RIEMANN COMPATIBLE TENSORS

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ABSTRACT. Derdzinski and Shen's theorem on the restrictions posed by a Codazzi tensor on the Riemann tensor holds more generally when a Riemann-compatible tensor exists. Several properties are shown to remain valid in this broader setting. Riemann compatibility is equivalent to the Bianchi identity of the new "Codazzi deviation tensor", with a geometric significance. The general properties are studied, with their implications on Pontryagin forms. Examples are given of manifolds with Riemann-compatible tensors, in particular those generated by geodesic mappings. Compatibility is extended to generalized curvature tensors, with an application to Weyl's tensor and general relativity.

1. INTRODUCTION

The Riemann tensor $R_{ijk}{}^m$ and its contractions, $R_{kl} = R_{kml}{}^m$ and $R = g^{kl}R_{kl}$, are the fundamental tensors to describe the local structure of a Riemannian manifold $(\mathscr{M}_n, g)$ of dimension n. In a remarkable theorem [9, 3] Derdzinski and Shen showed that the existence of a nontrivial Codazzi tensor poses strong constraints on the structure of the Riemann tensor. Because of their geometric relevance, Codazzi tensors have been studied by several authors, such as Berger and Ebin [1], Bourguignon [4], Derdzinski [7, 8], Derdzinski and Shen [9], Ferus [10], Simon [28]; a compendium of results is found in Besse's book [3]. Recently, we showed [21] that the Codazzi differential condition

$$\nabla_i b_{jk} - \nabla_j b_{ik} = 0, \tag{1.1}$$

sufficient for the theorem to hold, can be replaced by the more general notion of *Riemann-compatibility*, which is instead algebraic:

Definition 1.1. A symmetric tensor b_{ij} is Riemann compatible (R-compatible) if:

$$b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m = 0. \tag{1.2}$$

With this requirement, we proved the following extension of Derdzinski and Shen's theorem:

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Theorem 1.2. [21] *Suppose that a symmetric R-compatible tensor b_{ij} exists. Then, if X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point of the manifold, with eigenvalues λ, μ and ν, one has $R_{ijkl}X^i Y^j Z^k = 0$ provided that both λ and μ are different from ν.*

The concept of compatibility allows for a further extension of the theorem, where the Riemann tensor R is replaced by a generalized curvature tensor K, and b is required to be K-compatible [21].

This paper studies the properties of Riemann compatibility, and its implications on the geometry of the manifold. In section 2 R-compatibility is shown to be equivalent to the Bianchi identity of a new tensor, the *Codazzi deviation*. In section 3 the irreducible components of the covariant derivative of a symmetric tensor are classified in a simple manner, based on the decomposition into traceless terms. This is helpful in the study of various structures suited for R-compatibility. The general properties of Riemann compatibility are presented in section 4. In section 5 several properties of manifolds in the presence of a Riemann compatible tensor that were obtained by Derdzinski and Shen and Bourguignon for manifolds with a Codazzi tensor, are recovered. In particular, it is shown that R-compatibility implies pureness, a property of the Riemann tensor introduced by Maillot that implies the vanishing of Pontryagin forms. Manifolds that carry R-compatible tensors are presented in section 6; interesting examples are generated by geodesic mappings, that induce metric tensors that are R-compatible. Finally, in section 7, K-tensors and K-compatibility are presented, with applications to the standard curvature tensors. In the end, an application to general relativity is mentioned, that will be discussed fully elsewhere.

2. The Codazzi deviation tensor and R-compatibility

Since Codazzi tensors are Riemann compatible, for a non-Codazzi differentiable symmetric tensor field b it is useful to define its deviation from the Codazzi condition. This tensor satisfies an unexpected relation that generalizes Lovelock's identity for the Riemann tensor, and shows that Riemann compatibility is a condition for closedness of certain 2-forms.

Definition 2.1. The *Codazzi deviation* of a symmetric tensor b_{kl} is

$$\mathscr{C}_{jkl} =: \nabla_j b_{kl} - \nabla_k b_{jl} \tag{2.1}$$

Simple properties are: $\mathscr{C}_{jkl} = -\mathscr{C}_{kjl}$ and $\mathscr{C}_{jkl} + \mathscr{C}_{klj} + \mathscr{C}_{ljk} = 0$.

