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On elliptic Calabi–Yau threefolds in $\mathbb{P}^2$-bundles

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Introduction

The study of elliptic curves dates back to ancient Greece. As mathematics evolved through the centuries, so did the study of such curves, which now provides one of the best examples where different branches of mathematics coexist, in particular geometry and number theory.
From the number theoretical point of view, an elliptic curve (E, e) is a genus one curve E defined over an arbitrary field k, where a rational point e has been chosen. Thanks to the presence of this point, one can find an isomorphism with a cubic curve in $\mathbb{P}^2_k$, which will be called the Weierstrass model of E. The set $E(k)$ of k-rational points, which is actually a group, is called the Mordell–Weil group of E. In the case $k = \mathbb{C}(t)$ is the function field of $\mathbb{P}^1_{\mathbb{C}}$, we can give to an elliptic curve a more geometrical description, and in general when $k = \mathbb{C}(B)$ is the function field of a smooth curve.
An elliptic surface is a surface X with a morphism $\pi : X \longrightarrow B$ onto a smooth curve and with a section: its generic fibre is then an elliptic curve over $\mathbb{C}(B)$, providing then the link I told about in the previous paragraph. A study of this class of surfaces can be found in [Kod63], where one of the most important results in the theory of elliptic surfaces, and elliptic fibrations in general, is proved: Kodaira's theorem classifying all the possible singular fibres. This theorem is important for two reasons: first of all because it gives a full answer on the problem of classification, and second because joint with Tate's algorithm [Tat75] it gives a practical tool to distinguish between the different singular fibres in the list.
Another important result in this area is the Shioda–Tate formula [Shi90, Cor. 5.3] for computing the rank of the Mordell–Weil group of an elliptic surface. In fact, it's possible to define an analogue of the Mordell–Weil group of elliptic curves, which is isomorphic to the Mordell–Weil group of the generic fibre of the fibration, in terms of the fibration π and its sections. The formula relates then the geometric properties of the fibration, and in particular of its singular fibres, to this arithmetical object.
The case of elliptic threefolds, i.e. of morphisms $\pi : X \longrightarrow B$ where now X has dimension 3, shares many aspects with the elliptic surfaces, but is however very different. Many of the definitions one gives for the surfaces still work for the threefolds, but the precise results stated before may fail to have an analogue in this higher dimensional case.
This is not the case for the Shioda–Tate formula, which holds in great generality thanks to a generalization due to Wazir ([Waz04]). On the contrary, up to now there is no analogue of Kodaira's theorem and Tate's algorithm for elliptic threefolds which predicts the type of a singular fibre over *any* point of the base. I want to be precise, and justify this last statement: both in the cases of sur-

faces and threefolds, the locus in the base where we have singular fibres (the so called discriminant locus) has codimension one, and so it's possible to use the machinery for surfaces to find the type of singular fibres one has over the generic point of the irreducible components of the discriminant. In the case of surfaces, the base is a curve and so the discriminant is a finite number of distinct points, hence we have a precise description of each singular fibres; but in the threefold case the discriminant locus has dimension 1 and so it's possible that over some points the singular fibres change, becoming even of non-Kodaira type. Up to now (and to my knowledge) there is no way to determine a priori the type of these possible new fibres, nor a complete classification: this is the reason why the study of non-Kodaira fibres captures the interest of many people (see e.g. [Mir83], [EY], [EFY] and [GM12]).
The main problem sits in the theory of resolutions of singularities, which is well understood in the case of surfaces, but still incomplete in higher dimension. In the thesis I will show that not every genus 1 curve can be a fibre in an elliptic threefold: using the theory of threefold rational Gorenstein singularities I will prove that the possible non-Kodaira fibres of an elliptic threefold are contraction of Kodaira fibres.
Physics, and in particular string theory, gives another motivation for studying elliptic fibrations (in particular elliptic threefolds and fourfolds). In [Vaf96], [MV96a] and [MV96b] the study of F-theory (i.e. elliptically fibered varieties) was proposed as a part of string theory where Calabi–Yau elliptic fibrations play a crucial role. To give a physically consistent phenomenology, they showed that the fibrations must have further properties, one of them is that the total space is a Calabi–Yau manifold. For this reason the study of elliptically fibered Calabi–Yau manifolds grew, especially when these varieties are embedded as anticanonical hypersurfaces in some toric Gorenstein Fano ambient space. The reason for this last requirement is that toric varieties are a particularly large but relatively simple class of varieties, which can be studied by means of combinatorics in a very efficient way, especially when they are Fano.

For these reasons (and for personal interest), in the following I will study Calabi–Yau elliptic threefolds, stressing the points where the theory for threefolds differs from the one for surfaces. In particular, the elliptic fibrations I will describe are anticanonical hypersurfaces in a projective bundle over a surface B of the form $\mathbb{P}(\mathcal{L}^a \otimes \mathcal{L}^b \otimes \mathcal{O}_B)$ for $\mathcal{L}$ an ample line bundle on B. Observe that even in the case where the base B is toric, e.g. $B = \mathbb{P}^2$ which is a case I will analyse in great detail, the ambient bundle is typically not Fano, but also in this case it's possible to study these hypersurfaces by exploiting toric geometry.

In Chapter 1 I will give the basic definitions of elliptic fibration with section, of its Weierstrass model and Mordell–Weil group. I will also explain some of their properties, which I will use in the sequel, but since all of this material is well known, proofs will be omitted and are replaced with suitable references. I will also describe Tate's algorithm and illustrate with some examples that the presence of non-Kodaira fibres is quite common in the case of threefolds. All the examples and the varieties involved in this chapter and in all the other are defined over the complex numbers.
In Chapter 2 I will give a brief description of the singularities of the Weierstrass model of an elliptic fibration. In the case of threefolds I will use the work of Reid on canonical singularities ([Rei80], [Rei83] and [Rei87]) to show a necessary con-

dition that must be satisfied by the discriminant locus of an elliptic fibration on a smooth threefold. This is important since one of the most common ways to produce new examples of elliptic fibrations is to start with a singular fibration defined by a simple equation (e.g. a singular Weierstrass fibration) and then try to resolve it in a way that is compatible with the fibration. In particular, in Theorem 2.7 I will prove that the non-Kodaira fibres of elliptic threefolds are contraction of Kodaira fibres.
Chapter 3 is devoted to a survey of Calabi–Yau manifolds and the properties I will use in the sequel.
In Chapter 4, I will introduce the anticanonical subvarieties of the $\mathbb{P}^2$-bundle $Z = \mathbb{P}(\mathcal{L}^a \otimes \mathcal{L}^b \otimes \mathcal{O}_B)$ over a surface B. The generic such hypersurface is a fibration in curves of genus 1 over B, but there are two important things to keep in mind. The first is that it's not a priori clear if these varieties are smooth or not, and the second is that we have to find at least one section for them. Requiring the smoothness of the generic anticanonical subvariety has as effect to reduce the possible values for the pair (a, b) (the exponents in the definition of the $\mathbb{P}^2$-bundle Z) to only a finite number of cases. I will then determine these pairs in the cases where B is a del Pezzo surface and $\mathcal{L}$ a multiple of its anticanonical bundle, and where $B = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$.
In Chapter 5 I will give a description of the families found in the previous chapter over $\mathbb{P}^2$. It will turn out that not all of them admit a section, for the others I will describe the cubic intersection form, give equations for the Weierstrass model, study the Mordell–Weil group by finding suitable $\mathbb{Q}$-generators and determine the precise number of sections. Not all the families found here are new, some of them were already known in the literature, but with other descriptions.
In Chapter 6 I will show how the "classical" E_7 families are a part of this framework.
Since Calabi–Yau elliptic fibrations appear in the world of physics, in Chapter 7 I will deal with the problem of finding all the possible elliptic fibrations over a surface satisfying a relation which is assumed to hold in string theory, known as tadpole cancellation relation.
Finally, in Appendix A I will give a description of the algorithm I used to compute the Hodge diamonds of the elliptic fibrations which appeared in Chapter 5 and 6.

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Chapter 1

Elliptic fibrations

In this chapter I will introduce the basic definitions and properties concerning elliptic fibrations over a variety. In particular, I will show that it's possible to associate to each elliptic fibration with section another fibration birationally equivalent to the first, known as its Weierstrass model. By Tate's algorithm, the Weierstrass model of an elliptic fibration determines the configuration and the type of the singular fibres of the original fibration, at least generically. I will also define a more arithmetical object linked to elliptic fibrations: the Mordell–Weil group of rational sections.
The definitions given here have no bounds on the dimension of the total space of the fibration, but in many examples I will focus on the case of threefolds, being this case of special interest.
I recall that all the varieties in this thesis are always defined over $\mathbb{C}$.

Definition *We say that* $\pi : X \longrightarrow B$ *is a* smooth elliptic fibration *over* B *with section if*

1. X *and* B *are smooth projective varieties of dimension* n *and* $n-1$ *respectively;*
2. π *is a surjective holomorphic morphism;*
3. *the fibres of* π *are connected curves, and the generic fibre of* π *is a smooth connected curve of genus* 1*;*
4. *a section* $\sigma : B \longrightarrow X$ *of* π *is given.*

When $\pi : X \longrightarrow B$ *satisfies only the first three requirements above, we say that it's a* genus one fibration.

Elliptic fibrations are then an example of varieties over a given base B. A *morphism* between two elliptic fibrations $\pi : X \longrightarrow B$ and $\pi' : X' \longrightarrow B$ is then a morphism of varieties over B, i.e. a map $f : X \longrightarrow X'$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & B & \end{array}$$

commutes.

Definition *An elliptic fibration $\pi : X \longrightarrow B$ is* minimal *if for any morphism $f : X \longrightarrow X'$ of varieties over B such that f contracts at least one divisor, then $\pi' : X' \longrightarrow B$ is not an elliptic fibration.*

Remark 1.0.1 We use this definition to capture previous definitions of minimality. In the case $\dim X = 2$, a minimal elliptic surface is an elliptic surface having no (-1)-curves in its fibres. In the case $\dim X = 3$, a minimal elliptic threefold is an elliptic threefold which contains no contractible surface whose contractible fibres are contained in the fibres of π (compare to [Mir83, Sect. 0, (0.3)]).

I will denote the fibre over the point $b \in B$ with X_b. Of course, not every fibre of π needs to be smooth, and we define the *discriminant locus* of the fibration as the subset of B over which we have singular fibres:

$$\Delta = \{b \in B | X_b \text{ is singular}\} \subseteq B.$$

The following is an easy but important property of elliptic fibrations with section: a section can't pass through singular points in the fibres.

Proposition 1.1 *Let $\pi : X \longrightarrow B$ be a smooth elliptic fibration with section σ. Let $b \in \Delta$ and $q \in X_b$ be a singular point for the fibre. Then $\sigma(b) \neq q$.*

Proof Assume by contradiction that $\sigma(b) = q$. The differential

$$d\pi_q : T_q X \longrightarrow T_b B$$

gives us a decomposition

$$T_q X = \ker d\pi_q \oplus \operatorname{im} d\sigma_b \simeq \ker d\pi_q \oplus T_b B$$

because q lies on the section. Since $\ker d\pi_q$ is generated by the tangent vectors in q to X_b, and $\dim T_q X_b \geq 2$ because q is singular for X_b, we deduce that

$$\dim T_q X \geq 2 + (n-1) = n+1$$

hence that q is a singular point of X. □

There is obviously a relation between elliptic fibrations and elliptic curves. In fact, the presence of a section tells us that $(X_b, \sigma(b))$ is an elliptic curve for any $b \in B \setminus \Delta$. At a deeper level, the whole fibration can be seen as an elliptic curve since its generic fibre X_η is an elliptic curve over the function field $\mathbb{C}(B)$ of the base B:

$$\begin{array}{ccc} X_\eta & \xrightarrow{\ t\ } & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} \mathbb{C}(B) & \longrightarrow & B \end{array}$$

where the distinguished point is given by the restriction of the section. Since any elliptic curve (E, e) defined over a field k has a Weierstrass model (see e.g. [Sil09, Prop. 3.1]) and a Mordell–Weil group $E(k)$ of k-rational points, we can then define the Weierstrass model of the fibration and its Mordell–Weil group.

1.1 The Mordell–Weil group

Definition *Let* (E, e) *be an elliptic curve over a field* k. *Then the* Mordell–Weil group *of* E *is the group* $E(k)$ *of its* k*-rational points.*

As for any elliptic curve over a field, the generic fibre of an elliptic fibration has its Mordell–Weil group, but in this case we also have a more geometric description. Let $\pi : X \longrightarrow B$ be an elliptic fibration. A *rational section* of π is a rational map $s : B \dashrightarrow X$ such that $\pi \circ s = \mathrm{id}$ over the domain of s. Then the Mordell–Weil group of our fibration is

$$\mathrm{MW}(X) = \{s : B \dashrightarrow X | s \text{ is a rational section}\},$$

where the group law is given by addition fibrewise, i.e. given two rational sections s and s', we define $s + s'$ requiring that $(s + s')(b) = s(b) + s'(b)$ for any $b \in \operatorname{dom} s \cap \operatorname{dom} s'$.
The correspondence between rational sections of π and $\mathbb{C}(B)$-rational points of X_η is set up in this way: given a rational section s we have a $\mathbb{C}(B)$-rational point on X_η by taking the restriction of s to the generic fibre, while given a $\mathbb{C}(B)$-rational point $s_\eta : \operatorname{Spec} \mathbb{C}(B) \longrightarrow X_\eta$ we have a rational section s on an open dense subset U of B requiring that $\overline{s(U)} = \overline{t(s_\eta(\operatorname{Spec} \mathbb{C}(B)))}$.
We can then identify $MW(X) \simeq X_\eta(\mathbb{C}(B))$.

Remark 1.1.1 Given a rational section $s : B \dashrightarrow X$, with a slight abuse of language I will also call rational section the subvariety of X defined by $\overline{s(\operatorname{dom} s)}$. In the same spirit, I will call section the image $s(B)$ of a section $s : B \longrightarrow X$.

1.1.1 Estimating $\operatorname{rk} MW(X)$

Let E be an elliptic curve over any field, with zero e. Then we have a group law on E which allows us to calculate $a +^G b = c$. Since a and b are points on E, each of them also defines a divisor and so we can add them as divisors, finding $a +^D b$. By the very definition of the group law we have that

$$a +^G b = c \Longleftrightarrow a +^D b \equiv c +^D e \tag{1.1}$$

where $\equiv$ denotes linear equivalence of divisors. This relation is easy to generalize[1]:

$$\sum^G n_P P = Q \Longleftrightarrow \sum^D n_P P \equiv Q + \left(\sum n_P - 1\right) e.$$

To see that this is true we write $\sum^G n_P P$ as $\sum^G P_i$ where the P_i's need not be different, and proceed by induction on the number n of summands. Then we know that the first step, corresponding to $n = 2$, is true. Suppose that the result is true for $n-1$ summands and consider $\sum^G_{1 \le i \le n} P_i$. Then let $R = \sum^G_{1 \le i \le n-1} P_i$: by induction we have

$$\sum^D_{1 \le i \le n-1} P_i \equiv R +^D (n-2)e,$$

[1] I use $\sum^G$ and $\sum^D$ to denote the sum using the group law or the sum of divisors respectively.

while by (1.1)

$$R +^D P_n \equiv (R +^G P_n) + e.$$

This implies that

$$\sum_{1 \leq i \leq n}^{D} P_i - (n-2)e \equiv \left(\sum_{1 \leq i \leq n}^{G} P_i \right) +^D e$$

from which the result follows.

I now want to use this remark to give bounds on the rank of the Mordell–Weil group $\mathrm{MW}(X)$ of an elliptically fibred variety. The fibration X is assumed to have a section $S = \sigma(B)$, which we use to induce a group law on each fibre. Let $S \in \operatorname{Pic} X$ be the section, and call S_η its restriction to X_η. As element in $\operatorname{Div} X_\eta$ and in $\operatorname{Pic} X_\eta$, we have that S_η *is not* zero. But observe that since S is a section we have also $S_\eta \in \mathrm{MW}(X)$, and in this group S_η *is* the neutral element. In fact, using the notation of the previous paragraph we have as elliptic curve $(E, e) = (X_\eta, S_\eta)$. An element $P \in \mathrm{MW}(X)$ is a rational point of X_η, and the map

$$\begin{array}{ccc} \mathrm{MW}(X) & \longrightarrow & \operatorname{Pic} X_\eta \\ P & \longrightarrow & P - S_\eta \end{array} \tag{1.2}$$

is a group homomorphism, in fact it follows from (1.1) that

$$(P +^G P') - S_\eta \equiv (P - S_\eta) + (P' - S_\eta).$$

Remark 1.1.2 This homomorphism is injective. In fact, if $P - S_\eta \equiv 0$, then $P = S_\eta$ as points of X_η by [Sil09, Lemma 3.3].

So we can see $\mathrm{MW}(X)$ as a subgroup of $\operatorname{Pic} X_\eta$, and observing that the image of $\mathrm{MW}(X)$ is contained in the subgroup of degree 0 divisors of X_η, we have

$$\begin{array}{ccc} \mathrm{MW}(X) & \hookrightarrow & \operatorname{Pic} X_\eta \\ & \searrow \quad \nearrow & \\ & \operatorname{Pic}^0 X_\eta & \end{array}$$

It's possible to find the image of this map. In fact it follows from [Sil09, Rmk. 3.5.1] that $X_\eta(\mathbb{C}(B))$ is isomorphic (via this map) to the subgroup of $\operatorname{Pic}^0 X_\eta$ of divisors of degree zero defined over $\mathbb{C}(B)$.

This gives us the first estimate

$$\operatorname{rk} \mathrm{MW}(X) \leq \operatorname{rk} \operatorname{Pic}^0 X_\eta \leq \operatorname{rk} \operatorname{Pic} X_\eta.$$

Since the exact sequence defining $\operatorname{Pic}^0 X_\eta$

$$0 \longrightarrow \operatorname{Pic}^0 X_\eta \longrightarrow \operatorname{Pic} X_\eta \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 \tag{1.3}$$

is split by

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \operatorname{Pic} X_\eta \\ n & \longrightarrow & nS_\eta \end{array},$$

we have that

$$\begin{array}{ccc} \operatorname{Pic}^0 X_\eta \oplus \mathbb{Z} & \longrightarrow & \operatorname{Pic} X_\eta \\ (\sum n_P P, n) & \longrightarrow & \sum n_P P + nS_\eta \end{array} \tag{1.4}$$

is a group isomorphism, so $\operatorname{rk}\operatorname{Pic}^0 X_\eta = \operatorname{rk}\operatorname{Pic} X_\eta - 1$ and we have the estimate

$$\operatorname{rk}\mathrm{MW}(X) \leq \operatorname{rk}\operatorname{Pic} X_\eta - 1.$$

Let V be the subgroup of $\operatorname{Pic} X$ generated by the classes of the vertical divisors, i.e. those divisors whose support is contained in $\pi^* D$ for D a divisor on the base B. Then ([Mir89, Lemma VII.1.4]) we have the exact sequence

$$0 \longrightarrow V \longrightarrow \operatorname{Pic} X \xrightarrow{\text{res}} \operatorname{Pic} X_\eta \longrightarrow 0, \tag{1.5}$$

implying that

$$\operatorname{rk}\mathrm{MW}(X) \leq \operatorname{rk}\operatorname{Pic} X - \operatorname{rk} V - 1. \tag{1.6}$$

Remark 1.1.3 Since π has connected fibres, by Zariski's main theorem we have that $\pi_* \mathcal{O}_X \simeq \mathcal{O}_B$. Using the projection formula we can then show (see [Mir89, Lemma VII.1.1]) that $\pi^* : \operatorname{Pic} B \longrightarrow \operatorname{Pic} X$ is injective. In fact let $\mathcal{L}$ be any line bundle on B, then

$$\pi_* \pi^* \mathcal{L} \simeq \mathcal{L} \otimes \pi_* \mathcal{O}_X \simeq \mathcal{L}.$$

Of course $\pi^* \operatorname{Pic} B$ is a subgroup of V, so that

$$\operatorname{rk}\mathrm{MW}(X) \leq \operatorname{rk}\operatorname{Pic} X - \operatorname{rk}\operatorname{Pic} B - 1,$$

but $V \neq \pi^* \operatorname{Pic} B$ in general. For example, this happens if the fibration has reducible fibres over a codimension 1 locus in the base, and we will see examples of this fact in Sections 5.3.3 and 6.2.2.

1.1.2 The Shioda–Tate–Wazir formula

In the previous section we saw some estimates on $\operatorname{rk}\mathrm{MW}(X)$, but in fact there are formulae to compute this number. In the case of elliptic surfaces, i.e. elliptic fibrations $\pi : X \longrightarrow B$ where X is a surface, it's possible to compute the rank of V in terms of the singular fibres of the fibration. This leads to the Shioda–Tate formula for computing the rank of the Mordell–Weil group ([Shi90, Cor. 5.3]):

Theorem 1.2 (Shioda–Tate formula) *Let $\pi : X \to B$ be a smooth elliptic surface with section. Then*

$$\operatorname{rk}\mathrm{MW}(X) = \operatorname{rk}\mathrm{NS}(X) - 2 - \sum_{b \in \Delta} ((\#\ \textit{irreducible components of } X_b) - 1).$$

Observe that this theorem says that our estimate (1.6) is indeed an equality. This formula has been generalized in the setting of scheme theory by Wazir ([Waz04, Cor. 3.2]) for elliptic fibrations of any dimension:

Theorem 1.3 (Shioda–Tate–Wazir formula) *Let $\pi : X \longrightarrow B$ be an elliptic fibration with section over a field k. Then*

$$\operatorname{rk}\mathrm{NS}(X) = 1 + \operatorname{rk}\mathrm{MW}(X) + \operatorname{rk}\mathrm{NS}(B) + \operatorname{rk} F$$

where F is the vector space generated by the irreducible components of $\pi^* \Delta$ which don't intersect the section.

Roughly speaking, this theorem says that our estimates are quite sharp. In fact the rank of $\mathrm{MW}(X)$ depends on:

1. the geometry of the fibration, encoded in $\mathrm{NS}(X)$;
2. the geometry of the base, encoded in $\mathrm{NS}(B)$;
3. the presence of the section, which is given by the 1 in the formula;
4. the singular fibres, encoded by F, since if we have reducible fibres over some component of the discriminant locus, say Δ_{red}, then $\pi^{-1}(\Delta_{\mathrm{red}})$ splits into different irreducible components, giving the so called *fibral divisors*.

1.2 Weierstrass fibrations and Weierstrass models

In this section I want to define the notion of Weierstrass fibration and to show how we can associate to each elliptic fibration with section a Weierstrass fibration, which we call the Weierstrass model of the elliptic fibration. In the following definitions I will also set up the notation I will use in the sequel.

Definition *We say that $p : W \longrightarrow B$ is a* Weierstrass fibration *if*

1. *W and B are projective varieties, with B smooth (W not necessarily);*
2. *p is a surjective morphism whose fibres are smooth curves of genus 1, or rational curves with either a node or a cusp;*
3. *the generic fibre of p is smooth;*
4. *there is a section s of p such that $s(B)$ does not pass through the nodes or the cusps in the fibres.*

Definition *Let $p : W \longrightarrow B$ be a Weierstrass fibration over B, with section s. If $\Sigma = s(B)$ and $i : \Sigma \hookrightarrow W$ is the inclusion, then the* fundamental line bundle *of p is the line bundle $\mathcal{L}$ on B defined by*

$$\mathcal{L} = (p_* i_* \mathcal{N}_{\Sigma|W})^{-1}$$

where $\mathcal{N}_{\Sigma|X}$ is the normal bundle of Σ in W.

Observe that since Σ has codimension 1 in W, then $\mathcal{N}_{\Sigma|W}$ is a line bundle on Σ and so $\mathcal{L}$ is a line bundle on B because $p_{|\Sigma}$ is an isomorphism. Despite the fact that this definition depends on the section, the fundamental line bundle $\mathcal{L}$ is intrinsic to W since it's known that ([Mir89, Chap. II.3])

$$\mathcal{L}^{-1} \simeq R^1 p_* \mathcal{O}_W.$$

The following results are essential and explain the importance of the fundamental line bundle of a Weierstrass fibration. Proofs can be found in [Mir89, Sect. II.4 and III.1] in the case where W is a surface and B is a curve, but they still work in any dimension.

Proposition 1.4 *With the same notations as before, we have for any $n \geq 2$ a decomposition*

$$p_* \mathcal{O}_W(n\Sigma) = \mathcal{O}_B \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \oplus \ldots \oplus \mathcal{L}^{-n}.$$

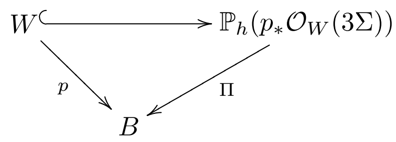
The role of the section in a Weierstrass fibration is essentially the same as that of the distinguished point on an elliptic curve: we can use it to embed the fibration in a $\mathbb{P}^2$-bundle over the base B.
Given any vector bundle E over B, I will denote with $\mathbb{P}_h(E)$ the projective space bundle of the hyperplanes in E as usual in algebraic geometry, while with $\mathbb{P}(E)$ I will denote the projective space of lines in E as it's customary in physics. Of course the two notions are equivalent, since they are linked by a duality relation

$$\mathbb{P}_h(E) \simeq \mathbb{P}(E^*).$$

Theorem 1.5 *The line bundle $\mathcal{O}_W(3\Sigma)$ defines a closed immersion*

$$W \hookrightarrow \mathbb{P}_h(p_* \mathcal{O}_W(3\Sigma))$$

in such a way that the diagram



commutes, where the morphism Π is the structure morphism of the projective space bundle.

For sake of simplicity I will call E_W the bundle $p_* \mathcal{O}_W(3\Sigma) \simeq \mathcal{O}_B \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$. Once we realize W as a hypersurface in a projective space bundle, we can ask for its equation. This is the usual equation of an elliptic curve in Weierstrass form, namely

$$y^2 z = x^3 + a_4 x z^2 + a_6 z^3, \tag{1.7}$$

where

$$\begin{aligned} x &\in H^0(\mathbb{P}_h(E_W), \mathcal{O}_{\mathbb{P}_h(E_W)}(1) \otimes \Pi^* \mathcal{L}^2), \\ y &\in H^0(\mathbb{P}_h(E_W), \mathcal{O}_{\mathbb{P}_h(E_W)}(1) \otimes \Pi^* \mathcal{L}^3), \\ z &\in H^0(\mathbb{P}_h(E_W), \mathcal{O}_{\mathbb{P}_h(E_W)}(1)) \end{aligned} \tag{1.8}$$

give coordinates in the fibres, and

$$\begin{aligned} a_4 &\in H^0(\mathbb{P}_h(E_W), \Pi^* \mathcal{L}^4) \simeq H^0(B, \mathcal{L}^4), \\ a_6 &\in H^0(\mathbb{P}_h(E_W), \Pi^* \mathcal{L}^6) \simeq H^0(B, \mathcal{L}^6) \end{aligned} \tag{1.9}$$

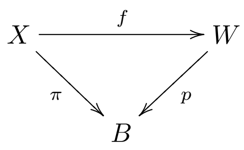
are coefficients. We can then give an equation for the discriminant locus:

$$\Delta : 4a_4^3 + 27a_6^2 = 0 \tag{1.10}$$

and so we see that $\Delta \in H^0(B, \mathcal{L}^{12})$.
Given a smooth elliptic fibration $\pi : X \longrightarrow B$ with section σ, we have already observed that $(X_b, \sigma(b))$ is an elliptic curve for any $b \in B \setminus \Delta$: as such it can be put in Weierstrass form. This pointwise fact globalizes to the whole fibration,

so that we can speak of the *Weierstrass model* $p : W \longrightarrow B$ of the fibration. To be more precise, we can put the generic fibre X_η of X in its Weierstrass form W_η, finding an equation whose coefficients are rational functions on B. After a birational change of variables we can then find an equation for W_η with coefficients defined everywhere on B, i.e. we find an equation for W in $\mathbb{P}_h(E_W)$.

Proposition 1.6 *Any smooth elliptic fibration with section $\pi : X \longrightarrow B$ admits a unique (up to isomorphism) Weierstrass model in such a way that*



commutes and f is a birational morphism. Moreover, all the fibres of f have dimension at most 1.

Proof For the existence, see [Nak88]. For the last part of the statement, just observe that every fibre of p and π is 1-dimensional, and so $f^{-1}(w)$ can be at most a curve. □

Geometrically, we can describe the morphism to the Weierstrass model as the contraction in the reducible fibres of all the irreducible components which don't intersect $\sigma(B)$. This implies that a smooth minimal elliptic fibration and its Weierstrass model have the same discriminant locus.

Example 1.2.1 In this example I want to show that the assumption of having a section is quite restrictive. In fact, let's consider the variety X defined in $\mathbb{P}^3 \times \mathbb{P}^2$ with coordinates $((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2))$ by the equations

$$X : \begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 \\ y_0^3 x_0 + y_1^3 x_1 + y_2^3 x_2 = 0. \end{cases}$$

Then X is smooth and can be viewed naturally as a fibration in elliptic curves in two different ways by restricting to X the projections onto the two factors.

1. In the first case X is a fibration over the Fermat cubic surface in $\mathbb{P}^3$. We observe that over the point $(0 : 0 : 1 : -1)$ we have a multiple fibre, with equation $y_2^3 = 0$, so any of its points is singular for the fibre. By Proposition 1.1 we can conclude that this fibre can't meet any section, hence that there is no section at all.

