UNIVERSITÀ DEGLI STUDI DI MILANO

Scuola di Dottorato in Fisica, Astrofisica e Fisica Applicata

and

UNIVERSITY OF PORTSMOUTH

Institute of Cosmology and Gravitation

# Modified Theories of Gravity

s.s.d. FIS/02

PhD Thesis of: **Fulvio Sbisà**

Ciclo **XXIII**

Thesis Director: **Prof. Dietmar Klemm**
Thesis Director: **Dr. Kazuya Koyama**

Director of the Doctoral School: **Prof. Marco Bersanelli**
Director of the Doctoral School: **Prof. David Wands**

A.A. 2012/2013

*To Dietmar*

# Abstract

The recent observational data in cosmology seem to indicate that the universe is currently expanding in an accelerated way. This unexpected conclusion can be explained assuming the presence of a non-vanishing yet extremely fine tuned cosmological constant, or invoking the existence of an exotic source of energy, dark energy, which is not observed in laboratory experiments yet seems to dominate the energy budget of the Universe. On the other hand, it may be that these observations are just signalling the fact that Einstein's General Relativity is not the correct description of gravity when we consider distances of the order of the present horizon of the universe.

In order to study if the latter explanation is correct, we have to formulate new theories of the gravitational interaction, and see if they admit cosmological solutions which fit the observational data in a satisfactory way. A necessary condition for the viability of a theory of "modified gravity" is that it has to reproduce to high precision the results of General Relativity in experimental setups where the latter is well tested. Quite in general, modifying General Relativity introduces new degrees of freedom, which are responsible for the different large distance behavior. For a modified gravity theory to be phenomenologically viable, it is necessary that the extra degrees of freedom are efficiently screened on terrestrial and astrophysical scales. One of the known mechanisms which can screen the extra degrees of freedom is known as the Vainshtein mechanism, which involves derivative self-interaction terms for these degrees of freedom.

In this thesis, we consider a class of nonlinear massive gravity theories known as dGRT Massive Gravity. These theories are candidates as viable models to modify gravity at very large distances, and, apart from the mass, they contain two free parameters. We investigate the effectiveness of the Vainshtein screening mechanism in this class of theories. There are two branches of static and spherically symmetric solutions, and we consider only the branch in which the Vainshtein mechanism can occur. We truncate the analysis to scales below the gravitational Compton wavelength, and consider the weak field limit for the gravitational potentials, while keeping all non-linearities of the mode which is involved in the screening. We determine analytically the number and properties of local solutions which exist asymptotically on large scales, and of local (inner) solutions which exist on small scales. We analyze in detail in which cases the solutions match in an intermediate region. Asymptotically flat solutions connect only to inner configurations displaying the Vainshtein mechanism, while non asymptotically flat solutions can connect both with inner solutions which display the Vainshtein mechanism, or with solutions which display a self-shielding behaviour of the gravitational field. We show furthermore that there are some regions in the parameter space where global solutions do not exist, and characterise precisely in which regions of the phase space the Vainshtein mechanism takes place.

# Contents

**Abstract** **iii**

**Conventions** **x**

**Abbreviations** **xii**

**1 Introduction** **1**

1.1 The Homogeneous and Isotropic Universe . . . . . . . . . . . . . . . . 2

1.1.1 The Robertson-Walker metric . . . . . . . . . . . . . . . . . 4

1.1.2 Perfect fluids . . . . . . . . . . . . . . . . . . . . . . . . . 6

1.2 The Friedmann-Robertson-Walker model . . . . . . . . . . . . . . . . 9

1.2.1 The Friedmann equations . . . . . . . . . . . . . . . . . . . 11

1.2.2 Generic observational features . . . . . . . . . . . . . . . . . 14

1.2.3 The Hot Big Bang cosmology . . . . . . . . . . . . . . . . . 20

1.3 The late time acceleration problem . . . . . . . . . . . . . . . . . . 26

1.3.1 The composition of our universe . . . . . . . . . . . . . . . . 27

1.3.2 The acceleration problem . . . . . . . . . . . . . . . . . . . 33

1.4 Thesis summary . . . . . . . . . . . . . . . . . . . . . . . . . . . 39

**2 Braneworlds and the DGP model** **41**

2.1 Introduction to braneworlds . . . . . . . . . . . . . . . . . . . . . 41

2.1.1 Historical introduction . . . . . . . . . . . . . . . . . . . . 41

2.1.2 Mathematical preliminaries . . . . . . . . . . . . . . . . . . 46

2.2 The DGP model . . . . . . . . . . . . . . . . . . . . . . . . . . . 51

2.2.1 Weak gravity in the DGP model . . . . . . . . . . . . . . . . 53

2.2.2 Nonlinearities and the Vainshtein mechanism . . . . . . . . . 59
2.2.3 Cosmology in the DGP model . . . . . . . . . . . . . . . . . 65

**3 dGRT massive gravity 69**
3.1 GR as an interacting massless helicity-2 field . . . . . . . . . . . . . 71
3.1.1 Linear GR as a free massless spin-2 field . . . . . . . . . . . 72
3.1.2 GR as an interacting massless spin-2 theory . . . . . . . . . 75
3.1.3 Propagator and relevant scales . . . . . . . . . . . . . . . . . 77
3.2 The Fierz-Pauli theory . . . . . . . . . . . . . . . . . . . . . . . . 79
3.2.1 The Fierz-Pauli action . . . . . . . . . . . . . . . . . . . . . 79
3.2.2 The VDVZ discontinuity and Vainshtein mechanism . . . . . 82
3.2.3 The Fierz-Pauli theory in the Stückelberg language . . . . . 89
3.3 Nonlinear extensions of the Fierz-Pauli theory . . . . . . . . . . . . 94
3.3.1 Generic nonlinear extension . . . . . . . . . . . . . . . . . . 94
3.3.2 The nonlinear Stückelberg formalism . . . . . . . . . . . . . 98
3.4 Stückelberg analysis of nonlinear massive gravity . . . . . . . . . . . 101
3.4.1 Interaction terms . . . . . . . . . . . . . . . . . . . . . . . . 102
3.4.2 Strong coupling scales and decoupling limit . . . . . . . . . . 103
3.5 dGRT massive gravity . . . . . . . . . . . . . . . . . . . . . . . . . 109
3.5.1 The $\Lambda_3$ theory . . . . . . . . . . . . . . . . . . . . . . . . . 110
3.5.2 Resummation of $\Lambda_3$ massive gravity . . . . . . . . . . . . . . 117

**4 The Vainshtein mechanism in dRGT massive gravity 125**
4.1 Spherically symmetric solutions . . . . . . . . . . . . . . . . . . . . 126
4.1.1 The two branches . . . . . . . . . . . . . . . . . . . . . . . . 126
4.1.2 The diagonal branch . . . . . . . . . . . . . . . . . . . . . . 128
4.1.3 Focusing on the Vainshtein mechanism . . . . . . . . . . . . 131
4.2 The quintic equation . . . . . . . . . . . . . . . . . . . . . . . . . . 134
4.2.1 The quintic equation . . . . . . . . . . . . . . . . . . . . . . 135
4.2.2 Symmetry of the quintic and dual formulation . . . . . . . . 137
4.3 Asymptotic and inner solutions . . . . . . . . . . . . . . . . . . . . 139
4.3.1 Asymptotic solutions . . . . . . . . . . . . . . . . . . . . . . 139
4.3.2 Inner solutions . . . . . . . . . . . . . . . . . . . . . . . . . 141

4.3.3 Existence of the asymptotic and inner solutions . . . . . . . 142
4.4 Characterization of the asymptotic and inner solutions . . . . . . . 144
4.4.1 Asymptotic solutions . . . . . . . . . . . . . . . . . . . . . . 144
4.4.2 Inner solutions . . . . . . . . . . . . . . . . . . . . . . . . . 145
4.4.3 Vainshtein mechanism and solutions matching . . . . . . . . 146
4.5 Phase space diagram for solutions matching . . . . . . . . . . . . . 147
4.5.1 Local solutions and the shape of the quintic . . . . . . . . . 147
4.5.2 Creation and annihilation of local solutions . . . . . . . . . . 148
4.5.3 Analysis strategy . . . . . . . . . . . . . . . . . . . . . . . . 150
4.5.4 Phase space diagram . . . . . . . . . . . . . . . . . . . . . . 151
4.6 Numerical solutions . . . . . . . . . . . . . . . . . . . . . . . . . . 155
4.6.1 Asymptotic flatness with Vainshtein mechanism . . . . . . . 156
4.6.2 Asymptotically non-flat spacetime with Vainshtein mechanism 157
4.6.3 Asymptotically non-flat spacetime with self-shielding . . . . 157
4.6.4 No matching . . . . . . . . . . . . . . . . . . . . . . . . . . . 159

**Conclusions** **162**

**A Total derivative combinations** **165**
A.1 Total derivative combinations of $\Pi_{\mu\nu}$ . . . . . . . . . . . . . . . . 165
A.2 The $X^{(n)}_{\mu\nu}$ tensors . . . . . . . . . . . . . . . . . . . . . . . . . 166

**B The implicit function theorem** **168**
B.1 Formulation of the theorem . . . . . . . . . . . . . . . . . . . . . . 168
B.1.1 The quintic equation and implicit functions . . . . . . . . . 169
B.1.2 Maximal extension of implicitly defined solutions . . . . . . 170

**C Useful properties of the quintic function** **171**
C.1 General properties . . . . . . . . . . . . . . . . . . . . . . . . . . . 171
C.2 Evolution with $A$ . . . . . . . . . . . . . . . . . . . . . . . . . . . 172
C.2.1 The quintic function . . . . . . . . . . . . . . . . . . . . . . 172
C.2.2 The first derivative . . . . . . . . . . . . . . . . . . . . . . . 173
C.2.3 The second derivative . . . . . . . . . . . . . . . . . . . . . . 174

**D Asymptotic structure of the quintic function** **176**
D.1 Study of the second derivative . . . . . . . . . . . . . . . . . . . . . . . 177
D.1.1 Inflection points at infinity . . . . . . . . . . . . . . . . . . . . . 177
D.1.2 Inflection points and fixed points . . . . . . . . . . . . . . . . . 179

**E Roots at infinity** **181**
E.1 Zeros of the asymptotic function . . . . . . . . . . . . . . . . . . . . . 181
E.1.1 Five-roots-at-infinity boundaries . . . . . . . . . . . . . . . . . . 182

**F Leading behaviors** **184**
F.1 Finite asymptotic and inner solutions . . . . . . . . . . . . . . . . . . 184
F.2 Asymptotic decaying solution **L** . . . . . . . . . . . . . . . . . . . . . 184
F.3 Inner diverging solution **D** . . . . . . . . . . . . . . . . . . . . . . . . 185

**Bibliography** **187**

# List of Figures


4.1 Phase space diagram for asymptotic solutions . . . . . . . . . . . . . 141
4.2 Phase space diagram for solutions matching . . . . . . . . . . . . . . 152
4.3 Numerical solution for the case $\mathbf{F}_- \leftrightarrow \mathbf{L}$. . . . . . . . . . . . . . . . . 156
4.4 Numerical solution for the case $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$ . . . . . . . . . . . . . . . 158
4.5 Numerical solution for the case $\mathbf{D} \leftrightarrow \mathbf{C}_+$ . . . . . . . . . . . . . . . 158
4.6 Numerical solution for the self-shielding case . . . . . . . . . . . . . 159
4.7 Numerical solution for the no-matching case . . . . . . . . . . . . . 160
4.8 Quintic function in the no-matching case. . . . . . . . . . . . . . . . . 161

D.1 Inflection points at infinity parabolas . . . . . . . . . . . . . . . . . . 179

E.1 Asymptotic function at $\alpha = 1$ for increasing values of $\beta$. . . . . . . . 183

# Conventions

Unless explicitly said, throughout this thesis we will use the following conventions:

For metric signature, connection, covariant derivative, curvature tensors and Lie derivative we follow the conventions of Misner, Thorne and Wheeler [1]. Explicitly, the metric signature is the "mostly plus" one

$$\eta_{AB} = diag(-1, +1, \ldots, +1) \tag{0.1}$$

so for example a spacelike unit vector $\mathbf{n}$ has positive norm ($n_A n^A = +1$). In a metric manifold with metric $\mathbf{g}$ we will always use the unique symmetric connection compatible with the metric (Levi-Civita connection). The sign convention for the covariant derivative associated to the connection is

$$\nabla_A V^B = \partial_A V^B + \Gamma^B_{AL} V^L \qquad\qquad \nabla_A \omega_B = \partial_A \omega_B - \Gamma^M_{AB}\, \omega_M \tag{0.2}$$

and the Riemann curvature tensor is defined as

$$R^A{}_{BMN} = \partial_M \Gamma^A_{NB} - \partial_N \Gamma^A_{MB} + \Gamma^A_{ML} \Gamma^L_{NB} - \Gamma^A_{NL} \Gamma^L_{MB} \tag{0.3}$$

while the Ricci curvature tensor is defined as

$$R_{MN} = R^L{}_{MLN} = \partial_L \Gamma^L_{MN} - \partial_N \Gamma^L_{ML} + \Gamma^S_{SL} \Gamma^L_{MN} - \Gamma^S_{NL} \Gamma^L_{SM} \tag{0.4}$$

The sign convention for the Einstein equation is

$$R_{MN} - \frac{1}{2} R\, g_{MN} = + \frac{8\pi G}{c^4}\, T_{MN} \tag{0.5}$$

The convention for the Lie derivative of a tensor $T^M_{AB}$ along a vector field $V^N$ is

$$\big(\mathcal{L}_{\mathbf{V}} \mathbf{T}\big)^M{}_{AB} = V^L \partial_L T^M{}_{AB} - (\partial_L V^M)\, T^L{}_{AB} + (\partial_A V^L)\, T^M{}_{LB} + (\partial_B V^L)\, T^M{}_{AL} \tag{0.6}$$

When dealing with models with one or two spatial extra dimensions, 6D indices are denoted by capital letters, so run from 0 to 5; 5D indices are denoted by latin letters, and run from 0 to 4, while 4D indices are denoted by greek letters and run from 0 to 3.