The following identity holds in general, and relates the Bianchi differential combination for $\mathscr{C}$ to the Riemann compatibility of b:

Proposition 2.2.

$$\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} = b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m \tag{2.2}$$

Proof.

$$\begin{aligned}\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} &= [\nabla_i, \nabla_j] b_{kl} + [\nabla_k, \nabla_i] b_{jl} + [\nabla_j, \nabla_k] b_{il} \\ &= b_{ml}(R_{ijk}{}^m + R_{kij}{}^m + R_{jki}{}^m) + b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m\end{aligned}$$

and the first term vanishes by the first Bianchi identity. □

Remark 2.3. *The identity* (2.2) *holds true if b_{ij} is replaced by $b'_{ij} = b_{ij} + \chi a_{ij}$, where a_{ij} is a Codazzi tensor and χ a scalar field. Then: $\mathscr{C}'_{jkl} = \mathscr{C}_{jkl} - (a_{kl}\nabla_j - a_{jl}\nabla_k)\chi$.*

The deviation tensor is associated to the 2-form $\mathscr{C}_l = \frac{1}{2}\mathscr{C}_{jkl} dx^j \wedge dx^k$. The closedness condition $0 = D\mathscr{C}_l = \frac{1}{2}\nabla_i \mathscr{C}_{jkl} dx^i \wedge dx^j \wedge dx^k$ (D is the exterior covariant derivative) is the second Bianchi identity for the Codazzi deviation: $\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} = 0$. This gives a geometric picture of Riemann compatibility:

Theorem 2.4. *b_{ij} is Riemann compatible if and only if $D\mathscr{C}_l = 0$.*

Remark 2.5. *The Codazzi deviation of the Ricci tensor is, by the contracted second Bianchi identity: $\mathscr{C}_{jkl} =: \nabla_j R_{kl} - \nabla_k R_{jl} = -\nabla_m R_{jkl}{}^m$. For the Ricci tensor the identity* (2.2) *identifies with Lovelock's identity* [17] *for the Riemann tensor:*

$$\begin{aligned}\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\ = -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m.\end{aligned} \tag{2.3}$$

A Veblen-like identity holds, that corresponds to (4.2) (For $b_{ij} = R_{ij}$ it specializes to Veblen's identity for the divergence of the Riemann tensor [19]):

Proposition 2.6.

$$\begin{aligned}\nabla_i \mathscr{C}_{jlk} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{lji} + \nabla_l \mathscr{C}_{ikj} \\ = b_{im} R_{jlk}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{lji}{}^m + b_{lm} R_{ikj}{}^m\end{aligned} \tag{2.4}$$

Proof. Write four versions of equation (2.2) with cyclically permuted indices i, j, k, l and sum up. Then simplify by means of the first Bianchi identity for the Riemann tensor and the cyclic identity $\mathscr{C}_{jkl} + \mathscr{C}_{klj} + \mathscr{C}_{ljk} = 0$. □

3. Irreducible components for $\nabla_j b_{kl}$ and R-compatibility

We begin with a simple procedure to classify the $O(n)$ invariant components of the tensor $\nabla_j b_{kl}$. They will guide us in the study of R-compatibility. If b is the Ricci tensor, this simple construction reproduces the seven equations linear in $\nabla_i R_{jk}$, invariant for the $O(n)$ group, that are enumerated and discussed in Besse's treatise "Einstein Manifolds" [3].
For a symmetric tensor b_{kl} with $\nabla_j b_{kl} \neq 0$, the tensor $\nabla_j b_{kl}$ can be decomposed into $O(n)$ invariant terms, where $\mathscr{B}^0_{jkl}$ is traceless ($\mathscr{B}^0{}_{jk}{}^j = \mathscr{B}^0{}_{kj}{}^j = 0$) [13, 16]:

$$\nabla_j b_{kl} = \mathscr{B}^0_{jkl} + A_j g_{kl} + B_k g_{jl} + B_l g_{jk} \tag{3.1}$$

$$A_j = \frac{(n+1)\nabla_j b^m{}_m - 2\nabla_m b^m{}_j}{n^2+n-2}, \qquad B_j = -\frac{\nabla_j b^m{}_m - n\nabla_m b^m{}_j}{n^2+n-2} \tag{3.2}$$