2. In the second case we have a fibration over $\mathbb{P}^2$, but even in this case we have no sections. In fact, if σ were a section then $\sigma(\mathbb{P}^2)$ would be contained in the Fermat cubic. But there is no non constant morphism from $\mathbb{P}^2$ to this surface.

Example 1.2.2 Let's consider the projective plane bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2})$ over $\mathbb{P}^2$. Denote with $(x : y : z)$ coordinates in the fibres and with $(t_0 : t_1 : t_2)$ coordinates in the plane, then the Weierstrass equation

$$y^2 z = x^3 + (t_0^6 + t_1^6 + t_2^6) z^3$$

defines a smooth elliptic threefold $\pi : X \longrightarrow \mathbb{P}^2$. The discriminant locus of this fibration is then

$$\Delta : 27(t_0^6 + t_1^6 + t_2^6)^2 = 0,$$

which is a Fermat sextic with multiplicity two, and over each point of the discriminant we have cuspidal cubics.

1.2.1 Adjunction in $\mathbb{P}_h(E_W)$

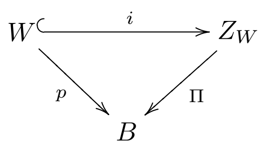
In this section I want to calculate the class of a *smooth* Weierstrass fibration $p : W \longrightarrow B$ in the Chow group of codimension 1 subvarieties of $\mathbb{P}_h(E_W)$. Using the adjunction formula we will be able to compute the class of a canonical divisor of W.
In this section I will use the projective space of lines, writing

$$Z_W = \mathbb{P}(E_W^*) = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$$

for the ambient space, and

$$L = c_1(\mathcal{L}), \qquad \xi_W = c_1(\mathcal{O}_{Z_W}(1)).$$

We have the following diagram



and the class $[W]$ of W is quite easy to compute: W is defined by the equation (1.7), which is a homogeneous cubic in the variables x, y and z with coefficients a_4 and a_6, and so by (1.8) and (1.9) this means that

$$[W] = 6\Pi^* L + 3\xi_W.$$

To compute the canonical class K_{Z_W} we first use the exact sequence defining the relative tangent bundle

$$0 \longrightarrow T_{Z_W|B} \longrightarrow T_{Z_W} \longrightarrow \Pi^* T_B \longrightarrow 0$$

to find the relation

$$c(Z_W) = c(T_{Z_W|B})\Pi^* c(B)$$

between the Chern polynomials, and then the exact sequence ([Ful98, App. B.5.8])

$$0 \longrightarrow \mathcal{O}_{Z_W} \longrightarrow (\Pi^* E_W) \otimes \mathcal{O}_{Z_W}(1) \longrightarrow T_{Z_W|B} \longrightarrow 0$$

to compute $c(T_{Z_W|B})$. What we find is that the first Chern class of Z_W is then the piece of degree one in the polynomial

$$c(Z_W) = (1 + \xi_W)(1 + 2\Pi^* L + \xi_W)(1 + 3\Pi^* L + \xi_W)(1 + \Pi^* c_1(B) + \Pi^* c_2(B)),$$

hence that

$$K_{Z_W} = -c_1(Z_W) = \Pi^* K_B - 5\Pi^* L - 3\xi_W.$$

We can now compute the canonical class of W by means of the adjunction formula: since

$$K_{Z_W} + [W] = \Pi^* K_B - 5\Pi^* L - 3\xi_W + 6\Pi^* L + 3\xi_W = \Pi^*(K_B + L), \quad (1.11)$$

we have that

$$K_W = (K_{Z_W} + [W])_{|W} = i^*\Pi^*(K_B + L) = p^*(K_B + L). \quad (1.12)$$

Observe that K_W the pull-back of a divisor in the base, hence K_W is vertical. In Proposition 2.5 we will see that (1.12) generalizes to singular Weierstrass models.

Example 1.2.3 Let X be the fibration of Example 1.2.2: its ambient bundle is of the form Z_W, with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$. Then we have that L is the class of a line in the base $\mathbb{P}^2$, and since $K_{\mathbb{P}^2} = -3L$ we can now justify our previous claim on the canonical bundle of X, in fact

$$K_X = p^*(K_{\mathbb{P}^2}) + p^* L = p^*(-2L).$$

Remark 1.2.4 The technique used here to compute $c_1(Z_W)$ and the Chern polynomial $c(Z_W)$ is quite general: if E is any rank 3 vector bundle on B and $Z = \mathbb{P}(E)$, it can be used to compute $c(Z)$.
The general result is that

$$c(Z) = c(T_{Z|B})\Pi^* c(B)$$

with

$$c(T_{Z|B}) = 1 + \Pi^* c_1(E) + 3\xi + \Pi^* c_2(E) + 2\Pi^* c_1(E)\xi + 3\xi^2.$$

In particular

$$c_1(Z) = \Pi^* c_1(B) + \Pi^* c_1(E) + 3\xi. \quad (1.13)$$

1.3 Singular fibres and Tate's algorithm

Let $\pi : X \longrightarrow B$ be a smooth elliptic fibration with section. We defined the discriminant locus of the fibration as the locus $\Delta \subseteq B$ over which we have singular fibres. Our interest is now on the type of these singular fibres.
In the case of surfaces, the situation is clear and well understood: the possible singular fibres were listed by Kodaira ([Kod63, Thm. 6.2]), who also named them.
In the following, I will refer to the singular fibres in the list as *Kodaira fibres.*

Theorem 1.7 (Kodaira's classification of singular fibres) *Given a smooth minimal elliptic surface with section $\pi : X \longrightarrow B$, the only possible singular fibres of π are the ones listed in Table 1.1.*

Table 1.1: List of Kodaira singular fibres

Name	*Description*
I_1	*Nodal rational curve*
I_2	*Two smooth rational curves meeting transversally at two points*
I_n *with* $n \geq 3$	n *smooth rational curves meeting with dual graph* $\tilde{A}_n$
I_n^* *with* $n \geq 0$	$n+5$ *smooth rational curves meeting with dual graph* $\tilde{D}_{n+4}$
II	*Cuspidal rational curve*
III	*Two smooth rational curves meeting at a point of order two*
IV	*Three smooth rational curves all meeting at a point*
IV^*	7 *smooth rational curves meeting with dual graph* $\tilde{E}_6$
III^*	8 *smooth rational curves meeting with dual graph* $\tilde{E}_7$
II^*	9 *smooth rational curves meeting with dual graph* $\tilde{E}_8$

Moreover, in the case of elliptic surfaces, there is an algorithm which allows one to determine the type of the singular fibre over a point b of the discriminant locus: first of all we put X in Weierstrass form, finding an equation of the form

$$W : y^2 z = x^3 + a_4 x z^2 + a_6 z^3,$$

and then we compute the multiplicities $\operatorname{mult}_b a_4$, $\operatorname{mult}_b a_6$ and $\operatorname{mult}_b \Delta$. The type of the singular fibre over b is then given by the following table ([Tat75, P. 46])

Name	$\operatorname{mult}_b a_4$	$\operatorname{mult}_b a_6$	$\operatorname{mult}_b \Delta$
I_1	0	0	1
I_n	0	0	n
I_0^*	2	3	6
	≥ 3	3	6
	2	≥ 4	6
I_n^*	2	3	$n+6$
II	≥ 1	1	2
III	1	≥ 2	3
IV	≥ 2	2	4
IV^*	≥ 3	4	8
III^*	3	≥ 5	9
II^*	≥ 4	5	10

This procedure to determine the type of a singular fibre is known as *Tate's algorithm*.
For minimal elliptic fibrations we can still run it: we put the fibration with section in Weierstrass form, and then we consider the irreducible components Δ_i's of Δ. Since the local rings $\mathcal{O}_{X,\Delta_i}$ are discrete valuation rings, it makes sense to compute the multiplicities $\operatorname{mult}_{\Delta_i} a_4$, $\operatorname{mult}_{\Delta_i} a_6$ and $\operatorname{mult}_{\Delta_i} \Delta$. From the previous table we can then deduce the type of the singular fibre *over the generic point* of Δ_i.
Observe that in the case of surfaces we have a precise description of any singular fibre, while in the case of higher dimensional elliptic fibrations what happens in codimension 2 (and greater) on the base is not yet well understood.

1.3.1 Examples of non-Kodaira fibres

As observed in the previous paragraph over codimension two loci in the base we can have singular fibres of non-Kodaira type. I want now to give some examples of non-Kodaira fibres in the case where $\pi : X \longrightarrow B$ is a local fibration of dimension 3, i.e. a fibration where B is a small disk in $\mathbb{C}^2$ centred in 0, and over the origin we will have the non-Kodaira fibre.
As a first example, in the paper [Mir83] the explicit desingularization of local Weierstrass fibrations satisfying some further assumption is given. In this work it's clear that when singular fibres of Kodaira type collide, the result can be of non-Kodaira type.
The second example I want to give is inspired by [EY]. Let $f(s,t)$ be a function which does not vanishes at the origin, and consider the Weierstrass fibration in $\mathbb{P}^2 \times B$

$$y^2 z = x^3 - \frac{1}{48}s^4 x z^2 + \left(\frac{1}{864}s^6 + f(s,t)t^5\right) z^3,$$

having

$$\begin{aligned} A &= -\tfrac{1}{48}s^4, \\ B &= \tfrac{1}{864}s^6 + ft^5, \\ \Delta &= \tfrac{1}{16}t^5 f(s^6 + 432ft^5). \end{aligned}$$

Over $t = 0$ we have nodal cubics, and the node of each is singular for the whole variety. We proceed by blowing-up this curve of singularities, and we obtain a fibration with I_3 fibres over $t = 0$: one edge of the triangle is the transform of the original nodal cubic, the other two edges have been introduced by the blow-up. Their intersection is singular again, hence we have to blow-up the curve of singular points again. Doing so, we find fibres of Kodaira type I_5 over $t = 0$, but in the fibre over the origin there is still a unique singular point of the whole variety. It's possible to resolve this isolated singularity by mean of a small blow-up, i.e. a blow-up which doesn't introduce exceptional divisors, and the resulting threefold is a smooth elliptic fibration with a non-Kodaira fibre over the origin. This fibre is made up of five irreducible components, each of which is a line. Three of them meet at a point, and have multiplicity 2, 3 and 4 respectively. The component of multiplicity 4 intersects one of the remaining components (which has multiplicity 2) and the component with multiplicity 2 meets the last component (which has multiplicity 1), which is the component intersecting the section. There is another way of describing this fibre: consider a Kodaira fibre of type II^*, and contract the four consecutive components of multiplicities 3, 4, 5 and 6 respectively.

Remark 1.3.1 If we consider a generic line through the origin, we have a local elliptic surface, and Tate's algorithm says that over that point we should have a fibre II^*. What we found in the threefold is a contraction of this fibre, and as we will see later (Proposition 2.7) this fact generalizes.

Remark 1.3.2 It's not necessary to start with a Weierstrass model if we want fibrations with I_5 fibres. For example (if we are looking for global models), we can start from three homogeneous cubic polynomials α, β, γ and consider the following singular fibrations, having a smooth resolution with I_5 fibres, as one can verify directly blowing-up the singular locus:

1. the fibration defined in $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(3) \oplus \mathcal{O})$ by
$$y^2 z = \alpha^2 x^3 + \frac{1}{4}\beta^2 x^2 z + \alpha\gamma z^3.$$
2. the fibration defined in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(6) \oplus \mathcal{O})$ by
$$y^2 z = \alpha x^3 + \frac{1}{4}\beta^2 x^2 z + \alpha^3\gamma z^3;$$

These two fibrations have the same Weierstrass model in $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(9) \oplus \mathcal{O})$, given by
$$y^2 z = x^3 - \frac{1}{48}\beta^2 x z^2 + \left(\frac{1}{486}\beta^6 + \alpha^5\gamma\right) z^3,$$
and in fact they are birationally equivalent:
$$(x : y : z) \longmapsto (\alpha x : \alpha y : z)$$
gives a birational map from the first to the second. The (smooth) fibrations in these bundles will be studied in detail in Chapter 6, in particular the fibration in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(6) \oplus \mathcal{O})$ is of E_7^0 type.

I want to give here a last example of non-Kodaira fibre. Consider in $\mathbb{P}^2 \times B$ the fibration
$$y^2 z = x^3 + t x^2 z + s^4 z^3,$$
if we write a Weierstrass model (1.7) we find that the coefficients are
$$\begin{aligned} a_4 &= -\tfrac{1}{3}t^2 \\ a_6 &= s^4 + \tfrac{2}{27}t^3 \\ \Delta &= s^4(27s^4 + 4t^3) \end{aligned}$$
and so by Tate's algorithm we expect to have I_4 fibres over the line $s = 0$. The curve $s = x = y = 0$ is singular for the threefold, hence we blow it up. The effect is that over $s = 0$ instead of nodal cubics now we have triangles, but one of its vertices is still singular for the whole variety. After a second blow-up of this curve of singular points we have a smooth threefold with I_4 fibres over $s = 0$. If

we restrict the fibration to a generic line through the origin, by Tate's algorithm on this elliptic surface we should have over the origin a fibre of Kodaira type I_1^*: what we see is not the whole fibre but a contraction of it (Proposition 2.7). To be precise, a I_1^* fibre has 6 irreducible components meeting with intersection graph $\tilde{D}_5$, the two central components have multiplicity 2 while the other four have multiplicity 1, and one of these intersects the section. Our non-Kodaira fibre is obtained by a I_1^* fibre as follows: let C denote the multiplicity one component which meets the section, then contract the multiplicity 2 component which does not intersect C, and the multiplicity 1 component intersecting the multiplicity 2 component which meets also C.

Remark 1.3.3 This last example is interesting for the following reason: let's call X the smooth threefold, and X_λ its restriction to the line $t = \lambda s$. As observed before, X_λ is singular over $s = 0$, since we does not have a Kodaira fibre, and so there are singular points whose coordinates depend on λ. In fact there are always two singular points: one of them is on the multiplicity two component of the non-Kodaira fibre and its coordinates depend on λ, while the second is at the point of the fibre where the multiplicity 2 component of the I_1^* fibre was blown down. This last has coordinates independent of λ, but nevertheless the threefold is smooth at this point.

Chapter 2

Singularities

Even though we are mainly interested in smooth varieties, as we saw when discussing the Weierstrass model of an elliptic fibration, we also have to deal with some kinds of singular varieties. In this section I recall the definition of rational Gorenstein singularities and their main properties, focussing in the case of threefolds. The main references for this chapter are [Rei80], [Rei83], [Rei87] and [KM98].
The canonical sheaf of a (smooth) variety X is $\Omega_X^n = \bigwedge^n \Omega_X^1$, the n-th exterior power of its cotangent sheaf, but to deal with singular varieties we need a new definition which generalizes the usual one.

Definition *Let X be a normal variety, with regular locus $X^{reg} = X \smallsetminus \operatorname{Sing} X$. Then we define*

$$\omega_X = j_* \left(\Omega^n_{X^{reg}}\right)$$

where $j : X^{reg} \hookrightarrow X$ is the inclusion.

In this way ω_X is a torsion-free sheaf, extending the usual canonical sheaf on X^{reg}. We also have a link between ω_X and Ω^n_X, in fact the first is the double dual of the second: $\omega_X \simeq (\Omega^n_X)^{**}$. There is also the possibility of defining a canonical Weil divisor, let $K_{X^{\mathrm{reg}}}$ be a canonical divisor in X^{reg}, then $K_X = \overline{K_{X^{\mathrm{reg}}}}$ is its closure in X. Observe that K_X may not be Cartier, and that

$$\omega_X = \mathcal{O}_X(K_X), \qquad \omega_X(U) = \left\{ f \in \mathbb{C}(X) | (f)_{|U} + K_{X_U} \geq 0 \right\},$$

is generated by the differential forms regular in codimension 1.

Definition *A normal variety X will be called* Gorenstein *if it's Cohen–Macaulay and its canonical sheaf ω_X is locally free of rank* 1 *(compare to [Rei80, P. 286]).*

The Cohen–Macaulay condition is local and algebraic in nature, and means that for any point $x \in X$ the local ring $\mathcal{O}_{X,x}$ have the property that there is a sequence of elements $f_1, \dots, f_r \in \mathcal{O}_{X,x}$ such that

1. all the f_i's lie in the maximal ideal of $\mathcal{O}_{X,x}$;
2. f_1 is not a zero-divisor in $\mathcal{O}_{X,x}$ and for all $i = 2, \dots, r$ the element $f_i \in \mathcal{O}_{X,x}/(f_1, \dots, f_{i-1})$ is not a zero-divisor;

3. $f_1, \dots, f_r$ has maximal length with respect to the previous properties;
4. $r = \dim \mathcal{O}_{X,x}$.

Example 2.0.4 Let X be a projective smooth variety, and $Y \subseteq X$ a (locally) complete intersection with $\operatorname{codim} \operatorname{Sing} Y \geq 2$. Then Y is regular in codimension 1 and so by [Har77, Prop. II.8.23] Y is also normal. Hence Y is Gorenstein by [Eis95, Cor. 21.19].

Definition *Let X be a Gorenstein variety, and $x \in X$. Then $x \in X$ is a* rational Gorenstein singularity *(compare to [Rei80, Def. 2.4]) if there exists a resolution of singularities (see p. 24 for the definition) $f : Y \longrightarrow X$ of x such that*

$$f_* \omega_Y = \omega_X.$$

In the case of threefolds, there is also a classification of rational Gorenstein singularities which can be found in [Rei80, Cor. 2.10]. The simplest are the so called compound Du Val singularities, which are an analogue for threefolds of the Du Val singularities for surfaces.
Let S be a normal surface, and $s \in S$ a singular point: s is a Du Val singularity if one of the following equivalent condition holds:

1. there exists a resolution of singularities $f : \tilde{S} \longrightarrow S$ such that $K_{\tilde{S}} = f^* K_S$;
2. in a neighbourhood of s we have that S is analytically isomorphic to one of the hypersurface singularities of $\mathbb{A}^3$ in the following list

Name	Equation
A_n	$x^2 + y^2 + z^{n+1}$
D_n	$x^2 + y^2 z + z^{n-1}$
E_6	$x^2 + y^3 + z^4$
E_7	$x^2 + y^3 + yz^3$
E_8	$x^2 + y^3 + z^5$

Observe that in this second point we also have a classification of Du Val singularities. The name refers to the Dynkin diagrams, in fact in the minimal resolution of a Du Val singularity, the exceptional divisors have the corresponding Dynkin diagram as incidence graph. The *minimal resolution* of a Du Val singularity is obtained by a sequence of blow-ups in the singular points, which ends as soon as the blown-up surface becomes smooth, and it's minimal in the sense that any other resolution factors through this.

Definition *A point x in a threefold X is a* compound Du Val singularity, *or cDV for short, if there is a hypersurface $H \subseteq X$ through x such that $x \in H$ is a Du Val singularity.*

Remark 2.0.5 We can then see X as a deformation of a Du Val singularity, since the definition is equivalent to ask that around x the variety X is locally analytically isomorphic to the hypersurface singularity in $\mathbb{A}^4$ given by

$$f(x, y, z) + t\, g(x, y, z, t) = 0,$$

where $f(x, y, z) = 0$ defines a Du Val singularity. Observe also that while Du Val singularities are isolated, *cDV* singularities can be isolated or not.

2.1 Resolution of the singularities

Even if I already mentioned the resolution of singularities, I want to give here a precise definition.

Definition *We say that $f : Y \longrightarrow X$ is a* resolution of the singularities *of X if*

1. *Y is smooth;*
2. *f is birational, and is an isomorphism between $f^{-1}(X^{reg})$ and X^{reg}.*

We will be interested in resolutions having other particular features.

Definition *Let $f : Y \longrightarrow X$ be a resolution of the singularities. We say that f is* small *if for any point $x \in X$ the fibre $f^{-1}(x)$ contains no divisors. If X is a threefold, this means that each fibre has dimension at most 1, i.e. it's a point or a curve.*

Example 2.1.1 Let $\pi : X \longrightarrow B$ be an elliptic fibration with section over a base B with $\dim B \geq 2$, and $p : W \longrightarrow B$ its Weierstrass model. Then we have the birational morphism $f : X \longrightarrow W$ from Proposition 1.6, which is a small resolution of the singularities of W.

When we have a resolution of the singularities $f : Y \longrightarrow X$, we can compare a canonical divisor K_Y on Y with f^*K_X: we have

$$K_Y \equiv f^*K_X + \sum_i a_i D_i$$

with D_i effective divisors which are contracted by f, satisfying $\operatorname{codim} f(D_i) \geq 2$. A divisor D_i for which $a_i = 0$ is called a *crepant divisor*, the other are called *discrepant*. The resolutions for which $a_i = 0$ for all i are hence particularly interesting.

Definition *Let $f : Y \longrightarrow X$ be a resolution of the singularities. We say that f is* crepant *if $K_Y = f^*K_X$.*

This definition makes sense since the exceptional divisors introduced by a resolution tends to become the zero locus for the pull-back of the differential forms on X. This happens for example if we want to desingularize a variety blowing-up its singular locus.

Remark 2.1.2 If $f : Y \longrightarrow X$ is a small resolution of an isolated singularity, then f is obviously crepant since it does not introduce exceptional divisors. It's a more difficult problem to understand when a small resolution of non-isolated singularities is crepant or not.

In the case of threefolds, there is a link between small and crepant resolutions, investigated by Reid in [Rei83]. In particular we are interested in the following two facts ([Rei83, Cor. 1.12] and [Rei83, Thm. 1.14]).

Proposition 2.1 *Let X be a singular Gorenstein threefold with a small resolution $f : Y \longrightarrow X$. Then the singularities of X are of cDV type.*

Proof It's not difficult to see that since f is small, then $f_*\omega_Y = \omega_X$ and so X has rational Gorenstein singularities. Let $P \in X$ be a singular point, then ([KM98, Thm 5.35]) the following are equivalent:

1. the general hypersurface section $P \in H \subseteq X$ is an elliptic singularity;
2. if $g : X' \longrightarrow X$ is any resolution of singularities then there is a crepant divisor $E \subseteq g^{-1}(P)$.

Since f is small, there is no divisor in $f^{-1}(P)$ and so the general hypersurface section through P is not an elliptic singularity. Then, by [KM98, Lemma 5.30] or [Rei80, Thm. (2.6)(I)], we have that the general hypersurface section through P must have a rational (i.e. Du Val) singularity, which proves that P is cDV.□

For threefolds with cDV singularities, we fully understand the link between small and crepant resolutions. Observe that the statement of Theorem 2.2 is more general, since it concerns *partial resolutions*, i.e. proper birational morphisms $Y \longrightarrow X$ where Y is assumed to be normal (not necessarily smooth).

Theorem 2.2 *Let X be a threefold with cDV singularities (not necessarily isolated), and let $f : Y \longrightarrow X$ be a partial resolution. Then the following are equivalent:*

1. *f is crepant;*
2. *f is small, and crepant above the general point of any 1-dimensional component of $\operatorname{Sing} X$;*
3. *for every x in X and hypersurface H through x for which $x \in H$ is a Du Val singularity, $H' = f^{-1}(H)$ is normal and $f_{|_{H'}} : H' \longrightarrow H$ is crepant. Thus the minimal resolution of $x \in H$ factors through H'.*

Proof See [Rei83, Thm. 1.14]. □

2.2 The singularities of Weierstrass fibrations

In this section I want to make some remarks on the singularities of Weierstrass fibrations over surfaces. Since all what I will say in this section is local in nature, here B will denote an open disc in $\mathbb{C}^2$ with coordinates (s, t). Hence a Weierstrass fibration is the variety defined in $\mathbb{P}^2 \times B$ by an equation of the form

$$y^2z = x^3 + a_4(s,t)xz^2 + a_6(s,t)z^3.$$

It's an easy computation with the Jacobian matrix to prove the following theorem

Theorem 2.3 *If W is a Weierstrass fibration over B with discriminant locus Δ, then*

1. *the point* $(0:1:0)$ *is never singular;*
2. *if* W *is singular at* $((x:y:z),(s,t))$*, then* $y=0$*;*
3. W *is singular at* $((0:0:1),(s,t))$ *if and only if* $a_4(s,t)=a_6(s,t)=0$ *and* a_6 *is singular in* (s,t)*;*
4. W *is singular at* $((x:0:1),(s,t))$ *with* $x\neq 0$ *if and only if* $a_4(s,t)\neq 0$, $a_6(s,t)\neq 0$, $\Delta(s,t)=0$ *and* Δ *is singular in* (s,t)*. Moreover in this case* $x=-\frac{3a_6(s,t)}{2a_4(s,t)}$.

Proof See [Mir83, Prop. 2.1]. □

This means that the singular locus of W consists at most of curves, hence $\operatorname{codim}\operatorname{Sing}W\geq 2$.

Remark 2.2.1 Consider the Weierstrass model $p:W\longrightarrow B$ of an elliptic fibration $\pi:X\longrightarrow B$ with $\dim X=3$. By Theorem 1.5, W is a hypersurface in a projective bundle over B, and we have just seen that its singular locus consists of isolated points or curves. So W is Gorenstein by Example 2.0.4.

Let $\pi:X\longrightarrow B$ be a smooth minimal elliptic threefold with Weierstrass model W, and call $f:X\longrightarrow W$ the morphism to the Weierstrass model. The following proposition summarizes the properties of f.

Proposition 2.4 *Let* $\pi:X\longrightarrow B$ *be a smooth minimal elliptic threefold, with Weierstrass model* $p:W\longrightarrow B$*. Then* W *is Gorenstein, has all singular points of cDV type and the morphism* $f:X\longrightarrow W$ *on the Weierstrass model is a small and crepant resolution.*

Proof We know that that W is Gorenstein by Remark 2.2.1 and that the morphism f on the Weierstrass model is a small resolution by Proposition 1.6. By Proposition 2.1 this means that W has *cDV* singularities. Finally, by minimality of $\pi:X\longrightarrow B$, we have that the resolution $f:X\longrightarrow W$ (outside a finite number of points in $\operatorname{Sing}W$) coincides with the minimal resolution of a Du Val singularity (compare to [Rei83, §2]), which is crepant. By Theorem 2.2 this means that $f:X\longrightarrow W$ is crepant. □

As a consequence, we have a way to compute a canonical divisor of the total space of the elliptic fibration.

Proposition 2.5 *Let* $\pi:X\longrightarrow B$ *be a smooth minimal elliptic threefold with section* S*. Then*

$$K_X=\pi^*K_B+\pi^*L$$

where L *is the first Chern class of the line bundle* $\mathcal{L}=(\pi_*i_*\mathcal{N}_{S|X})^{-1}$.

Proof Let $p:W\longrightarrow B$ be the Weierstrass model of X, with section Σ, and embed W in $Z=\mathbb{P}_h(p_*\mathcal{O}_W(3\Sigma))$ as in Theorem 1.5. In (1.11) we computed that

$$\omega_Z\otimes\mathcal{O}_Z(W)=\Pi^*(\omega_B\otimes\mathcal{L})$$

where $\mathcal{L}$ is the fundamental line bundle of W, i.e. $\mathcal{L}=(p_*\iota_*\mathcal{N}_{\Sigma|W})^{-1}$. Let $j:W^{\mathrm{reg}}\hookrightarrow W$ and $t:W\hookrightarrow Z$ be the inclusions, then

$$\omega_{W^{\mathrm{reg}}}=j^*t^*(\omega_Z\otimes\mathcal{O}_Z(W))=j^*t^*\Pi^*(\omega_B\otimes\mathcal{L})$$

and so, since $\operatorname{codim} \operatorname{Sing} W \geq 2$,

$$\omega_W = j_* \omega_{W^{\mathrm{reg}}} = j_* j^* t^* \Pi^* (\omega_B \otimes \mathcal{L}) = p^* (\omega_B \otimes \mathcal{L}).$$

Observe that the morphism to the Weierstrass model $f : X \longrightarrow W$ induces an isomorphism between open neighbourhoods of S and Σ respectively, so $\mathcal{L} = (p_* \iota_* \mathcal{N}_{\Sigma|W})^{-1} = (\pi_* i_* \mathcal{N}_{S|X})^{-1}$. Since f is a crepant resolution we finally have

$$\omega_X = f^* \omega_W = \pi^* (\omega_B \otimes \mathcal{L}). \qquad \square$$

The results on singularities exposed up to now, in particular the third condition in Theorem 2.2, make the proof of the following propositions straightforward. Observe that Theorem 2.7 gives a partial answer to the problem of classifying the non-Kodaira fibres.

Theorem 2.6 *Let $\pi : X \longrightarrow B$ be a smooth minimal elliptic threefold with section, with Weierstrass model W defined by the equation $y^2 z = x^3 + a_4 x z^2 + a_6 z^3$. Then there is no point $b \in B$ such that $\operatorname{mult}_b a_4 \geq 4$ and $\operatorname{mult}_b a_6 \geq 6$.*

Proof Assume that $b \in B$ is a point such that $\operatorname{mult}_b a_4 \geq 4$ and $\operatorname{mult}_b a_6 \geq 6$. From [Rei80, Cor. 2.10] we see that the singular point $(x : y : z) = (0 : 0 : 1)$ in the fibre over b is a rational Gorenstein singularity which is not *cDV*. So W can't have small resolutions, and in particular it can't be the Weierstrass model of a smooth minimal elliptic fibration $\pi : X \longrightarrow B$. $\square$

Theorem 2.7 *Let $\pi : X \longrightarrow B$ be a smooth minimal elliptic threefold with section. If $b \in B$ is a point such that the fibre X_b is of non-Kodaira type, then X_b is a contraction of the Kodaira fibre over b of the elliptic surface obtained restricting X to a generic smooth curve through b.*

Proof Thanks to the Theorem 2.6 we must have $\operatorname{mult}_b a_4 \leq 3$ or $\operatorname{mult}_b a_6 \leq 5$. By [Mir89, Prop. III.3.2], the restriction of W to the generic smooth curve C through b is then an elliptic surface W_C with only Du Val singularities and finally, by Theorem 2.2, the fibre X_b is a contraction of the fibre predicted by the smooth minimal elliptic surface corresponding to W_C. $\square$

Examples of Theorem 2.7 are given in Section 1.3.1 and Section 6.2.3.

