessario numerus. Superficies ergo ade est æqualis cubo ab, et quia cubus ab fit ex demonstratis ex cubis db et da, et triplo unius in quadratum alterius, et cubus bd est numerus, quia bd est numerus, ergo diviso cubo numero per bc numerum prodibit numerus. Sit igitur superficies ef æqualis cubo db, erit igitur superficies fg æqualis triplo bd in quadratum da et ad in quadratum db et cubo ad. Exemplum ergo erit (ut dixi) quod de sit 12 et bc 34, erit bd $\frac{6}{17}$, ab autem, ut binomium est 3 p: ℞ 7, et cubus bd $\frac{216}{4913}$, tota igitur superficies fc esset $12\frac{216}{4913}$. Propterea vides per eandem rationem, quod divisa fc per bc exit

Per 20am se 21am 10 Elementorum

fd numerus maior bd. Et rursus cubus ille componetur ex cubis bf, fa, et triplo mutuo dicto, et ita semper cubus fiet minor et numerus æquationis maior. Nam diviso $12\frac{216}{4913}$ per 34 exit $\frac{29586}{83521}$, et tanta est bf cuius cubum oporteret rursus addere ad superficiem be, et ita iuxta datam proportionem augetur numerus æquationis et cubus minuitur. Oportet igitur in hoc casu ita distinguere dicendo quod si per cubum intelligis priorem cubum, scilicet ab ille cum 12 numero, et non cum $12\frac{216}{4913}$, æquatur 34 rebus, licet enim contineat alios numeros, non sunt tamen de natura numeri æquationis, sed propria pars. Si vero dicas quod aliquis cubus p: $12\frac{216}{4913}$, qui erit minor cubo ab æquetur 34 rebus. Dico quod non, quia ille cubus erit cubus lineæ minoris ab, igitur si 34 ab æquantur cubo minoris lineæ, quam sit ab, et $12\frac{216}{4913}$ oportebit, tunc quod res tunc sit minor quæ est latus cubi, igitur oportebit quod sint plures res quam 34, quæ sint æquales cubo p: $12\frac{216}{4913}$, et ita omnia variantur uno variato.

Rursus ergo assumatur linea ah, quæ sit pars binomii et hb numerus, tunc cubus hb poterit solus esse numerus, ut cum ah fuerit quantitas absurda, velut gratia exempli RV: R 7 p: R 3 vel poterit esse cum cubo ah, cum ah fuerit Rcu: numeri, vel cum triplo hb in quadratum ah, ut in proposito posita ah R 7, nam cubus hb est 27, et triplum ha in quadratum ah est 63, ut totus numerus sit 90, quibus additis 12 fit 102, qui est æqualis 34, numero rerum ducto in 3, qui est numerus æstimationis seu binomii. In omni casu ergo ex his tribus constat quod numerus totus est superficies hc. Et quia numerus æquationis æquatur illi, dico quod non potest esse maior, nam sit pars æquaretur toti, nec æqualis ex demonstratione habita, nam bh tota esset numerus, ergo cubus eius esset numerus, ergo numerus æquationis hc, cum numero cubi hb esset maior numero, qui continetur in rebus, ergo res non possent esse æquales numero et cubo. Quia quantitas aloga esset æqualis numero, relinquitur igitur, ut numerus æquationis sit necessario minor numero qui continetur in rebus. Sit ergo numerus æquationis dc, et erit numerus cubi he necessario. Nam si[2] duo numeri pariter accepti sunt necessario æquales numero contento in rebus, quem supposuimus esse hc. Dico ergo quod ah non potest esse R simplex, quia non satisfacit per viam binomii, ut ostensum supra. Nec potest esse Rcu: nam cubus esset numerus, igitur 34 radices Capitulum
gratia exempli essent unum aggregatum radicum cub. quæ æquivalerent uni, et 10

[2] 1570 and 1663 have "*hi*".

hanc oporteret æquari tripla producti unius in quadratum alterius mutuo. At hoc esse non potest, quoniam illa solida sunt incommensa, quia sunt in proportione ah ad hb, id est R cub. ad numerum, quæ sunt incommensa inter se. Relinquitur ergo ut sic ah una quantitas alterius generis, quæ ducta vicissim cum hb una in quadratum alterius, additoq̀ue illius cubo sciat quantum ducta in 30 gratia exempli, qui est numerum rerum.

At quia in illo aggregato est etiam triplum quadrati hb in ah, oportebit ergo ut cubus ah cum triplo quadrati ah in hb sit æquale residuo tripli quadrati hb, et numeri rerum ducto in ah. Igitur divisis omnibus per ah, erit ut quadratum ah cum rebus triplo numeri hb, sit æquale numero simplici, qui est differentia numeri rerum et tripli quadrati hb. Exemplum ponatur hb 2 et bc numerus rerum 30, igitur triplum quadrati hb, quod est 12, detractum a 30, relinquitur 18, ergo 18 est æqualis 1 quad. p: triplo hb, id est b rebus. Quare res erit R 27 m: 3, id est ah, et tota hb R 27 m: 1, et erit 1 cu. p: 52 æqualis 30 rebus. Quantitas ergo ah oportet, ut sit generalis ad illam, et cum prædictis conditionibus. Quod si ab ponatur res, et hb numerus ut prius, sed m: operaberis et demonstrabis per ea quæ ostendimus in capite præcedenti. Nam cubus verus erit cubus ha, scilicet residui. Et quia additur numerus, et iam superficies hc est m: oportebit ut hg sit maior cubo ah quantum est numerus æquationis, quæ gratia exempli hk, ideo cubus ab, seu verius ah, erit ak, reliqua ut prius erunt examinanda.

Caput XXIV

Demonstratio ostendens quod caput nullum præter inventa generale sciri potest

Reliquum est ut ostendamus quod ab initio propositum est, cuius causa hæc scripsimus, scilicet non esse capitulum aliud generale, quod sciri possit, ultra ea quæ tradita sunt, quoniam ultra quatuor diversa genera nisi possit reduci ad pauciora, vel per divisionem, vel radicem, aut per mutationem, aut regulam propriam, vel deprimendo, aut ob originem, aut per demonstrationem geometricam, cum in singulis sint magnæ inæqualitates, quæ vix possunt intellegi in quatuor quantitatibus, nec in eis potuerit invenire perfectio quanto minus in illis. De his ergo, si sint quatuor usque ad cubum, iam doctus est[1] reducere ad tres quantitates, et capita trium quantitatum omnia ad cubum æqualem rebus et numero. Si igitur ostendero hoc non posse esse generale, etiam in parte ignota liquet propositum.

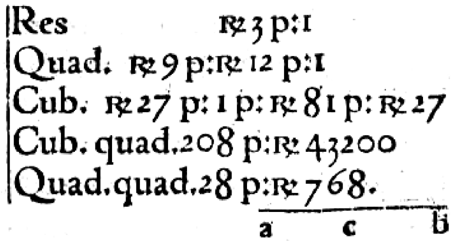
Assumamus igitur primam regulam capitulorum imperfectorum specialium, in qua 1 cu. æqualis est 20 rebus p: 32, et est rei æstimatio R 17 p: 1 binomium quintum. Et similiter in Arte magna visum est, quod duæ æstimationes capituli cubi et numeri æqualium rebus conficiunt æstimationem cubi æqualis totidem rebus, et eidem numero. Ex quibus liquet, quod oportet æstimationem generalem posse communicari numero, et quinto binomio, et quia simile quinto binomio est numerus, non quintum binomium oportet, ut sit in creatione eiusmodi. Quintum autem binomium hoc modo transit ad æquationem, ut pote R 3 p: 1, sic fiat cubus, erit R 108 p: 10, hic igitur æquatur 6 rebus necessario, quia R 108 continet R 3 sexies, et quia sex res non sunt nisi R 108 p: 6, igitur cubus æquatur sex rebus p: 4. Ergo cum illud quod potest esse ex ea natura, vel est R cubica cubi consimilis, vel quadrata quadrati, vel quadrata cubica quadrati cubi, vel differentia duorum, vel aggregatum necessarium est, ut talis æstimatio simpliciter

Capitulum 25 Artis magnæ

Capitulum 13 fine Corollario secundo [primo]

[1] 1570 and 1663 have "*es*".

sit una huiusmodi, si debet esse generalis, ut quandoque possit illi æquari, si occurrat quadratum, igitur R 3 p: 1 est, ut vides in margine differentiæ autem,


Res R 3 p: 1
Quad. R 9 p: R 12 p: 1
Cub. R 27 p: 1 p: R 81 p: R 27
Cub. quad. 208 p: R 43200
Quad. quad. 28 p: R 768.

et aggregata sunt infinitorum modorum, nam sit ab quævis quantitas, et cb ipsa æstimatio, si igitur detracta bc ex ab relinquitur ac, igitur detracta ac ex ab relinquetur bc æstimatio. Et similiter posita ab æstimatione, potes ab illa detrahere ac modo minor sit, ut relinquetur bc, igitur ex ac et cb iunctis fiet æstimatio. Iam ergo habes quod poterit esse radix quadrata trinomii, cuius una pars sit numerus, et R R binomii, aut quadrinomii, cuius una pars sit numerus, et Rcu: quadrata multinomii, scilicet tredecim partium aut pauciorum, quæ sint R quadratæ, ita ut in eis una sit numerus.

Pro aggregatis autem ac differentiis tradendis, volo tibi dare exemplum ex Arte magna, dixi quod[2] RV: 7 $\frac{3}{8}$ p: R 46$\frac{53}{64}$ p: 1 $\frac{3}{4}$ m: R 2 $\frac{5}{16}$ est æquale 3. Deducito partes ex partibus, ut videas si sit verum, et habebis 2$\frac{1}{4}$ p: R 2$\frac{5}{16}$ æqualia[3] RV: 7$\frac{3}{8}$ p: R 46$\frac{35}{64}$. Duc igitur utranque in se, et habebis idem ex utranque parte, id est 7$\frac{3}{8}$ p: R 46$\frac{53}{64}$, nam 2$\frac{1}{4}$ m: [per] se facit 5$\frac{1}{16}$, cui addit[4] [addito] 2$\frac{5}{16}$, fit 7 et $\frac{3}{8}$ et R 5$\frac{1}{16}$ in R 2$\frac{5}{16}$ fiunt $\frac{2997}{156}$, quæ duplicata faciunt $\frac{2997}{64}$, et sunt 46$\frac{53}{64}$, cuius radix addita ad 7$\frac{3}{8}$ facit 7$\frac{3}{8}$ p: R 46$\frac{53}{64}$. Unde in aliis eodem modo operaberis, dico ergo quod non potest esse R quadrata trinomii habentis duas R quad. et numerum unum, nam R quadrata R 6 p: R 2 p: 1, si posset esse ex genere binomii tertii vel sexti non possit satisfacere, ut demonstrandum est, neque si una pars sit numerus, et alia R nam eius quadratum erit binomium et non trinomium. Proponamus ergo R R 6 p: 1, et erit cuius quadratum 1 p: R 6 R R 96. Nam si capiamus R R 4 p: 1, fiet R 4 p: 1 p: R R 64, id est 3 p: R 8, si ergo capiamus[5] R R 12 p: R 3 p: 1, licet resolvatur in 160 p: R 432 p: R R 442368 p: R R 248832, hæ tamen non sunt

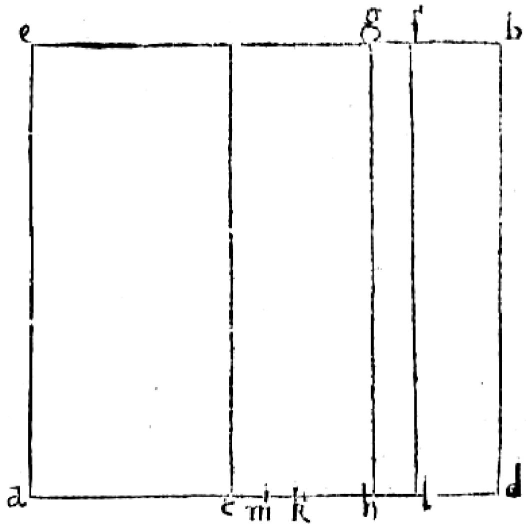
Per quartam secundi Elementorum Euclidi

[2]1570 has "*RV: 7* $\frac{3}{8}$ *p: R 46*$\frac{35}{64}$ *p: l* $\frac{3}{4}$ *m: R 2* $\frac{5}{16}$", while 1663 has "*RV: 7* $\frac{3}{8}$ *p: R 46*$\frac{35}{4}$ *p: l* $\frac{3}{4}$ *m: R 2* $\frac{5}{16}$".
[3]1570 has "*RV: 7*$\frac{3}{8}$ *p: R 46*$\frac{53}{46}$", while 1663 has "*RV: 7*$\frac{3}{8}$ *p: R 46*$\frac{53}{4}$".
[4]1663 has "*addito*".
[5]1570 and 1663 have "*R R 4 p: R 3 p: 1*".

commensæ, sed in proportione R R 256 ad R R 144, id est 4 ad 12, licet sit valde propinqua, nam R 432 est duodecupla R 3, et R R 248832 est duodecupla R R 12, et est mirum adeo quod si R R 442368 esset numerus, haberemus inventum. Cum ergo hoc trinomium non posset reduci ad pauciora multi minus reliqua, quare R cub. 460 p: R 432 p: R R 442368 p: R 248832 non potest esse æquatio quæsita,

R R 12 p: R 3 p: 1
4 p: R 48 p: R R 192 p: R R 1728
160 p: R 432 p: R R 248832 p: R R 442368

igitur oportet ut sit differentia duarum quantitatum, et fundamentum erit in prima regula dicta in Arte magna superius.



Sit cubus ab æqualis 29 rebus ad, et erit superficies bc 29 et ce 42 numeris, et erit corpus et iuxta altitudinem ad. Et cum ex 29 possint fieri partes, ut vides a latere ex quibus una ducta in alterius radicem fit numerus, poterit numerus rerum datus cum 28, 50, 60, 52, et 20 æquari cubo, dico modo quod etiam poterunt fieri aliæ partes non integræ, utpote 18 p: R 72 et 11 m: R 72, et ex 18 p: R 72 in 3 m: R 2, radicem 11 m: R 72, fit 42 alius numerus. Ex quo liquet quod oportet 11 m: R 72 esse binomium primum aut recisum.[6] Proponatur ergo ef 18 p: R 72, et cg 18, erit ergo hf p: R 72, igitur alia pars erit fd denominata per dg, scilicet 11 m: fh R 72, ita ut divisio vera bc, scilicet 29, fit vere in f, nam cf est 18, id est cg p: R 72, id est fh et df 11, id est dg m: R 72, id est fh. Divisio autem iuxta

[6] 1570 and 1663 add "*ut et etiam*".

nomen in g, quoniam cg est 18 et gd 11. Et quia proportio cb corporis ad ce est veluti cd ad ac, erit cd ad ca, velut 29 pos. ad 42, igitur velut 1 pos. ad $1\frac{13}{29}$, vel $\frac{29}{42}$ pos. ad 1 numerum.

18.1.1.28
25.4.2.50
20.9.3.60
13.10.4.52
4.25.5.20
18 p: R 72.11 m: R 72.3 m: R 2/42.

Rursus quia ex regula prima capituli cf in R fd fit ce corpus, et sit latus df, dk erit ad ad, dk veluti cf superficiei ad ce superficiem, quare veluti cl ad ca. Atque iterum cum lg fiat ex duplo partium dk, proponatur denominata per p: et m:, et sit quod est p: dm, et quod est m: km duplum, igitur mk in md producit hf. Ut in exemplo,

25 Artis magnæ

ef 18 p: R 72
cg 18
hf R 72
fd 11 m: R 72
dg 11
cd ad ca ut 1 pos. $1\frac{13}{29}$
dk R V 11 m: R 72
ad ad dk, ut cl ad ca
mk in md prod. dim. fh R 18
dm 3
mk R 2
dk 3 m: R 2
cg ad gl, ut dm ad mk.

cum ergo cg proponatur numerus 18, et gl R 72, et proportio partium dk ut denominatæ, id est ut dm, quæ est numerus m: mk, et eadem proportioni cg ad gl, sequamur ergo primum argumentum rei. Erit ad RV: $20\frac{3}{4}$ p: R $40\frac{1}{2}$ p: $1\frac{1}{2}$ m: R $\frac{1}{2}$ ex regula prima. Hæc igitur est vera æstimatio rei et eadem est 6, et fit experimentum, quia detracta $1\frac{1}{2}$ m: R $\frac{1}{2}$ ex 6, fit $4\frac{1}{2}$ p: R $\frac{1}{2}$, et hoc est æquale RV: $20\frac{3}{4}$ p: R $40\frac{1}{2}$ quod patet quia quadrata utriusque sunt $20\frac{3}{4}$ p: R $40\frac{1}{2}$. Igitur hoc genus æstimationis est generale, quia potest æquari numero, et non æquari, et posset æquari binomio, quia detractis partibus alligatis remaneret binomium aut recisum necessario, et tunc posset poni RV: radix binomii aut recisi primi.

Rursus proponatur bc 20 numerus rerum ce corpus 32, proponatur cf, cb, fd 4, ex ef in ℞ fd, quæ est 2, fit 32. Sit iterum cd ad ca ut 20 pos. ad 32, scilicet 1 pos. ad $1\frac{3}{5}$, et quia est iterum ad ad dk ut cf ad ce, quare ut cl ad ca. Et quia ad ex regula prima est ℞ 17 p: 1, et dk est 2, erit ak ℞ 17 m: 1, et ce ℞ 68 m: 2.

Caput XXV

De examine tertiæ regulæ Capituli XXV Artis magnæ

Proponamus quod cubus sit æqualis 18 rebus p: 108, tunc si fecero ex 18 numero rerum duas partes, ex quarum ductu unius in R alterius mutuo fiat 54 dimidium 108. Et manifestum est quod res est 6. Et per regulam generalem est RV: cub. 54 p: R 2700 p: RV: cu. 54 m: R 2700, et hæc vere est 3 p: R 3 p: 3 m: R 3, quod est 6 ut prius. Divisio autem non est secundum eum modum, sed R partium 18 sunt 3 et 3, et partes 9 et 9, et producta mutuo sic erunt 54. Et similiter assumptis 21 rebus et 90 numero, faciemus iuxta capitulum generale ex 90 duas partes, ex quarum ductu unius in alterum fiat 343, cubus 7, tertiæ partis 21 numeri rerum, et habebimus partes RV: cu. 45 p: R 1682 p: RV: cu. 45 m: R 1682, et est 3 p: R 2 p: 3 m: R 2, 6 ut prius, et ita augendo numerum rerum eximus extra capitulum, sed minuendo numerum rerum nimis non licet uti regula ut pote 1 cu. æqualis 15 rebus p: 126, non licet dividere 15 in duas partes, ex quarum ductu unius in R alterius mutuo, fiat 63 dimidium 126. Quia maximum in quo dividi possit, est quando dividitur in partes æquales, ut demonstratum est. Ergo tres sunt partes in hoc capitulo, prima quæ servit regulæ speciali non generali, cum numerus rerum est magnus in comparatione numeri æquationis. Secunda quæ servit regulæ generali non speciali cum numerus æquationis est magnus comparatione numeri rerum. Tertia quæ servit utrique, ut in exemplo non potest regula generalis attingere ad 1 cub. æqualem 22 rebus p: 84, quia 21 quarta pars 84 non facit quadratum, neque maius neque æquale cubo $7\frac{1}{3}$ tertiæ partis rerum. Similiter regula specialis non attingit ad 1 cub. æqualem 17 rebus p: 114, quoniam $8\frac{1}{2}$ ductum in R $8\frac{1}{2}$ producit R $614\frac{1}{8}$, quæ est minor $28\frac{1}{4}$, quarta parte 114 numeri propositi, ut mutua illa non possint componere 57 dimidium numeri propositi. Traducenda est ergo in toto illo spacio, in quo conveniunt una ad aliam faciendo ex re iam inventa duas partes, ex quarum ductu unius in quadrato alterius mutuo, fiat dimidium numeri propositi, et illæ erunt partes. Istud autem facile fiet dividendo numeri propositi, dimidium per rem inde

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dividendo rem in duas partes producentes id quod provenit. Exemplum, cubus æquatur 6 rebus p: 6, rei æstimatio est R cub. 4 p: R cub. 2, cum hoc dividemus 3 dimidium numeri æquationis, exit R cub. 2 m: 1 p: Rcu: $\frac{1}{2}$, ducam dimidium Rcu: 4 p: Rcu: 2 in se, fit[1] 1 p: Rcu: $\frac{1}{4}$ p: Rcu: $\frac{1}{16}$, a quo detraho Rcu: 2 m: 1 p: Rcu: $\frac{1}{2}$, relinquitur 2 m: R cub. $\frac{1}{4}$ m: Rcu: $\frac{1}{16}$, cuius RV: addita et detracta a dimidio prioris ostendit partes ut vides.

R cu. $\frac{1}{2}$ p R cu. $\frac{1}{4}$ p: R v: 2 m: R cu. $\frac{1}{4}$ m: R cu. $\frac{1}{16}$
R cu. $\frac{1}{2}$ p: R cu. $\frac{1}{4}$ p: R v: 2 m: R cu. $\frac{1}{4}$ m: R cu. $\frac{1}{16}$

Et modum etiam cum demonstratione superius docui. Quadrata ergo horum iuncta sunt 6, et mutuo producta sunt 3, quod patet experienti. Et est pulchra operatio.

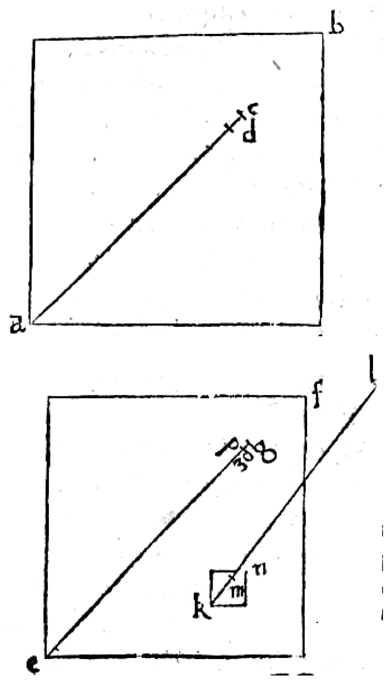
[1] 1570 and 1663 have "*1 p: Rcu:* $\frac{1}{4}$ *p: Rcu:* $\frac{1}{11}$".

Caput XXVI

De proportione cubi æqualis quadratis et numero ad cubum cum numero æqualem quadratis

Si[1] cubus sit æqualis quadratis et numero, alius vero cubus cum eodem numero sit æqualis aliquot quadratis, erit proportio differentiæ numeri quadratorum a sua æstimatione, dum cubus et numerus est æqualis quadratis ad differentiam æstimationis a numero quadratorum, dum cubus est æqualis quadratis et numero sicut æstimationis cubi æqualis quadratis et numero ad æstimationem cubi et numeri æqualium quadratis duplicata.

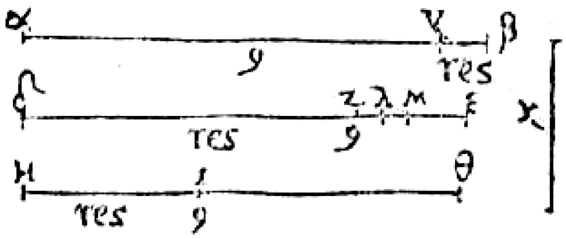
ac latus, ad 9 numerus quad., cd 8 divis. per ab, seu differentiam æstimationi a numero quad., eg 9 numerus quad. not. eh latus ign., hg 8 divis. seu per differentiam æstim. numero quad. ignom., kl 9 numerus quad., km latus secundæ æstim. ignom. lm 8 divisum per kn, seu differentiam æstimat. a numero quad. ignom.



Cum ex ab in cd, et ex ef in gh, et ex kn in lm fiat idem numerus, erit proportio cd ad gh, et cd ad lm, et gh ad lm, velut ef ad ab, et kn ad ab, et kn ad ef, quare ut eh ad hc, et km ad ac, et km ad gh duplicata. Veluti ponatur

[1]1570 and 1663 have "*[d]e propositione cubi æqualis quadratis et numero ad cubum cum numero æqualem quadratis*" as title.

eh R 24 p: 4, et km 1 in 1 cub. p: 8 æquali 9 quad.. Cum ergo nota eg, gh, he nota fiat sub eisdem terminis km et ml ex capite cubi et numeri æqualium quadratis, igitur nota ad dc, et paribus aliis erit nota eh et hg. Discrimen solum est quod in cubo æquali rebus et numero differentia est lateris, quod superat numerum quadratorum, in sequentibus figuris numerus quadratorum superat æstimationem rei seu latus quad. Liquet ergo quod inter eh et km intercedunt quatuor conditiones. Prima quod eh et km sunt ambæ æstimationes capituli propositi cubi et 8 numeri æqualium 9 quad.. Secunda quod eh et km sunt in proportione in qua est lm ad hg vicissim, sed hæc est duplicata. Tertia quod km est composita ex tetragonali gh in ep, posita oh dimidio hg, et ipsa ph dimidio oh. Quarta, quam diximus deesse comparando ac et cd ad eh et bg, est quod eh et km æstimationes sunt minores eg seu kl numero quadratorum. Cum ergo ex tertia conditione, quod fit ex gh in pe sit notum, quia gh et oe notæ sunt, et go nota, quia dimidium gh erit km composita ex eis nota. Deducitur ergo primum problema ad hoc, detrahe ex kl quantitatem quæ se habet in proportione duplicata ad hg in eh ad km, cum eg et km sint idem seu æquales. At secundum problema est, divide kl, quæ est eadem vel æqualis ad, ita ut proportio mi ad cd sit duplicata ei quæ est ac ad km. In utroque autem pariter deducitur res ad cubum et numerorum æqualem numero rerum, igitur æstimatio pariter ignota ex nota pendebit.



Sumantur ergo rursus $\alpha\gamma$, $\delta\epsilon$, $\eta\theta$ novem singulæ et æquales, et sit tota res, et in reliquis dum cubus et 8 æquantur quad. res sit $\delta\zeta$ et $\eta\iota$. Si ergo posuerimus $\delta\zeta$ R 24 p: 4, erit $\zeta\epsilon$ 5 m: R 24. Ponamus ergo $\iota\theta$ 1 quad. et sit medio in proportione inter $\iota\theta$ et $\zeta\epsilon$ κ, erit κ pos. RV: 5 m: R 24, igitur cum sit proportio κ ad $\zeta\epsilon$ ut $\delta\zeta$ ad $\eta\iota$, ducemus $\delta\zeta$, id est R 24 p: 4, in $\zeta\epsilon$, quæ est 5 m: R 24, fit R 24 m: 4, quam divido per κ, id est pos. 5 m: R 24, et exeunt pos. R $\frac{24\text{p:}4}{1\text{pos.}}$, et hæc est $\eta\iota$. Igitur tota $\eta\theta$ quæ est 9, est 1 quad. p: R $\frac{24\text{p:}4}{1\text{pos.}}$, quare 1 cub. p: R 24 p: 4 æquatur 9 pos. et quia notum est ex hoc capitulo iam dicto. Et assumo eodem modo $\alpha\gamma$ et $\gamma\beta$ notas ut sit $\gamma\beta$ 6 m: RV: cub. 31 p: R 934 p: RV: cub. 31 m: R 934, et dicamus quod fit $\frac{121}{100}$, nam est prope, et ponamus quod 10 ut prius sit 1 quad. erit κ pos. $\frac{11}{10}$, duc ergo $\frac{11}{10}$ $\beta\gamma$ in $\alpha\beta$, id est $\frac{121}{100}$ in $\frac{779}{100}$, et quia est dividendum productum per $\frac{11}{10}$, ideo sufficiet ducere per $\frac{11}{10}$, et fit $\frac{8569}{1000}$, dividendum per ipsos, nam κ fuit $\frac{11}{10}$ pos., quia fuit latus $\frac{121}{100}$ quadratum, habemus ergo ut prius $8\frac{569}{1000}$, dividendum per 1 pos. p: quadrata æqualia 9, igitur 1 cub. p: $8\frac{569}{1000}$ æqualia 9 pos..

Videtur ergo proprior modus demonstrationi, ut supponamus ad rei æstimationem, in qua ab numerus quadratorum et bd numerus æstimationis, divisus per quadratum ab. Et rursus ab numerus, id est quadratorum, et eb numerus æstimationis, idem cum priore, et divisum per quadratum ac vel ae, quia habet duas æstimationes, sed tunc æquatio erit diversa, quam oportebit invenire. Dico ergo quod, si cubus p: 200 est æqualis 100, erit ae res et ab 100, ponamus ergo ad æstimationem cubi æqualis 100 quad. p: 200, erit ergo ad nota, et ab est 100 numerus quadratorum, igitur bd differentia nota, et quia demonstratum est, quod proportio cb ad bd est duplicata ei quæ est ad ad ae, igitur proportio mediæ inter eb ad bd est ut ad ad ae, sit ergo bd 2, pro exemplo ut intelligas pones e $6\frac{1}{2}$ quad., et si bd esset 3 poneres e $6\frac{1}{3}$ quad., et si bd esset 4 poneres e $6\frac{1}{4}$ quad. ad hoc ut media sit 1 pos. quæ ducta in ae producit quantum ad in db, productum autem ad in db est notum, quia ad et db notæ sunt, et hoc est æquale mediæ ductæ in ae, quæ est numerus quadratorum communis, detracta eb quæ est pars illa quæ provenit divisa monade per bd, et est nota et est pars cubi. Sequitur igitur ex constructione, ut reducendo ad 1 cub. ut habeas cubum cum numero æqualem numero rerum. Et ut numerus rerum sit semper productum ex bd in ad. Et numerus æquationis compositus ex producto quadrati ab in ad, et cubo ipsius bd, veluti si ponatur (ut dixi) ab 100 et bd 2, erit 1 cub. p: 408 æqualis 200 rebus, fit autem 200 ex 2, quæ est bd, in 100, quæ est ab numerus quadratorum 408, autem componitur ex 400, producto 4, quadrati 2 et est bd, in 100, quæ est ab, et 8 cubo 2 bd. Et ita si bd esset 3, esset 1 cub. p: 927 æqualis 300 rebus, et eodem modo si bd esset 9, esset cubus cum 8829 æqualis 900 rebus, numerus enim rerum semper est productus ex æstimationis differentia a numero quadratorum in ipsum numerum quadratorum. Numerus autem æquationis, scilicet 8829, est compositus ex 8100 producto quadrati 9, id est 81, in 100 numerum quadratorum, et 729 cubo bd, quæ est 9. Cum igitur hoc capitulum sit speciale et circumscriptum habebit æstimationem notam, ut reliqua capitula specialia cubi et numeri æqualium rebus, et hæc æquivalebit generali cubi et numeri æqualium quadratis.

Ergo proposita quæstione cubi et numeri æqualium quadratis erit nota æstimatio cubi æqualis totidem quadratis et eidem numero, quare ad nota, et quia ab numerus quadratorum est notus, erit nota bd, ducemusque bd in ab et habebimus numerum rerum, ducemus etiam bd ad quadratum inde in ab, et producto addemus cubum db, et habebimus numerum æquationis cum regula, ergo speciali inveniemus æstimationem eius, et hæc prima æquatio, scilicet mediæ quantitatis inter eb et bd. Hanc igitur ducemus in se, et dividemus per bd, et exibit quantitas be secunda æquatio, quam detrahemus ex ab numero quadratorum proposito, et habebimus ae æquationem tertiam quæsitam. Unde patet quam difficilis sit hæc inventio et quam absurdum genus quantitatis proveniat per decem difficultates. Prima est inventio ad, quæ solet esse trinomium compositum cubicum, et ex radicibus universalibus, quia pendet ex capitulo generali. Secunda est residuum bd detracta ab. Tertia est productum ex ab in bd. Quarta est quadratum bd. Quintæ productum ex eodem quadrato in ab. Sexta cubus bd. Septima est æstimationis inventio cum operationibus capituli specialis. Octava est deductio inventæ æstimationis ad quadratum, nona est divisio producti per quantitatem bd. Decima est detractio proventus a numero quadratorum. Ex his facillimæ sunt tres, scilicet secunda, quinta et decima, pene impossibiles duæ, scilicet septima et nona, reliquæ valde difficiles.

Caput XXVII

De æstimatione data ut inveniatur numerus æquationis

Et cum in capitulis maioribus 1 cubi tum etiam in aliis ex tribus inveniatur quartum, utpote ex cubo æquali quadratis et numero datis invenimus æstimationem. Ita æstimatione et cubo et quadratis inveniemus numerum, aut ex eadem et cubo et numero inveniemus quadrata, nam de cubo non est, ut quæramus ipsum per quadrata et numerum datum cum sola æstimatio doceat, cum ergo sint sex capitula et duobus modis in singulis contingat inveniri, quarum erunt duodecim capitula.

1. Sit ergo primum

data ac æstimatio rei, et numerus quadratorum ab datus, qui cum numero aliquo æquatur cubo ac. Igitur quia ac data est, erit cubus ac datus et quadrata sub numero ab data, residuum ergo ad cubum est quod fit ex bc in quadratum ac, et hoc est notum, quia ac et ab notæ et quadratum ac, igitur numerus æquationis. Detrahe igitur numerum quadratorum ex æstimatione data, et quod relinquitur duc in quadratum æstimationis, productum est numerus æquationis. Exemplum æstimatio est 10 cubi æqualis 6 quadratis et numero cuipiam, detrahe 6 ex 10, relinquitur 4, duc in 100 quadratum 10, fit 400, igitur cubus æquatur 6 quad. p: 400.

2. Sit modo numerus æquationis, scilicet productum ex bc in quadratum ac, et dividam illum per quadratum ac, prodibit bc, detraho ex ac, relinquitur ab numerus quadratorum.

3. Sit cubus ab et quadrata bc data et æstimatio nota, erit ergo cubus notus, et bc ducta in quadratum ab etiam nota, iungendo utrunque habebis numerum æquationis.

4. Et sit cubus, et ab data sit et numerus æquationis datus, igitur detraham cubum ab datum ex æquationis numero dato, residuum dividam per quadratum ab datum, quia ab data est, quod prodit est bc numerus quadratorum.

5. Et sit ac numerus quadratorum datus, et ab æstimatio rei, et quadrata illa sint æqualia cubo et numero. Quia ergo ab data est, erit quadratum eius et cubus eius datus, ideo etiam productum ex ac in quadratum ab, a quo detracto cubo ab relinquitur numerus æquationis. Exemplum ac sit 6 numerus quadratorum, ab autem 4, cubus eius est 64, quadratum 16, igitur sex quadratum sunt 96, detrahe 64 cubum æstimationis, relinquitur 32 numerus æquationis, igitur 1 cu. p: 32 æquatur 6 quad. quando æstimatio rei est 4. Et in huiusmodi cum æstimatio media æquatur extremis, cave ne casus sit impossibilis.

6. Et sit modo numerus æquationis et æstimatione notus, et velim numerum quadratorum æqualium cubo et dicto numero æquationis. Quia ergo ab nota est æstimatio, erit cubus eius notus. Huic addam numerum æquationis iam notum, habebo totum numerum notum quem dividam per quadratum ab, iam notum prodibit ac numerus quadratorum.

7. Sit etiam æstimatio nota cubi et rerum æqualium numero, liquet quod cubus et res erunt notæ quæ iunctæ faciunt numerum æquationis notum.

8. Et rursus si a numero æquationis noto detrahas cubum æstimationis notæ, residuum erit notum, quod divisum per æstimationem ostendit numerum rerum.

9. Rursus si cubus æquatur rebus et numero, et res sint notæ et æstimatio, ducemus æstimationem in numerum rerum, et detrahemus a cubo rei, et residuum erit numerus æquationis.

10. Et ita, si a cubo iam noto æquationis numerus detrahatur residuum divisum per æstimationem, ostendit numerum rerum. Cave tamen ne casum proponas impossibilem, velut cubum æqualem rebus et 10 numero et æstimatio 2, nam oportet æstimationem semper esse maiorem Rcu: numeri æquationis, id est 10, et ita in aliis.

11. Sit etiam cubus p: 12 æqualis rebus, et sit æstimatio 2, tunc cubus 2 est 8, adde ad 12, fit 20, divide per 2, prodibit 10 numerus rerum.

12. Et iterum sit cubus cum numero æqualis 10 rebus et æstimatio 2, duco 2 in 10, fit 20, detraho 8 cubum, relinquitur 12 numerus æquationis, qui cum cubo 2 iunctus æquatur decuplo 2. Et quia in capitulis quadratorum vel rerum æqualium numero et cubo est duplex rei æstimatio, dico quod proposita quavis earum, sequitur idem. Veluti cubus p: 24 est æqualis quadratis et æstimatio una est 2, alia R 21 p: 3, duco 2 ad cubum, fit 8, addo ad 24, fit 32, divido per 4 quadratum 2, exit 8 numerus quadratorum. Similiter duco R 21 p: 3 ad cubum, fit R 48384 p: 216, adde 24, fit 240 p: R 48384, divide per 30 p: R 758 quadratum R 21 p: 3, exit 8.

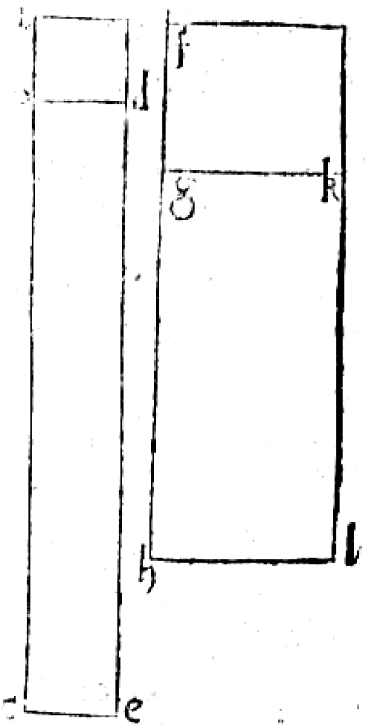
Caput XXVIII

Quod in proposito Capituli XXVI pervenitur ad cubum et res æqualia numero

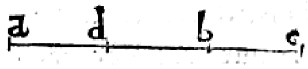
Cum vero iam conclusum sit quod, si quis possit invenire regulam specialem cubi et numeri æqualium rebus, quando numerus rerum fit ex ductu duorum numerorum invicem, et numerus æquationis ex ductu quadrati unius in aggregatum amborum, quod habebit æstimatio cubi et numeri æqualium quadratis. Dico quod hæc specialis regula est difficilis inventu, quia æquipollet uni generali, quoniam convenit ominibus casibus, in quibus cubus et numerus æquantur rebus. Exemplum, si dico cubus et 6 æquantur 8 rebus, dico quod hæc erit sub regula illa speciali quia ponam: quod una pars sit 1 pos., alia $\frac{8}{1\text{pos.}}$, duco igitur 1 pos. in se fit 1 quad., duco 1 quad. in aggregatum 1 pos. p: $\frac{8}{1\text{pos.}}$, fit cu. p: 8 pos. æqualia 6, at hoc habet capitulum generale, igitur regula illa non est proprie specialis.

Caput XXIX

De comparatione capitulorum cubi et rerum æqualium numero et cubi et numeri æqualium totidem rebus



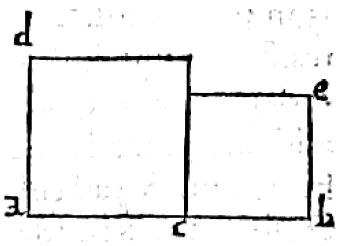
Et proponatur cubus ad cum rebus numero 10 æquales 12, et erit superficies bc 10, corpus autem ae 12. Dico primum quod, si sumatur fk cubus, qui cum 12 numero, et sit gl corpus iuxta altitudinem fg æqualia 10 rebus, erit ergo superficies fl ex supposito, et habebit duas æstimationes, quod singulæ illarum erunt in mutua proportione hoc modo bc ad fh ut fg ad ab, et iterum ae ad gh ut fg ad ab. Quare proportio ac ad gh, duplicata ec, quæ est be ad fh. Liquet etiam quod utraque æstimatio fg est maior ab, quia cum æqualiter sumatur est æqualis gl numero, qui est æqualis toti ac, et ultra etiam cubo fk per communem animi sententiam. Ex quo sequitur quod ac sit maior gh, igitur cum sit duplicata ei quæ est bc ad fh, erit be maior fh. Et etiam clare per se patet cum sit mutua ut fg ad ab. Et quia 10 res fh æquantur cubo fk et gl numero æquationis, et gl est æqualis cubo ad et be rebus, erit fh numerus rerum æqualis cubis fk, ad et rebus bc, detractis igitur rebus be ad rebus fh, quæ sunt numero æquales, erunt decem differentiæ fg et ab, æquales cubis ab et fg pariter acceptis.



Rursus proponatur duæ quantitates ab et bc, ut tota ac sit 2, gratia exempli, ut sit differentia illarum db, et decuplum db sit æquale cubis ab et bc, dividemus ac 2 in 1 pos. et 1 m: 1 pos., et cubi erunt 6 quad. p: 2, et hoc est æquale 20 rebus, id est decuplo db, quæ est differentia, igitur 1 quad. p: $\frac{1}{3}$ æquatur $3\frac{1}{3}$ rebus, et rei æstimatio est $1\frac{2}{3}$ p: R $2\frac{4}{9}$ vel $1\frac{2}{3}$ m: R $2\frac{4}{9}$.

Caput XXX

Qualis æqualitas cuborum partium lineæ divisæ



Sit ab divisa in c quadrata eius cd, ce, dico quod cubi ac, cb sunt æquales parallelipedo ex ab in aggregatum quadratorum cd, ce dempta superficie ac[1] in cb, nam quod fit ex ab in aggregatum quadratorum cd, ce est æquale ei quod fit ex ac in cd, ce et ex bc in ce, cd, quare duobus cubis ac et cb et eis quæ fiunt mutuo parallelipedis ae in ce, et cb in cd, at ac in ce quantum ex be in superficiem ac in cb, et ex cb in cd quantum ex ad in superficiem ac in cb. Quod igitur fit ex ab in cd et ce est æquale cubis ac, cb, et ei quod fit ex ad in superficiem ac in cb, et ex be in eandem, quod autem fit ex ad in superficiem ac in cb cum eo quod fit ex bc in eandem est æquale ei quod fit ex tota ab in superficiem ac in cb, eo quod ad est æqualis ac et be, æqualis bc. Igitur quod fit ex ab in cd, ce est æquale ei quod fit ex ab in superficiem ac in cb cum cubis ac et cb, igitur detracto eo quod fit ex ab in superficiem ac in cb ex eo quod fit ex ab in cd, ce, et est idem quod detrahere superficiem ac in cb ex quadratis ac, cb, erit parallelipedum ex ab in cd, ce detracta superficie ac in cb æquale cubis ac, cb quod erat demonstrandum.

[1]1570 and 1663 have ‘*ae*’.

Caput XXXI

De æstimatione generali cubi æqualis rebus et numero solida vocata, et operationibus eius

Et postquam non quærimus in æstimatione nisi demonstrationem operationem et propinquitatem, dico quod æstimatio cubi æqualis rebus et numero generalis in parte quæ non habetur est nota secundum tres modos propositos in solidis, per primam et tertiam regulam Capitulum 25 Artis magnæ. Et appropinquatio non est minor quam in reliquis radicibus quadratis aut cubicis, operationem autem nunc docebimus. Verum in tertia regula ob præcedentem videtur maior æqualitas atque notitia. Si quis ergo dixerit cubus est æqualis 13 rebus p: 60, igitur dicemus ex tertia regula quod res est R solida 13 in 30, qui est dimidium 60, id est, ut ita dividatur 13 ut ex partibus in radices suas mutuo ductis fiat 30. Dico ergo quod si volueris hanc R sol. ducere, gratia exempli in R duas, duces R 2 in se, fit 2, duc in 13, fit 26, inde duc R 2 ad cubum, fit R 8, duc R 8 in 30, fit R 7200, igitur R producta erit R sol. 26 in R 7200. Et ita si volueris eandem dividere per R 2, duc R 2 in se, fit 2, divide 13 per 2, exit $6\frac{1}{2}$, deinde divide 30 per R 8 cub. R 2, exit R $112\frac{1}{2}$, et erit quod provenit R sol. $6\frac{1}{2}$ in $112\frac{1}{2}$. Hæc autem facile demonstrari possunt in additione quoque similium, velut R sol. 13 in 30 cu. R sol. 52 in 240, divides singulos per suas correspondentes, et exibunt diviso 52 per[1] 13, et diviso 240 per 30, 8 et Rcu: 8 est eadem cum R quadrata 4, quia iam supponuntur similes partes, addam igitur ad 2 monadem, fiet 3, duc ad quadratum, fit 9, duc in 13, fit 117, duc et 3 ad cubum, fit 27, duc in 30, fit 810, erit ergo R coniuncta sol. 117 in 810. Idem dico de subtractione. Dividendo singulas partes per suas similis eius quod provenit capiendo R quad. vel cu. quæ erit una a qua detrahe, et residuum reducitur ad quadratum et cubum, et duc in suas partes quæ ei respondent. In dissimilibus autem adiiciemus aut detrahemus simpliciter, quod etiam facimus in R universalibus et anomalis. Possent et aliqua in huiusmodi subtiliora inveniri,

[1] 1570 and 1663 have "*134*".

sed satis sit, si aliquis dicat, habui cubum æqualem 6 rebus p: 1, dices igitur æstimatio rei est R sol. 6 in $\frac{1}{2}$, id est aggregatum duarum radicum quadratorum partium 6, ex quarum mutua multiplicatione in ipsas partes producatur $\frac{1}{2}$.

Et pro appropinquatione celeri ac brevi duces ad integras per numerum partes habentem, ducendo puta per 4, et habebis R sol. 96 in 32, igitur pars una erit $95\frac{8}{9}$ et alia $\frac{1}{9}$. Et hoc est maius, minus autem $95\frac{9}{10}$ et $\frac{1}{10}$, igitur propinqua una erit $95\frac{17}{19}$ alia $\frac{2}{19}$, huius ergo accipiemus quartam partem, et erunt numeri $5\frac{151}{152}$ et $\frac{1}{152}$.

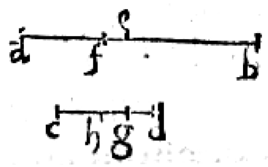
In inæqualibus autem iungendis, detrahendis, multiplicandis, ac dividendis eadem facimus quæ in R diversis, neque enim licet eas aliter iungere quam per p: et subtrahere quam per m:, velut Rcu: 10 cum R quadrata 8 dicemus R 8 p: Rcu: 10, vel detrahendo R 8 m: Rcu: 10. Et si quis dicat quod possumus etiam iungere hoc modo RV: 8 p: Rcu: 100 p: R Rcu: R 3276800, dico quod est hæc longior et difficilior. De longitudine patet sensu, de difficultate in ultima parte cogeris intelligere R quadratam et Rcu: ut in alia, et præter id etiam R cub. 100 inde totius aggregati R universalem, licet forsan quod ad propinquitatem attinet, forsan redderetur aliquanto exactior, quia esset una tantum et minoris aggregati, unde notandum quod, si quis velit Rcu: 10 p: R 8 et RV: Rcu: 10 p: R 8, quod prima sub eadem additione erit proxima $2\frac{17}{20}$ et $2\frac{3}{20}$, quod totum est 5, ut manifestum, sed RV: $2\frac{17}{20}$ et $2\frac{3}{20}$ est $2\ \frac{59}{225}$. Et hoc manifeste est proximius radici veræ R cub. 10 p: R 8 quam 5, quia R R 10 est proximior R R R 101 quam R 10, R R 101, et multo magis quam 10 ipsum R 101 differt m: pene per $\frac{1}{20}$, et R 10 differt eo modo sumpta a R R 101 per $\frac{1}{67}$ quod est multo minus quam $\frac{1}{20}$ in $\frac{1}{28}$ ferme. Sed tamen hoc contingit per se non habita proportionis ratione. Forsan in multiplicatione et divisione aliter dicendum esset, quoniam partes redduntur pauciores. Sed tamen cum incommensæ fuerint remanet numerus aggregati, ut R 6 p: R 5 in R 3 m: R 2, producit R 18 p: R 15 m: R 12 m: R 10, quid ergo refert si dicam R 18 p: R 15 m: R 12 m: R 10 et R 6 p: R 5 in R 9 m: R 2, cum enim oportebit illas addere, duplicare, dividere, dividam unamquamque seorsum, et post iungam eodem modo aut detraham. Sint ergo dissimiles R sol. 13 in 30 et R sol. 5 in 6, sic multiplicabo R sol. 13 in 30, productum in R sol. 5 in 6, sic dividam R sol. 13 in 30, et ita addam R sol. 13 in 30 p: R sol. 5 in 6, et ita detraham R sol. 13 in 30, R sol. 5 in 6 m: R sol. 5 in 6. Et accipiam RV: hoc modo, RV: R sol. 13 in 30, et est R 5, et ita accipiam Rcu: hoc modo, RV: cu. R sol. 13 in 30. Et in solidis radici cuicunque debet adiici v: id est nota universalis cum sit unum totum.

Et nota quod R sol. dicitur non tota sed comparative, velut cum dico RV: R 9 p: Rcu: 27 vult dicere accipi R 9 quæ est 3, et R cub. 27 quæ est etiam 3, iunge, fiunt 6, igitur RV: 9 p: Rcu: 27 est R 6. Sed non est sic de RV: R sol. 13 in 30, neque enim cum R 13 quadratorum aggregati sit 5, et R 30 ut parallelipeda sit rursus RV: est R 8 aggregati 5 et 5, sed est R simpliciter unius partis tantum, id est 5. Et ideo nota quod semper sunt æquales, igitur ducendo, dividendo R solidæ partes sunt æquales.

Et nota quod licet producti ex aggregato duorum quadratorum in aggregatum duorum quadratorum producant, semper aggregatum ex duobus quadratis, ut 5 in 5, et 5 in 13, et 5 in 8, et 5 in 18, et 13 in 25, et 13 in 8, et 8 in 25, et 8 in 50, et ita de aliis, tamen illæ partes non servant proportionem, velut 5 in 13, efficit 65, qui componitur ex 64 et 1 quadratis, qui nihil habent cum 15, qui vere producitur ex 5 R sol. 13 in 30 in 3 R sol. 5 in 6, nec etiam diviso 65 in 49 et 16, nam radices sunt 7 et 4, quæ iunctæ faciunt 11, qui etiam est diversus a 15. Ideo aliunde petenda est ratio cur componatur, constat enim esse longe plures qui non componuntur, ut usque ad 20 sunt 2, 5, 8, 10, 13, 17, 18, 20. Sunt ergo duodecim qui non componuntur et octo tantum qui componuntur. Et a 20 ad 40 sunt 25, 26, 29, 32, 34, 37, 40, adhuc pauciores a 40 ad 60 sunt 41, 45, 50, 52, 53, 58, pauciores.

Caput XXXII

De comparatione duarum quantitatum iuxta proportionem partium

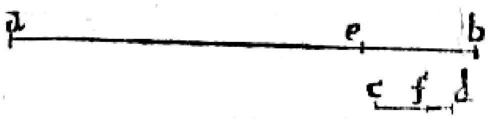


Et sumantur duæ quantitates ab maior, cd minor, dico quod poterunt dividi ita ut sit proportio unius partis ad aliam maioris inæqualitatis, et residui ad residuum usque ad infinitum, nam ablata ae æquali cd erit be ad residuum infinita, ergo ex regula dialectica semper licebit dividendo residuum, utpote facta af æquali cg, dividendo ef et dg per æqualia erit proportio residui usque ad b, ad residuum usque ad g perpetuo maior. Et ita usque in infinitum dividendo versus d, et assumendo aliquid maius in ab erit ut procedatur usque in infinitum in proportione residuorum. Per 10 quinti Elementorum

Dico præterea quod non poterunt ambæ proportiones esse minores proportione totius ad totum, quia si detrahatur minor proportio ut ae ad cg quam ab ad cd, fiat ae ad ch, æqualis ab ad cd, igitur ae ad cg minor quam ac ad ch, igitur ch minor cg; be ergo ad hd ut ab ad cd, igitur be ad gd maior quam ab ad cd. Per 19 quinti Elem.

Manifestum est ergo quod sub minima proportione ambæ partes erunt cum fuerint quantitates divisæ secundum proportionem totius ad totum. Hoc etiam infinitis modis, seu non fit varietas. Per 3 quinti eiusdem

Dico modo quod non poterunt in proportionem reduplicatam maiorem quam totius ad totum æqualem, nec minorem quam sit proportio media. Voco proportionem reduplicatam cum fuerit proportio partium ut residuorum duplicata.



Velut si proportio ab ad cd nonupla, dico quod non potest dividi ab et cd ut sit proportio maior nec æqualis nonupla, nec æqualis aut minor tripla. Nam, si sit ae ad ef nonupla, igitur cb ad fd nonupla, ergo nonupla nonuplæ duplicata erit quod esse non potest, et si maior nonupla ergo ex demonstratis eb ad fb minor nonupla, ergo non duplicata ad illam.