The traceless tensor can then be written as a sum of orthogonal components [17]:

$$\mathscr{B}^0_{jkl} = \frac{1}{3}\left[\mathscr{B}^0_{jkl} + \mathscr{B}^0_{klj} + \mathscr{B}^0_{ljk}\right] + \frac{1}{3}\left[\mathscr{B}^0_{jkl} - \mathscr{B}^0_{kjl}\right] + \frac{1}{3}\left[\mathscr{B}^0_{jlk} - \mathscr{B}^0_{ljk}\right] \tag{3.3}$$

The orthogonal subspaces classify the $O(n)$ invariant equations that are linear in $\nabla_j b_{kl}$. The trivial subspace: $\nabla_j b_{kl} = 0$. The subspace $\mathcal{I}$ (we follow Gray's notation, [12]) where $\mathscr{B}^0_{jkl} = 0$:

$$\nabla_j b_{kl} = A_j g_{kl} + B_k g_{jl} + B_l g_{jk}.$$

The complement $\mathcal{I}^\perp$ is characterized by $A_j, B_j = 0$ i.e. $\nabla_j b_{kl}$ is traceless. This gives two invariant equations: $\nabla_j b^j{}_l = 0$, and $\nabla_j b^m{}_m = 0$. Since $\nabla_j b_{kl} = \mathscr{B}^0_{jkl}$, the structure of $\mathscr{B}^0$ specifies two orthogonal subspaces, so that $\mathcal{I}^\perp = \mathcal{A} \oplus \mathcal{B}$. In $\mathcal{A}$:

$$\nabla_j b_{kl} + \nabla_k b_{lj} + \nabla_l b_{jk} = 0.$$

In $\mathcal{B}$:

$$\nabla_j b_{kl} - \nabla_k b_{jl} = 0.$$

The subspace $\mathcal{I} \oplus \mathcal{A}$ contains tensors with traceless part $\nabla_j b_{kl} - A_j g_{kl} - B_k g_{jl} - B_l g_{jk}$ that satisfies the cyclic condition:

$$[\nabla_j b_{kl} - \frac{1}{n+2}(\nabla_j b^m{}_m + 2\nabla_m b^m{}_j) g_{kl}] + cyclic = 0.$$

The subspace $\mathcal{I}\oplus\mathcal{B}$ contains tensors with traceless part that satisfies the Codazzi condition:

$$[\nabla_j b_{kl} - \frac{1}{n-1}(\nabla_j b^m{}_m - \nabla_m b^m{}_j)g_{kl}] = [\nabla_k b_{jl} - \frac{1}{n-1}(\nabla_k b^m{}_m - \nabla_m b^m{}_k)g_{jl}]$$

Accordingly, the Codazzi deviation tensor has the (unique) decomposition in irreducible components

$$\mathscr{C}_{jkl} = \mathscr{C}^0_{jkl} + \lambda_j g_{kl} - \lambda_k g_{jl}, \qquad \lambda_j = A_j - B_j = \frac{\nabla_j b^m{}_m - \nabla_m b^m{}_j}{n-1} \tag{3.4}$$

where $\mathscr{C}^0$ is traceless. Eq.(2.2) becomes

$$\begin{aligned} b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m &= \nabla_i\mathscr{C}^0_{jkl} + \nabla_j\mathscr{C}^0_{kil} + \nabla_k\mathscr{C}^0_{ijl} \\ &+ g_{il}(\nabla_j\lambda_k - \nabla_k\lambda_j) + g_{jl}(\nabla_k\lambda_i - \nabla_i\lambda_k) + g_{kl}(\nabla_i\lambda_j - \nabla_j\lambda_i) \end{aligned} \tag{3.5}$$

There are only two orthogonal invariant cases:
- $\mathscr{C}^0_{jkl} = 0$, then b is $R-$compatible if and only if λ is closed. If b is the Ricci tensor, this requirement gives nearly conformally symmetric $(NCS)_n$ manifolds, that were introduced by Roter [27].
-$\nabla_j b^m{}_m - \nabla_m b^m{}_j = 0$ then b is $R-$compatible if and only if $\mathscr{C} = \mathscr{C}^0$ satisfies the second Bianchi identity. If b is the Ricci tensor, this corresponds to $\nabla_j R = 0$.

Remark 3.1. *The decomposition* (3.4) *for the deviation of the Ricci tensor turns out to be*

$$\mathscr{C}_{jkl} = -\frac{n-2}{n-3}\nabla_m C_{jkl}{}^m + \frac{1}{2(n-1)}\left[g_{kl}\nabla_j R - g_{jl}\nabla_k R\right] \tag{3.6}$$

where $C_{jkl}{}^m$ is the conformal curvature tensor, or Weyl's tensor. In this case the λ covector is closed.