Remark 2.2.2 In a discussion, Prof. A. Grassi gave me an alternative proof of Theorem 2.6: in [Mir83] it's described how to desingularize a Weierstrass fibration, under further assumptions on the discriminant locus (e.g. its irreducible components must intersect transversally). If $b \in B$ is a point with $\operatorname{mult}_b a_4 \geq 4$ and $\operatorname{mult}_b a_6 \geq 6$, then following this procedure one has to blow-up the base surface at b, pull-back the fibration, and if necessary blow-up and pull-back again to reach Miranda's hypothesis. In this way we can desingularize the threefold, but this resolution is incompatible with the blow-ups performed on the base, and so it can't be blown-down to give a resolution of the original threefold.

Chapter 3

Calabi–Yau varieties

Calabi–Yau manifolds are the higher dimensional analogues of $K3$ surfaces. They are a class of varieties which are particularly simple, but having a great amount of interesting properties. Even if they are a subject of interest in its own, their study has been recently encouraged by physics, and in particular string theory: the mathematical models of F-theory are in fact all examples of Calabi–Yau varieties.

Definition *A* Calabi–Yau manifold *is a projective smooth manifold X with*

1. *trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$,*
2. *$h^{0,q} = 0$ for $q = 1, \ldots \dim X - 1$, where $h^{p,q} = \dim H^q(X, \Omega^p_X)$.*

Observe that if X is a Calabi–Yau manifold of dimension at least 3, then $\operatorname{Pic} X \simeq \operatorname{NS}(X) \simeq H^2(X, \mathbb{Z})$. This is a well known fact, arising from the long exact sequence induced in cohomology by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

on X. In fact we have

$$0 = H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \simeq \operatorname{Pic} X \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) = 0.$$

Example 3.0.3 If X is a Calabi–Yau variety of dimension 1, then X is a smooth Riemann surface of genus 1. In fact, since $\deg K_X = 2g(X)-2$ we have $g(X) = 1$, and viceversa any genus 1 Riemann surface has trivial canonical bundle. Observe that in this case, for dimensional reasons, the second condition in the definition above is empty.
In the case of dimension 2, X is Calabi–Yau if and only if it's a $K3$: this is due to the classification of surfaces.
In dimension 3, the Fermat quintic in $\mathbb{P}^4$, and in fact any smooth quintic, is a classical example of Calabi–Yau variety (see for instance [GHJ03] and [CK99]). Other Calabi–Yau threefolds which are complete intersections in projective spaces are the complete intersection of two hypersurfaces of degree 3 in $\mathbb{P}^5$, of a hyperquadric and a hypersurface of degree 4 in $\mathbb{P}^5$, of two hyperquadric and a hypercubic in $\mathbb{P}^6$ or the complete intersection of four hyperquadrics in $\mathbb{P}^7$. For other examples of Calabi–Yau manifolds, see e.g. [Hüb92].

Example 3.0.4 Let X be the variety in $\mathbb{P}^3 \times \mathbb{P}^2$ defined in Example 1.2.1, it's a smooth complete intersection defined by two equations of bidegree $(3,0)$ and $(1,3)$. We denote Y the variety of bidegree $(3,0)$, so by adjunction we have

$$K_Y = (K_{\mathbb{P}^3\times\mathbb{P}^2} + Y)_{|_Y} = ((-4,-3)+(3,0))_{|_Y} = (-1,-3)_{|_Y}.$$

Using adjunction again we can then compute the class of a canonical divisor on X. This last is defined as the restriction to Y of a variety of bidegree $(1,3)$ and so its class in the Chow group of codimension 1 cycles in Y is $(1,3)_{|_Y}$. This means that X is an anticanonical variety in Y, but then

$$K_X = (K_Y + X)_{|_X} = 0_{|_X} = 0,$$

and so X is Calabi–Yau.

There are some extremely useful intersection theoretic properties of Calabi–Yau manifolds, which I want to point out here. The first is that the adjunction formula becomes very easy: in fact if $i : Y \hookrightarrow X$ is a smooth hypersurface, then

$$K_Y = i^*(K_X + Y) = i^*Y \tag{3.1}$$

and so using the projection formula we have that the class of the codimension 2 cycle i_*K_Y in X is

$$i_*K_Y = i_*i^*Y = Y \cdot Y. \tag{3.2}$$

This implies that

$$Y^3 = Y^2 \cdot Y = i_*K_Y \cdot Y = i_*(i^*Y \cdot K_Y) = i_*(K_Y^2). \tag{3.3}$$

Also the Riemann–Roch formula becomes easier. In fact, if D is any divisor in a Calabi–Yau threefold X, then

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6}D^3 + \frac{1}{12}D \cdot c_2(X),$$

where $c_2(X)$ denotes the second Chern class of the tangent bundle of X.
The third fact is a result by Friedman ([Fri91, Lemma 4.4]), stating that for any smooth surface V in a Calabi–Yau threefold X the relation

$$V \cdot c_2(X) = \chi_{\text{top}}(V) - K_V^2 \tag{3.4}$$

holds. In particular this is true for a section S, where the right side of the equation is then $\chi_{\text{top}}(B) - K_B^2$.

3.1 Singular Calabi–Yau varieties

For many purposes it's convenient not to restrict only to the case of smooth varieties, but to admit also some class of singularities. In this section I want to give a definition of Calabi–Yau variety which generalizes the one of Calabi–Yau manifold I gave before. The point in the definition of Calabi–Yau manifold is that the canonical line bundle is trivial, and what may fail when we are dealing with singular varieties is that the canonical sheaf is not a line bundle. From Chapter 2, we can bypass this problem requiring the variety to be Gorenstein.

Definition *A* Calabi–Yau variety *is a normal projective variety X such that*

1. *X is Gorenstein;*

2. *$\omega_X \simeq \mathcal{O}_X$;*

3. *$h^{0,q} = 0$ for $q = 1, \dots, \dim X - 1$.*

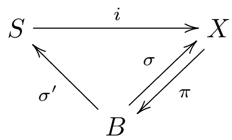
An important class of examples of such singular Calabi–Yau varieties is provided by the Weierstrass models $p : W \longrightarrow B$ of the elliptic fibrations $\pi : X \longrightarrow B$ for which X is a Calabi–Yau manifold.

3.2 Calabi–Yau elliptic threefold

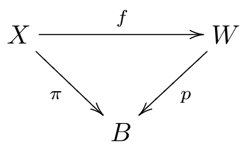
We now want to focus on the elliptic fibrations $\pi : X \longrightarrow B$ for which the total space X is a smooth Calabi–Yau threefold. In this case we can compute the fundamental line bundle of the Weierstrass model of our fibration in an easy way, and we will find that

$$\mathcal{L} = \mathcal{O}_B(-K_B).$$

Let $\pi : X \longrightarrow B$ be an elliptic threefold, with X and B smooth. Suppose also that X is a Calabi–Yau threefold, and that a section σ for π is given. We denote $S = \sigma(B)$ and so we have



We use the section S to put our fibration in Weierstrass form, finding a possibly singular elliptic threefold $p : W \longrightarrow B$ and a birational morphism $f : X \longrightarrow W$ such that $p \circ f = \pi$, so the diagram

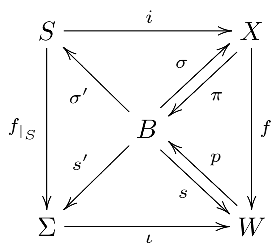


commutes. Recall from Section 2.2 that this construction has two features:

1. the dimension of the singular locus of W is at most 1;

2. we have that $f_{|_{f^{-1}(W^{\text{reg}})}} : f^{-1}(W^{\text{reg}}) \longrightarrow W^{\text{reg}}$ is an isomorphism.

Moreover, W comes with a section $\Sigma = s(B)$ so that



commutes, and we observe that f is an isomorphism between open neighbourhoods of S and Σ.
Using this diagram and this last observation, we have that $p_*\iota_*\mathcal{N}_{\Sigma|W} \simeq \pi_* i_* \mathcal{N}_{S|X}$, and I want to show that

$$\pi_* i_* \mathcal{N}_{S|X} \simeq \omega_B.$$

This follows from the adjunction and projection formulae and the fact that X is Calabi–Yau: since $i_* c_1(\mathcal{N}_{X|X}) = S^2$, using (3.2) we deduce that

$$i_* c_1(\mathcal{N}_{S|W}) = i_* K_S,$$

and so we finally have

$$\pi_* i_* c_1(\mathcal{N}_{S|X}) = \pi_* i_* K_S = \pi_* i_* \sigma'_* K_B = \pi_* \sigma_* K_B = K_B.$$

The fundamental line bundle $\mathcal{L}$ of $p : W \longrightarrow B$, which by definition is the inverse of $p_*\iota_*\mathcal{N}_{\Sigma|W}$, is then

$$\mathcal{L} = \mathcal{O}_B(-K_B).$$

This implies that we can map W isomorphically into the projective bundle

$$\mathbb{P}(\mathcal{O}_B(-2K_B) \oplus \mathcal{O}_B(-3K_B) \oplus \mathcal{O}) \longrightarrow B$$

as a hypersurface. By Section 1.2.1, the class of W in this bundle is $[W] = -6\Pi^* K_B + 3\xi_W = -K_Z$, and so $\omega_W \simeq \mathcal{O}_W$.

Remark 3.2.1 The Calabi–Yau condition can be used to check the results obtained in Section 2.2. In fact, $f : X \longrightarrow W$ is a small resolution of the singularities of W, and so by Zariski's main theorem [Har77, Thm. 11.4] f has connected fibres and then

$$f_*\omega_X = f_*\mathcal{O}_X = \mathcal{O}_W = \omega_W$$

meaning that W has rational Gorenstein singularities. We can also check that f is crepant, in fact

$$f^*\omega_W = f^*\mathcal{O}_W = \mathcal{O}_X = \omega_X.$$

Chapter 4

A finiteness result

Let B be a smooth projective surface, and $\mathcal{L}$ an ample line bundle on B. Having clear in mind that Weierstrass fibrations naturally embed into a projective plane bundle over B, I want now to study the *smooth* Calabi–Yau hypersurfaces in the bundle of projective planes $Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, where a and b are fixed integers. More in detail, I want to study the generic element in the anticanonical system $|-K_Z|$ of Z and to prove the following proposition

Proposition 4.1 *Let B be a smooth projective surface, and $\mathcal{L}$ an ample line bundle on B. Then only for a finite number of pairs (a, b) the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ is a smooth Calabi–Yau elliptic fibration over B.*

I want to recall some facts from the theory of surfaces which I will use in the following. We denote by L the first Chern class of $\mathcal{L}$. By Nakai–Moishezon criterion we have that $L^2 > 0$, so as a consequence of the Hodge index theorem we have that for any divisor D on B the inequality

$$(LD)^2 \geq L^2 D^2$$

holds, and moreover we have equality if and only if $rL = sD$ for some integer r and s.

4.1 The ambient bundle

Let's denote $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}$ and consider the bundle $Z = \mathbb{P}(E)$, so it's not restrictive to assume that $a \geq b \geq 0$. It's easy to see that $c_1(E) = (a + b)L$, and so by (1.13)

$$c_1(Z) = \Pi^* c_1(B) + (a + b)\Pi^* L + 3\xi$$

where $\xi = c_1(\mathcal{O}_Z(1))$ and Π is the bundle projection.
Since we want our variety to have class $-K_Z = c_1(Z)$, we see that it must be defined by a cubic equation in the variables x, y and z on the fibres because of the term 3ξ, say

$$F = \sum_{i+j+k=3} \alpha_{ijk} x^i y^j z^k,$$

and so we see that these varieties are also genus 1 fibrations over B. In this equation the coefficient of each monomial is a section of a line bundle on B according to Table 4.1.

Table 4.1: Weight of the coefficients

Monomial	Coefficient	Weight
x^3	α_{300}	$c_1(B) - (2a - b)L$
x^2y	α_{210}	$c_1(B) - aL$
xy^2	α_{120}	$c_1(B) - bL$
y^3	α_{030}	$c_1(B) + (a - 2b)L$
x^2z	α_{201}	$c_1(B) - (a - b)L$
xyz	α_{111}	$c_1(B)$
y^2z	α_{021}	$c_1(B) + (a - b)L$
xz^2	α_{102}	$c_1(B) + bL$
yz^2	α_{012}	$c_1(B) + aL$
z^3	α_{003}	$c_1(B) + (a + b)L$

4.2 Proof of the proposition

I will divide the proof of Proposition 4.1 in several steps. In the first step I will show that only for a finite number of pairs (a, b) it's not clear if the genus 1 fibrations in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ are elliptic fibrations, i.e. if they admit a section. The second step will reduce the problem of the smoothness of the generic anticanonical hypersurface to a problem concerning only the intersection form on the base. This step will split in two subcases, which will be analysed in the third and fourth step.

4.2.1 Step 1

Since L is an ample divisor, there exists a suitable integer n_0 such that $nL+K_B$ is ample for any $n \geq n_0$. Taking into account the limitation $a \geq b \geq 0$, there is only a finite number of pairs (a, b) such that $2a - b < n_0$: in these cases we have a genus 1 fibration, but since the equation F defining the variety is general, it's difficult to see if there are sections or not.
The other cases satisfy $2a - b \geq n_0$: in this case $(2a - b)L + K_B$ is ample, hence

$$H^0(B, (b - 2a)L - K_B) = H^0(B, -((2a - b)L + K_B)) = 0,$$

and so the coefficient of x^3 is identically 0. So the equation for our variety looks like

$$F = \alpha_{210}x^2y + \alpha_{201}x^2z + \ldots$$

and then we have a section, given by

$$b \longmapsto (1 : 0 : 0) \in X_b.$$

4.2.2 Step 2

We now focus on the infinitely many cases where $2a - b \geq n_0$, so that we can exploit the presence of a section.
Let ∂ be any (local) derivation on B, and observe that

$$(\partial F)_{|(b,(1:0:0))} = 0.$$

Since in a smooth fibration with a section the singularities in the fibres can't lie on the section (Proposition 1.1), the following system must have no solutions

$$\begin{cases} \frac{\partial F}{\partial x}_{|(b,(1:0:0))} & = 0 \\ \frac{\partial F}{\partial y}_{|(b,(1:0:0))} & = \alpha_{210}(b) = 0 \\ \frac{\partial F}{\partial x}_{|(b,(1:0:0))} & = \alpha_{201}(b) = 0. \end{cases} .$$

This is equivalent to require that the curves defined by $\alpha_{210} = 0$ and $\alpha_{201} = 0$ have no common points, i.e. that

$$(c_1(B) - aL)(c_1(B) - (a-b)L) = 0.$$

Observe that now we have a problem concerning only the base and its intersection theoretic properties. Computing the quantity on the left, we find the degree 2 polynomial in (a, b)

$$a(a-b)L^2 + (b - 2a)c_1(B)L + c_1(B)^2.$$

Thinking to $(a, b) \in \mathbb{R}^2$, the equation

$$a(a-b)L^2 + (b - 2a)c_1(B)L + c_1(B)^2 = 0 \tag{4.1}$$

defines a plane conic, which is reducible if and only if

$$L^2 = 0 \qquad \text{or} \qquad (c_1(B)L)^2 = L^2 c_1(B)^2.$$

The first case is impossible since we are assuming that L is ample.
Our next step is then to study the conic defined in (4.1) when it is irreducible and when it is reducible, and to show that in each of these two cases we have only a finite number of integral points (a, b) in the octant $a \geq b \geq 0$.

4.2.3 Step 3

Let's concentrate first on the case when the conic (4.1) is irreducible: it is a hyperbola, with asymptotes

$$a = \frac{c_1(B)L}{L^2} \qquad \text{and} \qquad b = a - \frac{c_1(B)L}{L^2}.$$

The change of variables

$$\begin{cases} a = a' + 2b' \\ b = b' \end{cases}$$

is represented by a matrix in $\mathrm{SL}(2, \mathbb{Z})$, hence preserves the integral lattice in $\mathbb{R}^2$ and the integral points on the hyperbola we are studying. In these new coordinates the equation of hyperbola satisfies the hypothesis of [Zel], so we have both that the number of integral points on the conic is finite and a way to compute them. Using the previous transformation and the (simple) algorithm in [Zel], we have that the integral points of the conic (4.1) are among the following:

$$\begin{aligned} a_i &= \tfrac{\pm 2L^2(c_1(B)L)^2 \mp 2(L^2)^2 c_1(B)^2 + d_i c_1(B)L}{d_i L^2}, \\ b_i &= \pm \tfrac{4(L^2)^2(c_1(B)L)^2 - 4(L^2)^3 c_1(B)^2 - d_i^2}{2d_i(L^2)^2}, \end{aligned}$$

where d_i runs through the (positive) divisors of $4(L^2)^2((c_1(B)L)^2 - c_1(B)^2L^2)$.

4.2.4 Step 4

We concentrate now in the case where the conic (4.1) is reducible, i.e. the case where $(c_1(B)L)^2 = L^2c_1(B)^2$. The equation for the conic (4.1) is

$$(L^2a - c_1(B)L)(L^2a - L^2b - c_1(B)L) = 0.$$

As we said before, in this case (and only in this case) we have $rL = sc_1(B)$ for suitable integer r and s, and so $sc_1(B)L = rL^2$, which is the same as $\frac{c_1(B)L}{L^2} = \frac{r}{s}$: we have then two further subcases according to whether $\frac{r}{s}$ is a positive integer or not.
If $\frac{r}{s} \notin \mathbb{N}$, the two lines

$$a = \frac{c_1(B)L}{L^2} \qquad \text{and} \qquad b = a - \frac{c_1(B)L}{L^2}$$

have no integral points at all. This means that we have no new smooth Calabi–Yau fibrations.
If instead $\frac{r}{s} \in \mathbb{N}$, then in the range $a \geq b \geq 0$ we have a finite number of pairs (a, b) on the line $a = \frac{c_1(B)L}{L^2}$, namely $\frac{c_1(B)L}{L^2} + 1 = \frac{r}{s} + 1$, and an infinite number of (a, b)'s on the line $b = a - \frac{c_1(B)L}{L^2}$. To give a limitation on the number of these last, we look at the coefficient of the first monomials in the equation $F = 0$. They are

Monomial	Weight of the coefficient
x^3	$-(b + \frac{r}{s})L$
x^2y	$-bL$
xy^2	$(\frac{r}{s} - b)L$
y^3	$(2\frac{r}{s} - b)L$

and so if $b - 2\frac{r}{s} > 0$, i.e. $b > 2\frac{r}{s}$, we have that $(b - 2\frac{r}{s})L$ is ample, hence

$$H^0\left(B, \left(2\frac{r}{s} - b\right)L\right) = H^0\left(B, -\left(b - 2\frac{r}{s}\right)L\right) = 0.$$

The same argument applies to the other three in the list since

$$b - 2\frac{r}{s} < b - \frac{r}{s} < b < b + \frac{r}{s}.$$

Hence the coefficients of x^3, x^2y, xy^2 and y^3 are necessarily identically zero, and so the equation F for the variety factors as $F(x,y,z) = z \cdot f(x,y,z)$. Then $F = 0$ can't define a smooth variety, and observe that $z = 0$ defines a divisor whose class is ξ, while $f(x,y,z) = 0$ defines a divisor of class $\Pi^* c_1(B) + (a+b)\Pi^* L + 2\xi$, which is neither a Calabi–Yau variety nor an elliptic fibration.
In particular, we have only a finite number of pairs (a,b) on the line $b = a - \frac{c_1(B)L}{L^2} = a - \frac{r}{s}$ such that the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ could define a Calabi–Yau elliptic fibration over B, and a limitation is

$$\frac{r}{s} \leq a \leq 3\frac{r}{s}, \qquad 0 \leq b \leq 2\frac{r}{s}. \tag{4.2}$$

If $\frac{r}{s} \in \mathbb{N}$ we have then at most

$$3\frac{r}{s} + 1 = \underbrace{\left(\frac{r}{s} + 1\right)}_{\substack{\text{Pairs on the line} \\ a = \frac{c_1(B)L}{L^2}}} + \underbrace{\left(2\frac{r}{s} + 1\right)}_{\substack{\text{Pairs on the line} \\ b = a - \frac{c_1(B)L}{L^2}}} - \underbrace{1}_{\substack{\text{The common case} \\ (a,b) = \left(\frac{c_1(B)L}{L^2}, 0\right)}}$$

such pairs (a,b).

4.2.5 Conclusion

Only for a finite number of pairs (a,b) the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O})$ is a smooth Calabi–Yau elliptic fibration, which completes the proof of Proposition 4.1.
We can summarize the results obtained in the following table.

$(2a-b)L+K_B$ is not ample	$(2a-b)L+K_B$ is ample		
	$(c_1(B)L)^2 \neq c_1(B)^2L^2$	$(c_1(B)L)^2 = c_1(B)^2L^2$	
		$\frac{r}{s} \notin \mathbb{N}$	$\frac{r}{s} \in \mathbb{N}$
Finite number of cases. These are a priori only genus one fibrations. It's not clear if they have at least a section or not.	The conic (4.1) is irreducible, and we have a finite number of cases.	No pairs.	Finite number of cases, at most $3\frac{r}{s}+1$.

4.3 Examples and remarks

I want now to run this program in some cases of interest: the case where the base B is a del Pezzo surface and L is a multiple of an anticanonical divisor, and the case where $B = \mathbb{P}^2$ and L is general.

4.3.1 The case of del Pezzo surfaces

Let B denote a del Pezzo surface and $\mathcal{L}$ a multiple of the anticanonical bundle (this is the most natural setting), say $\mathcal{L} = \omega_B^{-m}$, and observe that $\mathcal{L}$ is ample for any $m \geq 1$. We can assume that $m = 1$, since $\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B = \omega_B^{-ma} \oplus \omega_B^{-mb} \oplus \mathcal{O}_B = \omega_B^{-\alpha} \oplus \omega_B^{-\beta} \oplus \mathcal{O}_B$ with $\alpha = ma$, $\beta = mb$.
The divisor $nL + K_B = -(n-1)K_B$ is ample if $n - 1 \geq 1$, or equivalently if $n \geq 2$. With the notation of Section 4.2.1 we have $n_0 = 2$, and then the pairs (a, b) for which we can't ensure the presence of a section, satisfying $2a - b < n_0$, are

$$(a, b) = (0, 0), (1, 1).$$

For all the other pairs, we are in the case described in Section 4.2.4, so we observe that

$$rL = -sK_B \Longleftrightarrow \frac{r}{s} = 1.$$

The conic (4.1) has then this expression

$$(a - 1)(a - b - 1) = 0$$

and so we see only two points in the first octant on the line $a = 1$, namely $(a, b) = (1, 0), (1, 1)$, while on the line $b = a - 1$ we have to take care of the limitation (4.2) $b \leq 2$ and so we see two other pairs, i.e. $(a, b) = (2, 1)$ and $(a, b) = (3, 2)$. This means that we have at most 5 possibilities for (a, b):

$$(0, 0), \qquad (1, 1), \qquad (1, 0), \qquad (2, 1), \qquad (3, 2).$$

4.3.2 The case of $B = \mathbb{P}^2$

Observe that if B is a smooth surface with $\operatorname{Pic} B \simeq \mathbb{Z}$, then we are necessarily in the case described in Section 4.2.4. Take for example $B = \mathbb{P}^2$, and $L = dl$ for $d \in \mathbb{N}$ and l a line in $\mathbb{P}^2$. Now we compute the least integer n_0 such that $n_0 L + K_{\mathbb{P}^2}$ is ample: we find

$$n_0 = \begin{cases} 4 & \text{if } d = 1 \\ 2 & \text{if } d = 2, 3 \\ 1 & \text{if } d \geq 4, \end{cases}$$

so the cases where we can't apply the Kodaira vanishing theorem (Section 4.2.1), satisfying $2a - b < n_0$, are

$$\begin{array}{ll} (0,0), (1,0), (1,1), (2,1), (2,2), (3,3) & \text{if } d = 1 \\ (0,0), (1,1) & \text{if } d = 2, 3 \\ (0,0) & \text{if } d \geq 4 \end{array}.$$

Since $c_1(\mathbb{P}^2) = 3l$, we have

$$rdl = 3sl \Longleftrightarrow rd = 3s \Longleftrightarrow \frac{r}{s} = \frac{3}{d}.$$

This means that we have only two cases in which the ratio $\frac{r}{s}$ is an integer, and correspond to

$$d = 1 \qquad \text{and} \qquad d = 3,$$

i.e. when $L = l$ and $L = -K_{\mathbb{P}^2}$. For all the other cases, the only possible pair is then $(a, b) = (0, 0)$, with the exception of $L = 2l$, which has also $(a, b) = (1, 1)$. For $d = 3$, there are five possibilities: $\mathbb{P}^2$ is a del Pezzo surface and $d = 3$ corresponds to $L = -K_{\mathbb{P}^2}$, and so we can use the results of the previous section.

d	Possible (a, b)'s
2	$(0,0)$, $(1,1)$
3	$(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$, $(2,3)$
≥ 4	$(0,0)$

The only case left is $d = 1$ in the situation of Section (4.2.4). We have only to count the integral points on the conic

$$(a - 3)(a - b - 3) = 0$$

which are in the first octant and having $b \leq 6$ (estimate (4.2)): on the first line we have the points $(3, 2)$, $(3, 1)$ and $(3, 0)$, while on the second the points $(4, 1)$, $(5, 2)$, $(6, 3)$, $(7, 4)$, $(8, 5)$ and $(9, 6)$.
Then the pairs (a, b) such that the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2})$ could be a smooth elliptic fibration are the following 15:

$$\begin{array}{cccccc} (0,0), & (1,0), & (1,1), & (2,1), & (2,2), & (3,3), \\ & (3,2), & (3,1), & (3,0), & & \\ (4,1), & (5,2), & (6,3), & (7,4), & (8,5), & (9,6). \end{array}$$

4.3.3 Remarks

I want to stress the fact that we proved that the number of genus 1 fibrations whose total space is smooth lie in a finite number of $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, but we don't know if all of them are elliptic fibrations. In the finite number of cases detected in Section 4.2.1 it's not clear if there is at least a section; we will see this fact in Chapter 5, where we will analyse more in detail the 15 families over $\mathbb{P}^2$.

4.4 Chern classes

We want to compute the Chern classes of $X \in |-K_Z|$. We have $i : X \hookrightarrow Z$, and the normal bundle sequence

$$0 \longrightarrow T_X \longrightarrow i^*T_Z \longrightarrow \mathcal{N}_{X|Z} \longrightarrow 0,$$

which gives the following relation between the Chern polynomials

$$i^*c(Z) = c(X)c(\mathcal{N}_{X|Z}) = c(X)i^*(1 - K_Z).$$

The Chern polynomial of Z was computed in Remark 1.2.4 (p. 17), and $(1-K_Z)$ has as formal inverse

$$(1-K_Z)^{-1} = 1 + K_Z + K_Z^2 + K_Z^3 + K_Z^4.$$

Then

$$c(X) = i^* \left(\frac{c(T_{Z|B})}{1-K_Z} \Pi^* c(B) \right), \tag{4.3}$$

and an explicit computation shows that

1. the first Chern class of X vanishes, as expected since X is Calabi–Yau;
2. the second Chern class of X is the restriction to X of

$$3\xi^2 + (2\Pi^* c_1(E) + 3\Pi^* c_1(B))\xi + \Pi^*(c_1(E)c_1(B) + c_2(E) + c_2(B));$$

3. the third Chern class of X is the restriction to X of

$$-9\Pi^* c_1(B)\xi^2 - \Pi^*(2c_1(E)^2 + 6c_1(E)c_1(B) + 3c_1(B)^2 - 6c_2(E))\xi,$$

which is a codimension 3 cycle on X whose degree is

$$-6c_1(E)^2 - 18c_1(B)^2 + 18c_2(E). \tag{4.4}$$

In our case, with $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B$, the Chern classes of the vector bundle are then

$$\begin{aligned} c_1(E) &= (a+b)L, \\ c_2(E) &= abL^2, \\ c_3(E) &= 0, \end{aligned}$$

so we have the following formulae for the Chern classes of X:

$$\begin{aligned} c_1(X) = {} & 0, \\ c_2(X) = {} & 3\xi_{|X}^2 + \pi^*(2(a+b)L + 3c_1(B))\xi_{|X} + \\ & + \pi^*((a+b)Lc_1(B) + abL^2 + c_2(B)), \\ c_3(X) = \chi_{\text{top}}(X) = {} & -6(a^2 - ab + b^2)L^2 - 18c_1(B)^2. \end{aligned} \tag{4.5}$$

Chapter 5

Classification over $\mathbb{P}^2$

In this chapter I want to use the results in Chapter 4 and Appendix A to give a detailed description of the elliptic threefolds X which are anticanonical hypersurfaces in the projective bundle $Z = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2})$ over $\mathbb{P}^2$.[1] We have coordinates in the fibres of this bundles, which we call x, y and z; in particular (compare with (1.8))

$$\begin{aligned} &x \in H^0(Z, \mathcal{O}_Z(1) \otimes \Pi^*\mathcal{O}(a)),\\ &y \in H^0(Z, \mathcal{O}_Z(1) \otimes \Pi^*\mathcal{O}(b)),\\ &z \in H^0(Z, \mathcal{O}_Z(1)). \end{aligned} \tag{5.1}$$

We let $\xi = c_1(\mathcal{O}_Z(1))$, and since z is a section in $H^0(Z, \mathcal{O}_Z(1))$ then the class of the divisor $z = 0$ is ξ. We will use this fact to compute the Chern classes of anticanonical subvarieties, as shown in formula (4.5), and refer to ξ as an hyperplane section of Z.
In Section 4.3.2 we saw that a generic cubic equation

$$F = \sum_{i+j+k=3} \alpha_{ijk} x^i y^j z^k$$

defines a smooth Calabi–Yau elliptic threefold X in Z only if the pair (a, b) is one of the following 15

$$\begin{array}{cccccc} (0,0), & (1,0), & (1,1), & (2,1), & (2,2), & (3,3), \\ & & (3,2), & (3,1), & (3,0), & \\ & & (4,1), & (5,2), & (6,3), & \\ & & (7,4), & (8,5), & (9,6). & \end{array}$$

The coefficients α_{ijk} appearing in F are homogeneous polynomials of suitable degree, according to Table 4.1, p. 33. In the first 6 cases the polynomial F is general, so there is a priori no obvious section, while for the other cases $b \longmapsto (1:0:0) \in X_b$ defines a section, since α_{300} vanishes identically. In the Table 5.1 there are the degrees of the coefficients, computed using Table 4.1, and an empty cell means that the corresponding coefficient is necessarily identically 0.