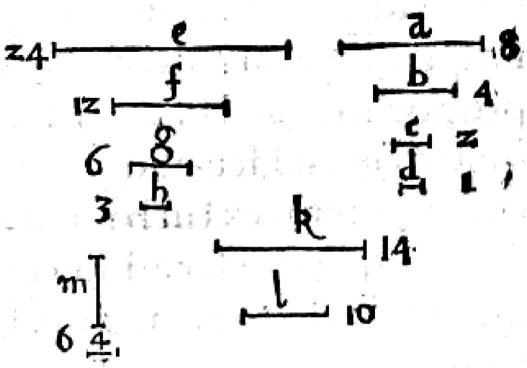
Nec potest dividi ab et cd ita ut sit minor quam tripla. Nam si sit tripla be ad fd, cum fit per demonstrata ae ad cf maior nonupla, eo quod eb ad fd est minor quam ab ad cd. Igitur ae ad cf maior duplicata eb ad fd, non ergo duplicata. Multo minus si sit proportio eb ad fd minor tripla poterit esse residui ad residuum duplicata.

Cum ergo quis dixerit divide 18 et 2 ita ut proportio partium sit reduplicata quadruplæ, tunc cum quadrupla sit minor nonupla et maior tripla, duc 4 numerus proportionis in se, fit 16, duc in 2 minorem quantitatem, fit 32, aufer maiorem scilicet 18, relinquitur 14, hunc divide per differentiam proportionis a suo quadrato, id est 12, qui est differentia quadrati 4, et ipsius 4, et exit $1\frac{1}{6}$, aufer ex 2, relinquitur $\frac{5}{6}$, aufer quadruplum $1\frac{1}{6}$, quod est $4\frac{2}{3}$ ex 18, relinquitur $13\frac{1}{3}$, quod est sexdecuplum ad $\frac{5}{6}$.

Ex hoc etiam patet quod seu maior maioris, ut hic seu minor habuerit rationem residui, id est partis quæ habet proportionem duplicatam, semper habebit ad minorem portionem minoris lineæ nunquam ad maiorem.

Caput XXXIII

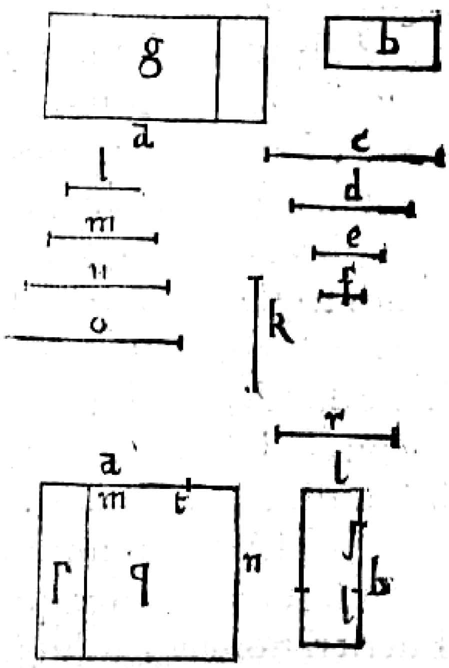
De duplici ordine quatuor quantitatum omologarum eiusdem proportionis ad duas alias



Sint a, b, c, d et e, f, g, h omologæ et in eadem proportione, et sint duæ aliæ k et l eiusdem generis, et ex differentia a et d in m producatur differentia productorum b in k et d in l, dico quod differentia productorum f et h in l producetur ex differentia e et h in eandem m. Et est generalis in similibus semper servando rationem assumptorum. Nam quia b ad d ut f ad h, erit b ad f ut d ad h permutando, quare productorum ex b et f in k invicem ut productorum d et h in l invicem, utraque enim ut b ad f et d ad h, quæ se habent eodem modo. Per 11 quinti Elementorum Permutando igitur productorum b in k et d in l ut f in k et h in l, quare et differentiarum veluti b ad f, at ut b ad f ita differentiæ a, d ad differentiam e, h. Igitur divisa differentia f in k et h in l per differentiam l, h exibit m. Per 19 quinti Elementorum

Caput XXXIV

De triplici divisione duarum quantitatum in mutuam reduplicatam



Cumque propositæ fuerunt duæ lineæ a et b, possumus imaginari ut dividamus utramque, ut sit proportio mutua reduplicata. Nam de recta superius locuti sumus. Et potest istud fieri per additionem eiusdem quantitatis ad utramque quantitatem, sed ut fiat media proportio, et potest fieri ut eadem quantitas addatur et detrahatur ab utraque, et residuorum proportio sit duplicata proportioni aggregatorum. Et hæc tria hic docebimus demonstrantes primum solum, nam reliquorum sufficiet docere operationem. Quartum autem est de quo posterius agetur quod est difficillimum. Volo ergo dividere a et b ut sit proportio secundæ partis b ad secundam partem, aut primæ partis a ad primam partem b duplicata iuxta proportionem datam inter c et d, statuo ef in continua proportione cum cd, et duco d in a et f in b, et fiant superficies a et b, et detraho b ex a, et relinquatur

Capitulum 31

Per 12 sexti Elementorum

g, et detraho f ex e, et relinquatur k, et fiat superficies super k æqualis g, cuius latus sit l, et iuxta proportionem f, e, d, c statuo l, m, n, o, et duco l in b, et n in a, et fiant superficies an, bl. Et rursus detraho bl ex an, et sit p, cuius residuum sit superficies q, aufero etiam l ex o, et relinquatur r, et super r statuo superficiem æqualem q, cuius secundum latus constat esse rursus l per præcedentem. Aufero l ex b, et relinquatur s et m, aufero ex a, et relinquatur t, dico ergo quod, cum proportio m ad l sit ut c ad d, quod proportio s ad t est duplicata ei quæ est m ad l, seu c ad d. Nam ex demonstratis p fit ex l in b, et q ex l in r, igitur a, n ex l in b, r, igitur n ad l ut b, r ad a, sed n ad l ut m ad l duplicata, igitur b, r ad a ut m ad l duplicata. Et ut c ad d pariter duplicata, constat autem b, r ex l, s, r. r autem, cum i facit o ex supposito, nam r fuit differentia o et l, igitur b, r sunt æquales ex communi animi sententia os. Igitur os ad a ut c ad d duplicata. At o ad m ut c ad d duplicata, quia sunt in continua proportione, igitur residui s ad residuum t ut c ad d duplicata, quod propositum erat. Operatio autem brevis est, ponamus ut in exemplo c 24, d 12, e 6, f 3, a si 10, b 8.

Per 44 primi Elementorum

Per 1 secundi Elementorum

Per 16 sexti Elementorum

Per 19 quinti Elementorum

24	
12	10
6	
3	8

Duco d in a fit 120, duco f in b fit 24, detrahe 24 ex 120, relinquitur 96, divide 96 per 21 differentiam c et f, exit $4\frac{4}{7}$ quantitas l, igitur ducendo l per e fit $36\frac{4}{7}$, divide per f, exit $9\frac{1}{7}$, quantitas m quæ est dupla ad l ut c ad d, detrahe ergo $4\frac{4}{7}$ ex 8, relinquitur $3\frac{3}{7}$, detrahe $9\frac{1}{7}$ ex 10, relinquuntur $\frac{6}{7}$, proportio $3\frac{3}{7}$ ad $\frac{7}{6}$ est quadrupla et duplicata ei quæ est c ad d.

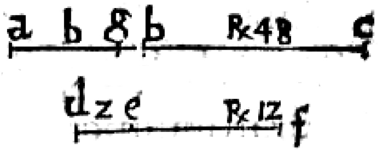
Propositis ergo duabus lineis rursus a et b, quibus volo addere communem c et detrahere rursus ut sit proportio residuorum duplicata ei quæ est aggregatorum. Duc differentiam quadratorum in se, et eius cape trigesimam sextam partem, cui adde tertiam partem producti unius in alteram, et a radice totius aggregati, detrahe sextam partem differentiæ dictorum quadratorum, residuum est quæsita tertia quantitas. Velut capio 5 et 4, differentia quadratorum est 9, eius quadratum 81, cuius $\frac{1}{36}$ est $2\frac{1}{4}$, cui adde $\frac{1}{3}$, producti 5 in 4, et est $6\frac{2}{3}$, qui est tertia pars 20, et fit $8\frac{11}{12}$, cuius a radice detrahe $\frac{1}{6}$, differentiæ quadratorum, id est $1\frac{1}{2}$, et relinquitur res quæsita $\mathrm{R}\ 8\frac{11}{12}$ m: $1\frac{1}{2}$. Igitur partes erunt $6\frac{1}{2}$ m: $\mathrm{R}\ 8\frac{11}{12}$, et $5\frac{1}{2}$ m: $\mathrm{R}\ 8\frac{11}{12}$ residua scilicet: aggregata autem $3\frac{1}{2}$ p: R $8\frac{11}{12}$, et R $8\frac{11}{12}$ p: $2\frac{1}{2}$.

Rursus sint propositæ duæ lineæ et sit una 4, alia 3, volo addere communem quantitatem utrique quod sit proportio aggregati media seu radix proportionis propositarum quantitatum. Et est quasi conversa præcedentis, duco 4 in 3, fit 12, huius R addo utrique, et habeo intentum, proportio enim R 12 ad 3 est velut 4 p: R 12 ad R 12 p: 3, nam 3 in 4 p: R 12 producit 12 p: R 108 et R 12 in R 12 p: 3 non minus producit idem 12 p: R 108.

Ex hoc sequitur quod proportio binomii ad aliud binomium alterius speciei potest esse quantitas potentia tantum rhete, velut si duco R 3 in R 12 p: 2, fit 6 p: R 12, igitur 6 p: R 12 est in proportione R 3 ad R 12 p: 2. Et ita de recisis 6 m: R 12, est in proportione R 3 ad R 12 m: 2, et hoc propter commutationem, quia R est media inter duos numeros et numerus inter duas R.

Scholium

Dico modo quod, si partes binomiorum non sint commensæ secundum eandem proportionem, quod si binomiorum binomio esset commensum, aut recisum reciso, numerus esset commensus potentia tantum rhete seu longitudine alogæ.



Et sint gratia exempli ab tripla de et bc dupla ef, dico quod si tota ac esset commensa toti df, essent partes ab, bc invicem commensæ itemque de et ef. Nam ut demonstravimus supra quia non est eadem omnium proportio, igitur unius par ad partem una maior altera minor, sit ergo minor bc ad ef quam ac ad df, et hæc minor quam ab ad de, fiat ut ac ad df ita ag ad de, commensum est igitur ag, de, et fuit etiam ab commensum de, igitur ag, gb commensæ. Similiter bc commensa fuit ef et ge eidem, quia in proportione ae ad df, gc igitur commensa be, quare gb ipsi bc fuerat etiam ba, igitur ab, bc commensæ sunt, quare etiam de et ef. Non est autem necessarium (ut dixi) quod si partes sint commensæ ut totum sit toti commensum ut dixi. Neque etiam si totum toti et pars parti, ut reliqua pars reliquæ parti, velut 10 et 9 sunt commensa et R 20 et R 5 commensa, non tamen

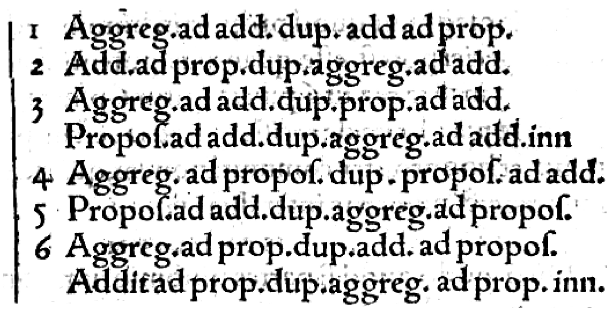
10 p: R 20 est commensum 9 p: R 5, aliter sequeretur quod 10 et R 20 essent commensa, quod est absurdum.

Ex hoc sequitur quod binomio non commensa non possunt esse in proportione numeri, possunt tamen esse in proportione unius simplicis quantitatis.

Caput XXXV

De sex proportionibus mutuis reduplicatis quæ oriuntur ex additione unius quantitatis ad unam aliam, et duabus inutilibus

Cum proposita fuerit una quantitas, puta 2, possum addere illi aliam quantitatem, octoq̀ue modi proportionis reduplicatæ consurgent, quorum duo sunt inutile. Modi ergo sunt ut quantitas addita ad propositum habeat duplicatam proportionem quam aggregatum ad additam secundus conversus ut aggregatum ad ad additam habeat duplicatam ad eam quæ est additæ ad propositum. Et ideo ponam eos ordinatim in tabula.


1 Aggreg.ad add.dup.add ad prop.
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4 Aggreg. ad propoſ. dup. propoſ. ad add.
5 Propoſ.ad add.dup.aggreg.ad propoſ.
6 Aggreg.ad prop.dup.add. ad propoſ.
Addit ad prop.dup.aggreg. ad prop. inn.

Prima igitur utilium duc 2 numerum propositum ad quadratum, fit 4, et ad cubum sit, et habebis 1 cub. æqualem 4 rebus p: 8.

Secunda, duc 2 ad cubum, fit 8, accipe R quæ est R 8, et accipe R 2, et ita habebis 1 cu. æqualem quadrat. R 2 et R 8, et quadratum æstimationis est res quæsita.

Tertia habet quadratum p: 2 pos. numero proposito æqualia 4 quadrato numeri propositi.

Quarta habebimus cub. p: quad. 2 numeri propositi æqualia 8 cubo numeri propositi.

Quinta, duc 2 ad cubum fit 8, et habebis 1 cub. p: rebus numero proposito, scilicet 2 æqualia R 8, et quadratum æstimationis est quantitas quæsita.

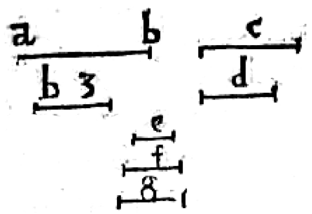
Sexta habebimus quadratum æquale rebus numero proposito, id est 2, et numero quadrato numeri propositi, id est 4, ut sit 1 quad. æquale 2 pos. p: 4.

Dico demum quod proportio confusa aggregati primæ et quartæ quantitatum omologarum ad aggregatum secundæ et tertiæ earundem est veluti quadrati p: 1, detracta proportione ad ipsam proportionem, ut alias demonstravi. Ex quo habetur confusa quarumlibet quatuor quantitatum recte intelligenti.

Caput XXXVI

De dividendis duabus lineis æqualibus secundum proportionem mutuam reduplicatam datam

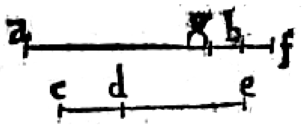
Istud docemus in Arte magna. Sed ibi adnotanda sunt illa verba ex quibus totum negocium pendet. Rursus quod fit ex ab et ad in ab, et ef est æquale ei quod fit ex ef et eg in aggregatum ab et ef, quia ex supposito ef et eg æquantur ab et ad, constat ergo ab quantitatem, et ef bis assumi, et cum hoc supponi ab et ad æquales esse ef et eg, ut primum potest supponi pro arbitrio, sed secundum non ita. Eo tandem venitur ut duæ et duæ quantitates sint in eadem proportione cum tertia. Et quod tertia illa, scilicet ab et ef, componitur ex secundis ab et ef. At duabus quibuslibet constitutis proportionibus, et manentibus duabus quantitatibus, licebit constituere communem illam quantitatem, et reliquas duas intervenire.



Exemplum, sint datæ duæ quantitates a 6, b 3, et aliæ duæ c, d, subiungo e ad a, b in continua proportione, et facio f ad e ut d ad c, et g ad f similiter, eritque g ad a ut f ad b duplicata. Eo igitur pervenire oportet cum proportione data loco æqualitatis. Constat etiam quod si proportio a ad c sit duplicata ei quæ est b ad d, quod hæ quatuor quantitates copulabuntur ad unam.

Caput XXXVII

De sex comparationibus quatuor quantitatum reduplicatæ proportionis



Et sint quatuor quantitates in reduplicata mutua proportione ab prima, cd secunda, de tertia, bf quarta, dico quod duabus ex his notis fiunt ut liquet sex coniugationes, et duæ harum neque cum aggregatis per se notis notæ sunt, scilicet nota prima et quarta, vel secunda et tertia, notisque af et ce. At reliquæ quatuor notam faciunt quantitatem modo aggregatum omnium notum sit. Sit ergo primum ab et cd nota, utpote ab 24, cd 6, aggregatum af et ce 47, tunc tu scis quod proportio de ad bf est ut ab ad cd duplicata, igitur ut 16 ad 1, igitur de et bf ad bf ut 17 ad 1, at de et bf sunt 17, igitur diviso 17 per 17, habebis bf unum et de sexdecim, nam de et bf sunt 30, ut dixi, quia ab et cd sunt 30, et af et ce 47, igitur residuum quod est de et bf est 17.

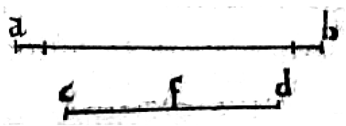
Sit rursus bf 1, de 16, aggregatum af et ce 47, igitur ab et cd sunt 30, proportio ab ad cd ut 4 ad 1, et ab cd ad cd ut 5 ad 1, divide 30 per 5 exit 6, et tanta erit cd et ab 24.

Proponatur modo ab et de notæ 40 totum, ut prius 47, et sit primo nota bf et sit 1, et cd 6, ponam ab 1 pos., erit tertia in proportione $\frac{1}{6}$ quad., duc in bf, fit 1 quad. divide per c, exit $\frac{1}{36}$ quad., igitur $\frac{1}{36}$ quad. p: 1 pos. æquantur 40, et 1 quad. p: 36 pos. æquantur 1440, et ita rei æstimatio est 24, cuius quadratum est 567, et eius pars trigesima sexta 16, seu detracto 24 a 40, relinquitur idem 16. Supponatur modo ab nota 24, de 16, totum 47 erit reliquum aggregatum cd et bf 7, ponatur cd 1 pos., erit tertia in proportione $\frac{1}{24}$ quad., duc $\frac{1}{24}$ quad. in 16, fit $\frac{1}{3}$ quad., divide per ab, id est 24, exit $\frac{1}{36}$ quad. æquantur igitur $\frac{1}{36}$ quad. p: 1 pos. ad 7. Igitur 1 quad. p: 36 pos. æqualia 252, et res est 6, et est cd residuum est 1 bf. At modo si ponatur ce 22 nota ita ut cd sit b, et de 16, et af 25. Ponemus ut in tertio casu ab 1 pos., erit tertia in proportione $\frac{1}{6}$ quad.. Igitur, si $\frac{1}{6}$ quad. producit 6, quid producet de quæ est 16, duc 6 in 16 fit 96, divide p: $\frac{1}{6}$ quad., exit $\frac{576}{1 \text{ quad.}}$ hoc ergo cum 1 pos. iunctum efficit 25, igitur 25 quad. æqualia 1 cub. p: 576.

Et hoc non continetur in capitulo. Sed quia in hoc casu supponimus numerum quadratorum esse 22, quia ce et æstimatio est cd 6, cuius cubus 216, qui cum 576 efficit 792, et hoc est æquale 22 quadratis, nam 22 in 36 efficit 792. Et supponimus ag numerum quadratorum, id est 22, et ab rei æstimationem, et quod ex bg in quadratum ab fiat 576, habebimus 1 cub. æqualem 22 quad. p: 576. Et hoc habet capitulum. Sed res non redit ad idem, nam æstimatio rei est minor 24, quia esset 24 cubus, esset æqualis 24 quadratis, igitur 22 quadratis et duplo unius quadrati, at unum quadratum est 576, igitur erit æqualis 22 quadratis, et 1152 non ergo 22 quadratis p: 576 solum. Et similiter notis ab et bf, et noto aggregato ce incidimus in eiusdem difficultates.

Caput XXXVIII

De conversa quantitatum in proportione reduplicata comparatione



Et ponamus ut sint duæ lineæ ab et cd, divisæ in e et f, et sit proportio fd ad cb velut ae ad cf duplicata, et sint ef secunda et cb quarta æquales et notæ, et totum aggregatum erit etiam notum. Nam in hoc casu proportio aggregati primæ et quartæ, id est ab ad aggregatum secundæ et tertiæ, id est cd, est ut quadrati proportionis p: 1 ad proportionem ipsam itidem p: 1, velut sit ab 15, cd 9, divido 15 per 9, exit $1\frac{2}{3}$, et hoc est quadratum proportionis p: 1 in comparatione ad 1 pos. p: 1, quare cum 1 pos. p: 1, hic habeat locum unius, erit ut ponamus 1 p: 1 pos., et ducamus p: $1\frac{2}{3}$, sit $1\frac{2}{3}$ pos. p: $1\frac{2}{3}$ æqualia 1 quad. p: 1, igitur 1 quad. æquatur $1\frac{2}{3}$ pos. p: $\frac{2}{3}$, ergo res est R$\frac{49}{36}$ p: $\frac{5}{6}$, quod est 2, et proportio erit dupla, pone igitur 1 pos., detrahe ex 9, fit 9 m: 1 pos., et ita etiam quia secunda est æqualis quartæ, erit[1] erit 15 m: 1 pos. dupla etiam 9 m: 1 pos., et 18 m: 2 pos. æqualia 15 m: 1 pos., et 15 p: 2 pos. æqualia 18 p: 1 pos., igitur res est 3. Ideo in hoc casu tres quantitates necessario sunt in continua proportione.

[1] 1570 has twice "15 m: 1 pos.".

Caput XXXIX

De dividendis duabus lineis secundum proportionem mutuam reduplicatam iuxta partes datas

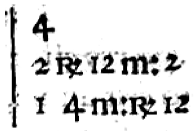
Hoc capitulum est pars duorum superiorum; et ex eo habetur capitulum generale cubi et numeri æqualium quadratis. Nam propositis, gratia exempli, 1 cu. p: 16 æqualibus 9 quadratis,

proponam lineam ae 9, et quæram æstimationem 1 cu. æqualis 9 quad. p: 16 quæ sit ab, igitur nota be, addam bg æqualem be, ergo ae, ab, ag, cb, bg, eg, cd æqualis ae omnes notæ. Propositum igitur est dividere cd in f ut sit fd ad bg duplicata ei qua est ab ad ef, qua inventa cum cubus ef, additis 16 ex supposito sit æqualis corpori ex cd in quadratum cf, quoniam totum est æquale suis partibus; et de sit 9, et quod fit ex df in quadratum cf 16, nam tantum fit ex bg in quadratum ab. Igitur 9 quadrata ef æquantur cubo p: 19. Ut ergo dividamus cd iuxta ut noscere oportet ordinem eorum quæ dicta sunt supra, scilicet quod quantitates ab, cf, fd, et b collocantur hoc ordine ut sunt mutuæ reduplicatæ. Alio ut sunt in continua proportione cum una et eadem, scilicet ab prima, fd secunda, cf tertia, bg quarta, prima et quarta manent in utroque ordine, sed secunda et tertia mutantur, nam cf est in reduplicata secunda, et fd tertia in recta fd est secunda, ef tertia.

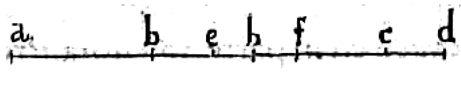
Capituli 26 et 27

Proponantur rursus notæ h, ab, et cd, et sint partes constitutæ ea, eb, fc, fd, quarum unam, si notam esset palam, est ob continuam proportionem quod essent omnes notæ, sed, si solæ h, ab, et cd, palam est quod erunt notæ partes per Artem magnam deveniendo ad capitulum cubi rerum et quadratorum æqualium numero. Ex qua pervenies ad cognitionem partium propositarum. Ut si h ponatur 4, ab 6 m: R 12 m: 1. Et partes se habebunt ut vides.

Cap. 39 regula secunda

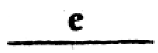


Est autem proportio 1 ad 4 m: R 12 duplicata ei quæ est 2 ad R 12 m: 2 et cave ne te confundas. Dico etiam quod si cubus et 24 sint æquales 8 et sit cd numerus quadratorum scilicet 8, ut sit æqualis cd, sciemus cd et bg, et erunt posita cb 1 quad. et bg, 1 cu. p: 8 pos. æqualia R 24 numeri propositi, et tam cb quam bg erunt quadrata æstimationis. Quia ergo notæ cb, bg, et per duo supposita nota, scilicet quantitatem cd seu ae, et numerum æquationis, id est 24. Et hic producitur ex supposito ex fd in quadratum fc, et fd et fe se habent necessitatem saltem alter, nam, quia dum cd et ac sunt 8 et numerus qui producitur 24 variatur ut sit 20 aut 22 aut 25, tunc variatur quantitas rei et quadratum eius eb, bg, igitur proposita quantitate cd vel ae quantitates eb seu bg habent connexionem cum cf et fd. Et quia, si non supponeretur numerus 24, haberetur ex partibus cf et fd, ducendo fd in quadratum fc, fiet ut inventa eb contraria ratione necessaria sit cognitio divisionis cd in f. Nam cum proposuerimus cf, fd cognitas per duo consequentia ad illa quæ sunt aggregatum earum et productum df in quadratum fc, consequimur duas alias ae et cb seu bg, igitur per ae, cb seu bg, et duo consequentia et sunt ab, bg et productum gb in quadratum ab cum uno ex tribus cf, fd, cd, inveniemus reliqua duo. At c nota est semper ex supposito cum sit æqualis ae, igitur cf et fd. Si ergo ponatur productum gb in quadratum ab 20, et cf 2, erit fd 5, diviso 20 per 4 quadratum 2 et, si fd ponatur 5, erit cf R 4, id est 2, nam diviso 20 per 5 exit 4, manifestum est ergo quod cf, fd, et cd habent consequentiam ad ab seu ae et eb seu bg. Concludo quod supposita cognitione ab, bg, quæ semper habetur necessaria, est connexio cum cf et fd, quia cd est differentia ab bg quæ non esset, si cd non esset illa differentia, sed solum 1 cub. p: 24 æquaretur 8 quadratis, et esset nota ab et bg, ex quarum ducta bg in quadratum ab, fieret 24, sed cd non esset 8, nec æqualis differentiæ ab et bg.



Proponatur ergo linea nota, et est rei æstimatio cubi æqualis quadratis numero ac, et numero æstimationis proposito qui fit ex cd in quadratum ad, et nota est ex hoc cd. Item nota est quia est differentia ad et ac numeri quadratorum atque notarum, iam vero ad divisa est bifariam in ac, cd notas, et ab et bd ignotas, quærendum est igitur an ex notis ac, cd (quia habent connexionem) haberi possint ab et bd, et ita erit quæsitum notum. Secundo an data divisione in ba magnitudo cd constituatur itemq̀ue ac. Et differt a præcedente quoniam per ae, cd in præcedente, et positionem quærimus quantitatem bd, et ea habita cognoscimus cd, bc, et ita ab et quæsitum. In hoc autem secundo constitutis ab, bd et habetur quantitas ad etiam, et est res, et eius quadratum etiam notum erit, ex quibus quærimus quantitatem ac, id est numeri quadratorum, et quod fit ex cd in quadratorum ab, et est numerus æstimationis, qui cum quadratis numero ac æquatur cubo ad. Tertio quæritur quam rationem[1] habet incrementum cd in comparatione ad bd, quia bc ad cd est duplicata ei quæ est ad ad ab ex supposito. Si ergo cd certa et data quantitas stuatuatur quo minor erit ab eo maior erit bc residuum, supponitur autem minor cd quam ab, maiorem autem oportet esse proportionem bc ad cd quam ad ad ab, quia duplicatam, igitur incrementum ab an semper augeat proportionem bc ad cd supra proportionem ad ad ab an minuat. Nam de æqualitate certum est quod non. Et an varietur hæc ratio mutare quantitate cd. Hoc igitur et quomodo certe est considerandum, loquamur igitur de secundo, quia est facillimum cum enim data sit ad et ab, data erit tertia linea quæ sit data, igitur proportio partium ad ad de divise ae, et data bd in divisa, ergo poterit dividi ut ad ad de, seu ad e,

Per 11 sexti Elementorum



et divisio illa cadet in c, cum igitur proportio ad ad e data sit, erit et bc ad cd, data est autem bd data ex supposito. Igitur utraque earum be, cd data quod erat demonstrandum. Nam data bc, cum sit data ab, erit data ac numerus quadratorum. Cumque sit data ad, erit illius quadratum datum. Et, quia cd data, erit productum cd in quadratum ad datum, is autem est numerus æstimationis quæsitus. Inveniamus etiam primum ut facilius, et proponamus ad 10, dc 1, erit ergo numerus 100, et sit bd 1 pos., et ab 10 m: 1 pos., cuius quadratum est 1 quad. p: 100 m: 20 pos., quod divisum per ad relinquit $\frac{1}{10}$ quad. p: 10 m: 2 pos., hæc est tertiæ quantitas quæ ducta in bc producit quamtum ad prima in cd quartam, quod productum est 10. Quia ergo bd est 1 pos., et cd 1, erit bc

Per 10 sexti Elementorum

[1]1570 has "*ratonem*".

1 pos. m: 1. Igitur productum tertiæ quantitatis est $\frac{1}{10}$ cu. p: 10 pos. m: 2 quad. m: $\frac{1}{10}$ quad. m: 10 p: 2 pos., et hoc totum est æquale 10. Quare reddendo vicissim fient $2\frac{1}{10}$ quad. p: 20 æqualia $\frac{1}{10}$ cu. p: 12 pos. et 1 cu. p: 120 pos. æqualia 21 quad. p: 200, et erit cu. æqualis 27 rebus p: 46, et ideo est in parte non nota. Pro tertio oportet præsupponere primum quod, si ad sit dividenda sic ut proportio ipsius ad ab sit ut bc ad bd, erit ca, cum maxima fuerit æqualis radicis radici octupli quadrati ad dempto duplo ad. Tunc si ae est minima, erit cd maxima. Et rursus cum fuerit proportio bc ad cd ut quadrati ad quadratum ab non poterit esse cd maior in comparatione ad ad, quam ut statuatur tertia pars ac æstimatio cubi p: unius rei æqualis quartæ parti quadrati ad. Et hoc pendet ex demonstratis in Libro de proportionibus. Exemplum, constituta ad 10, erit tertia pars ae æstimatio cubi et rei æqualis 25, qui est quarta pars 100, quadrati 10, erit ergo tertia pars ac RV: cu. R $256\frac{17}{64}$ p: $12\frac{1}{2}$ m: RV: cu. $256\frac{17}{64}$ m: $12\frac{1}{2}$, unde tota ac erit RV: cu. R $113917\frac{41}{64}$ p: $337\frac{1}{2}$ m: RV: cu. $11397\frac{17}{64}$ m: $337\frac{1}{2}$, et cd erit residuum. Considerandum est ergo quod, supposita cd minore, problema potest componi, quia primum proportio quadrati ad ad quadratum ab, quanto minor est ab, tanto maior est in comparatione ad proportionem bc ad cd, tum quia ad est maior bc, tum quia sumimus proportionem quadratorum in primis et linearum in secundis. Et ideo cum augetur ab minor fit differentia proportionis quadrati ad ad quadratum ab ad proportionem bc ad cd. Et quia rursus necesse est ut proportio quadrati ad ad quadratum ab sit maior proportione bc ad cd; quia bc poterit esse minor cd, quia cd data est, quadratum autem ad semper est maius quadrato ab, cum sit totum ad partem comparatum. Crescit ergo proportio bc ad cd in comparatione quadrati ad ad quadratum ad, donec fiat ei æqualis, inde fit maior, et rursus ut dixi minor, ergo rursus fiet æqualis, et hæc causa duarum æstimationum, oportet igitur invenire maximam proportionem bc ad cd in comparatione quadrati ad ad quadratum ab. Quia ergo maximum parallelipedum ae fit ex bc in quadratum ab, cum ab fuerit dupla bc, igitur tunc maxima erit proportio eius ad parallelipedum cd in quadratum ab, quare tum minima proportio quadrati ad ad quadratum ab in comparatione bc ad cd. Et ita si sumantur duo puncta e et f ita ut ce in quadratum ea, vel cf in quadratum fa, efficiant parallelipeda singula æqualia parallelipedo cd in quadratum ad, tunc punctum b erit inter e et f, sed non æqualiter distabit. Sed quia hoc est

Ex 5 secundi Elementorum

Propositio 135

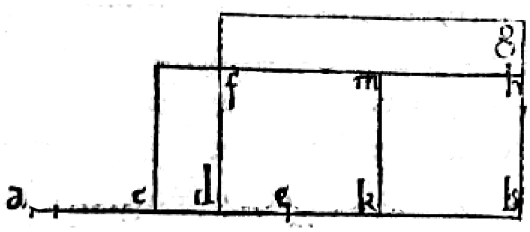
generale seu ae sit differentia ad et cd, seu quævis alia quantitas. Ideo oportet hoc invenire ex proprietate differentiæ coniuncta cum generali ratione dicta. Et ratione secundæ æstimationis inventa per primam sæpius dictam,

sit ergo ad, de data et punctum in ac maximæ proportionis bc ad cd in comparatione ad quadrati ad quadratum ab, b, et sit ce in quadratum ea datum ut sit æquale dc in quadratum da, et sit æstimatio data ce in quadratum ae, ut dixi, necessario ea, dico quod data est af similiter, et quod b est inter e et f, hoc enim est demonstratum. Tertio dico quod bf est minor be ita ut semper sit proximius b quam ipsum e. Cum igitur ex ratione inventionis secundæ æstimationis per primam ex tota ac numero quadratorum, oporteat detrahere ae primo inventam æstimationem et residuum, scilicet ec, ducere in ae cum quarta parte ec, quæ sit eg, ut ducatur ec in ag, et sumptum fuerit latus potens in illam superficiem, id est media inter ec et ag, et ei addita dimidia ce, quæ sit fh, et conflabitur af ex supposito, igitur ha est media inter ec et ag, ex his quæ dicta sunt, dico igitur quod f non poterit esse in ab, quia, si esset inter e et b, productum esset maius producto ec in quadratum ae et, si esset inter a et e, esset minus. Similiter si supponeretur cb et bf æquales, minus esset productum cf in quadratum fa quam ce in quadratum ea, ergo cum, ut demonstratum, quamto c prior est b tanto productum ce in quadratum ea est maius, igitur si debet minus quia in æquali distantia erat maius, necesse est ut eb sit maior bf, quod erat demonstrandum. Dico modo quod tota consideratio est in hoc, quia cd quæ assumpta est, variatur iuxta productum cf in quadratum fa, gratia exempli, et est numerus æstimationis, sed non sumitur a partibus ca, verum a tota solum, et ideo sumitur ca pro indivisa.

Si autem sumeretur pro divisa velut in e, vel b, vel f, non sumitur ae, ut differentia cd et da. Et iuxta hoc si, dicam proposita ab, volo eam dividere sic ut cubus ac sit æqualis ei quod fit ex ab in quadratum bc. Devenies ad cubum et res æqualia numero. Et eodem modo, si posita ab, bc, velis dividere ac in d ut cubus ad sit æqualis ductui, seu parallelipedo ab, bc, cd, pervenies ad 1 cu. et res æquales numero, et in ambobus supponitur quod latus cubi sit differentia laterum parallelipedi, adeo ut hic haberemus intentum, sed hic deficit unum, scilicet ut sit parallelipedum et non cubus.

Similiter notum est quod, cum fuerit proposita ab, quam velim dividere in c ut mutua parallelipeda sint decem. Gratia exempli, possum invenire parallelipedum ex bc in quadratum ca, quia, divisis 10 per ab, exit productum ac in cb notum, quare partes ac, cb. Igitur productum cb in quadratum ac notum erit. Et ponatur quod sit R cub. 10 mutuum, et ab sit 10. Gratia exempli, erit productum ac in cb Rcu: 100, quare ac 5 p: RV: 25 m: Rcu: 100 et cb 5 m: RV: 25 m: Rcu: 100. Inde habebis productum ut dixi. Et demonstratum est etiam quod eiusmodi producta sunt in proportione partium ae ad cb.

Per 5 secundi Elementorum



Et rursus, quia demonstratum est quod divisa quavis linea, puta ab, quomodolibet in c, proportio parallelipedorum mutuorum est ut partium. Et differentia illorum est parallelipedum ac in cb in differentiam ac et cb, sit igitur medium ab punctum e, erit ergo solidum ac in quadratum cb maius solido bc in quadratum ac solido dupli ce in superficiem ch, sit kb æqualis ac erit, ergo solidum ac in quadratum cb æquale solidis cubo bk et ae, quadratum ck, et parallelipedo ae in duplum ck in kb, quare cum ae sit æqualis kb, erit solidum ac in quadratum cb æquale solidis duobus uni quod constat ex ac in quadratum ck, solidum vero cb in quadratum ae, seu kb, est commune ei quod fit ex bc in quadratum ac, quoniam ac est æqualis kb et ak æqualis bc, igitur solidum ac in quadratum cb excedit solidum bc in quadratum ac in eo quod fit ex ck, in quadratum ca et ac in quadratum ck. Hoc autem est æquale ei quod fit ex duplo ce in superficiem ch. Quod enim fit ex ac in quadratum ck et ck in quadratum ac est æquale ei quod fit ex ak id quod fit ae in k. Dico ergo quod hoc est æquale ei quod fit ex duplo ce in ch. Id est ut proportio ch ad cm sit velut ak ad duplum ce. Nam ch ad cm est ut cb ad ck, ck autem est duplum ce, et ak æqualis cb, quia ac est æqualis kb, igitur per demonstrata ab Euclide proportio cb ad ck ut ak ad ce, quod fuit propositum.

Ex 143 propositione Libri de proportionibus

Per 1 sexti Elementorum

Ex quo patet maximum futurum parallelipedi ac in quadratum cb ad parallelipedum bc in quadratum ca, quoties proportio ce differentiæ ad de, differentia fuerit maxima in comparatione tetragoni partium rectanguli dg ad tetragonum rectangulum ch. Quo fit ut tale parallelipedum sit maximum, cum proportio ck ad ac facit maxima in comparatione quadrati ae ad ch, at proportio quadrati ae ad ch est ut quadrati ac ad quadratum ae detracta proportione confusa quadrati ae ad quadratum ce. Hæc autem duplicata ei quæ est ae ad ce. Maxima igitur differentia parallelipedorum, quoties proportio differentiæ partium ad dimidium quantitatis fuerit maxime propinqua proportioni quadrati dimidii ad seipsum detracta duplicata eiusdem dimidii ad dimidium illius differentiæ, nunquam autem potest esse ei æqualis. Et deducta ad numerum si ab ponatur 12, erit ae 6, et ac m: 1 pos., cb 6 p: 1 pos., et 1 cu. p: 108 æqualis 36 pos., et hoc esse non potest, igitur non potest æquari proportio. Ut ergo inveniamus maximum quod potest produci oportet, ut inveniamus numerum quem producit 24 in R 12 tertiæ partis, et producitur R 6912. Et hic est numerus maximus. Ideoq̀ue res R 12, scilicet tertiæ partis 36, igitur ac est 6 m: R 12, et bc 6 p: R 12, et parallelipedum R 27648, et est ferme 166, et partes quasi $9\frac{1}{2}$ et $2\frac{1}{2}$, et ideo in proportione, 24 divisi in 19 et 5. Et hoc non est mirum, sed quod mirum est, est quod cum parallelipedum ck in ch non sit annexum alteri aliorum, nam possum scire quodvis illorum ignorato parallelipedo ck ìn ch, et scire ck in ch, incognito utroque aliorum sicut etiam de parallelipedo ab in ch, hoc tamen sit notum aliud autem non. Et ideo id accidit, quia ab supponitur nota, sed ch præsupponitur incognita, est tamen magnum problema.

Per 7 quinti Elementorum

Ex 18 diffinitione Libri de proportionibus

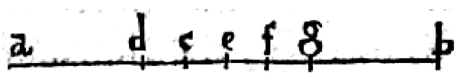
Iam vero habemus secundum modum principalem inventionis capituli cubi et numeri æqualium numero rerum. Posito enim quod velim scire 1 cub. p: 64 æquandum 36 rebus, ponam ad duplum R 36, et erit 12, et duplicabo 64 fit 128, et quæram divisionem ab in c ut ex ab in ch fiat 128, igitur diviso 128 per ab, quæ est 12, exibit ch $10\frac{2}{3}$, quare ac erit 6 m: R $25\frac{1}{3}$ et cb 6 p: R $25\frac{1}{3}$. Est autem divisa ba in c per æqualia, et propositum est dividere eam rursus in d per inæqualia ut sit proportio ae, dimidii ab, ad de, dimidium differentiæ db et da, velut dg ad ch, tunc enim erit parallelipedum ex de in db 64, et duplum de in dg 128, quemadmodum propositum est. Et ita proposito quovis numero qui possit produci ex 36, diviso in duas partes ita ut ex una in duplum R alterius fiat ille numerus, seu simpliciter

Per 5 secundi Elementorum

in R alterius producatur dimidium numeri propositi. Et ita habebimus capitulum generale. Constat autem in hoc casu quod ad erit 4, db 2 et dg 32, et cum de sit 2, erit duplum de, quod est 4, in dg 128 parallelipedum sensu inventum, sed hoc oportet invenire ratione. Habemus ergo, datam ab divisam per æqualia in e et per inæqualia in c, cognitas partes, et volumus dividere eam in d ut sit proportio da ad ca ad eam quæ est cb ad bd in proportione ae ad ed.

Caput XL

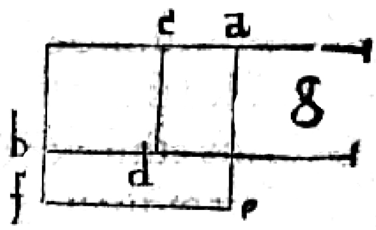
De tribus necessariis quæ præmittere oportet ad inventionem



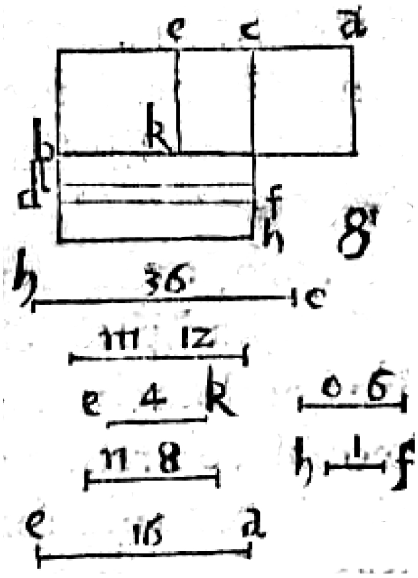
Si ergo de supponitur res, non potest esse numerus, et ad radix, quia esset ab tota R et non numerus propositus, neque radix, quia ad esset recisum, bd binomium, et producetur numerus simplex aut compositus cum radice per m: vel p:, igitur ductus in de R fieret binomium aut recisum aut R, igitur numerus æquationis non esset numerus. Pari ratione non potest de esse binomium aut recisum tertiæ nec sextæ speciei, qua non potest esse R simplex. Rursus proponatur de binomium, et sit dc numerus, et ef æqualis ec, et eg æqualis ed, erit ergo fg numerus, et cf R ex ad, ergo reciso in db binomium oportet ut fiat recisum simile binomio dg, ut ex eo in productum dg fiat numerus. Idem erit si ce ponatur numerus et dc R . Interest hoc solum an R sit maior numero an minor. Et in hac constitutione non potest de esse recisum, quia oportet assumere quantitatem maiorem de, et ita essemus extra casum regulæ et problematis. Semper ergo de est binomium. Et ponamus de 3 p: R 5, et erunt res 32 æquales cubo p: 24, et ae R 32, et similiter si de sit 3 m: R 5, sed non erit 3 contentum in de. Idem dico cum 1 cu. p: 12 æquatur 34 rebus, et est æstimatio 3 p: R 7 et 3 m: R 7, nam non potest verari nisi in binomio. Sed est aliud cum 1 cub. p: 8 æquatur 18 pos., nam res est R 6 m: 2, et non potest contingere in binomio. Igitur prima duo exempla sunt idonea. Et quia in his addere oportet aliquem numerum qui ductus in R totius producat numerum æquationis, et manifestum est quod non potest esse R neque binomium neque recisum, non enim conficeret numerum, ideo oportet ut sit numerus, nos autem iam supponimus hic esse quadratum. Proponamus ergo ae 8 et quadratum illius sit 64 numerus rerum; et sit ut addendo 17 fiat alius quadratum, scilicet 81, cuius R, quæ est 9, ducta in 17 additum faciat 153, erit igitur 1 cu. p: 153 æqualis 64 rebus, et rei æstimatio $4\frac{1}{4}$ p: R $3\frac{1}{4}$, id est dimidium

Vide supra Capitulum 5

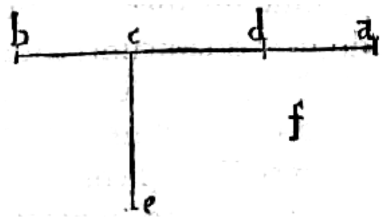
R totius p: R differentiæ numeri æquationis et $\frac{3}{4}$ numeri aggregati. Erit ergo, posita ce R $3\frac{1}{4}$, et cd [de?] $4\frac{1}{2}$, ad $3\frac{1}{2}$ m: R $3\frac{1}{4}$, et db $12\frac{1}{2}$ p: R $3\frac{1}{4}$, et dg 9 p: R 13. Est productum ad in db $40\frac{1}{2}$ m: R $263\frac{1}{4}$, hoc ergo ductum in de, scilicet $4\frac{1}{2}$ p: R $3\frac{1}{4}$, fit 153. Possumus ergo dividere etiam 64 in duas partes, ex quarum una in R alterius fiat 153, et quia R illa est res, et est de, ducemus de in se, et fit $23\frac{1}{2}$ p: R $263\frac{1}{4}$, igitur reliqua pars est $40\frac{1}{2}$ m: R $263\frac{1}{4}$, ecce quod res redit ad idem. Ex dg igitur in productum ad in db fit 306, quod divisum per 16 exit $19\frac{1}{8}$, igitur partes erunt 8 p: R $44\frac{7}{8}$ et 8 m: R $44\frac{7}{8}$. Reducuntur ergo hi duo modi ad unum,



velut si sit numerus rerum propositum ab, et g numerus æquationis, et seu diviseris ab in quadratum bc ut latus eius bd in superficiem da faciat g, seu addiderit bc superficiem ad ab ut tota af fiet quadrata et latus eius ae in additam be producat g, fient notæ æstimationes in prima quidem latus bd, in secunda dimidium ae addito aut detracto latere differentiæ dodrantis af et superficiei ab propositæ. At quoniam ae in cb est æquale g, et bd in da æquale eidem g, erit ae in cb æquale cd in da, ea igitur ad cd ut ad ad cb, at ae maior est cd, ergo ad maior eb, cumque bc et af quadratæ sint, erit ad æqualis df, igitur df maior cb, quod esse non potest. Non potest igitur divisio una esse.



Oportet igitur ut sit superficies ab æqualis numero rerum eiusmodi ut be pars quadrata sit, cuius latus ek in ka sit æquale g numero æquationis, et rursus sit cd æqualis ab, cui desit ad complendum quadratum superficies hd ita ut ex ch rursus in hd fiat idem g, et b erit in utroque casu nota res; scilicet in primo ek, in secundo dimidium ch cum ea quæ potest in superficiem fl posita lc, dodrante lc.



Quia ergo hc ad ek est ut ak ad dh, erit hc ad ek duplicata ad proportionem mediæ ac,[1] ek ad mediam inter hc et hf. Sit igitur m media inter hc et ck, igitur hc ad m ut mediæ inter ac, ek, quæ sit n ad mediam inter ac, ek, quæ sit n, ad mediam inter he, hf, quæ sit o, igitur hc ad m ut n ad o. Et erunt tres ordines coniuncti ad duo extrema ek, n, ea, ek, m ,hc, et hc, o, hf.

Et rursus cum dixerit quis divisi ab in c, et fuit cd differentia partium, ex qua in ce mediam inter partes producitur f. Dico quod habebo 1 quad. quad. p: quarta parte quadrati f æqualia quadratis numero, quadrati dimidii ab, et ideo f non potest excedere quadratum dimidii ab, seu quartam partem quadrati ab,

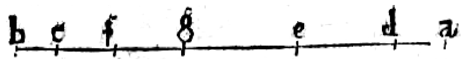
[1]1570 and 1663 have "*ae*".

velut si ab sit 8, eb, fb, ba, bc, bo, 1 quad. quad. p: 9 æqualia 16 (quadrato 4 dimidii ab) quadratis scilicet; quare res erit RV: 8 m: R 55 et duplum eius, id est RV: 32 R 880, erit quantitas cd differentiæ partium. Et ideo problema est ut, cum sciam quantitatem ab et modum inveniendi productum ex cd in ce ut sit æquale f, si invenero modum ut ex cd in productum bc in ac, quod est quadratum ce, fiat idem f, inventum erit capitulum. Sed variantur partes, scilicet cd et ce, in uno et altero problemate.

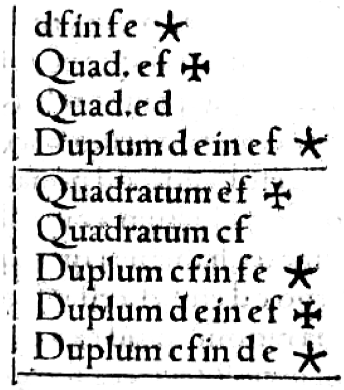
Rursus cum ex cd differentia partium in productum ac in ab fit f et bc sit æqualis ad, erit ut ex ac in ad, et post producto in cd fiat f. Ergo si cd esset media proportione inter ac et ad, esset ac divisa in d secundum proportionem habentem medium et duo extrema. Et si productum sic esset, esset cd Rcu: f; quoniam productum ac in ad est semper in aliqua proportione cum quadrato cd, vel maioris vel minoris, et ea sumitur in æquali proportione semper ab 1 quad. p: numero rerum lineæ divisæ æqualibus quadrato eiusdem. Aut 1 quad. p: quadrato numeri lineæ divisæ æqualibus rebus in triplo numeri rerum ut si linea divisa in 10, habebo 1 quad. p: 10 rebus æqualia 100, vel 1 quad. p: 100 æqualia 30 rebus, et æstimatio semper erit eadem. Et si quadratum cd sit duplum aut triplum producto ac in ad habebimus, id est quad. p: multiplici eiusdem numeri rerum æqualia multiplici quadrati eiusdem numeri, aut 1 quad. p: quadrato numeri eiusdem lineæ divisæ æqualia rebus ductis per conversum proportionis p: 2. Exemplum in quadrupla proportione antea fuit 1 quad. p: 10 rebus æqualia 100, vel 1 quad. p: 100 æqualia 30 rebus, hic habebo 1 quad. p: 40 rebus æqualia 400, vel 1 quad. p: 100 æqualia 60 rebus, qui numerus producitur ex 4 numero proportionis, et 2 assumpto ex regula. Et res seu æstimatio est eadem vel, si productum fuerit multiplex ex quadrato, assumemus contrario modo, vel 1 quad. cum rebus sumptis secundum illam partem æqualia parti eidem quadrati lineæ divisæ: vel 1 quad. p: quadrato eiusdem lineæ divisæ æqualia rebus duplo proportione eadem lineæ divisæ; et res redit ad idem. Et exemplum est clarum.

Ex quo tandem patet quod assumpta ab ut in præsenti capitulo, quæ sit 12, et ex cd differentia in productum ac in cb fiat 8, habemus 1 cu. p: 4 æqualia 36 pos., hoc enim demonstratum est. Ergo ac erit divisa in d, eo modo ut ex ac in ad, inde in cd fiat 8, et rei æstimatio erit dimidium cd. Ergo cd duplum æstimationis, et residui dimidium ad vel bc, si ergo cd esset Rcu: 8, id est 2, erit da R 5 m: et ca R 5 p: 1, et ideo tota ab R 20. Si quis ergo dicat fac ex R 20 duas partes ex quarum ductu rectanguli earum in differentiam fiunt 8, habebis partes bc R 5 m: 1, ca R 5 p: 1, productum quarum est 4, quod ductum in cd, quæ est differentia, et est 2, producit 8. Et habebimus 1 cu. p: 4 æqualia 5 rebus. Et fundamentum ab est potentia tantum rhete. Si ergo 1 cu. p: 6 æquatur 7 rebus, res potest esse 1 et 2, ut palam est. Ergo, si cd ponatur 2, habebimus posita da 1 pos., 2 quad. p: 4 pos. productum ac in ad et in ed [cd] æqualia 6, erit ad 1. Et si ponatur cd 1, habebis 1 quad. p: 1 pos. æqualia 6, igitur ad est 2, quando ergo cd est 2, da est 1, et quando cd est 1, da est 2.