4. Riemann compatibility: general properties

The existence of a Riemann compatible tensor has various implications. A first one is the existence of a new generalized curvature tensor. This leads to the generalization of the Derdzinski-Shen theorem and other relations that were obtained for Codazzi tensors.

We need the definition, from Kobayashi and Nomizu's book [15]:

Definition 4.1. A tensor K_{ijlm} is a *generalized curvature tensor* (or, briefly, a K-tensor) if it has the symmetries of the Riemann curvature tensor:
a) $K_{ijkl} = -K_{jikl} = -K_{ijlk}$,
b) $K_{ijkl} = K_{klij}$,
c) $K_{ijkl} + K_{jkil} + K_{kijl} = 0$ (first Bianchi identity).

It follows that the tensor $K_{jk} =: -K_{mjk}{}^m$ is symmetric.
In ref. [21], Lemma 2.2, we proved this interesting fact:

Theorem 4.2. *If b is R-compatible then $K_{ijkl} =: R_{ijpq}b^p{}_k b^q{}_l$ is a K-tensor.*

The next result remarks the relevance of the local basis of eigenvectors of the Ricci tensor. Another symmetric contraction of the Riemann tensor was introduced by Bourguignon [4]:

$$\mathring{R}_{ij} =: b^{pq}R_{pijq}. \tag{4.1}$$

Theorem 4.3. *If b is R-compatible then:*
1) $b_{im}R_j{}^m - b_{jm}R_i{}^m = 0$,
2) $b_{im}\mathring{R}_j{}^m - b_{jm}\mathring{R}_i{}^m = 0$

Proof. The first identity is proven by transvecting (1.2) with g^{kl}. The second one is a restatement of the symmetry of the tensor K_{ij}. □

Remark 4.4. *A) Identities 1 and 2 are here obtained directly from R-compatibility. Bourguignon* [4] *obtained them from Weitzenböck's formula for Codazzi tensors, and Derdzinski and Shen* [9] *from their theorem.*
B) As the symmetric matrices b_{ij}, R_{ij}, $\mathring{R}_{ij}$ commute, they share at each point of the manifold an orthonormal set of n eigenvectors.
C) If b' is a symmetric tensor that commutes with a Riemann compatible b, then it can be shown that $\mathring{R}'_{ij} =: b'^{pq}R_{pijq}$ commutes with b.

Finally, this Veblen-type identity holds:

Proposition 4.5. *If b is R-compatible, then:*

$$b_{im}R_{jlk}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{lji}{}^m + b_{lm}R_{ikj}{}^m = 0 \tag{4.2}$$

Proof. Write four versions of equation (1.2) with cyclically permuted indices i, j, k, l and sum up, and use the first Bianchi identity. □

5. Pure Riemann tensors and Pontryagin forms

Riemann compatibility and nondegeneracy of the eigenvalues of b imply directly that the Riemann tensor is *pure* and Pontryagin forms vanish.
We quote two results from Maillot's paper [18]:

Definition 5.1. In a Riemann manifold $\mathscr{M}_n$, the Riemann curvature tensor is pure if at each point of the manifold there is an orthonormal basis of n tangent vectors $X(1), \dots, X(n)$, $X(a)^i X(b)_i = \delta_{ab}$, such that the tensors $X(a)^i \wedge X(b)^j =: X(a)^i X(b)^j - X(a)^j X(b)^i$, $a < b$, diagonalize it:

$$R_{ij}{}^{lm} X(a)^i \wedge X(b)^j = \lambda_{ab} X(a)^l \wedge X(b)^m \tag{5.1}$$

Theorem 5.2. *If a Riemannian manifold has pure Riemann curvature tensor, then all Pontryagin forms vanish.*

Consider the maps on tangent vectors, built with the Riemann tensor,

$$\omega_4(X_1 \dots X_4) = R_{ija}{}^b R_{klb}{}^a (X_1^i \wedge X_2^j)(X_3^k \wedge X_4^l),$$
$$\omega_8(X_1 \dots X_8) = R_{ija}{}^b R_{klb}{}^c R_{mnc}{}^d R_{pqd}{}^a (X_1^i \wedge X_2^j) \cdots (X_7^p \wedge X_8^q),$$
$$\dots\dots$$