[1] In the rest of this chapter I will omit the subscript $\mathbb{P}^2$.

Table 5.1: Degrees of the coefficients

(a,b)	α_{300}	α_{210}	α_{120}	α_{030}	α_{201}	α_{111}	α_{021}	α_{102}	α_{012}	α_{003}
$(0,0)$	3	3	3	3	3	3	3	3	3	3
$(1,0)$	1	2	3	4	2	3	4	3	4	4
$(1,1)$	2	2	2	2	3	3	3	4	4	5
$(2,1)$	0	1	2	3	2	3	4	4	5	6
$(2,2)$	1	1	1	1	3	3	3	5	5	7
$(3,3)$	0	0	0	0	3	3	3	6	6	9
$(3,2)$		0	1	2	2	3	4	5	6	8
$(3,1)$		0	2	4	1	3	5	4	6	7
$(3,0)$		0	3	6	0	3	6	3	6	6
$(4,1)$			2	5	0	3	6	4	7	8
$(5,2)$			1	4	0	3	6	5	8	10
$(6,3)$			0	3	0	3	6	6	9	12
$(7,4)$				2	0	3	6	7	10	14
$(8,5)$				1	0	3	6	8	11	16
$(9,6)$				0	0	3	6	9	12	18

Using the algorithm described in Appendix A, we can compute the Hodge numbers of the anticanonical subvarieties of Z: since these are Calabi–Yau threefolds, the only interesting Hodge numbers are $h^{1,1}$ and $h^{2,1}$. In Table 5.2 I will list the following results:

1. $h^{1,1}(X)$, which equals $\operatorname{rk}\operatorname{Pic} X$ as observed in Chapter 3;
2. $h^{2,1}(X)$, which by Bogomolov–Tian–Todorov theorem is the dimension of the space of deformations of X;
3. the Euler–Poincaré characteristic of X, $\chi_{\text{top}}(X) = 2(h^{1,1}(X) - h^{2,1}(X))$. Observe that these results agrees with formula (4.5);
4. the rank of the Mordell–Weil group of X in the generic case, an empty cell means that X has no section.

Table 5.2: Hodge numbers

(a,b)	$h^{1,1}(X)$	$h^{2,1}(X)$	$\chi_{\text{top}}(X)$	$\operatorname{rk}\operatorname{MW}(X)$	Chapter
$(0,0)$	2	83	-162		5.2
$(1,0)$	2	86	-168		5.2
$(1,1)$	2	86	-168		5.2
$(2,1)$	2	92	-180		5.2
$(2,2)$	2	95	-186		5.2

$(3,3)$	4	112	-216	2	5.3
$(3,2)$	3	105	-204	1	5.4
$(3,1)$	3	105	-204	1	5.4
$(3,0)$	3	111	-216	1	5.4, 6.1, 6.5
$(4,1)$	3	123	-240	1	5.5
$(5,2)$	3	141	-276	1	5.5
$(6,3)$	3	165	-324	1	5.5, 6.3, 6.4
$(7,4)$	3	195	-384	0	5.6
$(8,5)$	3	231	-456	0	5.6
$(9,6)$	2	272	-540	0	5.7

This chapter is devoted to a case by case analysis of these 15 families. In particular, I will give the Weierstrass equation of each family with at least a section, compute the intersection form on the Picard group and describe (rational) sections which are Mordell–Weil generators. I will also check that all the deformations of our X's come from deformations of the defining polynomial F, and in some case I will give also canonical equations.
The sections of a fibration (if any) will be denoted by S or by $S_0, S_1, \ldots$, in this last case the Weierstrass equation will be computed with respect to S_0. The pull-back to X of a line l in $\mathbb{P}^2$ will be denoted by L, then L^2 is the class of a fibre.
Observe from Table 5.2 that $h^{1,1}(X) \geq 2$ for all the families we are considering. This is not a case since $\operatorname{rk}\operatorname{Pic} Z = 2$ and I will now show that the restriction map $i^* : \operatorname{Pic} Z \longrightarrow \operatorname{Pic} X$ is injective.

Proposition 5.1 *Let X be a smooth elliptic threefold which is an anticanonical divisor in $Z = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$, and denote by $i : X \hookrightarrow Z$ the inclusion. Then the restriction map*

$$i^* : \operatorname{Pic} Z \longrightarrow \operatorname{Pic} X$$

is injective.

Proof The Picard group of Z has rank 2 and is generated by $\Pi^* l$ and ξ (compare to Appendix A.1.2). The first of these divisors restricts to L, which is a non-trivial vertical divisor with $L^3 = 0$. It's then enough to show $i^*\xi^3 \neq 0$ to prove the linear independence of L and $i^*\xi$. The degree of $i^*\xi^3$ is the same as the degree of $i_* i^* \xi^3 = \xi^3 X$, and this cycle in Z is

$$\xi^3 X = \xi^3((3 + a + b)L + 3\xi) = (2a^2 + ab + 2b^2 - 3a - 3b)L^2\xi^2,$$

where $X = (3 + a + b)L + 3\xi$ was computed in (1.13), and we used the relation $\xi^3 + (a + b)L\xi^2 + abL^2\xi = 0$, which holds by the very definition of the Chern classes of Z. It's then straightforward to check that

$$i^*\xi^3 = 2a^2 + ab + 2b^2 - 3a - 3b$$

does not vanish when the pair (a, b) is one of the 15 we are dealing with. □

Remark 5.0.1 Actually, in the case $(a, b) = (0, 0)$ we find $i^*\xi^3 = 0$. To see that L and $i^*\xi$ are independent also in this case, we can use [Mir89, Lemma VII.1.4] and the fact that $i^*\xi$ is a divisor which intersects the generic fibre in three points to deduce that $i^*\xi$ is not linearly equivalent to a vertical divisor. This implies that L and $\xi_{|X}$ are linearly independent.

This result is interesting and also non-trivial, since we can't apply Lefschetz hyperplane theorem [Laz04, Thm. 3.1.17]: in fact our X's are not ample, since Z is typically not Fano.
Observe also that $\operatorname{rk}\operatorname{MW}(X) = h^{1,1}(X) - 2$ except for the cases $(a, b) = (7, 4), (8, 5)$. According to the Shioda–Tate–Wazir formula (Theorem 1.3), this means that the generic fibration in each family has no vertical divisors, except in these two cases where we expect to have a vertical divisor.

5.1 The number of parameters

In this section I want to prove the following result, concerning the number of moduli of the 15 families of Calabi–Yau elliptic fibrations I'm going to study. Roughly speaking, I want to show that all the deformations of a fibration $X \subseteq \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$ can be realized by deformations of the defining equation $F = 0$ for X. I will also give a practical algorithm to compute $h^{2,1}(X)$, which by Bogomolov–Tian–Todorov theorem is the dimension of the space of deformations of X.

Proposition 5.2 *Let X be an anticanonical hypersurface in $Z = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$. Then all the deformations of X comes from deformations of X inside Z. More precisely,*

$$h^{2,1}(X) = n(-K_Z) - 8 - n_{Z|\mathbb{P}^2},$$

where $n(-K_Z) = h^0(Z, -K_Z) - 1$ is the number of parameters for the generic equation of a threefold in the family, and $n_{Z|\mathbb{P}^2}$ is the dimension of the group $\operatorname{Aut}_{\mathbb{P}^2}(Z)$ of the automorphisms of Z over $\mathbb{P}^2$, i.e. those automorphisms $Z \longrightarrow Z$ such that

$$\begin{array}{ccc} Z & \longrightarrow & Z \\ & {\scriptstyle \Pi}\searrow \quad \swarrow{\scriptstyle \Pi} & \\ & \mathbb{P}^2 & \end{array}$$

commutes.

In fact, from Remark A.3.1 we have the exact sequence

$$0 \longrightarrow H^0(Z, \mathcal{T}_Z) \longrightarrow H^0(X, \mathcal{N}_{X|Z}) \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow 0,$$

which implies that

$$h^1(X, \mathcal{T}_X) = h^0(X, \mathcal{N}_{X|Z}) - h^0(Z, \mathcal{T}_Z).$$

Since sections in $H^1(X, \mathcal{T}_X)$ and in $H^0(X, \mathcal{N}_{X|Z})$ correspond to first order deformations of X and to first order deformations of X inside Z respectively, the first part of the proposition follows from the surjectivity of the last map in the

sequence.
By (A.1) we have $h^1(X, \mathcal{T}_X) = h^{2,1}(X)$ (which is the number of moduli of X), and if F is an equation for $X \subseteq Z$, then F depends on $h^0(Z, -K_Z)$ linear parameters. Finally, by [BC94, Rmk. 12.6] $h^0(Z, \mathcal{T}_Z)$ is the dimension of the group $\operatorname{Aut} Z$ of automorphisms of Z. In Appendix A, I also compute $h^0(X, \mathcal{N}_{X|Z})$ and $h^0(Z, \mathcal{T}_Z)$, finding that

$$\begin{aligned} h^0(X, \mathcal{N}_{X|Z}) &= h^0(Z, -K_Z) - 1, \\ h^0(Z, \mathcal{T}_Z) &= \textstyle\sum_{i=1}^6 h^0(Z, D_j) - 2, \end{aligned}$$

where $D_1, \ldots, D_6$ are the torus-invariant divisors of Z.
We can obviously write

$$h^0(Z, \mathcal{T}_Z) = \sum_{i=1}^{6} h^0(Z, D_j) - 2 = \left(\sum_{i=1}^{3} h^0(Z, D_j) - 1 \right) + \left(\sum_{i=4}^{6} h^0(Z, D_j) - 1 \right),$$

and since from Appendix A.1.2 the divisors D_1, D_2 and D_3 have class $\Pi^* l$, then $\sum_{i=1}^3 h^0(Z, D_j) - 1$ is the contribution of the automorphism group of the base $\mathbb{P}^2$, i.e. $\mathrm{PGL}(3, \mathbb{C})$. In fact (see Appendix A.2.4)

$$\sum_{i=1}^{3} h^0(Z, D_j) - 1 = 3 + 3 + 3 - 1 = 8.$$

It's also possible to see that $\sum_{i=4}^6 h^0(Z, D_j) - 1$ is the dimension of the group of automorphisms of Z acting fibrewise. We conclude that we can compute $h^{2,1}(X)$ as follows:

1. compute the number $n(-K_Z)$ of parameters of the generic anticanonical hypersurface in $Z = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$;
2. compute the dimension $n_{Z|\mathbb{P}^2}$ of the group $\operatorname{Aut}_{\mathbb{P}^2}(Z)$;
3. then $h^{2,1}(X) = n(-K_Z) - 8 - n_{Z|\mathbb{P}^2}$.

I will give an example on how this algorithm works in Section 6.1.1.

5.2 The families $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$, $(2,2)$

5.2.1 The cubic form

As it's clear from Table 5.2, all these families have $\operatorname{rk} \operatorname{Pic} X = 2$ and so, by Proposition 5.1, $L = \pi^* l$ and $R = \xi_{|X}$ are a $\mathbb{Q}$-basis for $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}$.[2]
Let's begin with a more precise description of R. By its definition it is given by

$$R : \left\{ \begin{array}{l} F = 0 \\ z = 0 \end{array} \right. \longrightarrow \left\{ \begin{array}{l} \alpha_{300} x^3 + \alpha_{210} x^2 y + \alpha_{120} x y^2 + \alpha_{030} y^3 = 0 \\ z = 0, \end{array} \right.$$

and in general the polynomial equation $\alpha_{300} x^3 + \alpha_{210} x^2 y + \alpha_{120} x y^2 + \alpha_{030} y^3 = 0$ has three distinct solutions. This number decreases when its discriminant

[2] By abuse of language, I will simply say that a set of divisors is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$.

vanishes, i.e. over the curve

$$C : \alpha_{210}^2\alpha_{120}^2 - 4\alpha_{300}\alpha_{120}^3 - 4\alpha_{210}^3\alpha_{030} + 18\alpha_{300}\alpha_{210}\alpha_{120}\alpha_{030} - 27\alpha_{300}^2\alpha_{030}^2 = 0, \quad (5.2)$$

Then $R = X \cap \{z = 0\}$ defines a $3 : 1$ covering of $\mathbb{P}^2$ branched along this curve, which is in general singular since it's the discriminant of a cubic polynomial.
Now we begin the calculation of the intersection numbers. Since R is a $3 : 1$ covering of the base, then $L^2R = 3$. So we have the intersection numbers

	L	R
L^2	0	3
LR	3	?

and since $\det \begin{pmatrix} 0 & 3 \\ 3 & ? \end{pmatrix} = -9 \neq 0$, we have a confirmation that $\{L, R\}$ is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$.
Now we want to calculate the intersection numbers LR^2 and R^3. Let $i : X \hookrightarrow Z$ be the inclusion, and $\Pi : Z \longrightarrow \mathbb{P}^2$ the bundle projection; then the fibration is $\pi = \Pi \circ i$, and so letting $\Lambda = \Pi^* l$ then $L = i^*\Lambda$ and $R = i^*\xi$.
This implies that

$$LR^2 = i^*(\Lambda\xi^2)$$
$$R^3 = i^*\xi^3$$

and so using the projection formula and (1.13)

$$i_*(LR^2) = \Lambda\xi^2 X = 3 - 2a - 2b$$
$$i_R^3 = \xi^3 X = 2a^2 + ab + 2b^2 - 3a - 3b$$

The whole intersection table is then

	L	R
L^2	0	3
LR	3	$3 - 2a - 2b$
R^2	$3 - 2a - 2b$	$2a^2 + ab + 2b^2 - 3a - 3b$

and we conclude that the cubic form is given by

(a, b)	$(\alpha L + \beta R)^3$
$(0, 0)$	$9\alpha\beta(\alpha + \beta)$
$(1, 0)$	$\beta(9\alpha^2 + 3\alpha\beta - \beta^2)$
$(1, 1)$	$\beta(9\alpha^2 - 3\alpha\beta - \beta^2)$

$(2,1)$	$3\beta(3\alpha^2 - 3\alpha\beta + \beta^2)$
$(2,2)$	$\beta(9\alpha^2 - 15\alpha\beta + 8\beta^2)$

5.2.2 Sections

As pointed out before, the presence of all the coefficients in F make it difficult to decide whether or not we have sections. We can use the results on the intersection form to solve this problem. In fact, any section S satisfies the following intersection theoretic requirements:

1. $L^2 S = 1$, since L^2 is the class of the generic fibre;
2. $c_2(X)S = -6$ by Friedman's result (3.4);
3. $S^3 = 9$, since $S^3 = K^2_{\mathbb{P}^2}$ by (3.3).

Let $\alpha L + \beta R$ be the class of a section, with $\alpha, \beta \in \mathbb{Q}$. Then we have

$$1 = L^2(\alpha L + \beta R) = \beta L^2 R = 3\beta,$$

which implies that $\beta = \frac{1}{3}$. The second condition gives a linear relation among α and β, since the pairing with $c_2(X)$ defines a linear form on $\operatorname{Pic} X$. Using (4.5) we can compute $c_2(X)$, and thanks to the intersection table above we can compute $c_2(X)(\alpha L + \beta R)$. The results are:

(a,b)	$c_2(X)(\alpha L + \beta R)$
$(0,0)$	$-18\alpha - 18\beta$
$(1,0)$	$-18\alpha - 10\beta$
$(1,1)$	$-18\alpha - 4\beta$
$(2,1)$	$-18\alpha + 6\beta$
$(2,2)$	$-18\alpha + 14\beta$

Since $\beta = \frac{1}{3}$, we can compute α using the second condition. Then computing $(\alpha L + \beta R)^3$ we can check if the result is 9 or not. We have

(a,b)	(α,β)	$(\alpha L + \beta R)^3$
$(0,0)$	$\left(0, \frac{1}{3}\right)$	0
$(1,0)$	$\left(\frac{4}{27}, \frac{1}{3}\right)$	$\frac{19}{243}$
$(1,1)$	$\left(\frac{7}{27}, \frac{1}{3}\right)$	$\frac{19}{243}$
$(2,1)$	$\left(\frac{4}{9}, \frac{1}{3}\right)$	$\frac{7}{27}$
$(2,2)$	$\left(\frac{16}{27}, \frac{1}{3}\right)$	$\frac{88}{243}$

We proved that the generic family in all of these cases has no sections.

5.3 The family $(3,3)$ and the E_6 family

The polynomial defining a generic fibration of type $(3,3)$, i.e. an anticanonical hypersurface in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(3) \oplus \mathcal{O})$, is of the form

$$F = \alpha_{300}x^3 + \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 + z(\ldots),$$

with $\alpha_{300}, \alpha_{210}, \alpha_{120}, \alpha_{030} \in \mathbb{C}$, so intersecting X with the hypersurface of Z given by $z = 0$ we fall into *one and only one* of the following possibilities:

1. the equation $\alpha_{300}x^3 + \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 = 0$ has one solution, i.e. $z = 0$ defines a section in X,
2. the equation $\alpha_{300}x^3 + \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 = 0$ has two solutions, i.e. $z = 0$ defines two sections in X,
3. the equation $\alpha_{300}x^3 + \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 = 0$ has three solutions, i.e. $z = 0$ defines three sections in X.

It's easy to see that the first two cases lead to singular X's, so we discard them and concentrate only on the third: such family is known in the literature as E_6 family (see e.g. [AE10]).
The fact that we have three *distinct* sections allows us to find a simple canonical form. The sections are given by

$$\begin{cases} F = 0 \\ z = 0 \end{cases} \longrightarrow \begin{cases} \alpha_{300}x^3 + \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 = 0 \\ z = 0, \end{cases}$$

and we call the three solution of the first equation $(x_0 : y_0)$, $(x_1 : y_1)$ and $(x_2 : y_2)$. Then the homography

$$\begin{cases} x = x_1(y_0x_2 - x_0y_2)x' + x_0(x_2y_1 - x_1y_2)y' \\ y = y_1(y_0x_2 - x_0y_2)x' + y_0(x_2y_1 - x_1y_2)y' \\ z = z' \end{cases}$$

puts these sections in a standard position, i.e. sends

$$\begin{array}{l} \mathbb{P}^2 \times \{(x_0 : y_0 : 0)\} \longmapsto S_0 = \mathbb{P}^2 \times \{(0 : 1 : 0)\} \\ \mathbb{P}^2 \times \{(x_1 : y_1 : 0)\} \longmapsto S_1 = \mathbb{P}^2 \times \{(1 : 0 : 0)\} \\ \mathbb{P}^2 \times \{(x_2 : y_2 : 0)\} \longmapsto S_2 = \mathbb{P}^2 \times \{(1 : -1 : 0)\} \end{array}$$

and so allows us to assume that the equation of the family X looks like (using not-primed letters)

$$x^2y + xy^2 + \alpha_{201}x^2z + \alpha_{111}xyz + \alpha_{021}y^2z + \alpha_{102}xz^2 + \alpha_{012}yz^2 + \alpha_{003}z^3.$$

Applying finally the homography

$$\begin{cases} x = x' - \alpha_{021} z' \\ y = y' + \alpha_{201} z' \\ z = z' \end{cases}$$

we obtain the canonical form of an E_6 family (I use not-primed variables again):

$$x^2 y + xy^2 + \alpha_{111} xyz + \alpha_{102} xz^2 + \alpha_{012} yz^2 + \alpha_{003} z^3 = 0. \tag{5.3}$$

Observe that this canonical form is not the same as that of the E_6 fibrations described in [AE10, Sect. 1.7], but it's equivalent to this last: the difference is in the choice of the "standard position" for the three sections.

5.3.1 The cubic form of the E_6 family

To compute the cubic form, we can assume that X is in canonical form (5.3)

$$X : F = x^2 y + xy^2 + \alpha_{111} xyz + \alpha_{102} xz^2 + \alpha_{012} yz^2 + \alpha_{003} z^3 = 0.$$

In this case, the intersection of X and the divisor $z = 0$ in Z gives *three distinct sections*, namely

$$\begin{cases} F = 0 \\ z = 0 \end{cases} \longrightarrow \begin{array}{l} S_0 = \mathbb{P}^2 \times \{(0:1:0)\} \\ S_1 = \mathbb{P}^2 \times \{(1:0:0)\} \\ S_2 = \mathbb{P}^2 \times \{(1:-1:0)\} \end{array}$$

I will now show that L and the three sections S_0, S_1 and S_2 give a $\mathbb{Q}$-basis for $\operatorname{Pic} X$. Since each S_i is a section, isomorphic to $\mathbb{P}^2$, we have that $L^2 S_i = 1$ and $S_i^3 = 9$ by (3.3). Observe also that $LS_i = L_{|S_i}$ is the class of a line in the section S_i: this means that[3]

$$LS_i^2 = LS_i \cdot S_i = (L_{|S_i})(S_{|S_i}) = -3,$$

because $S_{|S_i}$ is the class of a canonical divisor in S by (3.2). Finally, if $i \neq j$ then S_i is disjoint from S_j, hence $S_i S_j = 0$. We have then the intersection table of our variety:

	L	S_0	S_1	S_2
L^2	0	1	1	1
LS_0	1	-3	0	0
LS_1	1	0	-3	0
LS_2	1	0	0	-3
S_0^2	-3	9	0	0
$S_0 S_1$	0	0	0	0

[3] Observe the abuse of notation: to be precise, by the projection formula for the inclusion map $j_i : S_i \hookrightarrow X$ we have $LS_i^2 = LS_i \cdot S_i = j_{i*}((L_{|S_i})(S_{|S_i})) = j_{i*}(-3L^2_{|S_i})$, whose degree is then $\deg LS_i^2 = -3$.

S_0S_2	0	0	0	0
S_1^2	−3	0	9	0
S_1S_2	0	0	0	0
S_2^2	−3	0	0	9

Observe that the matrix obtained considering the first 4 rows of the previous table has non vanishing determinant:

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -3 \end{pmatrix} = -27 \neq 0,$$

this means that L, S_0, S_1 and S_2 are $\mathbb{Q}$-linearly independent, hence by Table 5.2 that they are a $\mathbb{Q}$-basis.
The cubic form is then

$$(\alpha L + \beta_0 S_0 + \beta_1 S_1 + \beta_2 S_2)^3 = \\ = 3\alpha^2(\beta_0 + \beta_1 + \beta_2) - 9\alpha(\beta_0^2 + \beta_1^2 + \beta_2^2) + 9(\beta_0^3 + \beta_1^3 + \beta_2^3).$$

Observe that after the change of coordinates

$$\begin{cases} \alpha = A \\ \beta_0 = B_0 + \frac{1}{3}A \\ \beta_1 = B_1 + \frac{1}{3}A \\ \beta_2 = B_2 + \frac{1}{3}A \end{cases}$$

the expression of the cubic form becomes basically a Fermat cubic:

$$(AL + B_0 S_0 + B_1 S_1 + B_2 S_2)^3 = A^3 + 9B_0^3 + 9B_1^3 + 9B_2^3.$$

5.3.2 The number of sections of the E_6 family

Thanks to the intersection table computed in the previous section, we can now show that the three sections S_0, S_1 and S_2 are the only sections of this type of fibrations.
Let $\alpha L+\beta_0 S_0+\beta_1 S_1+\beta_2 S_2$ be the class of a section. In Section 5.2.2 we discussed three intersection theoretic properties of a section, which give the system

$$\begin{cases} L^2(\alpha L + \beta_0 S_0 + \beta_1 S_1 + \beta_2 S_2) = 1 \\ (\alpha L + \beta_0 S_0 + \beta_1 S_1 + \beta_2 S_2)^3 = 9 \\ c_2(X) \cdot (\alpha L + \beta_0 S_0 + \beta_1 S_1 + \beta_2 S_2) = -6. \end{cases}$$

Using (4.5), we can compute that

$$c_2(X) = 3(S_0^2 + S_1^2 + S_2^2) + 21L(S_0 + S_1 + S_2) + 30L^2;$$

and so the previous system reduces to

$$\begin{cases} \beta_0+\beta_1+\beta_2=1 \\ 3\alpha^2(\beta_0+\beta_1+\beta_2)-9\alpha(\beta_0^2+\beta_1^2+\beta_2^2)+9(\beta_0^3+\beta_1^3+\beta_2^3)=9 \\ 36\alpha-6\beta_0-6\beta_1-6\beta_2=-6. \end{cases}$$

This last has as solution

$$\begin{cases} \alpha=0 \\ \beta_0=1-\beta_1-\beta_2 \\ (\beta_1-1)(\beta_2-1)(\beta_1+\beta_2)=0 \end{cases}$$

and so the class of a section can be only one of the following three:

Type 0	$S_0+\beta S_1-\beta S_2$
Type 1	$\beta S_0+S_1-\beta S_2$
Type 2	$\beta S_0-\beta S_1+S_2$

I now want to show that if we consider a class of type 0 which is the class of a section, then $\beta=0$, so that we obtain the class of S_0. The same argument for the other two types then shows that S_0, S_1 and S_2 are the only classes to which a section can belong. First of all observe that

$$(S_0+\beta S_1-\beta S_2)^2=S_0^2+\beta^2S_1^2+\beta^2S_2^2,$$

and so

$$K_{\mathbb{P}^2}=\pi_*((S_0+\beta S_1-\beta S_2)^2)=\pi_*(S_0^2+\beta^2S_1^2+\beta^2S_2^2)=(1+2\beta^2)K_{\mathbb{P}^2}.$$

Since $\operatorname{Pic}\mathbb{P}^2=\mathbb{Z}$ we can deduce that $\beta=0$. The last thing to do is to show that in each linear system $|S_i|$ there is only one element: assume that D_i is an effective divisor linearly equivalent to S_i, but different from S_i, then

$$0\le D_iS_iL=S_i^2L=-3,$$

which is absurd.