Sed supposita prima ratione quod ex ac, cd, da in continua proportione fiat 8, et cd sit Rcu: 8, scilicet 2, si ergo cd quadratum esset quadruplum rectangulo ac in ad hoc habet rationem, hoc modo quod enim fit ex ac in ad est æquale ei quod fit ex cd, da in ad, assumatur de dupla da et df quadrupla eidem quadratum, igitur de est æquale quadruplo quadrati ad, et quadratum cd est æquale quadratis ce, ed et duplo ce in ed, igitur duplum de in ec, et est df in ce cum quadrato ec, est æquale quadruplo cd in da, id est ei quod fit ex fd in dc semel, hoc autem est æquale ei quod fit ex fd, id, ce, et ed, detracto igitur communi, eo quod fit ex fd in ce relinquetur quadratum ce æquale ei quod fit ex fd in de, est autem fd quadrupla da et e [ed] dupla eidem da, igitur ce potest in octuplum da. Ponatur ergo ae qualiscunque numerus, puta 10, cum ea sit triplum da et ce R octupli quadrati da, erit tota ca 3 p: R 8 in numero rerum, et hoc æquatur 10, igitur res, scilicet da, est diviso 10 per 3 p: R 830 m: R 800, ergo cd residuum erit R 810 m: 20, ex tota igitur ac in da fit 300 m: R 80000, et hoc est quarta pars quadrati cd, scilicet 1200 m: R 320000, sicut proponebatur.



Rursus dicamus quod quadratum cd ad sexcuplum ei quod ex ca in ad, assumam df sexcuplam ut in priori quadruplam da, et similiter de media inter df et da, nam et in priori constitutione de fuit media inter fd et da, et assumam ge æqualem cd, sicut in priore, sed eg fuit ibi ipsa ef, hic autem est minor eo quod proportio est maior quadrupla, et tunc quadratum de est sexcuplum quadrato da, quia est æquales ei quod est ex fd in da, igitur ex supposito quod fit ex dg in ce cum quadrato ce est sexcuplum cd in da, seu æquale ei quod fit ex cd in df, seu quadrato df cum eo quod fit ex df in fe, dividamus ergo utranque, et fient partes (ut vides),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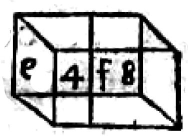
auferantur utrinque quadrata ef duplum de in ef, relinquentur df in fe, et quadratum ed æqualia quadrato cf, duplo cf in fe, et duplo cf in de, at df in fe est æquale ei quod fit ex ef in fe, et de semel eo quod df est æqualis fe et ed, iunctis igitur quadratum ed est æquale ei quod fit ex ef in se in fe,[2] et ed, et est tota ed. Quadratum autem ed est æquale sexcuplo quadrati da, igitur quod fit ex ef in cd est sexcuplum quadrati da. Ponatur ergo da 1 pos., d [df?] fecit 6 pos., tota fa 7 pos., si igitur ponamus ca 10, ut prius erit ef [cf] 10 m: 7 pos., cd autem 10 m: 1 pos., duc invicem fient 100 m: 80 pos. p: 7 quad. æqualia 6 quad., et ita vides quod res reducitur in quovis casu ad 1 quad., cum quadrato numeri propositi et numerus rerum semper fit ex numero proposito, utpote 10, in numerus proportionis p: 2, proportio fuit sexcupla, et ideo addito 2 fiet 8, et positiones 80, ergo reducetur ad regulam de modo sic. Proponitur linea ac 10 et proportio sexcupla, adde 2, fit 8, duc in 10, fit 80, accipe dimidium et est 40, duc in se, fit 1600, aufer 100 quadratum 10, relinquitur 1500, cuius R detracta a 40, efficit 40 m: R 1500 quantitatem da.

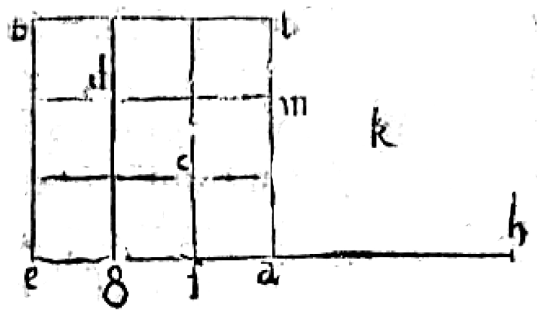
[2] 1663 has "ex ef in e in fe".



Ergo ut ad rem deveniam si quis dicat 1 cu. p: 4 æquatur 12 rebus, capiam ab duplum R 12 et est R 48, et f corpus duplum 4 et est 8, et dividam ab per æqualia in R 12, et addam et minuam 1 pos., et fiat eb R 12 p: 1 pos., R ae R 12 m: 1 pos., et productum erit 12 m: 1 quad., ducamque illud in ed differentiam ae et eb, facta db æquali ae, et fient 24 pos. p: 2 cub. æqualia 8, igitur 1 cub. p: 4 æqualis 12 pos., cum ergo dimidium ed sit rei æstimatio, et tota ab numerus aut potentia rhete. Erit primum ut ag sit numerus aut potentia rhete. Inde ut cum ex eb in bd et in de, fiat ut dixi f supposita bd numero, utpote 1, erunt quadrata et res æqualia 8, hoc enim est suppositum, et habebimus 1 q p: 2 pos. æqualia 2.

Corollarium. Constat autem quod proportio cubi cd ad parallelipedum cb, bd, dc est semper veluti quadrati cb ad rectangulum ex cd in db, quare cb ad latus parallelipedi eiusdem subtriplicata ei quæ est quadrati cb ad rectangulum cd in db, at cb ad mediam inter cd, db subduplicata ei quæ est quadrati cb ad rectangulum cd in db, lateris igitur solidi cb, cd, db, ad latus rectanguli ed in db, est ut R quad. $4\frac{1}{2}$ ad Rcu: $4\frac{1}{2}$.

Cum volueris dividere ba ut proportio eiusdem ad rectangulum ad in db, sit 24, gratia exempli, divide quadratum ba per 24, et quod exit detrahe ex quadrati dimidii ba, et R residui addita et detracta a dimidio ostendit partes, ut si ab sit 10, ducam in se, fit 100, divido per 24, exit $4\frac{1}{6}$, detraho ex 25 quadrato ag, relinquitur $20\frac{5}{6}$, cuius R addita 5, dimidio 10, et detracta ostendit

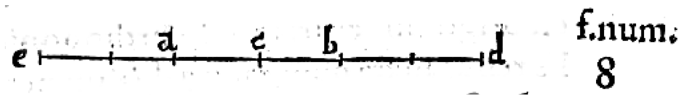


5 secundi Elementorum

partes ut pote ad, db, et habetur ex Euclide. Iam vero constituatur ab quadratum 7, et ac 1, et ad 4, erit ergo ae R 7, af 1, ag 2, et sit eh dupla ea, et erit R 28, et sit numerus kb, sit ergo cubus ac p: 6 æqualis 7 rebus, et item cubus ag p: eodem numero 6 æqualis 7 rebus. Quia ergo ab est 7, erit corpus ab, posita af altitudine et re 7 res, hoc autem corpus æquale est 1 cu., id est cubo af cum b, autem i gnomo lcbf iuxta altitudinem af, et similiter corpus ex ab in ag est æquale gnomoni ldbg in ag cum cubo ag, quare gnomo ldbg in ag est 6. Igitur divisa erit bifariam ab superficies, ut ex latere unius partis in reliquam fiat seu b. Et item divisa erit bifariam eh in a per æqualia, ut ex af et ag, ductis in quadratum ae, seu productum ah in ae fiant 7 res. Quia ab iam supponitur 7, et af et ag res. Et rursus divisa erit ch bifariam in f et in g ut productum bf in fe sit æquale gnomoni lcbf et in g ut productum lg in ge sit æquale gnomoni ldbg.

Per demonstratas 5 secundi Elementorum

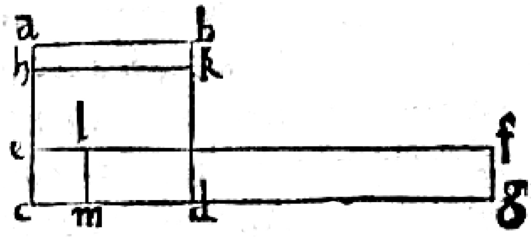
Unde unumquodque horum per primam partem huius ductum per differentiam a medietate, id est hf in fe per fa, et hg in ge per ga, producit eundem numerum k, seu b.



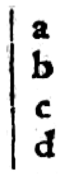
Iam vero sit cubus et 8 æqualis 8 rebus res 2, erit ut ducas 1 dimidium 2 in se fit

Per 12 Capitulum Artis magnæ

1, triplica fit 3, deducito numero rerum, relinquitur cuius R m: 1 dimidio prioris æstimationis R 5 m: 1 est secunda æstimatio. Ponam ergo f numerum 8, et ab 2 primam æstimationem, et bd R 5 m: 1 secundam æstimationem, et ideo posita bc, erit cd R 20, dupla ipsi cd, et ae R 5 m: 1 æqualis bd, ponam ergo ad R 5 p: 1 pos., ae R 5 m: 1 pos., ductæ invicem producunt 5 m: 1 quad., duco in ab, fiunt 10 pos. m: 2 cu. æqualia 8, igitur 1 cu. p: 4 æqualia 5, et res est eadem 2 et R 5 m: 1, ergo sub eidem æstimationibus fit transitus, sed non sine cognitione prioris æstimationis per quam devenio ad scientiam de, quæ est R 20. Dictum est etiam supra quod, si capiam duplum R numeri rerum, et est R 32, et dividam in R 8 p: 1 pos. et R 8 m: 1 pos., fiet 8 m: 1 quad., et ducto in 2 pos. fient 16 pos. m: 2 æqualia 16, et redibit ad 1 cu. p: 8 æqualia 8 rebus. In hac igitur per non nota invenitur aliquid novum in illo per nota invenitur aliquid, sed est idem, nam cd supponitur in priore R 32, hic R 20.

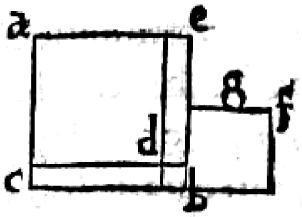


Rursus proponantur duæ superficies æquales rectangulæ abcd et cefg, et sint æquales numero rerum, et sint quadrata in eis chkd et celm ita ut ex latere illorum in reliquum suæ superficiei fiat numerus idem, qui sit n. Constat igitur tam ce quam ca esse rei æstimationem, cumque ex ce in lg fiat n, et ex ch in hb, idem enim fient etiam ex gm in mc, et ex ah in hd, quare gm ad ah duplicata ei quæ est hc ad ce. Igitur posita gm prima, ah quarta, ch secunda, ce tertia, erit ergo quod fit ex prima et tertia in tertiam, scilicet superficies eg, æqualis quod fit ex secunda et quarta in secundam, scilicet superficies ad. Et rursus quod fit ex prima in quadratum tertiæ æquale ei quod fit ex quarta in quadratum secundæ. Constituetur igitur problema sic.



Sunt quatuor quantitates ordinatim a, b, c, d, quarum proportio a ad d est duplicata ei quæ est b ad c. Et quod fit ex a, c in c est æquale ei quod ex d, b in b, et quod fit ex a in quadratum c est æquale ei quod fit ex d, b in b. Ex quibus sequitur quartum, quod proportio eius quod fit ex a in quadratum c ad id quod fit ex a, c in c est veluti eius quod fit ex d in quadratum b ad id quod fit ex d, b in b. Et permutando etiam, sed illud est perspicuum cum sit proportio æqualis ad æquale.

Dico præterea quod regula Artis magnæ quæ docet assumere radicem aggregati ex numero rerum et numero æquationis diviso per illam et sola est generalis illi capitulo, et est demonstrata ibi. Et est origo eius ex triangulo orthogonio,

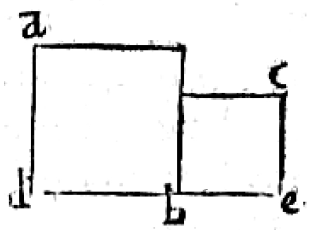


nam si sit cubus bc æqualis rebus iuxta numerum ad et numero g, erit ergo ex communi animi sententia g ex bc in gnomonem cde, fiat ergo bf quadratum æquale cde gnomoni, eritque cubus bc æqualis bc in ad et bf, sed quadratum bc, quod est ab, æquale est ad et bf, igitur latera ad et bf continent rectum contentum bc. Hæc igitur æstimatio satisfacit in omni æquatione seu numerus rerum sit parvus seu magnus.

Caput XLI

De difficillimo problemate quod facillimum videtur

Nihil est admirabilius quam cum sub facili quæstione latet difficillimus scrupulus, huiusmodi est hic.



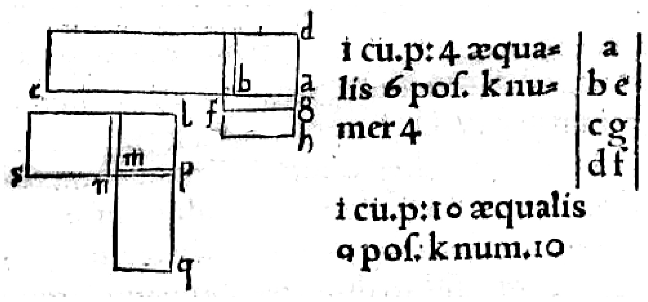
Quadratum ab cum latere be[1] est 10, et quadratum bc cum latere bd est 8, quæritur quantum sit unum horum seu latus seu quadratum? Quia ergo abe[2] est 10, et ab 1 quad., erit be 10 m: 1 quad., igitur bc 100 m: 20 quad. p: 1 quad. quad., igitur cbd erit 100 m: 20 quad. p: 1 pos. p: 1 quad. quad., et hoc est æquale 8, quare 1 quad. quad. p: 92 æquatur 20 quad. m: 1 pos., adde 19 quad. utrinque, fient 1 quad. quad. p: 19 quadrat. p: 92 æqualia 39 quad. m: 1 pos., detrahe $1\frac{3}{4}$, erunt 1 quad. quad. p: 19 quad. p: $90\frac{1}{4}$ æqualia 39 quad. m: 1 pos. m: $1\frac{3}{4}$, inde adde 2 pos. p: 1 quad. utrinque ut in Arte magna, et videbis difficillimam quæstionem.

[1] 1570 and 1663 have "*bc*".
[2] 1570 and 1663 have "*abc*".

Caput XLII

De duplici æquatione comparanda in capitulo cubi et numeri æqualium rebus

Et proponatur cubus et 4 æquales 6 rebus, et rei æstimatio est 2, et altera R 3 m: 1, et rursus ponatur cubus et 10 æqualia 9 rebus, et æstimatio est idem 2, altera R 6 m: 1, et manifestum quod prior æstimatio, scilicet maior, satisfacit diversis imo infinitis problematibus. At in reliqua fieri nullo modo potest ut neque in una cum neutra fuerit numerus, velut pro 1 cu. p: 12 æqualibus 34 rebus p: R 17, neque 3 m: R 7, nam posita re ut pote 3 m: R 7 cubus est semper 90 m: R 8092, ergo R non potest continere R nisi 34 vicibus, igitur cubus ille cum numero non potest æquari alteri numero rerum quam 34, et hoc est valde admiratione dignum.



Dispositis ergo fd et nl æqualibus, scilicet 4, quadrato 2, et ab R 3 m: 1, et mo R 6 m: 1, ponam a æqualem gd, b æqualem bc, c æqualem gh, d æqualem ab, e æqualem aq, f æqualem mo, g æqualem nf. Ex his sequuntur quinque principalia.

Corollarium. Si quadratum a auferatur ex numero rerum, et cum residuo dividatur numerus æquationis, prodibit ipsum a communis æstimatio, veluti 1 cu. p: 4 æquatur 6 rebus, et 1 cu. p: 10 æquatur 9 rebus, et communis æstimatio quæ est a est 2, duco o in se, fit 4, detraho ex 6 et 9 numeris rerum, relinquuntur 2 et 5, divido 4 numerum æquationis primæ per 2 et 10 numerus æquationis secundæ per 5, exit a in utroque, scilicet ipsum a.

Corollarium 2. Sequitur etiam quod cum ex dictis fiant ex g et c in quadratum ak et k numeri æquationis, ut sit g ad c ut q ad R et quia quod fit ex c in quadratum a est æquale ei quod fit ex b in quadratum d, et ex g in quadratum a æquale ei quod fit ex b in quadratum d, et ex g in quadratum a æquale ei quod fit ex e in quadratum f, erit quod fit ex b in quadratum d ad id quod fit ex e in quadratum f, velut c ad g.

4 p: R 12	1	R 3 m: 1
2 b	c	d
a e	g	f
		R 6 m: 1

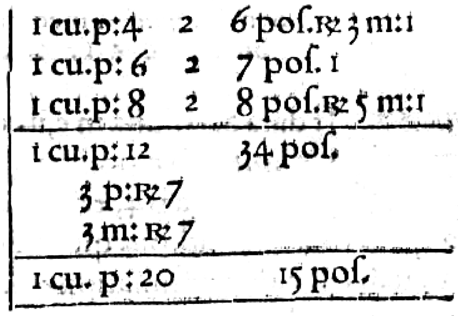
Et est probatum exemplum ex 7 m: R 24, quod est quadratum f in $3\frac{4}{3}$ p: R $3\frac{21}{25}$, fit 10.

Per eandem

Corollarium 3. Rursus quia quod fit ex c et a in a est æquale ei quod fit ex bd in d, et ex ga in a ei quod ex ef in f, erit quod fit ex bd in d ad id quod ex ef in f, velut c ad g, fit enim ex bd in d R 12, et ex ef in f R 75, et est proportio ut 1 ad $2\frac{1}{2}$.

Corollarium 4. Cumque æstimatio (ut dixi) non potuerit esse communis pluribus numeris rerum, et numeris æquationis commutabitur necessario si fuerit binomium in suum recisum, et ita habebis et secundam æquationem et numerum communem qui erit idem, velut 1 cu. p: 12 æqualis 34 rebus. Non se offert primo illa pars quæ ducta in R alterius efficit 12, sed est tamen 18 p: R 252, alia est 16 m: R 252, cuius R est 3 m: R , cum ergo habes 3 m: R , duc in se, et fit 16 m: R 252, et quia R 7 est sexta pars R 252, ideo oportet assumere numerum sexcuplum ad 3, et est 18 cum R 252 per p:, et addere ad 16 m: R 252, habes 34 ad unguem. Et vicissim si habueris 3 p: R 7, habebis quadratum 16 p: R 252, et ita reliquus erit sexcuplus ad 3 p: R 7, sed R erit m:, ideoque 18 m: R 252, et ita vicissim invenies ex æstimationes partes, una erit quadratum, alia erit multiplex, ut R radicis, sed contrario modo binomium pro reciso et recisum pro binomio.

Corollarium 5. Iam ergo habes duos ordines æstimationum. Primus cum eadem æstimatio est communis aliis numeris rerum et æquationum, et invenire licet illos ducendo in se, detrahendoque a quovis numero, et cum residuo dividere alium numerum, ut prodeat eadem æstimatio; ut in primo corollario. Secundus, cum æstimatio est binomium vel recisum, et ducitur in se, et detrahitur a numero aliquo ita ut residuum habeat eandem proportionem ad partem, quæ est numerus, quam R quæ est pars quadrati ad R , quæ est pars æstimationis. Et illa proportio est duplum numeri æstimationis semper, ideo numerus ille est semper duplum quadrati numeri æstimationis, ut in quarto seu præcedenti corollario.[1] Velut si numerus æstimationis fuerit 2, erit talis numerus 8, si 3, 18, si 4, 32, et ita deinceps, reliquus autem numerus erit compositus ex quadratis partium æstimationis, ut si partes sint 3 p: R 7 vel m: R 7, erit 16, igitur totus numerus erit 34. Ergo tertius modus qui quæritur erit diversus ab his, et non erit per viam recisi et binomii, neque ut eadem æstimatio serviat pluribus, velut in margine vides,



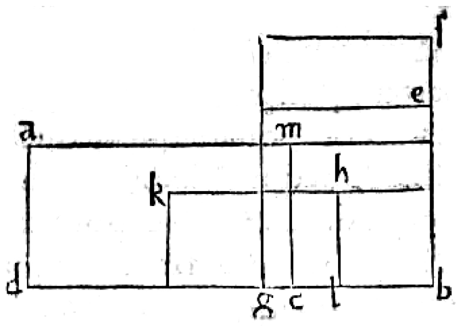
1 cu.p:4	2	6 pos.R 3 m:1
1 cu.p:6	2	7 pos. 1
1 cu.p:8	2	8 pos.R 5 m:1
1 cu.p:12		34 pos.
3 p:R 7		
3 m:R 7		
1 cu. p:20		15 pos.

quod singulis sunt duæ æstimationes 1 cu. p: 20 æquali 15 pos. neutrum contingit non primum, quia 2 est minus, et 3 est maius, neque potest esse pars numeri. Nec secundum, quia oportet ut addito 1 vel 10 ad 15 R 16 vel 25, dividendo 20, produceret idem 1 vel 10 et non fit, nam exeunt 4 vel 5.

[1]1570 and 1663 have "*corrolario*".

Caput XLIII

De comparatione numeri æquationis ad partes numeri rerum



Sit ab superius 12, et ex bc latere tertiæ partis in ca fit 16, maximum quod esse potest. Sit ergo bf æqualis ab, et quadrata superficies ge, ex cuius latere in residuum ef fiat 8, et hæc divisio est quam quærimus. Sit ergo bk, cuius tertia pars sit quadratum bh, ex cuius latere in residuum esset, fiat 8, erit ergo bl Rcu: 4, bh Rcu: 16, lk R cub. 128, qua ducta in bl fit Rcu: 512, scilicet 8. Igitur tota bk est cu. 432. Habemus ergo duo nota bc in ca, sed productum non est 8, bl in lk, quorum productum est 8, sed bk non est 12, et bg in ef, et est 8, et bf 12, sed non est nota divisio facta e. Proportio ergo ac ad kl est ut quadrati bc ad quadratum bl, quare ut bc ad bl duplicata. Cum vero proportio solidi bc in ca, sit dupla ad solidum ex bl in lk, erit ca ad lk velut quadrati proportionis ad R cub. quad. quad. proportionis, et bc ad bl ut proportionis ad Rcu: quad. quad. proportionis. Proportio autem kl ad ef est ut cb ad bl, quare be ad bl duplicata ei quæ est kl tetragonici ad ef tetragonicum. Habet ergo divisio bk per lh proportionem notam in omnibus partibus, ut liquet, cum ba divisa in c. Et habet etiam proportionem notam cum bf, divisa in e, quia ut dixi proportio kl ad ef ut eb ad bl est autem eg ad bh duplicata ei quæ est eb ad bl. Si ergo coniungantur hæ proportiones, quoniam extremorum componitur ex intermediis, et maxime quod differentia eg et ac est æqualis differentiæ quadrati bc et ef, seu

Per 34 undecimi Elementorum

gnomo emg æqualis differentiæ ac et fe.

Caput XLIV

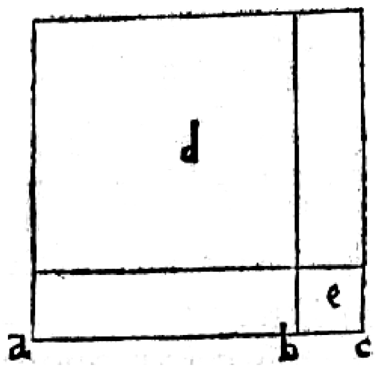
Quomodo dividatur data linea secundum proportionem habentem medium, et duo extrema in corporibus

Sit data ab divisa in c ut ex ab in quadratum ac fiat cubus, bc posita 1 quad., et ponamus ab 4, erit 1 cu. quad. æqualis 64 m: 32 quad p: 4 quad. quad., igitur ℞ radici 1 cu. p: 2 quad. æqualis 8, cuius æstimatione habita quadratum est quantitas bc quæ quærebatur.

Caput XLV

Quomodo partes divisæ lineæ corporibus et quadratis invicem comparentur

Et sint de quadrata 26, et de cubi 126, et compleatur superficies quadrata, et erit cubus p: 252, duplo 126, semper æqualis 78 rebus triplo numeri æqualis quadratis deiunctis.

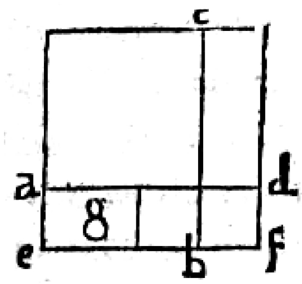


Et hoc ex regula, posita ac 1 pos., fient enim partes $\frac{1}{2}$ pos. p: RV: 13 m: $\frac{1}{4}$ quad. et $\frac{1}{2}$ pos. m: RV: 13 m: $\frac{1}{4}$ quad., quæ deductæ ad cubos ostendunt quod dixi. Et rursus si ponantur de quadrata 26, et corpora ex d in bc bis et b in ab bis 60, erit 1 cu. æqualis 26 rebus, numero quadratorum, et 60, duplo producti mutui, et res est in capitulo. Iam ergo ex hoc supposito sciemus quanta sit ac, quæ est b, et partes et æstimationem cubi p: 252 æqualium 78 rebus, quo proposito accipiemus $\frac{1}{3}$78 et $\frac{1}{2}$ de 252, et convertetur quæsitum in duo quadrata, quæ iuncta faciunt 26, et duo cubi, qui sunt 126. Et quia propositum est quod productum unius in alterum mutuo est 30, si hoc sciremus manifestum esset capitulum. Sunt ergo quatuor, quantitas ac et est 6, quantitas de et est 26, quantitas corporum mutuorum et est 30, quantitas cuborum et est 126. Illud accedit quod, si dicam quadrata sint 25, et cubi non poterunt esse maiores 125, cubo 5, R 25, igitur cum neque possint esse minores R 7812$\frac{1}{2}$ duplo, scilicet cubi R medietatis 25, quæ est R 12$\frac{1}{2}$, ut sit circumscripta inter 88, qui est R ferme 7812$\frac{1}{2}$ et 125, et præter id cum dico 1 cub. æquatur 6 rebus p: 9, manifestum est quod numerus 9 datur cubis non parallelipedis, ut etiam hic, ideo erit nota pars huius capituli cubi et numeri æqualium rebus. Et est valde dignum consideratione. Nam ut statuantur cubi æquales 126 et quadrata 26, ut dictum est, poterimus loco 26 assumere quemcunque numerum minorem pro quadratis usque ad 14, ut dicamus, quadrata de sint 14, vel 15, vel 16, et ita ad 25 usque et cubi sint 126, igitur ex regula præsenti cubus p: 252 æquabitur 42 rebus, vel 45, vel 48, et ita usque ad 78, et ita in intermediis eadem ratione scilicet 43, 44, 46, 47 rebus, et ita de singulis. Et variato numero 252 habebimus alios, ergo, habita hac regula, habebimus capitulum perfectum. Et tamen (ut dixi) in supposito habemus partem regulæ notam.

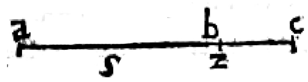
Et sane hoc est (ut in exemplo maneamus) iam notum quod, si quis dicat cubi ab, bc sunt 126, quadrata 26, quod numerus tribuitur cubis, et si 26 esset numerus rerum, aut numerus mutuorum solidorum, iam omnia essent nota. Et rursus, si dico quod 30 est numerus solidorum et 26 rerum iam habeo 1 cu. æqualem 26 rebus p: 60, et res est nota. Et si dico quadrata sunt 26 et parallelipeda 30, devenimus ad 1 cub. quad. p: 2028 quad. p: 3120 pos. æqualia 104 quad. quadrat. p: 3600, et hac via non habemus capitulum. Et mirum est quod cum assumimus 26 pro numero rerum et 60 pro solidis, aut 30, hic numerus transeat in cubos, quamvis sit mutuorum solidorum et, cum accipitur numerus pro cubis et quadrata pro alio numero, hæc transeant in res, et numerus cuborum in residuum rerum detracto cubo, quasi numerus rerum componatur ex tribus cubis.

Caput XLVI

Quomodo proposito rectangulo, et cubis laterum eius habeamus totum cubum



Et proponatur rectangulum ab, puta 4, et cubi laterum ac, bd 20, dico cubum notum esse, quia enim cubi ac, bd sunt 20, oportet facere ex 20 duas partes, quarum Rcu: ductæ invicem faciant 4, superficiem ab, igitur cubi invicem ducti facient cubum 4, qui est 64. Partes igitur, id est cubi ac, bd sunt 16 et 4 et Rcu: earum sunt latera ab igitur cubus totus est 20 p: Rcu: 27648 p: Rcu: 6912. Et si ponantur acbd nota ut quantitas rerum et corpora abcd iuxta altitudinem, erunt duo tantum, quia sub numero rerum acbd, ut pote 13, continent duo mutua et reliqua quatuor sub ab et cd, id est sub 60.



Igitur ac et bd numerus rerum si fuerit parvus, erit capitulum per se notum ex regula Artis magnæ. Si autem fuerit magnus, velut cu. 24 rebus p: 5, tunc ex præsenti problemate si possit reduci ad hoc, ut separentur mutua, erit propositum necessarium, scilicet ut accepto dimidio 5, et est $2\frac{1}{2}$, invenias duos numeros qui producant[1] $2\frac{1}{2}$, divisum per rem, et eorum cubi faciant 24 m: $2\frac{1}{2}$, id est $21\frac{1}{2}$, nam ut dixi in 24 continentur ambo acbd et duo mutua. Istud ergo non est per se notum. Invenias numerum qui divisis producat 6, tamquam superficiem ab, et ipse sit æqualis cubi ab et cd duobusque mutuis, aut quatuor, nam posito uno

[1]1570 has "$2\frac{1}{3}$".

1 pos., altero $\frac{6}{1\text{ pos.}}$, erunt 1 cu. p: $\frac{216}{1\text{ cu.}}$ cum 6 pos. p: $\frac{36}{1\text{ pos.}}$ vel cum 12 pos. p: $\frac{32}{1\text{ pos.}}$ æqualia 65 gratia exempli, igitur 1 cu. quad p: 6 quad. quad. p: 36 quad. p: 216 vel 1 cu. quad. p: 12 quad. quad. p: 72 quad. p: 216 æqualia sunt 65 cu., hoc ergo valde est obscurum, et oporteret ut haberet Rcu:. Verum quia ponitur 65 cu. ac et bd et duo mutua, et æquantur duo cubi cum duobus mutuis ac et bd in ef, ut nuper dixi, igitur ef quæ est res in ac et bd est 65, at ef in ab est 6 res ex supposito, et in cd 6 res, quoniam ab et cd sunt æquales, quia sunt supplementa circa diametrum, igitur ef in ab, cd sunt 12 res, et ef in ac, bd 65, et ef in ac, bd, ab, cd complet cubum ef, igitur cubus ef æquatur 12 rebus p: 65 et res est nota, puta 5, ex qua habetur æstimatio illa, fac de 5 duas partes quæ producant 6, et erunt 3 et 2, erit ergo res $2\frac{1}{2}$ p: R $\frac{1}{4}$ vel $2\frac{1}{2}$ m: R $\frac{1}{4}$, et hæc erit æstimatio 65 cuborum æqualium 1 cu. quad. p: 6 quad. quad. p: 36 quad. p: 216, nam 65 cu. sunt in una 1755, in alia 520, et tantundem sunt illæ quantitates. Proba et invenies.

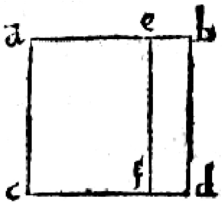
Corollarium. Ex hoc habetur quod cum 1 cu. quad. p: quad. quad. p: quad. p: numero in continua proportione fuerint æqualia cubis. Tunc habebis 1 cu. æqualem rebus duplo numeri quad. quad. cum numero cuborum. Et inventa æstimatione fac duas partes quæ producant numerum quad. quad. et partes utrique erunt æstimationes 1 cu. quad. p: quad. quad. p: quad. p: numero æqualibus numero cuborum. Velut si dicas 1 cu. quad. p: 9 quad. quad. p: 81 quad. p: 729 sunt æqualia 100 cu.. Dices ergo 1 cu. æqualis est 18 pos. p: 100, et rei æstimatio est RV: 50 p: R 2284 p: RV: 50 R 2284. Ex hac facito duas partes quæ invicem ductæ producant 9, et quælibet illarum partium est æstimatio quinomii illius propositi. Et proponatur rursus 1 cu. quad. p: 12 quad. quad. p: 72 quad. p: 216 æqualia 95 cu. et superficies ab sit b ut prius, et sit 95 æquale duobus cubis, et quatuor mutuis corporibus quæ fiunt ex ef in superficiem acdb, adeo ut ex ef in eam fiat 95, igitur ad complendum cubum deest quod fit ex ef in ab, et ab est 6, ideo et ab est 6, igitur quod fit ex ef in ab est 6 res, igitur 1 cu. æquatur 6 rebus p: 95, et res est 5, ut prius fac de 5 duas partes, ex quarum ductu unius in alteram fiat 6, dimidium 12 numeri quadratorum, et erunt partes 3 et 2, et ita 1 cub. quad. p: 12 quad. quad. p: 72 quad. p: 216 æqualia 95 cub., et res est 3 vel 2. Experiare et invenies.

Et eodem modo dicemus si corpus illud sit ex duobus cubis, et quatuor mutuis, et tertia parte duorum mutuorum, et sit gratia exempli 105 totum illud, et quia ex cb in bf sit ab, quod est 6, erit eg 4, igitur eg in ef 4 res, ergo 1 cub. æqualis 4 rebus p: 105, et res est 5, quia ducendo per primam viam pervenimus ad 1 cu. quad. p: 14 quad. quad. p: 84 quad. p: 216 æqualia 105 cu.. Ideo faciemus ex 5 re duas partes, ex quarum ductu producantur 6, qui 6 habentur ex 14, dividendo per $2\frac{1}{3}$ numerum mutuorum corporum duorum, vel ex 216, quia semper erit Rcu: eius, vel etiam diviso numero quadratorum, scilicet 84, per numerum quad. quad., qui est 14, et ita si numeri erunt dispositi hoc modo, ut secundus sit talis pars tertii, ut sit Rcu: quarti, erit regula generalis, sed ita ut quantitas eg varietur, ut oporteat problema ita construere. Sunt duæ quantitates ex quarum ductu producitur 6, et aggregatum cuborum cum duplo et sexta parte mutuorum est 100, tunc inveniemus superficiem eg 5, et erit cubus æqualis 5 rebus p: 100, et ita habebimus ef, partes producentes ab, et hic est primus modus et facilis. Sed si proponantur prius 1 cu. quad. p: 13 quad. quad. p: 78 quad. p: 216 æqualia 100, tunc quia tu nescis 100, quibus partibus æquetur, sed solum habes 6, Rcu: 3, seu quod provenit diviso 78 per 13, et diviso 13 per 6, exit $2\frac{1}{6}$, abiice, igitur relinquetur $\frac{1}{6}$, sume $\frac{1}{6}$ de 6, relinquetur 5, et habebis 1 cu. æqualem 5 rebus p: 100, ut prius, unde nota erit ef. Et ita si dixeris 1 cu. quad. p: 15 quad. quad. p: 216 æquatur 120 cu., accipe Rcu: 216, quæ est 6, seu diviso 90 per 15, et divide 15 per 6, exit $2\frac{1}{2}$, abiice 2, remanet $\frac{1}{2}$, sume dimidium 6, quod est 3, abiice 3 ex 6, relinquitur 3, dicemus ergo quod 1 cu. æquatur 3 pos. p: 120, igitur res erit Rcu: 60 p: 3599 p: Rcu: 60 m: R 3599, hanc ita dividemus ut producant 6 numerum primo inventum. Ut infra demonstrabimus.

Nota quod in huiusmodi æstimatione non solum necessarium est ut numerus, puta 65, vel 95, vel 100, aut 120, sit magnus comparatione numeri rerum quæ assumuntur, sed oportet ut res inventa possit in duas partes quæ producant R cub. numeri æquationis, quæ fuit in exemplis assumptis 6, aliter quæsitum est falsum et impossibile.

Caput XLVII

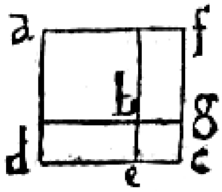
Quod divisa superficies seu corpus latera habet maiora latere totius



Sit quadratum abcd seu cubus, et sit divisum quomodolibet in ef, dico quod latera ce et ed, seu cubica seu quadrata pariter iuncta, sunt maiora ab, nam latus af est medium inter ac et ae, igitur cum ac sit maior, ae erit latus, af maius ae, et similiter latus de medium inter bd et df, igitur, cum bd sit maior df, erit latus de maius cb, quare latera af, fb iuncta maiora ae, cb simul iunctis, et hoc est quod voluimus. Similiter in cubo, nam latera sunt media secundo ordine inter ac et ce, et inter bd et df, ut demonstratum est ab Euclide in undecimo Elementorum, ideo erunt maiora ac et cb. Sed ex hoc sequitur quod in cubo æquali rebus et numero æstimatio rei est semper maior Rcu: numeri; et etiam quia talis æstimatio est Rcu: cubi qui est maior numero cum sit æqualis rebus ipsis etiam ultra numerum.

Caput XLVIII

De quadratorum quantitate et mutuis corporibus cognitis



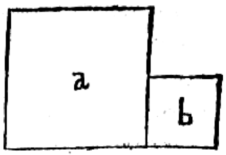
Animadvertendum quod si duo quadrata ab, bc sint nota, ut pote 13, et mutua quatuor sint 60, et velim efficere corpora solida ad altitudinem totius, illa erunt 13 res p: 60 æqualia cubo, et tunc 13 continebunt cubos ab, bc et insuper duo mutua. Sed quia ex capitulo proprio supponitur quod 13 res contineant tria mutua et cubos, ideo in æstimatione quærenda fiet res RV: cub. 30 p: R $808\frac{7}{8}$ p: R cub. 30 m: R $808\frac{7}{8}$. Et ideo non erunt 3 et 2, tamen totum erit. Cum autem dixero quod ex quadratorum ab, bc lateribus fiant mutua 30, tunc erit cd latus divisum aliter, scilicet 2 et 3. Ideo cum dicimus 1 cu. æquatur 13 rebus p: 60, istud servit eisdem quæsitis, ut 60 comprehendat duos cubos tantum, vel duos cubos cum duobus mutuis, vel duos cubos cum quatuor mutuis, vel cum quatuor mutuis, et dimidio duorum reliquorum, et generaliter cum omni parte. Sed ut dixi æquatio tamen capituli qua invenitur quantitas cd sumitur ac, si numerus ut 60 æqualis sit solis cubis, et hoc servit capitulo, quomodo proposito rectangulo et cubis laterum.

Si quis dicat 1 cu. p: 70 æquatur 39 rebus, dices tu, igitur duo cubi sunt 35, dimidium 70, et duo quadrata 13, tertia pars 39, et ita ex hoc pervenies ad 1 cu. p: 70 æqualia 39 rebus per regulam de modo.

Iterum ergo si quis dicat duo cubi sunt 35 productum unius in quadratum alterius mutuo est 30, triplicabis 30, fit 90, adde 35, fit 125, res est 5 Rcu: 125.

Et quoniam rursus ex dictis in Arte magna, cum fuerit cubus p: 70 æqualis 39 rebus, transmutatur in cubum æqualem totidem rebus et eidem numero, sed æstimatio prima habetur ducto dimidio secundæ æstimationis in se, et triplicato et deducto a numero rerum addita vel detracta a dimidio secundæ æstimationis ostendit primam.

Adhuc ergo sit cubus p: 70 æqualis 39 rebus, et res est 5 vel 2, et sub eadem æstimationem maiore cubus æqualis est 13 rebus p: 60,



et per primam consideratione quadrata a et b sunt tertia pars 39 et cubi 35, dimidium 70. Per secundam autem manentibus quadratis a et b 13, mutua corpora sunt 30, et æstimatio est eadem. Et est 5, et si esset $4\frac{1}{2}$, gratia exempli et quadrata ab 12. Igitur manente æstimatione eadem et numero quadratorum partes rei essent $2\frac{1}{4}$ p: R $\frac{15}{16}$ et $2\frac{1}{4}$ m: R $\frac{15}{16}$. Et mutua corpora erunt productum $4\frac{1}{2}$, aggregati in $4\frac{1}{8}$, productum laterum ad $18\frac{9}{16}$, igitur 1 cu. æquabitur 12 rebus p: $37\frac{1}{8}$, numerus vero rerum æqualium cubo et numero est 36, triplum 12, et numerus ipse $70\frac{7}{8}$ et cubi $35\frac{7}{16}$. Oportet ergo vel ex cubo et numero rerum eodem, et æstimatione eadem supposito numero æquationis invenire alterum, sed nondum cognita æstimatione, vel supposito numero æstimationis, et æquatione una invenire numerum rerum eundem. Exemplum 1 cub. p: 70 et 1 cu. æqualis 60, et oportet ut eadem quantitas, quæ est 13, satisfaciat utrique, scilicet 35, pro dimidio 70, et 30 pro dimidio 60. Hoc autem est notum per se, quoniam addo ad 60 dimidium ex dictis, fit 90, addo ad 90 35 dimidium 70, fit 125, cuius Rcu: est 5 æstimatio utrique satisfaciens, fac ex 5 duas partes, quarum 35 sint 35, ex dictis in arte erunt partes 3 et 2, quarum quadrata sunt 13, numerus rerum unus alter 39, triplum 13 pro altero.

Caput XLIX

De quibusdam æquationibus et modis extra ordinem

Cum fuerit cubus æqualis 6 rebus p: quovis numero, puta 40, tantum fit diviso 40 per 4 rei æstimationem, exit 10, quantum ducta æstimatione in se, fit 16, detracto numero rerum qui est 6, relinquitur 10. Ergo posito cubo æquali 6 rebus p: 20, æstimatio quæsita, si dividatur 20 per a erit quod provenit, et est æquali quadrato ipsius a m: 6, igitur diviso 40 per suam æstimationem, id est 10, se habet ad $\frac{20}{\mathrm{a}}$ sicut ducta æstimatione, quæ est 4, in se, et deducto a ad quadratum a deducto 6. Cum enim cubus fiat ex æstimationem in suum quadratum, igitur deducto quod fit ex divisione numeri per rem ex quadrato rei, relinquetur numerus rerum. Ergo vicissim deducto numero rerum ex quadrato æstimationis relinquitur quod exit. Si quis dicat divide 6 in duas partes quæ sint in proportione R cub. 3. Clarum est quod potest fieri ex tertio libro, dividendo per R cub. 3 p: 1, et est Rcu: 9 m: Rcu: 3 p: 1, et ductum in suum binomium producit 4, et in 6 fit Rcu: 19 m: Rcu: 648 p: 6, divide per 4, exit Rcu: $30\frac{1}{8}$ m: Rcu: $101\frac{1}{8}$ p: $1\frac{1}{2}$. Aliter ergo ponemus unam partem 6 m: 1 pos., aliam 1 pos., proportio R cub. 3, igitur duc 1 pos. in Rcu: 3, fit pos. Rcu: 3 duc ad cu. fit 3 cu. et hoc est æquale cubo 6 m: 1 pos., qui 216 p: 18 quad. m: 108 pos. m: 1 cu., igitur 1 cu. p: 27 pos. æqualia sunt $4\frac{1}{2}$ quad. p: 54. Igitur per regulam 1 cu. p: $20\frac{1}{4}$ pos. æquatur $20\frac{1}{4}$ numero, igitur cubus tertiæ partis rerum est $307\frac{35}{64}$, adde quadrato dimidii numeri æquationis, fit $410\frac{1}{16}$, igitur rei æstimatio est RV: cu. R $410\frac{1}{16}$ p: $10\frac{1}{8}$ m: RV: cu. R $410\frac{1}{16}$ m: $10\frac{1}{8}$ p: $1\frac{1}{2}$, at illæ radices æquivalent prædictis, quia R $410\frac{1}{16}$ est $20\frac{1}{4}$, igitur addendo et detrahendo $10\frac{1}{8}$, fiunt Rcu: $30\frac{3}{8}$ m: Rcu: $10\frac{1}{8}$ p: $1\frac{1}{2}$ ut prius.

Capitulum 18 Artis magnæ

Ex hoc patet quod cum habueris 1 cu. p: rebus æqualia quadratis p: numero, tum debes dividere numerum rerum per numerum quadratorum, et numerum qui exit duces ad cubum, et eum divides per numerorum æquationis, et cum eo multiplicabis totum quadrinomium, et quod superest in numero abiice ad uno, et illud serva deinde, divide R cub. numeri iam inventi in duas partes quæ se habent in proportione numeri abiecti per primum modum, et habebis æstimatione quæsitam. Exemplum habes iam propositum.

1 cu.p:27 æqual. | $4\frac{1}{2}$quad.p:54
4

cu.p:108 poſ. | 18 quad. p:216
3|216:p:18 quad.m:108 poſ.m:1 cu.

℞ cu.3 | 6 diuidendum

Sit 1 cub. p: 27 rebus æqualis $4\frac{1}{4}$ quad. p: 54, divide 27 per $4\frac{1}{4}$, exit 6, duc ad cubum, fit 216, tum divide per 54 numerum æquationis, exit 4, duc in superiorem, habes 4 cu. p: 108 pos. et 18 quad. p: 216, abiice quicquid est supra 1 cu., et est 3, et relinquentur 216 p: 18 quad. 108 pos. m: 1 cu., cape igitur Rcu: 3, et etiam Rcu: 3, et ei adde 1 pro regula, fit Rcu: 3 p: 1, divide 6 per Rcu: 3 p: 1 per priorem modum, exibit æstimatio quæsita 1 cu. p: 27 pos. æqualium $4\frac{1}{2}$ quad. p: 54. Sed hæc conversio non est generalis nisi cum deducto numero qui prodit ex divisione cubi in numerum quadratorum, consurgit numerus triplus ad R cub. numeri, seu ad numerum qui provenit ex prima divisione.

Caput L

De solidis radicibus et earum tractatione

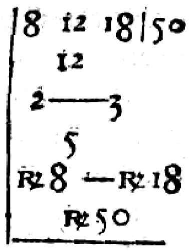
Regula prima. Cum voluero dividere 6 ut fiat R solida 9, duc 6 in se, fit 36, divide 9 per 36, exit $\frac{1}{4}$, et hæc prima pars, secunda igitur erit $5\frac{3}{4}$, nam ex cubo $\frac{1}{4}$, et duplo quadrati $\frac{1}{4}$ in $5\frac{3}{4}$ et quadrato $5\frac{3}{4}$ in $\frac{1}{4}$ iunctis fit 9.

2. Cum volueris habere radicem solidam 50 in proportione 3 ad 2, gratia exempli, cape 1 et $1\frac{1}{2}$ in proportione 3 ad 2, ita quod in illis sit unitas, iunge igitur 1 et $1\frac{1}{2}$, fit $2\frac{1}{2}$, duc in se, fit $6\frac{1}{4}$, divide 50 per $6\frac{1}{4}$, exit 8 cuius Rcu:, quæ est 2, est pars prima R solidæ 50.

3. Cum volueris, habita prima parte R solidæ, habere secundam in partibus cognitis primæ et secundæ, ut pote 8 ad 24, accipe Rcu: primæ, quæ est 2, et tum ea ducta in se et fit 4, divide dimidium 2, et quod exit est quæsitum 53, 3.

4. Cum volueris habita prima et tertia quantitate veluti 8 et 18 habere R solidam, ut scis quod prima pars est semper Rcu: primæ partis 8 quæ est 2, divide 18 exit 9, cuius R quadrata quæ est 3, est pars secunda.

5. Cum volueris, habita secunda et tertia parte, habere R solidam, tunc accipe dimidium secundæ partis, ut pote 12, quod est dimidium 24, et ex Capitulo 28 Artis magnæ habebis eas.



6. Cum volueris, habita prima parte et tertia et aggregato, comparare R invicem, scias quod R quadratæ partium extremarum, ut pote 8 et 18 sunt partium solidæ 1, 2 et 3, quæ sunt R solidæ 8 et 18, item ipsarum partium accipiendo dimidium secundæ pro secunda, nam proportio 18 ad 12, et 12 ad 8, et 3 ad 2, et R 18 ad R 8, sunt omnes sexquialtera.

7. Et sicut ex 3 et 2 partibus solidæ fit R 50, solida ita ex R 8 et R 18 quadratis fit R 50 quadrata.

8. Itaque cum volueris, habita prima parte, ut pote 8, et residuo aggregati, ut pote 42, habere radicem solidam totam, divide 50 aggregatum per 8, exit $6\frac{1}{4}$, cuius R quadratum, quæ est $2\frac{1}{2}$, accipe, et ab ea minue 1, fit $1\frac{1}{2}$, duc in Rcu: 8, quæ est 2, fit 3 pars reliqua, et est conversa secundæ regulæ.

9. Ex his manifestum est quod ubi cubus æquetur 36 rebus p: 36 dando duo solida cubo, alterum rebus alterum numero, proportio unius ad alterum erit 1 pos., quæ est ut 36 pos. ad 36, nam quilibet cubus ex duobus similibus solidis componitur ut 125, componitur ex solido 2 et 3, quod est 50, et 3 et 2 quod est 75, et proportio alterius ad alterum est sexquialtera, ut 3 ad 2.

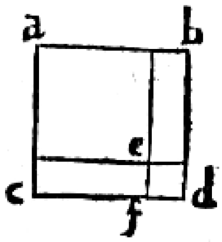
10. Quælibet duo solida similia cubum componunt, velut capio 24 p: 24 p: 6, quod totum est 54 solidum primum, aliud erit 12 et 12 et 3 quod est 27. Aggregatum est 81 cubus Rcu: 24 p: Rcu: 3, quod est dicere Rcu: 81, nam Rcu: 24 et Rcu: 3 componuntur Rcu: 81, et Rcu: 24 p: Rcu: 3, posita prima parte Rcu: 24, producit solidum 24 p: 24 p: 6, et, posita prima parte R cub. 3, producit solidum 12 p: 12 p: 6.

Caput LI

Regula quædam specialis atque item modus tractationis subtilis

Si fuerint duo numeri quod fit ex ductu unius in R alterius mutuo, inde aggregato in se ducto est æquale ei quod fit ex ductu unius in quadratum alterius, addito duplo R quadratæ producti unius in quadratum alterius invicem. Exemplum, capio 2 et 3, et producta mutua in R sunt R 18 p: R 12, quorum quadratum est 30 p: R 864, dico quod hoc est æquale producto unius in quadratum alterius, et est 30 cum duplo R 216, qui fit ex 12 in 18, mutuis 3 et 2. Ergo sint partes 6 p: 1 pos. et 6 m: 1 pos., et debeat esse quadratum mutui 100, id est mutuum R sit 10. Erunt ergo mutua quadratorum 432 m: 12 quad. p: R 186624 p: 432 quad. quad. m: 155552 quad. m: 4 cu. et hoc est æquale 100, igitur 332 p: illa R est æqua 12 quad., et 12 quad. m: 332 æqualia R illi 6, igitur quadrata 166 sunt æqualia R 46656 p: 108 quad. quad. m: 3888 quad. m: 1 cu. quad.. Igitur, partibus in se ductis, 1 cu. quad. p: 1896 quad. æquantur 19100 p: 72 quad. quad.. Sed æquatio non est in parte nota, est tamen pulchrum.

Proponatur rursus 6 divisum per Rcu: 4 p: Rcu: 2, et exibit Rcu: 16 m: 2 p: Rcu: 4, ut notum est,



et ponamus ce superficiem Rcu: 16 m: 2 p: Rcu: 4, et sint cubi ae, ed 40, igitur per dicta superius si velim assumere cubam trinomii, quadratum est 12 m: Rcu: 432, et cubos ob id R cub. 93312 m: 36, oportet autem ut hac quantitate quæ est Capitulum
40 et refert aggregatum cuborum, fiant duæ partes quæ invicem ductæ faciant 46
illum cubum. Erunt ergo partes 20 p: RV: 436 m: Rcu: 93312 et 20 m: RV: 436

m: Rcu: 93312, et RV: cu., harum partium ductæ invicem producunt Rcu: 93312 m: 536, et cubi sunt 40. Partes igitur sunt[1] RV: ma cu. 20 p: RV: quad. 436 m: Rcu: 93312, et[2] RV: cu. 20 m: RV: quad. 436 m: R 93312, cum ergo producant invicem ductæ ce, id est R cub. 16 m: 2 p: Rcu: 4, ubi esset R illa binomia proportionem habens, haberemus quæsitum cum sit ex natura binomii cubici. Hoc volui scribere ut intelligeres subtilitatem operationis. Et quod æstimatio non est in quantitate cognita, nisi ut divisum, scilicet velut dividendo quantitatem aliquam per virgulam quæ non habet nomen, et ita est et non est. Est tamen notior et magis habilis ad omnes operationes quantitate solida. Imo est quasi media inter solidam et per se notam, in quo genere sunt omnes R simplices et coniunctæ.

[1]In 1570 and 1663 editions, the following ‘*RV:*’ has ‘*ma*’ superscript.
[2]See footnote 1.