We define symmetrization and antisymmetrization without normalization

$$A_{(M|\cdots|N)} \equiv A_{M\cdots N} + A_{N\cdots M} \qquad\qquad A_{[M|\cdots|N]} \equiv A_{M\cdots N} - A_{N\cdots M} \tag{0.7}$$

and we indicate the trace of a rank (1,1) or (0,2) tensor by tr, so

$$\mathrm{tr} D^M_N = D^L_L \qquad\qquad \mathrm{tr} A_{MN} = g^{MN} A_{MN} \tag{0.8}$$

As for notation, abstract tensors are indicated with bold-face letters, while quantities which have more than one component but are not tensors (such as coordinates for example) are expressed in an abstract way replacing every index with a dot. For example, the sextet of coordinates $X^A$ are indicated in abstract form as $X^{\cdot}$, the quintet of coordinates $\xi^a$ are indicated in abstract form as $\xi^{\cdot}$, and the quartet of coordinates $x^\mu$ are indicated in abstract form as $x^{\cdot}$.

When studying perturbations, the symbol $\simeq$ indicates usually that an equality holds at linear order.

We use throughout the text the (Einstein) convention of implicit summation on repeated indices, and we will use unit of measure where the speed of light has unitary value $c = 1$. The reduced 4D Planck mass is defined as $M_P = (8\pi G)^{-1/2} \sim 2.43 \times 10^{18}$GeV.

# Abbreviations

Throughout this thesis we will use the following abbreviations:

GR: General Relativity
CDM: Cold Dark Matter
4D, 5D, 6D, ...: four dimensional, five dimensional, six dimensional, ...
FP: Fierz-Pauli
BD: Boulware-Deser

# Chapter 1

# Introduction

The universe displays a stunning variety of physical objects and phenomena. The (almost) empty and cold intergalactic space, the region around a black hole and a planet placed in one of the arms of a spiral galaxy are very different for average density, temperature and strength of the gravitational field, and bear little resemblance one to the other. The study of these objects and their properties is without doubt very interesting and important. However, from the point of view of a cosmologist, the questions that one would like to answer are more related to how these objects formed, how long ago this happened and what will happen to them in the future. More in general, one would like to understand if the universe itself, seen as a whole physical system, has always existed or not, how old is it in the latter case, and what will its final fate be. To be able to answer these questions, one should know what are the laws that govern its evolution and be able to solve the equations of motion. However, since we are not able to handle the complexity of a system as big and complicated as the universe, we are almost forced to tackle the problem trying to find a very simplified model, which still grasps the essence of the phenomena under study but is simple enough to be handled mathematically. As we shall see, this is made possible by the assumption (corroborated by the observations) that the universe is homogeneous and isotropic on very large scales. This approach has proved to be very fruitful, and has led to the so called standard cosmological model, where many observed phenomena like the redshifts of distant objects, the existence and spectrum of the Cosmic Microwave Background radiation (CMB), the relative abundance and the nucleosynthesis of light elements find a natural explanation.

## 1.1 The Homogeneous and Isotropic Universe

Despite the huge variety of physical configurations mentioned above (even if we concentrate just on mass, the density within a galaxy is tipically $10^5$ larger that the average density of the universe [2]), observing the universe at various length scales suggests that an averaged description on very large scales may be the simplified description we are looking for. In fact, once chosen a direction in the sky and averaged the observations over a solid angle of fixed opening $\vartheta$, it can be seen that progressively increasing the value of $\vartheta$ leads to a result which is independent of the direction we choose. In other words, on large scales the observable universe seems to be (spatially) highly isotropic around us. This is suggested by the number count of galaxies we see in the sky, but is also confirmed by the counting of radio sources we can detect, by the observations of X- and $\gamma$-ray backgrounds, and expecially by the striking smoothness ($\delta T/T \lesssim 10^{-5}$) of the Cosmic Microwave Background (to be discussed later) [2].

To be able to build a model of the universe, however, it is not enough to know how it looks like from our planet: we need more information, namely we need to know how the universe would look like from other positions as well. Since we cannot achieve that in practice, we have to make some assumptions: it is natural to assume that we don't occupy a special position in the universe (Copernican Principle), and therefore that the universe itself would look isotropic (in an averaged sense as previously mentioned) also when seen from every other point. This condition implies that, on large scales, we can describe the observable universe as being spatially homogeneous[1] and isotropic. Being impossible to prove it directly, this assumption has to be verified *a posteriori* comparing the predictions of the model we would obtain with the observations: it is indeed very well confirmed by several different kinds of observations.

In describing the dynamics of the universe as a whole, we rely heavily on the

[1]It can be seen that isotropy from every point implies homogeneity [3].

knowledge we have of physical phenomena on earth and in the solar system. It is in fact natural to start from the laws which we know describe well physics on energies/length scales we can study on and around our planet (in a lab, or with high precision measurements in the solar system), and extrapolate their validity to arbitrary large scales. We are of course not granted that this is the correct thing to do, since new degrees of freedom or even new dynamical laws may show up as we increase the length scales and the complexity of the system under study. On the other hand, it is a very reasonable guess to start with. We will therefore assume that the correct framework to use to model the universe is the one offered by Einstein's General Relativity (GR) [4], which is currently thought to describe correctly the gravitational interaction (up to very high energies), and that gravity is the only interaction responsible for the large scale structure of the universe. To be precise, we will consider an extension of the original theory, proposed by Einstein [5], where the cosmological constant is explicitly present in the equations of motion.

In this framework, gravity is seen as a geometrical effect, and the geometrical properties of the universe are encoded in the metric tensor $\mathbf{g}$. The curvature of the universe is sourced by the energy-momentum tensor of matter fields $\mathbf{T}$, and is determined by the Einstein equations[2]

$$\mathbf{G} + \Lambda \mathbf{g} = 8\pi G\, \mathbf{T} \tag{1.1}$$

where $G$ is the Newton constant, $\Lambda$ is the so called cosmological constant and $\mathbf{G}$ is the Einstein tensor. The large scale homogeneity and isotropy suggests to "approximate" the exact manifold $(\mathscr{M}, \mathbf{g})$ which describes our universe with a homogeneous and isotropic manifold. We suppose then that $(\mathscr{M}, \mathbf{g})$ is locally diffeomorphic to a homogeneous and isotropic manifold $(\bar{\mathscr{M}}, \bar{\mathbf{g}})$, where $\bar{\mathbf{g}}$ is the metric on $\bar{\mathscr{M}}$, and that (in a sense to be formalized later) they are very similar when we focus only on very large scales. We indicate with $\phi$ the diffeomorphism which relate the two manifolds

$$\phi : \bar{\mathscr{M}} \to \mathscr{M} \tag{1.2}$$

[2]We use units of measure where the speed of light $c$ is one.

We expect that the homogeneous and isotropic metric $\bar{\mathbf{g}}$ encodes the fundamental informations on the large scale geometry of the real universe, despite having (due to the high symmetry) fewer degrees of freedom compared to $\mathbf{g}$. The idea is to start from the Einstein equations for $\mathbf{g}$, and obtain a set of equations for $\bar{\mathbf{g}}$ which can be thought of describing the large scale dynamics of the real universe. This description turns out to be mathematically tractable, and very insightful. Furthermore, this approach allows us to approximately disentangle the large scale behavior of the universe from the dynamics of small scale structures which form inside it.

### 1.1.1 The Robertson-Walker metric

The condition of spatial homogeneity and isotropy is in fact highly stringent, and amounts to ask that there exist a class of observers (*comoving observers*) whose trajectories fill the universe, and to each of whom the universe appears spatially isotropic at every time. This implies that there is a natural 3+1 splitting of the spacetime $\bar{\mathscr{M}}$, and more precisely that $\bar{\mathscr{M}}$ can be foliated in three-dimensional spatial hypersurfaces $\Sigma_t$, parametrised by a timelike coordinate $t$, which are three-dimensional spaces of constant curvature. It can be shown that it is always possible to choose a coordinates system on $\bar{\mathscr{M}}$ such that the line element locally takes the form

$$ds^2 = -dt^2 + A^2(t)\left[\frac{dR^2}{1-KR^2} + R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{1.3}$$

where $\theta$ and $\phi$ are angular coordinates (therefore dimensionless), $R$ is a (dimensionful) radial coordinate, A is a dimensionless function of $t$ and $K$ is a real number which is proportional to the 3-dimensional curvature of the surfaces $\Sigma_t$. In this system of reference the comoving observes are at rest, and therefore the reference system itself is called the comoving reference. Note that in the cases $K > 0$ and $K < 0$ we can redefine the radial coordinate as

$$R \to r = \sqrt{K}R \qquad K > 0 \tag{1.4}$$

$$R \to r = \sqrt{-K}R \qquad K < 0 \tag{1.5}$$

and absorb the resulting multiplicative constant in $A(t)$ as

$$a(t) \equiv \frac{A(t)}{\sqrt{|K|}} \tag{1.6}$$

which means that all the $K > 0$ cases are equivalent, and the same holds for the $K < 0$ cases. There are therefore only three distinct physical cases, $K > 0$, $K = 0$ and $K < 0$: with the above mentioned coordinate redefinitions we arrive at the line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big)\right] \tag{1.7}$$

where $k$ can take on the values $+1$, $0$ and $-1$. In the case $k = +1$ the spatial curvature is positive and the spatial hypersurfaces $\Sigma_t$ are locally isomorphic to 3-spheres, while in the case $k = 0$ the spatial curvature is vanishing and the spatial hypersurfaces are locally isomorphic to a 3D flat Euclidean space. Finally, in the case $k = -1$ the spatial curvature is negative and the spatial hypersurfaces are locally isomorphic to 3D hyperboloids. If we assume that the isomorphism is global, then the universe is called closed in the case $k = +1$ (and $r$ is defined for $0 \leq r < 1$), flat in the case $k = 0$ ($0 \leq r < +\infty$) and open in the case $k = -1$ ($0 \leq r < +\infty$). Note that now the coordinate $r$ is dimensionless while $a(t)$ is dimensionful. It is useful sometimes to single out the part of the metric which is independent of the timelike coordinate $t$ (usually termed *cosmic time*) and define *spatial metric* the three-dimensional metric $\gamma_{ij}$ such that the Robertson-Walker line element takes the form

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}(x)\,dx^i dx^j$$

This metric defines a notion of distance on the three-dimensional hypersurfaces: taken any two points $P_1$ and $P_2$ on the same $\Sigma_t$, the distance calculated using $\gamma_{ij}$ is called *comoving distance* of the two points, and is indicated with $d_C(P_1, P_2)$. The spatial distance between $P_1$ and $P_2$ which is effectively measured is the one calculated using the full metric $g_{ij}$: it is called (instantaneous) *physical distance* and is related to the comoving distance *via* the relation $d_F(P_1, P_2) = a(t)\ d_C(P_1, P_2)$. Note furthermore that redefining the time coordinate in the following way

$$\eta(t) \equiv \int^t \frac{d\xi}{a(\xi)} \tag{1.8}$$

it is possible to factorize the dependence on the function $a$ and put the metric above in the form

$$ds^2 = a^2(\eta)\,\big(-d\eta^2 + \gamma_{ij}(x)\,dx^i dx^j\big)$$

The time coordinate $\eta$ defined in this way is called *conformal time.* A yet different way to write the line element (1.7) is obtained redefining the radial coordinate in order to have the radial-radial component of the metric independent of $k$: the line element reads in this coordinate system

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{1.9}$$

where

$$S_k(\chi) \quad \begin{cases} = \sin(\chi) & k = +1 \\ = \chi & k = 0 \\ = \sinh(\chi) & k = -1 \end{cases} \tag{1.10}$$

We can see that the requirement of homogeneity and isotropy drastically reduces the number of degrees of freedom: once specified the geometry of the spatial hypersurfaces (*i.e.* specified if $k = 0$, $k = 1$ or $k = -1$), the metric has just one degrees of freedom, the *scale factor* $a(t)$, which depends on just one of the four spacetime coordinates. The evolution of the universe is then constrained by the condition of homogeneity and isotropy to be just a uniform expansion/contraction of the three-dimensional spacelike hypersurfaces, encoded in the evolution of the scale factor. Its dynamics is determined by appropriate equations that are to be derived from the exact Einstein equations using the hypothesis of large scale homogeneity and isotropy.

### 1.1.2 Perfect fluids

The source term of the dynamical equations for the scale factor will involve (as we will see later) a spatial averaging procedure on the exact energy-momentum tensor of the universe. It is therefore important to understand what are the implications of spatial homogeneity and isotropy for the source term of Einstein equations.

Let us consider in general a tensor field $\bar{\mathbf{T}}$ of type $(1,1)$ defined on $\bar{\mathscr{M}}$ and let's impose the condition of homogeneity and isotropy on $\bar{\mathbf{T}}$: this implies that, in the comoving reference, the tensor is of the form

$$\bar{T}_\mu{}^\nu(t,\vec{x}) = \mathrm{diag}\big(-\rho(t), p(t), p(t), p(t)\big) \tag{1.11}$$

or equivalently, lowering one index,

$$\bar{T}_{00} = \rho \qquad \bar{T}_{0i} = 0 \qquad \bar{T}_{ij} = p\, g_{ij} \tag{1.12}$$

where $\rho$ and $p$ are functions of the cosmic time $t$ only. Note that this is true for every (homogeneous and isotropic) tensor of type (1,1), so $\rho$ and $p$ are completely arbitrary functions: the condition of homogeneity and isotropy does not tell anything about the time evolution of $p$ and $\rho$ and if they are independent one from the other or not. If we identify $\bar{\mathbf{T}}$ with the stress energy tensor, then this information is encoded in the continuity equation (which is implied by the equations of motion), and in the microscopic description of the system.