5.3.3 The Mordell–Weil group of the E_6 family

Thanks to the presence of at least one section, we can calculate the Weierstrass form of a Calabi–Yau elliptic fibration X of the $(3,3)$ family. After we put X in canonical form (5.3), we choose S_0 as 0-section, and using the methods described in [Cas91, Chap. 8] we find that the Weierstrass model has equation $y^2z=x^3+Axz^2+Bz^3$ with

$$\begin{aligned} A=\;& -\tfrac{1}{3}\alpha_{111}^4+\tfrac{8}{3}\alpha_{111}^2\alpha_{102}+\tfrac{8}{3}\alpha_{111}^2\alpha_{012}-\tfrac{16}{3}\alpha_{102}^2+ \\ & +\tfrac{16}{3}\alpha_{102}\alpha_{012}-\tfrac{16}{3}\alpha_{012}^2+8\alpha_{111}\alpha_{003}; \\ B=\;& \tfrac{2}{27}\alpha_{111}^6-\tfrac{8}{9}\alpha_{111}^4\alpha_{102}-\tfrac{8}{9}\alpha_{111}^4\alpha_{012}+\tfrac{32}{9}\alpha_{111}^2\alpha_{102}^2+\tfrac{16}{9}\alpha_{111}^2\alpha_{102}\alpha_{012}+ \\ & +\tfrac{32}{9}\alpha_{111}^2\alpha_{012}^2+\tfrac{8}{3}\alpha_{111}^3\alpha_{003}-\tfrac{128}{27}\alpha_{102}^3+\tfrac{64}{9}\alpha_{102}^2\alpha_{012}+\tfrac{64}{9}\alpha_{102}\alpha_{102}\alpha_{012}^2+ \\ & -\tfrac{128}{27}\alpha_{012}^3-\tfrac{32}{3}\alpha_{111}\alpha_{102}\alpha_{003}-\tfrac{32}{3}\alpha_{111}\alpha_{012}\alpha_{003}+16\alpha_{003}^2. \end{aligned}$$

It's then possible to compute the discriminant locus $\Delta = 4A^3 + 27B^2$, which is an irreducible curve over which we have nodal fibres.
Using the description of $\operatorname{Pic} X$ given in the previous section and the exact sequences (1.3) and (1.5), we have that the two sections S_1 and S_2 give a set of $\mathbb{Q}$-generators[4] for $\mathrm{MW}(X)$, which then has rank 2.
Observe that this is the rank of the Mordell–Weil group of the *generic* E_6 fibration, in fact we can specialize the equation to find subfamilies of type E_6 with rank of the Mordell–Weil group 1 or 0. The tangent line to the fibre X_b in $(0 : 1 : 0)$ is $x = 0$, this line intersects the fibre in another point, having coordinates $(0 : -\alpha_{003}(b) : \alpha_{012}(b))$. For the generic choice of the coefficients α_{012} and α_{003}, this point is different from $(0 : 1 : 0)$, but if we consider the subfamily with $\alpha_{012} = 0$ identically, then S_0 cuts each fibre in a flex point. The equation of the generic member of this subfamily is

$$x^2y + xy^2 + \alpha_{111}xyz + \alpha_{102}xz^2 + \alpha_{003}z^3 = 0$$

and its discriminant is

$$\Delta = \alpha_{003}^2(\alpha_{111}^4\alpha_{102} - 8\alpha_{111}^2\alpha_{102}^2 - \alpha_{111}^3\alpha_{003} + 16\alpha_{102}^3 + 36\alpha_{111}\alpha_{102}\alpha_{003} - 27\alpha_{003}^2).$$

Over the curve $\alpha_{003} = 0$, of degree 9, we now have fibres of type I_2, to be more precise over this curve the fibres have equation

$$x(xy + y^2 + \alpha_{111}yz + \alpha_{102}z^2).$$

The main difference with the generic family is the presence of two new vertical divisors: the divisor T defined by the line $x = 0$ over the curve $\alpha_{003} = 0$ and the divisor Q defined by the conic $xy + y^2 + \alpha_{111}yz + \alpha_{102}z^2 = 0$ over the same curve.
In $\operatorname{Pic} X$ we have $T = \alpha L + \beta_0 S_0 + \beta_1 S_1 + \beta_2 S_2$, and now I want to determine the coefficients. To do that I need the following informations:

1. $TL^2 = 0$ since T is vertical;
2. $T \cdot LS_1 = T \cdot LS_2 = 0$ since T is disjoint from S_1 and S_2;
3. $T \cdot LS_0 = 9$ since $T \cdot LS_0 = (T_{|S_0})(L_{|S_0})$ and $T_{|S_0}$ is a curve of degree 9, isomorphic to $\alpha_{003} = 0$.

Using the intersection table we computed previously, we can write a system in α, β_0, β_1 and β_2 whose solution gives

$$T = 3L - 2S_0 + S_1 + S_2,$$

and since $T + Q = 9L$ we also have

$$Q = 6L + 2S_0 - S_1 - S_2.$$

The Picard group of the generic fibration in this subfamily has then a $\mathbb{Q}$-basis given by L, S_0, T and S_2. Using this basis, the exact sequence (1.5) tells us that this subfamily has $\operatorname{rk} \mathrm{MW}(X) = 1$, and that a generator is given by S_2.

[4]By a set of $\mathbb{Q}$-generators for $\mathrm{MW}(X)$, I mean a set of generators for $\mathrm{MW}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

In the same way it's possible to show that the fibrations in the subfamily with $\alpha_{012} = \alpha_{102} = 0$ identically, defined by

$$x^2y + xy^2 + \alpha_{111}xyz + \alpha_{003}z^3 = 0,$$

where each of the three sections cuts the fibres in flex points, has $\operatorname{rk}\mathrm{MW}(X) = 0$. In this case in fact the discriminant locus is

$$\Delta : 256\alpha_{003}^3(\alpha_{111}^3 + 27\alpha_{003}) = 0,$$

and so over the curve $\alpha_{003} = 0$ the fibres are of type I_3, with equation

$$xy(x + y + \alpha_{111}z) = 0.$$

The divisor $\pi^*\alpha_{003} = 0$ splits in three irreducible components, giving three extra vertical divisors, and so by the Shioda–Tate–Wazir formula (Theorem 1.3) the rank of the Mordell–Weil group of the generic fibration in this subfamily is 0.

5.4 The families $(3, 2)$, $(3, 1)$, $(3, 0)$

As we can see from Table 5.1, in the equation F defining such families the term x^3 is missing, so we have an obvious section

$$S : b \longmapsto (1 : 0 : 0) \in X_b.$$

The tangent line to the curve X_b at the point $(1 : 0 : 0)$ is $\alpha_{210}y + \alpha_{201}z = 0$, so by Proposition 1.1 we must have $\alpha_{210} \neq 0$ in order to have a smooth X. We can then make the following change of coordinates

$$\begin{cases} x = x' - \frac{\alpha_{120}}{2\alpha_{210}^2}y' + (\frac{\alpha_{120}\alpha_{201}}{\alpha_{210}^2} - \frac{\alpha_{111}}{2\alpha_{210}})z' \\ y = \frac{1}{\alpha_{210}}y' - \frac{\alpha_{201}}{\alpha_{210}}z' \\ z = z' \end{cases}$$

which allows us to assume that the equation F has the following simpler form (I drop the primes)

$$x^2y + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{102}xz^2 + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0. \qquad (5.4)$$

The hyperplane section ξ of Z when restricted to X then gives

$$\begin{cases} F = 0 \\ z = 0 \end{cases} \longrightarrow y(x^2 + \alpha_{030}y^2) = 0,$$

the vanishing of the first factor gives the section, the second defines a double cover R of the base and the ramification locus for the covering map $\rho = \pi_{|R}$ is the curve $\alpha_{030} = 0$.

5.4.1 The cubic form

From Table 5.2, these families have all $\operatorname{rk}\operatorname{Pic}X = 3$ and so we search for three generators. We try with L, the section S and the double cover R.
First of all, since R and S are disjoint we have

$$SRL = S^2R = SR^2 = 0.$$

Assuming that α_{030} defines a smooth curve, a canonical divisor on R is given by ([BHPVdV04, Sect. I.17])

$$K_R = \rho^* K_{\mathbb{P}^2} + \frac{m}{2}\rho^* l = (\frac{m}{2} - 3)\rho^* l,$$

where m is the degree of α_{030}. Then, thanks to (3.2) and (3.3), we have

$$LR^2 = (L_{|R})(R_{|R}) = \rho^* l \cdot \left(\frac{m}{2} - 3\right)\rho^* l = m - 6, \qquad R^3 = K_R^2 = 2\left(\frac{m}{2} - 3\right)^2.$$

So the intersection table is

	L	S	R
L^2	0	1	2
S^2	-3	9	0
R^2	$m-6$	0	$2\left(\frac{m}{2}-3\right)^2$
LS	1	-3	0
LR	2	0	$m-6$
SR	0	0	0

and since

$$\det \begin{pmatrix} 0 & 1 & 2 \\ 1 & -3 & 0 \\ 2 & 0 & m-6 \end{pmatrix} = 18 - m$$

is non-zero (in our cases) we deduce that $\{L, S, R\}$ is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$. Moreover the cubic form is

(a,b)	$(\alpha L + \beta S + \gamma R)^3$
$(3,2)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 6\alpha^2\gamma - 12\alpha\gamma^2 + 8\gamma^3$
$(3,1)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 6\alpha^2\gamma - 6\alpha\gamma^2 + 2\gamma^3$
$(3,0)$	$3(\alpha^2\beta - 3\alpha\beta^2 + 3\beta^3 + 2\alpha^2\gamma)$

5.4.2 The number of sections

I want to determine the number of sections for the generic fibration of these three families. Let $\alpha L + \beta S + \gamma R$ be the class of a section. The first two requirements discussed in Section 5.2.2, give the relations between α, β and γ

$$\begin{cases} 1 = L^2(\alpha L + \beta S + \gamma R) = \beta + 2\gamma \\ -6 = c_2(X)(\alpha L + \beta S + \gamma R), \end{cases}$$

and using (4.5) to compute $c_2(X)$ we obtain the three systems

$$\begin{cases} \beta + 2\gamma = 1 \\ 36\alpha - 6\beta - 4\gamma = -6 \end{cases}, \qquad \begin{cases} \beta + 2\gamma = 1 \\ 36\alpha - 6\beta + 8\gamma = -6 \end{cases},$$

$$\begin{cases} \beta + 2\gamma = 1 \\ 36\alpha - 6\beta + 24\gamma = -6. \end{cases}$$

We can solve them expressing α and β in function of γ, finding

$$(\alpha, \beta) = \left(1 - 2\gamma, -\frac{2}{9}\gamma\right), \left(1 - 2\gamma, -\frac{5}{9}\gamma\right), (1 - 2\gamma, -\gamma)$$

for the cases $(a, b) = (3, 2), (3, 1), (3, 0)$ respectively. The last condition I use to detect a section is that $S^3 = K^2_{\mathbb{P}^2} = 9$ by (3.3), so since we know the cubic self-intersection form for the three cases, we can solve for γ the three equations: the results are

$$\begin{array}{l} -\frac{4}{27}\gamma(360\gamma^2 - 676\gamma + 351) = 0 \\ -\frac{7}{27}\gamma(180\gamma^2 - 343\gamma + 189) = 0 \\ -3\gamma(12\gamma^2 - 25\gamma + 15) = 0 \end{array}$$

which have only $\gamma = 0$ as rational solution. Hence in all of these three cases we have only one section.

5.4.3 Weierstrass form and Mordell–Weil group

Putting the equation (5.4)

$$x^2y + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{102}xz^2 + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0$$

in Weierstrass form with respect to the section S gives the equation $y^2z = x^3 + Axz^2 + Bz^3$ with

$$\begin{array}{rl} A = & 81\alpha_{030}\alpha_{102}^2 - 27\alpha_{012}^2 + 81\alpha_{021}\alpha_{003}, \\ B = & \frac{729}{4}\alpha_{021}^2\alpha_{102}^2 - 486\alpha_{030}\alpha_{102}^2\alpha_{012} - 54\alpha_{012}^3 + \\ & +243\alpha_{021}\alpha_{012}\alpha_{003} - 729\alpha_{030}\alpha_{003}^2. \end{array}$$

We can then see that the generic fibre over the discriminant locus is a nodal rational curve.
Thanks to the exact sequence (1.5) we have that $\operatorname{Pic} X_\eta$ is generated by the restriction of the classes of S and R, and so by (1.4) $\operatorname{Pic}^0 X_\eta$ has as generator $R_\eta - 2S_\eta$ (observe in fact that R_η is a degree 2 divisor on X_η). $R_\eta - 2S_\eta$ is in the image of the map (1.2) and so $R_\eta - S_\eta$ is a $\mathbb{Q}$-generator for the Mordell–Weil group of X_η. I now want to find a divisor R' in X such that R'_η is linearly equivalent to $R_\eta - S_\eta$ in $\operatorname{Div} X_\eta$: such R' is a better choice for a $\mathbb{Q}$-generator of $\mathrm{MW}(X)$ since, as we will see in a moment, it is defined by a rational section of the fibration.
The tangent line to X_b in the point $(1 : 0 : 0)$ is $y = 0$; this line intersects X_b in $z^2(\alpha_{102}x + \alpha_{003}z) = 0$, defining the divisor $2S + R'$ where R' is the rational section defined by

$$R' : b \longmapsto (\alpha_{003}(b) : 0 : -\alpha_{102}(b)) \in X_b.$$

Then R' is the blow-up of $\mathbb{P}^2$ in $(\deg \alpha_{003})(\deg \alpha_{102})$ points. The following table lists the number of the blown-up points, the self-intersection of a canonical divisor and the Euler–Poincaré characteristic of R': I will use them to determine the class of R' in $\operatorname{Pic} X$.

(a,b)	Number of points	$K^2_{R'}$	$\chi_{\text{top}}(R')$
$(3,2)$	40	-31	43
$(3,1)$	28	-19	31
$(3,0)$	18	-9	21

Observe now that $LSR' = \deg \alpha_{102}$ since the generic line in the base $\mathbb{P}^2$ does not pass through the points where we blow-up. To find the class $R' = \alpha L + \beta R + \gamma R$ we have to solve the system

$$\begin{cases} L^2 R' = 1 \\ LSR' = \deg \alpha_{102} \\ c_2(X)R' = \chi_{\text{top}}(R') - K^2_{R'} \end{cases}$$

where the third equation follows from Friedman's result (3.4). The solution of this system is

(a,b)	$R' = \alpha L + \beta S + \gamma R$
$(3,2)$	$R' = 2L - S + R$
$(3,1)$	$R' = L - S + R$
$(3,0)$	$R' = -S + R$

The result is that in any case we can use L, S, R' as a $\mathbb{Q}$-basis for $\operatorname{Pic} X$, and R' as a $\mathbb{Q}$-generator for $\operatorname{MW}(X)$. Observe also that in $\operatorname{Pic} X_\eta$ we have $R'_\eta = R_\eta - S_\eta$.

5.4.4 The family $(3,0)$

I will give a more detailed description of this type of family in Section 6.1 and Section 6.2.

5.5 The families $(4,1)$, $(5,2)$, $(6,3)$

All the fibrations of these families have a section given by

$$S : b \longmapsto (1:0:0) \in X_b$$

since $\alpha_{300} = 0$ identically, and from Table 5.1 we see that the same is true for α_{210}.
In each projective plane Z_b, the tangent line to X_b in $(1:0:0)$ has equation

$\alpha_{201}z = 0$: our smoothness assumption on X forces the constant α_{201} to be non-zero, hence

$$\begin{cases} x = x' - \frac{\alpha_{111}}{2\alpha_{201}}y' - \frac{\alpha_{102}}{2\alpha_{201}^2}z' \\ y = y' \\ z = \frac{1}{\alpha_{201}}z' \end{cases}$$

is a well defined homography, putting the equation in the simpler form (I drop the primes)

$$\alpha_{120}xy^2 + \alpha_{030}y^3 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0. \tag{5.5}$$

The hyperplane section ξ in Z described by $z = 0$ cuts X in the image of the section S (with multiplicity 2) and in another divisor R, given by the rational section

$$b \longmapsto (\alpha_{030}(b) : -\alpha_{120}(b) : 0) \in X_b,$$

which is the blow-up of the base $\mathbb{P}^2$ in $\{\alpha_{120} = 0\} \cap \{\alpha_{030} = 0\}$.

Remark 5.5.1 In the case $(a, b) = (6, 3)$ we have $\deg \alpha_{120} = 0$, i.e. $\alpha_{120} \in \mathbb{C}$. This means that generically (i.e. for $\alpha_{120} \neq 0$) this type of fibration has *two* disjoint sections.

5.5.1 The cubic form

All these fibrations have $\operatorname{rk} \operatorname{Pic} X = 3$ (compare with Table 5.2), and we have the following three obvious divisors:

1. the divisor L, coming from the base;
2. the section $S = \mathbb{P}^2 \times \{(1 : 0 : 0)\}$;
3. the rational section R.

Observe that $\xi_{|X} = 2S + R$ since $z = 0$ cuts on X both S and R. We know all about S, to study R we begin with the diagram

$$\begin{array}{ccc} R & \overset{j}{\hookrightarrow} & X \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{P}^2 & \end{array}$$

where now ρ is the blow-up of $\mathbb{P}^2$ along $\{\alpha_{120} = 0\} \cap \{\alpha_{030} = 0\}$: let $m = \deg \alpha_{120}$, then this locus consists of $m(m + 3)$ points. We thus have

$$K_R = \rho^* K_{\mathbb{P}^2} + E_1 + \ldots + E_{m(m+3)}$$

where the E_i's are the exceptional divisors introduced by ρ: then

$$R^3 = K_R^2 = 9 - m(m + 3),$$

and since $S \cap R$ is a curve of degree m in S we get

$$SR^2 = (R_{|S})(R_{|S}) = m^2.$$

So the intersections are given in this table

	L	S	R
L^2	0	1	1
S^2	-3	9	$-3m$
R^2	-3	m^2	$9-m(m+3)$
LS	1	-3	m
LR	1	m	-3
SR	m	$-3m$	m^2

and in particular

$$\det \begin{pmatrix} 0 & 1 & 1 \\ -3 & 9 & -3m \\ -3 & m^2 & 9-m(m+3) \end{pmatrix} = -6(m-3)(m+3),$$

which has no zeros in our cases ($m = 0, 1, 2$). We conclude that $\{L, S, R\}$ is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$, and that the cubic form is

(a,b)	$(\alpha L + \beta S + \gamma R)^3$
$(4,1)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 3\alpha^2\gamma + 12\alpha\beta\gamma - 18\beta^2\gamma - 9\alpha\gamma^2 - \gamma^3 + 12\beta\gamma^2$
$(5,2)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 3\alpha^2\gamma + 6\alpha\beta\gamma - 9\beta^2\gamma - 9\alpha\gamma^2 + 5\gamma^3 + 3\beta\gamma^2$
$(6,3)$	$3(\alpha^2\beta - 3\alpha\beta^2 + 3\beta^3 + \alpha^2\gamma - 3\alpha\gamma^2 + 3\gamma^3)$

5.5.2 The number of sections

Let $\alpha L + \beta S + \gamma R$ be the class of a section. Then we have two linear relations between the coefficients, given by $L^2(\alpha L + \beta S + \gamma R) = 1$ and $c_2(X)(\alpha L + \beta S + \gamma R) = -6$ as discussed in Section 5.2.2. Explicitly we have

$$\begin{cases} \beta + \gamma = 1 \\ 36\alpha - 6\beta + 14\gamma = -6 \end{cases}, \quad \begin{cases} \beta + \gamma = 1 \\ 36\alpha - 6\beta + 2\gamma = -6 \end{cases},$$

$$\begin{cases} \beta + \gamma = 1 \\ 36\alpha - 6\beta - 6\gamma = -6 \end{cases},$$

and so we can express α and β as functions of γ, finding

$$(\alpha, \beta) = \left(-\frac{5}{9}, 1-\gamma\right), \left(-\frac{2}{9}, 1-\gamma\right), (0, 1-\gamma)$$

for the three cases respectively. Using the expression of the cubic self-intersection we can solve for γ since we know that $(\alpha L + \beta S + \gamma R)^3 = 9$ by (3.3). The only rational solution to this equation is $\gamma = 0$ for the first two cases, while we have two solutions, $\gamma = 0$ and $\gamma = 1$, for the case $(a,b) = (6,3)$. So S is the only section, apart in the case where there is also another obvious section, which is given by R.

5.5.3 Weierstrass form and Mordell–Weil group

Putting the equation (5.5)

$$\alpha_{120}xy^2 + \alpha_{030}y^3 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0$$

in Weierstrass form with respect to the section S, we find the equation $y^2z = x^3 + Axz^2 + Bz^3$ with

$$\begin{aligned} A =\ & \alpha_{120}^2\alpha_{003} - \tfrac{1}{3}\alpha_{021}^2 + \alpha_{030}\alpha_{012}, \\ B =\ & \tfrac{1}{4}\alpha_{120}^2\alpha_{012}^2 - \tfrac{2}{3}\alpha_{120}^2\alpha_{021}\alpha_{003} - \tfrac{2}{27}\alpha_{021}^3 + \tfrac{1}{3}\alpha_{030}\alpha_{021}\alpha_{012} - \alpha_{030}^2\alpha_{003}. \end{aligned}$$

By Tate's algorithm the generic fibre over the discriminant locus is a nodal rational curve.
Using $\{L, S, R\}$ as $\mathbb{Q}$-basis for $\operatorname{Pic} X$ is then easy to see that $R \in \mathrm{MW}(X)$ gives a $\mathbb{Q}$-generator.

5.5.4 The $(6,3)$ family

The family of elliptic Calabi–Yau threefolds in $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(3) \oplus \mathcal{O})$ will be studied in detail in Chapter 6. In fact I will show there that the generic element of this family is a classical E_7 fibration ([AE10]), and that there are also some interesting fibrations of non-generic type. In fact for the generic equation

$$F = \alpha_{120}xy^2 + \alpha_{030}y^3 + \alpha_{201}x^2z + \alpha_{111}xyz + \alpha_{021}y^2z + \alpha_{102}xz^2 + \alpha_{012}yz^2 + \alpha_{003}z^3$$

we have two possibilities (we have $\alpha_{201} \neq 0$ in both cases):

1. if $\alpha_{120} \neq 0$ then the change

$$\begin{cases} x = -\frac{2}{\alpha_{120}}x' - \frac{\alpha_{030}}{\alpha_{120}}y' + \left(\frac{\alpha_{120}\alpha_{111}^2}{32\alpha_{201}^2} - \frac{\alpha_{030}^2}{8\alpha_{120}} - \frac{\alpha_{120}^2\alpha_{102}}{8\alpha_{201}^2}\right)z' \\ y = y' + \left(\frac{\alpha_{030}}{4} - \frac{\alpha_{120}\alpha_{111}}{8\alpha_{201}}\right)z' \\ z = \frac{\alpha_{120}^2}{4\alpha_{201}}z' \end{cases}$$

   lead us to the canonical form of the E_7 fibrations (compare to Section 6.3)

$$-2xy^2 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0;$$

2. if $\alpha_{120} = 0$ we apply the change

$$\begin{cases} x = x' - \frac{\alpha_{111}}{2\alpha_{201}}y' - \frac{\alpha_{102}}{2\alpha_{201}^2}z' \\ y = y' \\ z = \frac{1}{\alpha_{201}}z' \end{cases}$$

   and have the form (with non-primed variables)

$$\alpha_{030}y^3 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{030}z^3 = 0.$$

   I will study this subfamily more in detail in Section 6.4.

5.6 The families $(7, 4)$, $(8, 5)$

As in the other cases, we have the section

$$S : b \longmapsto (1:0:0) \in X_b,$$

and it must be $\alpha_{201} \neq 0$ for otherwise we will find singular families. Then we can use the homography

$$\begin{cases} x = x' - \frac{\alpha_{111}}{2\alpha_{201}} y' - \frac{\alpha_{102}}{2\alpha_{201}^2} z' \\ y = y' \\ z = \frac{1}{\alpha_{201}} z' \end{cases}$$

to put the general equation in the easier form (using not-primed variables)

$$\alpha_{030} y^3 + x^2 z + \alpha_{021} y^2 z + \alpha_{012} y z^2 + \alpha_{003} z^3 = 0. \tag{5.6}$$

The restriction to X of the divisor $\xi : z = 0$ gives

$$\begin{cases} F = 0 \\ z = 0 \end{cases} \longrightarrow \alpha_{030} y^3 = 0.$$

So we are cutting on X the section S (with multiplicity 3) and a vertical divisor T, defined by

$$T = \{(x : y : 0) \in X_b \,|\, \alpha_{030}(b) = 0\}$$

i.e. T is a $\mathbb{P}^1$-bundle over the curve $\alpha_{030} = 0$, whose fibres are the lines in the reducible I_2 fibres $z(x^2 + \ldots) = 0$ we have over that curve.

5.6.1 The cubic form

I now want to determine $\mathbb{Q}$-generators for $\operatorname{Pic} X$. From Table 5.2 we look for three divisors, and I want to show that L, S and T is a good choice. Let $m = deg\alpha_{030}$, then LT is given by the lines in the m fibres over $\{\alpha_{030} = 0\} \cap l$, so we conclude that $LTS = m$. ST is a curve of degree m in S, and so

$$ST^2 = (T_{|S})(T_{|S}) = m^2.$$

Observe that $T_{|L}$ is the disjoint union of m lines in the reducible fibres of the elliptic surface L; since each of these is a -2-curve ([Mir89, Chap. I.6]) we then have

$$LT^2 = (T_{|L})(T_{|L}) = (T_{|L})^2 = -2m.$$

Since T is a $\mathbb{P}^1$-bundle over the curve $\alpha_{030} = 0$ we have ([Bea96, Prop. III.21])

$$T^3 = (K_T)^2 = 8\left(1 - \frac{(m-1)(m-2)}{2}\right) = 4m(3-m).$$

Another important fact is that the generic fibre does not intersect T, so that $TL^2 = 0$, and finally that

$$S^2T = (S_{|S})(T_{|S}) = K_S(T_{|S}) = -3m.$$

So we have the intersection table

	L	S	T
L^2	0	1	0
S^2	-3	9	$-3m$
T^2	$-2m$	m^2	$4m(3-m)$
LS	1	-3	m
LT	0	m	$-2m$
ST	m	$-3m$	m^2

The following result

$$\det \begin{pmatrix} 0 & 1 & 0 \\ -3 & 9 & -3m \\ -2m & m^2 & 4m(3-m) \end{pmatrix} = 6m(6-m)$$

tells us that if we want $\{L, S, T\}$ to be a $\mathbb{Q}$-basis for $\operatorname{Pic} X$ it's necessary that $\deg \alpha_{030} \neq 0, 6$, which doesn't occur in our cases. Hence the cubic form is given by

(a,b)	$(\alpha L + \beta S + \gamma T)^3$
$(7,4)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 12\alpha\beta\gamma - 18\beta^2\gamma - 12\alpha\gamma^2 + 12\beta\gamma^2 + 8\gamma^3$
$(8,5)$	$3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 6\alpha\beta\gamma - 9\beta^2\gamma - 6\alpha\gamma^2 + 3\beta\gamma^2 + 8\gamma^3$

5.6.2 The number of sections

I want to show now that S is the only section for that type of families. In fact, let denote by $\alpha L + \beta S + \gamma T$ the class of a section. Then we have three linear relations among the coefficients, discussed in Section 5.2.2, which allow us to determine α, β and γ:

$$\begin{cases} L^2(\alpha L + \beta S + \gamma T) = 1 \\ LT(\alpha L + \beta S + \gamma T) = \deg \alpha_{030} \\ c_2(X)(\alpha L + \beta S + \gamma T) = 36\alpha - 6\beta - 4\gamma = -6. \end{cases}$$

From (4.5) we can compute $c_2(X)$, and the solution of the system is given by $(\alpha, \beta, \gamma) = (0, 1, 0)$.

5.6.3 Weierstrass form and Mordell–Weil group

If we put the equation (5.6)

$$\alpha_{030}y^3 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0$$

in Weierstrass form with respect to the section S, we will find the equation $y^2z = x^3 + Axz^2 + Bz^3$ with

$$\begin{aligned} A =&\ 9\alpha_{030}\alpha_{012} - 3\alpha_{021}^2, \\ B =&\ 9\alpha_{030}\alpha_{021}\alpha_{012} - 2\alpha_{021}^3 - 27\alpha_{030}^2\alpha_{003}. \end{aligned}$$

Then the discriminant locus has equation

$$\alpha_{030}^2(-\alpha_{021}^2\alpha_{012}^2 + 4\alpha_{030}\alpha_{012}^3 + 4\alpha_{021}^3\alpha_{003} - 18\alpha_{030}\alpha_{021}\alpha_{012}\alpha_{003} + 27\alpha_{030}^2\alpha_{003}^2),$$

hence we have naturally the presence of nodal rational curves and I_2 fibres. These last are over the component $\alpha_{030} = 0$ of Δ and have equation

$$z(x^2 + \alpha_{021}y^2 + \alpha_{012}yz + \alpha_{003}z^2) = 0.$$

Observe that in this case $\operatorname{Pic} X$ is generated by two vertical divisors and the section, hence none of them gives something in the Mordell–Weil group, which has then rank 0.

5.7 The $(9, 6)$ family

As in the previous section, we must have $\alpha_{201} \neq 0$. We observe moreover that if $\alpha_{030} = 0$, then the equation splits, giving us a singular variety. So we also have $\alpha_{030} \neq 0$ and the homography

$$\begin{cases} x = x' - \frac{a_2\alpha_{111}}{2\alpha_{201}}y' + (\frac{\alpha_{111}^3}{24\alpha_{030}\alpha_{201}^3} - \frac{\alpha_{111}\alpha_{021}}{6\alpha_{030}\alpha_{201}^2} + \frac{\alpha_{102}}{2\alpha_{201}^2})z' \\ y = a_2y' + (\frac{\alpha_{021}}{3\alpha_{030}\alpha_{201}} - \frac{\alpha_{111}^2}{12\alpha_{030}\alpha_{201}^2})z' \\ z = -\frac{1}{\alpha_{201}}z' \end{cases}$$

where a_2 satisfies $a_2^3\alpha_{030} = 1$, puts the equation into the well known Weierstrass form

$$x^2z = y^3 + \alpha_{012}yz^2 + \alpha_{003}z^3.$$

5.7.1 The cubic form

Looking at Table 5.2 we see that $\operatorname{Pic} X$ has rank 2, and by Proposition 5.1, a $\mathbb{Q}$-basis is given by L and the section

$$S : b \longmapsto (1 : 0 : 0) \in X_b.$$

The intersection table is particularly simple, and is given by

	L	S
L^2	0	1
S^2	-3	9
LS	1	-3

Thus the cubic form is

$$(\alpha L + \beta S)^3 = 3\beta(\alpha^2 - 3\alpha\beta + 3\beta^2).$$

Also in this case S is the only section: denoting with $\alpha L + \beta S$ the class of any section, we have

$$\begin{cases} L^2(\alpha L + \beta S) = 1 \\ c_2(X)(\alpha L + \beta S) = 36\alpha - 6\beta = -6 \end{cases}$$

whose solution is $(\alpha, \beta) = (0, 1)$.