Caput LII

De modo omnium operationum in quantitatibus medio modo notis

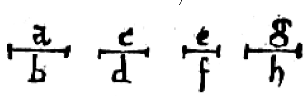
Debes scire quod omnes operationes multiplicatio, divisio, additio, detractio et R inventio in huiusmodi est velut in partibus numerorum, velut volo multiplicare $\frac{3}{\text{R } 6 \text{ p: R } 5 \text{ p: R } 3 \text{ m: R } 2 \text{ m: } 1}$ per $\frac{\text{Rcu: } 7 \text{ m: Rcu: } 2}{\text{Rcu: } 5 \text{ m: Rcu: } 3 \text{ p: R } 2}$ oportet ut ducas denominatores simul et fiet hoc

$$\frac{\text{Rcu: } 189 \text{ m: Rcu: } 54}{\text{R } 12 \text{ p: R } 10 \text{ p: R } 6 \text{ m: } 2 \text{ m: R } 2 \text{ p: Rcu: quad. } 5400 \text{ p: Rcu: quad. } 3125 \text{ p:}}$$
$$\frac{\text{Rcu: } 189 \text{ m: Rcu: } 54}{\text{Rcu: quad. } 675 \text{ m: Rcu: quad. } 200 \text{ m: Rcu: } 5 \text{ m: Rcu: quad. } 1944 \text{ m:}}$$
$$\frac{\text{Rcu: } 189 \text{ m: Rcu: } 54}{\text{Rcu: quad. } 1125 \text{ m: R } 243 \text{ p: Rcu: quad. } 72 \text{ p: Rcu: } 3}$$

Et similiter facies in divisione additionibus ac detractionibus reducendo ad idem genus quantitates simplices, et similiter in capiendo radicem. Velut capio radicem

$$\frac{25}{14 \text{ p: R } 120 \text{ p: R } 2 \text{ m: R } 48 \text{ m: R } 24 \text{ m: R } 10 \text{ m: R } 5},$$

capio Rcu: 25, et est 5, et capio radicem infra scripti denominatoris, et est R 6 p: R 5 m: R 2 m: 1, et habeo $\frac{5}{\text{R } 6 \text{ p: R } 5 \text{ m: R } 2 \text{ m: } 1}$, ductum hoc ad veram quantitatem per sua contraria fiet divisor,



qui fit b, et qui dividitur multorum nominum a, et 4 divisus c, et R 6 p: R 5 m: R 2 m: 1 dicatur d, et dicatur 25 numerator primus, et suus denominator septem nominum f. Quia ergo a ad b ut c ad d, et e ad f ut c ad d duplicata erit e ad f ut a ad b duplicata. Igitur si ducantur a et b in se et producantur g et h, erit h numerus, et g, h proportio nota, et est g ad h ut e ad f, igitur g ad f nota. Et hæc est sexta operatio propria quantitatibus mediis.

Per 20 sexti Elementorum

Caput LIII

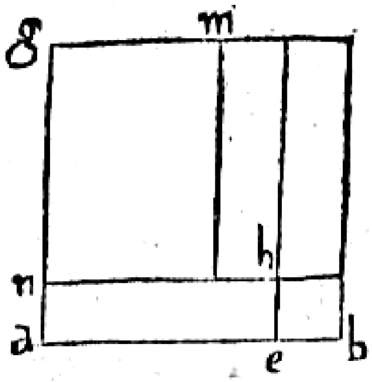
De diligenti consideratione quorundam superius dictorum Capitulo 7

Et iam dicamus quod cubus æqualis sit 12 rebus p: 20, et rei æstimatio est R cub. 16 p: Rcu: 4, et hæc potest tribui dando 20 numerum cubis similiter, et potest idem numerus dari ambobus cubis et duobus mutuis, et etiam ambobus cubis et quatuor mutuis parallelipedis, et ita trifariam. Consideremus ergo postquam capituli inventio ac regula cum demonstratione sumpta fuit per primum modum. Sumemus ergo cubum dimidii æstimationis, id est Rcu: 2 p: Rcu: $\frac{1}{2}$, et est $2\frac{1}{2}$ p: Rcu: 54 p: Rcu: $13\frac{1}{2}$, et duplum eius quod est minimum, quod possit produci ex divisione æstimationis, est[1] 5 p: Rcu: 432 p: Rcu: 108, liquet igitur non posse dividi sic hanc R propter numeri parvitatem, nam cubus totius esset 20 p: Rcu: 27648 p: Rcu: 6912. Sin autem capiamus 1 cu. æqualem 12 rebus p: 34, erit æstimatio Rcu: 32 p: Rcu: 2, et duplum cubi dimidii[2] $8\frac{1}{2}$ p: Rcu: 864 p: Rcu: 54, et hoc totum est proximum[3] $21\frac{1}{2}$, ideo duo mutua poterunt contineri in[4] $\frac{25}{2}$, divides ergo 34 per Rcu: 32 p: Rcu: 2, exit R cub. 1024 m: Rcu: 64, quod est[5] 4, p: R cub. 4, et hoc oportet esse æquale duobus quadratis, fac ergo ex Rcu: 32 p: Rcu: 2 duas partes, quarum quadrata sint æqualia trinomio illi. Accipe ergo dimidium trinomii, et est Rcu: 128 m: 2 p: R cub. $\frac{1}{2}$, a quo aufer quadratum dimidii dividendi, id est quadratum Rcu: 4 p: R cub. $\frac{1}{4}$, et est Rcu: 16 p: 2 p: Rcu: $\frac{1}{16}$, detrahe, relinquetur[6] Rcu: 16 m: 4 p: Rcu:$\frac{1}{16}$, huius igitur RV: addita et detracta ostendit partes hoc modo.

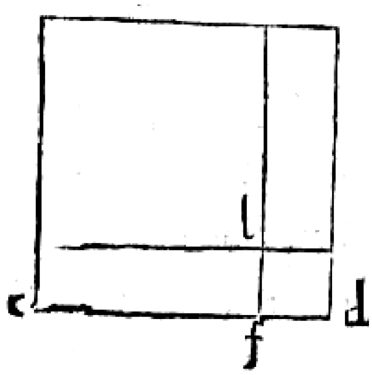
[1]1570 and 1663 have "*5 p: R 432 p: Rcu: 108*".
[2]1570 and 1663 have "*$8\frac{1}{2}$ p: Rcu: 1024 p: Rcu: 54*".
[3]1570 and 1663 have "*$22\frac{1}{2}$*".
[4]1570 and 1663 have "*$11\frac{1}{2}$*".
[5]1570 and 1663 have "*4, m: R cub. 4*".
[6]1570 and 1663 have "*Rcu: 54 m: 4 p: Rcu: $\frac{1}{16}$*".

℞ cu.4 p:℞ cu.$\frac{1}{4}$ p:℞ V^{m1} ℞ cu.54 p:℞ cu.$\frac{1}{16}$ m:4
℞ cu 4 p:℞ cu.$\frac{1}{4}$ m:℞ V^{m1} ℞ cu.54 p:℞ cu.$\frac{1}{16}$ m:4

Iam ergo vides quod cubus æquatur 34, ita quod 34 numerus est æqualis duobus cubis cum duobus mutuis partium, et quia residuum est numerus rerum, et est duplum mutuorum diviso eo per rem, exibit numerus rerum quem constat esse eundem.

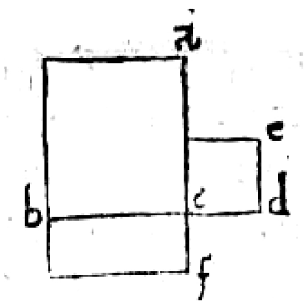


Proponatur ergo ab et cd 4, et sint res, et sint earum quadrata bg, dk, sit autem ab divisa in e ut cubi gh, hb sint quadraginta, et erunt b res p: 40 æqualia toti cubo, et ideo auferatur mh æqualis ah, erunt igitur tres illæ superficies b, et iuxta altitudinem ab, b res, et ex ab in mn et hb 40. Et ae[7] erit RV: cu. 20 p: R 392, et eb RV: cu. 20 m: R 392, et sit cf[8] 3, et fd erit 1, et cubi kl et ld cum duobus mutuis corporibus et hoc est quantum fit ex cd in kl, ld iterum 40, et erunt superficies kl et ld 10, et æquales necessario superficiebus mn et hb, quia ipsæ ductæ in ab, quæ est æqualis cd producit 40.



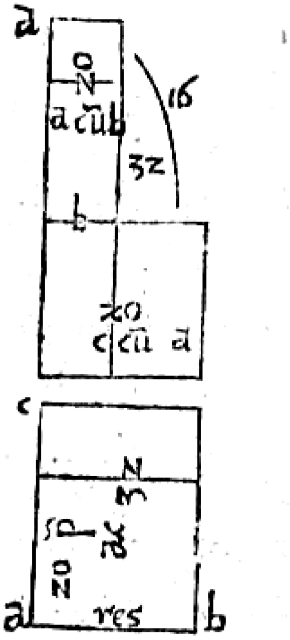
[7]1570 and 1663 have "*ac*".
[8]1570 and 1663 have "*ef*".

Igitur quia volo in prima superficie quod soli cubi æquales sint 40, et in secunda quod cubi cum duobus corporibus mutuis efficiant idem 40, et quod æstimatio sit eadem, igitur necesse est ut in secunda figura 1 cub. æquetur 6 rebus etiam p: 40, sed divisio in f est proximior medio quam in prima figura e, nec regula illa servit huic æquationi sic intellectæ, ergo oporteret invenire aliam ei propriam. Idem igitur dico de exemplo superiore, ponatur ab Rcu: 32 p: Rcu: 2, et sit divisio binomii in e, et 1 cu. æqualis 12 rebus p: 34 et erunt nm et hb 12. In secunda autem figura erunt itidem kl, ld 12, sed divisio erit ut propositum est in f, nec licebit cum æquatione 1 cu. æqualis 12 rebus p: 34, invenire cd ut composita est ex cf et fd, sed ex alia regula. Sed inveniemus ab ut est divisa in partes ae, ae, eb[9] postmodum si noluerimus cf, fd.[10] Hoc tamen satis est ut intelligamus dari quantitatem mutuam, quæ possit eo modo ducta producere numerum. Si fuerint duæ quantitates quod fit ex prima in quadratum secundæ est æquale ei quod fit ducta secunda in R primæ in se. Hoc autem conmutandi causa.



32 p:lat.a.

[9] 1570 and 1663 have “*ae, eb, ec*”.
[10] 1570 and 1663 have “*cf, fd*”.

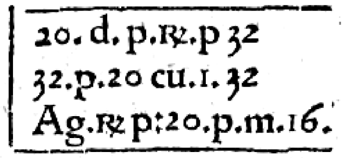


Sit prima ab quadratum, secunda cd, fiat ergo ex bc in cd, bf, dico bf esse latus ba in cd.[11] Quia enim ex ab in cd fit quantum ex bc in bf, eo quod utrobique ducitur cd in quadratum bc, erit proportio corporis cd in ab ab bf superficiem linea bc. Similiter proportio corporis ducti in cd[12] ad ab est quadratum cd. Igitur proportio producti ab in ce ad bf est ipsa bf, igitur bf in se ducta producit ab in ce.

Scholium

Ex visis hic et superius apparet liquido quod omnes regulæ vigesimiquinti Capitulum 2 capituli Artis magnæ, quas vocant speciales, sunt generales, et dicuntur speciales solum ratione generis æstimationis, et ideo si quis dicat cu. æqualis est 20 rebus p: 32, dicemus quod æstimatio est 20 d. p. R p. 32, id est divisum in partem et radicem producentes 32. Et similiter erit 32 p. 20 cum p. 32, id est producentis 20 cum producente 32. Et similiter dicetur Ag. R p: 20 p: n: 16. id est aggregatum radicum partium 20, quæ mutuo ductæ producunt 16 dimidium 32.

[11] 1570 and 1663 have "*ce*".
[12] 1570 and 1663 have "*bc*".


20. d. p. ℞. p 32
32. p. 20 cu. 1. 32
Ag. ℞ p: 20. p. m. 16.

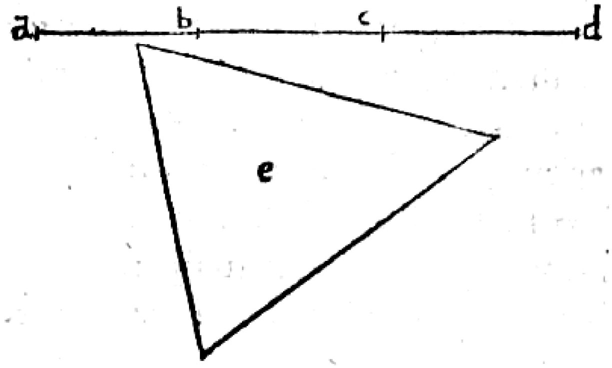
Dicemus etiam ex superius dictis hoc idem, ut res redigatur ad tres æstimationes, nam aliæ sunt confusæ. Ex quibus sequitur quod istæ æstimationes inter se erunt æquales. Et similiter cum operatus fuerint in illis, transibis ex uno in aliud capitulum, ut cum æstimatione. Et nota quod in figura a variat magnitudinem iuxta singulas regulas.

Capitulum 28

Vide supra Capitula 31 et 40 in fine, et infra Capitulum 57

Caput LIV

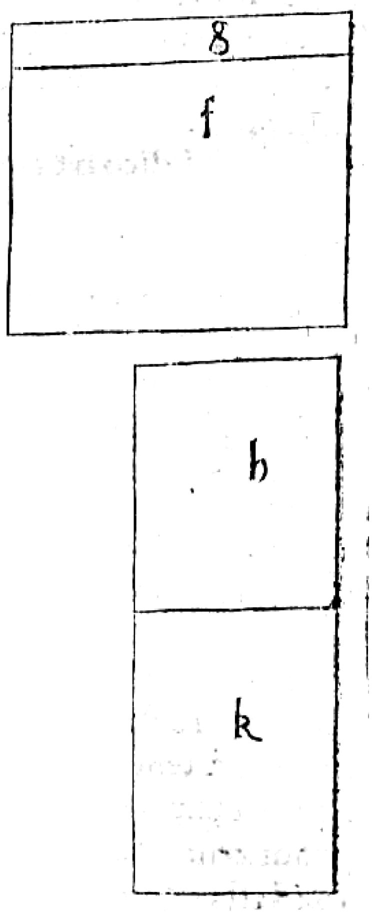
De perpetua additione quantitatum



Dico quod si capias duas quantitates ab, bc, et iungas eas, et fit producti ba in ac, aggregatum a quadrato bc differentia superficies e, dico quod, si addatur ac tanquam parti cd æqualis bc, quod differentia quadrati ac conversa ratione a producto cd in ad, et hoc semper procedet, id est posita ad una parte addemus æqualem ac, et fiet ac in aggregatum ad et ac differentia a quadrato ad idem e. Ostensa prima patent reliquæ. Et semper fit commutatio, nam, si in prima quadratum bc sit maius eo quod ex ab in ac, erit in secunda quod fit ex cd in ad maius quadrato ac. Quod ergo fit ex cb in se cum eo quod fit ex ca in se est æquale duplo quadrati cb in se, et duplo cb in ab, et quadrato ab; quod etiam fit ex ab in ac et cd in da est æquale eisdem quinque superficiebus. Igitur quadrata cb et ca sunt æqualia duabus superficies ab in ac et dc in da, sint ergo quadrata bc, ca superficies fg, ita ut f sit æqualis quadrato ac, g quadrato bc, superficies autem hk æqualis ab in ac et cd in da, erit igitur ut demonstratum kh æqualis fg, sit autem h æqualis ab in ac, k autem æqualis cd in da, quantum igitur h excedit g tantum fk, vel contra quantum g excedit h tantum k excedit f, sed differentia g et h ex supposito est e, igitur e est etiam differentia f et k, sed f est æquale quadrato ac, et k producto ex cd in da, igitur constat propositum.

Per 4 secundi Elementorum

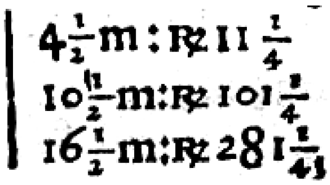
Per 1 secundi Elementorum


g
f
h
k

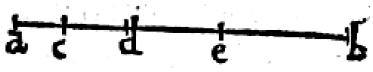
Caput LV

Quæstio generalissima, per quam ex tribus conditionibus universalibus ad unam devenimus quantitatem specialem, et est admirabilis

Est quantitas cuius latus ductum in residuum producti latus tanto maius est latere aggregati quanto residuum totius detractis duobus lateribus maius est hoc ipso latere. Quantitas est 1 quad., latus 1 pos., residuum 1 quad. m: 1 pos., latus igitur producti Rcu: m: 1 quad., habemus igitur 1 pos. R 1 cu. m: 1 quad. et 1 quad. m: R 1 cu. m: 1 quad. et m: 1 pos. quæ se æqualiter excedunt. Igitur ut in proportionibus æqualibus multiplicatio ita in excessibus, coniuncto 1 quad. m: R 1 cu. m: 1 quad., duplum erit R 1 cu. m: 1 quad.. Et ideo 1 quad. æquale triplo R 1 cu. m: 1 quad. quod est R 9 cu. m: 9 quad.Igitur 1 quad. quad. æquale 9 cu. m: 9 quad. et 1 quad. p: 9 æqualia 9 pos., igitur res est $4\frac{1}{2}$ m: R $11\frac{1}{4}$. Aggregatum $31\frac{1}{2}$ m: R 9 $11\frac{1}{2}$, detrahe $4\frac{1}{2}$ m: R $11\frac{1}{4}$. Relinquitur aggregatum secundæ et tertiæ 27 m: R 720, hanc divide ut æqualiter se excedant, detrahe duplum $4\frac{1}{2}$ m: R $11\frac{1}{4}$ ex 27 m: R 720, relinquuntur 18 m: R 405, cuius sume tertiam partem quæ est 6 m: R 45, adde primæ habebis $10\frac{1}{2}$ m: R $101\frac{1}{4}$, tertia fiet simili ex additione $16\frac{1}{2}$ m: R $281\frac{1}{4}$.



Quæstio II



Linea ab est decem divisa in quatuor quantitates æqua proportione et differentiæ illarum, simul iunctæ sunt quinque. Sit igitur ae 1, et cd 1 pos., de erit 1 quad., et eb 1 cu.. Et quia ex regulis generalibus quantitatum differentiæ ac, cd, de, eb sunt æquales differentiæ ac et eb, in quotvis quantitatibus quolibet modo et ordine sumptis, erit differentia ac, ab, cb ad ab 1 cu. m:1 ad 1 cu. p: 1 quad. p: 1 pos. p: 1, igitur dupla, quare 1 cu. p: 1 quad. p: 1 pos. p: 1 æqualia 2 cu. m: 2 et 1 cu. æqualis 1 quad. p: 1 pos. p: 3, et est in capitulo et clarum. Habebimus ergo aggregatum 1 cu. p: 1 quad. p: 1 pos. p: 1, at nos volebamus non illud, sed 10. Dicemus ergo si ab aggregatum esset 10, quanta esset ac, duc 10 in 1, fit 10, divide per aggregatum, exibit quantitas ac in linea ab, quæ est 10, et ea quantitas ducta per rem producet cd, eadem ducta in rem producet de, deductis ae, cd et de ex ab, relinquetur nota etiam be.

Quæstio III

Quod si dicat differentias ac et cd, itemque de et eb esse 5 cum tota ab sit 10. Ponemus ut prius, et erunt differentiæ cd et ea 1 pos. m: 1, et eb et ed 1 cu. m: 1 quad., igitur 1 cu. p: 1 quad. p: 1 pos. p: 1 sunt dupla 1 cub. m: 1 quad. p: 1 pos. m: 1. Quia ergo 1 cub. p: 1 quad. se habet ad 1 pos. p: 1 ut 1 cu. m: 1 quad. ad 1 pos. m: 1, nam utriusque proportio est 1 pos., erit permutando 1 cu. p: 1 quad. ad 1 cu. m: 1 quad. ut 1 pos. p: 1 ad 1 pos. m: 1, igitur iungendo erit proportio 1 cu. p: 1 quad. p: 1 pos. p: 1 ad 1 cu. m: 1 quad. p: 1 pos. m: 1 ut 1 pos. p: 1 ad 1 pos. m: 1. At illa proportio fuit dupla, duplum igitur est 1 pos. p: 1 ad 1 pos. m: 1 et 1 pos. p: 1 æqualis 2 pos. m: 2 igitur 1 pos. æqualis 3 proportio, igitur quantitatum tripla est. Erunt igitur quantitates 1, 3, 9, 27, tota igitur ab est 40. At nos supponimus eam esse decem, solum igitur cum 40, sit quadruplum ad 10 erunt ac, cd, de, eb quarta pars 1, 3, 9, 27. Quare erunt $\frac{1}{4}$, $\frac{3}{4}$, $2\frac{1}{4}$, $6\frac{3}{4}$. Et differentia $\frac{1}{4}$, et $\frac{3}{4}$, et $2\frac{1}{4}$, et $6\frac{3}{4}$ sunt 5, 1, 1, $\frac{1}{2}$, et $4\frac{1}{2}$.

Corollarium 1. Ideo nota quod aggregatum quatuor quantitatum ad aggregatum illarum duarum differentiarum proportionem habet quam proportio ipsa monade, addita habet ad proportionem ipsam detracta unitate, ut ita liceat latine loqui tamen. Velut 8, 12, 18, 27, aggregatum 65, aggregatum differentiarum 13 proportio quintupla, et est ut $2\frac{1}{2}$ ad $\frac{1}{2}$ est autem $2\frac{1}{2}1$ p: proportione sexquialtera, quæ scribitur $1\frac{1}{2}$ et $\frac{1}{2}$ m: eadem proportione.

Corollarium 2. Ex hoc etiam sequitur quod cum proportio aggregati ad duas differentias primæ et secundæ, itemque tertiæ et quartæ, fiat detracta et addita monade ad proportionem partium, et in omnibus quantitatibus eadem maneat, quod si voluero aliquam proportionem, utpote nonuplam, inter aggregatum quantitatum et duarum differentiarum, accipiam 1 m: in proportione 1 octuplam, et accipiam partem octavam 2, et est $\frac{1}{4}$, cui addam 1, et est $1\frac{1}{4}$, et hæc erit proportio, scilicet sexquiquarta, etsi voluero decuplam, aufero 1, sit nonupla, et capio nonam partem 2, quæ est $\frac{2}{9}$, et ei addo 1, fit $1\frac{2}{9}$, proportio 1 superbipartiens duas nonas, et si voluero supertripartientem decimas[1] 1 $1\frac{3}{10}$ inter quantitates ut habeam proportionem aggregati ad aggregatum ut 23 ad 3, et habebis quantitates ut vides,

Quantit.	1000
	1300
	1690
	2197
Ag.q.	6187
Ag.d.	807
Propor. 23 ad 3	

ideo detrahe 3 a 23, relinquitur 20, divide 2 per 20, exit[2] $\frac{1}{10}$ sumo triplum, et est[3] $\frac{3}{10}$, cui addo 1, et fit[4] $1\frac{3}{10}$, proportio partium quæsita, et idem in aliis.

Corollarium 3. Quantitatum proportiones ad aggregatum manent eadem dico, ad aggregatum differentiarum omnium, et primæ et tertiæ, et ad differentiam secundæ a tertia. Et tamen una est facillima inventu, scilicet ad differentiam primæ et secundæ et tertiæ et quartæ, alia difficillima, scilicet ad aggregatum omnium, ut visum est in quæstione secunda, alia ferme impossibilis scilicet ad differentiam secundæ et tertiæ. Nam cum proportio ad aggregatum omnium sit ut vides,

[1]1570 has "$\mathit{1}\ \mathit{1}\frac{3}{01}$".
[2]1570 has "$\frac{1}{01}$".
[3]1570 has "$\frac{3}{01}$".
[4]1570 has "$\mathit{1}\frac{3}{01}$".

1 cu. p: 1 quad. p: 1 pos. p: 1
1 cu. m: 1 quad. p: 1 pos. m: 1
1 cu. p: 1 quad. p: 1 pos. p: 1
1 cu. m: 1
1 cu. p: 1 quad. p: 1 pos. p: 1
1 quad. m: 1 pos.

et similiter ad aggregatum duarum differentiarum detracta una ab alia, seu in prima positione relinquetur proportio aggregati ad differentiam secundæ et tertiæ, ut 1 cu. p: 1 quad. p: 1 pos. p: 1 ad 1 quad. m: 1 pos., et ita fiet æquatio cubi rerum et numeri æqualium quadratis, quantum ad generalem modum. Quia vero proportiones se habent invicem, ut 1 pos. 1 quad. et 1 cu. proportionis, proportio enim secundæ ad primam est simplex et una, et tertiæ ad secundam ut quad., et quartæ ad tertiam ut cubus. Velut vides in exemplo,

8.	12.	18.	27
$1\frac{1}{2}$.	$2\frac{1}{4}$.	$3\frac{3}{8}$	
4	6	9	

differentiæ vero sunt in eadem proportione. Ideo si quis dicat divide 10 in quatuor quantitates quarum proportio differentiarum extremarum sit tripla ad mediam, facile invenies, nam habebis 1 cu. m: 1 quad. p: 1 pos. m: 1 tripla ad 1 quad. m: 1 pos. divide per 1 pos. m: 1, habebis 1 quad. p: 1 æqualem 3 pos., igitur res est $1\frac{1}{2}$ m: R $1\frac{1}{4}$, et hæc erit proportio quantitatum iuxta quam dividemus postea 20, et semper differentia primæ a secunda et tertiæ a quarta tripla erit differentiæ secundæ a tertia.

Quæstio IIII

Iuxta quam faciemus quatuor quantitates in continua proportione quarum differentia secundæ a tertia sit 2, et primæ a secunda et tertiæ a quarta 6. Erunt igitur illæ differentiæ in ea proportione, ut pote $1\frac{1}{2}$ m: R $1\frac{1}{4}$ $3\frac{1}{2}$ m: R $11\frac{1}{4}$ 9 m: R 80. Sed media differentia non est 2, dic ergo hi $3\frac{1}{2}$ m: R $1\frac{1}{4}$ esset 2, quid erit $1\frac{1}{2}$ m: R $1\frac{1}{2}$ et 9 m R 80. Duc 2 in eas quantitates, fient ut vides.

3 m: R 5
18 m: R 320
———
$3\frac{1}{2}$ m: R $11\frac{1}{4}$
$3\frac{1}{2}$ p: R $11\frac{1}{4}$

Divide eas per $3\frac{1}{2}$ m: R $11\frac{1}{4}$, et est ut multiplices per binomium omnia fietque divisor 1, et est ac, si non divideres quantitates, ergo erunt ut vides,

1
———
3 p: R 5
2
3 m: R 5

sed hæ sunt differentiæ quantitatum. Pones ergo primam 1 pos., secundam 1 pos. p: 3 m: R 5, tertiam 1 pos. p: 5 m: R 5, quartam 1 pos. p: 8, duc primam in ultimam, fiunt 1 quad. p: 8 pos., æqualia ductui secundæ in tertiam, qui est 1 quad. p: 8 pos. m: pos. R 20 p: 20 numero m: 320. Igitur pos. R 20 æquantur 120 m: R 328, divide numerum per numerum positionum, erit rei æstimatio R 20 m: 4. Igitur quantitates erunt ut vides.

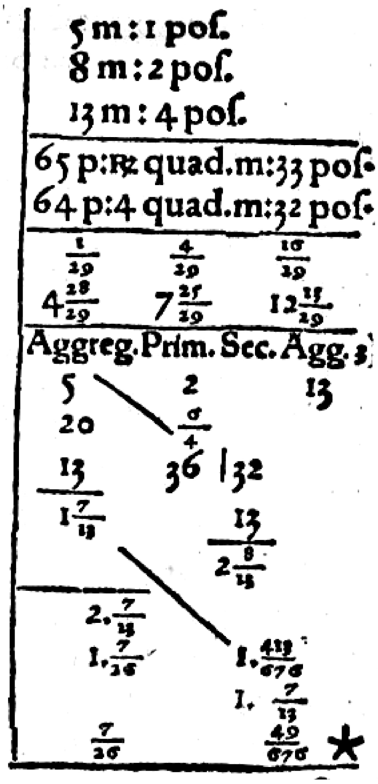
R 5 m: 1	1 pos.
R 5 m: 1	1 pos. p: 3 m: R 5
R 5 p: 1	1 pos. p: 5 m: R 5
R 20 p: 4	1 pos. p: 8
R 180	

Et constat quod sunt in continua proportione. Nam ex prima in tertiam fit 6 m: R 20, quod est quadratum secundæ. Et differentia primæ a secunda est 3 m: R 5, et tertiæ a quarta 3 p: R 5, quæ iunctæ faciunt 6, et differentia secundæ a tertia est 2, ut propositum est.

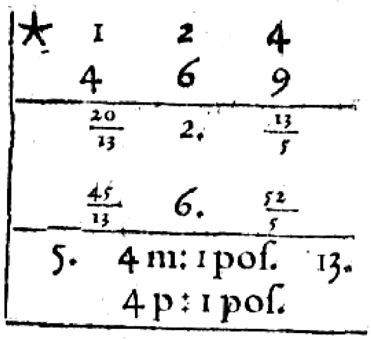
Caput LVI

De duabus quæstionibus pulchris sed impertinentibus

Cum fuerint tres quantitates et volueris eas dividere in duos ordines quantitatum eiusdem proportionis primum, divides secundum, pro arbitrio 1 mediam, quia innumeris modis poterit solvi quæstio, ut etiam sub certa proportione quantitatibus ut libet variatis iuxta proportionis naturam, erunt ergo duo generales modi, scilicet quantitas et proportionis.



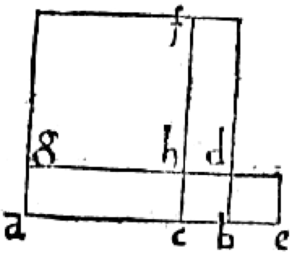
Sint ergo quantitates 5, 8, 13, et proportionem assumamus duplam, erunt igitur 5 m: 1 pos., 8 m: 2 pos., 13 m: 4 pos. in continua proportione, quare ut vides extrema invicem conveniunt ducta cum media in se, et abiecto numero quadratorum utrinque qui semper erit idem erit 1 pos. æqualis 1, igitur quantitates erunt 1, 2, 4 et reliquæ 4, 6, 9, et hic modus est facilis. Etenim si posuisses in proportione quadrupla fuissent ut vides.



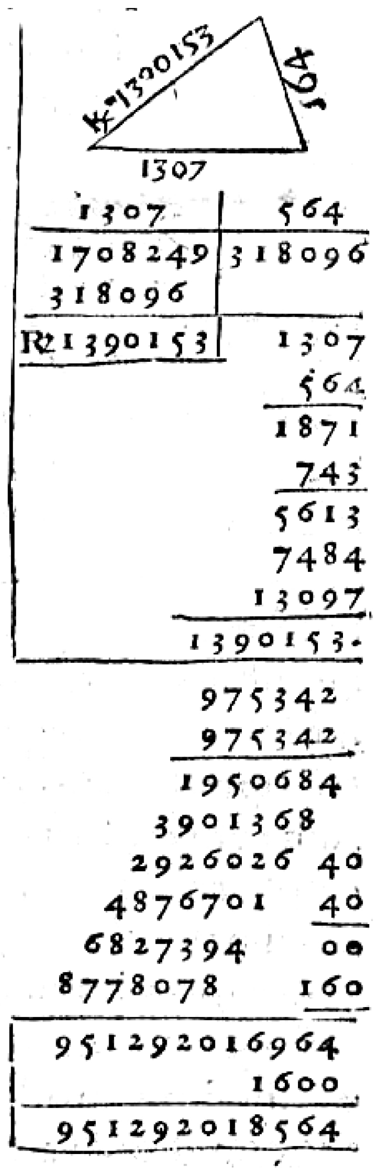
★	1	2	4
	4	6	9
	$\frac{20}{13}$	2.	$\frac{13}{5}$
	$\frac{45}{13}$	6.	$\frac{52}{5}$
5.	4 m: 1 pos.		13.
	4 p: 1 pos.		

At si quantitates mediæ iam distinctæ supponatur, velut in primo exemplo a latere vides. Duc 5 primum aggregatum in 4 quadratum mediæ minoris, fit 20, divide per 13 aggregatum maiorum, exit $1\frac{7}{13}$, detrahe inde 4 quadratum mediæ minoris ex 36 quadrato mediæ maioris, relinquitur 32, divide per 13, exit $2\frac{6}{13}$, detrahe ex 5 minore aggregato, relinquitur $2\frac{7}{13}$, cuius dimidio in se ducto cum fiat $1\frac{413}{676}$, detrahes iam servatum primum proventum et est $1\frac{7}{13}$, relinquetur $\frac{49}{676}$, cuius R $\frac{7}{26}$ addita vel detracta ab $1\frac{7}{26}$ dimidio residui minoris aggregati, ostendit partes 1 vel $1\frac{7}{13}$. Igitur partes erunt secundum primam æstimationem 1, 2, 4 et 4, 6, 9 et iuxta secundam. Quod si aggregata sint mutua 1, ut prima cum tertia coniungatur, erunt gratia exempli 8, et 2, et 6, et 10, pervenies ad notitiam eodem modo 4, 2, 1, et 4, 6, 9, et $\frac{4}{5}$, 2, 5, et 7 $\frac{1}{5}$, 6, 5, et ideo duplex ordo videtur ex his haberi.

Regula secunda Pomponii de Bolognetis



Sint duæ lineæ ab et bc, gnomo, qui est differentia quadratorum cgfd, et producatur bc æqualis bc, dico quod rectangulum ex ac differentia in ae aggregatum laterum est æquale gnomoni dicto. Nam ex prima secundi Elementorum quod fit ex ac in ae est æquale ei quod



fit ex ac in se et in cb et bc. Sed quod fit ex ac in seipsum est æquale quadrato gf, et quod fit ex ac in bc est æquale rectangulo cg ex diffinitione data in initio secundi Elementorum. Et quod fit ex ac in be est æquale rectangulo df, quia be est æqualis ch, etenim supposita est æqualis bc et df est æquale ab, ex his quæ dicta sunt in primo Elementorum, igitur liquido patet propositum. Propositio 4

Cum ergo soleamus invenire ex basi orthogonii et altero latere reliquum latus hoc modo, sit latus recto oppositum 1307, alterum 564, ducuntur in se, et fiunt 1708249 et 318096, detrahe unum ex altero, fit 1390153, cuius R est alterum latus.

Sed ex præcedenti demonstratione longe brevius iunge 564 et 1307, fiunt 1871, detrahe etiam unum ex altero, fit 743, duc 743 in 1871, fiunt ut supra.

In hac operatione ingrediuntur figuræ 43, in priore autem figuræ 72. Maius etiam est discrimen et licentia errandi maior in maioribus numeris. At vero ex demonstratione simili poterimus iungere latera, nam si magna sint ambo, ut pote 975342 et 975362, ducemus maiorem in se, et duplicabimus, et ei addemus quadratum differentiæ, et habebimus quadratum lateris oppositi angulo recto. Fit ergo hæc operatio tota cum 75 figuris, at alio modo 120 figuris indiget. Præterea operationes addendi in hac sunt 16, in alia 34, quod si quantitas minor parva sit, et differentia magna erit, tunc ordinarium modum sequemur.

Modus multiplicandi noster ut 87 in 89, duc 90 in 90 proximum denarium fit 8100, duc defectum seu differentiam, in differentiam fit 3 totum 8103, iunge 3 et 1, fit 4, duc in 90, fit 360, detrahe ex 8103, relinquitur 7743, si vero volueris ducere 87 in 93, duc 90 in 90 fit 8100, relinquitur 8091. Duc tertio 88 in 94, duc 90 in 90, fit 8100, duc 2 minus in 4 excessum, fit 8, detrahe ex 8100, relinquitur 8092, detrahe 2 minus a 4 plus, fit 2, plus, duc in 90, fit 180, adde ad 8092, fit 8272. Duc demum 49 in 93, duc 50 in 90, fit 4500. Et 1 in 3, fit 3, detrahe, habes 4497, duc 1 in 90, fit 90, duc 3 in 50, fit 150, detraho 90 a 150, relinquitur 60, adde ad 4497, habes 4557. Vel ducas 47 in 88. Duc 90 in 50, fit 4500, duc 3 in 2, fit 6, iunge fiunt 4506, duc 3 in 90, fit 270, et 2 in 50, fit 100, iunge, fiunt 370, detrahe ex 4506, relinquuntur 4136, semper autem oportebit duo iungere tantum aut quatuor aut duo iungere et duo minuere. Et utilis est ad supputationem quæ mente sola fit.

Caput LVII

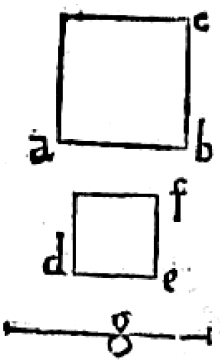
De tractatione æstimationis generalis capituli cubi æqualis rebus et numero

Iam docui te quod æstimatio generalis capituli cubi æqualis rebus et numero non est habita, neque per regulam generalem, neque specialem, nisi per illam ut invenias quantitatem quæ ducta in secundam producat numerum æquationis, et illa secunda quantitas gerit vicem gnomonis, et sit prima radix seu latus aggregati ex numero rerum, et secunda illa quantitate inventa. Et est hoc secundum naturam (ut dixi), quia linea ponitur latus aggregati duarum superficierum quadratarum, et ideo erit opposita angulo recto a lateribus illorum duorum quadratorum contento. Et dixi iam quod hæc quantitas describitur, ut in exemplo cubi æqualis 20 rebus p: 32, sic 32 p: 20 c. p. 32, 1 producens 20 cum producente 32, seu melius R 20 p: d. 32, id est R 20 p: diviso 32 per ipsam radicem. Aliter R 20 f. 32, id est R 20 cum fragmento 32, supple per eandem radicem divisi. Fragmentum enim est quod ex divisione prodit. Hoc igitur nomine utemur deinceps si cui aliorum aliquid arrideat, vel etiam novum imponat, modo res constet non gravabor. Igitur R 20 f. 32 est æstimatio cubi æqualis 20 rebus p: 32 numero, ut dictum est.

Capitula 40 in fine et 53 in fine.

Per 47 primi Elementorum

Dico ergo primum quod hæc æstimatio non potest esse, neque ex natura binomii, nisi ut mutantur, neque recisi,



sint abc et def quadrata illa, et ac numerus rerum, df quod provenit diviso numero per g rem ipsam. Quia ergo g, si est binomium, def est recisum, igitur cum abc sit numerus, erit aggregatum ex abc et def recisum, igitur latus eius est recisum. Non ergo g fuit binomium et, si ponas quod g sit recisum, erit def binomium, et aggregatum abc, def binomium, igitur latus eius binomium primum et non recisum.

Ex 54 decimi Elementorum et sequentibus

Capitulum 25 Artis magnæ

Cum igitur cubus æqualis rebus et numero, ut in exemplo præcedenti, ut supra visum est, habeat æstimationem R 17 p: 1, et hoc est binomium, et necesse est ut sit R 20 f 32, divido 32 per R 17 p: 1, et sufficit ducere R 17 m: 1 in 2, fit R 68 m: 2, quod additum ad 20, efficit 18 p: R 68. Et ita vides quod redit ad binomium, cuius R est R 17 p: 1 rei æstimatio, constat ergo quod nullum recisum potest esse eiusmodi. Neque etiam binomium cuius prima pars sit numerus, nam fragmentum erit necessario cum secunda parte m: et R , igitur totum esset recisum. Est igitur quærenda quantitas eius generis ut diviso numero per eam illius possit esse radix, et constat in binomio quinto (ut dixi) et in secundo sit R 12 p: 3, ut supra volo invenire cubum æqualem rebus et numero, fac ut in regula de modo, et videbis quod solum convenit secundo binomio et quinto. Regula ergo de modo duplica numerum æquationis seu æstimationis habitæ, et duc utrunque in se, et differentiam adde quadrato R æstimationis, et habebis numerum rerum. Inde accipe R quadrati rei et ab ea minue differentiam numeri rerum, et numeri quadrati rei, et hoc duc in rem ipsam, et producetur numerus æquationis. Exemplum proponitur R 7 p: 2 pro æstimatione, duplica 2, fit 4, duc 2 et 4 in se, fiunt 16 et 4, quorum differentia est 12, adde 7 quadratum R 7, fit 19 numerus rerum. Inde accipio R 112 quadrati R 7 p: 2, et ab ea minue 8, differentiam 19 numeri rerum et 11 numeri quadrati R 7 p: 2, nam ducta in se producit 11 p: R 112, igitur numerus illius quadrati est 11, hanc ergo differentiam minue a R 112 iam servatam, et est R quadrati, rei fiet R 112 m: 8, duc in rem quæ est radix 7 p: 2, habebis numerum 12, igitur 1 cu. æquatur 19 rebus p: 12 numero. Constat vero quod æstimatio non potest augeri nec minui stante numero rerum et æquationis eodem, nam si augeatur quod exit, minuitur igitur et R aggregati quæ est res, et si minuitur quod exit, augetur igitur et R aggregati quæ est res. Et ita dum augetur, minuitur et dum minuitur, augetur, quod esse non potest. Constat etiam quod talis æstimatio est communis binomio cubico invento in parte capituli et binomio superficiali hic declarato, et communis quantitas est

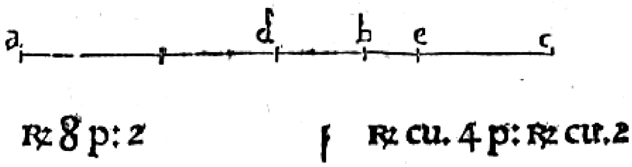
Capitulum 4

æstimatio generalis.

Caput LVIII

De communi quantitate duabus incommensis quot modis dicatur

Sunt ergo iam notæ duæ æstimationes cubi æqualis rebus et numero, una parte in maiore[1] numeri, et est binomii cubici, alia in parte minoris numeri binomii ex R quadratis secundi vel quinti, et communis æstimatio, quæ non potest esse incommensis[2], essent enim inter se commensæ, et quarta scilicet quæ intelligitur in parte minoris numeri, deficere igitur commune oportet, ut dicatur per coniunctionem.



Sint igitur ab et bc incommensæ, et sint coniunctæ ita ut medium earum sit d, id est aggregati, ut gratia exempli, ab sit R 8 p: 2 et bc Rcu: 4 p: Rcu: 2. Postquam igitur non potest esse communis æstimatio per commensum commune. Ita enim essent eiusdem naturæ inter se, aut erunt ergo per viam additionis et detractionis ut sit ad, igitur ad erit R 2 p: 1 p: Rcu: $\frac{1}{2}$ p: Rcu: $\frac{1}{4}$, quare bd erit R 2 p: 1 m: Rcu: $\frac{1}{2}$ m: Rcu: $\frac{1}{4}$, quam convenit addere quadrinomio, et ita potuissemus ab initio invenire ab et bc, sicut duo hæc quadrinomia eiusdem generis. Ponamus rursus quod primum inventum, gratia exempli, sit ae quod addat super ab Rcu: 2, ut eam oporteat detrahere, aut sit minus ce in Rcu: 2, igitur oporteret invenire ae et ec prius quæ sunt inæquales, et una est quantitas trinomia alia Rcu: simplex, hoc autem absurdum. Ideo via operationis nulla est. Necesse est igitur ut sit quantitas communis genere non ab nec bc. Et hoc esse potest, nam animal est commune homini et asino et bovi et equo, ita ab et bc continentur sub communi aliqua quantitate, quæ donec communis est

[1] 1663 has “*una in parte maiore*”.
[2] 1663 has “*in commensis*”.

omnibus habet solam eam proprietatem, quod cum dividitur numerus simplex æquationis, per illam ipsam est R numeri rerum cum eo quo prodit. Huic accidere potest ut sit numerus, ut binomium secundi et quinti generis; ut sit Rcu: binomia simplex, ut hic vel binomii cum suo reciso, vel ut sit alia quantitas semper cum illa proprietate. Dividamus ergo 16 per R 8 p: 2, exit R 128 m: 8, addo ad 20, fit 12 p: R 128, quadratum R 8 p: 2, nam cubus fuit æqualis 20 rebus[3] $* * *$ 2, exit Rcu: 16 m: 2 p: Rcu: 4, hoc adde ad 6 numerum rerum, fit Rcu: 16 p: 4 p: Rcu: 4, et hoc est quadratum Rcu: 4 p: Rcu: 2. Commune est ergo, ut vides, in utraque divisione prodire recisum, quod additum numero rerum, transeat in naturam similem quadrato rei. Numerus igitur rerum mutat naturam eius, quod provenit ex divisione numeri æquationis per rem.

[3]A part of the text seems to be missing here. In fact, in the 1570 edition there is a page break at this point and the remainder at the bottom of page 106 ("*bus*") does not match the beginning of page 107 ("*2, exit*"). The same is in the 1663 edition, even without page break.

Caput LIX

De ordine et exemplis in binomiis secundo et quinto

Cum semper incrementum numeri, et primus numerus incipiat a R primi numeri rerum, et dimidium eius R sit secunda pars binomii stabilis, quæ est numerus æstimationis, et primæ partis quadratum incipit a quarta parte primi numeri rerum, et inde tam numerus rerum quam etiam incrementa quadratorum primæ partis binomii, quæ est R augeantur per monades. Quæ facilius patent in suppositis exemplis primis quatuor, cum quintum sit extra ordinem manente æstimatione, velut in tertio exemplo primus numerus rerum est 9, cuius R est 3, a quo incipit primus numerus æquationis, et eius dimidium est $1\frac{1}{2}$ pars secunda æstimationis,[1] quæ remanet immobilis, et prima quæ est R $2\frac{1}{4}$ cuius quadratum est quarta pars primi numeri rerum, id est 9. Et augentur talia quadrata post modum per monadem, seu unum, ut etiam numerus rerum, ut in figura vides.

[1]1570 and 1663 have "*æquationis*".

Exemplum primum incrementi per 1.	Exemplum ſecundum incrementi per 2.
1 cu. 0 p: 1 poſ. ℞ $\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 0 p: 4 poſ. ℞ 1 p: 1
1 cu. 1 p: 2 poſ. ℞ 1$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 2 p: 5 poſ. ℞ 2 p: 1
1 cu. 2 p: 3 poſ. ℞ 2$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 4 p: 6 poſ. ℞ 3 p: 1
1 cu. 3 p: 4 poſ. ℞ 3$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 6 p: 7 poſ. ℞ 4 p: 1
1 cu. 4 p: 5 poſ. ℞ 4$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 8 p: 8 poſ. ℞ 5 p: 1
1 cu. 5 p: 6 poſ. ℞ 5$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 10 p: 9 poſ. ℞ 6 p: 1
1 cu. 6 p: 7 poſ. ℞ 6$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 12 p: 10 poſ. ℞ 7 p: 1
1 cu. 7 p: 8 poſ. ℞ 7$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 14 p: 11 poſ. ℞ 8 p: 1
1 cu. 8 p: 9 poſ. ℞ 8$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 16 p: 12 poſ. ℞ 9 p: 1
1 cu. 9 p: 10 poſ. ℞ 9$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 18 p: 13 poſ. ℞ 10 p: 1
1 cu. 10 p: 11 poſ. ℞ 10$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 20 p: 14 poſ. ℞ 11 p: 1
1 cu. 11 p: 12 poſ. ℞ 11$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 22 p: 15 poſ. ℞ 12 p: 1
1 cu. 12 p: 13 poſ. ℞ 12$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 24 p: 16 poſ. ℞ 13 p: 1
1 cu. 13 p: 14 poſ. ℞ 13$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 26 p: 17 poſ. ℞ 14 p: 1
1 cu. 14 p: 15 poſ. ℞ 14$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 28 p: 18 poſ. ℞ 15 p: 1
1 cu. 15 p: 16 poſ. ℞ 15$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 30 p: 19 poſ. ℞ 16 p: 1
1 cu. 16 p: 17 poſ. ℞ 16$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 32 p: 20 poſ. ℞ 17 p: 1
1 cu. 17 p: 18 poſ. ℞ 17$\frac{1}{4}$ p: $\frac{1}{2}$	1 cu. 34 p: 21 poſ. ℞ 18 p: 1

Exemplum tertium incrementi per 3.		Exemplum quartum incrementi per 4.	
1 cu.	0 p: 9 pos. ℞ $2\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	0 p: 16 pos. ℞ 4 p: 2
1 cu.	3 p: 10 pos. ℞ $3\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	4 p: 17 pos. ℞ 5 p: 2
1 cu.	6 p: 11 pos. ℞ $4\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	8 p: 18 pos. ℞ 6 p: 2
1 cu.	9 p: 12 pos. ℞ $5\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	12 p: 19 pos. ℞ 7 p: 2
1 cu.	12 p: 13 pos. ℞ $6\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	16 p: 20 pos. ℞ 8 p: 2
1 cu.	15 p: 14 pos. ℞ $7\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	20 p: 21 pos. ℞ 9 p: 2
1 cu.	18 p: 15 pos. ℞ $8\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	24 p: 22 pos. ℞ 10 p: 2
1 cu.	21 p: 16 pos. ℞ $9\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	28 p: 23 pos. ℞ 11 p: 2
1 cu.	24 p: 17 pos. ℞ $10\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	32 p: 24 pos. ℞ 12 p: 2
1 cu.	27 p: 18 pos. ℞ $11\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	36 p: 25 pos. ℞ 13 p: 2
1 cu.	30 p: 19 pos. ℞ $12\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	40 p: 26 pos. ℞ 14 p: 2
1 cu.	33 p: 20 pos. ℞ $13\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	44 p: 27 pos. ℞ 15 p: 2
1 cu.	36 p: 21 pos. ℞ $14\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	48 p: 28 pos. ℞ 16 p: 2
1 cu.	39 p: 22 pos. ℞ $15\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	52 p: 29 pos. ℞ 17 p: 2
1 cu.	42 p: 23 pos. ℞ $16\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	56 p: 30 pos. ℞ 18 p: 2
1 cu.	45 p: 24 pos. ℞ $17\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	60 p: 31 pos. ℞ 19 p: 2
1 cu.	48 p: 25 pos. ℞ $18\frac{1}{4}$ p: $1\frac{1}{2}$	1 cu.	64 p: 32 pos. ℞ 20 p: 2
1 cu.	51 p: 26 pos. ℞ $19\frac{1}{4}$ p: $1\frac{1}{2}$		

Exemplum quintum ubi res eadem est.

1 cu.	216 p: 0 pos. 6	1 cu.	162 p: 9 pos. 6
1 cu.	210 p: 1 pos. 6	1 cu.	156 p: 10 pos. 6
1 cu.	204 p: 2 pos. 6	1 cu.	150 p: 11 pos. 6
1 cu.	198 p: 3 pos. 6	1 cu.	144 p: 12 pos. 6
1 cu.	192 p: 4 pos. 6	1 cu.	138 p: 13 pos. 6
1 cu.	186 p: 5 pos. 6	1 cu.	132 p: 14 pos. 6
1 cu.	180 p: 6 pos. 6	1 cu.	126 p: 15 pos. 6
1 cu.	174 p: 7 pos. 6	1 cu.	120 p: 16 pos. 6
1 cu.	168 p: 8 pos. 6		

Ex quibus sequuntur quatuor corrolaria.

Corollarium 1. Ex hoc igitur odine habemus primum, quod oportet ut cum dimidium R sit pars secunda æstimationis, et R sit necessario numerus par vel impar, ut secunda pars sit numerus integer aut numeri dimidium.

Corollarium 2. Secundo, sequitur quod capitulum non potest esse generale, quia primus numerus necessario est quadratus, nam si non sit cum incrementa fiant per radicem numeri, igitur vel primus numerus, ut pote in tertio ordine, erit integer et non quadratus, aut quadratus sed non integer. Si quadratus et non integer, igitur cum alii numeri rerum fiant per additionem continuam, unius

Per ultimam decimi Elementorum

erunt omnes numeri rerum fracti, igitur non serviet capitulum cubo æquali rebus integris et numero ulla ex parte, quod est absurdum. Sin autem fuerit numerus et non quadratus, igitur cum incrementa fiant per R illius, nunquam prodibit numerus verus æquationis, et ita capitulum erit inutile.

Corollarium 3. Ex hoc sequitur etiam quod, nunquam numerus æquationis potest adeo augeri, ut quadratum dimidii eius sit maius cubo tertiæ partis numeri rerum. Nam tunc per primam regulam fieret æstimatio binomium cubicum. Et per hanc regulam binomium quadratum, et ita unum æquale esset alteri. Quod licet esse possit, ut in hoc exemplo[2] RV: cu. 20 p: R 392 p: RV: cu. 20 m: R 392, et est 2 p: R 2 et 2 m: R 2, quod est 4, non potest tamen continuari, et æstimatio resolvitur in numerum integrum.