They are antisymmetric under exchange of vectors in the single pairs, and for cyclic permutation of pairs. The *Pontryagin forms* [25] Ω_{4k} result from total antisymmetrization of ω_{4k}: $\Omega_{4k}(X_1 \dots X_{4k}) = \sum_P (-1)^P \omega_{4k}(X_{i_1} \dots X_{i_{4k}})$ where P is the permutation taking $(1 \dots 4k)$ to $(i_1 \dots i_{4k})$. $\Omega_{4k} = 0$ if two vectors repeat, intermediate forms Ω_{4k-2} vanish identically.
Pontryagin forms on generic tangent vectors are linear combinations of forms evaluated on basis vectors.
If the Riemann tensor is pure, all Pontryagin forms on the basis of eigenvectors of the Riemann tensor vanish. For example, if X, Y, Z, W are orthogonal: $\omega_4(XYZW) = \lambda_{XY}\lambda_{ZW}(X^a \wedge Y^b)(Z_b \wedge W_a) = 0$ and $\Omega_4(XYZU) = 0$.

A consequence of the extended Derdzinski-Shen theorem 1.2 is the following:

Theorem 5.3. *If a symmetric tensor field b_{ij} exists that is R-compatible and has distinct eigenvalues at each point of the manifold, then the Riemann tensor is pure and all Pontryagin forms vanish.*

Proof. At each point of the manifold the symmetric matrix $b_{ij}(x)$ is diagonalized by n tangent orthonormal vectors $X(a)$, with distinct eigenvalues. Since b is R-compatible, theorem 1.2 holds and, because of antisymmetry of R in first two indices:

$$0 = R_{ij}{}^{kl} X(a)^i \wedge X(b)^j X(c)_k, \quad a \neq b \neq c.$$

This means that all column vectors of the matrix $V(a,b)^{kl} = R_{ij}{}^{kl} X(a)^i \wedge X(b)^j$ are orthogonal to vectors $X(c)$ i.e. they belong to the subspace spanned by $X(a)$ and $X(b)$. Because of antisymmetry in indices k, l, it is necessarily $V(a,b) = \lambda_{ab} X(a) \wedge X(b)$, i.e. the Riemann tensor is pure. □

This property has been checked by Petersen [26] in various examples with rotationally invariant metrics, by giving explicit orthonormal frames such that $R(e_i, e_j)e_k = 0$.

6. Structures for Riemann compatibility

Some differential structures are presented that yield Riemann compatibility. Of particular interest are geodesic mappings, which leave the condition for R-compatibility form-invariant, and generate R-compatible tensors. Other examples where b is the Ricci tensor are discussed in [20, 22]

6.1. **Pseudo-K-symmetric manifolds.** They are characterized by a generalized curvature tensor K such that ([5, 23])

$$\nabla_i K_{jkl}{}^m = 2A_i K_{jkl}{}^m + A_j K_{ikl}{}^m + A_k K_{jil}{}^m + A_l K_{jki}{}^m + A^m K_{jkli},$$

The tensor $b_{jk} =: K_{jmk}{}^m$ is symmetric. It is $R-$compatible if its Codazzi deviation $\mathscr{C}_{ikl} = A_i b_{kl} - A_k b_{il} + 3A_m K_{ikl}{}^m$ fulfills the second Bianchi identity. This is ensured by A_m being concircular, i.e. $\nabla_i A_m = A_i A_m + \gamma\, g_{im}$.

6.2. **Generalized Weyl tensors.** A Riemannian manifold is a $(NCS)_n$ [27] if the Ricci tensor satisfies $\nabla_j R_{kl} - \nabla_k R_{jl} = \frac{1}{2(n-1)}[g_{kl}\nabla_j R - g_{jl}\nabla_k R]$. As such, the Ricci tensor is the Weyl tensor, and the left-hand side is its Codazzi deviation. This condition, by (3.6), is equivalent to $\nabla_m C_{jkl}{}^m = 0$. This suggests a class of deviations of a symmetric tensor b with $\mathscr{C}^0_{jkl} = 0$ in (3.4):

$$\mathscr{C}_{jkl} = \lambda_j g_{kl} - \lambda_k g_{jl} \tag{6.1}$$

Proposition 6.1. *b is R-compatible if and only if λ_i is closed.*

Proof. Transvect (3.5) with g^{kl} and obtain: $-b_i{}^m R_{jm} + b_j{}^m R_{im} = (n-2)(\nabla_i \lambda_j - \nabla_j \lambda_i)$. Then b commutes with the Ricci tensor if and only if λ is closed and, by the previous equation, b is R-compatible. □

An example is provided by spaces with

$$\nabla_j b_{kl} = A_j g_{kl} + B_k g_{jl} + B_l g_{jk}, \tag{6.2}$$

where $\mathscr{C}_{jkl} = \lambda_j g_{kl} - \lambda_k g_{jl}$ with $\lambda_j = A_j - B_j$. Sinyukov manifolds [29] are of this sort, with b_{ij} being the Ricci tensor itself.