Chapter 6

New and old families

In this chapter I want to present a detailed description of the Calabi–Yau elliptic fibrations we can find in the two bundles $\mathbb{P}(\mathcal{O}(3)\oplus\mathcal{O}\oplus\mathcal{O})$ and $\mathbb{P}(\mathcal{O}(6)\oplus\mathcal{O}(3)\oplus\mathcal{O})$ over $\mathbb{P}^2$.
Elliptic Calabi–Yau threefolds in $\mathbb{P}(\mathcal{O}(3)\oplus\mathcal{O}\oplus\mathcal{O})$ are interesting because we can find subfamilies having in a natural way Kodaira singular fibres degenerating to non-Kodaira fibres. Moreover, in Chapter 7 we will see that these fibrations enjoy another interesting property.
In the bundle $\mathbb{P}(\mathcal{O}(6)\oplus\mathcal{O}(3)\oplus\mathcal{O})$, the generic anticanonical hypersurface turns out to be a classical elliptic fibration, named E_7. I will give a brief description of E_7 families, and then focus on the non-generic elliptic fibrations in this bundle, which can be viewed as E_7 fibrations where we try to collapse its two sections into one. I will call these non-generic fibrations of type E_7^0.

6.1 Families in $\mathbb{P}(\mathcal{O}(3)\oplus\mathcal{O}\oplus\mathcal{O})$

I want to describe the anticanonical varieties X in the projective bundle $Z = \mathbb{P}(\mathcal{O}(3)\oplus\mathcal{O}\oplus\mathcal{O})$ over $\mathbb{P}^2$.
I will show that the generic Calabi–Yau elliptic fibration X in this bundle is smooth, compute the cubic intersection form in terms of two different $\mathbb{Q}$-basis for $\operatorname{Pic} X$ and study the group of automorphisms of X. I will also give a description of the Weierstrass model and the singular fibres of the generic fibration.
Finally, I will focus on a particular subfamily to show two interesting facts:

1. even if the generic member of the family of Calabi-Yau elliptic threefolds in $\mathbb{P}(\mathcal{O}(3)\oplus\mathcal{O}\oplus\mathcal{O})$ has Mordell–Weil group of rank 1, it's possible to find subfamilies for which the rank drops to 0;

2. it's possible to find elliptic fibrations with non-Kodaira fibres which are defined by a cubic homogeneous polynomial in $\mathbb{P}^2$.

6.1.1 The number of parameters and the canonical form

I will compute the number of effective parameters of the family using the formula in Proposition 5.2. The most general equation defining an anticanonical hypersurface X in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O} \oplus \mathcal{O})$ is

$$\begin{aligned} F = \ & \alpha_{210}x^2y + \alpha_{120}xy^2 + \alpha_{030}y^3 + \alpha_{201}x^2z + \alpha_{111}xyz + \\ & + \alpha_{021}y^2z + \alpha_{102}xy^2 + \alpha_{012}yz^2\alpha_{003}z^3 = 0, \end{aligned}$$

where the degree of the coefficients is as in Table 5.1 (p. 41). We can count the total number of parameters involved, finding that our equation depends on $2+3\cdot 10+4\cdot 28 = 144$ parameters. Since F is defined only up to a scalar multiple, the equation actually depends on $144-1 = 143$ parameters. To have the number of *effective* parameters, we have to take in account the action of automorphisms. In particular, the group automorphism of the base is $\mathrm{PGL}(3, \mathbb{C})$, and so we loose 8 parameters. Then there are the automorphisms on the fibres of the form

$$\begin{cases} x' = c_{11}x + c_{12}y + c_{13}z \\ y' = c_{22}y + c_{23}z \\ z' = c_{32}y + c_{33}z \end{cases} \quad \text{with} \quad \begin{array}{l} c_{11}, c_{22}, c_{23}, c_{32}, c_{33}, \in \mathbb{C}, \\ c_{12}, c_{13} \in H^0(\mathbb{P}^2, \mathcal{O}(3)) \end{array} .$$

These automorphisms lower the number of effective parameters by 24 (the coefficients depend on 25 parameters, but we loose one since x, y and z are projective coordinates). Then we reach $143-8-25 = 111$ parameters, which coincides with $h^{2,1}(X)$ as shown in Table 5.2 (p. 41). It's known that $h^{2,1}(X)$ is the dimension of the space of deformations of X (Bogomolov–Tian–Todorov theorem), the fact that an equation for X depends on exactly $h^{2,1}(X)$ parameters then means that any deformation of X comes from a deformation of the defining polynomial.
There is an obvious section for these varieties: it's defined by

$$S : b \longmapsto (1:0:0) \in X_b.$$

Given a point $b \in \mathbb{P}^2$, the tangent line to the curve X_b in $(1:0:0)$ is defined by $\alpha_{210}y + \alpha_{201}z = 0$ and so we see that $(\alpha_{210}, \alpha_{201}) \neq (0, 0)$ for otherwise our section will pass through singular points for the fibres and this is possible only if X is singular by Proposition 1.1. So at least one among α_{210} and α_{201} must be non-zero. Since the bundle automorphism

$$\begin{cases} x' = x \\ y' = z \\ z' = y \end{cases}$$

has the effect of exchanging

$$\alpha_{210} \text{ with } \alpha_{201}, \qquad \alpha_{120} \text{ with } \alpha_{102}, \qquad \alpha_{030} \text{ with } \alpha_{003}, \qquad \alpha_{021} \text{ with } \alpha_{012},$$

we can always assume that the non vanishing parameter is α_{210}. We can then define a canonical form for this kind of families, sending the tangent line $\alpha_{210}y + \alpha_{201}z = 0$ to the line $y = 0$: we can achieve this with the change of coordinates

$$\begin{cases} x' = x \\ y' = \alpha_{210}y + \alpha_{201}z \\ z' = z \end{cases} \longleftrightarrow \begin{cases} x = x' \\ y = \frac{1}{\alpha_{210}}y' - \frac{\alpha_{201}}{\alpha_{210}}z' \\ z = z' \end{cases}$$

and then assume that our family is defined by the equation (using x, y, z instead of x', y', z')

$$F = x^2y+\alpha_{120}xy^2+\alpha_{111}xyz+\alpha_{102}xz^2+\alpha_{030}y^3+\alpha_{021}y^2z+\alpha_{012}yz^2+\alpha_{003}z^3 = 0.$$

We can also use the change of coordinates

$$\begin{cases} x' = x - \frac{\alpha_{120}}{2}y - \frac{\alpha_{111}}{2}z \\ y' = y \\ z' = z \end{cases}$$

to make the equation simpler: it allows us to assume that the defining equation is (using x, y, z as before)

$$F = x^2y + \alpha_{102}xz^2 + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0. \qquad (6.1)$$

Observe that this equation depends on 122 parameters, and that we can't lower this number any more. In fact the only automorphisms we are left are given by $\mathrm{PGL}(3, \mathbb{C})$ and by those of the form

$$\begin{cases} x' = \frac{1}{\sqrt{\alpha}}x \\ y' = \alpha y \\ z' = \beta z, \end{cases} \qquad (6.2)$$

depending on $8 + 2 = 10$ parameters. The homogeneity of the equation lowers the number of parameters by 1, so we loose $10+1 = 11$ parameters, which is the number we need to reach 111. Observe also that the families with $\alpha_{030} = 0$ or $\alpha_{003} = 0$ identically are necessarily singular: they have another obvious section, which is $\{(0 : 1 : 0)\} \times \mathbb{P}^2$ in the first case and $\{(0 : 0 : 1)\} \times \mathbb{P}^2$ in the second, but these meet some singular point in the fibres, to be more explicit, the point $(0 : 1 : 0)$ is singular for the fibres over the sextic curve $\alpha_{021} = 0$ in the first case while $(0 : 0 : 1)$ is singular over $\alpha_{012} = 0$ in the second. As we will see S is the unique section for these families.
We want now to know when the point $(1 : 0 : 0)$ is a flex point. The tangent line in $(1 : 0 : 0)$ is $y = 0$, and the system

$$\begin{cases} F = 0 \\ y = 0 \end{cases}$$

has two solutions: one is obviously $(1 : 0 : 0)$ with multiplicity two, the other is

$$(\alpha_{003} : 0 : -\alpha_{102}).$$

So we see that $(1 : 0 : 0)$ is a flex point if and only if $\alpha_{102} \equiv 0$.
An interesting divisor in X is the one cut by $z = 0$: from the system

$$\begin{cases} F = 0 \\ z = 0 \end{cases} \longrightarrow \begin{cases} z = 0 \\ y(x^2 + \alpha_{030}y^2) = 0 \end{cases}$$

we find the section S, corresponding to $y = 0$, and a double cover of $\mathbb{P}^2$ defined by

$$R : x^2 + \alpha_{030}y^2 = 0.$$

Observe that since the branch locus for this covering is the sextic curve $\alpha_{030} = 0$, if it is smooth then R is a $K3$ surface, so by (3.2) we have

$$R^2 = i_* K_R = 0.$$

As a by-product, letting $H = \xi_{|X}$ denote the hyperplane section of X, then

$$H = S + R.$$

6.1.2 Weierstrass model and singular fibres

Using the methods described in [Cas91, Chap. 8], it's possible to put our families in Weierstrass form. Let

$$X : F = x^2 y + \alpha_{102} x z^2 + \alpha_{030} y^3 + \alpha_{021} y^2 z + \alpha_{012} y z^2 + \alpha_{003} z^3 = 0$$

be a generic family in the bundle Z as in (6.1). Then the Weierstrass model W of X with respect to the section S is the elliptic fibration in $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(9) \oplus \mathcal{O})$ with coordinates $(s : t : u)$ defined by

$$\begin{aligned} t^2 u = s^3 &+ \left(16\alpha_{030}\alpha_{102}^2 - \tfrac{16}{3}\alpha_{012}^2 + 16\alpha_{021}\alpha_{003}\right) s u^2 + \\ &+ \left(16\alpha_{021}^2\alpha_{102}^2 - \tfrac{128}{3}\alpha_{030}\alpha_{102}^2\alpha_{012} - \tfrac{128}{27}\alpha_{012}^3 + \tfrac{64}{3}\alpha_{021}\alpha_{012}\alpha_{003} - 64\alpha_{030}\alpha_{003}^2\right) u^3. \end{aligned}$$

We computed the equation of the discriminant locus, and observed that it is an irreducible curve with multiplicity one, over the generic point of which we have a rational nodal curve by Tate's algorithm. It's also possible to write the birational morphism from X to W, which is given by

$$\begin{cases} s = -4\alpha_{102} x z - \frac{4}{3}\alpha_{012} y z - 4\alpha_{003} z^2 \\ t = 8\alpha_{102} x^2 + 8\alpha_{030}\alpha_{102} y^2 + 8\alpha_{003} x z + 4\alpha_{021}\alpha_{102} y z \\ u = yz. \end{cases}$$

The situation changes if we consider the subfamily with $\alpha_{102} = 0$ identically. In fact in this case, the section S cuts each fibre in a flex point. The discriminant locus in this case splits in two curves:

$$\Delta : \alpha_{003}^2 (4\alpha_{030}\alpha_{012}^3 - \alpha_{021}^2\alpha_{012}^2 + 4\alpha_{021}^3\alpha_{003} - 18\alpha_{030}\alpha_{021}\alpha_{012}\alpha_{003} + 27\alpha_{030}^2\alpha_{003}^2)$$

and over the curve $\alpha_{003} = 0$ the fibre is generically of type I_2. In the intersection of these two components of the discriminant the fibre changes, more precisely

1. over $\alpha_{003} = \alpha_{012} = 0$ the fibre is of type III;
2. over $\alpha_{003} = 4\alpha_{030}\alpha_{012} - \alpha_{021}^2 = 0$ the fibre is of type I_3.

I will study the subfamily with $\alpha_{102} \equiv 0$ more in detail in Section 6.2. Observe that this is a description of the singular fibres we find for a generic choice of the coefficients. As we will see in Section 6.2.3 it's possible to find smooth fibrations of this type with different singular fibres, also of non-Kodaira type.

6.1.3 Is the general family smooth?

Here we want to show that the generic member of family defined by the equation (6.1)

$$F = x^2y + \alpha_{102}xz^2 + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0$$

in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O} \oplus \mathcal{O})$ is smooth. The singular points (q, b) with $q = (x : y : z)$ and $b = (t_0 : t_1 : t_2)$ satisfy

$$\begin{cases} 2xy + \alpha_{102}z^2 = 0 \\ x^2 + 3\alpha_{030}y^2 + 2\alpha_{021}yz + \alpha_{012}z^2 = 0 \\ 2\alpha_{102}xz + \alpha_{021}y^2 + 2\alpha_{012}yz + 3\alpha_{003}z^2 = 0 \\ (\partial_n\alpha_{102})xz^2 + (\partial_n\alpha_{030})y^3 + (\partial_n\alpha_{021})y^2z + (\partial_n\alpha_{012})yz^2 + (\partial_n\alpha_{003})z^3 = 0 \end{cases} \tag{6.3}$$

where we write $\partial_n = \frac{\partial}{\partial t_n}$ for $n = 0, 1, 2$. So we want the set

$$V_{\alpha_{102},\alpha_{030},\alpha_{021},\alpha_{012},\alpha_{003}} = \left\{ (q, b) \in \mathbb{P}^2 \times \mathbb{P}^2 | \begin{array}{l} \text{the system (6.3)} \\ \text{has } (q, b) \text{ as solution} \end{array} \right\}$$

to be empty for the generic choice of the coefficients. To show that this is true we consider the subvariety

$$V = \{(q, b, \alpha_{102}, \alpha_{030}, \alpha_{021}, \alpha_{012}, \alpha_{003}) | (q, b) \in V_{\alpha_{102},\alpha_{030},\alpha_{021},\alpha_{012},\alpha_{003}}\}$$

of $\mathbb{P}^2 \times \mathbb{P}^2 \times H^0(\mathbb{P}^2, \mathcal{O}(3)) \times H^0(\mathbb{P}^2, \mathcal{O}(6))^4$, and the projection over the last 5 factors. This map is projective and has $V_{\alpha_{102},\alpha_{030},\alpha_{021},\alpha_{012},\alpha_{003}}$ as fibre over $(\alpha_{102}, \alpha_{030}, \alpha_{021}, \alpha_{012}, \alpha_{003})$, so it suffices to show that for a particular choice we have $V_{\alpha_{102},\alpha_{030},\alpha_{021},\alpha_{012},\alpha_{003}} = \varnothing$, i.e. that there exists a smooth X of this kind, to deduce that the same holds for the general choice of parameters. For example we can take

$$\alpha_{102} = 0, \qquad \alpha_{012} = 0, \qquad \alpha_{021} = \alpha_{003} = t_0^6 + t_1^6 + t_2^6, \qquad \alpha_{030} = t_0^6 + 2t_1^6 + 4t_2^6.$$

6.1.4 The cubic form

We computed the intersection matrix for this type of fibrations in Section 5.4.1, explicitly it is

	L	S	R
L^2	0	1	2
LS	1	-3	0
LR	2	0	0
S^2	-3	9	0
SR	0	0	0
R^2	0	0	0

and so the cubic form has expression

$$(\alpha L + \beta S + \gamma R)^3 = 3\alpha^2\beta - 9\alpha\beta^2 + 9\beta^3 + 6\alpha^2\gamma. \tag{6.4}$$

Observe that in the projective plane with coordinates $(\alpha : \beta : \gamma)$ this equation defines a cuspidal cubic, with the cusp in $(0 : 0 : 1)$ and tangent $\alpha = 0$. Moreover this cubic has one flex point in $(18 : 6 : -1)$.

Remark 6.1.1 Apart the cases $(a, b) = (9, 6), (3, 3)$, all the other families of elliptic Calabi–Yau threefolds we have over $\mathbb{P}^2$ have $\operatorname{rk}\operatorname{Pic} X = 3$. Among these, the family in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O} \oplus \mathcal{O})$ is the only one for which the cubic intersection form of its generic member defines a singular cubic curve in $\mathbb{P}^2$. In the case $(a, b) = (9, 6)$ we have three distinct points, while in the case $(a, b) = (3, 3)$ we have a smooth cubic surface in $\mathbb{P}^3$, isomorphic to a Fermat cubic.

A rational section

We call R' the divisor defined by the rational section

$$b \longmapsto (\alpha_{003}(b) : 0 : -\alpha_{102}(b)) \in X_b.$$

We know that if we cut X with $z = 0$ what we see is the section S and the double covering R. So we have that the hyperplane section H of X is simply

$$H = S + R.$$

Cutting now with $y = 0$ we see the section S with multiplicity two, and the rational section R'. But then we can express R' in terms of the $\mathbb{Q}$-generators of $\operatorname{Pic} X$:

$$2S + R' = H = S + R \longrightarrow R' = -S + R.$$

So we can use $\{L, S, R'\}$ as a $\mathbb{Q}$-basis of $\operatorname{Pic} X$, and it may be useful to know the intersection table in terms of this basis. This is

	L	S	R'
L^2	0	1	1
LS	1	-3	3
LR'	1	3	-3
S^2	-3	9	-9
SR'	3	-9	9
R'^2	-3	9	-9

So we can rewrite the cubic form (6.4) as

$$(\alpha L+\beta S+\gamma R')^3 = 3(\alpha^2\beta - 3\alpha\beta^2 + 3\beta^3 + \alpha^2\gamma + 6\alpha\beta\gamma - 9\beta^2\gamma - 3\alpha\gamma^2 + 9\beta\gamma^2 - 3\gamma^3).$$

As we will see in Section 6.2.2 the $\mathbb{Q}$-basis $\{L, S, R'\}$ of $\operatorname{Pic} X$ is more convenient for the study of the Mordell–Weil group.

6.1.5 The automorphism group

We know from Section 5.4.2 that the generic Calabi–Yau variety defined by equation (6.1) have only one section. This fact has an interesting consequence on the group $\operatorname{Aut}_B X$ of automorphisms of X over B, i.e. isomorphisms preserving the projection on the base or equivalently such that

$$\begin{array}{ccc} X & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^2 & \end{array}$$

commutes. I will prove the following proposition, which gives a description of $\operatorname{Aut}_B X$.

Proposition 6.1 *Let $\pi : X \longrightarrow \mathbb{P}^2$ be an elliptic fibration, which is an anticanonical hypersurface in $\mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O} \oplus \mathcal{O})$ defined by equation (6.1). Then $\operatorname{Aut}_B X$ has at most two elements,* id *and* $-\operatorname{id}$, *where $-\operatorname{id}(P) = -P$ is the inverse of P in the group law we have on the fibres of π. Moreover*

$$\operatorname{Aut}_B X = \{\operatorname{id}, -\operatorname{id}\} \iff \alpha_{102} \equiv 0.$$

Proof Since an automorphism $X \longrightarrow X$ sends sections to sections and we have only one of them in our case, any automorphism of X over B fixes the section and then induces a group homomorphism on each fibre. Generically these have only two automorphisms, the identity and the inversion with respect to the group law, so we can conclude that the same is true for the whole X, i.e.

$$\operatorname{Aut}_B X \leq \{\operatorname{id}, -\operatorname{id}\}.$$

To complete the proof I have to show that $\operatorname{Aut}_B X = \{\operatorname{id}, -\operatorname{id}\}$ if and only if $\alpha_{102} \equiv 0$.
Observe that any $\varphi \in \operatorname{Aut}_B X$ induces an automorphism

$$\varphi^* : \operatorname{Pic} X \longrightarrow \operatorname{Pic} X$$

which is the identity map. In fact $\varphi^* L = L$ because φ acts fibrewise, and $\varphi^* S = S$ because S is the only section. From [OVdV95, Prop. 5], we have that $\varphi^*(R) = R$, since φ^* induces an automorphism on the cubic curve defined by the cubic self-intersection form, which leaves fixed the singular point $(0 : 0 : 1)$ corresponding to R since it's the only singular point for the cubic. Moreover, from equation (6.2), we have that φ maps R to R.
Let assume that $-\operatorname{id}$ is an automorphism on X, then $-\operatorname{id}$ restricted to R must be an automorphism of R. Remember that R is defined by

$$\begin{cases} z = 0 \\ x^2 + \alpha_{030} y^2 = 0 \\ (y \neq 0) \end{cases}$$

and then a point $(x : y : 0) \in R$ has image

$$\begin{cases} X = \alpha_{102}^3 x^2 + (\alpha_{102}^2 \alpha_{012} - \alpha_{003}^2) xy + \alpha_{102}\alpha_{021}\alpha_{003} y^2 \\ Y = y(\alpha_{102}^3 x + (\alpha_{102}^2 \alpha_{012} + \alpha_{003}^2) y) \\ Z = \alpha_{102} y (2\alpha_{003} x - \alpha_{102}\alpha_{021} y). \end{cases}$$

This point still lies on R if and only if $Z = X^2 + \alpha_{030}Y^2 = 0$, and in particular $Z = 0$ can happen only in the following three cases:

1. when $\alpha_{102} \equiv 0$. We have that $-\mathrm{id}_{|R}$ is the covering automorphism;
2. when $\alpha_{003} \equiv 0$ and $\alpha_{102} \equiv 0$, but in this case the polynomial F defining the family is reducible and the family is singular;
3. when $\alpha_{003} \equiv 0$ and $\alpha_{021} \equiv 0$. In this case the equation for X is

$$X : x^2y + \alpha_{102}xz^2 + \alpha_{030}y^3 + \alpha_{012}yz^2 = 0,$$

   and we see that we have another section, given by $(0 : 0 : 1)$. But this section passes through singular points in the fibres over $\alpha_{102} = \alpha_{012} = 0$, so that these families are always singular.

Viceversa, let assume that $\alpha_{102} \equiv 0$. Then a direct computation shows that

$$\begin{array}{rccc} -\mathrm{id}: & X & \longrightarrow & X \\ & (x : y : z) & \longmapsto & (-x : y : z), \end{array}$$

which is an automorphism of X over B. □

Remark 6.1.2 Another way to show this result is to write explicitly the equations for $-\mathrm{id}$, and observe that its indetermination locus is empty if and only if α_{102} vanishes identically.

6.2 The subfamily with two automorphisms

In this section I want to study the subfamily we get when in the generic equation (6.1) we set $\alpha_{102} \equiv 0$, i.e

$$X : F = x^2y + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0. \tag{6.5}$$

From the previous sections we know that

1. the point $(1 : 0 : 0)$ is a flex point, with tangent line given by $y = 0$;
2. the discriminant locus splits in two curves, over one of them we have I_1 fibres and over the other I_2 fibres;
3. the group of automorphisms of X over B has order two.

The generic member of this subfamily is a smooth elliptic fibration: by the same argument used to prove the analogous statement for the general family in Section 6.1.3, we have only to find a smooth element in the subfamily to conclude that the generic element of the subfamily is smooth. We can use the example given in Section 6.1.3, which has $\alpha_{102} \equiv 0$ and so defines an elliptic fibration in the subfamilies we are studying now.

6.2.1 The cubic form

The subfamily defined by equation (6.5) is special in the sense that the section S passes through flex point in each fibre. Over the curve $\alpha_{003} = 0$ we see reducible fibres of Kodaira type I_2, whose equation is

$$y(x^2 + \alpha_{030}y^2 + \alpha_{021}yz + \alpha_{012}z^2) = 0.$$

This means that the divisor $\pi^*\alpha_{003} = 0$ splits in two irreducible components: one gives the divisor T defined by the lines $y = 0$ in the fibres over this curve, and the other gives the divisor Q defined by the conics. Since α_{003} is a polynomial of degree 6 we can claim that

$$T + Q = 6L.$$

If we intersect T with the curves in X, defined by

$$L^2, \qquad LS, \qquad LR, \qquad S^2, \qquad SR, \qquad R^2,$$

we find

1. $L^2T = 0$ since L^2 is a generic fibre of our fibration;
2. $LST = 6$ since ST is the curve defined by α_{003} in S;
3. $LRT = 0$ since T does not meet R, in fact points in T have $y = 0$ while points in R have $z = 0$ and $(1 : 0 : 0)$ is not a point of R;
4. $S^2T = -18$ since $S^2 = -3LS$;
5. $SRT = 0$ since $RT = 0$ as we have just observed;
6. $R^2T = 0$ since $R^2 = 0$.

Let now write $T = \alpha L + \beta S + \gamma R$, with $\alpha, \beta, \gamma \in \mathbb{Q}$. Since we know the intersection table in X, we find by the first three conditions above that

$$\begin{cases} 0 = L^2T = \alpha L^3 + \beta L^2S + \gamma L^2R = \beta + 2\gamma \\ 6 = LST = \alpha L^2S + \beta LS^2 + \gamma LSR = \alpha - 3\beta \\ 0 = LRT = \alpha L^2R + \beta LSR + \gamma LR^2 = 2\alpha \end{cases} \longrightarrow \begin{cases} \alpha = 0 \\ \beta = -2 \\ \gamma = 1, \end{cases}$$

so that

$$T = -2S + R, \qquad Q = 6L + 2S - R.$$

Another (and simpler) way to have this result is to consider the hyperplane section H of this kind of families: the general equation (6.5) is

$$F = x^2y + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0,$$

so we see that cutting with $y = 0$ we find $H = 3S + T$, while cutting with $z = 0$ we have $H = S + R$. Equating the two expressions give us $T = R - 2S$.
We can then compute all the intersection numbers involving T by mean of the expression $T = R - 2S$, but it's not so difficult to find them directly. In fact $RT = 0$ as we observed, and $ST^2 = (T_{|S})(T_{|S}) = 36$ since $(T_{|S})$ is the curve

defined by $\alpha_{003} = 0$ in S. Moreover we have that $T^3 = K_T^2 = -72$ since T is a geometrically ruled surface over $\alpha_{003} = 0$, and so ([Bea96, Prop. III.21])

$$K_T^2 = 8(1 - 10) = -72,$$

and finally that $T^2L = -12$ since $T^2L = (T_{|L})(T_{|L})$ and $T_{|L}$ consists of the 6 lines in the reducible I_2 of the fibration $X_{|L}$, since each of them is a (-2)-curve we have $(T_{|L})^2 = -12$.
The complete intersection table is then

	L	S	R	T
L^2	0	1	2	0
LS	1	-3	0	6
LR	2	0	0	0
S^2	-3	9	0	-18
SR	0	0	0	0
R^2	0	0	0	0
LT	0	6	0	-12
ST	6	-18	0	36
RT	0	0	0	0
T^2	-12	36	0	-72

Observe that the rank of this matrix is 3, as we expected. We make also another remark: the linear transformation sending L, S, R to L, S, T respectively is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

whose determinant is 1. So L, S, T is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$, and with respect to this basis we have the cubic intersection form

$$(\alpha L+\beta S+\gamma T)^3 = 3\alpha^2\beta-9\alpha\beta^2+9\beta^3+36\alpha\beta\gamma-54\beta^2\gamma-36\alpha\gamma^2+108\beta\gamma^2-72\gamma^3.$$

6.2.2 The Mordell–Weil group

In Section 5.4.3 we saw that the generic fibration defined by equation (6.1) has Mordell–Weil group of rank 1, with a generator given by the class of R.
I now want to concentrate on $\mathrm{MW}(X)$ when X is a member of the subfamily with $\alpha_{102} = 0$ identically, and show that in this case the Mordell–Weil group has rank 0. I use the $\mathbb{Q}$-basis of $\operatorname{Pic} X$ given by L, S, T, in terms of which we showed that

$$T = -2S + R \text{ meaning that } R = T + 2S.$$

So $R = 0$ in $\mathrm{MW}(X)$ since T is vertical, implying that $\operatorname{rk} \mathrm{MW}(X) = 0$ as claimed.

6.2.3 A smooth family with non-Kodaira fibres

Let's consider the subfamily defined by

$$x^2y + \alpha_{030}y^3 + \alpha_{021}(y^2z + z^3) = 0,$$

where α_{030} and α_{021} are smooth sextics, intersecting transversally. With these requirements the threefold X is smooth. Thanks to our previous work, we can see that the Weierstrass model $y^2z = x^3 + Axz^2 + Bz^3$ of such fibrations has coefficients

$$A = 16\alpha_{021}^2, \qquad B = -64\alpha_{021}^2\alpha_{030},$$

and so the discriminant locus of the fibration has equation

$$\Delta : \alpha_{021}^4(27\alpha_{030}^2 + 4\alpha_{021}^2) = \alpha_{021}^4(2\alpha_{021} + 3\sqrt{-3}\alpha_{030})(2\alpha_{021} - 3\sqrt{-3}\alpha_{030}) = 0.$$

Over each component, by Tate's algorithm, we can compute the type of singular fibre:

1. over the curve $\alpha_{021} = 0$ we have IV fibres;
2. over each of the curves $2\alpha_{021} \pm 3\sqrt{-3}\alpha_{030}$ we have I_1 fibres.

The three curves intersect in the 36 distinct points given by $\alpha_{021} = \alpha_{030} = 0$, and over these points the equation of the fibre is $x^2y = 0$, which is not of Kodaira type.

Remark 6.2.1 A cubic curve in the plane can be of nine types, and we ask if we can find all of them as fibres of a smooth elliptic threefold with section. Of course we can find smooth cubics, so we concentrate only on the singular ones. The presence of the section tells us that one case is impossible to find, it's the case of a triple line (but observe that if we do not require the presence of the section we can have it). Among the remaining 7, we can find 6 in Kodaira's list: they are I_1, I_2, I_3, II, III and IV. Only one case is left, the one with a double line intersecting another line, which is the case I have just described.