Corollarium 4. Ex hoc habetur æstimatio, proposito numero rerum et æquationis, invenias omnia quadrata contenta sub numero rerum, et suas R cum quibus duces istas in differentiam numeri rerum et numeri quadrati, et si producatur numerus æquationis, tunc differentiæ illius et quartæ partis numeri quadrati inventi R est prima pars binomii, et dimidium R illius inventæ pars secunda binomii. Exemplum, 1 cu. æqualis est 30 p: 19 pos., sub 19 numero rerum continentur quadrati numeri, ut a latere vides.

16—4	3	12
9—3	10	30
4—2	15	30
1—1	18	18

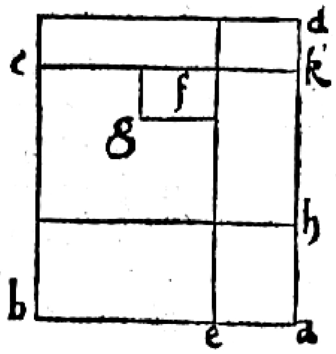
Cum vero differentia 9 a singulis sit ducta in R numeri bifariam, producitur 30 numerus æquationis. In posteriore accipiemus 1, quartam partem 4, et addemus ad 15 differentiam, fit 16, cuius R, quæ est 4, addito, constituit æstimationem 5. In priore addemus $2\frac{1}{4}$, quartam partem 9, ad 10 differentiam, fit $12\frac{1}{4}$, cuius R, quæ est $3\frac{1}{2}$, addito $1\frac{1}{2}$, dimidio 3, R 9, fit 5, ut prius rei æstimatio.

[2]1570 and 1663 have "*RV: cu. 20 p: R 329 p: RV: cu. 20 m: R 392*".

Caput LX

Demonstratio generalis capituli cubi æqualis rebus et numero

Et cum sit regula hæc quod ad æstimationem attinet specialis, ideo etiam non mirum est si sit etiam specialis in modo inveniendi, cum supponat numerum quadratum.

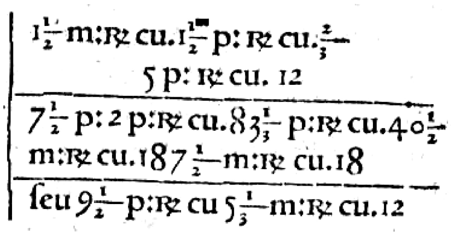


Ergo ut generaliter consideretur proponamus rem ipsam ab, et eius quadratum ac, quod constat ex aliquo numero diviso per ab, et proventu addito numero rerum, numerus igitur divisus nunc ponatur superficies. Ideoque poterit esse maior, et minor, et æqualis ipsi ac,[1] proponatur primum quod sit æqualis. Igitur quod provenit erit ba latus, et hoc est notum. Quippe numerus notus ideo nota, velut 1 cub. æqualis 25 p: 20 rebus, res est 5; et æqualis 36 p: 30 rebus, res est 6. Sit[2] modo bd maior quadrato ae in de,[3] et sit ae[4] unum, et quia ed erit quantum ad, et addita cd, constituit quadratum ac ex demonstratis, si ergo adderetur sola af ut fieret ac, esset numerus rerum ad unguem ec, sed quia additur df plus, constituatur fg æqualis fd, igitur superficies egc erit numerus rerum, puta 8, et superficies bd est numerus ex supposito, et differentia earum erit 24, qui est

[1] 1570 and 1663 have "*ae*".
[2] From now on, up to the end of the paragraph 1663 has capital letters to refer to the diagram.
[3] 1663 has "*AC in DE*".
[4] 1570 and 1663 have "*ac*".

dodrans 32, et triplum numeri rerum ab. Et ideo ed fit ex ea, id est uno, in ad, seu ak, cum adiecta kd. Igitur adiecto quadrato kd commune erit productum ex ab adiecta ad in kd monade addita æquale differentiæ numeri æquationis et numeri rerum cum quadrato kd. Si vero proponatur bh numerus parvus, et qui exit ah et monade ducta in ah fit eh superficies, quæ adiecta numero rerum constituit quadratum ac, igitur numerus rerum est superficies hce, et sit gratia exempli 18, et hb 8, igitur differentia erit 10, talis autem differentia est hc m: he. Hc fit ex hk in ab, he ex ha in ae. Igitur est divisa ak æqualis ab, ut tota in unam partem, altera detracta relinquatur 10.

Quando ergo superficies dividenda, et est numerus æquationis, fuerit magna, tunc in pluribus satisfaciet pars illa capituli iam inventi per binomia ex R cubicis. Quandoque etiam non. Sed quando superficies fuerit minor quadrato, non poterit. Postquam ergo supponimus monadem illa nota est et quia supponimus ak potentia etiam alogam capiamus. Gratia exempli, quod sit Rcu: 12 p: 2, cuius quadratum ac est Rcu: 144 p: Rcu: 768 p: 4. Volumus ergo dividere Rcu: 12 p: 2, ut ducta in unam partem, et addita reliqua sit æqualis 3. Gratia exempli et alteri parti; sit ergo pars una 1 pos., et erunt partes 1 pos. et Rcu: 12 p: 2 m: 1 pos., duc ergo 1 pos. in Rcu: 12 p: 2, fiunt pos. Rcu: 12 p: 2, et hoc est æquale Rcu: 12 p: 5 m: 1 pos., quare pos. Rcu: 12 p: p: 3 æquabuntur Rcu: 12 p: 5, divide numerum æquationis per numerum pos. inveniendo recisum Rcu: 12 p: 3, seu Rcu: 27 p: Rcu: 12, et est Rcu: $3\frac{3}{8}$ m: Rcu: $1\frac{1}{2}$ p: Rcu: $\frac{2}{3}$, duc in ipsum, fit $6\frac{1}{2}$, ducito Rcu: 12 p: 5 per $1\frac{1}{2}$ m: Rcu: $1\frac{1}{2}$ p: Rcu: $\frac{1}{3}$. Hoc igitur productum divide per $6\frac{1}{2}$, exit res ipsa $1\frac{6}{13}$ p: Rcu: $\frac{128}{6591}$ m: Rcu: $\frac{96}{2197}$. Hæc est una pars, alia igitur erit $\frac{7}{13}$ p: Rcu: 12 p: Rcu: $\frac{96}{2197}$ m: Rcu: $\frac{128}{6591}$, ducta igitur Rcu: 12 p: 2 in $1\frac{6}{13}$ p: Rcu: $\frac{182}{6591}$ m: Rcu: $\frac{69}{2197}$, et a producto detrahendo $\frac{7}{13}$ p: Rcu: $\frac{96}{2197}$ p: Rcu: 12 m: Rcu: $\frac{128}{6591}$, relinquetur 3 ad unguem. Nos autem quærimus simul quod ex ductu ab, id est R 12 p: 2, in ha, id est residuum quod fuit $\frac{7}{13}$ p: Rcu: 12 p: Rcu: $\frac{96}{2197}$ m: Rcu: $\frac{128}{6591}$, fiat numerus. Et hæc erit quantitas.


$1\frac{1}{2}$ m: ℞ cu. $1\frac{1}{2}$ p: ℞ cu. $\frac{2}{3}$
5 p: ℞ cu. 12
$7\frac{1}{2}$ p: 2 p: ℞ cu. $83\frac{1}{3}$ p: ℞ cu. $40\frac{1}{2}$
m: ℞ cu. $187\frac{1}{2}$ m: ℞ cu. 18
ſeu $9\frac{1}{2}$ p: ℞ cu $5\frac{1}{3}$ m: ℞ cu. 12

Clarum est igitur quod problema construitur hoc modo, et componitur ex regula de modo et positione. Invenias quantitatem quæ possit dividi in duas partes ut ductum totum in unam producat 3, gratia exempli, et in reliquam partem addito priore producat 8 pro exemplo. Quoniam ergo liquet quod genus æstimationis illius est quantitas ex genere, vel forma divise[5] ut $\frac{a}{b}$ superius, n [*sic*] est demonstratum quod non licet dividere nisi per quadrinomium in R quadratis in cubicis per binomium aut trinomium analogum, vel per regulam specialem, cum ergo in cæteris non liceat, dico quod adeo sunt notæ hæ quantitates ut illæ. Nam quod ad essentiam attinet ita aloga est R 2 ut Rcu: 7 p: R R 3 m: R R 5,[6] vel etiam totum hoc

$$\frac{\text{Rcu: 7 p: R R 3 m: R R 5}}{\text{Rcu: quad 10 p: R R 3 m: R 2}}.$$

Quod ad propinquitatem attinet nihil refert cum perpetuo liceat appropinquare. Quo vero ad operationes illæ sunt notissimæ, ideo propono eas. Sit ergo ut velim R $\frac{\text{a}}{\text{b}}$, capio R numeratoris et denominatoris, et est R b et R a, et superpono unam alteri eadem ordine, et habeo R $\frac{\text{a R a}}{\text{b R b}}$ et similiter $\frac{\text{Rcu: a}}{\text{Rcu: b}}$, et ita Rcu: $\frac{10}{\text{R R 5 p: Rcu: 2}}$ est $\frac{\text{Rcu: 10}}{\text{RV: cu. R R 5 p: Rcu: 2}}$, et ita volo ducere

$$\frac{10}{\text{R R 5 p: Rcu: 2}} \quad \text{in} \quad \frac{\text{R 2}}{\text{R R 5 m: R R R 2}}$$

fit

$$\frac{\text{R 200}}{\text{R R R 1953125 p: Rcu: R 4000 m: R R 10 m: R R Rcu: 128}}$$

et ita dividendo multiplicabimus crucis modum, et habebimus

$$\frac{\text{R R 500000 }^{\circ}\text{ R 20000}}{\text{R R 20 p: Rcu: R 32}}.$$

Et contrario modo contrario dividendo. Et ita in additione[7]

$$\frac{\text{R R 500000 [p:] R R 20 p: Rcu: R 32 m: R R 20000}}{\text{R R R 1953125 p: Rcu: R 40000 m: R R 10 m: R Rcu: 128}}$$

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[5] 1663 has "*divisæ*".

[6] 1570 and 1663 have "*Rcu: 7 p: R regula 3 m: R R 5*".

[7] 1570 and 1663 have "

$$\frac{\textit{R R 500000 R R 20 p: Rcu: R 32 m: R R 20000}}{\textit{R R R 1953125 p: Rcu: R 40000 m: R R 10 m: R Rcu: 128}}$$

".

et in detractione pariter

$$\frac{\text{R R } 20 \text{ p: Rcu: R } 32 \text{ p: R R } 20000 \text{ m: R R } 500000}{\text{R R R } 1953125 \text{ p: Rcu: } 4000 \text{ m: R R } 10 \text{ m: R R Rcu: } 128}.$$

Hæc igitur eo usque acta sint.

Finis

Basileæ, ex Officina Henricpetrina, anno salutis MDLXX, mense martio

BY GIROLAMO CARDANO
OF MILAN, CITIZEN OF BOLOGNA,

philosopher, physician, and famous mathematician,

OPUS NOVUM
ABOUT THE PROPORTIONS OF NUMBERS

OF MOTIONS, OF WEIGHTS, OF SOUNDS,

and of other things that are to be measured, not only established by geometry,
but also by various experiences and observations of natural things,
illustrated by an ingenious demonstration, adapted to many uses,
and distributed over five books.

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treasure very abstruse and unexhausted
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that is the necessary crowning of his algebraic logistic
investigating the numbers trough the hidden fineness of counting,
according to geometrical quantities, now finally brought to light

The work [is] useful and necessary
chiefly for physicists and mathematicians

With the favour and privilege of his illustrious majesty

BASEL

BY GIROLAMO CARDANO
OF MILAN, CITIZEN OF BOLOGNA,
VERY FAMOUS PHYSICIAN AND MATHEMATICIAN,

booklet on the Aliza rule,

THAT IS,

the necessary crowning

OF HIS OPUS PERFECTUM
OR OF THE ALGEBRAIC LOGISTIC

investigating the numbers trough the hidden fineness of counting,
according to geometrical quantities,
now finally brought to light

Note. The English translation has been made according to the following criteria.

In order for the reader to easily get oriented in the original Latin text, the translation has been made as literal as possible. When multiple interpretations can be given and one cannot decide which is the good one, the literal translation has the advantage to manage in maintaining the ambiguity. Even though the punctuation has been at best adapted to the contemporary habits, the English translation still sometimes mirrors the flowing of the Latin sentences, especially when they turn out to be long and burdensome.

Since the translation – as well as the transcription – aims to make Cardano's text more easily readable, whenever the Latin text clearly contains a mistake, both a grammatical or a mathematical one, it has directly been proofread in the English translation. The original version (with the mistakes) can still be checked in the Latin transcription. When instead I have been led to suppose a mistake since the text is not coherent, but the mistake cannot be clearly proofread, I have put in a footnote a likely hypothesis. Also the diagrams have been emended, especially when some letters were missing.

Moreover, for a vocabulary of Cardano's mathematical terms in the *De regula aliza*, see in the first part, Appendix B at page 359.

Table of contents

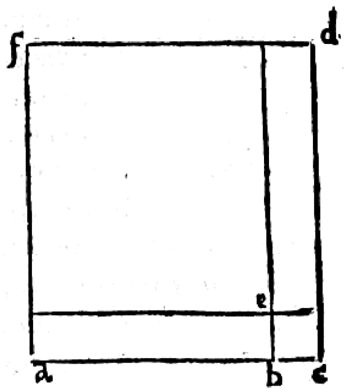
Chapter I

On the supposed [things] and on the ways

Since I have already demonstrated in the *Ars magna* that all the chapters are transformed, provided that the two principal [chapters] had been discovered [to be] general and not by transformation, it is plain that, having discovered another general chapter in addition to the chapter of the cube and some things equal to a number and [the chapter] of the cube equal to some squares and a number, which is deduced by transformation from the preceding [one], even if [the other general chapter] was general, all the chapters either of three or of four terms would not only be known in general, but also demonstrated, provided that this very same [chapter] is discovered by a demonstration. But in truth [the chapters] that consist of three terms are more easy to be discovered than [the chapters] that [consist] of four [terms], not only because these consist of several parts, but also since the latter ones are obtained by the former ones. But among these [chapters] that consist of three terms the easiest one is the chapter of the cube equal to some things and a number, the greatest part of which is already obtained by the first rule, and which stands according to the reasoning of the chapter already discovered, and also which we will show [to be] from that to the others not contrary to transformation. We will deal with that due to all these [things].

Therefore in the chapter of the cube equal to some things and a number, two [numbers] are proposed in particular, the number of the equality and also the number of the things. But more generally let now the cube of a certain line or quantity to be equal to the proposed numbers, the simple [one] and [the one] of the things. But it is necessary that we discover this generally, demonstratively, and the most easily. Being therefore the cube equal to two quantities of a different kind (otherwise this would not be general, if it was extended to the numbers alone and to their parts), it is necessary both that it is unravelled in just two parts, one of which is a number and [is] equal to an assigned [one], the other [part] contains as many parts different in nature as there are in the things equal to these.

Therefore it is necessary that the cube consists of at least two different parts, and therefore its side or the things, and indeed several quantities of different kinds cannot be made by a nature of a single kind through a multiplication however often repeated, as it is demonstrated by Euclid in the tenth Book (Proposition 20).[1] Truly, if two quantities alien to the number were contained in the thing, it would be necessary that they were incommensurable one to the other, otherwise they would have been equivalent to one [quantity]. But in this way it is necessary that the cube, which contains three parts, a number and two irrationals,[2] is made in order that it can be equal to some things and the number. Having divided the thing in two parts, it is necessary that the cube produces three of this sort and that, if [the thing is divided] in three, it produces four and so on, and (as it is said) in order that in those the number [is] equal to the proposed number, [and] that in truth the remaining parts are from the nature of the parts of the sides and [are] equal to the aggregates of these.



Again, in order to repeat [the things] which are said, let the cube on AD of the line AC be divided in three [parts], and it is agreed that in that four different principal parts there will be the cube of AB, the cube of BC, three times AB by the square of BC, and three times BC by the square of AB. Therefore it is

[1] *Elements* X.20: "If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied", see [Heath 1956c, page 49].

[2] Pay attention not to merge into the term 'irrational [*irrationalis*]' our modern meaning, even though Cardano's sense largely covers ours. Note that in the *Elements* you choose a magnitude to be rational and then define as 'irrational' the ones the measure of which is incommensurable with the measure of the first magnitude. Nevertheless, Euclid's meaning is narrower than ours. In fact, a magnitude is rational also if it is commensurable in square with the reference magnitude.

necessary to adjust the number and it can be done in seven easiest ways, as we have said. If we cannot find the value in the easier [ways], how will we find [it] in the more difficult [ways]? Then the first way is that the number is assigned to the cubes. And this way is discovered and is that part of the chapter that is considered in which we take the cubic roots of the parts of the number for the parts of the thing, and so the cubes of these are numbers, which joined are equal to the aggregate of the cubes on CE and on EF and the things themselves are equal to the six parallelepipeds that remain from the cube on AD. But, since the cubes of AB and BC can never be smaller than one-fourth of the whole cube AD (for the 9th [Proposition] of the second [Book] of Euclid[3] and for some Dialectic rules[4]), and indeed this does not happen except when AC will be divided in equal [parts] by B. Therefore being the number smaller than one-fourth of the whole cube on AD, it could not be equal to the cubes of AB, BC. And therefore the chapter was not general in this part.

Therefore the second way follows, and it is that all the parallelepipeds are given to the number and the cubes [are given] to the things. And, since the parallelepipeds cannot be bigger than three-fourths of the whole cube, because the cubes taken together cannot be smaller than one-fourth, therefore this chapter cannot be general, since, when the number will be bigger than three-fourths of the whole cube, it cannot be assigned to the parallelepipeds. Then, since neither the first nor the second of these chapters can be general by itself, nevertheless the both of them joined constitute a general chapter. Indeed, the first [way]

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[3]*Elements* II.9: "If a straight line be cut into equal and unequal segments, the square on the unequal segments of the whole are double of the square on the half and on the square on the straight line between the points of section", see [Heath 1956a, page 392].

[4]Note that both 1570 and 1663 have "*[p]er 9 secundi Elementi et regula Dialectica*". Since the proposition "'*per*' takes the accusative case, we would have expected to find either '*regulam*' if singular or '*regulas*' if plural, which is not the case. In these uncertain cases I prefer to leave the interpretation as loose as possible and this is why I chose to translate it using the plural. By the way, Cardano also wrote a *Dialectica* (which in volume I of the *Opera omnia*, pages 293-308). It contains a lot of classical logical rules and also some mathematics.

Pietro Cossali interprets the term '*regula*' to be a singular. He affirms that the "dialectic" rule is the fact that in an equality one can adds the same quantity on both sides of the equal, leaving the equality unchanged. "*Quanto alla Regola Dialettica, si tratta della nota proprietà delle equazioni secondo la quale è possibile aggiungere o sottrarre ad ambo i membri dell'equazione una stessa espressione, proprietà che consente di spostare un termine da un membro all'altro dell'equazione*", see [Cossali 1966, Chapter I, paragraph 3, footnote 20, page 30].

helps when the number will not be smaller than one-fourth of the whole cube or one-third of the things, which is the same. The second [way helps] when the number will not be bigger then three-fourths of the whole cube or bigger than three times the quantity of the things, which reaches the same. From this it is clear that, when the number will be between one-fourth and three-fourths of the whole cube, and of course the biggest width is at one half, then we can have a value by means of both rules, because the number can be related to the cubes and the parallelepipeds [can be related] to the things, and reciprocally the number [can be related] to the parallelepipeds and the things [can be related] to the cubes. Then the value will be discovered at the same time and by two ways. Or one may see by [the things] demonstrated in the Book on proportions (Proposition[5] 146) that the ratio of the aggregate of the cubes to the aggregate of the six parallelepipeds is as [the ratio] of the aggregate of the squares of the parts AB and BC, the product of AB times BC [being] subtracted from three times the product or the surface of AB times BC, or three times the surface AE.

The third, the fourth, and the fifth ways are not elegant to such a degree, although the previous two have something particular. Accordingly the third is when four parallelepipeds are given to the number, the remaining two [parallelepipeds] with the cubes [are given] to the things. But in these bodies this is particular, that the ratio of those bodies is as [the ratio] of the squares of the parts joined together to the products from both. For example, I take 7 things divided in 5 and 2, four parallelepipeds are 140, the cube of the parts with two parallelepipeds is 203, the ratio of 203 to 140 is as 29, [which is] the aggregate of the squares of 5 and 2, to 20, [which is] two times the product of 5 and 2. And similarly in the fourth way, the number is given only to the two mutual parallelepipeds. Nevertheless this has of particular, that it is extended from one-fourth of the number to perfection, whence it is seen that the chapter perfectly matches with the first rule. Then it is plain that in the second way it is necessary to produce the third part of the number by the mutual parallelepipeds, in the third [way it is

[5] *De proportionibus* 146: "The body that is made by a divided line times the surface equal to the squares of both parts having subtracted the surface of one part times the other, is equal to the aggregate of the cubes of both parts" or "*[c]orpus quod fit ex linea divisa in superficiem æqualem quadratis ambaru partium detracta superficie unius partis in alteram, est æquale agggregato cuborum ambarum partium*", see [Cardano 1570c, page 140r].

necessary to produce] the half [of the number], but in this [way it is necessary to produce] the whole number. And the aggregate of two mutual parallelepipeds is always equal to the product of the parts reciprocally multiplied by the aggregate of these or by the thing. But in the fifth way we give the number to one cube, the remaining seven bodies to the things, therefore it is really difficult and it falls back on a chapter of four terms, thence from that to the first [way]. Therefore it is the worst of all. Nevertheless, if it could be discovered, it would have been general as the two [ways] that follows.

The sixth way is that we give the number to one cube and three parallelepipeds that are made by the side of that cube times the square of the side of the other cube and [we give] the remaining four bodies, that is the cube with three parallelepipeds, to the things. And therefore it is misshaped, since what is made equal is similar, namely the cubes with the three opposite parallelepipeds, to which a dissimilar [aggregate] is made equal, in fact one aggregate is made equal to the number, the other [is made equal] to the things. Nevertheless in these bodies the particularity is that the difference of the aggregates is equal to the cube of the difference of the sides, as in the given example the first aggregate is 185, the second 158, the difference is 27, [which is] the cube of 3, which is the difference between 5 and 2. And this chapter, if it is discovered, is general.

The seventh way is when we give the number to the aggregate from the cube and two connected parallelepipeds with the opposite [one], and [we give] the things to the remaining four bodies, as in the example I add to 125, [which is] the cube of 5, 100, [which is] the double of the parallelepiped from 2 and 25, [which is] the square of 5, and 20, [which is] the parallelepiped from 5 and 4, [which is] the square of 2, and the whole makes 245. And similarly the remainder will be 8 plus the cube and 40 plus the double of 5 times 4, [which is] the square of 2, and 50, [which is] the parallelepiped from 2 by the square [of 5], that all gives 98. The particularity in this is that, both parts have the ratio of the square, and the square root is made by 75 times the root of a single part, as $\sqrt{245}$ is made by 7 times $\sqrt{5}$ and 98 [is made] from 7 times $\sqrt{2}$. And the ratio of such bodies is as [the ratio] of the parts of the thing, that is as 5 to 2. And nevertheless [the seventh way] undergoes to the same difficulty as before, namely that bodies by a similar generation are compared to natures different by themselves in kind, as number and things. The remaining [ways] are either compositions or anomalous, as if we had given the number to one parallelepiped, to three [parallelepipeds], to five [parallelepipeds], to two [parallelepipeds] not mutual, to four [parallelepipeds] two of which are not mutual, or to a cube and a parallelepiped, to two or to three [parallelepipeds] not of the same kind. The other [ways] are useless, as if we had given the number to the aggregate from both cubes and two or four parallelepipeds in whatever way. In fact, if it is not enough for the aggregate of the cubes, since the number is small, in what way is it enough for the same if the parallelepipeds are added?

Chapter II

On the special rules of Chapter XXV of the *Ars magna* of the cube equal to some things and a number

By[1] the first [rule] here, [let] the number of the things [to be] AD, the rectangular surface constructed according to that reasoning so that its height CD multiplied by the remaining, and the square of BC, which is AE, [being] subtracted, produces the number f. Then one may make the cube of CD, or CB, with the number f equal to the number of the things, that is to the parallelepiped from BC times AD. Then BL, [which is] the fourth part of BD and of the whole AL, [which is] the side of AM, is taken, to which MN, [which is] the half of BC, [which is] in BL, is added. Then AN, [which is] the body from AN times AD, will be again put RE the same thing. I say that the cube of AN is made equal to all the things AN, that is according to the number AD, and to the number f. In fact the square of AN is equal to the square of AM and of MN, hence (by the fourth [Proposition][2] of the second [Book] of the *Elements*) [it is equal] to the surfaces AL and BL by hypothesis and to the double of AM times MN, which is equal to the double of the square of MN produced by MN times MO (MO [being] taken equal to MN) and to the double of the product of MN times AO. But the double of the product of MN times MO, or of the square of MN, is by hypothesis equal to the double of BL. Therefore the square of AN is equal to the surface AL and to three times BL and to the double of AP times MN. But AL with three times BL is the whole surface AD. Therefore the square of AN is equal to the surface AD and to the double of MN times AO, but CD is the double of MN, therefore [the square of AN is equal] to the surface from CD times AO. Then the cube of AN, which is made by AN times the square of AN,

[1] Note that no diagram is present in 1570 and 1663 editions.

[2] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].

is equal to the parallelepiped from AN times AD and times [*sic*] CD times AO, but the same as from CD times the surface from AN times AO, which is equal to the surface AE, is made by AN, CD, AO. In fact the square of AM is made (by the fifth [Proposition][3] of the second [Book] of the *Elements*) by AN times AO with the square of MN, which is BL, for the reason that MN and MO are equal, the common surface BL [being] subtracted from AN times AO, AC is made. And the number f is made by hypothesis by CD times AO, therefore the cube of AN is equal to the parallelepipeds from AN times AD and from CD times the surface AE, which is equal to the number f, therefore the cube of AN is equal to the parallelepiped from AN times AD with the number f. But the number of the things, [which has been] put supposing that AN [is] the thing, because the number of the things was near, is from AN times AD. Therefore the cube is equal to the proposed things and number. This demonstration shows that this chapter does not arise from those seven ways [in Chapter I], but from another reason. From this it also follows that the chapter of the cube and a number equal to some things is simpler and from it [it is] more easy to [get to] the knowledge in the chapter of the cube equal to some things and a number. In fact in that it is enough to discover the part in the number of the things, the root of which, [being] multiplied by the remaining part, produces the proposed number. Furthermore this rule is not general by itself to the whole chapter of the cube equal to some things and a number, because, where the number had been bigger, it would have not satisfied. But it is general to the chapter of the cube and a number equal to some things. Also the second rule is not demonstrated from this [the seven ways in Chapter I]. But, no matter which part of the cube [being] taken for the number, the remaining [part] is equal to the things by hypothesis, therefore the surface is equal to the number of the things. If then that surface with what comes from the number divided by the same quantity was the square of that quantity, therefore that quantity is the thing. But this does not pertain to this purpose, since it does not depend on that division of the thing. And, thought according to the division of the square, which is the side of the cube, the thing is imagined

[3] *Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [HEATH 1956a, page 382].

as divided, but yet that division pertains more to the chapter of the cube and a number compared to some squares than [to the chapter of the cube and a number equal] to some things. The third rule arises from the third way of the previous division [the third way in Chapter I]. Similarly the fourth rule is demonstrated by the fourth way. In fact two cubes with four parallelepipeds to the two remaining parallelepipeds obtain the ratio that the squares of the parts of the divided thing with the surface of one part by the other [obtains] to the other surface that completes the square. Then, since the two squares and the surface from one part by the other are three quantities in continued proportion and [since] the roots of the extremes, or the sides of the squares reciprocally multiplied by hypothesis by the same squares, produce the number of the equality and also the parallelepipeds, therefore the parallelepipeds are equal to the number, moreover the remaining bodies [are equal] to the things. Then, being the cube equal to those eight bodies, it will also be equal to the things and the number. Since in truth those six bodies are equal to the things, this is agreed by the disposition of the cube and by [the things] demonstrated in the book on Proportions. The fifth rule originates from the second way [in Chapter I]. Being truly demonstrated there, I will not repeat it. The sixth rule of that [*Ars magna*, Chapter XXV] does not agree with the purpose, in fact is it special. The seventh [rule] arises from the the fifth [one]. But it is seen that [it is] different from that, because in that [fifth rule] the whole root, namely $\sqrt{28-3x^2}$, is supposed, in this [seventh rule] the half [of the root], [namely] $\sqrt{7-\frac{3}{4}x^2}$, [is supposed]. And because in one [fifth rule] the parts are reciprocally multiplied by the squares, in the other [seventh rule] the aggregate [is multiplied] by the product of the parts. Nevertheless they are the same as it is demonstrated in the book on Proportions. But, when $x^3 = 7x + 90$, it depends on the rule *de modo*. From 7 minus the square of the difference, which is x^2, x [being] taken the difference (as the product of the parts and this is always equal to the difference of the aggregate of the squares from the product of one part by the other) from $\sqrt{28-3x^2}$, [which has been] found by the rule *de modo*, 30, the third part of the number of the equality, is made. Thus the eight [rule] depends on the third [one] by the same procedure. But, since they are only proposed in order to discover the kind of the quantities that multiplied by some squares or roots produces the number, therefore I omit [it].

Chapter III

On the way to discover the quantities that help to the chapters by the product of one part by the other and by the square of the difference of the parts

Having said that $x^3 + 8 = 7x^2$, then divide 7 in two parts, one of which times the square of the other makes the number. And it is always necessary that the remaining part, which is not multiplied by itself, is the first *binomium* or *recisum*. Therefore, by Euclid, if [one] must produce the number [that is] multiplied by the remaining part, it is necessary that the second part is the first *binomium* or *recisum*. Then the first part can be the first, the second, and the third *binomium* or *recisum*, and can also be the fourth, the fifth, and the sixth [*binomium* or *recisum*]. Nevertheless [it can]not [be] a *recisum*, because, being the first [part] a root, the second part is necessarily a *binomium*, because the first [part] is a *recisum*, therefore in both there is a positive root, therefore it cannot be that number that is divided at the beginning. Then I say that, the number of the squares [being] divided in two parts, which we call 'principal' (and we call principal [a part] that is multiplied by itself), multiply the first [part] by the double of the second [part] to find the other two parts. And you will deduce the square of the first [part] from the product and the root of the *residuum* is the part to add or to subtract to the principal [parts] with the agreed conditions. And similarly, to produce a positive number, multiply the difference of the principal [parts] between them and the product by the double of the first principal [part], and what is produced is the searched number. Therefore in the proposed [example], I divide 7 in 4 and 3, I multiply 4 by the double of 3, 24 is made, I subtract CB, [which is] the square of 4, 8 is left, the root of which added or subtracted, and thus added or subtracted to 3 in a reciprocal way, it constitutes the parts $4 + \sqrt{8}$ and $3 - \sqrt{8}$, or $4 - \sqrt{8}$ and $3 + \sqrt{8}$. And therefore in the equality it is to be noted that the first principal part and the second [principal part] make the same by the means of the *binomium* and *recisum*. And indeed each of those values, namely $4+\sqrt{8}$ and $4-\sqrt{8}$ is the value of $x^3+8=7x^2$. Then, in order to have the number of the value or what is produced, take 1, [which is] the difference of the principal parts 3 and 4, and multiply [it] by itself, 1 is made, multiply [it] by the double of the first principal [part] 8, the searched number 8 is made. Therefore another example, I divide 7 in 3 and 4, and 3 is the first part, I multiply [it] by the double of 8, 24 is made, I take away the square of the first [part] 9, 15 is made. And the part that is going to be subtracted from 4 and added to 3 will be $\sqrt{15}$, because of [the things] that have been said. In order to [have] the number take the difference, which is 1, multiply [it] by itself, 1 is made, multiply [it] by the double of the first principal [part] 3, the searched number 6 is made. I have therefore $x^3+6=7x^2$. And similarly I divide 7 in $\frac{3}{2}$ and $\frac{11}{2}$ and I multiply the double of $\frac{11}{2}$ by $\frac{3}{2}$, $\frac{33}{2}$ is made, I subtract $\frac{9}{4}$, [which is] the square of $\frac{3}{2}$, $\frac{57}{4}$ is made. Therefore add $\frac{3}{2}$ to the root of that, and subtract [it] from $\frac{11}{2}$, and the parts $\sqrt{\frac{57}{4}}+\frac{3}{2}$ and $\frac{11}{2}-\sqrt{\frac{57}{4}}$ are made. But the product by the difference of $\frac{11}{2}$, and $\frac{3}{2}$ by itself is made, and 16 is made and, 16 [being] multiplied by 3, [which is] the double of $\frac{3}{2}$, 48 is made. In a similar way take two parts from 8, the product of one of which times the square of the other makes 9. One [part being] taken $\frac{9}{2}$, the other [part] $\frac{7}{2}$, the purpose is well known by the second rule, namely that we will produce 9 or 7. In fact the square of the difference is 1, and the product by the double of $\frac{9}{2}$ constitutes 9, and the double of $\frac{7}{2}$ constitutes 7, and thus, if [8] is divided in 5 and 3, 40 or 24 will be produced. Therefore in the first division there will be the parts $\frac{9}{2}\pm\sqrt{\frac{45}{4}}$ and $\frac{7}{2}\pm\sqrt{\frac{45}{4}}$. And thus, if you had divided [8] in $\frac{5}{2}$ and $\frac{11}{2}$, you will have 45 and the parts will be $\sqrt{\frac{85}{4}}+\frac{5}{2}$ and $\frac{11}{2}-\sqrt{\frac{85}{4}}$. In fact it cannot be otherwise, as we have said at the beginning. And if you had divided [8] in $\frac{3}{2}$ and $\frac{13}{2}$, we will have 75. And by this operation it is well known that no division except [the one in] half of integers can be useful. In such a way indeed, multiply [the thing] divided by a and b, and let c to be the difference. Since it is divided by $\frac{9}{2}$ and $\frac{11}{2}$, or by $\frac{13}{2}$ and $\frac{7}{2}$, and in such a way the difference from each [parts] is an integer number, therefore, since a and b are neither integers nor half [of integers], they will be bigger or smaller [than an integer], therefore c is bigger or smaller than an integer. And, since the number that must be produced is necessarily made by the square of c times the double of a, where a is the first part and a is now neither an integer number nor the half [of an integer], therefore the double of a is not a number, but something from the same denomination as c. But, since it is multiplied by itself and is beyond an integer, the product will be something else made by the kind of the squared denomination, as, if it was $\frac{5}{2}$, it will be $\frac{49}{9}$, but, $\frac{49}{9}$ [being] multiplied by the double of the smaller $\frac{23}{3}$, since the denomination to $\frac{49}{9}$ is composite, it produces the number of the kind of the fractions, the denominator of which is 27, namely $\frac{1127}{27}$ as it is demonstrated in its place. Therefore a certain integer number cannot be the product, but 10 cannot be divided in integers and half [of integers] except in ten ways, therefore the number of the equality cannot be nothing but in ten [ways]. But 10 squares equal to the cube and a number can be made equal for each integer number up to 148, as it is demonstrated in the Book on proportions. Therefore the division of the *binomia* and *recisa* and by the integers does not satisfy, but 138 integer numbers are lacking, except those in which the same parts of the numbers, in which these quantities cannot satisfy for the same reason, are added. But for now it is enough to show [it] for the integers and, since all the chapters are transformed, it is clear that there is the same lack in those.

In addition it [is] to be considered that the ratio of the number with the addition of the product to the number without addition is had by this rule, as if 8 is divided in 3 and 5, it produces 45 and, the parts that are by the rule [being] added, 24 is produced, as I have said before. As I have said above for the root by the second rule, I say that the ratio of 45 to 24, as it is well known that the demonstration is had, [is] as 15, the product of 5 by 3, to the double of the square of the difference. And thus, dividing 8 in 6 and 2, 24 is made by the first way and 64 is made by the second rule, and the ratio of 24 to 64 [is] as [the ratio of] 12, [which is] the product of 6 times 2, to 32, [which is] the double of the square of the difference 4.

Chapter IV

On the way to reduce all the quantities that are called 'first sides' by the tenth [Book] of Euclid in short

Euclid has set up twenty-four *alogai* or irrational lines and one *rhete* or rational [line], which plays the role of the number.

The rational [is] as 6 or 7. The simple irrational, which is commensurable to a number and to an irrational only to the power, [is] as $\sqrt{6}$ or $\sqrt{7}$, which is the tetragonical side of the rational surface, but not of the squared [surface]. The medial irrational is the tetragonical side of the simple *alogai* surface, as $\sqrt{\sqrt{6}}$ or $\sqrt{\sqrt{7}}$. Moreover, six kinds of *binomia*, about which we are going to talk, rise by the comparison of the irrational [lines] between them or with the rational [ones].

The characteristic of the first *binomium*, and similarly of the *recisum*, is that the first part is a number and the second [part] an irrational, and that the squares of those differ by a squared number, as $3+\sqrt{5}$, the squares of which are 9 and 5, the difference is 4. Moreover the sides of 9 and 4 are 3 and 2, and thus $3-\sqrt{5}$ is a first *recisum*.[1] And thus $4+\sqrt{12}$. Moreover, as it is said in the third Book [of the *Opus arithmeticæ perfectum*],[2] the *binomia* as much as the *recisa* agree that the first part of the first and of the fourth [*binomia*] is a number, the second [one is] a root, [and that] the second part of the second and of the fifth [*binomium*] is a number, the first [is] a root, both parts of the third and of the sixth [*binomia*] are roots. But the first order, that is 1, 2, and 3, differs from the second order, that is 4, 5, and 6, because in the first order the bigger part, or the first [one], is always more powerful than the smaller part, or than the quantity commensurable to the first part according to the square, in the second order [the quantities are] incommensurable. Moreover the general way to have the root for all the *binomia* and *recisa* is that you multiply the second part by itself, as $\sqrt{12}$ by itself makes

[1]Note that Cardano uses '*residuum*' to mean '*recisum*'.
[2]See [Cardano 1663c, Chapter III, pages 307-308].

12, of which you always take the fourth part, and it is 3. Make from the first part 4 two parts that produce 3, and they will be 3 and 1, the joined roots of which make $4 + \sqrt{12}$. And, while producing the squares, a rational is always produced and an irrational is made by the double of one part by the other. And it is the rule that we have put in the third Book of the *Opus perfectum* (Chapter 19, Rule 2).[3] Since in truth Euclid did not put the side of the line, he discovered that he took the surface by the rational and the first *binomium*, the tetragonical side of which is said to be a certain *binomium*. Moreover it is agreed that such a surface is also a first *binomium* and it is the third rule, which is taken out from Euclid (Book 10, Proposition[4] 54) as said above. Indeed each quantity commensurable to a first *binomium* is a first *binomium* (Proposition[5] 66). Moreover they are commensurable, since their ratio was as a number to a number by [the things] said by him [Euclid] in the tenth Book (Proposition[6] 6). But all the *binomia* of the same type are not commensurable between them, and it is the fourth rule. In fact, as it is seen, $3 + \sqrt{5}$ and $4 + \sqrt{7}$ are first *binomia* and nevertheless they are not commensurable between them by the same Euclid (Proposition[7] 17). The same happens in each kind of irrationals, namely that the commensurable [ones] are of the same type, nevertheless those that are of the same type are not reciprocally commensurable. Afterwards I have discovered that multiplying $\sqrt{5}$ by 3, and it is $\sqrt{45}$, which, if it is multiplied by $\sqrt{5}$, is 15 is the same as if we multiplied $\sqrt{5}$ by itself, $\sqrt{25}$ is made, and 5 by 3, 15 is made. You see how easy it is and this is the fifth rule. Moreover the sixth [rule] contains four propositions that are reciprocally transformed. And it is that, when there will be two quantities, AD the bigger and C the smaller, and if AB was divided in D so that the half, [which is] the half of the part C, falls between BD and DA, therefore, if AB is

[3]See [Cardano 1663c, Chapter XIX, pages 323-324].

[4]*Elements* X.54: "If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called *binomial*", see [Heath 1956c, pages 116-120].

[5]*Elements* X.66: "A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order", see [Heath 1956c, pages 145-147].

[6]*Elements* X.6: "If two magnitudes have to one another the ratio which a number has to a number, the magnitude will be commensurable", see [Heath 1956c, pages 26-27].

[7]*Elements* X.17: "If to any straight line there be applied a parallelogram deficient by a square figure, the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application", see [Heath 1956c, pages 41-45].

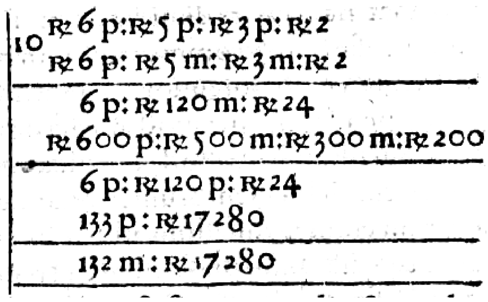
more powerful than C, the square of the commensurable BD is commensurable to DA, otherwise not. And if BD is commensurable to DA, the whole AB is more powerful than C, [which is] commensurable to the same AB, otherwise not. But this depends on that, that, DE [being] removed equal to DA, [which is] the square of AB, it surpasses the square of C, which is equal to four times BD times DA times the square of BE by the eighth [Proposition][8] of the second [Book] of the *Elements*. But the use of those lines is, as it is plain, to discover the *binomia*, the *recisa*, and both the said bimedial [lines]. The seventh rule is assumed by Euclid and it is that, if the surface equal to the square of the *binomium* is joined to a rational, the second side is a first *binomium*. And it is the same as the third rule, it is allowed to be seen [that is it] the transformed. Indeed, as we have said in those, since the side of the first *binomium* is a certain *binomium*, therefore the squares of all the *binomia* are first *binomia*, but that rational side is just as a ratio and it does not change type, therefore those other side is necessarily a first *binomium*. And all [the things] that are said here about the *binomia* are realised for their *recisa* and are general in all the quantities [while] comparing the aggregate to a *recisum* or to a remainder. Therefore by this eight rule we will only deal with six kinds of *binomia* and with five others, and we will put the terms separately here in the side.

Rhete	Aloga	Media	
Binom 1	Binom	Ref. 1	Refid
Binom 2	Bimed 1	Ref. 2	Ref.med 1
Binom 3	Bimed 2	Ref. 3	Ref.med 2
Binom 4	linea ma.	Ref. 4	linea mi.
Binom 5	Pot.in Rat & Med	Ref. 5	cũ Rat & Med
Binom 6	Pot.in duo Med	Ref. 6	cũ Med & Med

Also the way to join these roots, as it appears in the same third Book, is more suitable, as you divide the bigger root by the smaller [one], and take the root of what comes forth, to which add, that is, [*sic*] and multiply by itself. And the product by the square of the smaller root is searched for. The tenth rule is that the divisions of this sort are much more clearly seen in a figure that in numbers,

[8]*Elements* II.8: "If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line", see [HEATH 1956a, pages 389-391].

namely, put the root of the biggest *alogai*, it is agreed by the [Proposition][9] 44 of the first [Book] of the *Elements* that the surface equal over the given rational can be made equal to that. Therefore, as I have said in the seventh rule, its second side will be of the same nature as the surface is, and I could divide the rational or *rhete* surface by that side. And suddenly by the first Definition[10] of the second [Book] of the *Elements* I will have the second side, which can hardly be obtained in numbers and, when it is obtained [in numbers, it is] by a great effort, there in truth it is immediately clearly seen, but the general method is not yet discovered in numbers. Moreover, according to the eleventh rule, to discover the *recisum* up to four quantities, as if I want to divide 10 by $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$. Put the *recisum* by equal opposite parts and you will have the dividend and the divisor, which is $6+\sqrt{120}-\sqrt{24}$, to which you add the *trinomium* that, [being] multiplied by the *recisum*, produces $132+\sqrt{17280}$. Furthermore multiply that *trinomium* by the *quadrinomium* and you will have the dividend, which I omit for weariness and for the sake of brevity.


10 ℞ 6 p: ℞ 5 p: ℞ 3 p: ℞ 2
℞ 6 p: ℞ 5 m: ℞ 3 m: ℞ 2
6 p: ℞ 120 m: ℞ 24
℞ 600 p: ℞ 500 m: ℞ 300 m: ℞ 200
6 p: ℞ 120 p: ℞ 24
133 p: ℞ 17280
132 m: ℞ 17280

Again I place the *recisum* near, and I multiply [it] by the *binomium*, and 144 is made according to the division. Moreover, the *recisum* [being] multiplied by the quantity that consists of twelve terms, twenty-four terms are made dividing. They can perhaps be reduced to fewer [terms], because those roots are commensurable. And, if they go over four non commensurable quantities, still in two cases it can be divided. Or when it will have a root, as[11] $6+\sqrt{12}+\sqrt{24}+\sqrt{8}$, or when it will have a divisor, as if someone says to divide 10 by $\sqrt{24}+\sqrt{15}+\sqrt{15}+$

[9] *Elements* I.44: "To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle", see [Heath 1956a, pages 341-345].
[10] *Elements* II Definition 1: "Any rectangular parallelogram is said to be contained by the two straight lines containing the right angle", see [Heath 1956a, page 370].
[11] Note that $6+\sqrt{12}+\sqrt{24}+\sqrt{8}=(\sqrt{3}+\sqrt{2}+1)^2$.

$\sqrt{12}+\sqrt{10}+\sqrt{5}+\sqrt{2}+\sqrt{2}$. Since this is produced by[12] $\sqrt{6}+\sqrt{5}+\sqrt{2}$ times $\sqrt{3}+\sqrt{2}+1$, we will divide by the other of those by the given rule, thence what comes by the remaining [one]. And therefore we can reduce [it] to one case, when the divisor can be divided by a *multinomium*, so to say, such that the number of the terms decreases. In fact the discovery of the side is a certain division. At last the reason of the ratio of the surface to the line is to be considered. And I say that the surface[13] A is to the line BC as the rectangular surface equal to A is made over BC. I say that the second side of CD is the ratio, or what is made equal to the ratio of A to BC, because one[14] is made by the ratio multiplied by the boundary, the other [is] the boundary as we see in the numbers. And it is by the definition of a ratio, and A is made by the side of CD times BC, because BD is equal to A, therefore CD is said [to be] the true ratio of the surface A to the line BC. And it is more clearly seen than [the ratio] of a surface to a surface and [than the ratio] of a line to a line. Nevertheless (as I have said) Euclid overlooked [it], since he considered the lines to be indivisible in width. Nevertheless we say that it is composed by lines just as the line is made by the flow of a point and the time [is made by the flow] of an instant, and the others in the same way. And this is the twelfth rule.

The side of the second *binomium* is the first bimedial, as Euclid teaches in his way. Therefore it could be showed either under the term of a universal root or by a right reasoning. Then we take $\sqrt{72}+8$, and by the second rule we multiply 8 by itself, 64 is made, the fourth part of which is 16. Make two parts out of $\sqrt{72}$, from the reciprocal product of which 16 is made. I take the half of $\sqrt{72}$, which is $\sqrt{18}$, by the fifth [Proposition][15] of the second [Book] of the *Elements*, and I multiply [it] by itself, 18 is made, I throw away 16, 2 remains, the root of which [is] added and subtracted to $\sqrt{18}$, and it makes $\sqrt{32}$ and $\sqrt{8}$ by our rule. Therefore the roots of those parts, that is $\sqrt[4]{32}+\sqrt[4]{8}$, constitute the first bimedial and, [being] subtracted one from the other, [they constitute] the remainder of

[12]Actually, $(\sqrt{6}+\sqrt{5}+\sqrt{2})(\sqrt{3}+\sqrt{2}+1)=\sqrt{32}+\sqrt{24}+\sqrt{15}+\sqrt{12}+\sqrt{10}+\sqrt{5}+\sqrt{4}$.

[13]Note that no diagram is present in 1570 and 1663 editions.

[14]1570 and 1663 have "*quia n ex proportione ducta in terminum fit, alter terminus*".

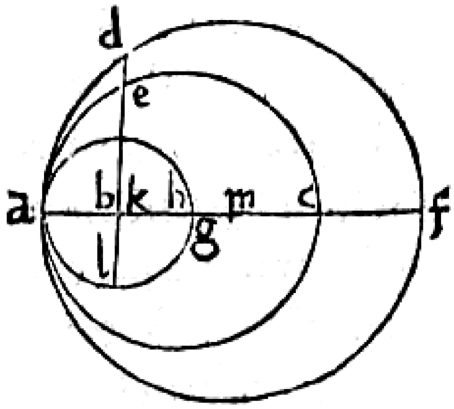
[15]*Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [Heath 1956a, page 382].

the first bimedial. They are indeed commensurable only to the power, in fact the square of $\sqrt[4]{8}$ is the half of the square of $\sqrt[4]{32}$, in fact $\sqrt{8}$ is the half of $\sqrt{32}$. And they contain a rational surface, and the bigger is more powerful than the smaller times the square of $\sqrt{8}$, the side of which is $\sqrt[4]{8}$, [which is] incommensurable in length with $\sqrt[4]{32}$, which is the purpose. And this way is by far easier than all the others and is explained by us according to the example.

The root of the third *binomium*, as $\sqrt{32} + \sqrt{24}$, is taken and it will be $\sqrt[4]{18} + \sqrt[4]{2}$, which are commensurable only to the power and contain the medial surface $\sqrt[4]{36}$ or $\sqrt{6}$, and it is different from the preceding [one] in this, and the bigger is more powerful than the smaller times the square of $\sqrt[4]{8}$, which is $\sqrt[4]{18}$, [which is] incommensurable in length, as if the root of the half to the root of the whole, or the root of 3 to $\sqrt{2}$, and this is called second bimedial.

I find the root of the fourth *binomium*, as $3 + \sqrt{6}$. And, in order to avoid the fractions, I take $6 + \sqrt{24}$, I take the root of it, which is $\sqrt{3+\sqrt{3}} + \sqrt{3-\sqrt{3}}$, you will multiply the first $\sqrt{3+\sqrt{3}}$ by itself, $3+\sqrt{3}$ is made, multiply $\sqrt{3-\sqrt{3}}$ [by itself], $3-\sqrt{3}$ is made, join [them], 6 is made. Multiply $\sqrt{3+\sqrt{3}}$ by $\sqrt{3-\sqrt{3}}$, the square is made by multiplying a certain one separately by itself, $3+\sqrt{3}$ and $3-\sqrt{3}$ are made, thence reciprocally, 6 is made, thence taking the root, which will be $\sqrt{6}$, and this will be the product of $\sqrt{3+\sqrt{3}}$ times $\sqrt{3-\sqrt{3}}$. But, since it is necessary to make twice the multiplication, multiply $\sqrt{6}$, which are $\sqrt{24}$. But, since by the fourth [Proposition][16] of the second [Book] of the *Elements*, the square of $\sqrt{3+\sqrt{3}} + \sqrt{3-\sqrt{3}}$ is equal to the square of the parts with the double of one times the other. Therefore the square of $\sqrt{3+\sqrt{3}} + \sqrt{3-\sqrt{3}}$ is $6+\sqrt{24}$.

[16] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].



Moreover we take AB in the figure, to which we make the triple BC directly contiguous and we take the half of AC, which is H, we delineate on it the circle AEC and we draw BE, which will be the side or $\sqrt{3}$, perpendicularly from AB, we directly add AC to that and we cut [it] off from AC, and BF $3+\sqrt{3}$ and BG $3-\sqrt{3}$ are made on CA. Therefore AF [is] divided in equal in K and M, we draw the circles ADF and AL, and we prolong BED, [which is] the side of BF, from the part E and [we prolong] BL on the opposite [side], [which is] the side of BG. And so it will be equal to $\sqrt{3+\sqrt{3}}+\sqrt{3-\sqrt{3}}$. From this it is well known that the Geometrical presentation is clearer than the arithmetical [one]. And to say it more evident, what in truth is made through the numbers is more faithful, certain, and safe than what is proved by the experience, as I did above. And it is the thirteenth rule. Furthermore another beautiful fourteen [rule] follows, namely in succession (thought not really according to the theory) and it is that, just as one is the beginning in the natural things, so also the monad, which somebody calls the 'unit', [is the beginning] in the passage from the arithmetic to the geometrical figures, and it is the necessary beginning of the discovery on which the whole theory is founded. I only say that all the properties of the bigger lines fit to $\sqrt{3+\sqrt{3}}+\sqrt{3-\sqrt{3}}$ (by the 39$^{\text{th}}$ [Proposition][17] of the tenth [Book of the *Elements*]). In fact they are two quantities incommensurable

[17] *Elements* X.39: "If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, the whole straight line is irrational: and let it be called *major*", see [HEATH 1956c, pages 87-88].

in power, each *binomium* is incommensurable to its *recisum* (by the Proposition[18] 16). Furthermore it is true in all the irrationals, and it is easily demonstrated, and it is the fifteenth rule. And both the squares equally taken are rational and the product of one by the other [is] medial, in fact the joined squares make 6 and the product of one by the other is $\sqrt{6}$ (by the Proposition[19] 33). It is to be noted that in [the writings of] Euclid one operation is added, namely that the parts are multiplied by themselves and the square of the medial smaller part is added, thence the universal root is taken. And this operation is superfluous in numbers, because we can take the root of whatever quantity. [To find the root] of the fifth *binomium*, and it is as $\sqrt{24}+4$, we multiply the half of the smaller by itself by the second rule, 4 is made. Make out of $\sqrt{24}$ two parts that produce 4 and they will be the halves of $\sqrt{6}$, which multiplied by themselves make 6, add 4, 2 is left, the root of which added to $\sqrt{6}$ makes $\sqrt{6}+\sqrt{2}$ and subtracted makes $\sqrt{6}+\sqrt{2}$, and the universal root of these quantities constitutes the quantity that can [be] in rational and medial (by the Proposition[20] 40). The squares of those indeed are $\sqrt{24}$ and the product of one by the other is $\sqrt{4}$, which is 2, the double of which is the number 4 of the *binomium*. And they are commensurable to the power, since they are, as I said, the *binomium* and the *recisum*, $\sqrt{6}+\sqrt{2}$ with $\sqrt{6}-\sqrt{2}$. Therefore, since all those quantities are roots of *binomia* multiplied by themselves, they produce their *binomium*. And it is the sixteenth rule. Likewise, as I said, the *recisa* and the remainders, which are the roots of the *recisa*, are to be understood. [To find the root] of the sixth *binomium*, and it is as $\sqrt{24}+\sqrt{12}$, we multiply $\sqrt{12}$ by itself, 12 is made, the fourth part of which is 3, we make out of $\sqrt{24}$ two parts that produce 3 and we multiply $\sqrt{6}$, [which is] the half of $\sqrt{24}$, by itself, 6 is made, remove 3, 3 is left, the root of which added and subtracted to $\sqrt{6}$ makes $\sqrt{6}+\sqrt{3}$ and $\sqrt{6}-\sqrt{3}$. The joined universal roots of those quantities constitutes

[18] *Elements* X.16: "If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable", see [Heath 1956c, pages 40-41].