There is a well known class of physical systems which is described by an energy-momentum tensor of this form: *perfect fluids*. A fluid living in a Minkowski spacetime is said to be perfect if, whatever its four-velocity profile $u^\mu(x)$, the heat conduction is always absent and there are no shear stresses (*i.e.* its viscosity is zero). Therefore (apart from its velocity profile) a perfect fluid is characterised by only two macroscopic quantities, its rest frame energy density $\rho(x)$ and pressure $p(x)$: this implies that its energy momentum tensor is of the form

$$T^{\mu\nu} = \big(\rho + p\big) u^\mu u^\nu + p\, \eta^{\mu\nu} \tag{1.13}$$

where $\rho$, $p$ and $u^\mu$ in general depend on all the four coordinates $x^\mu$. It follows that a perfect fluid living in a curved spacetime has a (lowered indices) energy-momentum tensor of the form

$$T_{\mu\nu}(x) = \big(\rho(x) + p(x)\big) u_\mu(x) u_\nu(x) + p(x)\, g_{\mu\nu}(x) \tag{1.14}$$

Considering now equation (1.12), we can see that an homogeneous and isotropic fluid always behaves as a perfect fluid which is at rest in the comoving reference and whose energy density and pressure are constant on the spatial hypersurfaces $\Sigma_t$. If we relax the assumption of homogeneity and isotropy, it is not necessarily true that we can describe the matter-energy content of the universe as a (inhomogeneous and anisotropic) perfect fluid with nontrivial velocity profile, because heat conduction and viscosity may play a role. However, it turns out to be very fruitful to model the energy-matter content of the universe as a *collection* of perfect fluids, so it is worthwhile to spend some more words on it.

There are many physical systems that can be macroscopically described as fluids. Their (different) microscopic structure shows up at macroscopic level *via* relations, which are called *equations of state*, that link together the thermodynamical parameters of the fluid. A particular importance in cosmology is given to perfect fluids which are characterised by the very simple equation of state $p = w\rho$, where $w$ is a constant. Among this class of fluids, there are three special cases which deserve a more detailed discussion: the cases $w = 0$, $w = 1/3$ and $w = -1$. The case $w = 0$ is suitable to describe a gas of nonrelativistic particles, in other words particles whose kinetic energy is negligible compared to their rest energy, and can be used for example to describe the matter which constitutes galaxies. The case $w = 1/3$ instead is suitable to describe a gas of ultrarelativistic particles, that is particles whose rest energy is negligible with respect to their kinetic energy, such as neutrinos. Note that also a system like the electromagnetic field can be described as a perfect fluid with the equation of state $p = (1/3)\rho$: this follows from the well known fact that the energy-momentum tensor of the electromagnetic field is traceless, and is consistent with the idea that we may see the electromagnetic field as a collection of photons (which are by definition ultrarelativistic being massless). Finally, the case $w = -1$ can be used to describe the so-called *vacuum energy*. Quantum Field Theory suggests that also the vacuum state (that is, a configuration devoid of particles) possesses a nonzero energy (which is actually divergent unless we put a cutoff to the theory): the contribution of a quantum field to the classical energy-momentum tensor is expected to be the expectation value $\langle 0 \,|\, \hat{T}^{\mu\nu} \,|\, 0 \rangle$ on the vacuum state $|\, 0 \rangle$. On flat space, the requirement that the quantum theory and likewise the vacuum state are invariant with respect to Lorentz transformations imply that the above mentioned expectation value has the form $\langle 0 | \, \hat{T}_{\mu\nu} \, | 0 \rangle \propto \eta_{\mu\nu}$: it follows that on curved spacetime

$$\langle 0 | \, \hat{T}_{\mu\nu} \, | 0 \rangle \propto g_{\mu\nu} \tag{1.15}$$

We can conclude that vacuum energy can be treated as a perfect fluid with the equation of state $p(x) = -\rho(x)$. Note that the cosmological constant term in equation (1.1) is precisely of the form above. The cosmological constant in fact can be alternatively thought of as a second characteristic energy/length scale of the gravitational field (beside $G$) which show up only at ultra large scales, or from another point of view can be thought of as describing the effect of vacuum energy of quantum fields in the cosmological context. From the latter point of view, it is more logical to consider it as a source term, and move the cosmological constant term to the right hand side of the Einstein equations defining

$$T_{\mu\nu}^{(\Lambda)} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \tag{1.16}$$

This energy-momentum tensor is characterised by a pressure $p = -\Lambda/8\pi G$ and an energy density $\rho = \Lambda/8\pi G$. In the following we adopt this point of view and include the contribution of a (possibly nonzero) cosmological constant in the total energy-momentum tensor: we don't constrain *a priori* the sign of $\Lambda$ and allow it to have positive or negative value.

## 1.2 The Friedmann-Robertson-Walker model

Before deriving the equations that govern the evolution of the scale factor, it is useful to specify how the large scale spatial homogeneity and isotropy is expressed in our formalism. Using the diffeomorphism $\phi$ which maps the reference manifold $(\bar{\mathscr{M}}, \bar{\mathbf{g}})$ into the manifold $(\mathscr{M}, \mathbf{g})$ which describes the "real" universe (or at least its observable part), we can pull-back the exact metric $\mathbf{g}$ obtaining the metric $\phi_\star(\mathbf{g})$ which is defined on $\bar{\mathscr{M}}$. We can define now the deviation from spatial homogeneity and isotropy as the difference of the two metrics on $\bar{\mathscr{M}}$, which in comoving coordinates reads as

$$h_{\mu\nu}(t, \vec{x}) = \big(\phi_\star(\mathbf{g})\big)_{\mu\nu}(t, \vec{x}) - \bar{g}_{\mu\nu}(t, \vec{x}) \tag{1.17}$$

where $t$ is the cosmic time and $\vec{x}$ indicates the spatial coordinates on the spacelike hypersurfaces $\Sigma_t$. Note that since homogeneity and isotropy provide a natural way of splitting space and time on $\bar{\mathscr{M}}$ (which is explicitly realized in the comoving reference), it makes sense to talk about operations which involve just the spatial coordinates. The tensor $h_{\mu\nu}(x)$ is not a perturbation and does not need to be small, actually it can be huge: the condition of large scale spatial homogeneity and isotropy is translated in the fact that $h_{\mu\nu}(x)$ gives approximately a vanishing contribution to the Einstein tensor when the latter is averaged on spatial volumes

$\mathscr{V}$ large enough to render the homogeneity apparent (to be quantitative, spheres with diameter bigger than[3] 100 Mpc [2]). To be more precise, let's indicate with $\hat{\mathbf{G}}$ the operator which associates to any metric the Einstein tensor built with the metric itself, and for every point $\vec{x}$ on $\Sigma_t$ let's consider a large enough volume $\mathscr{V}(\vec{x})$ centered around it. The large scale spatial homogeneity and isotropy at a fixed time $t$ is expressed by the fact that, performing some spatial average over $\mathscr{V}(\vec{x})$ of the pull-back of the "real" Einstein tensor, one gets approximately the Einstein tensor built with the homogeneous and isotropic metric

$$\langle \big[\phi_\star\big(\hat{\mathbf{G}}(\mathbf{g})\big)\big]_{00} \rangle_{\mathscr{V}(\vec{x})} \simeq \big(\hat{\mathbf{G}}(\bar{\mathbf{g}})\big)_{00}(t, \vec{x}) \tag{1.18}$$

$$\langle \mathrm{tr}\big[\phi_\star\big(\hat{\mathbf{G}}(\mathbf{g})\big)\big]_{ij} \rangle_{\mathscr{V}(\vec{x})} \simeq \mathrm{tr}\big(\hat{\mathbf{G}}(\bar{\mathbf{g}})\big)_{ij}(t, \vec{x}) \tag{1.19}$$

Here $\mathrm{tr}[\ ]_{ij}$ stands for the trace over spatial components. Imposing that the large scale homogeneity and isotropy holds at every $t$, amounts to ask that the equations above hold at every $t$. This implicitly defines the time evolution of the scale factor: to obtain it, we should calculate the evolution of the full metric and then take the spatial average at every time. However, this is not doable in practice, and we would like to obtain some dynamical (differential) equations for the scale factor itself. Therefore, we consider the equations

$$\big(\hat{\mathbf{G}}(\bar{\mathbf{g}})\big)_{00}(t, \vec{x}) = 8\pi G \, \langle \big(\phi_\star(\mathbf{T})\big)_{00} \rangle_{\mathscr{V}(\vec{x})} \tag{1.20}$$

$$\mathrm{tr}\big(\hat{\mathbf{G}}(\bar{\mathbf{g}})\big)_{ij}(t, \vec{x}) = 8\pi G \, \langle \mathrm{tr}\big(\phi_\star(\mathbf{T})\big)_{ij} \rangle_{\mathscr{V}(\vec{x})} \tag{1.21}$$

which are written in terms of the scale factor, its derivatives and the averaged energy-momentum tensor. Note that these equations are not exactly compatible with the validity of (1.18)-(1.19) at every time: if we start at time $t_i$ with a scale factor which satisfies (1.18)-(1.19), its time evolution according to (1.20)-(1.21) will not exactly satisfy (1.18)-(1.19) at subsequent times. In other words, the time evolution of the complete metric (including deviations from from homogeneity and isotropy) does not commute with the operation of spatial averaging. The actual difference depends on the explicit form of the real metric as well as the details of the spatial averaging procedure. We decide to neglect this difference for the

[3]One megaparsec (Mpc) is approximately $3.1 \times 10^{19}$ km.

moment, therefore studying the evolution of the scale factor according to (1.20)-(1.21), leaving the possibility to study the effect of this approximation later.

### 1.2.1 The Friedmann equations

We define $\bar{T}_{\mu\nu}(t, \vec{x})$ as the homogeneous and isotropic tensor (therefore of the form (1.12)) whose nonzero components are obtained by spatial averaging the pullback of the real energy-momentum tensor

$$\bar{T}_{00}(t, \vec{x}) \equiv \langle \big(\phi_\star(\mathbf{T})\big)_{00} \rangle_{\mathscr{V}(\vec{x})} \tag{1.22}$$

$$\operatorname{tr} \bar{T}_{ij}(t, \vec{x}) \equiv \langle \operatorname{tr}\big(\phi_\star(\mathbf{T})\big)_{ij} \rangle_{\mathscr{V}(\vec{x})} \tag{1.23}$$

Note that we can then write the equations (1.20)-(1.21) in a more familiar way as

$$\big(\hat{\mathbf{G}}(\bar{\mathbf{g}})\big)_{\mu\nu} = 8\pi G\, \bar{T}_{\mu\nu} \tag{1.24}$$

since, out of the 10 components of this equation, just two of them are linearly independent due to the high symmetry of the system. Taking a suitable linear combination of these two equations one gets the *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \tag{1.25}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p) \tag{1.26}$$

and it is customary to refer to the first one simply as the Friedmann equation, and to the second one as the acceleration equation[4]. Note that these two equation imply the *continuity equation*

$$\dot{\rho} = -3\,\frac{\dot{a}}{a}\,(\rho + p) \tag{1.27}$$

which actually expresses the fact that energy is conserved and can be obtained from $\nabla_\mu \bar{T}^{\mu\nu} = 0$. It is customary to define the *Hubble parameter*

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{1.28}$$

[4] We indicate derivatives with respect to the cosmic time with an overdot $\dot{a} \equiv da/dt$.

and the *deceleration parameter*

$$q(t) \equiv -\frac{a(t)\ddot{a}(t)}{\dot{a}^2(t)} \tag{1.29}$$

which are independent of the overall normalization of the scale factor. The Hubble parameter has the dimension of inverse time, and its value today $H_0$ can be taken to be a rough measure of the inverse of the age of the universe, as we shall see. It is also useful to define the *critical density* of the universe $\rho_{crit} \equiv 3H^2/\,8\pi G$ (which is a time dependent quantity) and the *density parameter* $\Omega \equiv \rho/\rho_{crit}$ : using these two quantities, the Friedmann equation reads as

$$\Omega(t)\, - 1 = \frac{k}{a^2(t)H^2(t)} \tag{1.30}$$

and it is easy to see that the sign of $k$ is determined by the fact that $\rho$ is larger, smaller or equal to the critical density. In fact we have

$$\begin{aligned} \rho < \rho_{crit} \quad &\Leftrightarrow \quad k < 0 \\ \rho = \rho_{crit} \quad &\Leftrightarrow \quad k = 0 \\ \rho > \rho_{crit} \quad &\Leftrightarrow \quad k > 0 \end{aligned}$$

and this implies that the total value of the density of energy (relatively to the the square of the Hubble parameter) is directly linked to the spatial geometry of the universe.

To study the evolution of the scale factor, we should solve equations (1.25)-(1.26) with appropriate initial conditions. This system of differential equations is however not closed, since there are two equations and three unknowns ($a$, $\rho$ and $p$): to be able to solve it, we need an additional equation, such as one which tells us how the average pressure $p$ of the universe is related to the average energy density $\rho$ and to the scale factor $a$. If we knew the precise distribution and thermodynamic properties of all matter in the universe, we may construct an equation of state $p = p(\rho, a)$ which expresses the "global" thermodynamic properties of the universe. In practice, we model the matter/energy content of the universe as the sum of few contributions whose thermodynamic properties are simple and easy to handle. In fact, we consider a model in which the universe is filled with three components, which are nonrelativistic matter (which from now on will be simply called "matter"), radiation (which comprises also ultrarelativistic matter) and vacuum energy. As said previously, all these components are perfect fluids which obey the simple equation of state $p = w\rho$ with $w$ respectively equal to 0, 1/3 and $-1$. Note that the evolution of the scale factor influences differently the energy density of every component since the continuity equation implies that

$$\rho(t) \propto a^{-3(1+w)}(t) \tag{1.31}$$

In particular, for matter the energy density scales as $a^{-3}$, *i.e.* inversely proportional to the spatial volume, while for radiation we have $\rho \propto a^{-4}$, which is consistent with idea that a dilatation/contraction of the spatial volume influences both the number density and the wavelength of photons. Instead, the dilatation/contraction of the spatial volume does not influence the energy density of the vacuum. It follows that, in order to determine the evolution of scale factor and therefore the history of the universe, it is essential to know not only the overall energy density, but also the relative abundances of the three different components.