6.3. **Geodesic mappings.** Riemann compatible tensors arise naturally in the study of geodesic mappings, i.e. mappings that preserve geodesic lines [24, 11]. Their importance arise from the fact that Sinyukov manifolds are $(NCS)_n$ manifolds and they always admit a nontrivial geodesic mapping. Geodesic mappings preserve Weyl's projective curvature tensor [29]. We show that they also preserve the form of the compatibility relation.

A map $f : (\mathscr{M}_n, g) \to (\mathscr{M}_n, \overline{g})$ is *geodesic* if and only if Christoffel symbols are related by $\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k X_j + \delta_j^k X_i$ where, on a Riemannian manifold, X is closed ($\nabla_i X_j = \nabla_j X_i$). The condition is equivalent to:

$$\nabla_k \overline{g}_{jl} = 2X_k \overline{g}_{jl} + X_j \overline{g}_{kl} + X_l \overline{g}_{kj} \tag{6.3}$$

which has the form (6.2). The corresponding relation between Riemann tensors is

$$\overline{R}_{jkl}{}^m = R_{jkl}{}^m + \delta_j^m P_{kl} - \delta_k^m P_{jl} \tag{6.4}$$

where $P_{kl} = \nabla_k X_l - X_k X_l$ is the *deformation* tensor. The symmetry $P_{kl} = P_{lk}$ is ensured by closedness of X.

Proposition 6.2. *Geodesic mappings preserve R-compatibility:*

$$b_{im}\overline{R}_{jkl}{}^m + b_{jm}\overline{R}_{kil}{}^m + b_{km}\overline{R}_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{6.5}$$

where b is a symmetric tensor.

Proof. Let's show that the difference of the two sides is zero. Eq.(6.4) gives:
$b_{im}(\delta_j^m P_{kl} - \delta_k^m P_{jl}) + b_{jm}(\delta_k^m P_{il} - \delta_i^m P_{kl}) + b_{km}(\delta_i^m P_{jl} - \delta_j^m P_{il})$
$= b_{ij}P_{kl} - b_{ik}P_{jl} + b_{jk}P_{il} - b_{ji}P_{kl} + b_{ki}P_{jl} - b_{kj}P_{il} = 0$ □

Since $\overline{g}$ is trivially $\overline{R}$-compatible (first Bianchi identity), form invariance implies:

Corollary 6.3. *$\overline{g}$ is R-compatible.*

7. Generalized curvature tensors.

Several results that are valid for the Riemann tensor with a Riemann compatible tensor, extend to generalized curvature tensors K_{ijkl} (hereafter referred to as K-tensors) with a K-compatible symmetric tensor b_{jk}. The classical curvature tensors are K-tensors. The compatibility with the Ricci tensor is then examined.

Definition 7.1. A symmetric tensor b_{ij} is *K-compatible* if

$$b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = 0. \tag{7.1}$$

The metric tensor is always K-compatible, as (7.1) then coincides with the first Bianchi identity for K.

Proposition 7.2. *If K_{ijlm} is a K-tensor and b_{kl} is K-compatible, then $\hat{K}_{ijkl} =: K_{ijrs}b_k{}^r b_l{}^s$ is a K-tensor.*

We quote without proof the extension of Derdzinski and Shen theorem for generalized curvature tensors [21]:

Theorem 7.3. *Suppose that K_{ijkl} is a K-tensor, and a symmetric K-compatible tensor b_{ij} exists. Then, if X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point x of the manifold, with eigenvalues λ, μ and ν, it is $X^i\, Y^j\, Z^k\, K_{ijkl} = 0$ provided that both λ and μ are different from ν.*

Proposition 7.4. *If b is $K-$ compatible, and b commutes with a tensor h, then the symmetric tensor $\mathring{K}_{kl} =: K_{jklm}h^{jm}$ commutes with b.*