[19] *Elements* X.33: "To find two straight lines incommensurable in square which make the sum of the squares on the rational but the rectangle contained by them medial", see [Heath 1956c, pages 75-78].

[20] *Elements* X.40: "If two straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational; and let it be called *the side of a rational plus a medial area*", see [Heath 1956c, pages 88-89].

the quantity that can [be] in two medial, in fact they are at first incommensurable to the power, because the squares of those are a *binomium* and a *recisum*. Next, because the compound by the squares is the medial $\sqrt{24}$, and the product of one by the other is $\sqrt{3}$, and $\sqrt{3}$ is incommensurable to $\sqrt{24}$, in fact the ratio of one to the other [is] $\sqrt{8}$. The purpose is established by Euclid. In addition to this he proves that the said irrational quantities cannot be differently divided in order to be of the same kind that they were before the separation (by the Proposition[21] 41). Namely $6+\sqrt{20}$ is divided in 6 and $\sqrt{20}$ and it constitute the first *binomium*, it cannot be differently divided in order to constitute the same. When the rational quantities will be divided by a certain remainder, a *binomium* of the same order commensurable in its parts to that remainder will come out, although a certain remainder of the same order similarly commensurable in its parts comes out by a *binomium*, and they will be those parts of the *binomia* with the *recisa*, and moreover in the same ratio of the *binomia* and *recisa*, and a rational [line] is always produced by the product of the remainder by the *binomium*. And these [things] are demonstrated by Euclid at the end of Book 10 (Propositions[22] 112, 113, and 114). From this it is agreed that a rational [line] is always produced by a *binomium* commensurable to a *binomium* of a remainder, or by a remainder commensurable to the remainder of a *binomium*. Likewise, if the second side of a surface equal to the square of a line rational only to the power is divided by a *binomium* or by a remainder, it comes out the *binomium* or the *recisum* with the same ratio of the parts, sometimes of the same order, sometimes of a different [one]. As dividing $\sqrt{24}+\sqrt{3}+\sqrt{2}$, $\sqrt{72}-\sqrt{48}$ comes out. But the ratio of $\sqrt{72}$

[21]*Elements* X.41: "If two straight lines incommensurable in square which make the sum of the squares on them medial, and the rectangle contained by them medial ans also incommensurable with the sum of the squares on them, be added together, the whole straight line is irrational; and let it be called *the side of the sum of two medial areas*", see [Heath 1956c, pages 89-92].

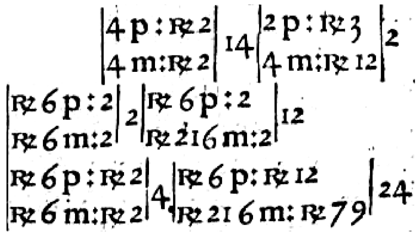
[22]*Elements* X.112: "The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio; and further the apotome so arising will have the same order as the binomial straight line", *Elements* X.113: "The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome", and *Elements* X.114: "If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, the "side" of the area is rational" see [Heath 1956c, pages 243-254].

to $\sqrt{48}$ is as [the ratio of] $\sqrt{3}$ to $\sqrt{2}$ and they are of the same order. But, if you divide the same $\sqrt{24}+2+\sqrt{2}$, $\sqrt{24}-\sqrt{12}$ will come out, which is allowed to have the parts in the same ratio, then the same [parts] are a *binomium* with a *recisum* of the same order, but the *binomium* is of the fourth order and the *recisum* [is] of the sixth [order]. From this it is well known one wonderful [thing], that it is allowed that four quantities, three of which are numbers and the fourth [of which] is rational only to the power, cannot be in the same ratio. Nevertheless four quantities, three of which are rational only to the power and one [of which] is a number, can [be in the same ratio], and it could be the ratio of an *alogai* quantity to a number, namely [the ratio] of the other *alogai* to the *alogai*, or [the ratio] of two *alogai*. Furthermore it follows that two incommensurable quantities have both parts commensurable, as $2+\sqrt{3}$ and $5+\sqrt{12}$, in fact, since the *binomia* of the first or fourth order are incommensurable, and nevertheless 2 and 5 are commensurable, and similarly $\sqrt{12}$ and $\sqrt{3}$ [are commensurable], since one is the double of the other.

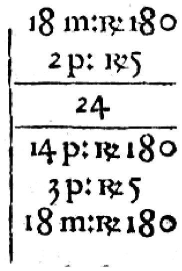
Chapter V

On the consideration of the *binomia* and *recisa* that contain a rational figure, whence on the value of the chapters

Since each *binomium* and *recisum* can be the side of a surface reduced to the number, therefore I will distinguish [them] except on the basis of the parts. In some [*binomia* and *recisa*] the bigger part is a number, in some the smaller [part is a number], in some none of the two. Moreover I display an example for each.



I say that the value by a *binomium* or *recisum* in which there is no number is not suitable in this case, because, [the value being] subtracted from the number, three non-composite quantities, a number and two roots, are left, and from those roots multiplied by themselves it is nothing but a number and a root of a number, then in the product they cannot erase each other. Therefore we examine the thing one by one and we say that, if $x^3 + 24 = 32x$, the value of the thing, being double, is $3 + \sqrt{5}$ and $3 - \sqrt{5}$. The same joined make 6, the value of $x^3 = 32x + 24$, and, since it is necessary to make two parts from 32, one of which by the root of the other makes 24, therefore I multiply $3 + \sqrt{5}$ by itself, it makes $14 + \sqrt{180}$, I subtract [it] from 32, $18 - \sqrt{180}$ is left. From this, multiplied by $3 + \sqrt{5}$, it must be produced 24, the number of the value.


18 m: ℞ 180
2 p: ℞ 5
24
14 p: ℞ 180
3 p: ℞ 5
18 m: ℞ 180

Therefore in this first example it appears that it is necessary that the division is made by the first *binomium*. In fact, $18-\sqrt{180}$ and $14+\sqrt{180}$ are first *binomia*,[1] because they have a root, and it is also necessary that this [the root] is a first *binomium*, because, [being] multiplied by the first *binomium*, it produces the number. And if the *recisum*[2] had not been the first *binomium*, but the fourth, also the root of the first *binomium* would have been a fourth *binomium*, otherwise it cannot produce the number. Therefore the second example is $x^3+12=34x$. The values of the thing are $3+\sqrt{7}$ and $3-\sqrt{7}$, which compose 6, [which is] the value of $x^3=34x+12$. Therefore I multiply for example $3+\sqrt{7}$ by itself, $16+\sqrt{252}$ is made, I subtract [it] from 24, $18-\sqrt{252}$ is left, by which and by $3+\sqrt{7}$ is produced perfectly 12. These are plain. The third is $x^3+8=18x$ and the value is $\sqrt{6}-2$ (I omit now the integer [value]). The square of $\sqrt{6}-2$ is $10-\sqrt{96}$, the remaining part is $\sqrt{96}+8$. The fact that a first *binomium* remains as a remainder is also cause of the contrary. Then, since the root $[\sqrt{6}-2]$ is a fifth *recisum*,[3] the thing behaves well. I say the same for the second *binomium* and *recisum*. And in this kind [one] has nearly more examples than in the first [case], as you see.

[1] Note that in this chapter Cardano sometimes ambiguously uses the term '*binomium*' to generally refer to both a *binomium*' or a '*recisum*'.

[2] Note that Cardano uses '*residuum*' to mean '*recisum*'.

[3] Again, note that Cardano uses '*residuum*' to mean '*recisum*'.

℞ 96 p:8
℞ 6 m:2

8

1 cu.p:8 æqual.18 pos.
æstim.℞ 6 m:2

1 cu.p:48 æqual 25 rebus
æstim.℞ $3\frac{1}{4}$ m:$1\frac{1}{2}$

1 cu.p:21 æqual. 16 rebus
æstim.℞ $9\frac{1}{4}$ m:$1\frac{1}{2}$

1.cu.p:18 æqual.19 rebus
æstim.℞ $17\frac{1}{4}$ m:$\frac{1}{2}$

1 cu.p:18 æqual.15 rebus
æstim.℞ $8\frac{1}{4}$ m:$1\frac{1}{2}$

1 cu.p:18 æqual.39 rebus
℞ 12 m:3

1 cu.p:12 æqual.34 rebus
æstim.3 p:℞ 7,uel 3 m:℞ 7

1 cu.p:24 æqual 34 rebus
æstim.3 p:℞ 5,uel 3 m:℞ 5

And $\sqrt{12} - 3$ is a second *recisum*, $\sqrt{6} - 2$ is a fifth *recisum*, $3 - \sqrt{5}$ [is] a first *recisum*, and $3 - \sqrt{7}$ [is] a fourth *recisum*. Therefore you have all the examples.

Therefore it is first to be considered that the value of the thing can be a *binomium* and a *recisum* in the first and fourth [type], as you see in the last two examples, but it can be nothing but a *recisum* in the second and fifth [type]. It is proved in fact [that], if it was a first *binomium*, therefore the *recisum*[4] will be the *recisum*[5] of the first or fourth way, therefore by the preceding [things], [being] multiplied by the *recisum* of the second or fifth kind, it cannot produce the number. Therefore there are six kinds of values, the first and fourth *binomium* and the *recisum* of the first and fourth [way] and also of the second and fifth way. The second is that, being two values (either equal or unequal) in this chapter of the cube and number equal to some things and [being] in the second or fifth

[4]Again, note that Cardano uses '*residuum*' to mean '*recisum*'.
[5]Again, note that Cardano uses '*residuum*' to mean '*recisum*'.

recisum, its *binomium* cannot be. And the second value is had through the first [one] (the half of that [being] multiplied by itself, [what is] produced [being] triplicated [and] subtracted from the number of the things), and the half of the first value [being] subtracted from the root of the remaining, there is the second value, as $x^3 + 18 = 39x$ and the value of the thing is $\sqrt{12} - 3$, I multiply $\sqrt{3} - \frac{3}{2}$ by itself, $\frac{21}{4} - \sqrt{27}$ is made, triplicate, $153 - \sqrt{243}$ is made, subtract [it] from 39, [which is] the number of the things, $\frac{133}{4} + \sqrt{243}$ is left, subtract $\sqrt{3} - \frac{3}{2}$, [which is] the half of the first value, from its universal root, the second value will be $\sqrt{\frac{93}{4} + \sqrt{243}} - \sqrt{3} + \frac{3}{2}$, and is it agreed [that] all this is equal to 6. It is necessary that the second value is the number or something that is to the previous value as $\sqrt{\frac{93}{4} + \sqrt{243}} - \sqrt{3} + \frac{3}{2}$ [is] to the second *recisum*. And thus one might find the second value by the same rule. The third is that, being discovered more values in the same number (say 8), namely think $\sqrt{\frac{69}{4}} - \frac{1}{2}$ and $\sqrt{\frac{33}{4}} - \frac{3}{2}$ e $\sqrt{12} - 3$, thus it is necessary to make the same number of the things under the same number. And this was appropriate to a greater extent to understand the thing.

Fourth, that we see [that] the number of the things [is] not only in integer numbers, but also in fractional [numbers], as in the fourth example 19 is divided in $\frac{35}{2}$ and $\frac{3}{2}$. Moreover in the fifth [example] 15 [is divided] in $\frac{21}{2}$ and $\frac{9}{2}$, in the second [example] 25 [is divided] in $\frac{7}{2}$ and $\frac{43}{2}$, and in the third [example] 16 [is divided] in $\frac{23}{2}$ and $\frac{9}{2}$. Therefore it is necessary now to consider others.

Fifth, that we see that, if the number of the equality is composed as 18, 12, and 24, [one] easily has the value and also several [values], but, if it is prime,[6] it is difficult to find one single [value].

[6]'*Primus*'.

Chapter VI

On the operations plus and minus according to the common usage

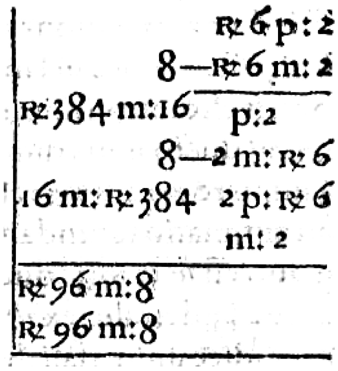
1. In the multiplication and division the plus is always made by similar [things], the minus by contrary [things], whence the plus multiplied by the plus and divided by the plus, and the minus multiplied by the minus and divided by the minus produces always the plus. And thus the plus times the minus or the minus times the plus, or the plus divided by the minus, or the minus [divided] by the plus produces the minus.

2. They all retain their nature in addition, they change [it] in subtraction, as the added plus becomes the plus, the subtracted minus becomes the plus, the subtracted minus bears instead the plus. But, if it is defeated, it is left what from which it was subtracted, as -2 is left by [the subtraction of] -4 from -6, because the minus from which the plus is subtracted [is] bigger.

3. The square root of the plus is the plus. The square root of the minus is nothing according to the common use, but we will deal with this below.[1] There is no doubt on the cubic [root], in fact $\sqrt[3]{-8}$ is -2.

4. If someone says to divide $8 + 2 + \sqrt{6}$ or $\sqrt{6} + 2$, then find both the *recisa* $\sqrt{6} - 2$ and $2 - \sqrt{6}$, which is the true minus, then, in order to divide the quantity, multiply the *recisa* by 8, $\sqrt{384} - 16$ and $16 - \sqrt{384}$, which is the minus, are made. Therefore, the first being to be divided by $+2$, it comes manifestly $\sqrt{96} - 8$. The second [being] to be divided by -2, it comes by the first rule the same, that is $\sqrt{96} - 8$.

[1]See *De regula aliza* Chapter XXII, page 699.


℞ 6 p: 2
8 — ℞ 6 m: 2
℞ 384 m: 16 p: 2
8 — 2 m: ℞ 6
16 m: ℞ 384 2 p: ℞ 6
m: 2
℞ 96 m: 8
℞ 96 m: 8

But the *recisum* that is composed by the plus and the minus can have a root and that consists of the plus and the minus, as of the same $\sqrt{5-\sqrt{24}}$ is $3-\sqrt{2}$.

Chapter VII

On the examination of the values assumed by the second and third rules of the first chapter

We[1] propose $x^3 = 18x + 30$, whence the value of the thing $\sqrt[3]{18} + \sqrt[3]{12}$ is discovered according to the part of the chapter, and it is extended to infinite increasing beyond the number. And if we gave all the parallelepipeds to the number, it would be necessary to make two parts from this value, from the reciprocal product of which times the squares, 10, [which is] the third part of the number, is made. Hence the same is also made by the product of the aggregate or of the value times the product. Then I will divide 10 by $\sqrt[3]{18} + \sqrt[3]{12}$, the product $\sqrt[3]{12} - 2 + \sqrt[3]{\frac{16}{3}}$ comes out. I will divide $\sqrt[3]{18} + \sqrt[3]{12}$ in two parts that, reciprocally multiplied, produce $\sqrt[3]{12} - 2 + \sqrt[3]{\frac{16}{3}}$, and (by the fifth [Proposition][2] of the second [Book] of the *Elements*) the parts will be

$$\sqrt[3]{2 + \frac{1}{4}} + \sqrt[3]{1 + \frac{1}{2}} + \sqrt{5 - \sqrt[3]{\frac{3}{16}} - \sqrt[3]{\frac{1}{12}}}$$

and

$$\sqrt[3]{2 + \frac{1}{4}} + \sqrt[3]{1 + \frac{1}{2}} - \sqrt{5 - \sqrt[3]{\frac{3}{16}} - \sqrt[3]{\frac{1}{12}}}.$$

Then I say that, since two parallelepipeds with a similar value can be made equal also to 30, six parallelepipeds can be made equal to 90 and further more as, if $x^3 = 18x + 58$, the value of the thing is $\sqrt[3]{54} + \sqrt[3]{4}$. And, if $x^3 = 18x + 75$, the

[1]A remark on the title of this chapter. In 1570 and 1663 the title would make reference to the second and the third rules of Chapter II. Actually, there are no values – or solutions – assumed in those propositions. I would rather say that the reference is to the Chapter I, since in all those propositions we have assumed that $x = y + z$. Indeed, in the following Cardano uses the second rule of Chapter I. Anyway, this is the only proposition explicitly mentioned, so that the reference to the third rule is empty.

[2]*Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [Heath 1956a, page 382].

value of the thing will be $\sqrt[3]{72}+\sqrt[3]{3}$. And, if somebody says $x^3=18x+33$, the value of the thing will be $\sqrt[3]{24}+\sqrt[3]{9}$ by the same rule. Then I will divide 11 by $\sqrt[3]{24}+\sqrt[3]{9}$, $\sqrt[3]{\frac{64}{3}}-\sqrt[3]{24}$ comes out. Therefore the parts will be

$$\sqrt[3]{3}+\sqrt[3]{1+\frac{1}{3}}+\sqrt{5-\sqrt[3]{\frac{1}{3}}-\sqrt[3]{\frac{3}{64}}}$$

and

$$+\sqrt[3]{3}+\sqrt[3]{1+\frac{1}{3}}-\sqrt{5-\sqrt[3]{\frac{1}{3}}-\sqrt[3]{\frac{3}{64}}}.$$

Similarly if $x^3=18x+42$ the value will be $\sqrt[3]{36}+\sqrt[3]{6}$. And 14 [will be] the parallelepipeds, then divide 14 by $\sqrt[3]{36}+\sqrt[3]{6}$, $\sqrt[3]{48}-2+\sqrt[3]{\frac{4}{3}}$, multiply $\sqrt[3]{\frac{9}{2}}+\sqrt[3]{\frac{3}{4}}$ by itself, $\sqrt[3]{\frac{81}{4}}+3+\sqrt[3]{\frac{9}{16}}$ is made. Subtract what you want to be produced from this, that is the aggregate, $5-\sqrt[3]{\frac{3}{4}}-\sqrt[3]{\frac{1}{48}}$ is left. The parts will be

$$\sqrt[3]{4+\frac{1}{2}}+\sqrt[3]{\frac{3}{4}}+\sqrt{5-\sqrt[3]{6+\frac{3}{4}}-\sqrt[3]{\frac{1}{48}}}+$$

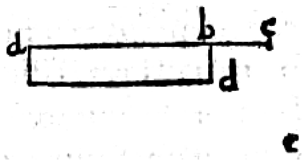
and

$$\sqrt[3]{4+\frac{1}{2}}+\sqrt[3]{\frac{3}{4}}-\sqrt{5-\sqrt[3]{6+\frac{3}{4}}-\sqrt[3]{\frac{1}{48}}}.$$

Chapter X

In what way the parts with the proposed line are appropriate to the parallelepiped

1. Let firstly be proposed



the line AC divided in B so that the parallelepiped from the whole AC times the surface AD from AB times BC is equal to the number, and let at first AC be the number. It is agreed that it is necessary that AD is a number and [that] the parts AB, BC [are] numbers or a *binomium* with a *recisum*, and it can be demonstrated that the difference of the parts is necessarily the root of a number or a number itself, so that the square of the half, which is a number since the whole AC is a number, exceeds the rectangle AD, which is a number since the number E [of the equality] is by that times the number AC (by the fifth [Proposition][1] of the second [Book] of the *Elements*). For example, let the whole AC be b and [let] the number E [be] 36. By the second way I can assign the number to six parallelepipeds and then two [parallelepipeds] will be 12. Divide 12 by AC, 2, [which is] the surface AD, comes out, therefore the parts will be $3 + \sqrt{7}$ and $3 - \sqrt{7}$. And two cubes will be $90 + \sqrt{8092}$ and $90 - \sqrt{8092}$, which all is 180, whence 36 is left by the parallelepipeds. According to the third way I can give the half of E to one parallelepiped and the parts will be $3 + \sqrt{3}$ and $3 - \sqrt{3}$. According to the fourth way I can assign the whole 36 to two parallelepipeds and the parts will be $3 + \sqrt{6}$ e $3 - \sqrt{6}$. And in the first case $x^3 = 30x + 36$, in the second [case] also $x^3 = 30x + 36$. Again in the third [case] in the same way, but

[1] *Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [Heath 1956a, page 382].

the distinction is that in the first case $30x$ are made equal to the cubes only, in the second [case] to the cubes and to two parallelepipeds, in the third [case] to two cubes and to four parallelepipeds.

2. Again E is put 12, AC [is put] $\sqrt{24}$. I divide the whole E, but it is better to lead back to the third way by dividing the half, namely 6, by $\sqrt{24}$, $\sqrt{\frac{3}{2}}$ is made, multiply $\sqrt{6}$, [which is] the half of $\sqrt{24}$, by itself, 6 is made, throw away $\sqrt{\frac{3}{2}}$, and $6 - \sqrt{\frac{3}{2}}$ is made, the universal root of which added and subtracted to $\sqrt{6}$ shows the parts, but I propose only the single ways:

$$\text{second way } \sqrt{6} + \sqrt{6 - \sqrt{\tfrac{2}{3}}}\ \sqrt{6} - \sqrt{6 - \sqrt{\tfrac{2}{3}}},$$
$$\text{third way } \sqrt{6} + \sqrt{6 - \sqrt{\tfrac{3}{2}}}\ \sqrt{6} - \sqrt{6 - \sqrt{\tfrac{3}{2}}},$$
$$\text{fourth way } \sqrt{6} + \sqrt{6 - \sqrt{6}}\ \sqrt{6} - \sqrt{6 - \sqrt{6}}.$$

It is agreed that those values are useless, in fact we will have the cube through the two parallelepipeds, or [through] four [parallelepipeds], or [through] six [parallelepipeds] equal to the number. Moreover the remaining [part] is then a certain root, namely $\sqrt{13824} - 12$, or $\sqrt{13824} - 6$, or $\sqrt{13824} - 4$, but this divided by $\sqrt{24}$, which is the thing, cannot produce any number, therefore the cube cannot be made equal to the things according to a certain integer or fractional number.

3. By a similar reasoning but yet by another cause I show that, if AC is the simple cubic root of a number, it cannot satisfy. Then we propose that AC is $\sqrt[3]{40}$ and E is 2, and in this case I take the fourth way as [it is] easier and simpler. I divide 2 by $\sqrt[3]{40}$, the surface AD $\sqrt[3]{\frac{1}{5}}$ comes out, I divide AC in equal [parts], $\sqrt[3]{5}$ is made, I multiply [it] by itself, $\sqrt[3]{25}$ is made, I subtract $\sqrt[3]{\frac{1}{5}}$, $\sqrt[3]{\frac{64}{5}}$ is left, I add and subtract its root to $\sqrt[3]{5}$, the parts will be $\sqrt[3]{5} + \sqrt[3]{\frac{64}{5}}$ and $\sqrt[3]{5} - \sqrt[3]{\frac{64}{5}}$. Therefore only two parallelepipeds of these parts make 2, therefore it is plain that it is necessary that the remaining [part] from the cube $\sqrt[3]{40}$ is a number and it is 38. Therefore 38, divided by the thing, which [is] $+\sqrt[3]{40}$, necessarily produces a cubic root.[2] Therefore the number of the things cannot be a real number, but a cubic root, as if someone says $x^3 = \sqrt[3]{100}x + 10$, but this is not useful. Indeed we search for the cube equal to a number of the things and a number, either integer

[2] 1570 and 1663 has "*38 igitur divisum per rem quæ p: Rcu: 40 necessario producit Rcu:*".

or fractional. And, given that we fall into such a case, it is rare and we do not have a general rule, but it can be discovered, as in the *binomia* and *recisa* just as we will now add.

4. Now (seeing that the three preceding ways are very little useful, in fact the first [way] is well known, even if without the chapters, and it is easy for everybody, and it is not general and at least [it is] not [general] in many [cases], the two remaining [ways] are entirely useless, and no other simple root, or fourth root, or fourth root plus a square root can be useful by the same reason, because its cube is necessarily of the kind of the first root and, E [being] subtracted, then the *recisum* [is] divided by the thing, no number can come forth) it is proposed that AC is two square roots. I say that this way is useless. In fact, subtracted the number E, the cut cube is left, and thus it cannot be divided by the thing in order for the number to come forth. In fact, as it is agreed, the number cannot be produced in the cube of a *binomium* or *recisum* where both parts are roots.

5. And AC cannot be a fourth root or a sixth root, because such [roots] come to roots of the same kind, whence, the number [being] subtracted, the *recisa* are made, but a *recisum* cannot be divided by any root of the same kind in order for the number to come forth, therefore there cannot be a real number of the things in the equality. But neither from $\sqrt{2}$ and $\sqrt[4]{18}$, nor from $\sqrt[8]{8}$ and $\sqrt[4]{2}$. In fact whatever three parallelepipeds in the first [case] are 18 and in the second [case are] 6, nevertheless three [parallelepipeds] of a different nature are left, as in the first [case] $\sqrt{8}$, $\sqrt[4]{5832}$, and $\sqrt[4]{23328}$, but it is agreed that a fourth root cannot be to a greater extent commensurable to a simple root than a simple root to a number. Then [neither] $\sqrt[4]{23328}$ nor $\sqrt[4]{5832}$ can be commensurable with $\sqrt{8}$ and not even between them, because $\sqrt[4]{5832}$ is made by $\sqrt{18}$ times $\sqrt[4]{18}$ and $\sqrt[4]{23328}$ is made by 6 times $\sqrt[4]{18}$, but 6 and $\sqrt{18}$ are not commensurable. One may divide 23328 by 5832, 4 comes out and therefore it happens that, divided $\sqrt[4]{23328}$ by $\sqrt[4]{5832}$, $\sqrt[4]{2}$ comes out, therefore they are not commensurable. Since then in the thing there is nothing but two kinds of quantities while dividing and [since] the remaining [part] of the cube [has] three [kinds of quantities], the number of the things cannot come forth. And thus in all the similar [cases].

Corollary 1. From this it is well known that it is general. One may explain [it] about the parallelepiped, no matter which part of the same cube is given to the number, the remaining [part of the cube] will be of the several, non commensurable parts that are in the thing, therefore the things cannot be according to a certain number.

Corollary 2. Moreover from this it follows that the more the incommensurable parts of this sort are, the more the number of the parts of the cube, a number [being] subtracted, differs from the parts of the root. Therefore what is left, [being] divided by the thing, could much less return the number, as it was proposed.

6. It cannot be neither a universal square root, nor a cubic [universal root], nor a [universal root] of another kind. In fact, if it was a square root, the same follows as in the second rule. But if the cubic [root] is dissolved, therefore it could not contain the thing, that is the universal cubic root, according to any number. And neither a universal square root joined to another simple root, in fact – as I said in the previous second corollary – where there are several incommensurable parts in the thing, for that reason there will be more parts in the cube in comparison to the remaining [ones].

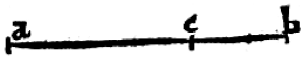
Therefore, it is necessary that the universal value of this sort is either under the *binomium* in which there is a number or not, or under the *trinomium* in which there is the number or not, or in more terms in which there is [the number] or not, or in a wild[3] quantity, namely [a quantity] that is not in any kind of roots, not even composed by those, nor left by subtraction, as the quantity the square root of which multiplied by the remainder from 12 produces 2, whence the chapter cannot be discovered.

[3]"*Sylvester*".

Chapter XI

How many and what parts of the cube, and on the necessity of those, and what incommensurable [parts are]

Therefore we repeat and we say that, regarding to the side of the cube the quantity of which is searched, if the cube must be equal to two [parts], some things and a number, it is necessary that the cube [is] divided in such a way at least in two, then its side. In fact, from one it comes nothing but one. Then, at least in two. Being then two parts, while the cube is made, it is necessary that the square of all is made, and this consists of three parts different in nature and ll incommensurable between them, if AC and AB are incommensurable to the first power.



In fact the ratio of the square of AC to what is made by AC times CB is as AC to CB, and similarly the ratio of what is made by AC times CB to the square of BC [is] in the same way. Then what is made by AC times CB is incommensurable to both. Also the squares of AC and CB are incommensurable between them. Then by [the things] shown by Euclid (fourth [Proposition][1] of the second [Book] of the *Elements*) there will be in the square of AB three surfaces all incommensurable: the square of AC, [the square of] CB, twice AC times CB. But the cube of AB consists of AC and CB times the three mentioned surfaces and four kinds of bodies are made: one by AC, another by BC times the square of BC, three [bodies] by AC times the square of AB and three [bodies] by BC times the square of AC. (But I do not recite these because I want to recall to the memory the construction of the cube, but in order to manage to show what is useful, since they are elsewhere demonstrated,[2]). Hence at first [the things]

[1] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].
[2] See [Cardano 1570a, Chapter XXX, pages 59-60].

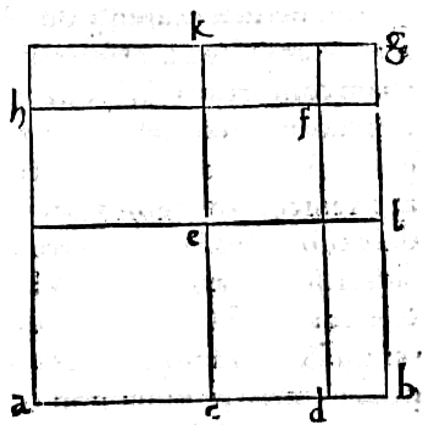
that are made by AC times the square of BC are incommensurable to the cube of BC, because they are as AC to CB, and by the same reason [the things] that are made by BC times the square of AC and similarly [the things] that are made by AC times the square of BC to those [things] that are made by BC times the square of AC, in fact they all are – as I said – in the ratio of AC to CB, as you see in the side.

d	e	f	g
8	12	18	27

But I set the numbers in order to see more clearer the ratio, and the same [numbers] reciprocally multiplied only once [are] representing the parallelepipeds by AC times the square of CB and CB times the square of AC. I also say that by the absurd the parts AC and CB are to be taken reciprocally commensurable, because so they were equally AC, if they were only one quantity. This being so, you have that the nearest are incommensurable. But the ratio of f to d is duplicated of [the ratio] that is of AC to CB, where then AC and CB were commensurable to the power, d and f are commensurable, and e and g. In truth, if [they are commensurable] to the second power, as if one had been the number [and] the other a cubic root or both cubic root commensurable [to the power], then the cubes would have been reciprocally commensurable, but the parallelepipeds [would have been] both incommensurable. Also the same parallelepipeds are incommensurable between them, as it is clear, since they are in the ratio of AC to CB. And similarly $\sqrt{2}$ and $\sqrt[6]{32}$, and $\sqrt{3}$ and $\sqrt[6]{108}$ are commensurable to the second power. In fact the first [ones] raised to the cube produce $\sqrt{8}$ and $\sqrt{32}$, the second [ones] $\sqrt{27}$ and $\sqrt{180}$, which are reciprocally doubled. From where it is to remark that one thing is that [two] cubic roots are commensurable to the second power, in fact all [the above ones] are such and the cube of those necessarily are numbers, another thing is that the same are commensurable as $\sqrt[3]{16} + \sqrt[3]{2}$ or $\sqrt[3]{16} - \sqrt[3]{2}$. In fact only those are the ones by which the parallelepipeds are numbers, which is demonstrated. In fact, if AC and CB were not commensurable, therefore neither g and f, nor d and e [would be commensurable], but d and g are numbers, because [they are] cubes of cubic roots, therefore e and f cannot be numbers. Then AB cannot be composed by two incommensurable cubic roots, because the parallelepipeds were not numbers. And [AB can]not even [be composed by two] incommensurable [cubic roots], because they were [only] one cubic root and the whole cube a number. Then, in any case, if the number was not near, none of the two quantities cannot satisfy the parallelepipeds according to the number in order for the remaining [part] of the cube to satisfy the things. In fact, if they were entirely incommensurable in first length and to the second power, four incommensurable [parts] will be produced. Therefore, given that one [part] was the number, those three remaining [parts] cannot contain the two [parts] that are in the things according to the number. Therefore the number of the things is not given. But, if they were commensurable in length, they would be one quantity, therefore they would not satisfy (by Chapter X). In truth if they were commensurable to the second [third] power and they were cubic roots, the cubes would be numbers but not the parallelepipeds, but, if they were not cubic roots, e will be to f as AC to CB, but AC is not commensurable to CB, therefore not even e to f. Therefore two drawbacks follow. Firstly that, if a parallelepiped is a number, the other will not be [a number], because the general rule cannot be made. Secondly, that the aggregate of the cubes, which will be of the same nature, will not be able to fit a part according to the number, because it is in a commensurable ratio to whatever part with the square of one of those, then being the square not a number except [when] the quantity is a square root and, if it was, then it is against [the things] said, it is agreed that the equality cannot be made. The said example is $\sqrt[6]{32}+\sqrt{2}$. According to the single parallelepipeds to avoid the effort, the cube is $\sqrt{72}+\sqrt[6]{2048}+\sqrt[6]{8192}$. Here it is agreed that no number is made, therefore it cannot be appropriate.

a—b—c
d—e—f

I only say that no parallelepiped can be a number under these hypotheses. Otherwise a and b are not cubic roots commensurable to the second power and, if the number can be made, they produce the parallelepiped c and, since they are a second power of commensurable [numbers]. I take two cubic roots in the same ratio of d to e that produce the parallelepiped f, then by [the things] said f will be the cubic root of the number, but, if c is the number, then the ratio of c to f, [which is] as [the ratio] of the number to such a cubic root, is triplicated of the one that is a to d by Euclid, but d is the cubic root of a certain number, therefore a is a number or a cubic root of a number, which is against [the things] supposed. But the parts cannot be commensurable to the first power, because in such a way f were to g and g to e as the number to the number. Then there were these two quantities, therefore, if one is the number, the remaining [ones] could not contain AC and CB, which [are] incommensurable in length, if none [is the number], then the cube would not be made equal to the number. Neither these two parts could be universal cubic roots. Since, if [the parts are] commensurable, they will be one, but it has been demonstrated that it cannot be and, if [they are] incommensurable, four incommensurable parts will be made in the cube, therefore one will be superfluous. In the end it is left that one [part] is a number, the other [is] a square root, as we have seen, or that there are more than two parts. Then we see that, firstly, out of three parts all cubic and incommensurable, as $\sqrt[3]{6}$, $\sqrt[3]{5}$, and $\sqrt[3]{2}$ are, moreover you will recognise that [they are] incommensurable in length, when (as I said) the numbers of each one of those by the square of the other do not produce the cubic number, and then the cubic root of one multiplied by the square of the other does not produce a number, then CRUX and those roots are commensurable. And the opposites of those are also transformed. In the end it is deduced from this that that part of the chapter of the cube equal to some things and a number cannot consist of a quantity composed by two simple or universal cubic roots or by the number and a cubic root. In fact in the number and cubic root it will be necessary to give the cubes to the number, because [the cubes] will be numbers, then they will not satisfy a small number. Besides the parallelepipeds will be incommensurable, and [they will be] two cubic roots, and in the thing there is nothing but one part that is a cubic root, therefore the number of the things will not be [given]. And not even if both parts are cubic roots, since if you gave the parallelepipeds to the number, firstly they will not necessarily fit the cubes if [the cubic roots] are not commensurable, then the cubic roots will not be a number. Besides the cubes will be numbers, then they could not contain the thing according to the number, since the thing consists of two cubic roots multiplied by the number, they would produce the number. And we cannot even give both cubes, when they are positive, to the number, where the number was not smaller than the fourth part of the whole cube, as we taught.[3] But where it is bigger or equal, we give [both cubes to the number] and that part of the chapter of the cubes equal to some things and a number that is already known is made. Therefore the remaining part in that equality never takes place. And we cannot even give the difference of the cubes to the number, as in a positive and a negative cubic root like $\sqrt[3]{6} - \sqrt[3]{2}$, because the negative parallelepipeds will be bigger and the positive [parallelepipeds] [will be] smaller. Then, since in the thing the positive cubic root are necessarily bigger than the negative cubic root, the things could in no way be contained in the parallelepipeds according to the number, but, joining well the positive [part] of the things with the negative [one] and the negative [part] with the positive [one] with the cube, the chapter of the cube and some things equal to a number is perfectly made. But the value cannot consist of a number and a square root in order to be general, in fact this is shown above[4] and it cannot consist of a number and a universal square root, because two incommensurable parts will be in the cube besides the number (because the universal root is not a first power commensurable to the number, see Chapter III at the end) and only one [part] in the thing, then the number of the things does not correspond. And not even by a number and a universal cubic root. It will be necessary to give the cubes to the number and the parallelepipeds will be two incommensurable, then as before, being only one universal cubic root, the thing could not be contained in the cube under any number. Then now one necessarily comes to three [parts].



[3]See the first way in *Aliza*, Chapter I.
[4]See Chapter I.

Therefore let AB be divided in three parts that are all incommensurable cubic roots and not in the same ratio, and it is agreed that eight kinds of bodies are made. One that will be the number [and] that consists of the cube of each part. In fact, being AC, CD, DB cubic roots of numbers, the cubes of those will be numbers and hence the aggregate of those [will be] a number. The second body consists of six bodies the sides of which are all parts, namely AC, CD, DB. You have already nine bodies. The remaining eighteen [bodies], being three by three equal, will then be six: the first [body] will consist of EC, CD times three times the square of AC, the second [one] of BD times three times the square of AC, the third [one] of AC times three times the square of CD, the fourth [one] of BD times three times the square of CD, the fifth [one] of AC times three times the square of BD, the sixth [one] of CD times three times the square of BD. Being then seven incommensurable parts in the cube and only three in the thing, the cube will not be equal to some things under any number. I only show that it is so. In fact in the surface AG there are three quantity AE, EF, FG, and six surfaces of which two by two and two by two are equal DE and EH, DL and HK, LF and FK (by the [Proposition][5] 43 of the first [Book] of the *Elements*). But three cubes are made from AC, CD, DB times their squares, what is from CD times DL, HK and from BD times DE, EH is in truth the same of [what is made] from AC times FL, FK, therefore it already consists of nine bodies driven back to two [bodies]. I only say that three bodies are made by one part times the square of the other, namely the three parallelepipeds by CD times the square of AC are made by AC times ED, EH and by CD times [*sic*], therefore, being the remaining of the double quantity multiplied by the square of the third, six aggregates are made from three parallelepipeds, therefore all the twenty-seven [bodies are] reduced to eight.

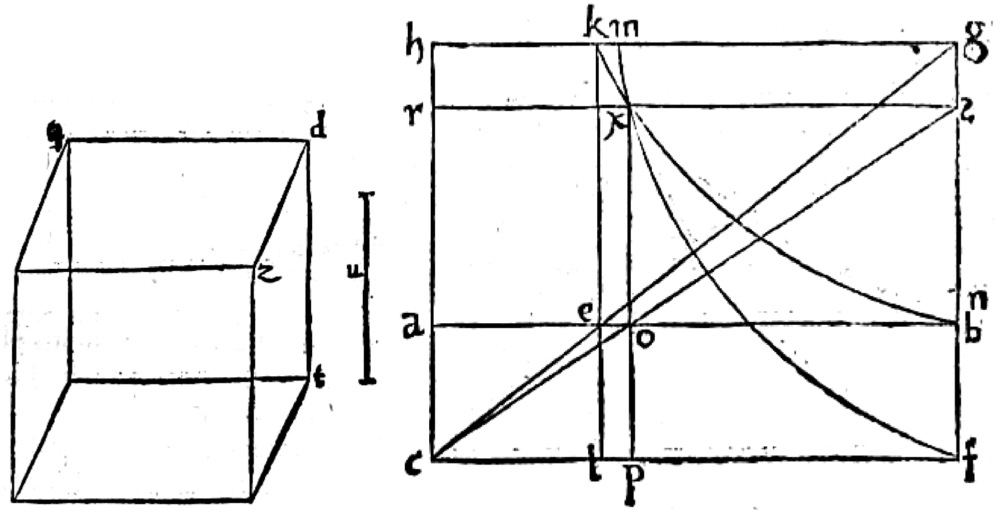
[5] *Elements* I.43: "In any parallelogram the complements of the parallelograms about the diameter are equal to one another", see [HEATH 1956a, page 340-341].

Chapter XII

On the way to geometrically prove the value of a cube and a number equal to some squares

Let for example 12 squares be equal to a cube and to the number 192. It is agreed by the things said above (Proposition[1] 27) that, if the number is greater than 256, then the purpose is false and, if the same number [is] 256, then the side of the cube is 8 or two-thirds of the number of the squares. Therefore we have proposed a [number] smaller than that number. It is agreed by the same things that, if the number is the half of the maximum number, which is 128, which is the third part of the number of the proposed squares, that the ratio of the square of two-thirds [of the number of the proposed squares] to the square of one-third [of the number of the proposed squares] is as [the ratio] of two-thirds [of the number of the proposed squares] to what comes from the division of 128, [which is] the proposed solid, by the square of two-thirds [of the number of the proposed squares], which is 64. Indeed two, which is the fourth part of 8, comes, as 16, the square of 4, one-third [of the number of the proposed squares] is the fourth part of 64, [which is] the square of 8, [which is] two-thirds of the proposed number of the squares. Then we took a different number from this as I said.

[1]There is no 'Proposition 27' in the *Ars magna*. Actually, there is one in the *De proportionibus*, but it does not concern the same topic.



Then it is proposed the rectilinear, solid body $DQTZ$ with sides and surfaces at equal distance, the bottommost squared surface of which is DQT, and the whole solid is 192, which is the proposed number, and its height is the line DZ. AB is given equal to 12, which is the number of the proposed squares, and it is divided such that BE is the double of EA. And a certain line U is subtended under the two CB and DQ, and let DZ to AC as EB to U. Then [the ratio] of the square of EB will be to QT as EB to U, hence as DZ to AC. Therefore the solid that [is] under AC and the square of EB [is] equal to the solid $DQTZ$. Therefore the purpose is to divide AB such that the solid by one part times the square of the other is equal to the solid by AC times the square of EB. And Eutocius of Ascalon teaches us how to do this twice in the second [Book] on the *Sphere and cylinder*, but it is enough to bring the first demonstration of that. But I will not bring the so-to-speak most well-known propositions by Euclid. Then I raise AC perpendicularly over AB, and I complete the surface $ABCF$ and draw CE up to meeting FG in G and complete similarly the surface $HGCF$ with sides at equal distance, and I draw [the segment] by E at equal distance from CH, which is LEK, and GM, [which is] equal to DQ, is cut, and FN is subtended under the two lines AB and EB in continued proportion. Then the parabola over the axis GF that passes through M is drawn – as I will show – and similarly the hyperbola by B around the coincident [segments] HC and CF that passes through K is drawn again by means of the things that are demonstrated by Apollonius in the second [Book] of the *Conic elements*. Then, where KB

and MF divide themselves in X, I draw RXS at equal distance from AB and [I draw] XP at equal distance from RC, which will cut AB in O, which I say to be the point I looked for. Then I will draw CO, which I will show to reach S. Then, since as EA [is] to AC, so the square of BE [is] to the square of GM, therefore [the ratio] of the rectangle by CF times FN [is] to the same, but, as EA [is] to AC, so CF [is] to FG and, as CF to [is] FG, so the square of CF [is] to what [is] under CF, FG, hence [it is] as what [is] under CF, FN to the square of GM so the square of CF [is] to what [is] under CF, FG. Therefore the square of CF [is] to what [is] under CF, FN as what [is] under CF, FG [is] to the square of GM. But as the square of CF [is] to the thing contained under CF, FN, so CF [is] to FN and, as CF [is] to FN, so [the ratio] of the thing contained under CF, FG [is] to the thing contained under FG and FN. Therefore, as CF, FG [is] to the square of GM, so [the ratio] of the thing contained CF, FG [is] to the thing contained under GF, FN. Therefore the square of GM [is] equal to what is made by GF times FN (by the ninth [Proposition][2] of the eight [Book] of the *Elements*). Therefore GM is the mean proportional between GF and FN. Then, the parabola [being] drawn by F with axis GF by the first [Book] of the *Conics* by Apollonius, it falls on the point M, which is the first. And, because HE is equal to EF, they are indeed supplement, HL and AF will be equal and HC, CF [will be] coincident. Then the hyperbola drawn by B will cut FB with a ratio corresponding to the same GF from GH. Therefore it falls on K, which is the second. Being then HC and CF coincident and [the hyperbola] touching the rectangle RXP and ABF the hyperbola, therefore they are reciprocally equal. Therefore, the common $AOPC$ [being] subtracted, the two surfaces $ARXO$ and $OBPF$ will be equal. And, since they are supplementary, they will be around the same diameter. Therefore CO falls on S, which is the third. Then, since CF [is] to FS, as CP [is] to PO and therefore as AO [is] to AC, and as CF [is] to FS, so [the ratio] of the thing contained under CF, FN, which is the square of EB, [is] to the thing contained under SF, FN, [the ratio] of the square of EB will be to the thing contained under SF, FN as OA [is] to AC, and the thing contained

[2]*Elements* VIII.9: "If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion", see [Heath 1956b, page 358].

under SF, FN will be equal to SX because of the parabola assumed over NF (by the 34th [Proposition][3] of the eleventh [Book] of the Elements). Therefore [the ratio] of the square of EB to the square of OB, which is equal to the square of XS, [is] as OA to AC. Therefore the solid by AO times the square of OB is equal to the solid by AC times the square of EB, which was to be demonstrated. And it was the fourth. But it is clear that the reason of the construction of this problem depends on these two [things]. First, that, the point N [being] taken equally distant from the vertex of the parabola, which is F, so that the parabola cuts an [segment] equal to the same NF on the perpendicular drawn from N up to the parabola itself, [being] always drawn perpendicularly from that axis GF, no matter how big the parabola is, that mean is between NF and the line from the vertex to the point from which you led the perpendicular out. The second depends on the construction of the hyperbola. In fact, [a line being] drawn perpendicularly to the axis and the axis falling perpendicularly into two straight [lines], those in which they fall into are called 'coincident' and they always make equal surfaces outside of [the things] contained. As in the example, the point K with the vertex [B] [being] taken, KC is equal to CB and, taken the point X with the vertex [B], XC is made equal to the same CB. From this it manifestly follows that XC is equal to KC. Therefore it turns out that KR is always equal to XL no matter where the point X is placed. Then it is apparent that, AB [being] set 12, FN will always be the same, because [it is] in the ratio to AB and EB, and therefore $\frac{16}{3}$. And, if $DQTZ$ is set 192, AC will be 3 and, if [$DQTZ$ is set] 128, AC will be 2 and, if [$DQTZ$ is set] 64, AC will be 1. And in the first case XC will always be 36, in the second 24, and in the third 12. Then, FS [being] set a square, XS will be in every case the thing, the number [being] $\sqrt{\frac{16}{3}}$. Therefore RX will be $12 - \frac{16}{3}x$. Then, since XP is equal to SF, the surface RP, and therefore AF, will be $12x^2 - \sqrt{\frac{16}{3}}x^2$, and this can be equal to 36, or to 24, or to 12, or to any other number. Then, when we will reduce the cube to one, in each case 8 [$3\sqrt{3}$] square will be made, that is the quantity AB, which is 12, by FN, which is $\sqrt{\frac{16}{3}}$, 64 is made, then it is 8. Therefore, AB [being] only supposed

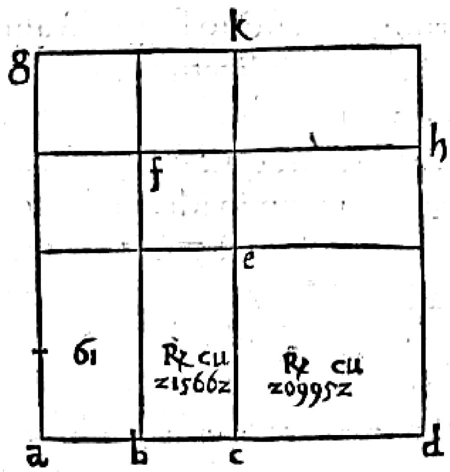
[3] *Elements* XI.34: "In equal parallelepipedal solids the basis are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal", see [HEATH 1956c, page 345].

12, no matter how big is $DQTZ$, 8 squares will always be equal to the cube and the number that is produced multiplying AC and AB and to the product times $\sqrt{\frac{16}{3}}$. Then, if BC is made 36, the number will be $\sqrt{6912}$, and, if it was 24, it will be $\sqrt{3172}$, and, if it was 12, it will be $\sqrt{168}$. And the value multiplied by itself produces SF, which, [being] multiplied by NF and taken the root, becomes SX, [which is] the part I looked for. In fact, the same is equal to OB. Then, [being] multiplied by itself and subtracted by AB and multiplied one by the other, the solid $DQTZ$ comes forth. Therefore, the easy Geometrical operation is the most difficult in arithmetic and it does not even satisfy.

Chapter XIII

On the discovery of the parts of the cubic *trinomium*, which produces the cube with two only cubic parts

And I only say that, if the cubic *trinomium*, the parts of which [being] multiplied produce the number, was assumed, only two parts that are cubic roots would be produced, but there are three incommensurable parts in the thing – as it is said, therefore the parts of the cube cannot contain the parts of the things according to the number. From this it follows that the thing will be by three cubic roots in continued proportion, because the same follows.



In fact $\sqrt[3]{12}+\sqrt[3]{6}+\sqrt[3]{3}$ produces how much the middle [square root produces being] raised to the cube, therefore, [being] reciprocally multiplied, they produce $\sqrt[3]{216}$, which is 6. Therefore it is generally demonstrated in such a way (by [Proposition][1] 17 of the sixth Book of the *Elements*). The cubic *trinomium* $ABCD$ is supposed only with this condition, that the body by AB, BC, CD is a number, then it is agreed that there are nine bodies that are equal to the number. The eighteen

[1]*Elements* VI.17: "If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and, if the rectangle contained by the extremes be equal on the square on the mean, the three straight lines will be proportional", see [Heath 1956b, page 228-229].

remaining [bodies] are three – as it is said[2] – by AB times the square of BC, by CD times the square of AB, which I say [to be] commensurable, as [*sic*]

℞ cu.3 ℞ cu.4 ℞ cu.18.

1296.	216	972.	729
162.	27	288.	216
48	8	36	27

and by BC times the square of CD. In fact [the ratio] of what is made by AB times the square of BC to what is made by CD times the square of AB is as the square of BC to what is made by AB times CD, but the square BC is to what is made by AB times CD as the number [is] to the number, and by BC times the rectangle by AB times CD is made the parallelepipeds equal to the number. Therefore the ratio of the square of BC to the surface AB times CD is as [the ratio] of a number to a number (by [Proposition][3] 32 of the eleventh [Book] of the *Elements*). Therefore also that ratio that [is] by BC times the square of CD. Therefore only the commensurable cubic roots are made, but there are three roots in the root, therefore some things cannot be made equal to the cubes taken as the things. But if you have displayed a part [to be] a number, as $\sqrt[3]{32} + \sqrt[3]{16} + 2$, $152 + \sqrt[3]{131072} + \sqrt[3]{128000}$ comes. Therefore, since the ratio is by far smaller than the parts of the thing, the cube could certainly not be made equal to the number of things. You see [it] also below (Chapter LI).

[2]See Chapter XI.

[3]*Elements* XI.32: "Parallelepipedal solids which are of the same height are to one another as their bases", see [Heath 1956c, page 341-342].