Note that, once we specify the composition of the universe thereby fixing its equations of state, in principle to solve the system (1.25)-(1.26) we need the initial conditions[5] $a_0$, $\dot{a}_0$, $k$, $\rho_0^M$, $\rho_0^R$, $\rho_0^\Lambda$. However, the overall value of the scale factor is not physically observable, so to find $H(t)$, $\rho^M(t)$, $\rho^R(t)$ and $\rho^\Lambda(t)$ it is enough to know $H_0$, $k$, $\rho_0^M$, $\rho_0^R$, $\rho_0^\Lambda$. A nice way to parametrize the initial conditions for the Friedmann equations, and therefore to parametrize the cosmological models, is to introduce separate density parameters for every component type of perfect fluid which composes the energy-momentum tensor: we define

$$\Omega_M(t) \equiv \frac{8\pi G}{3}\frac{\rho^M}{H^2} \quad , \quad \Omega_R(t) \equiv \frac{8\pi G}{3}\frac{\rho^R}{H^2} \quad , \quad \Omega_\Lambda(t) \equiv \frac{8\pi G}{3}\frac{\rho^\Lambda}{H^2} \tag{1.32}$$

It is also useful to incorporate the dependence on the sign of the spatial curvature in another density parameter, which however does *not* come from an energy density and is therefore only a way of keep track of spatial curvature: we define

$$\Omega_K(t) \equiv -\frac{k}{a^2H^2} \tag{1.33}$$

[5]We indicate with the pedix $_0$ the quantities evaluated today.

In term of these cosmological parameters the Friedmann equations take the suggestive form

$$1 = \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t) \tag{1.34}$$

$$q(t) = \frac{1}{2}\,\Omega_M(t) + \Omega_R(t) - \Omega_\Lambda(t) \tag{1.35}$$

### 1.2.2 Generic observational features

The Friedmann-Robertson-Walker model, provided with information about the composition of the universe, gives very distinctive observational features. These features are indeed observed in the real universe and provide a strong support in favor of the assumptions we made and on the validity of the model.

#### Kinematic in a Robertson-Walker spacetime

Let us study the (free) motion of test particles in a Friedmann-Robertson-Walker universe. In General Relativity free test particles move along geodesics (timelike geodesics for massive particles and null geodesics for massless particles), and it is always possible to parametrize the trajectory of a particle $x^\mu(\lambda)$ so that its tangent vector $v^\mu(\lambda) = dx^\mu/d\lambda$ satisfies the *geodesic equation*

$$\frac{D}{d\lambda}\,v^\mu(\lambda) = \frac{dv^\mu(\lambda)}{d\lambda} + \Gamma^\mu_{\alpha\beta}(x(\lambda))\,v^\alpha(\lambda)\,v^\beta(\lambda) = 0$$

where the parameter $\lambda$ is called *affine parameter*, and $\Gamma^\mu_{\alpha\beta}$ are the connection coefficients relative to the only symmetric connection compatible with the metric. There are two useful quantities which are conserved along the geodesics: the first is the squared module of the tangent vector

$$g_{\mu\nu}(x(\lambda))\,v^\mu(\lambda)\,v^\nu(\lambda) \tag{1.36}$$

which is conserved in every spacetime, and the second is the quantity

$$B_{\mu\nu}(x(\lambda))\,v^\mu(\lambda)\,v^\nu(\lambda) \tag{1.37}$$

whose conservation is instead characteristic of the Robertson-Walker spacetime. The tensor $B_{\mu\nu}$ is defined (in comoving coordinates) as

$$B_{\mu\nu}(x) = a^2(t)\big(g_{\mu\nu}(x) + U_\mu U_\nu\big) \tag{1.38}$$

where $U^\mu = (1,0,0,0)$ is the four-velocity field of the comoving observers, and satisfies

$$\nabla_{(\sigma} B_{\mu\nu)}(x) = 0 \tag{1.39}$$

where $\nabla$ is the covariant derivative associated to the metric, while the round parenthesis mean symmetrization over all the indices.

Let's consider massive particles: in this case it is useful to use the proper time of the particle as the affine parameter, so that the tangent vector to the trajectory is the actual four-velocity of the particle $v^\mu(\lambda)$ and the conserved quantity (1.36) takes the value

$$g_{\mu\nu}(x(\lambda))\, v^\mu(\lambda)\, v^\nu(\lambda) = -1 \tag{1.40}$$

We call *peculiar velocity* the spatial part $v^i$ of the four-velocity expressed in the comoving reference. This name is motivated by the fact that $v^i$ is the "excess" (spatial) velocity of the test particle compared to the comoving observers' one (which is zero in the comoving reference). Indicating $|\vec{v}|^2 \equiv g_{ij} v^i v^j$, the conservation of the quantity (1.37) implies

$$|\vec{v}|(t) \propto \frac{1}{a(t)}$$

This implies that, if the scale factor is increasing (and so the universe is expanding), the peculiar velocity of a particle is destined to kinematically decrease and eventually die off, while a gas of particles in thermal equilibrium will get cooler and cooler. The opposite would happen if the universe is contracting. For massless particles, the concept of proper time cannot be defined and we parametrize the tangent vector $k^\mu(\lambda)$ using a generic affine parameter $\lambda$. In this case the conserved quantity (1.36) reads

$$g_{\mu\nu}(x(\lambda))\, k^\mu(\lambda)\, k^\nu(\lambda) = 0 \tag{1.41}$$

and the conservation of the quantity (1.37) implies

$$k^0(t) \propto \frac{1}{a(t)}$$

We conclude that, if the universe expands, the energy of a massless particles kinematically decreases, while it increases if the universe is contracting.

### The cosmological redshift

Consider a photon (a light ray in practice) which is emitted in the comoving reference at cosmic time $t_i$ with frequency $\omega_i$: since the frequency is proportional to $k^0$, the frequency for the same photon observed in the same reference at time $t_f$ is

$$\omega(t_f) = \frac{a(t_i)}{a(t_f)}\, \omega(t_i)$$

The expansion/contraction of the universe therefore determines an overall shift in the frequency of the electromagnetic radiation between its emission (for example by a galaxy) and its detection (for example by a telescope). This is not due to the peculiar motion of the particle, but just to expansion/contraction of the universe: the received frequency is lower than the emitted one if the universe expands, while it is higher if the universe contracts. The quantity used to express a generic frequency shift is the *redshift* $z$ defined as $z \equiv \frac{\lambda_f - \lambda_i}{\lambda_i}$, where $\lambda$ is the wavelength of the radiation: the redshift due to the cosmological expansion is called *cosmological redshift* and reads

$$z = \frac{a(t_f)}{a(t_i)} - 1$$

In general a frequency shift can be due to different effects, for example it can be due to the relative motion between emitter and observer (Doppler effect): we expect the total redshift to include also a Doppler component due to peculiar velocities. Therefore, the Friedmann-Robertson-Walker model implies that if the universe is expanding we should observe that the radiation coming from most of the celestial bodies is redshifted, and going to higher redshifts we should observe less or none contributions from the (conventional) Doppler effect. We instead expect to observe to opposite if the universe is contracting. Experimentally, the observations are in extremely good agreement with the predictions of an *expanding* Robertson-Walker universe.

### The Hubble's law

In an expanding Robertson-Walker universe one expects that the further away from us an object is, the more redshifted it appears to us. Roughly speaking, this is due to the fact that the more distant an object is, the more time it takes for its radiation to reach us: therefore, the cosmological redshift is bigger. However, to be precise we have to refine this argument, since the concept of distance is not so well defined in cosmology: in fact, to measure the physical distance $d_F$ defined above we should perform an instantaneous measurement, while the only thing we can do in cosmology is to study the light signals which reach us after travelling throughout the universe. Therefore we define the *luminosity distance* of a light source

$$d_L^2 \equiv \frac{L}{4\pi F} \tag{1.42}$$

where $L$ is the absolute luminosity of the source and $F$ is the energy flux measured by the observer. This definition is motivated by the fact that, in a Minkowski spacetime, the flux of incoming light is the ratio between the intrinsic luminosity and the surface area of a sphere of radius $d_F$, where $d_F$ is the (instantaneous) spatial distance between the emitter and the observer: this is just a consequence of energy conservation. Therefore in a Minkowski spacetime the luminosity distance and the instantaneous spatial distance are coincident. While in the Minkowski spacetime the cosmological redshift is by definition vanishing, we expect that in an expanding universe there is a relation between the luminosity distance of an object $d_L(z)$ and its (cosmological) redshift, and we expect that the bigger the distance the bigger the redshift.

Let's consider for simplicity the case of a spatially flat universe. As we will see in section (1.3), the luminosity distance of an object of redshift $z$ and whose comoving distance from us in the coordinate system (1.9) is $\chi$ can be expressed as

$$d_L = a_0 \chi (1+z) \tag{1.43}$$

where $a_0$ is the value of the scale factor today (when the radiation is received). However, the comoving distance $\chi$ itself is determined by the redshift: $\chi$ is linked to $\Delta t = t_0 - t_e$ by the conservation of the quantity (1.41) which implies

$$\chi = \int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{1.44}$$

and therefore

$$\chi = a_0^{-1} \Big[ \Delta t + \frac{1}{2} H_0 \, \Delta t^2 + \mathcal{O}\big(\Delta t^3\big) \Big] \tag{1.45}$$

On the other hand, $\Delta t$ is linked to $z$ by the Friedmann equation: by Taylor expanding $a(t)$ around $t = t_0$ we have

$$\frac{1}{1+z} = \frac{a_e}{a_0} = 1 - H_0\,\Delta t - \frac{1}{2}\,q_0\,H_0^2\,\Delta t^2 + \mathcal{O}\big(\Delta t^3\big) \tag{1.46}$$

and inverting this relation and inserting in (1.45) we arrive at

$$\chi = \frac{1}{a_0 H_0}\Big(z + \frac{1}{2}(1-q_0)z^2 + \mathcal{O}(z^3)\Big) \tag{1.47}$$

Using this expression in (1.43) one finally gets

$$d_L = H_0^{-1}\Big(z + \frac{1}{2}(1-q_0)z^2 + \mathcal{O}(z^3)\Big) \tag{1.48}$$

We notice that, when the redshift is small, the luminosity distance-redshift relationship is *linear*

$$d_L = H_0^{-1} z \tag{1.49}$$

This relation is known as *Hubble's Law*, and is indeed confirmed by observations: the geometrical explanation of the distance-redshift relation is one of the major successes of the Standard Cosmological Model. Notice furthermore that measurements of luminosity distances and redshifts of many objects in a suitable range of redshifts allows us to estimate both the present value of the Hubble parameter and the present value of the acceleration parameter.

### Horizons

A fundamental concept in Cosmology is the one of *horizon*. Since interactions cannot propagate faster than light, which would violate causality, it is important to know if the physical configurations at two different spacetime points had the possibility to influence each other, or if they are causally disconnected. This is crucial in cosmology since, as we will argue in the next section, the universe may not be infinitely old, and therefore we cannot assume that every spatial point $\vec{x}$ has been able to communicate *via* light signals with any other spatial point $\vec{x}'$ during the entire cosmic history. Taken an event labelled by the coordinates $(\vec{x}, t)$, we define *particle horizon* of the event the surface which divides the part of the universe with which the event had been able to communicate with light signals since the birth of the universe (*i.e.* with which the event is in causal contact), from the rest of the universe. In a Friedmann-Robertson-Walker spacetime, the horizon of an event $(\vec{x}, t)$ is (by symmetry reasons) a 3-sphere, whose physical radius is the physical distance travelled by the light since the birth of the universe till $t$. Considering for simplicity the case of a spatially flat universe, the trajectory of a light ray propagating radially obeys $-\left(\frac{dt}{d\lambda}\right)^2 + a^2\left(\frac{dr}{d\lambda}\right)^2 = 0$, and if we assign $t = 0$ as the time of the birth of the universe we have that the physical radius of the particle horizon is

$$d_H(t) = a(t) \int_0^t dt' \frac{1}{a(t')} \tag{1.50}$$

If the universe if radiation- or matter-dominated we have respectively $d_H(t) = 2t = H^{-1}(t)$ e $d_H(t) = 3t = 2H^{-1}(t)$: in this cases the inverse of the Hubble parameter fixes the scale of the causal horizon.