I want now to perform a local analysis around the points over which we have non-Kodaira fibres: in particular I want to show that the restriction of the fibration to a generic smooth curve through one of these points is a singular elliptic surface, whose desingularization has fibres of type I_0^*. The non-Kodaira fibres of the threefolds are contraction of the I_0^* fibres, according to Proposition 2.7. Let (s, t) be local coordinates centred in these points and such that the curves α_{030} and α_{021} are given respectively by $t = 0$ and $s = 0$. So the local equation for the fibration is

$$x^2y + ty^3 + sy^2z + sz^3 = 0,$$

and I want to take the restriction of this family to the generic line through the origin, of equation $t = \lambda s$. So we are interested in the elliptic surface

$$x^2y + \lambda sy^3 + sy^2z + sz^3 = 0.$$

Over $s = 0$ there must be singular points for the surface, as the corresponding fibre is of non-Kodaira type. By Tate's algorithm we expect to have a fibre I_0^*, the fibre we actually have is obtained from a I_0^* fibre after a contraction of the three multiplicity 1 components which don't intersect the section. The points to which these components are contracted are the singular points for the surface. In fact we can find them in the following way: for the generic choice of λ, the equation $\lambda y^3 + y^2 z + z^3 = 0$ has three distinct solutions, and so if we let $(\bar{y} : \bar{z})$ be one of them, we see that $(0 : \bar{y} : \bar{z})$ defines a (local) section of the fibration. This section intersects the fibre over $s = 0$ in a point of the line $x = 0$, which is a multiple component (of multiplicity 2) for the fibre, so by Proposition 1.1 the intersection point must be singular.

Remark 6.2.2 Observe that if X is an elliptic threefold over the surface B, and $x \in X_b$ is a singular point for X, then the restriction of X to any smooth curve through b is an elliptic surface singular in x. In our case, since the singular points for the elliptic surface vary when we vary the base curve, we have points which are singular for the fibre and for the surface, but not for the threefold. There are also examples where every restriction has the same singular point, but this is not a singular point for the threefold, see e.g. Example 1.3.1 and Remark 1.3.3.

6.3 E_7 families in $\mathbb{P}(\mathcal{O}_B(-2K_B) \oplus \mathcal{O}_B(-K_B) \oplus \mathcal{O}_B)$

We want to study in detail the Calabi–Yau elliptic threefolds of type E_7 over a base B, focusing in particular in the case $B = \mathbb{P}^2$.
An E_7 family ([AE10, Sect. 1.7]) is a smooth hypersurface in a bundle of weighted projective planes $\mathbb{P}^{(1,1,2)}(\mathcal{O}_B \oplus \mathcal{O}_B(-K_B) \oplus \mathcal{O}_B(-2K_B))$ with coordinates $(z : x : y)$, given by the vanishing of a homogeneous equation of (weighted) degree 4

$$a_0 y^2 + a_1 x^2 y + a_2 x^4 + b_0 xyz + b_1 x^3 z + c_0 x^2 z^2 + c_1 y z^2 + dxz^3 + ez = 0,$$

where $a_i \in \mathbb{C}$, $b_i \in H^0(B, -K_B)$, $c_i \in H^0(B, -2K_B)$, $d \in H^0(B, -3K_B)$ and finally $e \in H^0(B, -4K_B)$. Observe that if we cut this hypersurface with the hyperplane $z = 0$ we will find one or two (constant) sections according to the case where $a_0 y^2 + a_1 x^2 y + a_2 x^4 = 0$ has one or two solutions. However, in the first of these cases it's easy to see that the section passes through some singular point of the fibres hence that these families are all singular.
We can then assume that $a_0 y^2 + a_1 x^2 y + a_2 x^4 = z = 0$ has two distinct solutions, and putting them in standard position $(x : y : z) = (1 : 1 : 0)$ and $(1 : -1 : 0)$, the equation defining the E_7 families is

$$x^4 - y^2 + b_0 xyz + b_1 x^3 z + c_0 x^2 z^2 + c_1 y z^2 + dxz^3 + ez = 0.$$

The following transformation

$$\begin{cases} x = x' - \frac{1}{4} b_1 z' \\ y = y' + \frac{1}{2} b_0 x' z' + \frac{1}{2} \left(c_1 - \frac{1}{4} b_0 b_1 \right) z'^2 \\ z = z' \end{cases}$$

allows us to simplify the equation further (using non-primed variables):

$$y^2 = x^4 + c_0 x^2 z^2 + dxz^3 + ez^4.$$

Since the weighted projective plane $\mathbb{P}^{1,1,2}$ can be embedded in $\mathbb{P}^3$, an E_7 family can as well be defined as the intersection in the bundle of (*non*-weighted) projective spaces $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(-K_B) \oplus \mathcal{O}_B(-2K) \oplus \mathcal{O}_B(-2K))$ of the two hypersurfaces defined by

1. $X_0X_2 - X_1^2 = 0$, which is the image of $\mathbb{P}^{(1,1,2)}$ under the standard embedding
$$(z : x : y) \longmapsto (X_0 : X_1 : X_2 : X_3) = (z^2 : zx : x^2 : y);$$
2. a quadric in $\mathbb{P}^3$, defined by the vanishing of the degree two polynomial corresponding to the weighted polynomial of degree four:
$$X_3^2 = X_2^2 + c_0X_1^2 + dX_0X_1 + eX_0^2$$
with $c_0 \in H^0(B, -2K_B)$, $d \in H^0(B, -3K_B)$ and $e \in H^0(B, -4K_B)$.

6.3.1 Plane form

I want to describe a procedure which puts our family in a planar form, i.e. I want to define an isomorphism between the variety X defined in the bundle $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(-K_B) \oplus \mathcal{O}_B(-2K) \oplus \mathcal{O}_B(-2K))$ by

$$\begin{cases} x_0x_2 = x_1^2 \\ x_3^2 = x_2^2 + c_0x_1^2 + dx_0x_1 + ex_1^2 \end{cases}$$

and a hypersurface $\bar{X}$ in a suitable bundle over B with $\mathbb{P}^2$-fibres. To obtain this I follow the method described in [Cas91, Chap. 8].
We have two obvious sections for our family E_7, defined by

$$\sigma_+ : b \longmapsto (0 : 0 : 1 : 1) \in X_b$$
$$\sigma_- : b \longmapsto (0 : 0 : 1 : -1) \in X_b$$

and the first step is to make a transformation which maps $S_+ = \sigma_+(B)$ to $S'_+ = B \times \{(0 : 0 : 0 : 1)\}$:

$$\begin{cases} z_0 = x_0 \\ z_1 = x_1 \\ z_2 = x_2 - x_3 \\ z_3 = x_3 \end{cases} \longleftrightarrow \begin{cases} x_0 = z_0 \\ x_1 = z_1 \\ x_2 = z_2 + z_3 \\ x_3 = z_3. \end{cases}$$

So our family is defined by

$$\begin{cases} z_3z_0 + z_0z_2 - z_1^2 = 0 \\ 2z_3z_2 + z_2^2 + ez_1^2 + fz_0z_1 + gz_0^2 = 0 \end{cases}$$

and the two sections are

$$S'_+ = B \times \{(0 : 0 : 0 : 1)\} \text{ and } S'_- = B \times \{(0 : 0 : 2 : -1)\}.$$

The hypersurface described by $z_3 = 0$ gives us the $\mathbb{P}^2$-bundle $Z = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(-K_B) \oplus \mathcal{O}_B(-2K_B))$, and we project X from the constant section $(0 : 0 :$

$0 : 1)$ on $z_3 = 0$. The image of this projection in the coordinates $(X_0 : X_1 : X_2)$ of $\mathbb{P}^2$ is given by the equation

$$eX_0^3 + dX_0^2X_1 + c_0X_0X_1^2 + 2X_1^2X_2 - X_0X_2^2 = 0.$$

This equation defines $\bar{X}$, and we have the projection

$$\begin{array}{cccc} \psi : & X & \longrightarrow & \bar{X} \\ & (x_0 : x_1 : x_2 : x_3) & \longmapsto & \begin{cases} X_0 = x_0 \\ X_1 = x_1 \\ X_2 = x_2 - x_3. \end{cases} \end{array}$$

The image of S_+ is the section $S_1 = B \times \{(0 : 1 : 0)\}$, while S_- has image $S_2 = B \times \{(0 : 0 : 1)\}$.
The inverse of the previous morphism is

$$\begin{array}{cccc} \psi^{-1} : & \bar{X} & \longrightarrow & X \\ & (X_0 : X_1 : X_2) & \longmapsto & \begin{cases} x_0 = X_0^2 \\ x_1 = X_0X_1 \\ x_2 = X_1^2 \\ x_3 = X_1^2 - X_0X_2. \end{cases} \end{array}$$

Remark 6.3.1 In the case $B = \mathbb{P}^2$, what we found is (up to a sign) the equation of the generic smooth elliptic fibration in the bundle $\mathbb{P}(\mathcal{O}(6)\oplus\mathcal{O}(3)\oplus\mathcal{O})$ obtained in Section 5.5.4. So if $B = \mathbb{P}^2$ then the E_7 family is the $(6, 3)$ family of Section 5.5.4.

6.3.2 Weierstrass model and singular fibres

Let's consider an E_7 family X defined over $\mathbb{P}^2$ and embedded in $\mathbb{P}(\mathcal{O}(6)\oplus\mathcal{O}(3)\oplus\mathcal{O})$ with coordinates $(x : y : z)$. Then X has equation

$$X : -2xy^2 + x^2z + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0,$$

and I want to describe the Weierstrass model with respect to the section $S = \sigma(\mathbb{P}^2)$ with

$$\sigma : b \longmapsto (1 : 0 : 0) \in X_b.$$

This is given by $y^2z = x^3 + Axz^2 + Bz^3$ with

$$\begin{aligned} A &= 4\alpha_{003} - \tfrac{1}{3}\alpha_{021}^2 \\ B &= \alpha_{012}^2 - \tfrac{2}{27}\alpha_{021}^3 - \tfrac{8}{3}\alpha_{021}\alpha_{003} \end{aligned}$$

and discriminant locus

$$-4\alpha_{021}^3\alpha_{012}^2 + 16\alpha_{021}^4\alpha_{003} + 27\alpha_{012}^4 - 144\alpha_{021}\alpha_{012}^2\alpha_{003} + 128\alpha_{021}^2\alpha_{003}^2 + 256\alpha_{003}^3.$$

The generic fibre over the discriminant locus is a rational nodal cubic, of Kodaira type I_1, and these specialize to I_2 fibres over $\alpha_{012} = \alpha_{021}^2 + 4\alpha_{003} = 0$ and to II fibres over $27\alpha_{012}^2 - 8\alpha_{021}^2 = \alpha_{021}^2 - 12\alpha_{003} = 0$. If for some point $\alpha_{021} = \alpha_{012} = \alpha_{003} = 0$, over that point we have fibres III: to see with an example that this is possible, consider a fibration having $\alpha_{012} \equiv 0$ and α_{021}, α_{003} defining smooth curve intersecting transversally.

6.3.3 The number of sections

We know that intersecting X with the hyperplane section $z = 0$ gives two distinct sections, namely S and S_1. I want now to show that over $\mathbb{P}^2$ these two are the only sections of such fibrations. Using (4.5) we can compute the second Chern class of X, and we also know the expression of the cubic form (see Section 5.5.1). What we have to do is to solve the following system. Let $\alpha L + \beta S + \gamma S_1$ be the class of a section, then

1. it intersects the class of a fibre only in a point;
2. its intersection with $c_2(X)$ is $\chi_{\mathrm{top}}(\mathbb{P}^2) - K^2_{\mathbb{P}^2} = -6$ by Friedman's result (3.4);
3. the value of the cubic self-intersection is $K^2_{\mathbb{P}^2} = 9$ by (3.3).

So we have to solve the system

$$\begin{cases} L^2(\alpha L + \beta S + \gamma S_1) = 1 \\ c_2(X)(\alpha L + \beta S + \gamma S_1) = -6 \\ (\alpha L + \beta S + \gamma S_1)^3 = 9, \end{cases}$$

which has only two solution, i.e. $(\alpha, \beta, \gamma) = (0, 1, 0), (0, 0, 1)$.

6.4 Limits of E_7 families over $\mathbb{P}^2$

In Section 5.5.4 we saw that the anticanonical hypersurfaces in $\mathbb{P}(\mathcal{O}(9) \oplus \mathcal{O}(3) \oplus \mathcal{O})$ can be of two different types. The equation defining such varieties is

$$F = \alpha_{120}xy^2 + \alpha_{030}y^3 + \alpha_{201}x^2z + \alpha_{111}xyz + \alpha_{021}y^2z + \alpha_{102}xz^2 + \alpha_{012}yz^2 + \alpha_{003}z^3,$$

and the generic one, having $\alpha_{120} \neq 0$, belongs to the E_7 family. I now want to study in detail the subfamily of the family of elliptic Calabi–Yau threefolds in $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(3) \oplus \mathcal{O})$ having $\alpha_{120} = 0$ (for this reason I propose to call E_7^0 this subfamily). In Section 5.5.4 we showed that a canonical equation for these varieties is

$$F = x^2z + \alpha_{030}y^3 + \alpha_{021}y^2z + \alpha_{012}yz^2 + \alpha_{003}z^3 = 0,$$

where each coefficient is as in Table 5.1 (p. 41). To see that we have a smooth example, consider the following choice for the coefficients: $\alpha_{012} = 0$ identically, and α_{030}, α_{021}, α_{003} define smooth curves with no point common to the three. The first thing we observe is that we can easily calculate the class of the hyperplane section $H = \xi_{|X}$ of X:

$$\begin{cases} z = 0 \\ F = 0 \end{cases} \longrightarrow \begin{cases} z = 0 \\ \alpha_{030}y^3 = 0. \end{cases}$$

So we have $H = 3S + T$, where S is the section $(1 : 0 : 0)$ and T is the irreducible component of $\pi^*\alpha_{030} = 0$ containing the line $z = 0$ in the reducible fibres over the curve $\alpha_{030} = 0$ (the equation of such fibres is $z(x^2 + \alpha_{021}y^2 + \alpha_{012}yz +$

$\alpha_{003}z^2) = 0)$.
So we have three divisors arising in a natural way,

$$L, \qquad S, \qquad T:$$

we know that $\operatorname{rk}\operatorname{Pic} X = 3$ and we will soon see that $\{L, S, T\}$ is a $\mathbb{Q}$-basis for $\operatorname{Pic} X$.

6.4.1 Weierstrass model and singular fibres

It's easy to see that the Weierstrass model of these families with respect to the section S is given by

$$t^2u = s^3 + \left(\alpha_{030}\alpha_{012} - \frac{1}{3}\alpha_{021}^2\right) su^2 + \left(\frac{1}{3}\alpha_{030}\alpha_{021}\alpha_{012} - \alpha_{030}^2\alpha_{003} - \frac{2}{27}\alpha_{021}^3\right) u^3$$

and the discriminant locus has equation

$$\alpha_{030}^2(4\alpha_{030}\alpha_{012}^3 - \alpha_{021}^2\alpha_{012}^2 + 4\alpha_{021}^3\alpha_{003} - 18\alpha_{030}\alpha_{021}\alpha_{012}\alpha_{003} + 27\alpha_{030}^2\alpha_{003}^2).$$

This means that the generic fibration of type E_7^0 has a curve in B over which we have I_2 fibres: the cubic $\alpha_{030} = 0$. Over this curve the equation of the fibre is

$$z(x^2 + \alpha_{021}y^2 + \alpha_{012}yz + \alpha_{003}z) = 0.$$

6.4.2 The cubic form

We want to write the intersection table for this kind of families. We observe that

- $L^2T = 0$ since the generic fibre is not over $\alpha_{030} = 0$;
- $LST = L_{|S}T_{|S} = 3$ since it's the intersection of a line with a cubic;
- $LT^2 = T_{|L}T_{|L} = -6$ since $T_{|L}$ is given by three (-2)-curves in a reducible fibre of the elliptic surface $X_{|L}$;
- $S^2T = S_{|S}T_{|S} = -9$ since it's the intersection of the canonical class of $\mathbb{P}^2$ with a cubic;
- $ST^2 = T_{|S}T_{|S} = 9$ since it's the intersection of two cubics in $S \simeq \mathbb{P}^2$;
- $T^3 = K_T^2 = 0$ since T is a ruled surface over a cubic curve in $B = \mathbb{P}^2$.

Then the complete table is

	L	S	T
L^2	0	1	0
LS	1	−3	3
LT	0	3	−6
S^2	−3	9	−9
ST	3	−9	9
T^2	−6	9	0

Since the matrix obtained considering the first three rows has non-zero determinant, we conclude that the three divisors L, S, T are linearly independent, so that they form a $\mathbb{Q}$-basis for $\operatorname{Pic} X$.
The cubic form thus has this expression:

$$(\alpha L+\beta S+\gamma T)^3=-9\alpha\beta^2-18\alpha\gamma^2+3\alpha^2\beta+18\alpha\beta\gamma+9\beta^3+27\beta\gamma^2-27\beta^2\gamma.$$

Observe that in the projective plane $\mathbb{P}^2$ with coordinates $(\alpha:\beta:\gamma)$ the above expression defines a smooth cubic curve.

6.4.3 The number of sections

I want to show that these families have only one section. Since I will use Friedman's result (3.4), I need to compute $c_2(X)$: by (4.5) we have that

$$c_2(X)=48L^2+81LS+27LT+27S^2+18ST+3T^2.$$

Then we can use Friedman's result (3.4) as first step in searching for sections: if $\alpha L+\beta S+\gamma T$ is the class of a section, then

$$c_2(X)(\alpha L+\beta S+\gamma T)=\chi_{\text{top}}(\mathbb{P}^2)-K^2_{\mathbb{P}^2}\Longrightarrow 36\alpha-6\beta=-6.$$

Another thing characterizing sections is the fact that they intersect the fibres in one point, i.e.

$$\beta=L^2(\alpha L+\beta S+\gamma T)=1.$$

So we deduce that $\beta=1$, $\alpha=0$ which means that a section has class $S+\gamma T$.
Finally, I exploit the fact (3.3) that

$$9-27\gamma+27\gamma^2=(S+\gamma T)^3=K^2_{\mathbb{P}^2}=9$$

to deduce $\gamma=0$ or $\gamma=1$. In the first case, we have the class S, in the second the class $S+T$.
I want to show that $S+T$ can't be the class of a section. In fact, a section intersects the I_2 fibres over the cubic $\alpha_{030}=0$ either in a point of the line or in a point of the conic: in the first case the intersection of the section with LT is 3, in the second it's 0. Since

$$LT(S+T)=-3,$$

we deduce that no section can have class $S+T$, hence that S is the only possible class for a section.
I now want to study the linear system $|S|$ to show that $|S|=\{S\}$ so that S is the only section of the fibration. Let D be an effective divisor linearly equivalent to S but different, then we would have

$$0\le LSD=LS^2=-3,$$

which is absurd. There is another way to see this: from the long exact sequence of

$$0\longrightarrow\mathcal{O}_X(-S)\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_{\mathbb{P}^2}\longrightarrow 0$$

we have $H^3(X,\mathcal{O}(-S))=\mathbb{C}$, then by Serre duality with $K_X=0$ we have the result.

6.5 Elliptic fibrations with 2 rational sections

In the recent paper [MP, App. B] the authors deal with the problem of describing the elliptic fibrations over $\mathbb{P}^2$ with two rational sections. To be more precise, they show that it's possible to embed an elliptic curve defined over a field K and having two rational points in the weighted projective space $\mathbb{P}^{1,1,2}_K$ as a quartic, with a particularly simple equation. When $K = \mathbb{C}(s,t)$ is the function field of $\mathbb{P}^2$ we are then dealing with the elliptic fibrations with two rational sections. In particular, the E_7 fibrations naturally situate in this framework, and in [MP, App. C] we also have a description of the E_7^0 families.
The more natural ambient space for fibrations with two rational sections is then a $\mathbb{P}^{1,1,2}$-bundle over $\mathbb{P}^2$, but this have the disadvantage of being singular. We can then desingularize this, finding a fibration in the Hirzebruch surface $\mathcal{F}_2$. The fact that $\mathbb{P}^2$, $\mathbb{P}^{1,1,2}$ and $\mathcal{F}_2$ are toric varieties allows us to use the powerful machinery of toric geometry, and hopefully to describe all the possible cases where we have smooth elliptic fibrations with two rational sections.
This is still a work in progress, we hope to obtain results in the near future.

Chapter 7

An application to physics

One of the main reasons for studying Calabi– Yau elliptic fibrations comes from physics. In the context of string theory, F-theory was first formulated by Vafa ([Vaf96]), and provides a sort of dictionary which allows one to deduce the physical phenomenology arising from an elliptic fibration. Since the results must be compatible with the ones observed in the "real" world, it's natural to restrict the attention only to particular subclasses of Calabi–Yau elliptic fibrations, satisfying some more requirements.
From a physical perspective, the generation of fluxes is a consequence of the self-interactions of particles or fields (the Feynman diagram of such a self-interaction is called a *tadpole*). By Gauss's theorem these fluxes must vanishes, so there are only two possibilities: one is to introduce extra-fluxes as compensation, the other is to allow only configurations for which these fluxes naturally vanish. This second possibility is known as *tadpole cancellation*.
In mathematical terms, we can say that a Calabi–Yau elliptic threefold $\pi : X \longrightarrow B$ embedded in $Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ satisfies the tadpole cancellation if it's possible to find a suitable section $h \in H^0(B, \mathcal{L}^2)$ and curves D_i for $i = 0, \ldots r$ in the double cover $\rho : Y \longrightarrow B$ branched along $h = 0$ such that

1. each D_i has class $m_i \rho^* L$, i.e. D_i comes from B;
2. $m_i > 0$ and $\sum_{i=1}^r m_i = 12$, because in this way $D_1 + \ldots + D_r$ has class $12\rho^* L$ which is the same as the class of $\rho^*\Delta$. So we can think of the D_i as the pull-back of components of the discriminant locus of $\pi : X \longrightarrow B$;
3. the tadpole relation

$$\pi_* c(X) = \frac{1}{2} \sum_{i=1}^{r} \rho_* j_{i*} c(D_i) \tag{7.1}$$

holds, where $j_i : D_i \hookrightarrow Y$ is the inclusion.

This definition is inspired from results showing how to connect the physical aspects of tadpole cancellation to the geometric properties of the Calabi–Yau threefold (or fourfold) representing the theory. These relations involve the topological Euler–Poincaré characteristic of divisors in the base that are interpreted as components of a "limit discriminant locus" for the fibration. See e.g. [AE10, Sect. 1.6 and 4.1] for a more precise description of tadpole cancellation and [Sen98] for the limit process.

7.1 The left hand side of the tadpole relation

Even if in physics it's customary to focus on the case of elliptic fourfolds, through these sections I will assume the total space of the fibration to be a threefold. In particular, I will concentrate on the situation of Chapter 4, where the threefold X sits inside the projective bundle $Z = \mathbb{P}(E)$, and $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B$ with $\mathcal{L}$ an ample line bundle on B.
Recall formula (4.3)

$$c(X) = i^* \left(\frac{c(T_{Z|B})}{1 - K_Z} \Pi^* c(B) \right),$$

so we can see that $\pi_* c(X)$ is a multiple of $c(B)$:

$$\pi_* c(X) = \Pi_* \left(\frac{c(T_{Z|B})}{1 - K_Z} X \right) c(B).$$

In the case of threefolds, we can work this out as follows:

$$\begin{aligned} \tfrac{c(T_{Z|B})}{1-K_Z} X = \; & 3\xi + \Pi^* c_1(E) + \Pi^* c_1(B) + \\ & -3\Pi^* c_1(B)\xi - \Pi^* c_1(E)\Pi^* c_1(B) - \Pi^* c_1(B)^2 + \\ & 12\Pi^* c_1(B)\xi^2 + 2\Pi^*(c_1(E)^2 + 4c_1(E)c_1(B) + 3c_1(B)^2 - 3c_2(E))\xi + \\ & +6(3c_2(E) - 5c_1(B)^2 - c_1(E)^2)\xi^2 \end{aligned}$$

and so, using the projection formula and keeping in mind that

$$\Pi_* 1_Z = \Pi_* \xi = 0, \qquad \Pi_* \xi^2 = 1_B,$$

we finally obtain

$$\pi_* c(X) = (12c_1(B) + 18c_2(E) - 30c_1(B)^2 - 6c_1(E)^2)c(B). \tag{7.2}$$

Remark 7.1.1 Observe that this result is consistent with the one we found in Section 4.4. In fact, since $c(B) = 1 + c_1(B) + c_2(B)$, we have

$$\pi_* c(X) = 12c_1(B) + 18c_2(E) - 18c_1(B)^2 - 6c_1(E)^2$$

whose degree coincides with the Euler–Poincaré characteristic of the threefold computed in (4.4).

Specializing the previous formula in the case with $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B$, we get

$$\pi_* c(X) = (12c_1(B) + 6(ab - a^2 - b^2)L^2 - 30c_1(B)^2)c(B), \tag{7.3}$$

which means

$$\pi_* c(X) = 12c_1(B) + 6(ab - a^2 - b^2)L^2 - 18c_1(B)^2. \tag{7.4}$$

Thus we know the left hand side of the tadpole relation (7.1), and we can focus on the right one.

7.2 The right hand side of the tadpole relation

Let $\rho : Y \longrightarrow B$ be the double covering of X branched along the zero locus of $h \in H^0(B, \mathcal{L}^2)$, and let D be a curve in Y of class $m\rho^* L$. Then we have

$$\begin{array}{ccc} D & \overset{j}{\hookrightarrow} & Y \\ & {\scriptstyle\tilde{\rho}} \searrow \quad \swarrow {\scriptstyle\rho} & \\ & B & \end{array}$$

and we want to compute $\tilde{\rho}_* c(D)$.
Using the normal bundle sequence for D in Y we have

$$c(D) = j^* \left(\frac{c(Y)}{1 + m\rho^* L} \right),$$

and so using the projection formula

$$\tilde{\rho}_* c(D) = \frac{mL}{1 + mL} \rho_* c(Y).$$

Using the same technique for the inclusion $i : Y \hookrightarrow \mathcal{L}$ we have

$$c(Y) = i^* \left(\frac{c(T_{\mathcal{L}})}{1 + 2p^* L} \right)$$

where $p : \mathcal{L} \longrightarrow B$ is the bundle projection. So our previous formula becomes

$$\tilde{\rho}_* c(D) = \frac{mL}{1 + mL} \frac{2L}{1 + 2L} p_* c(T_{\mathcal{L}}),$$

and using the inclusion of B as the 0-section of $\mathcal{L}$ we finally obtain

$$\tilde{\rho}_* c(D) = \frac{mL}{1 + mL} \frac{2L}{1 + 2L} \frac{1 + L}{L} c(B) = 2 \frac{mL}{1 + mL} \frac{1 + L}{1 + 2L} c(B).$$

Letting now $D_1, \ldots, D_r$ be curves of class $m_1 \rho^* L, \ldots, m_r \rho^* L$ respectively, we can then compute the right hand side of the tadpole relation (7.1):

$$\frac{1}{2} \sum_{i=1}^{r} \rho_* j_{i*} c(D_i) = \left(\sum_{i=1}^{r} \frac{m_i L}{1 + m_i L} \right) \frac{1 + L}{1 + 2L} c(B).$$

This equation holds in every dimension, in the case where B is a surface we can then make an explicit computation, which gives us

$$\frac{1}{2} \sum_{i=1}^{r} \rho_* j_{i*} c(D_i) = \left(12L - \left(12 - \sum_{i=1}^{r} m_i^2 \right) L^2 \right) c(B), \tag{7.5}$$

using the assumption that $\sum_{i=1}^{r} m_i = 12$.

7.3 Tadpole cancellation for threefolds

As we computed before, the tadpole cancellation (7.1) for elliptic Calabi–Yau threefolds can be written as

$$(12c_1(B)+6(ab-a^2-b^2)L^2-30c_1(B)^2)c(B) = \left(12L - \left(12 - \sum_{i=1}^{r} m_i^2\right) L^2\right) c(B). \tag{7.6}$$

The simplest way we can satisfy this equation is to require that

$$12c_1(B) + 6(ab - a^2 - b^2)L^2 - 30c_1(B)^2 = 12L - \left(12 - \sum_{i=1}^{r} m_i^2\right) L^2, \tag{7.7}$$

in fact in this way we have tadpole cancellation over any base surface. For this reason we say that (7.7) gives the *universal tadpole cancellation* relation. Observe that this last condition is purely numerical.
I will say that $\pi : X \longrightarrow B$ satisfies the tadpole relation (or the universal tadpole relation) if X sits in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ and equation (7.6) (or (7.7)) is satisfied for suitable integers m_i.
As observed before, the difference between tadpole cancellation and universal tadpole cancellation is that the second is only a numerical relation while in the first there is also a contribution of the base B. Nevertheless in the case we are studying the two concepts are equivalent.