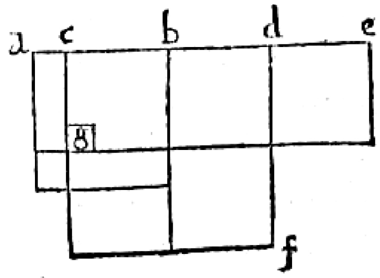
Chapter XIV

On the discovery of the kind of the value

Since it has been established (Chapter 13 of the *Ars magna*) that the value of the cube and the number [equal to some things] is double, either a first *binomium* and its *recisum* or a fifth *recisum*, and [that] the value of the cube equal to some things and a number [is] by both, it is necessary that, since the number is made by the *binomium* and its *recisum* and the fifth *binomium* [is made] by the fifth *recisum* and a number (see above Chapter V), the value of the fifth *binomium* of the cube equal to some things and a number is only discovered beyond the number. Therefore at first we search [the value] when the value of the cube and a number equal to some things is had, but the number of the things and of the equality is not given. Then, proposed a first or four *binomium*, or the *recisum*[1] of those, or a second or fifth *recisum*,[2] the part that is a number is multiplied by itself and is triplicated, and the square of the part that is a root is added to it, and the number of the things is brought together. Then, for the number of the equality, double the part that is a number, and multiply [it] by itself, and the remainder from the number of the things is multiplied by the same double of the number, and the number of the equality is made. For example $\sqrt{7} - 2$. Firstly multiply 2 by itself, 4 is made, triplicate [it], 12 is made, add the square of $\sqrt{7}$, 7 is made, and the whole 19 [is] the number of the things. Then, for the number of the equation, double 2, 4 is made, multiply [it] by itself, the difference from the number of the things 19 is 3, multiply [it] by 4, [which is] the double of 2, the number of the equality 12 is made. Therefore $x^3 + 12 = 19x$.

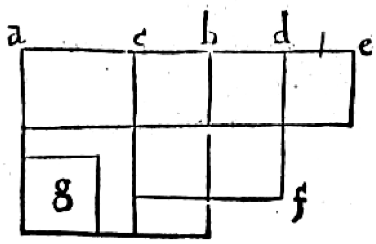
[1]Note that Cardano uses '*residuum*' to mean '*recisum*'.

[2]Again, note that Cardano uses '*residuum*' to mean '*recisum*'.



The cause of this is that, AB [being] put the thing divided in whatever way in C so that BC is a number and AC the other quantity, and two other squares equal to it are added close to the square BC, and the same [squares] with the square of AC are taken for the number of the things, those things will be equal to the cube of the assumed line AB with what is made by the double BC times the difference of the number of the things from the double of the square of BC, which is BD. In fact, if three squares of BC with the square of AC are the number of the things, then the things are equal to the three cubes of BC and to three times AC times the square of BC and to the cube of AC and to the parallelepiped from BC times the square of AC. Therefore, I subtract the cube of AB from those bodies, the difference of the double of the cube of BC to the double BC is left, but in truth the number is by hypothesis from the double times the difference of EF from the number of the things, which is the square of BC minus the square of AC. In fact the square CF contains four times the square of BC and the number of the things contains three times the square of BC and in addition the square of AC. Therefore, made G equal to the square of AC, the double of the square of BC exceeds the number of the things by the gnomon G, but it is demonstrated that the things exceed the cube of AB by the difference of the double of the cube of AB to what is made by BC times the square of AC, which is G, and the same by the double BC times the gnomon, therefore this number added to the cube is equal to some things. You will say the same if the number is smaller than the root, but AB [is] a *recisum*.[3] Indeed the demonstration proceeds in the same way, but it is necessary to change the figures.

[3]Again, note that Cardano uses '*residuum*' to mean '*recisum*'.



In truth the same happens where the cube equal to some things and a number, and the second or fifth *binomium* is such that the number of the things is the triple of the square of BC with the square of AC and it will be the number, that is what is made by the double CB times the difference of the square AC from the square of the double BC, which is CF. Indeed they will be the triple of the cube of BC, the triple of the parallelepiped of AC times the square of BC, the cube of AC, and the parallelepiped from BC times the square of AC. These eight bodies are subtracted from the cube of AB, the difference of the cube of AB from the eight bodies, [which is] the double of BC times the square of AC from the double of the cube of BC, is left. But the number is made by hypothesis from the double BC times the difference CF of the square from the three squared surfaces of BC with the square of AC, but this is how much the difference of the square of AC from the square of BC [is]. Since the triple of the square BC is common to both the quantities, therefore G, [which is] the square of BC times the square of AC is made. Therefore, since it is minus, the number is made equal to the double BC times the gnomon G. But the cube of AB exceeds the things, as it is demonstrated, by the difference of the double BC times the square of AC from the double of the cube of BC, but the double BC times the gnomon G is equal to the double of the excess of BC times the square of AC from the double of the cube of BC, since the heights and the surfaces are the same, then the cube is make equal to the assumed things and to the number.

Corollary. From this it is clear that the equality is unequal, therefore the rule cannot be generally delivered. In fact it is given a number [that is] double of the parallelepiped of the smaller part times the gnomon, which is the difference of the squares of the parts. Furthermore it is clear that such a gnomon is in every case equal to the things alone, where the supposed parts are the half of the number plus one thing and minus one thing.

Chapter XVI

That a *quadrinomium* from cubic roots is reduced to three parts, two of which are only cubic roots, or by far many

And firstly, if the *quadrinomium* by cubic roots in continued proportion, in which there is no number [is considered], as $\sqrt[3]{3}+\sqrt[3]{6}+\sqrt[3]{12}+\sqrt[3]{24}$ is. Then $\sqrt[3]{24}$ to $\sqrt[3]{3}$ [is] a triplicated ratio, but 6 to 3 [is] as $\sqrt[3]{3}$ to $\sqrt[3]{3}$ triplicated, therefore $\sqrt[3]{3}$ is the half of $\sqrt[3]{24}$, therefore they make the cubic root, that is $\sqrt[3]{81}$, but $\sqrt[3]{81}$ multiplied by $\sqrt[3]{72}$, [which is] the product by $\sqrt[3]{6}$ times $\sqrt[3]{12}$, produces $\sqrt[3]{729}$, namely 9 doubled. Therefore by [the things] said above that *quadrinomium* raised to the cube has only two cubic roots, therefore the things cannot be equated to the cube. I only show that the number is made from the product by $\sqrt[3]{72}$ times $\sqrt[3]{81}$, because by [the things] said the product by $\sqrt[3]{3}$ times $\sqrt[3]{6}$, thence times $\sqrt[3]{12}$, are in continued proportion. Equally by $\sqrt[3]{6}$ times $\sqrt[3]{12}$ and after that times $\sqrt[3]{24}$. Therefore some numbers are made by the product of $\sqrt[3]{6}$ times $\sqrt[3]{12}$ times $\sqrt[3]{24}$, then the number is made in the aggregate, which was to be demonstrated.

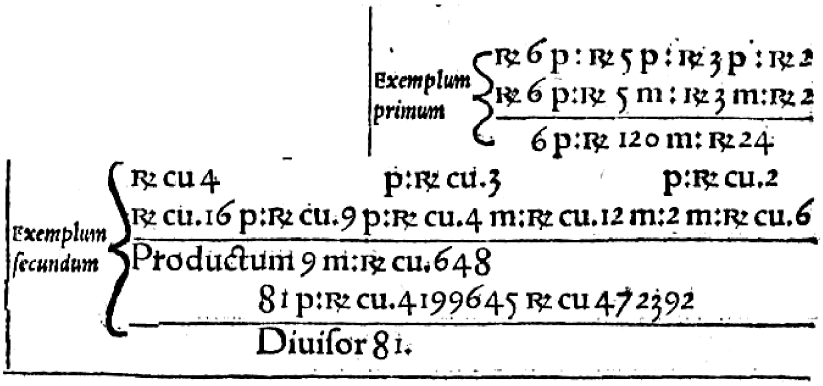
If in truth there is the number among those cubic roots, as $\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[3]{8}+\sqrt[3]{16}$, the same will happen as it is plain. In fact $\sqrt[3]{128}$ and 4 are produced, and all the remaining [parts] are commensurable to those, as it can be easily demonstrated. The same is made in the *trinomium* only by $\sqrt[3]{16}+\sqrt[3]{4}+\sqrt[2]{2}$. And for that reason there is no doubt to a greater extent, because we are in the preceding case.

If in truth they are such that two [cubic roots] times the square of another produce a number, they are already commensurable. And therefore they are not more than four [parts], but three. If a root does not carry anything of another, then in this case we take $\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[2]{4}+\sqrt[3]{5}$. And it is plain that, raising [it] to the cube, more than four non commensurable cubic roots are made. Then no *quadrinomium* by cubic roots is suitable.

Chapter XVII

In how many ways the number can be produced by the non-number

At first whatever number is produced by these numbers by which it can be divided. And thus, if I want to divide 10, it can be divided by an irrational number that consists of four incommensurable roots, yet of no more [parts], even if one part is a number. And, if they are cubic root, [it can be divided by an irrational number that consists] of no more than three [parts]. And this is made through the *recisa*, as in the third Book of the *Opus perfectum*, as if the dividend is $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$, find its *recisum*, as you see.


Exemplum primum
℞ 6 p: ℞ 5 p: ℞ 3 p: ℞ 2
℞ 6 p: ℞ 5 m: ℞ 3 m: ℞ 2
6 p: ℞ 120 m: ℞ 24

Exemplum secundum
℞ cu 4 p: ℞ cu. 3 p: ℞ cu. 2
℞ cu. 16 p: ℞ cu. 9 p: ℞ cu. 4 m: ℞ cu. 12 m: 2 m: ℞ cu. 6
Productum 9 m: ℞ cu. 648
81 p: ℞ cu. 4199645 ℞ cu 472392
Diuisor 81.

And thus, if I want to divide by $\sqrt[3]{4}+\sqrt[3]{3}+\sqrt[3]{2}$, as Scipione del Ferro of Bologna taught. And it is plain that the square of those are assumed [as] plus and the reciprocal products for minus. Moreover the product is the aggregate of the cubes of the parts 4, 3, and 2, which is 9, minus the triple cubic root of what is made by the first times the extreme, namely $\sqrt[3]{24}$, which triplicated produces $\sqrt[3]{648}$. And by the same reason you will find its *trinomium* in the same way, as you see multiplying the parts by themselves and between them. And thus you will compose the number that divides. But, as I said, this pertains to another [thing].

This same is what I wanted to teach, namely that where I cannot divide the number because of the multitude of the parts, it is enough to suppose [it], as if

I want to divide 10 by $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}+$ [*sic*] it is enough to suppose the divisor to be divided and you will have $\frac{10}{\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}+1}$. And, with this you will be able to perfectly perform operations by multiplying, dividing, adding, and subtracting, just as one is accustomed to make with fractional numbers. As if I want to divide 20 by that number, $\sqrt{24}+\sqrt{20}+\sqrt{12}+\sqrt{8}+2$ comes out. And thus you will have that 10 is produced by whatever number that divides with its alternate.

But particularly it is made at first from whatever *binomia* and *recisa* the difference of the parts of which is multiplied by itself, that same is a number, as by $\sqrt{11}+1$ and $\sqrt{11}-1$, and by $\sqrt{12}-\sqrt{2}$ and $\sqrt{12}+\sqrt{2}$, and thus by the others, and thus it can be produced $4+\sqrt{6}$ and $4-\sqrt{6}$, and $5+\sqrt{15}$ and $5-\sqrt{15}$.

Secondly it can be produced by a *binomium* and a *recisum* that have a ratio, as $4+\sqrt{12}$ and $2-\sqrt{3}$ produce indeed 2, then, whatever *binomium* or *recisum* [being] proposed, you find its alternate, and multiply reciprocally, and divide the proposed number by the product, and multiply what comes out by what you have discovered at the second place, and you will have what is searched for, as if I want to find the number that, multiplied by $3+\sqrt{7}$, produces 10. $3-\sqrt{7}$ [being] discovered, and reciprocally multiplied [by $3+\sqrt{7}$], 2 is made, divide 10 by 2, 5 comes out, multiply 5 by $3-\sqrt{7}$, $15-\sqrt{175}$ is made, multiply $15-\sqrt{175}$ by $3+\sqrt{7}$, $45-\sqrt{1225}$ is made, the root of which is 35, subtract [it] from 45, 10 is left.

Thirdly it is made by fractions in the same way, where in the first [case] $\sqrt{\frac{86}{5}}+\sqrt{\frac{16}{5}}$ times $\sqrt{\frac{52}{3}}-\sqrt{\frac{22}{3}}$ produces 10, and thus in numbers as $\frac{19}{6}+\sqrt{\frac{1}{36}}$ and $\frac{19}{6}-\sqrt{\frac{1}{36}}$, and thus in irrational [numbers] as $\frac{10}{3}+\sqrt{\frac{6}{25}}$ and $\frac{10}{3}-\sqrt{\frac{6}{25}}$.

Fourthly, if you want two numbers the squares of which differs by 10, this is easy with whatever number. For example I take 2 and 10, multiply 2 by itself, 4 is made, subtract [it] from 10, 6 remains, divide [it] by 2, 3 comes out, the half of which taken by itself and added to 2 produces the squares of the numbers the difference of which is 10. In fact the squares $\frac{7}{2}$ and $\frac{3}{2}$ differ by 10. Similarly I take $\frac{11}{5}$, I multiply [it] by itself, $\frac{121}{25}$ is made, I subtract [it] from 10, $\frac{129}{25}$ is left, divide [it] by $\frac{11}{5}$, and it is AC, if you divide $\frac{129}{25}$ by $\frac{21}{5}$, which is AC, if you would divide 1295 by 55, then $\frac{129}{55}$ comes out, the half of which is $\frac{129}{110}$, add [it] to $\frac{11}{5}$, $\frac{344}{110}$ is made, [this] and $\frac{129}{110}$ produce the squares the difference of which is 10.

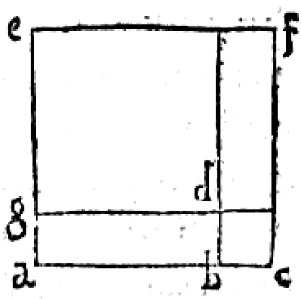
In addition it is produced by no matter which simple number that is in the same order and [that] produces the number according to its nature, as by $\sqrt{10}$ by itself, and by whatever fourth root that multiplied by another fourth root produces $\sqrt{100}$, and by analogous [things], as $\sqrt{20}$ times $\sqrt{5}$ produce 10, since the ratio of $\sqrt{20}$ to $\sqrt{10}$ is as [the ratio] of $\sqrt{10}$ to $\sqrt{5}$. And the ratio of $\sqrt{50}$ to $\sqrt{10}$ [is] as [the ratio] of $\sqrt{10}$ to $\sqrt{2}$. And [it is produced] no matter which cubic root that reciprocally multiplied produces 1000, [which is] the cube of 10, as $\sqrt[3]{1000}$ times $\sqrt[3]{10}$, $\sqrt[3]{200}$ times $\sqrt[3]{5}$, and $\sqrt[3]{50}$ times $\sqrt[3]{20}$. And thus 10 is produced by all the fifth roots that produce 100000, the first root[1] of 10.

[1] 1570 and 1663 have "*R primum 10*".

Chapter XXII

On the contemplation of plus and minus, and that minus times minus makes minus, and on the causes of these according to the truth

When I say $6+2$, it is clear that it is 8 according to the thing, but according to the term is composed by 6 and 2. Similarly when I say $10-2$, it is 8 following the thing, but is 10 subtracted according to the term. And therefore, in the operation that concerns the goal, $6+2$ must produce 64, because 8 multiplied by itself produces 64, and so $10-2$, which is 8, must produce the same 64. But, regarding to the way of performing the operation that is 8 divided by $6+2$, or by $10-2$, it is necessary to make operations by the fourth [Proposition][1] of the second [Book] of Euclid.



And in $6+2$ it is plain that in the figure AB is set 6, and BC 2, AD is made 12, DC [is made] 4, DF [is made] 12, DE [is made] 36, therefore all together [is made] 64, and there is no doubt on this. But, if AC is set 10 and BC -2, the square of AC will be none the less 64, which is the square DE, because AB is truly 8. Then there is AC. If someone says: you have a ten-feet, squared field two feet of which are of another [owner], and the square of your part is yours,

[1] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].

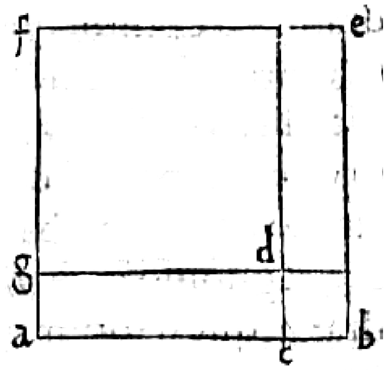
[and] all the remainder is to the other, therefore you only have DE, which is 64, and the gnomon of that GBF was of the other, and it was 36 as it is clear.

Then the cause of the division of plus or minus is two-folded. In fact, if they are of the same nature, like 6 and 2, or 10 and 2, or 6 and $\frac{1}{2}$, or $10 - \frac{10}{3}$, it would be stupid and superfluous to say $6 + 2$, or $6 + \frac{1}{2}$, or $10 - 2$, or $10 - \frac{10}{3}$, but we must say 8 [for] $6 + 2$ or $10 - 2$, or $\frac{13}{2}$ [for] in $6 + \frac{1}{2}$ or $10 - \frac{7}{2}$ and it is easiest for the multiplication and division. And especially since in the division it is always necessary to reduce the quantity indicated by more terms, either plus or minus, to a simple quantity. But the cause of such terms plus or minus is that the quantity that is either added or subtracted is not of the same nature of the first, like $6 + \sqrt{2}$, otherwise the *binomium* would have been rational or irrational,[2] that is the number or the root of the number, which it is shown by Euclid that cannot be. And thus [for] $6 - \sqrt[3]{2}$. Since they [6 and $\sqrt[3]{2}$] are of different nature and cannot be indicated by a term, it was necessary to join those quantities by plus or minus, and moreover they cannot be indicated by a term by means of a root. In fact $6 + \sqrt{2}$ and $\sqrt{38 + \sqrt{288}}$, and it is allowed to be seen, nevertheless the simple is a root of one composite number or quantity, that is $\sqrt{38 + \sqrt{288}}$. The second cause is when the second or third quantity [that is] added or subtracted is unknown, as if we say $6 + x$. Indeed one may set that x is 2, and thus that the whole is truly 8, because nevertheless we know how much x is. Therefore it is gathered to say $6 + x$, $10 - x$, from which it is agreed that, in the first case, what is multiple can never be reduced to one nature except by chance. In fact, as said, $6 + \sqrt{2}$ can make neither one number nor of one nature, but in the second case [it] sometimes can, sometimes not. As if we say $10 - x$, and x is 2, then it is equivalent to 8. But, if x is $\sqrt{2}$ or $\sqrt[3]{3}$, it is clear that it cannot never be reduced to one nature, but it is always equivalent to a *binomium*, or to a *recisum*, or to another incommensurable[3] quantity, as $6 - \sqrt{2}$ or $6 - \sqrt[3]{3}$. In the first case I have said that a multiple quantity is nevertheless equivalent to a simple [one], and this mostly happens with universal and abstruse roots, as it is declared in the *Ars magna* (Chapter 11) that $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$ is the same as 2. And this

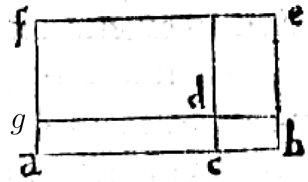
[2] "*[R]hete aut alogum*".
[3] "*[A]loga*"

also happens with square [roots], as $\sqrt{6+\sqrt{9}}$ is 3. Therefore, as I have said, it was necessary to set plus and minus for the sake of those two causes.

Having seen this, the operation plus being clear and by [the things] demonstrated by Euclid in the second [Book] of the *Elements* (Proposition[4] 4), it remains to show the same about the minus, and AB is set 10 as before and BC -2.



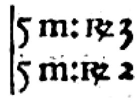
Then it is clear that AC is truly 8 and its square DF will be 64, but the gnomon is the whole remainder, as I have similarly said, and if BC is of someone else, therefore the whole gnomon also [belongs] to him, as I will show. And it is agreed that that gnomon is made by AC times CB twice, and they are the rectangles AD, DE, with the square of BC by the same proposition (fourth [Proposition] of the second [Book] of the *Elements*). But that whole gnomon is 36, because the square of AB is 100 and FD [is] 64, therefore the remaining gnomon GCE is 36, and AD and DE are minus and are 32, and the gnomon is -36. Therefore the square of BC, which is 4, is also minus. In fact, if it would have been plus, the gnomon was not minus, if not 28, and DF 72, and AC $\sqrt{72}$ and not $\sqrt{64}$, which is 8. Therefore the square of BC is minus and is made by minus multiplied by itself. Therefore minus multiplied by itself produces minus.



[4]*Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].

And similarly [if] AB was set up 10 and BC -2, then AC will truly be 8 and [if] for example AF was set 4 and AG 1, therefore FG will truly be 3, hence FD [will be] 24, but the whole surface AE is 40, therefore the gnomon is the remainder 16, and it is made by AC times CD, and therefore the surface is near to 8, and by BC times GF, the surface DE, and by BC times CD, and it is 2, everything that is 16, but this is minus because it is the difference of the products of 10 times 4 and of $10-2$ times $4-1$. Therefore minus times minus, that is BC times CD no matter which of the two is minus, produces $-BD$, which is 2, as much as $+AC$ times $-CD$ and $-BC$ times $+FG$, which produce minus by the thing acknowledged on those. And therefore it is well known the common error of saying that minus times minus produces plus, and not rather indeed minus times minus produces plus more than plus times plus produces minus. And, because everywhere we have said the contrary, therefore I will teach the cause of this, hence [that] in the operation minus times minus it is seen to produce plus, and in what way it must be understood. Then, in the second figure, let AB be 10 as before and BC be $-x$, it is plain that it is necessary to multiply AC by itself and AC by BC twice according to that operation. But, since AC is unknown, it is $10-x$, let us take AB which is known, it is indeed 10, in order to perform operations with AB and BC 9. And, since the square of AB with the square of BC is equal to the square of AC with two times AB times BC, therefore we subtract two times AB times BC from the squares of AB and BC. And, since two times AB times BC surpasses the gnomon GCE by the square of BD as it is agreed, therefore we subtract as much as the square of BC plus [what] is necessary, and we set minus, since only the gnomon is truly a minus. Then, since we subtract as much as the square of BC plus what we would have [to subtract] from the square of AB as if [it was] plus, therefore, in order to rebuild that minus that we subtract beyond measure, it is necessary to add as much as plus the square of BC. And therefore, since BC is minus, we say that -2 is the transformed square in plus, therefore that minus times minus produced plus. But it is not true. But we have added as much as plus the square of BC. [It is] not that the square of BC is plus, but another quantity, assumed arbitrarily equal to BC, is added and made plus. I say the same in the third figure, because we perform operations through AB and AF instead of through

AC and FG. Therefore in performing operations it is seen that minus times minus produces plus. And then, as in the second example AB is 10, BC [is] x and thus 2, AC will truly be 8, therefore DF will be 64 and the gnomon GCE $16x + 136x^2$, in fact $16x$ are 32, and x^2 [is] 4, [so] that everything is 36. And then it must be said [that] AB joined or separated [is] not properly minus. If in truth we perform operations with the whole AB and BC, we will have $100 + x^2 - 20x$, but –look! – that in the previous equality you had nothing but $-16x$, here in truth you have 20. Therefore, since we have $-x^2$ in the previous equality and we have here $+x^2$, it is necessary to add $4x$ to the number from 16 to 20. Or rather, because you have added $-4x$ more than necessary, therefore you have subtracted $-x^2$, and also instead of that you have added $+x^2$, and therefore you came to this, as it would have been said [that] minus times minus produces plus, which is nevertheless false. In fact it it not produced by the operation of multiplication, but, in order for you to come to a greater acquaintance through that seventh Proposition[5] of the second [Book] of Euclid, I similarly say [that], if you multiply $3 - \sqrt{2}$ by $5 - \sqrt{3}$,



it is truly necessary to multiply[6] AC by FG, and you have the true product. But, because neither $3 - \sqrt{2}$ is truly known under one term, nor $5 - \sqrt{3}$ is known under one term, and each multiplication and division is separately by simple quantities, therefore in the *recisa* it is necessary to perform operations by the seventh Proposition of the second [Book] of Euclid instead of the fourth [Proposition]. And thus, since that addition of minus the square is included in that, in the multiplication of one integer part by the part subtracted twice over the gnomon, therefore it is necessary to add to plus as much as the square of that part that is minus. Therefore, in order to perform operations in the *binomia* by the fourth Proposition and according to the substance of the composite quantity, thus concerning the substance and truly we perform operations also in the *recisa*

[5] *Elements* II.7: "f a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment", see [Heath 1956a, page 388-389].

[6] Note that Cardano employs '*multiplico*' for numbers and '*duco*' for geometrical quantities, which are both translated as 'to multiply'.

by the same, but we perform operations by the seventh [Proposition] of the same to know the terms.

Fourth and last that was to be considered is why only plus times plus makes plus, and minus times minus and plus makes minus. And I say (as I said) that it is necessary to suppose minus not just as if is made of the same plus, in fact it is alien. Therefore it is necessary to assume a lot [of things] in order to construct, in order to destroy it is enough [to assume] one [thing]. Then, for that, for the plus to be setup, it is necessary that plus is multiplied by plus, in fact plus times minus, or the alien, makes minus, because nothing is able beyond its powers, then plus is able of as much as itself, therefore, when it is multiplied outside of itself, it produces minus, otherwise it could have produced more than [what] his power is. But, when it is multiplied by another plus, it is able of nothing but as much as it is able [being multiplied] by the parts of that plus. For example, 6 is multiplied by 10, therefore by 6 and 4, but, as [6] times 6 is not able beyond 36 by [the things] demonstrated, Moreover as 4 is multiplied by 6, it is able of nothing but 4 and 2, and as [it is able of] nothing but as by itself, therefore it is able of nothing but up to 16, and as the remainder 2 by 4 [is able of] nothing but by 2 and 2, therefore it is able of nothing but 4, and there is 4, but 36, 16, 4, and 4 produces 60. Therefore 6 times 10 is able to produce nothing but 60. Therefore minus times minus, or the alien times the alien, and minus times plus or plus times minus, or what is times the alien or the alien times what is, produce only minus, or the alien, which was to be demonstrated.

Corollary 1 on Chapter 6. From that you realise the true reason of multiplying by minus, and dividing by minus, and taking the square root as well as the cubic [root] (in fact there is no doubt about the cubic root, which is minus), which before were unknown.

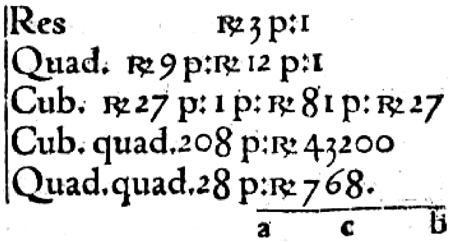
Corollary 2. From that it is also well known that, divided minus by plus, it comes minus. In fact, multiplied minus by plus, minus is made, therefore divided minus by plus, minus is made. And, divided minus by minus, minus and plus come, because minus is made from minus by plus and minus, therefore, that product, which is minus, [being] divided, the other of the two, which is plus or minus, comes. But, plus [being] divided by minus, nothing comes, in fact either plus or minus would come from minus times the same plus or minus is produced by plus, which is against [the things] demonstrated.

Chapter XXIV

Demonstration that shows that no chapter except the discovered [ones] can be generally known

It remains to show what has been proposed at the beginning, because of which we wrote this, namely that there is no other general chapter that can be known beyond those [ones] that are bequeathed. Since, more than four different kinds [of terms] cannot be reduced to fewer, either by a division, or [by taking the] root, or by a change, or [by] a own rule, or [by] suppressing, or for the sake of the origin, or by a Geometrical demonstration, because there are great inequalities in the single [terms] that can hardly be understood in four quantities and, if the perfection cannot be found in these, the less [it can be found] in those. Then on this, if there are four quantities up to the cube, it is now wise to reduce to three quantities and [to reduce] all the chapters of three quantities to the cube equal to some things and a number. Therefore, if I will show that this cannot be general, the purpose is also clear in the unknown part.

Therefore we assume the first rule of the imperfect special chapters (Chapter 25 of the *Ars magna*), in which $x^3 = 20x + 32$, and the value of the thing is the fifth *binomium* $\sqrt{17} + 1$. And similarly it is seen in the *Ars magna* (Chapter 13 in the end, first Corollary) that two values of the cube and a number eqial to some things compose the value of the cube equal to some things and a number. From these it is clear that it is necessary that the general value can be shared by the number and by the fifth *binomium* and, since the number is similar to the fifth *binomium*, the fifth *binomium* is not necessary that it is of this sort in the creation. Moreover in this way the fifth *binomium* crosses the equality, namely $\sqrt{3} + 1$. Let the cube $\sqrt{108} + 10$ be done, therefore this is made necessarily equal to $6x$, because $\sqrt{108}$ contains six times $\sqrt{3}$ and because $6x$ are nothing else than $\sqrt{108} + 6$, therefore $x^3 = 6x + 4$. Then, since what can be by the same nature is either the cubic root of the very similar cube, or the square [root] of the square, or the sixth [root] of the sixth [power], or the difference or the aggregate of two, it is necessary that such a simple value is one of this sort, if it must be general, so that whenever it can be made equal to that, if the square comes to mind, therefore $\sqrt{3}+1$ is the difference as you see in the margin.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Quad. ℞ 9 p:℞ 12 p:1
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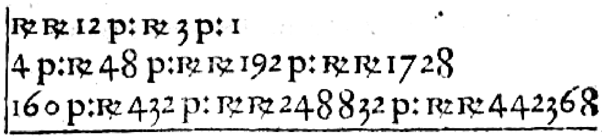
Quad.quad. 28 p:℞ 768.

But the aggregates are in infinite ways. In fact let AB be equal to any whatever quantity and CB [be] the same value, therefore, BC [being] subtracted from AB, AC is left. Therefore, AC [being] subtracted from AB, the value BC will be left. And similarly, AB [being] put the value, you can subtract AC from that, if only it is smaller, so that BC will be left, therefore the value is made from the joined AC and CB. Then you already have that it can be the square root of a *trinomium* one part of which is a number, and the fourth root of the *binomium* or of the *quadrinomium* one part of which is a number, and the sixth root of a *multinomium*, namely of thirteen parts or of less that are square roots so that one of those is a number.

Moreover to relate the aggregates and the differences, I want to give you an example from the *Ars magna*. I have said that $\sqrt{\frac{59}{8}+\sqrt{\frac{2997}{64}}}+\frac{3}{4}-\sqrt{\frac{37}{16}}$ is equal to 3. You will deduce the parts from the parts, as you see if it is true, and you will have $\frac{9}{4}+\sqrt{\frac{37}{16}}=\sqrt{\frac{59}{8}+\sqrt{\frac{2997}{64}}}$. Therefore multiply both by themselves and you will have the same in both parts, that is $\frac{59}{8}+\sqrt{\frac{2997}{64}}$. In fact $-\frac{9}{4}$ by itself makes $\frac{81}{16}$, to which, $\frac{37}{16}$ [being] added, $\frac{59}{8}$ is made, and $\sqrt{\frac{81}{16}}$ times $\frac{37}{16}$ makes $\frac{2997}{156}$, which duplicated makes[1] $\frac{2997}{64}$, the root of which, added to $\frac{59}{8}$, makes $\frac{59}{8}+\sqrt{\frac{2997}{64}}$. Whence you will perform operations in others in the same way, therefore I say that it cannot be the square root of the *trinomium* that have two square roots and a number. In fact, if the square root of $\sqrt{6}+\sqrt{2}+1$ could have been from the kind of the third or sixth *binomium*, it cannot satisfy as it is demonstrated, and neither if one part is a number and the other a root, in fact its square will be

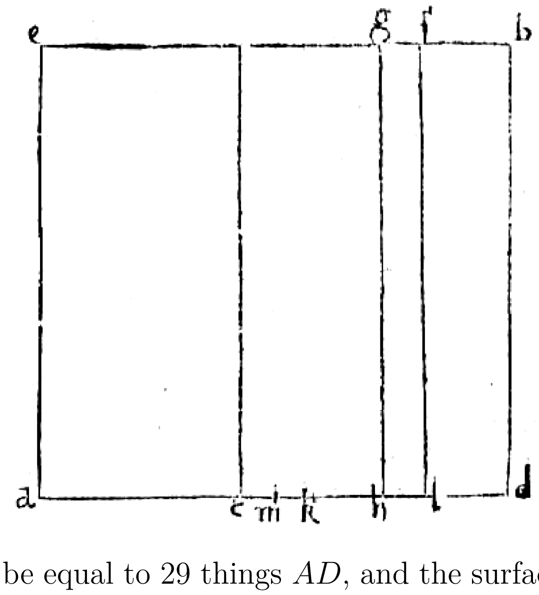
[1] 1570 and 1663 have "$\frac{2997}{64}$, *et sunt* $46\frac{53}{64}$".

a *binomium* and not a *trinomium*. Therefore we propose $\sqrt[4]{6}+1$, and its square will be $1+\sqrt{6}+\sqrt[4]{96}$ (by the fourth [Proposition][2] of the second [Book] of the *Elements* of Euclid). In fact, if we take $\sqrt[4]{4}+1$, $\sqrt{4}+1+\sqrt[4]{64}$ is made, that is $3+\sqrt{8}$, then, if we take $\sqrt[4]{12}+\sqrt{3}+1$,



℞℞ 12 p: ℞ 3 p: 1
4 p: ℞ 48 p: ℞℞ 192 p: ℞℞ 1728
160 p: ℞ 432 p: ℞℞ 248832 p: ℞℞ 442368

one may solve [it] in $160+\sqrt{432}+\sqrt[4]{442368}+\sqrt[4]{248832}$, nevertheless these are not commensurable, but in the ratio of $\sqrt[4]{256}$ to $\sqrt[4]{144}$, that is 4 to 12. It is allowed that it is very near, in fact $\sqrt[4]{432}$ is twelve times $\sqrt{3}$, and $\sqrt[4]{442368}$ is twelve times $\sqrt[4]{12}$, and it is remarkable to a such degree that, if $\sqrt[4]{442368}$ is a number, we will have the discovery. Therefore, since this *trinomium* cannot be reduced to lesser [terms], the remaining [*multinomia*] much less, because $\sqrt[3]{460}+\sqrt{432}+\sqrt[4]{442368}+\sqrt[4]{248832}$ cannot be the searched equality. Therefore it is necessary that it is the difference of two quantities, and the foundation is in the first rule in the *Ars magna* said above.



Let the cube of AB be equal to 29 things AD, and the surface BC will be 29 and CE [will be] the number 42, and the body will be according to the height AD. And, since the parts can be made from 29 as you see in the side,

[2] *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments", see [Heath 1956a, page 379].

28.1.1.28
25.4.2.50
20.9.3.60
13.10.4.52
4.25.5.20
18 p:℞72.11 m:℞72.3 m:℞2/42.

the number is made from one of these multiplied by the root of the other, the given number of the things with $28, 50, 60, 52$, and 20 could be made equal to the cube. I only say that also the other non-integer parts can be made, namely $18 + \sqrt{72}$ and $11 - \sqrt{72}$, and the other number 42 is made from $18 + \sqrt{72}$ times $3 - \sqrt{2}$, [which is] the root of $11 - \sqrt{72}$. From this it is allowed that it is necessary that $11 - \sqrt{72}$ is the first *binomium* or *recisum*. Then EF is proposed $18 + \sqrt{72}$ and CG 18, then HF will be $+\sqrt{72}$, therefore the other part will be FD denominated by[3] DG, namely $11 - FH$, [which is] $\sqrt{72}$, so that the true division of BC, namely 29, is in truth made in F. In fact CF is 18, that is $CG + \sqrt{72}$, that is FH, and DF [is] 11, that is $DG - \sqrt{72}$, that is FH. Moreover the division in G [is] according to the term, since CG is 18 and GD is 11. And since the ratio of the body on CB to CE is as [the ratio] of CD to AC, namely $29x$ to 42, therefore [it is] as x to $\frac{42}{29}$, or [as] $\frac{29}{42}x$ to the number 1. Again, since by the first rule of the chapter ([Chapter] 25 of the *Ars magna*) the body on CE is made by CF times the root of FD, and let DK be the side of DF, [the ratio] of AD to DK will be as [the ratio] of the surface CF to the surface CE, hence as CL to CA. And for the second time, since LG is made by the double of the parts of DK, it is proposed [to be] denominated by plus and minus, and it is what is $+DM$ and what is the double of $-KM$, therefore MK times MD produces HF. As in the example.

[3]1570 and 1663 have "*denominata per*".

ef 18 p: ℞ 72
cg 18
hf ℞ 72
fd 11 m: ℞ 72
dg 11
cd ad ca ut 1 pos. 1 $\frac{13}{19}$
dk ℞ v 11 m: ℞ 72
ad ad dk, ut cl ad ca
mk in md pd. dim. fh ℞ 18
dm 3
mk ℞ 2
dk 3 m: ℞ 2
cg ad gl, ut dm ad mk.

Then since CG is proposed [to be] the number 18, and GL to be $\sqrt{72}$, and the ratio of the parts of DK as the denominated [ones], that is as DM, which is a number, to $-MK$, and in the same of CG to GL, then we follow the first argument of the thing. AD will be $\sqrt{\frac{83}{4}+\sqrt{\frac{81}{2}}}+\frac{3}{2}-\sqrt{\frac{1}{2}}$ by the first rule. Therefore this is the true value of the thing and the same is 6. And let the experience be made, because, $\frac{3}{2}-\sqrt{\frac{1}{2}}$ [being] subtracted from 6, $\frac{5}{2}+\sqrt{\frac{1}{2}}$ is made, and this is equal to $\sqrt{\frac{83}{4}+\frac{18}{4}\sqrt{2}}$, which is well known because the squares of both are $\frac{83}{4}+\frac{18}{4}\sqrt{2}$. Therefore this kind of value is general, because it can be made equal to a number or not and [it can be] made equal to the *binomium*, because, the tied parts [being] subtracted, a *binomium* or a *recisum* were necessary left, and then the universal root of a first*binomium* or *recisum* can be put.

Again it is proposed the number of the things BC [to be] 20, the body CE [to be] 32. It is proposed CF, CB, FD [to be] 4, 32 is made from EF times the root of FD, which is 2. For the second time CD is to CA as $20x$ to 32, namely x to $\frac{8}{5}$, and since for the second time AD is to DK as CF [is] to CE, hence as CL [is] to CA. And since by the first rule AD is $\sqrt{17}+1$ and DK is 2, AK will be $\sqrt{17}-1$ and CE [will be] $\sqrt{68}-2$.

Chapter XXV

On the examination of the third rule of Chapter XXV of the *Ars magna*

We propose $x^3 = 18x + 108$. Then, if I will make two parts from the number of the things 18, by the reciprocal product of one of which times the root of the other is made 54, the half of 108, it is plain that the thing is 6. And by the general rule [the thing] is $\sqrt[3]{54 + \sqrt{2700}} + \sqrt[3]{54 - \sqrt{2700}}$, and this is in truth $3 + \sqrt{3} + 3 - \sqrt{3}$, which is 6 as before. But the division is not according to that way, but the root of the parts of 18 are 3 and 3, and the parts [are] 9 and 9, and thus the products will reciprocally be 54. And similarly, 21 [being] assumed for the things and 90 for the number, we will make according to the general chapter two parts from 90, by the reciprocal product of one of which times the root of the other is made 343, [which is] the cube of 7, [which is] the third part of the number of the things 21. And we will have the parts $\sqrt[3]{45 + \sqrt{1682}} + \sqrt[3]{45 - \sqrt{1682}}$, and it is $3 + \sqrt{2} + 3 - \sqrt{26}$, 6 as before. And thus, increasing the number of the things, we go outside the chapter and, diminishing too much the number of the things, a rule is not permitted in this way. Namely $x^3 = 15x + 126$. It is not permitted to divide 15 in two parts, by the reciprocal product of one of which times the root of the other is made 63, [which is] the half of 126, because the maximum in which it can be divided is when it is divided in equal parts, as it is demonstrated (by [the Proposition] 209 of the Book on proportion[1]). Therefore

[1] *De proportionibus*, Proposition 209: "If it appears [that] a rectangular surface [is] divided in two equal parts, which are both squared, likewise in two unequal [parts], the parallelepiped from the side of the middle part times the whole surface will be bigger than the aggregate of the parallelepipeds from the unequal parts times the sides of the other part by what is reciprocally made from the difference of the side of the smaller part from the side of the middle [part] times twice the difference of the bigger part of the surface from the middle [part] of the surface and from the difference both of the unequal sides joined to both sides and of the equal [sides] joined times the smaller part of the surface" or "*[s]i superficies rectangula in duas partes æquales divisa intelligatur, quæ ambę quadratæ sint, itemque in duas inæquales, erit parallelepipedum ex latere mediæ partis in totum [totam] superficiem maius aggregato parallelipedorum ex partibus inæqualibus, in latera alterius partis mutuo in eo, quod fit ex differentia lateris minoris partis a*

there are three parts in this chapter, the first that serves the special, non-general rule when the number of the things is bigger in comparison to the number of the equality. The second that serves the general, non-special rule when the number of the equality is bigger in comparison the the number of the things. The third that serves both, as in the example the general rule cannot reach $x^3 = 22x + 84$, because 21, [which is] the fourth part of 84, makes a square neither bigger nor equal[2] to the cube of the third part of the things $\frac{22}{3}$. Similarly the special rule does not reach $x^3 = 17x + 114$, since $\frac{17}{2}$ multiplied by $\sqrt{\frac{17}{2}}$ produces $\sqrt{\frac{4913}{8}}$, which is smaller than $\frac{114}{4}$, [which is] the fourth part of the proposed number 114, so that those mutual cannot compose the half of the proposed number 57. Therefore it is to be brought over in all that space, in which one fits to the other, making two parts from the thing already discovered, by the reciprocal product of one of which times the square of the other is made the half of the proposed number, and those will be the parts. But this is easily made by dividing the half of the proposed number by the thing, thence by dividing the thing in two parts that produce what comes forth. [For] example $x^3 = 6x + 6$. The value of the thing is $\sqrt[3]{4} + \sqrt[3]{2}$. We divide the half of the number of the equality 3 by this, $\sqrt[3]{2} - 1 + \sqrt[3]{\frac{1}{2}}$ comes out, I multiply the half of $\sqrt[3]{4} + \sqrt[3]{2}$ by itself, $1 + \sqrt[3]{\frac{1}{4}} + \sqrt[3]{\frac{1}{16}}$ is made, from which I subtract $\sqrt[3]{2} - 1 + \sqrt[3]{\frac{1}{2}}$, $2 - \sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{16}}$ is left, the universal root of which added and subtracted from the half of the preceding shows the parts, as you see.

$$\sqrt[3]{\frac{1}{2}} + \sqrt[3]{\frac{1}{4}} + \sqrt{2 - \sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{16}}} \quad \sqrt[3]{\frac{1}{2}} + \sqrt[3]{\frac{1}{4}} - \sqrt{2 - \sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{16}}}$$

And I taught above the way with the demonstration. Therefore the joined square of those are 6, and the reciprocally products are 3, which is well known by having been experienced. And the operation is beautiful.

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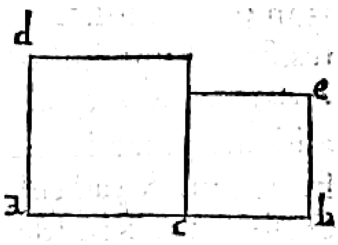
mediæ latere in differentiam maioris partis superficiei á media superficie bis, et ex differentia amborum laterum inæqualium iunctorum ad ambo latera, æqualia iuncta in minorem partem superficiei", see [CARDANO 1570c, pages 241-242]. This means that, if we take three real numbers u, v, w such that $2u = v + w$, then

$$2u\sqrt{u} = v\sqrt{w} + w\sqrt{v} + \big(2(\sqrt{u} - \sqrt{v})(w - u) + (2\sqrt{u} - \sqrt{v} - \sqrt{w})v\big)\,.$$

[2]Actually, $\left(\frac{84}{2}\right)^3 > \left(\frac{22}{3}\right)^3$.

Chapter XXX

What [is] the equality of the cubes of the parts of a divided line

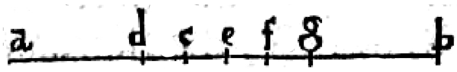


Let AB be divided in C, [let] their squares [be] CD, CE. I say that the cubes of AC, CB are equal to the parallelepiped from AB times the aggregate of the squares CD, CE, the surface AC times CB [being] taken away. In fact what is made by AB times the aggregate of the squares CD, CE is equal to what is made by AC times CD, CE and by BC times CE, CD, hence [it is equal] to the two cubes of AC and CB and to what is reciprocally made by the parallelepipeds AE times CE and CB times CD. But AC times CE [is] how much [is made] by BE times the surface AC times CB and CB times CD [is] how much [is made] by AD times the surface AC times CB.Therefore what is made by AB times CD and CE is equal to the cubes of AC, CB and to what is made by AD times the surface AC times CB and by BE times both [of them]. Moreover what is made by AD times the surface AC times CB with what is made by BC times both is equal to what is made by the whole AB times the surface AC times CB. For that reason, since AD is equal to AC and BE, it is equal to BC. Therefore what is made by AB times CD, CE is equal to what is made by AB times the surface AC times CB with the cubes of AC and CB. Therefore, what is made by AB times the surface AC times CB [being] subtracted from what is made by AB times CD, CE, and it is the same as subtracting the surface AC times CB from the squares of AC, CB, the parallelepiped from AB times CD, CE, the surface

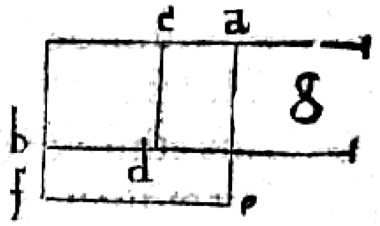
AC times CB [being] subtracted, will be equal to the cubes of AC, CB, what was to be demonstrated.

Chapter XL

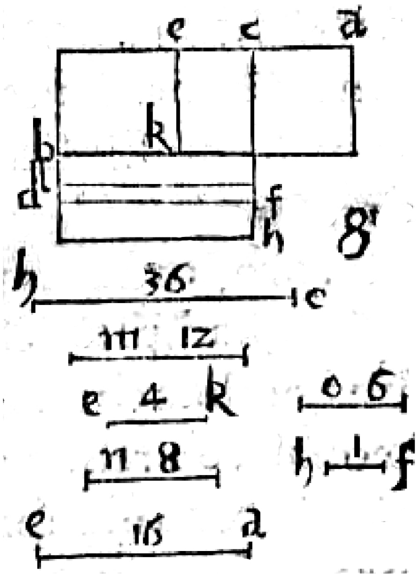
On three necessary [things] that is necessary to put before the discovery



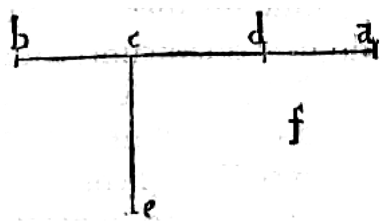
Then if DE is put the thing, it cannot be a number and, [if] AD [is put] a root, since the whole AB is a root and not the proposed number and not a root, because AD is a *recisum* [and] BD [is] a *binomium*, and a simple number is produced or [a number] composed with the root through plus and minus [is produced], therefore, the root [being] multiplied by DE, a *binomium* or a *recisum* or a root are made, therefore the number of the equality is not a number. For an equal reason DE cannot be a *binomium* or *recisum* of the third or sixth type, whence it cannot be a simple root. Again DE is proposed [to be] a *binomium*, and let DC be a number, and EF [be] equal to EC, and EG [be] equal to ED, then FG will be a number and CF a root, then it is necessary that a *recisum* similar to the *binomium* DG is made from the *recisum* AD times the *binomium* DB so that the number is made from that times the product DG. The same will be if CE is put a number and DC is put a root. This only matters, whether the root is bigger than the number or smaller. And in this disposition DE cannot be a *recisum*, because it is necessary to assume the quantity [to be] bigger than DE, and thus we were outside of the case of the rule and of the problem. Then DE is always a *binomium*. And we put DE $3+\sqrt{5}$, and it will be $32x = x^3 + 24$, and AE [will be] $\sqrt{32}$. And similarly, if DE is $3-\sqrt{5}$, but 3 will not be contained in DE. I say the same when $x^3 + 12 = 34x$, and the value is $3+\sqrt{7}$ and $3-\sqrt{7}$, in fact it cannot be true except in a *binomium* (see above, Chapter 5). But it is different when $x^3 + 8 = 18x$, in fact the thing is $\sqrt{6} - 2$ and it cannot happen in a *binomio*. Therefore the first two examples are suitable. And, since in those it is necessary to add a certain number that, [being] multiplied by the root of the whole, produces the number of the equality, and it is clear that it cannot be a root or a *binomium* or a *recisum*, in fact the number is not constructed, therefore it is necessary that it is a number, but we now suppose that this is a square. Then we propose AE 8, and its square is the number of the things 64, and it is such that, adding 17, another square is made, namely 81, the root of which (which is 9) multiplied by 17 makes 153. Therefore it will be $x^3 + 153 = 64x$ and the value of the thing [will be] $\frac{17}{4} + \sqrt{\frac{13}{4}}$, that is the half of the whole root plus the root of the difference of the number of the equality and of the number of the aggregate $\frac{3}{4}$. Then it will be put CE $\sqrt{\frac{13}{4}}$, and CD [DE?] $\frac{9}{2}$ [$\frac{9}{2} - \sqrt{\frac{13}{4}}$], [and] AD $\frac{7}{2} - \sqrt{\frac{13}{4}}$, and DB $\frac{25}{2} + \sqrt{\frac{13}{4}}$, and DG $9 + \sqrt{13}$. The product of AD times DB is $\frac{81}{2} - \sqrt{\frac{1053}{4}}$, then, this [being] multiplied by DE, namely $\frac{9}{2} - \sqrt{\frac{13}{4}}$, 153 is made. Then we can also divide 64 in two parts, 153 is made from one of which times the root of the other, and, since that root is the thing and it is DE, we multiply [it] by itself, and $\frac{47}{2} + \sqrt{\frac{1053}{4}}$ is made, therefore the remaining part is $\frac{81}{2} - \sqrt{\frac{1053}{4}}$, here that the thing falls back on the same. Therefore 306 is made from DG times the product of AB times DB, since, divided by 16, $\frac{153}{8}$ comes out, therefore the parts will be $8 + \sqrt{\frac{359}{8}}$ and $8 - \sqrt{\frac{359}{8}}$. Then these are reduced to one in two ways,



namely if the proposed number of the things is AB, and G [is] the number of the equality, and either you will divide AB by the square BC so that its side BD times the surface DA makes G or you will add the surface BC to AB so that the whole AF is squared and its side AE times the added BD produces G, the values are made well known, indeed in the first [case] the side BD, in the second [case] the half of AE, the side of the difference of three-fourths AF [being] added or subtracted to the proposed surface AB. But, since AE times CB is equal to G and BD times DA [is] equal to the same G, AE times CB will be equal to CD times DA, therefore AE [is] to CD as AD [is] to CB, but AE is bigger than CD, then AB [is] bigger than EB and, since BC and AF are squares, AD will be equal to DF, therefore DF [is] bigger than CD, which cannot be. Therefore there cannot be one division.



Therefore it is necessary that the surface AB is equal to the number of the things in such a way that BE is a squared part, the side of which EK times KA is equal to the number of the equality G. And again let CD be equal to AB, to which the surface HD is missing to complete the square, so that again the same G is made from CH times HD, and B will be in both cases the known thing, namely in the first [case] EK, in the second [case] the half of CH with what can [be produced] times the surface FL, put LC three-fourths of LC [*sic*]. Then, since HD to EK is as AK to DH, [the ratio] of HC to EK will be duplicated of the ratio of the mean of AE, EK to the mean between HC and HF. Therefore let M be the mean between HC and CK, therefore [the ratio] of HC to M [is] as [the ratio] of the mean between AC, EK, which is N, to the mean between HE, HF, which is O, therefore HC [is] to M as N to O. And the three orders will be connected to the two extrema EK, N, EA, [and] EK, M, HC, and HC, O, HF.



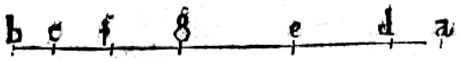
And again, when I will say to someone that AB is divided in C, and CD were the difference of the parts, from which times the mean CE between the parts F is produced, I say that I will have x^4+ the fourth part of the square of F equal to the squared number of the square of the half of AB. And therefore F cannot exceed the square of the half of AB, or the fourth part of the square of AB, namely if AB is 8, EB, FB, BA, BC, BO [*sic*], namely $x^4 + 9 = 16x^2$ ([16 being] the square of 4, the half of AB), hence the thing will be $\sqrt{8 - \sqrt{55}}$ and its double, that is $\sqrt{32 - \sqrt{880}}$, will be the quantity CD of the difference of the parts. And therefore the problem is that, since I know the quantity AB and the way to discover the product from CD times CE such that it is equal to F, if I discover the way to make the same F from CD times the product BC times AC, which is the square of CE, the chapter will be discovered. But in one and in the other problems the parts are changed, namely CD and CE.