Another information (related to causality) which is very useful to know is if, during a time interval $[t_1, t_2]$, the (instantaneous) physical distance between two spatial points has grown more or less than the distance travelled by the light in that time interval. Light has travelled a equal or bigger distance if

$$d_F(t_2) - d_F(t_1) \leq a(t_2) \int_{t_1}^{t_2} dt' \frac{1}{a(t')}$$

and, in the limit $t_2 \to t_1 = t$ where the time interval becomes infinitesimal, we get

$$d_F(t) \leq H^{-1}(t) \tag{1.51}$$

We say that a physical distance $d_F(t)$ is inside the horizon at the time $t$ if $d_F(t) \leq H^{-1}(t)$, while we say it is outside the horizon if $d_F(t) > H^{-1}(t)$. It follows that, if the physical distance between two particles is outside the horizon in a time interval $[t_1, t_2]$, then these two particles did not have the possibility to communicate via light signals during that time interval, or more realistically no physical process had the possibility to correlate the physical states of the two particles during $[t_1, t_2]$. This concept turns out to be very useful when studying cosmological perturbations, since the time evolution of Fourier modes of the perturbations evolve in a qualitatively different way according to whether the physical length scale associated to the wavevector is inside or outside the horizon. Considering the Fourier transform of a quantity $\delta(t,\vec{x})$ with respect to the comoving coordinates $\vec{x}$

$$\delta(t,\vec{x}) = \int dk^3\, \delta(t,\vec{k})\ e^{i\vec{k}\cdot\vec{x}} \tag{1.52}$$

where the vector $\vec{k}$ is the comoving wavevector of the mode $\delta(t,\vec{k})$, while the physical wavevector at time $t$ is defined as $\vec{k}_F(t) = \vec{k}/a(t)$. Correspondingly, the physical length scale corresponding to $\vec{k}$ is

$$\lambda_F(t) = \frac{2\pi a(t)}{k} = \frac{2\pi}{k_F(t)} \tag{1.53}$$

where $k \equiv \|\vec{k}\|$ is the wavenumber. Therefore the mode $\delta(t,\vec{k})$ is said to be inside the horizon if

$$\frac{k}{2\pi a(t)} > H(t) \tag{1.54}$$

while is said to be outside the horizon if

$$\frac{k}{2\pi a(t)} < H(t) \tag{1.55}$$

### 1.2.3 The Hot Big Bang cosmology

As we shall discuss in detail later on, the cosmological observations have reached a degree of precision which enables us to characterise precisely the values of the cosmological parameters, and therefore determine the composition of our universe. In fact, the observations tell us that

$$H_0 \simeq 70\,\mathrm{km/s/Mpc} \qquad \Omega_{M_0} \simeq 0.3 \qquad \Omega_{R_0} \simeq 10^{-4} \qquad \Omega_{\Lambda_0} \simeq 0.7 \qquad \Omega_{K_0} \simeq 0$$

We will discuss in the next section the implications of these results in relation to our understanding of the universe. For the time being, we just want to use our knowledge about the composition of the universe to study qualitatively the past evolution of the universe and point out the main prediction and successes of the Standard Cosmological Model.

### Qualitative evolution of the scale factor

The information about the composition of the universe summed up above implies that the energy density is positive definite: this means that the scale factor is a monotonically increasing function of cosmic time. Note that, as already mentioned, the densities of the different components of our universe scale differently with the scale factor, and more precisely we have

$$\frac{\Omega_\Lambda}{\Omega_M} \propto \left(\frac{a}{a_0}\right)^3 \quad , \quad \frac{\Omega_M}{\Omega_R} \propto \frac{a}{a_0} \tag{1.56}$$

Therefore, apart from the transition periods when the energy density of two (or in principle several) components are comparable, one of the components is always much bigger than the others, and so effectively dominating the total energy density. It is then useful to solve approximately the equations for the scale factor neglecting the energy density of the components which are not dominating, and to patch together these solutions at the transition times. We will say that the universe is *matter dominated* when $\Omega_\Lambda$, $\Omega_R$ and $\Omega_K$ are negligible with respect to $\Omega_M$, and analogous definitions hold for *radiation dominated*, *curvature dominated* and *vacuum dominated* universe. Under this approximation, we can explicitly solve the Friedmann and continuity equations for the different domination cases, and for example for a spatially flat universe we obtain

$$\text{radiation} \qquad \rho \propto a^{-4} \qquad a(t) \propto t^{1/2} \qquad H(t) = \frac{1}{2}\, t^{-1} \tag{1.57}$$

$$\text{matter} \qquad \rho \propto a^{-3} \qquad a(t) \propto t^{2/3} \qquad H(t) = \frac{2}{3}\, t^{-1} \tag{1.58}$$

$$\text{vacuum} \qquad \rho = \text{cost} \qquad a(t) \propto e^{Ht} \qquad H(t) = \text{cost} \tag{1.59}$$

The observations then tell us that the universe was radiation dominated in the past, then at redshift $z = z_{eq} \sim 3 \times 10^3$ it became matter dominated, and it has (just) passed the transition between matter and vacuum domination, which happened at $z \sim 0.3$. Note furthermore that the pressure of matter and radiation is non-negative, while a positive energy vacuum has negative pressure: the second Friedmann equation tells us that the second derivative of the scale factor has been negative in the past till $z \sim 0.7$, and is now positive (equivalently, the deceleration parameter $q$ was positive in the past and is now negative). Therefore, the universe has been expanding in a decelerated way until very recently, and is now expanding in an accelerated way.

Following the evolution of the scale factor backwards in time, the universe seems to approach a singular state, since $a \to 0$, $\rho \to +\infty$ and the curvature of spacetime diverges: this singularity is usually called *Big Bang*. Note that the Big Bang is a fictitious singularity, in the sense that we do not expect General Relativity to be a reliable description of gravity and of the geometry of spacetime when curvature and energy are so high. We expect in fact GR to be the effective theory of a quantum theory of gravity, whose details are not clear yet, and that at least at energies higher than the *Planck energy* $E_{pl} \simeq 1.2 \times 10^{19}\,\text{GeV}$ we cannot make reliable calculations without taking into account the quantum aspects of gravity. Nonetheless, it is useful to fix the origin of time assigning the value $t = 0$ to the fictitious singularity: with this convention, if we assume that quantum gravity effects are under control for energies below the Planck energy, then the Standard Cosmological Model describes our universe for $t \geq t_{pl}$, where $t_{pl} = 10^{-43}\,\text{s}$ is the *Planck time*. Even if we don't know what happens before the Planck time, we may think that in some sense the Big Bang actually marks the birth of our universe. From this point of view, we can use the Friedmann equations to estimate the age of our universe. We can get an upper limit to this value extrapolating linearly the evolution of the scale factor back in time (since $\dot{a}(t)$ is negative for most of time in the past, the actual age will be lower): this procedure gives the value $H_0^{-1}$, which corresponds roughly to $10^{10}$ years. A more careful treatment using the actual solutions of the Friedmann equations shows that this rough estimate gives the correct timescale for the age of the universe.

### The Cosmic Microwave Background

From the study of the kinematics of particles and radiation in a Robertson-Walker universe, we expect that a gas of particles which is now at a temperature $T_0$ becomes hotter and hotter as we go back in time, and that photons belonging to the cosmological backgrounds we see today were more and more energetic. This implies that, going enough back in time, we reach a time $t_{dec}$ when the photons are energetic enough to ionize the atoms: before that time, the universe was forcefully made by a sea of ions (nuclei) and electrons coupled to photons. Reversing this argument and now going forward in time starting from $t < t_{dec}$, when $t_{dec}$ is reached the photons decouple from the electrons, and go on propagating in the universe with their energy redshifting as $1/a(t)$ because of the cosmological redshift. Since before decoupling the electrons and photons were in thermodynamic equilibrium at every point in space, the photons had a Planck spectrum which should remain untouched apart the overall redshift which make the temperature of the spectrum decrease. Therefore, if this model is correct we should observe a cosmological background radiation with nearly perfect Planckian spectrum at some very low temperature: such a radiation at a temperature $T \simeq 2.73K$ has indeed been observed by Penzias and Wilson in 1967 [6] and successively studied in detail by several missions including the Wilkinson Microwave Anisotropy Probe (WMAP) and the ongoing mission PLANCK. This radiation is usually called the Cosmic Microwave Background (CMB): the prediction of its existence is one of the major successes of the Standard Cosmological Model, and is nowadays one of the most powerful tools in understanding our universe.

### Deviations from homogeneity and isotropy

So far we have dealt mainly with the homogeneous and isotropic approximation of the universe, however for the model to be really successfull it should also qualitatively explain why and how there are (huge) deviations from homogeneity and isotropy on smaller scales. It is for example essential to understand if these structures were present also in the far past or has formed during the universe evolution, and to identify the mechanism responsible for their formation. The CMB gives unvaluable indications in this sense. In fact, the photons at the decoupling had a Planck spectrum which however may have had a different peak temperature at different places in space: the presence of any structure should have been reflected in local variations of the temperature of the Planckian spectrum, which should have been remained imprinted in the CMB we see today (apart from being overall redshifted). In fact the CMB is not perfectly uniform, but is incredibly smooth: relative variations of temperature at different directions in the sky are as small as $\delta T/T \simeq 10^{-5}$. This fact tells us that the structures we see nowadays were not present at decoupling, but there were very small density perturbations. In other words, the deviations from homogeneity and isotropy were small at *all* scales at the decoupling. This suggests the following general picture: small density perturbations which were already present at decoupling grew because of their self-gravity and eventually formed the huge inhomogeneities we observe nowadays *via* gravitational instability. This picture also justifies a *posteriori* the choice to study the evolution of the universe singling out an "homogeneous and isotropic" evolution from the exact evolution.

### The formation of structures

Since the deviations from homogeneity and isotropy were small on all scales, we can safely study their evolution at first order in perturbations. There are two different kinds of perturbations: adiabatic perturbations, which are perturbations in the total energy density, and isocurvature perturbations, where the total energy density is not modified but the ratio between the densities of different species is perturbed. The study of the spectrum of temperature anisotropies in the CMB reveals that isocurvature perturbation, if present, were very suppressed compared to adiabatic perturbations, so we consider only the latter ones in what follows. Performing a Fourier decomposition, qualitatively we can distinguish between superhorizon modes and subhorizon modes: to study the former modes one needs to use the relativistic equations, while for the latter modes one can safely use the Newtonian equations. The Newtonian analysis reveal that the stability properties of subhorizon perturbations is influenced by two different effects: the gravitational attraction, which favors the growth of perturbations, and the pressure due to the photons, which obstacolate it. As a result, there is a characteristic wavenumber, the (comoving) *Jeans wavenumber* $k_J$, which discriminate between perturbation modes which are unstable and can grow ($k < k_J$), and modes which oscillate acoustically and do not grow ($k > k_J$). Indicating with $c_s$ the sound speed[6] of perturbations and with $\bar{\rho}$ the density of the homogeneous and isotropic solution

[6] The sound speed is defined as $c_s^2 = \partial p/\partial \rho$.

(the unperturbed density), we have

$$k_J^2 = \frac{4\pi G\bar{\rho}\, a^2}{c_s^2} \tag{1.60}$$

Although the expansion rate of the universe does not influence the stability of a mode, it does influence the growth rate of the unstable modes. It can be shown in fact that when the universe is radiation dominated even the unstable modes does not grow appreciably (they grow logaritmically), because the expansion of the universe suppresses the growth of perturbations. Therefore, roughly speaking only after the matter-radiation equality can unstable modes indeed grow [2].

Note that the Jeans wavenumber depends on the effective sound velocity of perturbations. If we assume that different species of particles are present, each one will have its own sound velocity and therefore its own Jeans scale. If we assume that after the matter-radiation equality the matter is composed of baryonic matter and cold dark matter (CDM, to be introduced shortly), the two Jeans scales will be very different since the baryons are still strongly coupled to the photons, while the CDM is not. As a result, before the electron-photon decoupling the physical Jeans length for baryons is outside the horizon, while the physical Jeans length for density perturbations of CDM is inside the horizon; after the decoupling, instead, the sound velocity for baryons drops abruptly, since the photons cannot provide pressure anymore, and the Jeans length for baryons drops inside the horizon as well. Therefore the picture is roughly the following: CDM density perturbations start growing as soon as the universe becomes matter dominated, while density perturbations in baryonic matter start growing ony after decoupling. After that, their growth is “guided” by the potential wells created by CDM density perturbations and their amplitude soon reach the same amplitude of the latter ones, and then the pertubations in the two species grow together [2].

When the amplitude of the perturbations ceases to be very small, the analysis above is not accurate enough since we should take into account nonlinear effects. Roughly speaking, when a perturbation reaches the amplitude $\delta \simeq 1$ it ceases to behave as a perturbation in the expanding fluid (expanding means that it follows the expansion of the universe), and becomes a virialized system which decouples from the overall expansion [7]. Each separate system undergoes gravitational collapse, and starts forming the high density structures we observe nowadays. This general picture has proved to be self-consistent and seem to link naturally the features of the CMB with the presence of structures in the large scales homogeneous and isotropic universe suggested by observations today. Note that the presence or CDM is crucial in the picture, because the baryon perturbations feel the potentials wells created by CDM perturbations and grow more rapidly then they would otherwise: without CDM, they growth of baryon perturbations wouldn't have been rapid enough to be able to form structures now.

## 1.3 The late time acceleration problem

### The $\Lambda$CDM model

In the framework of the cosmological model we described in the previous sections, the evolution of the scale factor and so the large scale evolution of our universe is determined once we specify the composition of our universe and the Hubble parameter today. However, apart from the macroscopic equation of state of the perfect fluid components of the energy-momentum tensor, to study the thermodynamic history of the universe and the evolution of inhomogeities it is important to know also the details of how different components interact (the interaction rates between different species, for example). It turns out to be very useful to separate two components inside the (nonrelativistic) matter fluid: the *baryonic matter* and the *dark matter*, which have the same macroscopic equation of state ($p = w\rho$ with $w = 0$) but have very different interaction properties. The former indicates matter whose building blocks are baryons (protons and neutrons at low energy), which is the matter we are most familiar with, and comprises all particles we have observed in colliders so far and are nonrelativistic today (therefore, despite the name, it comprises also electrons, which however does not contribute appreciably to the "baryonic" mass). The latter instead is a type of matter which interacts very weakly with all the other species via the electromagnetic, weak and strong interactions, and shows its existence basically only through its gravitational effects. Despite the fact that there is no observed particle (yet) which has the right properties to consistently provide a realization of dark matter (although there are several candidates), its existence is strongly suggested by several observations. In fact, the presence of dark matter has first been proposed to explain puzzling observational features of galaxies, and (as we mentioned in the previous section) it later turned out to be crucial in the cosmological structure formation mechanism.