Proof. Multiply relation of K compatibility for b by h^{kl}. The last term vanishes for symmetry. The remaining terms give the null commutation relation. □

In ref.[19] (prop.2.4) we proved that a generalization of Lovelock's identity (2.3) holds for certain K-tensors, that include all classical curvature tensors:

Proposition 7.5. *Let $K_{jkl}{}^m$ be a K-tensor with the property*

$$\nabla_m K_{jkl}{}^m = \alpha\, \nabla_m R_{jkl}{}^m + \beta\, (a_{kl}\nabla_j - a_{jl}\nabla_k)\, \varphi, \tag{7.2}$$

where α, β are non zero constants, φ is a real scalar function and a_{kl} is a Codazzi tensor. Then:

$$\begin{aligned} \nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m \\ = -\alpha(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \end{aligned} \tag{7.3}$$

7.1. **ABC curvature tensors.** A class of curvature tensors with the structure (7.2) are the ABC curvature tensors. They are combinations of the Riemann tensor and its contractions (A, B, C are constants unless otherwise stated):

$$\begin{aligned} K_{jkl}{}^m = R_{jkl}{}^m \;+\; A(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl}) + B(R_j{}^m g_{kl} - R_k{}^m g_{jl}) \\ +C(R\delta_j{}^m g_{kl} - R\delta_k{}^m g_{jl}) \end{aligned} \tag{7.4}$$

The canonical curvature tensors are of this sort:

- *Conformal* tensor C_{ijkl}: $A = B = \frac{1}{n-2}$, $C = -\frac{1}{(n-1)(n-2)}$;
- *Conharmonic* tensor N_{ijkl}: $A = B = \frac{1}{n-2}$, $C = 0$;
- *Projective* tensor: P_{ijkl}: $A = \frac{1}{n-1}$, $B = C = 0$;
- *Concircular* tensor: $\tilde{C}_{ijkl}$: $A - B = 0$, $C = \frac{1}{n(n-1)}$.

ABC tensors are generalized curvature tensors (in the sense of Kobayashi and Nomizu, Def. 4.1) only for $A = B$. If $A \neq B$ the (0,4) tensor is not antisymmetric in the last two indices.

Proposition 7.6. *Let $K_{jkl}{}^m$ be an ABC tensor (A, B, C may be scalar functions) and b_{ij} a symmetric tensor;*

1) if b is R-compatible then b is K-compatible.
2) if b is K-compatible and $B \neq \frac{1}{n-2}$ then b is R-compatible.

Proof. The following identity holds for ABC tensors and a symmetric tensor b:

$$b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \\ +B\left[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)\right]. \tag{7.5}$$

1) by theorem 4.3, if b is R-compatible then it commutes with the Ricci tensor, and K-compatibility follows.
2) if b is K-compatible it commutes with K_{ij}. Contraction with g^{kl} gives:

$$b_{im}K_j{}^m - b_{jm}K_i{}^m = (b_{im}R_j{}^m - b_{jm}R_i{}^m)[1 - B(n-2)],$$

then, if $B \neq \frac{1}{n-2}$, b commutes with the Ricci tensor and by (7.5) it is R-compatible. □

The first statement of the proposition was proven for $A = B$ in [21], Prop. 3.4.

Proposition 7.7. *Let K be an ABC tensor with constant $A \neq 1$ and B. If*

$$\nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m = 0 \tag{7.6}$$

then the Ricci tensor is K-compatible.

Proof. If A and B are constants, one evaluates

$$\nabla_m K_{jkl}{}^m = (1-A)\nabla_m R_{jkl}{}^m + \frac{1}{2}(B+2C)\left(g_{kl}\nabla_j R - g_{jl}\nabla_k R\right), \tag{7.7}$$

Lovelock's identity (2.3) for the Riemann tensor implies

$$\nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m \\ = -(1-A)(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \tag{7.8}$$

In the right-hand side the Riemann tensor can be replaced by the tensor K by (7.5) written for the Ricci tensor. □

Sufficient conditions are: K is harmonic, K is recurrent (with closed recurrency 1-form, see eq.(3.13) in [19]). Note that prop. 7.7 remains valid for the Weyl conformal tensor, which is traceless.