Again, since F is made from the difference CD of the parts times the product of AC times AB, and BC is equal to AD, it will be as from AC times AD, and F is made after the product by CD. Then, if CD had been the mean proportional between AC and AD, AC would have been divided in D according to the proportion that has a mean and two extreme. And, if there had been such a product, CD would have been $\sqrt[3]{F}$ and, since the product of AC times AD is always in a certain ratio with the square CD, either bigger or smaller, and it is taken always in the same ratio of AB, [which is] x^2, + the number of the things of the divided line equal to its square. But x^2+ the square of the number of the divided line equal to the things times the triple of the number of the things [is] as if the line [is] divided by 10, I will have $x^2 + 10x = 100$ or $x^2 + 100 = 30x$, and the value will always be the same. And, if the square of CD was the double or the triple of the product of AC times AD, we would have that x^2+ a multiple of the same number of the things equal to a multiple of the square of the same number, or x^2+ the square of the number of the same divided line equal to the things multiplied by the reciprocal of the ratio $+2$. For example in the fourth ratio before there was $x^2 + 10x = 100$ or $x^2 + 100 = 30x$, I will have this $x^2 + 40x = 400$ or $x^2 + 100 = 60x$, whereby the number is produced by the number of the ratio 4 and [by the number] 2 taken according to the rule. And the thing or the value is the same, or, if the product is a multiple from the square, we will assume [it] in the opposite way, or x^2 with some things taken according to that part equal to the part of the same square of the divided line, or x^2+ the square of the same divided line equal to some things in the double ratio of the same divided line. And the thing falls back on the same. And the example is clear.

In the end from this it is well known that, assumed AB, which is 12, as in the present chapter and 8 is made from the difference CD times the product AC times CB, we will have $x^3+4=36x$, indeed this is demonstrated. Therefore AC will be divided in D in such a way that 8 is thence made from AC times AD, and the value of the thing will be the half of CD, therefore CD [will be] the double of the value, and AD or BC [will be] the half of the remainder. Therefore, if CD were $\sqrt[3]{8}$, which is 2, DA will be $\sqrt{5}-[1]$ and CA [will be] $\sqrt{5}+1$, and therefore the whole AB [will be] $\sqrt{20}$. Therefore if someone says: make two parts out of $\sqrt{20}$, from the product of the rectangle times their difference 8 is made, you will have the parts BC $\sqrt{5}-1$ [and] CA $\sqrt{5}+1$, the product of which is 4, which, [being] multiplied by CD, which is the difference and is 2, produces 8. And we will have $x^3+4=5x$. And the foundation of AB is only the rational power. Therefore, if $x^3+6=7x$, the thing can be 1 and 2, as it is plain. Therefore, if CD is put 2, put DA x, we will have $2x^2+4=6$ (where $2x^2+4$ [is] the product of AC times AD and times CD), AD will be 1. And, if CD is put 1, you will have $x^2+x=6$, therefore AD is 2. Therefore, when CD is 2, DA is 1 and, when CD is 1, DA is 2.

But, supposed by the first reason that 8 is made by AC, CD, DA in continued proportion, and let CD to be $\sqrt[3]{8}$, namely 2, therefore, if the square of CD was four times the rectangle AC times AD, this has the reason in this way, indeed what is made from AC times AD is equal to what is made by CD, DA times AD. DE is assumed [to be] the double of DA and DF [is assumed to be] four times the square, therefore DE is equal to four times the square AD, and the square of CD is equal to the squares CE, ED and to the double of CE times ED. Therefore the double of DE times EC, and it is DF times CE with the square of EC, is equal to four times CD times DA, that is [it is equal] to what is made by FD times DC once. Moreover this is equal to what is made by FD, that is

CE, and ED, therefore, the common [being] subtracted from what is made by FD times CE, the square CE, [which is] equal to what is made by FD times DE, is left. But FD is four times DA, and DE [is] the double of the same DA, therefore CE can [be] in eight times DA. Therefore AE is put whatever number, suppose 10, since EA is the triple of DA and CE [is] the root of eight times the square of DA, the whole CA will be $3+\sqrt{8}$ times the number of the things, and this is made equal to 10. Therefore the thing, namely DA, is from the division of 10 by $3+\sqrt{830}-\sqrt{800}$, therefore the remainder CD will be $\sqrt{810}-20$, therefore from the whole AC times DA $300-\sqrt{80000}$ is made, and this is the fourth part of the square of CD, namely $1200-\sqrt{320000}$, just as it was proposed.

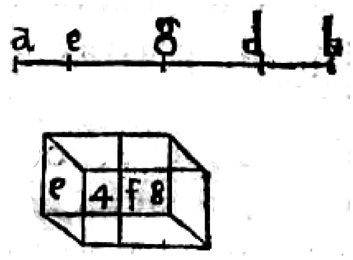


Again we say that the square of CD [is] to six times of what [is made] from CA times AD. I assume DF [to be] six times, as in the previous [case] it [was] four times DA. And similarly DE [is] the mean between DF and DA, in fact in the previous disposition DE was the mean between FD and DA and I assume GE equal to CD just as in the previous [case]. But there EG was the same EF, while here is smaller than that, because the ratio is bigger than four times, and then the square of DE is six times of the square of DA, since it is equal to what is from FD times DA. Therefore by hypothesis what is made by DG times CE with the square of CE is six times CD times DA, or [it is] equal to what is made by CD times DF, or to the square of DF with what is made by DF times FE. Then we divide both and the parts (as you see) are made.

d f in f e ★
Quad. e f ✠
Quad. e d
Duplum d e in e f ★

Quadratum e f ✠
Quadratum c f
Duplum c f in f e ★
Duplum d e in e f ✠
Duplum c f in d e ★

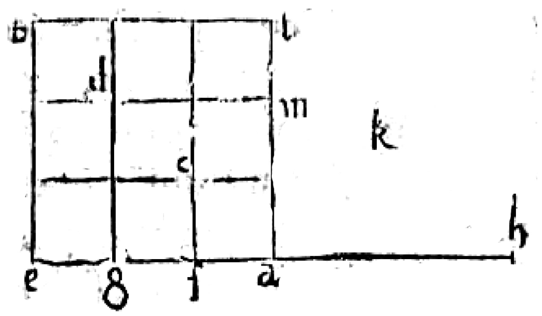
On both parts the squares of EF, [which is] the double of DE times EF, are taken away, DF times FE and the square of ED equal to the square of CF, the double of CF times FE and the double of CF times DE are left. But DF times FE is equal to what is made by EF times FE and one times DE, for that reason DF is equal to FE and ED joined, therefore the square of ED is equal to what is made by EF by itself times FE and ED, and it is the whole ED. Moreover the square of ED is equal to six times the square of DA, therefore what is made by EF times CD is six times the square of DA. Then DA is put x, D [DF?] makes $6x$, the whole FA [makes] $7x$. Therefore, if we put CA 10 as previously, EF [CF] will be $10 - 7x$, moreover CD [will be] $10 - x$, multiply [them] reciprocally, $100 - 80x + 7x^2 = 6x^2$ is made. And thus you see that the thing is reduced in any case to x^2 with the square of the proposed number, and [that] the number of the thing is always made by the proposed number, namely 10 times the number of the ratio +2, [and that] the ratio is six times. Therefore, 2 [being] added, $8 + 80x$ is made, then it is reduced to the rule *de modo* in such a way. The line AC is proposed [to be] 10 and the ratio [is proposed to be] six times. Add 2, 8 is made, multiply by 10, 80 is made, take the half and it is 40, multiply by itself, 1600 is made, remove 100, the square of 10, 1500 is left, the root of which, [being] subtracted from 40, produces the quantity DA $40 - \sqrt{1500}$. Then, in order to arrive at the thing, if someone says $x^3 + 4 = 12x$,



I take AB the double of $\sqrt{12}$, and it is $\sqrt{48}$, and the body F [to be] the double of 4, and it is 8, and I divide AB in equal [parts] in $\sqrt{12}$, and I add and subtract x, and EB is made $\sqrt{12} + x$ [and] $\sqrt{AE}$ [is made] $\sqrt{12} - x$, and the product will be $12 - x^2$, and I multiply that by the difference of AE and EB, DB [being] made equal to AE, and $24x + 2x^3 = 8$ is made, therefore $x^3 + 4 = 12x$, since then the half of DE is the value of the thing and the whole AB [is] a number or rational to the power. The first will be that AG is a number or rational to the power. Thence that, since F is made (as I said) from EB times BD and times DE, BD

[being] supposed [to be] a number, namely 1, the squares and the things will be equal to 8, in fact this is supposed, and we will have $x^2 + 2x = 2$.

Corollary. Moreover it is agreed that the ratio of the cube of CD to the parallelepiped from CB, BD, DC is always as [the ratio] of the square of CB to the rectangle from CD times DB, because [the ratio] of CB to the side of the parallelepiped of the same [is] subtriplicated to what is [the ratio] of the square of CB to the rectangle CD times DB. But [the ratio] of CB to the mean between CD, DB [is] subtriplicated to what is [the ratio] of the square of CB to the rectangle CD times DB, therefore [the ratio] of the side of the solid CB, CD, DB to the side of the rectangle ED times DB is as [the ratio] of $\sqrt{\frac{9}{2}}$ to $\sqrt[3]{\frac{9}{2}}$.

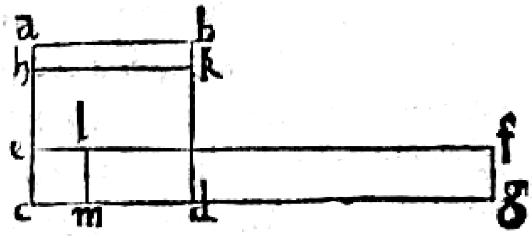


When you want to divide BA so that the ratio of the same to the rectangle AD times DB is twenty-four times, for example, divide the square BA by 24, and subtract what comes out from the square of the half of BA, and the root of the remainder added and subtracted from the half shows the parts. As if AB were 10, I multiply [it] by itself, 100 is made, I divide [it] by 24, $\frac{25}{6}$ comes out, I subtract [it] from 25, the square of AG, $\frac{125}{6}$ is left, the root of which added and subtracted to 5, the half of 10, shows the parts, namely AB, DB, and it is had by Euclid ([Proposition][1] 5 of the second [Book] of the *Elements*). Now in truth the square of AB is set up 7, and AC 1, and AD 4, then AE will be $\sqrt{7}$, AF 1, AG 2, and let EH to be the double of EA, and it will be $\sqrt{28}$, and let KB to be a number, then let the cube of $AC + 6$ to be equal to $7x$, and likewise the cube of $AG+$ the same number 6 [to be] equal to $7x$. Then, since AB is 7, there will be the body AB, AF [being] put the height and the thing $7x$, but this body is

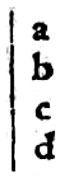
[1] *Elements* II.5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half", see [HEATH 1956a, page 382].

equal to x^3, that is [it is equal] to the cube of AF with b [*sic*], moreover I [*sic*] is the gnomon $LCBF$ according to the height AF, and similarly the body from AB times AG is equal to the gnomon $LDBG$ times AG with the cube of AG, because the gnomon $LDBG$ times AG is 6. Therefore the surface AB will be divided twice so that it, or b [*sic*], is made from the side of one part times the remaining [one]. And likewise EH will be divided twice in A in equal [parts] so that $7x$ is made by AF and AG multiplied by the square AE, or by the product AH times AE, since AB has already be supposed 7 and AF and AG the thing. And again CH will be divided twice in F and in G so that the product BF times FE is equal to the gnomon $LCBF$, and [CH will be divided] in G so that the product LG times GE is equal to the gnomon $LDBG$. Whence each one of those by its first part multiplied by the difference from the half, that is HF times FE by FA, and HG times GE by GA, produces the same number k or b (by [the things] demonstrated in [Proposition] 5 of the second [Book] of the *Elements*). Now in truth let $x^3 + 8 = 8x$, the thing will be 2, so that you multiply 1, [which is] the half of 2, by itself, 1 is made, triplicate [it], 3 is made, the number of the things [being] subtracted, it is left [what] the root of which -1, [which is] the half of the previous value, $\sqrt{5} - 1$ is the second value (by Chapter 12 of the *Ars magna*).

Then I put F the number 8, and the first value AB 2, and the second value BD $\sqrt{5} - 1$ and therefore, BC [being] put, CD will be $\sqrt{20}$, [which is] the double of the same CD, and AE $\sqrt{5} - 1$ [will be] equal to BD. Then I put AD $\sqrt{5} + x$, AE $\sqrt{5} - x$, they produce reciprocally multiplied $5 - x^2$, I multiply [it] by AB, $10x - 2x^3 = 8$ is made, therefore $x^3 + 4 = 5x$. And the thing is the same 2 and $\sqrt{5} - 1$, then the passage is made under the same values, but not without the knowledge of the previous value through which I arrive to know DE, which is $\sqrt{20}$. Moreover it is said above that, if I take the double of the root of the number of the things, and it is $\sqrt{32}$, and I divide [it] in $\sqrt{8} + x$ and $\sqrt{8} - x$, $8 - x^2$ is made, and, multiplied by $2x$, $16x - 2x^3 = 16$ is made, and it falls back on $x^3 + 8 = 8x$. Therefore in this something new is discovered through the [things that are] not known, in that something is discovered through the [things that are] known, but it is the same, in fact CD is supposed in the previous [case] $\sqrt{32}$, here $\sqrt{20}$.

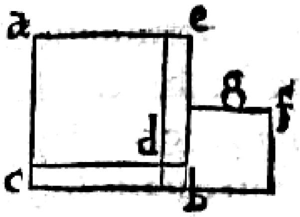


Again two surfaces are proposed equal to the rectangles $ABCD$ and $CEFG$, and they are equal to the number of the things, and in those there are the squares $CHKD$ and $CELM$ so that the same number, which is n, is made by the sides of those times the remaining of its surface. Therefore it is agreed that CE as well as CA are the value of the thing and, since n is made by CE times LG and by CH times HB, in fact the same are also made by GM times MC and by AH times HD, because [the ratio] of GM to AH [is] duplicated of [the ratio] which is HC to CE. Therefore, put GM the first [proportional], AH the fourth [proportional], CH the second [proportional], CE the third [proportional], then what is made by the first and the third [proportionals] times the third [proportional], namely the surface EG, will be equal to what is made by the second and the fourth [proportionals] times the second [proportional], namely the surface AD. And again what is made by the first [proportional] times the square of the third [proportional will be] equal to what is made by the fourth [proportional] times the square of the second [proportional]. Therefore the problem is established as follows.



There are four quantities, a, b, c, d in order, such that the ratio of a to d is duplicated of the one which is b to c, and [such that] what is made by a c [*sic*] times c is equal to what [is made] by d b [*sic*] times b, and [such that] what is made by a times the square of c is equal to what is made by d b times b. From these follows the fourth, that the ratio of what is made by a times the square of c to what is made by a c times c is as [the ratio] of what is made by d times the square of b to what is made by d b times b. And moreover exchanging [them], but this is transparent, since it is the ratio of an equal [thing] to an equal [thing].

Besides I say that the rule of the *Ars magna* that teaches to assume the root of the aggregate from the number of the things and from the number of the equality divided by that is the only general of that chapter and it is demonstrated there. And its origin is from the orthogonal triangle,



in fact, if the cube of BC is equal to the things according to the number AD and to the number g, then by the common opinion of the mind g will be from BC times the gnomon CDE, then the square BF is made equal to the gnomon CDE and the cube of BC will be equal to BC times AD and BF, but the square of BC, which is AB, is equal to AD and BF, therefore the sides of AD and of BF contain the right content of BC. Therefore this value satisfies in each equality, [when] the number of the things is either small or big.

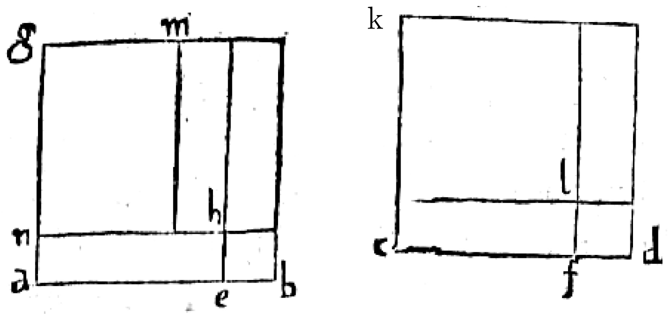
Chapter LIII

On the accurate consideration of [the things] above said in Chapter VII

And we now say that $x^3 = 12x + 20$, and the value of the thing is $\sqrt[3]{16} + \sqrt[3]{4}$, and this can be assigned giving similarly the number 20 to the cubes, and the number can also be given to both cubes and to two mutual [parallelepipeds], and [the number can] also [be given] to both cubes and to four mutual parallelepipeds, and thus in three ways. Then we consider [it], after that the discovery of the chapter and the rule with the demonstration are assumed according to the first way. Then we will take the cube of the half of the value, that is $\sqrt[3]{2} + \sqrt[3]{\frac{1}{2}}$, and it is $\frac{5}{2} + \sqrt[3]{54} + \sqrt[3]{\frac{27}{2}}$, and its double, which it is the smallest that can be produced by the division of the value, is $5 + \sqrt[3]{432} + \sqrt[3]{108}$. Therefore it is clear that that root cannot be divided because of the smallness of the number, in fact the cube of all is $20 + \sqrt[3]{27648} + \sqrt[3]{6912}$. But, if we take $x^3 = 12x + 34$, the value will be $\sqrt[3]{32} + \sqrt[3]{2}$, and the double of the cube of the half $\frac{17}{2} + \sqrt[3]{864} + \sqrt[3]{54}$, and all this is the nearest to $\frac{43}{2}$. Therefore two mutual [parallelepipeds] could contain $\frac{25}{2}$. Then divide 34 by $\sqrt[3]{32} + \sqrt[3]{2}$, $\sqrt[3]{1024} - 4 + \sqrt[3]{4}$ comes out, and it is necessary that it is equal to the two squares. Then make two parts from $\sqrt[3]{32} + \sqrt[3]{2}$ the squares of which are equal to that *trinomium*. Then take the half of the *trinomium*, and it is $\sqrt[3]{128} - 2 + \sqrt[3]{\frac{1}{2}}$, from which you take away the square of the half of [the number] to be divided, that is the square of $\sqrt[3]{4} + \sqrt[3]{\frac{1}{4}}$, and it is $\sqrt[3]{16} + 2 + \sqrt[3]{\frac{1}{16}}$, subtract [it], $\sqrt[3]{16} - 4 + \sqrt[3]{\frac{1}{16}}$ is left, the universal root of which, added and subtracted, shows the parts in this way:

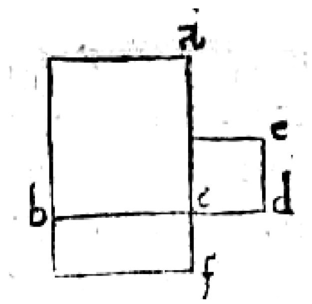
$$\sqrt[3]{4} + \sqrt[3]{\frac{1}{4}} + \sqrt{\sqrt[3]{16} + \sqrt[3]{\frac{1}{16}} - 4},$$
$$\sqrt[3]{4} + \sqrt[3]{\frac{1}{4}} - \sqrt{\sqrt[3]{16} + \sqrt[3]{\frac{1}{16}} - 4}.$$

Then you see now that the cube is made equal to 34, so that the number 34 is equal to two cubes with two mutual [parallelepipeds] of the parts. Since what is left is the number of the things, and it is the double of the mutual [parallelepipeds] divided by the thing, the number of the things, which it is agreed to be the same, comes out.



Then it is proposed AB and CD [both equal to] 4 and they [both] are the thing, and let their squares to be BG, DK. But let AB be divided in E so that the cubes GH, HB are forty and $bx + 40$ will be equal to the whole cube. Therefore MH equal to AH is taken away, therefore those three surfaces will be b, and according to the height AB they [will be] bx, and 40 by AB times MN and [by AB times] HB. And AE will be $\sqrt[3]{20+\sqrt{392}}$ and EB [will be] $\sqrt[3]{20-\sqrt{392}}$. And let CF to be 3, and FD will be 1. And the cubes KL, LD with the two mutual bodies, and this is how much is made by CD times KL, again by [CD times] LD [and it is] 40, and the surfaces KL and LD will be 10 and necessarily equal to the surfaces MN and HB, because the same multiplied by AB, which is equal to CD, produce 40. Therefore, since I want in the first surface that only the cubes are equal to 40, and in the second that the cubes with the two mutual bodies make the same 40, and that the value is the same, therefore it is necessary that in the second figure $x^3 = 6x + 40$. But the division in F is nearer to the half than [the division in] E in the first figure, and that rule does not help to that equation so understood. Therefore it is necessary to discover another [equation] typical of that [rule]. Therefore I say the same of the previous example. AB is put $\sqrt[3]{32} + \sqrt[3]{2}$, and let the division of the *binomium* to be in E, and $x^3 = 12x + 34$, and NM and HB will be 12. Moreover in the second figure KL, LD will be in the same way 12, but the division will be as proposed in F. And with the equality $x^3 = 12x + 34$ one may not discover CD

so that it is composed by CF and FD, but by the other rule. But we will find AB when it is divided in the parts AE, EB, EC, afterwards if we wish not [in the part] CF, FD. This is nevertheless enough to understand that the mutual quantity, which, [being] multiplied in that way, can produce the number, is given. If there were two quantities, what is made from the first by the square of the second is equal to what is made by the second multiplied by the root of the first by itself. Moreover this [is] the cause of the changing.



Let the first be the square AB, the second CD. Then BF is made by BC times CD. I say that BF is the side of BA times CD. Since how much [is made] by BC times BF is indeed made by AB times CD, for the reason that on one side and on the other CD is multiplied by the square of BC, the ratio of the body CD times AB to the surface BF will be the line BC. Similarly the ratio of the body [CD times AB] multiplied by DC to AB is the square of CD. Therefore the ratio of the product AB times CE to BF is the same BF. Therefore the same BF multiplied by itself produces AB times CE.

Note

From [the things] seen here and above it appears clearly that all the rules of Chapter XXV of the *Ars magna* (Chapter 2), which they call 'special', are general, and they are said [to be] special for the only reason of the kind of the value. Therefore, if someone says $x^3 = 20x + 32$, we will say that the value is

20 d. p. R p. 32,

that is [20] divided in the part and the root that produce 32. And similarly it will be

32 p. 20 with p. 32,

that is what produces 20 with what produces 32. And it is similarly said

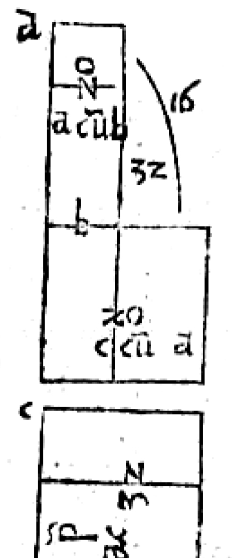
Ag. ℞ p: 20 p: n: 16,

that is the aggregate of the roots of the parts of 20 that reciprocally multiplied produce 16, the half of 32. We also say by [the things] above said (Chapter 28) that the thing is reduced to three values, in fact the others are confused.

20. d. p.℞.p 32
32.p.20 cu.1.32
Ag.℞ p:20.p.m.16.

From these it follows (see above Chapters 31 and 40 at the end and below Chapter 57) that those values are equal one to the other. And similarly, when the operations will be performed in those, you pass from one chapter to the other as with the value.

32 p:lat.a:



And remark that in the figure a varies in size according to each rule.

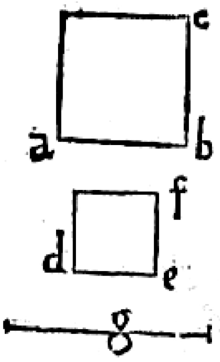
Chapter LVII

On the treatment of the general value of the cube equal to some things and a number

I have already taught (Chapter XL at the end and LIII at the end) that the general value of the chapter of the cube equal to some things and a number is had neither by a general nor by a special rule, except by that [which is to] find a quantity that multiplied by a second [one] produces the number of the equality. That second quantity behaves like the gnomon, and is the first root or the side of the aggregate by the number of the things, and [is] discovered according to that quantity. And this is according to the nature (as I have said), because the line is put the side of the aggregate of two squared surfaces, and therefore it will be opposite to the right angle contained in the sides of those two squares (by the [Proposition][1] 47 of the first [Book] of the *Elements*). And I have said that this quantity is described as in the example $x^3 = 20x + 32$, in such a way 32 p: 20 c. p. 32, [that is] what produces 20 with what produces 32, or better $\sqrt{20}$ p: d. 32, that is the root of 20 plus 32 divided by the same root. Otherwise $\sqrt{20}$ f. 32, that is the root of 20 with a fragment of 32. Complete by both roots of the divided [thing]. A fragment is indeed what comes forth from a division. Therefore we will use these names, thereafter if some of others is satisfactory to someone or [if] he also establishes a new [one], at least it is agreed that I will not have burdened the things. Therefore $\sqrt{20}$ f. 32 is the value of $x^3 = a_1x + a_0$ as it is said.

Then I say firstly that that value can be neither by the nature of the *binomium*except when it is changed, nor by the nature of the *recisum*.

[1] *Elements* I.47: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle", see [Heath 1956a, pages 349-368].



Let ABC and DEF to be those squares. And AC [is] the number of the things, DF what comes from the division of the number by G, the thing itself. Since then G is a *binomium*, DEF is a *recisum.* Therefore, since ABC is a number, the aggregate by ABC and DEF will be a *recisum.* Therefore its side is a *recisum*(by the [Proposition][2] 54 of the tenth [Book] of the *Elements*). Then G was not a *binomium.* And if you put that G is a *recisum*, DEF will be a *binomium*, and the aggregate ABC, DEF a *binomium*, therefore its side a first *binomium* and not a *recisum.*

Therefore, since as in the preceding example – as it is seen above (Chapter 25 of the *Ars magna*, Corollary) – $x^3 = a_1x + a_0$ has the value $\sqrt{17} + 1$, and this is a *binomium*, and it is necessary that it is $\sqrt{20}$ f. 32. I divide 32 by $\sqrt{17} + 1$ and it is enough to multiply $\sqrt{17} - 1$ by 2, $\sqrt{68} - 2$ is made, which [being] added to 20 makes $18 + \sqrt{68}$. And thus you see that it falls back on the *binomium*, the root of which is the value of the thing $\sqrt{17} + 1$. Then it is agreed that no *recisum* can be in this way, and also no *binomium* the first part of which is a number, in fact the fragment will be necessarily with the second part a minus and a root, therefore it is all a *recisum.* Therefore the quantity of such a kind that, the number [being] divided by it, it can be the root, is to be searched. And it corresponds to the second and fifth *binomium* (as I have said, Chapter 4). Let $\sqrt{12} + 3$ as before. I want to discover the cube equal to some things and a number. Make as in the rule *de modo* and you will see that it only fits a second and fifth *binomium.* Therefore by the rule *de modo* double the number of the

[2]*Elements* X.54: "If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called *binomial*", see [Heath 1956c, pages 116-120].

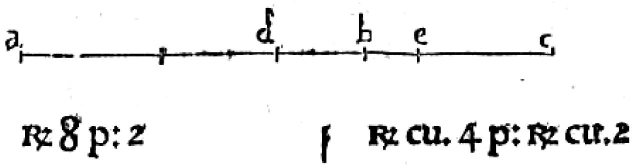
equality or [the number] of the value [that you have] had, and multiply both by themselves. Add the difference to the square of the root of the value and you will have the number of the things. Thence take the root of the square of the thing, and write off from that the difference of the number of the things and of the number of the square of the thing, and multiply this by the thing itself, and the number of the equality will be produced. It is proposed the example $\sqrt{7}+2$ for the value. Double 2, 4 is made, multiply 2 and 4 by themselves, 16 and 4 are made, the difference of which is 12, add 7, the square of $\sqrt{7}$, the number of the things 19 is made. Thence I consider $\sqrt{112}$ from the square of $\sqrt{7}+2$ and from that write off 8, [which is] the difference of the number of the things 19 and the number of the square of $\sqrt{7}+2$ (in fact, multiplied $\sqrt{7}+2$ by itself, it produces $11+\sqrt{112}$, therefore the number of that square is 11). Then write off that difference from the already stored $\sqrt{112}$, and the root of the square of the thing is made $\sqrt{112}+8$. Multiply [it] by the thing, which is $\sqrt{7}+2$, you will have the number 12, therefore $x^3 = 19x + 12$. In truth it is agreed that, the number of the things and [the number] of the equality remaining the same, the value can be neither increased nor diminished. In fact, if it had been increased, therefore what comes would have been diminished, and [it is] the root of the aggregate that is the thing, and if it had been diminished, therefore what comes would have been increased, and [it is] the root of the aggregate that is the thing. And thus, as long as it is increased, it is diminished and, as long as it is diminished, it is increased, which cannot be. It is also agreed that this value is common to the cubic *binomium* discovered in a part of the chapter and to the surface[3] *binomium* here indicated, and the common quantity is the general value.

[3]"*[B]inomio superficiali*".

Chapter LVIII

On the common quantity from two incommensurable [quantities], in how many ways it is said

Then two values of the cube equal to some things and to a number are already known, one in the biggest part of the number and is [the value] of the cubic *binomium*, and the other in the smallest part of the number [and is the value] of the second or fifth *binomium* by square roots, and the common value that cannot be incommensurable, in fact they would have been commensurable between them, and the fourth [value is known], namely [the one] that is realised in the smallest part of the number. Therefore it is necessary that the common fails, as it is said, by conjunction.



Therefore let AB and BC be incommensurable and connected so that the middle of them, that is [the middle] of the aggregate, is D. For example, let AB be $\sqrt{8}+2$ and BC [be] $\sqrt[3]{4}+\sqrt[3]{2}$. Therefore the common value cannot be by a common commensurable, thus they are indeed of the same nature between them, or rather they will then be by the way of the addition and subtraction, as AD is. Therefore AD will be $\sqrt{2}+1+\sqrt[3]{\frac{1}{2}}+\sqrt[3]{\frac{1}{4}}$, because BD will be $\sqrt{2}+1-\sqrt[3]{\frac{1}{2}}-\sqrt[3]{\frac{1}{4}}$, which is appropriate to add to the *quadrinomium.* And thus we could have found from the beginning AB and BC, as if these two *quadrinomia* were of the same kind. We put again what [was] discovered as first for example. Let AE to be what is added to AB $\sqrt[3]{2}$, so that it is necessary to subtract it, or let $-CE$ times $\sqrt[3]{2}$. Therefore it is necessary to discover first of all AE and EC, which are different, and one is a quantity by three terms, the other is a simple cubic root, but this [is] absurd. Therefore the way of the operation is none. Therefore it is necessary that the common quantity is neither of the kind of AB nor of BC. And this can be, in fact the animal is common to the human being, to the the donkey, to the ox, and to the horse, thus AB and BC are contained in a certain common quantity, which as long as it is common to all [AB and BC] has only this property, that, the simple number of the equality [being] divided by that same, it is the root of the things with what comes out. [One] can fall upon this when there is a number, a *binomium* of the second or fifth kind, a simple *binomium* from cubic roots, or this *binomium* with its *recisum*, or another quantity always with that property. Then we divide 16 by $\sqrt{8}+2$, $\sqrt{128}-8$ comes out. I add [it] to 20, $12+\sqrt{128}$, [which is] the square of $\sqrt{8}+2$, is made. In fact, the cube was equal to 20 things[1] 2, $\sqrt[3]{16}-2+\sqrt[3]{4}$ comes out. Add this to the number of the things 6, $\sqrt[3]{16}+4+\sqrt[3]{4}$ is made and this is the square of $\sqrt[3]{4}+\sqrt[3]{2}$. Therefore, in both divisions, as you see, it is common that the *recisum* comes forth and, the number of the things [being] added, it goes through a nature similar to the square of the thing. Therefore the number of the things changes its nature, which comes from the division of the number of the equality by the thing.

[1]A part of the text seems to be missing here. In fact, in the 1570 edition there is a page break at this point and the remainder at the bottom of page 106 ("*bus*") does not match the beginning of page 107 ("*2, exit*"). The same is in the 1663 edition, even without page break.
Cardano is indeed performing here the following calculation. Consider $x^3 = 20x + 16$. One of its solution is $x_1 = \sqrt{8} + 2$. He checks that $x_1 = \sqrt{20 + \frac{16}{x_1}}$ holds in this case. The same for $\sqrt[3]{16} - 2 + \sqrt[3]{4}$, which is a solution of $x^3 = 6x + 6$.

Chapter LIX

On the order and on the examples in the second and fifth *binomia*

The increase of the number and the first number always starts from the root of the first number of the things, and the half of that root is the second fixed part of the *binomium* that is the number of the value, and the square of the first part starts from the fourth part of the first number of the things, and thence both the number of the things and the increases of the squares of the first part of the *binomium*, which is the root, are increased by the monad. They are more easily known in the first four supposed examples, since the fifth is outside of the order in the remaining value, as in the third example the first number of the things is 9, the root of which is 3, from which the first number of the equality starts, and its half $\frac{3}{2}$ is the second part of the value, which remains fixed, and the first [part of the value], which is $\sqrt{\frac{9}{4}}$, the square of which is the fourth part of the first number of the things, that is 9. And such squares are increased by the monad, or one, so that also the number of the things [is] as you see in the figure.

First example of the increase by 1.

$x^3 = x;$	$x = \sqrt{\frac{1}{4}} + \frac{1}{2}$	$x^3 = 10x + 9;$	$x = \sqrt{9\frac{1}{4}} + \frac{1}{2}$
$x^3 = 2x + 1;$	$x = \sqrt{1\frac{1}{4}} + \frac{1}{2}$	$x^3 = 11x + 10;$	$x = \sqrt{10\frac{1}{4}} + \frac{1}{2}$
$x^3 = 3x + 2;$	$x = \sqrt{2\frac{1}{4}} + \frac{1}{2}$	$x^3 = 12x + 11;$	$x = \sqrt{11\frac{1}{4}} + \frac{1}{2}$
$x^3 = 4x + 3;$	$x = \sqrt{3\frac{1}{4}} + \frac{1}{2}$	$x^3 = 13x + 12;$	$x = \sqrt{12\frac{1}{4}} + \frac{1}{2}$
$x^3 = 5x + 4;$	$x = \sqrt{4\frac{1}{4}} + \frac{1}{2}$	$x^3 = 14x + 13;$	$x = \sqrt{13\frac{1}{4}} + \frac{1}{2}$
$x^3 = 6x + 7;$	$x = \sqrt{5\frac{1}{4}} + \frac{1}{2}$	$x^3 = 15x + 14;$	$x = \sqrt{14\frac{1}{4}} + \frac{1}{2}$
$x^3 = 7x + 6;$	$x = \sqrt{6\frac{1}{4}} + \frac{1}{2}$	$x^3 = 16x + 15;$	$x = \sqrt{15\frac{1}{4}} + \frac{1}{2}$
$x^3 = 8x + 7;$	$x = \sqrt{7\frac{1}{4}} + \frac{1}{2}$	$x^3 = 17x + 16;$	$x = \sqrt{16\frac{1}{4}} + \frac{1}{2}$
$x^3 = 9x + 8;$	$x = \sqrt{8\frac{1}{4}} + \frac{1}{2}$	$x^3 = 18x + 17;$	$x = \sqrt{17\frac{1}{4}} + \frac{1}{2}$

Second example of the increase by 2.

$x^3 = 4x;$	$x = \sqrt{1} + 1$	$x^3 = 13x + 18;$	$x = \sqrt{10} + 1$
$x^3 = 5x + 2;$	$x = \sqrt{2} + 1$	$x^3 = 14x + 20;$	$x = \sqrt{11} + 1$
$x^3 = 6x + 4;$	$x = \sqrt{3} + 1$	$x^3 = 15x + 22;$	$x = \sqrt{12} + 1$
$x^3 = 7x + 6;$	$x = \sqrt{4} + 1$	$x^3 = 16x + 24;$	$x = \sqrt{13} + 1$
$x^3 = 8x + 8;$	$x = \sqrt{5} + 1$	$x^3 = 17x + 26;$	$x = \sqrt{14} + 1$
$x^3 = 9x + 10;$	$x = \sqrt{6} + 1$	$x^3 = 18x + 28;$	$x = \sqrt{15} + 1$
$x^3 = 10x + 12;$	$x = \sqrt{7} + 1$	$x^3 = 19x + 30;$	$x = \sqrt{16} + 1$
$x^3 = 11x + 14;$	$x = \sqrt{8} + 1$	$x^3 = 20x + 32;$	$x = \sqrt{17} + 1$
$x^3 = 12x + 16;$	$x = \sqrt{9} + 1$	$x^3 = 21x + 34;$	$x = \sqrt{18} + 1$

Third example of the increase by 3.

$x^3 = 9x;$	$x = \sqrt{2\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 18x + 27;$	$x = \sqrt{11\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 10x + 3;$	$x = \sqrt{3\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 19x + 30;$	$x = \sqrt{12\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 11x + 6;$	$x = \sqrt{4\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 20x + 33;$	$x = \sqrt{13\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 12x + 9;$	$x = \sqrt{5\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 21x + 36;$	$x = \sqrt{14\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 13x + 12;$	$x = \sqrt{6\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 22x + 39;$	$x = \sqrt{15\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 14x + 15;$	$x = \sqrt{7\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 23x + 42;$	$x = \sqrt{16\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 15x + 18;$	$x = \sqrt{8\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 24x + 45;$	$x = \sqrt{17\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 16x + 21;$	$x = \sqrt{9\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 25x + 48;$	$x = \sqrt{18\frac{1}{4}} + 1\frac{1}{2}$
$x^3 = 17x + 24;$	$x = \sqrt{10\frac{1}{4}} + 1\frac{1}{2}$	$x^3 = 26x + 51;$	$x = \sqrt{19\frac{1}{4}} + 1\frac{1}{2}$

Fourth example of the increase by 4.

$x^3 = 16x$;	$x = \sqrt{4} + 2$	$x^3 = 25x + 36$;	$x = \sqrt{13} + 2$
$x^3 = 17x + 4$;	$x = \sqrt{5} + 2$	$x^3 = 26x + 40$;	$x = \sqrt{14} + 2$
$x^3 = 18x + 8$;	$x = \sqrt{6} + 2$	$x^3 = 27x + 44$;	$x = \sqrt{15} + 2$
$x^3 = 19x + 12$;	$x = \sqrt{7} + 2$	$x^3 = 28x + 48$;	$x = \sqrt{16} + 2$
$x^3 = 20x + 16$;	$x = \sqrt{8} + 2$	$x^3 = 29x + 52$;	$x = \sqrt{17} + 2$
$x^3 = 21x + 20$;	$x = \sqrt{9} + 2$	$x^3 = 30x + 56$;	$x = \sqrt{18} + 2$
$x^3 = 22x + 24$;	$x = \sqrt{10} + 2$	$x^3 = 31x + 60$;	$x = \sqrt{19} + 2$
$x^3 = 23x + 28$;	$x = \sqrt{11} + 2$	$x^3 = 32x + 64$;	$x = \sqrt{20} + 2$
$x^3 = 24x + 32$;	$x = \sqrt{12} + 2$		

Fifth example where the thing is the same.

$x^3 = 216$;	$x = 6$	$x^3 = 9x + 162$;	$x = 6$
$x^3 = x + 210$;	$x = 6$	$x^3 = 10x + 156$;	$x = 6$
$x^3 = 2x + 204$;	$x = 6$	$x^3 = 11x + 150$;	$x = 6$
$x^3 = 3x + 198$;	$x = 6$	$x^3 = 12x + 144$;	$x = 6$
$x^3 = 4x + 192$;	$x = 6$	$x^3 = 13x + 138$;	$x = 6$
$x^3 = 5x + 186$;	$x = 6$	$x^3 = 14x + 132$;	$x = 6$
$x^3 = 6x + 180$;	$x = 6$	$x^3 = 15x + 126$;	$x = 6$
$x^3 = 7x + 174$;	$x = 6$	$x^3 = 16x + 120$;	$x = 6$
$x^3 = 8x + 168$;	$x = 6$		

Four corollaries follow from these.

Corollary 1. Therefore from this order we have first that it is necessary that, since the half of the root is the second part of the value, the second part is an integer number or the half of a number.

Corollary 2. Second it follows that the chapter cannot be general, because the first number is necessarily a square. In fact, if it was not, since the increases are made by the root of the number, therefore either the first number will be an integer and not a square, namely in the third order, or [the first number will be] a square but not an integer. If it was a square and not an integer, therefore, being the other numbers of the things made by a continued addition, they will all be fractions of one number of the things. Therefore it does not help to the chapter of the cube equal to some integer things and a number according to any part, which is absurd. But if [the first number] were a number and not a square, therefore, the increases [being] made by the root of those, the true number of the equality will never come out (by the last [Proposition][1] of the tenth [Book] of the *Elements*), and thus the chapter will be useless.

Corollary 3. From this it also follows that the number of the equality can never be increased to such a point that the square of the half of that is bigger than the cube of the third part of the number of the things. Then in fact by the first rule the value would be the cubic *binomium*, and by that rule the squared *binomium*, and thus one would be equal to the other. Which can be allowed, as in the example $\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}$, and it is $2+\sqrt{2}$ and $2-\sqrt{2}$, which is 4. Nevertheless it cannot be extended and the value is loosened in an integer number.

Corollary 4. Given a number of the things, the value of the equality is had by this. And you discover all the squares contained in the number of the things and their roots, with which you multiply these by the difference of the number of the things and of the squared number. And, if the number of the equality is produced, then the difference of that and of the fourth part of the squared number, the root [being] discovered, is the first part of the *binomium* and the half of the discovered root of that [is] the second part of the *binomium*. For example, $x^3 = 19x + 30$. The squared number, as you see in the side, are contained in the number of the things 19.

16—4	3	12
9—3	10	30
4—2	15	30
1—1	18	18

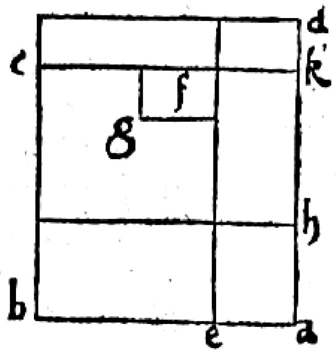
[1] *Elements* X.115: "From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as nay of the preceding", see [Heath 1956c, page 254].

Since in truth the difference 9 from each is multiplied by twice the root of the number, the number of the equality 30 is produced. In the following, we will take 1, [which is] the fourth part of 4, and we will add [it] to the difference 15, 16 is made, the root of which is $+4$. It constitutes the value 5. In the previous, we will add $\frac{9}{4}$, [which is] the fourth part of 9, to the difference 10, $\frac{49}{4}$ is made, the root of which is $\frac{7}{2}$, $\frac{3}{2}$, [which is] the half of 3, [which is] $\sqrt{9}$, [being] added, the value of the thing 5 is made as before.

Chapter LX

General demonstration of the chapter of the cube equal to some things and a number

And, since there is this special rule that concerns the value, therefore it is also not remarkable if it is also special in the way of the discovery, supposing a squared number.

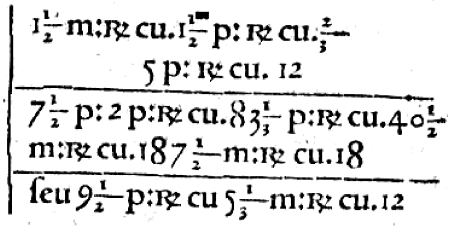


Therefore, in order to be generally considered, we propose AB the same thing and its square AC, which consists of a certain number divided by AB, added the number of the things to the outcome. Therefore the divided number is now put the surface. Therefore it can be bigger, or smaller, or equal to the same AC. First it is proposed that it is equal. Therefore what comes forth is the side BA and this is known. Of course the number [is] known, therefore [it is] known, as [in] $x^3 = 25 + 20x$ the thing is 5, and [in] $[x^3] = 36 + 30x$ the thing is 6. Let only BD be bigger than the square AC[1] and AE to be one, and, since ED will be as much as AD, and CD [being] added, it constitutes the square AC by [the things] demonstrated. Then, if AF alone was added, in order for AC to be made, the number of the things would perfectly be EC, but, since DF is

[1]1570 has "*quadrato ae in de*" and 1663 has "*quadrato AC in DE*". In any case, it would be a multiplication between a square and a parallelepiped, which sounds awkward. Since from the context it is clear that here Cardano is dealing with the case $a_0 > x^2$, I have assumed that he takes $a_0 = (\overline{BD})$ and $(\overline{BD}) > \overline{AB}^2$, as it appears from the diagram and coherently with what he will do in the case $a_0 < x^2$.

furthermore added, FG is constituted equal to FD, therefore the surface EGC will be the number of the things, suppose 8. And the surface BD is the number by hypothesis, and their difference will be 24, which is three-fourth of 32 and the triple of the number of the things AB. And therefore ED is made by EA, that is one, times AD or AK with KD added. Therefore, the square of KD [being] added, the product by AB times KD, [which is] the added unity equal to the difference of the number of the equality and of the number of the things with the square KD, will be common. If in truth BH is put a small number, and what comes out [is] AH, and, multiplied the unity by AH, the surface EH, which [being] added to the number of the things constitutes the square AC, is made. Therefore the number of the things is the surface HCE, and it is for example 18, and HB [is] 8. Therefore the difference is 10. Moreover such a difference is $HC - HE$, HC is made by HK times AB, HE [is made] by HA times AE. Therefore AK, equal to AB, is divided so that 10 is left by the whole times one part, the other [being] subtracted.

Therefore, when the surface to be divided is the number of the equality and [it] was big, then the part of that chapter already discovered by the *binomia* from cubic roots satisfies. And at some times [it] also [does] not [satisfy]. But when the surface was smaller than the square, it could not. And afterwards we suppose the unity, this is known, and, since we suppose AK to the power, we also take [it] irrational. For example, let $\sqrt[3]{12}+2$, the square of which AC is $\sqrt[3]{144}+\sqrt[3]{768}+4$. Then we want to divide $\sqrt[3]{12}+2$ so that, [being] multiplied by one part and the remaining [being] added, it is equal to 3, for example, and to the other part. Therefore let one part be z, and the parts will be z and $\sqrt[3]{12}+2-z$. Therefore multiply z by $\sqrt[3]{12}+2$, $z(\sqrt[3]{12}+2)$ is made, and this is equal to $\sqrt[3]{12}+5-z$, whereby $z(\sqrt[3]{12}+3)=\sqrt[3]{12}+5$. Divide the number of the equality by the number to discover z. The *recisum* of $\sqrt[3]{12}+3$, or $\sqrt[3]{27}+\sqrt[3]{12}$, is $\sqrt[3]{\frac{27}{8}}-\sqrt[3]{\frac{3}{2}}+\sqrt[3]{\frac{2}{3}}$. Multiply [it] by itself, $\frac{13}{2}$ is made, multiplied $\sqrt[3]{12}+5$ by $\frac{3}{2}-\sqrt[3]{\frac{3}{2}}+\sqrt[3]{\frac{2}{3}}$. Therefore divide this product by $\frac{13}{2}$, the thing itself $\frac{19}{13}+\sqrt[3]{\frac{128}{6591}}-\sqrt[3]{\frac{96}{2197}}$ comes out. This is one part, therefore the other will be $\frac{7}{13}+\sqrt[3]{12}+\sqrt[3]{\frac{96}{2197}}-\sqrt[3]{\frac{128}{6591}}$. Therefore $\sqrt[3]{12}+2$ [being] multiplied by $\frac{19}{13}+\sqrt[3]{\frac{128}{6591}}-\sqrt[3]{\frac{96}{2197}}$, and subtracting $\frac{7}{13}+\sqrt[3]{\frac{96}{2197}}+\sqrt[3]{12}-\sqrt[3]{\frac{128}{6591}}$ from the product, 3 is perfectly left. But we ask at the same time that a number is made by the product AB, that is $\sqrt[3]{12}+2$, times HA, that is the *residuum* that was $\frac{7}{13}+\sqrt[3]{\frac{96}{2197}}+\sqrt[3]{12}-\sqrt[3]{\frac{128}{6591}}$. And this will be the quantity.


$1\frac{1}{2}$ m:℞ cu.$1\frac{1}{2}$ p: ℞ cu.$\frac{2}{3}$
5 p: ℞ cu. 12
$7\frac{1}{2}$ p: 2 p:℞ cu.$83\frac{1}{3}$ p:℞ cu.$40\frac{1}{2}$
m:℞ cu.$187\frac{1}{2}$ m:℞ cu.18
ſeu $9\frac{1}{2}$ p:℞ cu $5\frac{1}{3}$ m:℞ cu.12

It is therefore clear that the problem is constructed in this way and it is settled by the rule *de modo* and *positione*: find the quantity that can be divided in two parts so that the whole product by one produces, for example, 3 and [the product of the whole] by the remaining [part] added the preceding [one] produces, for example, 8. Then, since it is allowed that the kind of that value is the quantity from the divided kind or shape as $\frac{a}{b}$ above, it is shown that it is not permitted to divide either by a *quadrinomium* from square roots, by a *binomium* from cubic [roots], or by a similar *trinomium*, or by a special rule. Since then it is not permitted in others, I say that these quantities are known to such a degree as those. In fact, since it concerns the essence, thus $\sqrt{2}$ is irrational, as $\sqrt[3]{7}+\sqrt[4]{3}-\sqrt[4]{5}$ or also all this

$$\frac{\sqrt[3]{7}+\sqrt[4]{3}-\sqrt[4]{5}}{\sqrt[6]{10}+\sqrt[4]{3}-\sqrt{2}}.$$

What concerns the nearness returns nothing, since one may perpetually come near. Where in truth those are very renown to the operations, therefore I propose those. Therefore let $\sqrt{\frac{a}{b}}$ as I wish, I take the root of the numerator and of the denominator, and it is $\sqrt{b}$ and $\sqrt{a}$, and I place one over the other in the same order, and I have $\sqrt{\frac{a}{b}}\ \frac{\sqrt{a}}{\sqrt{b}}$, and similarly $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$. And thus $\sqrt[3]{\frac{10}{\sqrt[4]{5}+\sqrt[3]{2}}}$ is $\frac{\sqrt[3]{10}}{\sqrt[3]{\sqrt[4]{5}+\sqrt[3]{2}}}$. And thus I want to multiply

$$\frac{10}{\sqrt[4]{5}+\sqrt[3]{2}} \quad \text{by} \quad \frac{\sqrt{2}}{\sqrt[4]{5}-\sqrt[6]{2}}$$

it makes

$$\frac{\sqrt{200}}{\sqrt[8]{1953125}+\sqrt[6]{4000}-\sqrt[4]{10}-\sqrt[24]{128}}.$$

And thus dividing we will multiply in a crossing way and we will have

$$\frac{\sqrt[4]{500000}-\sqrt[4]{20000}}{\sqrt[4]{20}+\sqrt[6]{32}}.$$

And in the opposite way, dividing to the contrary. And thus in the addition

$$\frac{\sqrt[4]{500000}+\sqrt[4]{20}+\sqrt[6]{32}-\sqrt[4]{20000}}{\sqrt[8]{1953125}+\sqrt[6]{40000}-\sqrt[4]{10}-\sqrt[12]{128}}$$

and equally in the subtraction

$$\frac{\sqrt[4]{20}+\sqrt[6]{32}+\sqrt[4]{20000}-\sqrt[4]{50000}}{\sqrt[8]{1953125}+\sqrt[6]{40000}-\sqrt[4]{10}-\sqrt[12]{128}}.$$

Therefore these acts are up to that.

End

Basel, by the workshop of Henricus Petrus, in the year of salvation 1570 in the month of March

The telling of the unattainable attempt to avoid the *casus irreducibilis* for cubic equations: Cardano's *De Regula Aliza*.

With a compared transcription of 1570 and 1663 editions and a partial English translation

Abstract

Solving cubic equations by a formula that involves only the elementary operations of sum, product, and exponentiation of the coefficients is one of the greatest results in 16^{th} century mathematics. This was achieved by Girolamo Cardano's *Ars magna* in 1545.

Still, a deep, substantial difference between the quadratic and the cubic formula exists: while the quadratic formula only involves imaginary numbers when all the solutions are imaginary too, it may happen that the cubic formula contains imaginary numbers, even when the three solutions are anyway all real (and different). This means that a scholar of the time could stumble upon numerical cubic equations of which he already knew three (real) solutions and the cubic formula of which actually contains square roots of negative numbers. This will be lately called the '*casus irreducibilis*'. Cardano's *De regula aliza* (Basel, 1570) is (at least, partially) meant to try to overcome the problem entailed by it. Its (partial) analysis is the heart of this dissertation.