Therefore, we split the matter component $\Omega_M = \Omega_B + \Omega_C$ as a sum of a baryonic matter component $\Omega_B$ and a dark matter component $\Omega_C$. The dark matter is furthermore assumed to be "cold", which means that it was non-relativistic when it decoupled from the thermal bath. The phenomenological model of the universe that results from the assumptions we made so far is usually termed the ΛCDM cosmological model: it is basically the combination of the observational evidence for large scale homogeneity and isotropy and the laws of physics we formulated to explain phenomena on earth and in the solar system, with the addition of cold dark matter and a (possibly nonzero) cosmological constant. It is a remarkable success that such a model is indeed able to account for almost all the existing observational data in cosmology.

### 1.3.1 The composition of our universe

The estimation of the cosmological parameters

$$H_0 \qquad \Omega_{B_0} \qquad \Omega_{C_0} \qquad \Omega_{R_0} \qquad \Omega_{\Lambda_0} \qquad \Omega_{K_0} \tag{1.61}$$

is something that has to be done observationally, comparing theoretical predictions with observations. The observational estimation of these parameters has recently become a very active field of research: on one hand this is due to the fact that the theoretical framework just described is flexible enough to account for different kinds of observations, but at the same time simple enough to permit its predictions to be tested with precision. On the other hand, it is due to the fact that the amount and precision of observational data has recently reached a previously undreamed-of level. It is also a quite technical field, therefore we give in following just the basic underlying ideas.

#### Standard candles and standard rulers

One of the most important concepts in modern observational cosmology is the notion of *standard candle* and *standard ruler*. A standard candle is an (astrophysical) object whose absolute luminosity is precisely known, while a standard ruler is an absolute length scale which is accurately known and which is imprinted in one or several cosmological features. By absolute luminosity we mean the flux of energy (in form of light) per unit time across a sphere which closely surrounds the emitting object, divided by the surface area of the sphere. The importance of standard candles in cosmology lies in the fact that the observed luminosity of a source is influenced both by its absolute luminosity and by the evolution history of the Hubble parameter, so if we know the absolute luminosity we can gain informations on the evolution history. Likewise, the observed length scale corresponding to the absolute length of a standard ruler is influenced by the evolution history of the Hubble parameter, and therefore a precise knowledge of the absolute length enables us to characterise the evolution history.

The astrophysical objects which come closer to be standard candles are Type Ia supernovae. They are quite rare objects, since we expect to see few of them per century in a Milky-Way-sized galaxy, but have the advantage to be very bright (their brightness is comparable to their host galaxy's one) and so potentially observable at high redshift ($z \sim 1$). This is important to test the evolution history of the Hubble parameter, as can be seen looking at (1.48): low redshift supernovae ($z \ll 1$) enables to estimate just the Hubble parameter today, while observing also high redshift ones enables to estimate also the deceleration parameter. They are however not perfect standard candles, since nearby type Ia supernovae display a scatter of about 40% in their peak brightness [8]. However, the observed differences in their peak luminosities turns out to be very closely correlated with observed differences in the shapes of their light curves: type Ia supernovae explosions can then be considered a one-parameter family of events, and observing both the peak brightnesses and the light curves enables to compensate for the difference and standardize their peak brightness, significantly reducing the scatter. In this sense, type Ia supernovae are "standardizable candles".

The standard ruler in cosmology is instead provided by the characteristic scale of acoustic oscillations in the photon-baryon fluid. As we already mentioned, before decoupling the nuclei and electrons were tightly coupled with photons: in this regime, baryons and photons moved in unison and can be treated as a single fluid [9]. Since the perturbations from homogeneity and isotropy were small, it is sufficient to work at first order in perturbations, and it is useful to decompose the relative perturbation $\delta$ of the density of the baryon-photon fluid in Fourier modes

$$\delta(\eta, \vec{x}) = \int dk^3 \, \delta(\eta, \vec{k}) \; e^{i\vec{k}\cdot\vec{x}} \tag{1.62}$$

where $\eta$ indicates the conformal time. For modes inside the horizon, a Newtonian analysis suffices and it can be shown that every mode $\delta(\eta, \vec{k})$ obeys a forced and damped harmonic oscillator equation, where the damping is due to the expansion of the universe, the forcing to the gravitational potential, and the harmonic force to the pressure exerted by the photons. Neglecting the damping term, the solution to the associated homogeneous equation is approximately given by

$$\delta(\eta, \vec{k}) \supset A_k \sin(k\, c_s \eta) + B_k \cos(k\, c_s \eta) \tag{1.63}$$

where $c_s$ is the sound speed of the baryon-photon fluid, while for modes inside the horizon the damping term introduces only a smooth modulation which does not significantly distort the oscillating pattern of the solution (1.63). The coefficients $A_k$ and $B_k$ are to be determined by the initial conditions, and comparison with the CMB anisotropy spectrum tells that $A_k \ll B_k$ and $B_k$ is nearly independent of $k$. Therefore we approximately have a pure oscillating contribution in the density perturbations

$$\delta(t, \vec{k}) \supset B \cos(k\, c_s \eta) \tag{1.64}$$

Focusing on a fixed mode $k$, this tells us that the amplitude of every mode oscillates periodically in time. Focusing on a fixed time, on the other hand, this contribution to the amplitudes of the modes displays a periodic oscillation in $k$. The acoustic oscillations of the baryon-photon fluid therefore fix a characteristic scale in Fourier space when the density perturbation is studied at a fixed time: this scale is set by the physics of a tightly coupled baryon-photon plasma, which is quite well understood, and therefore we can predict this scale with great accuracy. The periodicity scale set by acoustic oscillations remains imprinted in both the CMB anisotropies spectrum and in the large scale distribution of galaxies.

### Observations and cosmological parameters

To understand why standard candles and standard rulers can allow us to determine observationally the cosmological parameters, suppose to begin with that we are able to observe several objects which have different redshifts and belong to the same class of standard candles. From Earth, we can determine the redshift $z$ and the flux of light $F$ received from each object, but not its instantaneous physical distance. The flux $F$, apart from the absolute luminosity $L$ which is the same for every object by hypothesis, depends both on the comoving distance between the object and us, and on the expansion history of the universe during the propagation of the light signal. Since the comoving distance is determined by the redshift (neglecting peculiar velocities), informations on the dependence $F(z)$ allows us to probe the expansion history and therefore the value of the cosmological parameters.

To be quantitative, we consider the (square root of the) ratio between the absolute luminosity and the received flux

$$d_L = \sqrt{\frac{L}{4\pi F}} \tag{1.65}$$

which we've already encountered in section (1.2.2) and is called the *luminosity distance* of the source, since in flat space is exactly equal to the physical distance. In an expanding universe, instead, it is a function of redshift and is different from the instantaneous physical distance. The flux of energy across a spherical surface of comoving radius $\chi$ due to isotropic radiation can be expressed as

$$F_\chi = \frac{E_\chi N_\chi}{\Delta t_\chi \, A_\chi} \tag{1.66}$$

where $N_\chi$ is the number of photons (which for simplicity we assume to have the same energy) which pass across the surface in a time $\Delta t_\chi$, $A_\chi$ is the area of the surface and $E_\chi$ is the energy of every photon. The number of photons is conserved during the propagation, however the time it takes for $N$ photons to pass across the surface $A_\chi$ is higher of a factor $1 + z$ compared to the time it takes for them to pass across a surface surrounding the source. Furthermore, the energy gets redshifted of a factor $1 + z$ during the propagation. Therefore, the ratio between the absolute luminosity of a source and the flux of energy detected by an observer whose comoving distance from the source is $\chi$ reads

$$\frac{L}{F_\chi} = \frac{E_S}{E_\chi} \frac{\Delta t_\chi}{\Delta t_S} \frac{N}{N} A_\chi = (1+z)^2 \, A_\chi \tag{1.67}$$

where $E_S$ is the energy of the photons when emitted while $\Delta t_S$ is the time interval needed for the $N$ photons to pass across a surface which closely surrounds the source. Since the area of a surface of comoving radius in the system of coordinates (1.9) is $A_\chi = 4\pi\, a_0^2\, S_k^2(\chi)$, we get

$$d_L(z) = (1+z)\, a_0 S_k(\chi) \tag{1.68}$$

The comoving radial distance $\chi$ is in turn determined by the redshift, in fact we have

$$\chi = \int_{t_e}^{t_r} \frac{dt}{a(t)} = \int_{a_e}^{a_r} \frac{da}{a^2 H(a)} = a_0^{-1} \int_0^z \frac{d\zeta}{H(\zeta)} \tag{1.69}$$

where, in deriving these identities, we have changed variables twice, used the fact that $a$ and $t$ are in one to one correspondence and used $a/a_0 = 1/(1+z)$. We then have

$$d_L(z) = (1+z)\, a_0\, S_k\!\left(\frac{1}{a_0}\int_0^z \frac{d\zeta}{H(\zeta)}\right) \tag{1.70}$$

In the spatially flat case the $a_0$ factors cancel out, while in the spatially curved cases we can use the definition of curvature density parameter $\Omega_K = -k/a^2H^2$ to get

$$d_L(z) \quad \begin{cases} = (1+z) \displaystyle\int_0^z \frac{d\zeta}{H(\zeta)} & k=0 \\ = (1+z) \displaystyle\frac{H_0^{-1}}{\sqrt{|\Omega_{K0}|}}\, S_k\!\left(\sqrt{|\Omega_{K0}|}\int_0^z \frac{H_0}{H(\zeta)}\, d\zeta\right) & k=\pm 1 \end{cases} \tag{1.71}$$

Each choice of cosmological parameters gives a unique evolution history $H(z)$: therefore, if we know $d_L(z)$ we can characterise exactly the cosmological parameters, and if we have just some experimental points about $d_L(z)$ we can still put constraints on the values of the parameters.

For standard rulers, the situation is very similar. Considering an astrophysical object, we can never measure its real length $l$ just observing the light which comes from it, but we can measure the angle $\vartheta$ subtended by the object. Suppose we can measure the angle subtended by the same length scale $l$ at different redshifts: analogously to the case of standard candles, the angle $\vartheta$ depends on the length $l$ as well as from the expansion history of the universe, so informations on the dependence $\vartheta(z)$ allow us to probe the expansion history of the universe and therefore the value of the cosmological parameters. Again, to be quantitative we consider the ratio between the length scale $l$ and the subtended angle $\vartheta$

$$d_A = \frac{l}{\vartheta} \tag{1.72}$$

This quantity is called the *angular diameter distance*, since in flat space a source of length $l$ whose distance from us is $D$, subtends an angle $\vartheta = l/D$. In an expanding universe, the angular diameter distance is a function of redshift and is different from the instantaneous physical distance. It turns out that the angular diameter distance and the luminosity distance are related, in fact we have [10]

$$d_L(z) = (1+z)^2 d_A(z) \tag{1.73}$$

so formulas very similar to (1.71) hold also for $d_A(z)$. Therefore, if a length scale is imprinted in some features of the universe and we are able to observe it a different redshifts (in practice, we observe the footprint of the baryon acoustic oscillations in the large scale structure of galaxies at different redshifts), we can gain information on the evolution history of the universe and therefore on the value of the cosmological parameters.

As we already mentioned, the field of observational cosmology is at present very active. A real breakthrough came at the end of last century, when the Supernova Search Team [11] and the Supernova Cosmology Project [12] using data on the luminosity distance-redshift relation for type Ia supernovae indipendently provided evidence for a nonzero cosmological constant and a negative value of $q_0$. For this very surprising and important result the Nobel Prize in Physics 2011 was awarded to S. Perlmutter, B. P. Schmidt, and A. G. Riess. Using data from luminosity distance of type Ia supernovae [13], from the large scale distribution of galaxies [14] and from the angular spectrum of anisotropies of the CMB from the satellite WMAP [15] it is possible to rigorously test the $\Lambda$CDM cosmological model, and the model shows to provide a consistent fit to the data. Recently a general agreement in the community has been reached on the values of the cosmological parameters, providing the values [15]

$$\begin{aligned} &h \sim 0.702 \qquad &&\Omega_{B_0} h^2 \sim 0.02246 \qquad &&\Omega_{C_0} h^2 \sim 0.1120 \\ &\Omega_{R_0} \sim 10^{-4} \qquad &&\Omega_{\Lambda_0} \sim 0.728 \qquad &&\Omega_{K_0} \sim 0 \end{aligned} \tag{1.74}$$

where we have defined $H_0 = 100\, h$ km/s/Mpc.

### 1.3.2 The acceleration problem

We may conclude that the $\Lambda$CDM model is very satisfactory since it gives a consistent description of all the cosmological observations up to date. Note that, as we already mentioned, the observed values of the cosmological parameters (1.74) imply that the universe is at present vacuum dominated, and it is expanding in an *accelerated* way $q_0 < 0$. A closer look to (1.74), on the other hand, gives a somewhat strange feeling. It seems in fact that 70% of the energy density in the universe is in the form of a mysterious component with negative pressure, a property which we never observe in particle colliders and in earth-based labs experiments. Also, the elusive dark matter hasn't been observed in colliders yet, but nevertheless seems to be the dominant component of nonrelativistic matter and in fact significantly more abundant that the "normal" baryonic matter ($\Omega_{DM} \sim 6.5\, \Omega_B$). Instead of confirming the picture we had about how nature works, and enriching it with new details, the recent cosmological observations suggest a radically different picture. This, although unexpected, is not *a priori* wrong or worrying, and we may just accept it as an observational evidence.