Proposition 7.1 *Let $\pi : X \longrightarrow B$ be an elliptic threefold, with $X \subseteq \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ for $\mathcal{L}$ an ample line bundle on B. Then X satisfies the tadpole cancellation if and only if it satisfies the universal tadpole relation.*

Proof Let assume that the non-universal relation holds. We have to prove that the universal relation (7.7) is satisfied. Writing explicitly $c(B) = 1 + c_1(B) + c_2(B)$ we see that (7.6) reduces to

$$\begin{aligned} 12c_1(B) + 6(ab - a^2 - b^2)L^2 - 18c_1(B)^2 = \\ = 12L - \left(12 + \sum_{i=1}^{r} m_i^2\right) L^2 + 12c_1(B)L, \end{aligned}$$

and so the left side of the universal tadpole relation (7.7) equals

$$12L - \left(12 + \sum_{i=1}^{r} m_i^2\right) L^2 + 12c_1(B)L - 12c_1(B)^2.$$

Since we are assuming that the tadpole relation holds, we have $12c_1(B) = 12L$, it follows that $12c_1(B)^2 = 12Lc_1(B)$, i.e. that universal tadpole cancellation holds.
The other part of the proposition is obvious. □

In the case of elliptic threefolds we have then reduced the problem of tadpole cancellation to a numerical problem. Our next goal is to solve this problem, finding all the possible bundles and curves configurations.
The next proposition will help us.

Proposition 7.2 *Let $\pi : X \longrightarrow B$ be an elliptic threefold with tadpole cancellation in $X \subseteq \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ for $\mathcal{L}$ an ample line bundle on B. Then $\mathcal{L} = \omega_B^{-1}$ and so B is a del Pezzo surface.*

Proof From the universal tadpole relation we have that $12L = 12c_1(B)$, which means that $c_1(B) = L+T$ with T a torsion divisor satisfying $12T = 0$. Since L is ample, we then deduce that $c_1(B)$ is also ample. So B is a del Pezzo surface, and since these surfaces have torsion-free Picard group we have that $c_1(B) = L$. □

In Section 4.3.1 we computed that if B is a del Pezzo surface, then only for

$$(a, b) = (0, 0), (1, 1), (1, 0), (2, 1), (3, 2),$$

the generic anticanonical hypersurface in $\mathbb{P}(\omega_B^{-a} \oplus \omega_B^{-b} \oplus \mathcal{O}_B)$ is a smooth elliptic fibration. This means that with a case by case analysis of the partitions of 12 we solve the problem of finding all the cases with tadpole cancellation. This is the result.

Proposition 7.3 *Let B be a del Pezzo surface, and $X \subseteq \mathbb{P}(\omega_B^{-a} \oplus \omega_B^{-b} \oplus \mathcal{O}_B)$ a generic anticanonical hypersurface. Then X satisfies the tadpole cancellation only in the cases listed in the following table:*

(a, b)	$(m_1, \ldots, m_r)$
$(0, 0)$	$(2, 2, 2, 1, 1, 1, 1, 1, 1)$
$(1, 0)$	$(4, 1, 1, 1, 1, 1, 1, 1, 1)$
$(1, 0)$	$(3, 3, 1, 1, 1, 1, 1, 1)$
$(1, 0)$	$(3, 2, 2, 2, 1, 1, 1)$
$(1, 0)$	$(2, 2, 2, 2, 2, 2)$
$(1, 1)$	$(4, 1, 1, 1, 1, 1, 1, 1, 1)$
$(1, 1)$	$(3, 3, 1, 1, 1, 1, 1, 1)$
$(1, 1)$	$(3, 2, 2, 2, 1, 1, 1)$
$(1, 1)$	$(2, 2, 2, 2, 2, 2)$
$(2, 1)$	$(5, 2, 2, 1, 1, 1)$
$(2, 1)$	$(4, 4, 1, 1, 1, 1)$
$(2, 1)$	$(4, 3, 3, 1, 1)$
$(2, 1)$	$(3, 3, 3, 3)$
$(3, 2)$	$(7, 3, 1, 1)$

From the above table we can see that any of the five admissible cases for (a, b) has at least a configuration determining tadpole cancellation. However, to construct explicitly an example for each of them it's a quite arduous problem. In fact many constructions which are possible locally on the base, fail to globalize; see e.g. [KMSNS11].
It's my personal opinion that in order to find an example it's necessary the presence of at least a 1 in the sequence $(m_1, \ldots, m_r)$, even if I do not have a

proof for that.

7.3.1 The case of fourfolds

It's possible to perform analogous computations assuming that X is an elliptically fibered Calabi–Yau fourfold over a threefold B, as it's usual in physics. In particular, the analogous table of possible cases for the universal tadpole cancellation reduces to the following:

(a, b)	$(m_1, \dots, m_r)$
$(1, 0)$	$(2, 2, 2, 2, 2, 2)$
$(1, 1)$	$(3, 3, 1, 1, 1, 1, 1, 1)$
$(2, 1)$	$(4, 4, 1, 1, 1, 1)$

In this higher dimensional setting, the equivalence between tadpole cancellation and universal tadpole cancellation breaks, so the table above *is not* a list of all the configurations giving tadpole cancellation. It's possible to find examples of elliptic Calabi–Yau fourfolds of type $(1, 1)$ and $(1, 2)$ in the original paper [AE10]. In [CCvG11], we give a detailed proof of the fact that these three are the only possible configurations for Calabi–Yau elliptic fibrations X with universal tadpole cancellation and $\dim X \geq 3$. The family of fibrations of type $(1, 0)$ is the higher-dimensional analogous of the family of elliptic threefolds in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ studied in Chapter 6.1, but, as observed above, we couldn't find a geometric example with a configuration realizing the universal tadpole cancellation relation.

Appendix A

Computing Hodge numbers

In this appendix I will give a brief description of the algorithm introduced in [BJRR10], [RR10] and [BJRR12], and the computation made to determine the Hodge numbers in Chapter 5. There is also an independent proof of the same algorithm, which can be found in [Jow11].
Since all the line bundles $\mathcal{O}_{\mathbb{P}^2}(n)$ are over the projective plane, I will write $\mathcal{O}(n)$ to simplify the notation.
Observe that if $\pi : X \longrightarrow B$ is an elliptic Calabi–Yau fibration, then ([Ogu93, Main Theorem]) B is a rational surface. So the methods used in this appendix can be used to compute, for example, the Hodge diamonds of the Calabi–Yau elliptic fibrations in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ where $\mathcal{L}$ is an ample line bundle on a rational toric surface B. The main class of such surfaces is given by the Hirzebruch surfaces $\mathcal{F}_r$.
Observe that there is another (and well known) algorithm to compute the Hodge numbers of anticanonical hypersurfaces in toric varieties, described in [CK99]. The reason why I will use the algorithm in [BJRR12] is that the one in [CK99] only works for anticanonical hypersurfaces in *Gorenstein Fano* toric varieties, and our $\mathbb{P}^2$-bundles are Gorenstein and toric, but typically not Fano.

A.1 Toric description of $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$

The $\mathbb{P}^2$-bundle over $\mathbb{P}^2$

$$Z = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$$

is a toric variety, so we can find many information on it. Since I will use results from [Ful93] and [CLS11], I'll use mainly $\mathbb{P}_h$, but every result for $\mathbb{P}_h(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O})$ is also a result for $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O})$ provided we replace α with $-a$ and β with $-b$.

A.1.1 The fan

I want to determine the fan for the projective space bundle $\mathbb{P}_h(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O})$. The fibration is locally trivial, hence we begin with the cone of vertices

$$v_0 = (1, 0, 0, 0) \quad w_1 = (0, 0, 1, 0)$$
$$v_1 = (0, 1, 0, 0) \quad w_2 = (0, 0, 0, 1)$$

defining $\mathbb{C}^4$. To be more precise, associating coordinates to each ray in the following way

$$\begin{array}{ll} v_0 \rightsquigarrow x_1 & w_1 \rightsquigarrow y_1 \\ v_1 \rightsquigarrow x_2 & w_2 \rightsquigarrow y_2 \end{array}$$

we have $\mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[x_1, x_2, y_1, y_2]$. Defining $w_3 = (0, 0, -1, -1)$, we have that the toric variety defined by the fan with maximal cones

$$\operatorname{cone}(v_0, v_1, w_0, w_1), \qquad \operatorname{cone}(v_0, v_1, w_0, w_2), \qquad \operatorname{cone}(v_0, v_1, w_1, w_2)$$

is $\mathbb{C}^2 \times \mathbb{P}^2$, where the variable x_i's are coordinates in the base $\mathbb{C}^2$ and the y_j's are coordinates in $\mathbb{P}^2$. Coming back to the first cone, we consider the map

$$\begin{array}{ccc} \mathbb{C}^4 & \longrightarrow & \mathbb{C}^4 \\ (x_1, x_2, y_1, y_2) & \longmapsto & (x_1 x_2^{-1}, x_2^{-1}, x_2^{\alpha} y_1, x_2^{\beta} y_2) \end{array}$$

where the first two components are the glueing for the base $\mathbb{P}^2$, and the last two are the transition functions of $\mathbb{P}(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O})$. The target $\mathbb{C}^4$ is then the toric variety $\operatorname{Spec} \mathbb{C}[x_1 x_2^{-1}, x_2^{-1}, x_2^{\alpha} y_1, x_2^{\beta} y_2]$, which is given by the cone with generators

$$(1, 0, 0, 0), \qquad (-1, -1, \alpha, \beta), \qquad (0, 0, 1, 0), \qquad (0, 0, 0, 1).$$

Defining $v_2 = (-1, -1, \alpha, \beta)$, we have just showed that $\operatorname{cone}(v_0, v_2, w_1, w_2)$ is in the fan for $\mathbb{P}(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O})$.
With the same argument as before, it's possible to find all the cones in the fan: its set of edges $\Sigma(1)$ has as elements the rays generated by the six vectors of $\mathbb{R}^4$

$$\begin{array}{ll} v_0 = (1, 0, 0, 0) & w_1 = (0, 0, 1, 0) \\ v_1 = (0, 1, 0, 0) & w_2 = (0, 0, 0, 1) \\ v_2 = (-1, -1, \alpha, \beta) & w_3 = (0, 0, -1, -1) \end{array}$$

and its nine maximal cones have as generators two rays v_i and two rays w_j.

A.1.2 Invariant divisors

To each ray in $\Sigma(1)$ is associated a torus-invariant divisor. What we said before allows us to conclude that D_{v_0}, D_{v_1} and D_{v_2} are given by $p^* l_i$, being l_i the line in $\mathbb{P}^2$ associated to v_i, but then

$$\mathcal{O}(D_{v_0}) = \mathcal{O}(D_{v_1}) = \mathcal{O}(D_{v_2}) = p^* \mathcal{O}_{\mathbb{P}^2}(1).$$

We can verify this since ([Ful93, P. 61])

$$(\chi^u) = \sum_i < u, v_i > D_i :$$

using this relation on the four rational functions x_1, x_2, y_1 and y_2 we can conclude that

$$\begin{array}{ll} (x_1) = D_{v_0} - D_{v_2} & (x_2) = D_{v_1} - D_{v_2} \\ (y_1) = \alpha D_{v_2} + D_{w_1} - D_{w_3} & (y_2) = \beta D_{v_2} + D_{w_2} - D_{w_3} \end{array}$$

In particular D_{v_0}, D_{v_1} and D_{v_2} are linearly equivalent divisors, and we denote by L their class. Let ξ be the class of D_{w_3}, then

$$D_{w_1} \equiv D_{w_3} - \alpha D_{v_2} = \xi - \alpha L, \qquad D_{w_2} \equiv D_{w_3} - \beta D_{v_2} = \xi - \beta L$$

and so L and ξ are generators for $\operatorname{Pic} Z$: L is the class of the pull-back of a line in $\mathbb{P}^2$ and ξ is the hyperplane section of Z.

A.2 Cohomology of $\mathcal{O}_Z(-K_Z)$

We want to use the algorithm described in [BJRR12] to calculate the dimension of the cohomology groups $H^i(Z, -K_Z)$: then we can describe the Hodge diamond of $X \in |-K_Z|$.
Let Z be the projective bundle over $\mathbb{P}^2$ defined by

$$Z = \mathbb{P}_h(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O}),$$

which we analysed in the previous section. The Stanley–Reisner ideal of Z is the ideal of $\mathbb{Q}[u_1, \dots, u_6]$ generated by the square-free monomials $u_{i_1} \dots u_{i_t}$ for which the corresponding edges generate a cone which *is not* in the fan for Z, so it's given by[1]

$$\mathrm{SR}(Z) = \langle u_1u_2u_3, u_4u_5u_6 \rangle.$$

A.2.1 A brief description of the algorithm

Let Z be a toric variety, with homogeneous coordinates $u_1, \dots, u_n$ and Stanley–Reisner ideal $\mathrm{SR}(Z) = \langle S_1, \dots, S_t \rangle$. Let D be a divisor on Z: the following algorithm ([BJRR12, Sect. 2.1]) will compute the dimension of the cohomology spaces $H^i(Z, \mathcal{O}_Z(D))$. First of all we define:

1. for any subset $A = \{a_1, \dots, a_k\} \subseteq \{1, \dots, t\}$ of k elements, the set
$$S^k_A = \{S_{a_1}, \dots, S_{a_k}\};$$
2. $Q(S^k_A)$, the square-free monomial obtained multiplying all the variables appearing in the monomials in S^k_A;
3. the "degree" of $Q(S^k_A)$
$$M(S^k_A) = (\text{degree of } Q(S^k_A)) - k.$$

For any monomial Q it's possible to define a chain complex $\mathcal{C}_\bullet(Q)$ such that

$$\dim \mathcal{C}_i(Q) = \#\{S^k_A | Q(S^k_A) = Q,\ M(S^k_A) = i\},$$

[1] The coordinates u_i correspond to the rays in $\Sigma(1)$ in the following way:

$$\begin{array}{ll} u_1 \rightsquigarrow v_0 & u_4 \rightsquigarrow w_1 \\ u_2 \rightsquigarrow v_1 & u_5 \rightsquigarrow w_2 \\ u_3 \rightsquigarrow v_2 & u_6 \rightsquigarrow w_3 \end{array}$$

and the first step in the algorithm is to compute the *multiplicity factors*

$$h_i(Q) = \dim H_i(\mathcal{C}_\bullet(Q)),$$

which are then quantities depending only on Z and not on D.
Given any monomial Q, we consider the *rationomials* (rational monomials) associated to Q, i.e. rational functions in the homogeneous variables on Z of the form

$$R^Q(u_1, \dots, u_n) = \frac{T}{Q \cdot W}$$

where

1. T is a monomial which does not depend on the variables appearing in Q;
2. W is a monomial which depends only on the variables appearing in Q.

We then define

$$N_D(Q) = \dim\{R^Q | \deg R^Q = D\},$$

and compute ([Jow11, Cor. 1.2] and [BJRR12, Sect. 2.1])

$$\dim H^i(Z, \mathcal{O}_Z(D)) = \sum_Q h_i(Q) N_D(Q).$$

A.2.2 Computing the multiplicity factors

Since in our case $\mathrm{SR}(Z)$ has only two generators, $S_1 = u_1u_2u_3$ and $S_2 = u_4u_5u_6$, we have the following four sets:

$$\begin{aligned}
&S^0_\varnothing = \{0\}\\
&S^1_{\{1\}} = \{u_1u_2u_3\}\\
&S^1_{\{2\}} = \{u_4u_5u_6\}\\
&S^2_{\{1,2\}} = \{u_1u_2u_3, u_4u_5u_6\},
\end{aligned}$$

to which we associate the monomials

$$\begin{aligned}
&Q(S^0_\varnothing) = 1\\
&Q(S^1_{\{1\}}) = u_1u_2u_3\\
&Q(S^1_{\{2\}}) = u_4u_5u_6\\
&Q(S^2_{\{1,2\}}) = u_1u_2u_3u_4u_5u_6.
\end{aligned}$$

To each of them is associated the number

$$\begin{aligned}
&M(S^0_\varnothing) = 0 - 0 = 0\\
&M(S^1_{\{1\}}) = 3 - 1 = 2\\
&M(S^1_{\{2\}}) = 3 - 1 = 2\\
&M(S^2_{\{1,2\}}) = 6 - 2 = 4,
\end{aligned}$$

and these numbers are needed to calculate the dimensions of the vector spaces in the chain complexes associated to each of the four monomials. In our case these complexes are extremely easy, and we can compute the dimension of the

i-th homology vector space of each complex, which gives us a multiplicity factor h_i:

$$h_i(1) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad h_i(u_1u_2u_3) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$
$$h_i(u_4u_5u_6) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases} \qquad h_i(u_1u_2u_3u_4u_5u_6) = \begin{cases} 1 & \text{if } i = 4 \\ 0 & \text{otherwise.} \end{cases}$$

The algorithm then implies that for any divisor D we have

$$\begin{aligned} h^0(D) &= N_D(1), \\ h^1(D) &= 0, \\ h^2(D) &= N_D(u_1u_2u_3) + N_D(u_4u_5u_6), \\ h^3(D) &= 0, \\ h^4(D) &= N_D(u_1u_2u_3u_4u_5u_6). \end{aligned}$$

A.2.3 Computing the number of rationomials

The next step is to calculate the number of rationomials associated to a given divisor D. The rationomials we are interested in are:

$$\begin{aligned} R^1(u_1,\dots,u_6) &= T(u_1,\dots,u_6) \\ R^{u_1u_2u_3}(u_1,\dots,u_6) &= \tfrac{T(u_4,u_5,u_6)}{u_1u_2u_3W(u_1,u_2,u_3)} \\ R^{u_4u_5u_6}(u_1,\dots,u_6) &= \tfrac{T(u_1,u_2,u_3)}{u_4u_5u_6W(u_4,u_5,u_6)} \\ R^{u_1u_2u_3u_4u_5u_6}(u_1,\dots,u_6) &= \tfrac{1}{u_1u_2u_3u_4u_5u_6W(u_1,\dots,u_6)}. \end{aligned}$$

The number $N_D(Q)$ we want to determine is the number of all possible rationomials R^Q whose associated divisor is D. Denoting by e_i the exponent of u_i in T and W, then

1. to R^1 we associate the divisor
$$(e_1 + e_2 + e_3 - \alpha e_4 - \beta e_5)L + (e_4 + e_5 + e_6)\xi;$$
2. to $R^{u_1u_2u_3}$ we associate
$$(-3 - e_1 - e_2 - e_3 - \alpha e_4 - \beta e_5)L + (e_4 + e_5 + e_6)\xi;$$
3. to $R^{u_4u_5u_6}$ we associate
$$(e_1 + e_2 + e_3 + \alpha + \beta + \alpha e_4 + \beta e_5)L + (-3 - e_4 - e_5 - e_6)\xi;$$
4. to $R^{u_1u_2u_3u_4u_5u_6}$ we associate
$$(-3 + \alpha + \beta - e_1 - e_2 - e_3 + \alpha e_4 + \beta e_5)L + (-3 - e_4 - e_5 - e_6)\xi.$$

A.2.4 Cohomology of invariant divisors

The choice $D = L$ gives

$$N_L(1) = 3 \qquad N_L(u_1u_2u_3) = 0$$
$$N_L(u_4u_5u_6) = 0 \quad N_L(u_1u_2u_3u_4u_5u_6) = 0$$

and then for $k = 1, 2, 3$ we have

$$\dim H^i(Z, \mathcal{O}_Z(D_k)) = \begin{cases} 3 & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $D = \xi$ we have[2]

$$N_\xi(1) = \binom{\alpha+2}{2} + \binom{\beta+2}{2} + 1 \quad N_\xi(u_1u_2u_3) = \binom{-1-\alpha}{2} + \binom{-1-\beta}{2}$$
$$N_\xi(u_4u_5u_6) = 0 \qquad N_\xi(u_1u_2u_3u_4u_5u_6) = 0$$

hence

$$\dim H^i(Z, \mathcal{O}_Z(D_6)) = \begin{cases} \binom{\alpha+2}{2} + \binom{\beta+2}{2} + 1 & i = 0 \\ \binom{-1-\alpha}{2} + \binom{-1-\beta}{2} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $D_4 = \xi - \alpha L$ we have

$$N_{D_4}(1) = \binom{-\alpha+2}{2} + \binom{\beta-\alpha+2}{2} + 1 \quad N_{D_4}(u_1u_2u_3) = \binom{\alpha-1}{2} + \binom{\alpha-\beta-1}{2}$$
$$N_{D_4}(u_4u_5u_6) = 0 \qquad N_{D_4}(u_1u_2u_3u_4u_5u_6) = 0$$

and so

$$\dim H^i(Z, \mathcal{O}_Z(D_4)) = \begin{cases} \binom{-\alpha+2}{2} + \binom{\beta-\alpha+2}{2} + 1 & i = 0 \\ \binom{\alpha-1}{2} + \binom{\alpha-\beta-1}{2} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

In the same way

$$\dim H^i(Z, \mathcal{O}_Z(D_5)) = \begin{cases} \binom{-\beta+2}{2} + \binom{\alpha-\beta+2}{2} + 1 & i = 0 \\ \binom{\beta-1}{2} + \binom{\beta-\alpha-1}{2} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

A.2.5 Cohomology of $-K_Z$

Remembering that

$$-K_Z \equiv (3 - \alpha - \beta)L + 3\xi,$$

we have

$$N_{-K_Z}(u_4u_5u_6) = 0, \qquad N_{-K_Z}(u_1u_2u_3u_4u_5u_6) = 0$$

and then

$$\dim H^i(Z, -K_Z) = \begin{cases} N_{-K_Z}(1) & i = 0 \\ N_{-K_Z}(u_1u_2u_3) & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

[2] Form here the binomial $\binom{n}{m}$ is zero if $n < m$ (in particular this holds for $n < 0$).

and each of them is the sum of $10 = \binom{3+2}{2}$ binomial terms. An explicit computation then shows that

$$\begin{aligned} h^0(Z, -K_Z) = & \binom{2\alpha-\beta+5}{2} + \binom{\alpha+5}{2} + \binom{\beta+5}{2} + \binom{-\alpha+2\beta+5}{2} + \binom{\alpha-\beta+5}{2} + \\ & + \binom{5}{2} + \binom{-\alpha+\beta+5}{2} + \binom{-\beta+5}{2} + \binom{-\alpha+5}{2} + \binom{\alpha-\beta+5}{2}; \\ h^2(Z, -K_Z) = & \binom{-2\alpha+\beta-4}{2} + \binom{-\alpha-4}{2} + \binom{-\beta-4}{2} + \binom{\alpha-2\beta-4}{2} + \binom{-\alpha+\beta-4}{2} + \\ & + \binom{-4}{2} + \binom{\alpha-\beta-4}{2} + \binom{\beta-4}{2} + \binom{\alpha-4}{2} + \binom{\alpha+\beta-4}{2}. \end{aligned}$$

A.3 Hodge numbers of $X \in |-K_Z|$

Let $X \in |-K_Z|$ be a smooth hypersurface. Our aim is to describe the Hodge diamond of X using what we know up to now. Since X is a Calabi–Yau variety, the only interesting numbers are $h^{1,1}(X)$ and $h^{1,2}(X)$. By Serre duality with $\omega_X \simeq \mathcal{O}_X$ we have

$$H^{1,q}(X) = H^q(X, \Omega^1_X) = H^q(X, \Omega^1_X \otimes \omega_X) = H^{3-q}(X, \mathcal{T}_X), \tag{A.1}$$

where $\mathcal{T}_X$ is the tangent bundle of X. Our aim is then to compute $H^p(X, \mathcal{T}_X)$, and I will do it in several steps.

1. Tensoring the exact sequence defining X in Z

$$0 \longrightarrow \mathcal{O}_Z(K_Z) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_X \longrightarrow 0$$

   with $\mathcal{O}_Z(-K_Z)$ we get

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(X) \longrightarrow \mathcal{O}_X(X) \longrightarrow 0,$$

   which allows us to compute the cohomology of the normal bundle $\mathcal{N}_{X|Z}$: $H^p(X, \mathcal{N}_{X|Z}) = H^p(X, \mathcal{O}_X(X))$. In particular, the result is

$$\begin{aligned} & h^0(X, \mathcal{N}_{X|Z}) = h^0(Z, -K_Z) - 1, \\ & h^1(X, \mathcal{N}_{X|Z}) = 0, \\ & h^2(X, \mathcal{N}_{X|Z}) = h^2(Z, -K_Z), \\ & h^3(X, \mathcal{N}_{X|Z}) = 0, \end{aligned}$$

   and we computed $h^p(Z, -K_Z)$ before.

2. Thanks to the algorithm we know the cohomology $H^p(Z, D_j)$ for any torus invariant divisor D_j, and by Serre duality we know also $H^p(Z, K_Z - D_j)$. So we can compute $h^p(D_j, \omega_{Z|_{D_j}}) = h^p(D_j, \mathcal{O}_{D_j}(K_Z))$ from the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(K_Z - D_j) \longrightarrow \mathcal{O}_Z(K_Z) \longrightarrow \mathcal{O}_{D_j}(K_Z) \longrightarrow 0,$$

   finding

$$\begin{aligned} & h^0(D_j, \mathcal{O}_{D_j}(K_Z)) = 0, \\ & h^1(D_j, \mathcal{O}_{D_j}(K_Z)) = h^2(Z, D_j), \\ & h^2(D_j, \mathcal{O}_{D_j}(K_Z)) = 0, \\ & h^3(D_j, \mathcal{O}_{D_j}(K_Z)) = h^0(Z, D_j) - 1, \\ & h^4(D_j, \mathcal{O}_{D_j}(K_Z)) = 0. \end{aligned}$$

3. Tensor the sequence [Ful93, P. 87]

$$0 \longrightarrow \Omega^1_Z \longrightarrow \mathbb{Z}^4 \otimes \mathcal{O}_Z \longrightarrow \bigoplus_{j=1}^{6} \mathcal{O}_{D_j} \longrightarrow 0$$

by ω_Z to find

$$0 \longrightarrow \Omega^1_Z \otimes \omega_Z \longrightarrow \mathbb{Z}^4 \otimes \omega_Z \longrightarrow \bigoplus_{j=1}^{6} \mathcal{O}_{D_j}(K_Z) \longrightarrow 0,$$

and then compute $H^p(Z, \Omega^1_Z \otimes \omega_Z)$. The result is

$$\begin{aligned}
&h^0(Z, \mathcal{T}_Z) = \textstyle\sum_{j=1}^6 h^0(Z, D_j) - 2,\\
&h^1(Z, \mathcal{T}_Z) = 0,\\
&h^2(Z, \mathcal{T}_Z) = \textstyle\sum_{j=1}^6 h^2(Z, D_j),\\
&h^3(Z, \mathcal{T}_Z) = 0,\\
&h^4(Z, \mathcal{T}_Z) = 0,
\end{aligned}$$

where by Serre duality $H^p(Z, \mathcal{T}_Z) = H^{4-p}(Z, \Omega^1_Z \otimes \omega_Z)$.

4. Exploiting the fact that

$$H^p(Z, \mathcal{T}_Z(-X)) = H^p(Z, \mathcal{T}_Z \otimes \omega_Z) = H^{4-p}(Z, \Omega^1_Z)$$

we can compute the cohomology $H^p(X, \mathcal{T}_{Z|_X})$ from the exact sequence

$$0 \longrightarrow \mathcal{T}_Z(-X) \longrightarrow \mathcal{T}_Z \longrightarrow \mathcal{T}_{Z|_X} \longrightarrow 0.$$

In particular,

$$\begin{aligned}
&h^0(X, \mathcal{T}_{Z|_X}) = \textstyle\sum_{j=1}^6 h^0(Z, D_j) - 2,\\
&h^1(X, \mathcal{T}_{Z|_X}) = 0,\\
&h^2(X, \mathcal{T}_{Z|_X}) = \textstyle\sum_{j=1}^6 h^2(Z, D_j) + 2,\\
&h^3(X, \mathcal{T}_{Z|_X}) = 0.
\end{aligned}$$

5. Finally, from the normal bundle sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{Z|_X} \longrightarrow \mathcal{N}_{X|Z} \longrightarrow 0$$

we can compute $H^p(X, \mathcal{T}_X)$:

$$\begin{aligned}
&h^0(X, \mathcal{T}_X) = 0,\\
&h^1(X, \mathcal{T}_X) = h^0(Z, -K_Z) + 1 - \textstyle\sum_{j=1}^6 h^0(Z, D_j),\\
&h^2(X, \mathcal{T}_X) = \textstyle\sum_{j=1}^6 h^2(Z, D_j) + 2 - h^2(Z, -K_Z),\\
&h^3(X, \mathcal{T}_X) = 0.
\end{aligned}$$

In this way we can compute $h^{1,1}(X)$ and $h^{1,2}(X)$: the results are in Table 5.2.

Remark A.3.1 The long exact sequence of

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{Z|_X} \longrightarrow \mathcal{N}_{X|Z} \longrightarrow 0$$

begins with

$$0 \to H^0(X, \mathcal{T}_X) \to H^0(X, \mathcal{T}_{Z|_X}) \to H^0(X, \mathcal{N}_{X|Z}) \to H^1(X, \mathcal{T}_X) \to H^1(X, \mathcal{T}_{Z|_X}).$$

From the previous steps we know that $H^0(X, \mathcal{T}_X) = 0$, $H^0(X, \mathcal{T}_{Z|_X}) = 0$ and also that there is an isomorphism

$$H^0(X, \mathcal{T}_{Z|_X}) \simeq H^0(Z, \mathcal{T}_Z).$$

We can then deduce that the short sequence

$$0 \longrightarrow H^0(Z, \mathcal{T}_Z) \longrightarrow H^0(X, \mathcal{N}_{X|Z}) \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow 0$$

is exact.

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