8. Weyl-compatibility and General Relativity

In general relativity, the Ricci tensor is related to the energy-momentum tensor by the Einstein equation: $R_{jl} = \frac{1}{2}Rg_{jl} + kT_{jl}$ with scalar curvature $R = -2kT/(n-2)$ ($T = T^k{}_k$).
The contracted second Bianchi identity gives

$$\nabla_m R_{jkl}{}^m = k\left(\nabla_k T_{jl} - \nabla_j T_{kl}\right) + \frac{1}{2}\left(g_{jl}\nabla_k R - g_{kl}\nabla_j R\right).$$

Let K be an ABC tensor, with constant A, B, C. Its divergence (7.7) can be expressed in terms of the gradient of the trace of the energy momentum tensor T_{ij}. In the same way Einstein's equations and (7.8) give an equation which is local in the energy momentum tensor:

$$\begin{aligned}(8.1)\qquad &\nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m \\ &= -(1-A)k\left(T_{im}K_{jkl}{}^m + T_{jm}K_{kil}{}^m + T_{km}K_{ijl}{}^m\right).\end{aligned}$$

The Weyl tensor $C_{jkl}{}^m$ is the traceless part of the Riemann tensor, and it is an ABC tensor. There are advantages in discussing General Relativity by taking the Weyl tensor as the fundamental geometrical quantity [2, 14, 6].
The first equation (7.7)

$$\nabla_m C_{jkl}{}^m = k\frac{n-3}{n-2}\left[\nabla_k T_{jl} - \nabla_j T_{kl} + \frac{1}{n-1}\left(g_{jl}\nabla_k T - g_{kl}\nabla_j T\right)\right]$$

is reported in textbooks, such as De Felice [6], Hawking Ellis [14], Stephani [30], and in the paper [2]. Instead, a further derivation yields a Bianchi-like equation for the divergence, eq.(8.1), which contains no derivatives of the sources

$$\begin{aligned}(8.2)\qquad &\nabla_i\nabla_m C_{jkl}{}^m + \nabla_j\nabla_m C_{kil}{}^m + \nabla_k\nabla_m C_{ijl}{}^m \\ &= -k\frac{n-3}{n-2}\left(T_{im}C_{jkl}{}^m + T_{jm}C_{kil}{}^m + T_{km}C_{ijl}{}^m\right).\end{aligned}$$

It can be viewed as a condition for Weyl-compatibility for the energy momentum tensor.
In view of prop.7.4 and the previous equation, the following holds:

Proposition 8.1. *If T_{ij} is Weyl-compatible, the symmetric tensor $\mathring{C}_{kl} =: T^{jm}C_{jklm}$ commutes with T_{ij}.*

In 4 dimensions, given a time-like velocity field u^i, Weyl's tensor is projected in longitudinal (electric) and transverse (magnetic) tensorial components [2]

$$E_{kl} = u^j u^m C_{jklm}, \qquad H_{kl} = \frac{1}{4}u^j u^m\left(\epsilon_{pqjk}C^{pq}{}_{lm} + \epsilon_{pqjl}C^{pq}{}_{km}\right)$$

that solve equations that resemble Maxwell's equations with source. Therefore, the tensor $E_{kl} = \mathring{C}_{kl}$ can be viewed as a generalized electric field. It coincides with the standard definition if $T_{ij} = (p+\rho)u_i u_j + p g_{ij}$ (perfect fluid). The generalized magnetic field is $H_{kl} = \frac{1}{4}T^{jm}(\epsilon_{pqjk}C^{pq}{}_{lm} + \epsilon_{pqjl}C^{pq}{}_{km})$.

Proposition 8.2. *If T_{kl} is Weyl compatible then $H_{kl} = 0$.*

Proof. From the condition for Weyl compatibility we obtain $\epsilon_{ijkp}[T^{im}C^{jk}{}_{lm} + T^{jm}C^{ki}{}_{lm} + T^{km}C^{ij}{}_{lm}] = 0$. The first and the second term are modified as follows:

$$\epsilon_{ijkp}T^{im}C^{jk}{}_{lm} = \epsilon_{kijp}T^{km}C^{ij}{}_{lm} = \epsilon_{ijkp}T^{km}C^{ij}{}_{lm}$$
$$\epsilon_{ijkp}T^{jm}C^{ki}{}_{lm} = \epsilon_{jkip}T^{km}C^{ij}{}_{lm} = \epsilon_{ijkp}T^{km}C^{ij}{}_{lm}.$$

Then, since the sum becomes $\epsilon_{ijkp}T^{km}C^{ij}{}_{lm} = 0$, the magnetic part of Weyl's tensor is zero. □

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