However, if we are to accept a radically new picture of how nature works, we would like to understand it both from the phenomenological and the fundamental point of view. The problem is that we don't understand at a fundamental level why the $\Lambda$CDM model should be correct. As we said, we haven't yet observed directly the particles which should constitute the dark matter. More importantly, the observed value of the cosmological constant $\Omega_\Lambda \neq 0$, $\Omega_\Lambda \sim \Omega_M$ is actually very puzzling and difficult to understand, as we will see soon. It is therefore reasonable to wonder if instead some of the assumptions at the core of the $\Lambda$CDM model are maybe not correct, and if we are maybe misinterpreting the observational data. It is in fact possible that gravity is not described by GR at very large scales, or that there exist new degrees of freedom (or even new laws of nature!) which show up only when we increase enormously the length scales and the complexity of the system under study. Or it may be that the Copernican principle is not really valid (which however would be puzzling from a philosophical point of view). If one or several of this things are true, then the conclusion that $\Lambda$ is nonzero maybe ill based. It seems indeed worth exploring these other routes, before concluding that the picture of the universe drawn by the $\Lambda$CDM model is reliable.

**The cosmological constant problem**

The invariance with respect to general coordinate transformations and the energy conservation, which are at the heart of the formulation of GR, allow the addition of a term $\Lambda\, g_{\mu\nu}$ to the (1915) Einstein equations [4] which does not alter the structure of the theory, as first recognized by Einstein himself [5]. Although we are not forced to keep such a term, since we don't observe its effects in the solar system or on earth, it is not obvious that we should set it to zero either: it may in fact describe a second characteristic constant of the gravitational force [16]. A nonzero value of $\Lambda$ introduces into the theory a length scale

$$r_\Lambda \sim \sqrt{\frac{1}{|\Lambda|}} \tag{1.75}$$

above which the cosmological constant term would strongly affect the spacetime: the gravitational interaction would then be characterised by two parameters, one which describes the strength of the interaction (Newton's constant $G$) and one which describe its large scale behavior ($\Lambda$). There is however a problem, coming from the fact that cosmological observations imply that today $\Omega_{\Lambda_0} \sim \Omega_{M_0}$. The energy density of matter and vacuum scale very differently with the scale factor $\rho_\Lambda/\rho_M = a^3$, so the time when these densities are comparable is a very special and rare one in the history of the universe: for most of the time, vacuum energy is either dominating or negligible compared to matter. On the other hand, the time when astrophysical structures form is another very special moment is the cosmic history, and is correlated with the time of matter-radiation equality. The fact that $\Omega_{\Lambda_0} \sim \Omega_{M_0}$ today means that matter-vacuum equality and the formation of structures happens roughly at the same time: however this is *a priori* highly unlikely to happen, since we don't expect correlations between $\Omega_\Lambda/\Omega_M$ and $\Omega_R/\Omega_M$. To say the same thing differently, an extreme fine tuning in initial conditions would be necessary for this to happen: this problem is known as the *coincidence problem* (or also as the "new" cosmological constant problem). It is fair to say that, in this approach, the small and fine tuned value of $\Lambda$ is no more a mystery than the fine tuning in other constants of nature [17]. Furthermore, anthropic arguments may provide a way out of this problem [18, 19].

The situation is in any case deeply worsened by the fact that we expect a contribution of exactly the same form coming from the source term of the Einstein equations. As we already mentioned, in Quantum Field Theory the vacuum state $|\,0\rangle$ seems to possess a nonzero energy and pressure, and if the field theory is Lorentz invariant it should produce a contribution to the energy momentum tensor of the form

$$T^{(vac)}_{\mu\nu} = \langle 0\,|\,\hat{T}_{\mu\nu}\,|\,0\rangle = -\rho_{vac}\, g_{\mu\nu} \tag{1.76}$$

Despite the fact that this is an expectation value in quantum theory, while GR is a classical theory, we expect that such a term should be included as a source in the Einstein equations, since vacuum energy has shown to have measurable effects at classical level (consider for example the Casimir effect). To understand what may be a reasonable value for $\rho_{vac}$, let's consider as an example a free (*i.e.* non interacting) scalar field in a Minkowski spacetime. In a canonical quantization approach, every Fourier mode $\vec{k}$ of the field is equivalent to a quantum harmonic oscillator, which is known to possess a nonzero vacuum energy $E_0(\vec{k}) = \hbar\,\omega(\vec{k})/2$ where $\omega(\vec{k}) = \sqrt{m^2 + k^2}$. Therefore, summing up the contributions of every single mode, we find that the total vacuum energy of the field diverges. However, we may assume that the quantum field theory description is reliable only below a momentum cut-off scale $k_{cut}$: we definitely expect the description not to be adequate for energies above the Planck energy $E_{pl} = \sqrt{\hbar c^5/G} \sim 10^{19}$ GeV, but to be conservative we may lower the cutoff at the TeV energy scale $\sim 10^{-16} E_{pl}$. Summing the vacuum energy of the modes up to the cutoff, we have that the vacuum energy scales as the cutoff energy scale at the fourth power [18]

$$\rho_{vac} \sim \frac{E^4_{cut}}{\hbar^3 c^3} \tag{1.77}$$

where we have explicitly shown the $c$ and $\hbar$ coefficients for dimensional clarity. Note that if we assume that the value of $\Lambda$ estimated by the cosmological observations is due to vacuum energy, we have

$$\rho^{(obs)}_{\Lambda} \sim 10^{-8}\ \text{erg/cm}^3 \tag{1.78}$$

while using (1.77) we get the theoretical estimates

$$\rho_{vac}^{(th)} \sim 10^{112}\mathrm{erg/cm^3} \quad \text{(Planck)} \qquad\qquad \rho_{vac}^{(th)} \sim 10^{48}\mathrm{erg/cm^3} \quad \text{(TeV)} \tag{1.79}$$

We can see that, if we take the cutoff to the Planck scale, there is a difference of about 120 orders of magnitude between the observed value and the theoretical expectation, and even in the case of the TeV cutoff scale the difference is nearly 60 orders of magnitude. This extreme clash between predictions and observations is sometimes called the "old" cosmological problem, and can be restated as the fact that vacuum energy seems to gravitate much less then expected.

In general, we expect that the only observable signature of both vacuum energy and a "true" cosmological constant is its effect on spacetime, and therefore the two in principle very different contributions cannot be distinguished by observations [17]. Therefore, we should write the cosmological constant present in the Einstein equations as an "effective" constant which is the sum of a "bare" cosmological constant and of a vacuum energy contribution

$$\Lambda_{eff} = \Lambda + 8\pi G\rho_{vac} \tag{1.80}$$

To match the observed value, we need that the two term cancel with a relative precision which is almost incredible: $(\Lambda - \Lambda_{vac})/\Lambda \sim 10^{-56}$ in the TeV scale cutoff case, and even more so in the Planck scale cutoff case. Therefore an extreme fine-tuning between the two contributions is needed to be consistent with the observations.

It is natural to wonder whether the two problems we have highlighted above are two faces of the same problem or are two different problems. It may well be that the reason why vacuum energy does not gravitate (almost), and the reason why cosmological observations suggest a nonzero $\Lambda$ are in some sense independent. It is in fact reasonable to expect that, since vacuum energy gravitate so much less than expected, it may actually does not gravitate at all. This may be due to a symmetry which prevents that or to a completely different reason, and understanding that seems one of the most difficult problems in contemporary physics. Nevertheless, we may take the point of view that, however difficult to solve, this problem is disentangled from the implications of cosmological observations. This is the point of view we take in this thesis: without addressing the problem of why vacuum energy does not gravitate, we try to understand why in cosmology we observe a nonzero and fine tuned $\Lambda$.

### Backreaction, dark energy and modified gravity

If we want to explain the cosmological observations without resorting to a nonzero cosmological constant, some of the hypothesis which underlie the $\Lambda$CDM model have to be relaxed. Despite the fact that all of them may not be correct, for simplicity we can study what happens if we relax in turn just one of these assumptions, namely the large scale homogeneity and isotropy, the assumption that the universe is filled only with CDM and standard model particles, and the fact that gravity is described by GR at all scales. In the following, we describe briefly the main advantages/disadvantages of the different cases.

As we said previously, while large scale isotropy is very well tested observationally, homogeneity is not. It is usually assumed that we don't occupy a special place in the universe (the Copernican principle), which implies homogeneity, but since this is a philosophical assumption, it may be wrong after all. In fact, if the Earth was situated near the center of a huge, nearly spherical structure, the supernovae observations may be explained as an due to the inhomogeneity, without having a nonzero $\Lambda$ ([20, 21]). However, apart from being philosophically puzzling, this scenario poses another fine tuning problem, regarding the characteristic of the spherical structure and our position inside it. Moreover, it is not so clear whether it is consistent with all the cosmological observations, not just supernovae [17]. A different possibility is that the fact that inhomogeneities go nonlinear produce a sizable effect on the evolution of the scale factor. As we said in section (1.2), the time evolution does not commute with the averaging procedure on the Einstein equations. Therefore, the "real" scale factor that describe our universe is different from the one we get by solving the Friedmann equations, and it may be that this difference is crucial in judging if $\Lambda$ is zero or not: the universe may seem to accelerate at late times just because we don't take into account properly this effect. The influence of inhomogeneities on the evolution of the scale factor is known as *backreaction* (see for example [22]) and references in [17]): this would provide a dramatic resolution of the coincidence problem, since in this case the formation of structures and the apparent acceleration are correlated since they are both a consequence of the fact that inhomogeneities go nonlinear. However, there is no convincing demonstration that the backreaction is indeed able to explain the apparent acceleration. It should be noted anyway that it may significantly affect the estimation of cosmological parameters, even if it does not lead to acceleration [17].

Alternatively, if we take $\Lambda = 0$, neglect backreaction and assume that large scale homogeneity and isotropy hold, we are forced to admit that either gravity is not described exactly by GR, or that there is a new degree of freedom whose contribution to the energy-momentum tensor is responsible for the acceleration of the universe. The situation is somewhat similar to what happened when deviations from the predicted orbits were observed for some planets in the solar system: in the case of the anomalies of the orbits of Uranus and Neptune, the existence of a new, unobserved planet was postulated. Pluto was indeed discovered later on. On the other hand, the anomalous precession of the perihelion of Mercury could not be explained as the effect of a yet unobserved object (originally called Vulcan): the discrepancy was shown to be due to the inadequacy of the Newtonian theory of gravity, and the resolution of the problem was the result of the development of a new theory of gravity, General Relativity. If we consider GR to be the correct theory of gravitational interaction, even at extremely large scales, then the cosmological observations can be explained by adding a source term in the Einstein equations, which by equation (1.26) have to satisfy $\rho + 3p < 0$. This is a very unusual property, since at the classical level the matter we observe in Earth-based experiments has positive energy and non-negative pressure. Therefore, not only we have to introduce an ad-hoc matter which we don't observe on Earth and in the solar system, but this matter has to have very exotic properties. On the other hand, at quantum level such a property is not so strange, and can be enjoyed also by a very simple system such as a (classical) scalar field. This new component of the energy-momentum tensor is usually termed *dark energy*, and there are several different models/scenarios (such as for example quintessence models, K-essence and others, see [22]) which address the late time acceleration problem following this idea. However, most of them are not well motivated (so far) from the point of view of fundamental physics, and in general do not solve the coincidence problem, since some sort of fine tuning seems to be required anyway [17].

Finally, we may assume that there is not such a thing as dark energy, but the observations just signal the breakdown of the validity of GR at ultra large scales. From this point of view, the explanation of the apparent acceleration is to be found in formulating a new theory of gravity, which should reproduce very well the results of GR at scales from a micron up to astrophysical scales, but should deviates from it at ultra large scales. This approach is usually called *modified gravity*: for an extensive review, see [23]. There are several modified gravity scenarios which have been studied, among which $f(R)$ gravity, braneworld models, and massive gravity. Braneworld models have the appealing feature to be in a loose sense motivated by fundamental physics, since the existence of extra dimensions and "branes" where matter is localised is a important ingredient in string theory. However, quite in general, braneworld models which modify gravity at large distances are mainly phenomenological, in the sense that there are usually no precise indications about how to embed them into string theory. Overall, one of the crucial points is that it is very difficult to modify gravity at large distances, without introducing changes at intermediate and small distances: typically, the modifications can be traced back to the presence of new (gravitational) degrees of freedom, which however seems to contribute also at small scales. In order this not to happen, it is necessary that there is a "screening" mechanism which efficiently suppresses the contributions of the new degrees of freedom in the contexts where GR results have to be reproduced. Another problem is that modifying gravity at large scales quite often produces new degrees of freedom which have (at least in some configurations) negative kinetic energy (in which cases they are called *ghosts*). This is usually regarded as unacceptable, since at quantistic level the vacuum would be unstable.

## 1.4 Thesis summary

In this thesis, we adopt the modified gravity approach to tackle the late time acceleration problem. In particular we focus on the problem of finding a model which exibits an efficient screening mechanism which permits to recover GR results at small and intermediate scales. For definiteness, we consider the massive gravity approach to modify gravity. The thesis is therefore structured as follows: in chapter 2 we introduce braneworld models and the DGP models, which, although not providing itself a modified gravity solution to the late time acceleration problem, have been studied extensively and provided ideas and tools which turned out to be useful to propose new models. We then consider the case of massive gravity, which we discuss in detail in chapter 3, and concentrate on the recently proposed class of models known as ghost-free massive gravity, which as the name suggests have been shown to be free of ghost instabilities. Finally, in chapter 4 we study in detail the efficiency of the screening mechanism known as "Vainshtein mechanism" in this class of models. We consider spherically symmetric solutions, and select one of the two branches of solutions which have been found. We provide a complete characterisation of the phase space of these theories in relation to the way the Vainshtein mechanism works, which is an important step in establishing the viability of such theories.

# Chapter 2

# Braneworlds and the DGP model

In the framework of modified gravity, theories with extra spatial dimensions and in particular the so called braneworld models have attracted a lot of attention. Apart from providing a geometrical mechanism of modifying gravity at large distances, they have played a crucial role in the recent construction of a class of ghost-free massive gravity theories. Therefore, we dedicate this chapter to a general introduction to braneworld theories and in particular to the DGP model.

## 2.1 Introduction to braneworlds

### 2.1.1 Historical introduction

#### Kaluza-Klein theories

The idea that there may be some spatial dimensions in addition to the three we have experience of is in fact not a recent one. Already in 1921, Theodor Kaluza [24] (reprinted with English translation in [25]) studied a five dimensional extension of General Relativity, and noticed that the degrees of freedom of the metric associated with the extra dimension could be interpreted as a vector field in our four dimensional world (plus an additional scalar). Recognizing in this vector the 4-potential of electromagnetic theory, the Einstein equations for the 5D metric would produce respectively the Einstein equations and the Maxwell equations for gravity coupled to the electromagnetic field, thereby geometrically giving a unified description of these two forces. Oskar Klein in 1926 [26] (also reprinted with English translation in [25]) proposed that, if the extra dimension is compact and of radius $r$, deviations to the known laws would not show up for length scales larger than $r$, or for energies less than $1/r$, thereby we wouldn't be able to observe them if $r$ is small enough (say $r < 10^{-19}$m, corresponding to an energy $E \sim 1$TeV). This idea of the extra dimensions being rolled up and small is usually referred to as the Kaluza-Klein (KK) scenario: it has been almost universally adopted for a long time to explain why we don't observe the extra dimensions, despite their existence, and tipically the characteristic radius of the extra dimensions was assumed to be incredibly small, of the order of the Planck length $l_{pl} = \sqrt{\hbar G/c^3} \sim 10^{-35}$ m. The very idea of the existence of extra dimensions had a big push by the discovery in the 1970's that string theory, one of the most promising candidates for unifying general relativity and quantum mechanics as well as providing a unification of all the forces, is only consistent if there is a suitable number of extra dimensions (10 for superstring theory).

### Braneworlds and large extra dimensions

A conceptual revolution began around 1960 [27, 28] when the idea that matter and force fields, instead of propagating in all the space, could be confined to a surface in a higher dimensional space started being discussed. At the beginning of the 1980's, Akama [29] and independently Rubakov and Shaposhnikov [30] proposed an explicit particle physics realisation of the localization phenomenon, while Visser [31] proposed a gravitational realization of the same phenomenon. The idea of matter being localized on a surface, or on a "brane", became much more popular with the discovery in the 1990's that extended objects, called p-branes, are of fundamental importance in string theory. In particular there are objects called D-branes to which the ends of open strings are attached, while closed strings can propagate in the bulk. The idea that gravity could propagate in the extra dimensions (in string theory it is described by closed strings) while matter and standard model interactions could be confined to a brane, led Arkani-Ahmed, Dimopolous and Dvali [32, 33] (ADD) to propose that the characteristic length of compact extra dimensions could be much bigger than the Planck length, and in fact macroscopic (even at at millimeter scale). The crucial observation is that while particle interactions are probed by high energy colliders at energies up to the TeV scale, and therefore for length scales down to $10^{-19}$ m, gravity is tested only for length scales down to $10^{-6}$ m. This idea led to the proposal that the observed Newton constant $G$ may not be the fundamental strength of gravity, but it is an effective strength related to the fundamental strength $G_\star$ *via* the relation $G \propto G_\star/V$ where $V$ is the volume of the compact extra dimensions. This idea opened up the fascinating possibility of having a fundamental (Planck) scale for gravity as low as 1 TeV (with the possibility of realistically observing quantum gravity effects in particle colliders) [34], and from another point of view of explaining the observed weakness of gravity compared to the other interactions as an effect of the ability of gravity to propagate in all the spatial dimensions.

### Non factorizable geometry and localization of gravity

In the braneworld picture, more often than not it is assumed that some mechanism (the presence of a bulk soliton in QFT, or the very existence of D-branes in string theory) localizes matter and the standard model interactions. Once assumed the existence of such a mechanism, explaining why the extra dimensions are not observed reduces to explain why gravity behaves as in the (4D) GR despite propagating in more than four dimensions. Despite the widespread belief that compact (although not necessarily extremely small) extra dimensions are needed to reproduce 4D gravity in a suitable distance range, it was shown by Randall and Sundrum in a famous series of two papers [35, 36] that, if the bulk metric is not factorizable, this is not the case. In particular, a flat 4D brane with nonzero tension $T$ in a 5D bulk with negative cosmological constant $\Lambda$ causes the bulk to become an anti-deSitter space (if $T$ and $\Lambda$ are appropriately tuned), with warped metric $ds^2 = e^{-|y|/L}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$ where $y$ is the extra dimension and $L \propto \sqrt{1/\Lambda}$. In particular, they showed how the warping in the bulk metric between two flat branes could be used to explain the hierarchy between the electroweak mass scale and the gravitational Planck scale [35], and how the warping could effectively localize gravity on one brane even if the extra dimension is not compact [36]. However, in the Randall-Sundrum model the extra dimension is not truly infinite since its *volume* is still finite due to the warping. As a result, the relation between the fundamental Planck scale of gravity and the 4D effective one is very similar to the one which holds in the ADD model, with the radius of the extra dimension replaced by the characteristic length $L$ of $AdS_5$. Likewise, in both ADD and RS models the modifications to the Newton law happen at *small* distances, where the critical length is set by the characteristic length of the extra dimensions.

**Infinite volume extra dimensions**

In 2000, Gregory, Rubakov and Sibiryakov (GRS) [37] instead showed that it is possible to construct a braneworld model where gravity looks like GR at observable scales, but behaves differently both at smaller *and larger* scales. In their system, made up of three flat 4D branes with negative bulk cosmological constant between the branes and zero outside, the metric is flat outside the branes and therefore the extra dimension is truly infinite (its volume is infinite indeed). Soon after that, it was proposed [38, 39] that the ability of some theories with one infinite volume extra dimension to reproduce 4D gravity can be understood as if gravity were mediated by a *metastable* 4D graviton, or in other words by a continuous superposition of 4D massive gravitons peaked around $m = 0$ with a finite width. The GRS model in fact was shown to belong to this class of models. Later in the same year, the celebrated DGP model [40] was proposed. In this case, there is just one 4D brane in an infinite 5D bulk, and its distinctive feature is the presence of an induced gravity term localized on the brane, which is responsible for the peaked profile in the mass space.

The DGP model inspired a lot of activity, both to establish its phenomenological viability [41, 42, 43, 44, 45, 46] and to explore its potential ability to address long standing theoretical problems like the cosmological constant problem [18] and more recent ones as the late time acceleration problem of cosmology (see section 1.3). In the cosmological context, a breakthrough came when Deffayet [47] showed that the DGP model admits "self-accelerating" solutions, opening the door to the idea of explaining the late time acceleration as a purely geometric and "modified gravity" phenomenon [48], without resorting to the idea of dark energy. Concerning the cosmological constant problem, it has been shown [49] that infinite volume extra dimensions provide a way to bypass the no-go theorem formulated by Weinberg [18], and therefore are extremely appealing from that point of view.

However, the attempts were not crowned by success. It has been shown that the self accelerating cosmological solution contains a ghost [50, 45, 46, 51] and therefore cannot be quantum mechanically stable. Furthermore, a careful analysis has shown that there is strong tension between the theoretical predictions and the cosmological data, which in practice rule out the DGP self-accelerating solution as an explanation for the late time acceleration [52]. From another point of view, it has been shown that the DGP model cannot solve the Cosmological Constant problem by "degravitating" sources with very large characteristic length scales, since its gravitational potential does not decay fast enough at large distances [53].

### Generalizations of the DGP model

Nevertheless, the richness of ideas and approaches to several problems of modern physics which were conceived by studying the DGP model, even if it is not successful itself, suggest that it may be worth trying to find generalizations of the DGP model which may be similar enough to its original formulation to preserve the good features, and different enough to be free of its shortcomings. For example, it is conceivable that a generalization of the DGP model may still contain cosmological self-accelerated solutions, but the effective Friedmann equations in this case will be necessarily modified and may fit the data better. Furthermore, more sofisticated constructions may provide a mechanism to get rid of the ghost.

A quite natural way to generalize the DGP model is to consider a higher codimension setup. Higher codimension branes are notoriously very delicate to deal with since the thin limit of a brane is not well defined for codimension $\geq 2$ [54]. In fact, different regularization procedures give different answers in the thin limit, which is then not unique, or in other words any effective description in which the internal degrees of freedom are not excited, nevertheless remembers of the details of the internal structure of the brane, and displays distinctive features. This is not necessarily a problem as long as one keeps in mind that the regularization procedure is a important and central part of the model [55]. On the other hand, the ultra large distance behavior of the gravitational field depends markedly on the codimension, which is promising both for the degravitation phenomenon and for the cosmological solutions. Furthermore, pure codimension-2 models are known to possess the peculiar feature that pure tension on the brane (equivalent to vacuum energy) does not produce curvature on the brane, but merely create a deficit angle in the extra dimensions (in other words, it curves just the extra dimensions). This is extremely interesting for the cosmological constant problem.

In reality, increasing the codimension seems to worsen the problem of appearance of ghosts, since a codimension-2 formulation of the DGP model seems to have a ghost even among perturbations around the flat Minkowski solution [56]. However, this is not necessarily a general feature of any codimension-2 extensions of DGP, since a model defined by a different regularization of the same codimension-2 setup has been shown to be ghost free [57]. An intriguing possibility has been proposed few years ago, in which there is a recursive embedding of branes into branes of increasing dimensionality, and which produces a gravitational field which "cascades" from $N$-D to $(N-1)$-D and so on down to 4D going from large to small distances ($N$ here is the dimension of the ambient space) [58]. This model, named *Cascading DGP*, has been shown to admit self-accelerating solutions [59] and seem to provide a promising setup for the degravitation mechanism [60, 61]. Furthermore, it has been claimed that, in the minimal setup where the bulk is 6D, there is a critical value for the brane tension above which there are no ghosts around the Minkowski-flat solution [62]. However, it is not so clear it these results are general or depend on the regularization procedure chosen to perform the thin limit.

### 2.1.2 Mathematical preliminaries

Let $\mathscr{M}$ be a $N$-dimensional ($N \geq 4$) manifold. We call a $D$-dimensional brane (or a $(D-1)$-brane for short) a $D$-dimensional submanifold $\Sigma$ of $\mathscr{M}$. We define *codimension* of the brane the number $N - D$. Despite being a subset of $\mathscr{M}$, we can equivalently consider $\Sigma$ to be a separate manifold equipped with an *embedding function*

$$\varphi : \Sigma \to \mathscr{M} \tag{2.1}$$

which specifies the "position" of $\Sigma$ inside $\mathscr{M}$ when seen as a subset. Being the dimensionalities of $\mathscr{M}$ and $\Sigma$ different, $\varphi$ is not invertible, and can be used to pull-back to $\Sigma$ tensors of type $(0,k)$ defined on $\mathscr{M}$ and push-forward to $\mathscr{M}$ tensors of type $(n,0)$ defined on $\Sigma$. In particular, for every $p \in \Sigma$, if $\{\mathbf{w}_{(j)}\}_j$ $(j = 1, \ldots, D)$ is a basis of tangent vectors in $T_p\Sigma$, then $\{\varphi^\star(\mathbf{w}_{(j)})\}_j$ is a linearly independent set of vectors in $T_{\varphi(p)}\mathscr{M}$, where $\varphi^\star$ indicates the push-forward with respect to the embedding function. We define the $D$-dimensional subset of $T_{\varphi(p)}\mathscr{M}$ spanned by this set of vectors to be the tangent space to $\Sigma$ (seen as a subset of $\mathscr{M}$) and we will denote it as $T_{\varphi(p)}\Sigma$.

We will in general consider two different atlases of maps, one which defines coordinates on $\mathscr{M}$ and another one which defines coordinates on $\Sigma$. Indicating with $X^M$ the coordinates on $\mathscr{M}$ and with $\xi^m$ the coordinates on $\Sigma$, the embedding function reads in coordinates $\varphi^M(\xi^m)$, and a basis of $T_{\varphi(p)}\Sigma \subset T_{\varphi(p)}\mathscr{M}$ is given by the directional derivatives of the embedding function

$$v^A_{(a)}(p) \equiv \left\{ \frac{\partial}{\partial \xi^a}\bigg|_p \varphi^A \right\}_a \qquad a = 0, \ldots, D-1 \tag{2.2}$$

where the derivative is evaluated in the coordinates corresponding to $p$. If the ambient manifold $\mathscr{M}$ is a metric manifold $(\mathscr{M}, \mathbf{g})$, the embedding induces a metric structure on the brane $\Sigma$ as well: we define the *induced metric* $\tilde{\mathbf{g}}$

$$\tilde{\mathbf{g}} : T\Sigma \times T\Sigma \to \mathbb{R} \qquad \tilde{\mathbf{g}} \equiv \varphi_\star(\mathbf{g}) \tag{2.3}$$

where $\varphi_\star$ indicates the pullback with respect to the embedding function. In coordinates the previous relation reads

$$\tilde{g}_{ab}(\xi^\cdot) = \mathbf{g}\big(\mathbf{v}_{(a)}, \mathbf{v}_{(b)}\big)(\xi^\cdot) \tag{2.4}$$

and explicitly

$$\tilde{g}_{ab}(\xi^\cdot) = \frac{\partial \varphi^A(\xi^\cdot)}{\partial \xi^a} \frac{\partial \varphi^B(\xi^\cdot)}{\partial \xi^b}\, g_{AB}(X^\cdot)\Big|_{X^\cdot = \varphi^\cdot(\xi^\cdot)} \tag{2.5}$$

where we used the notational convention of indicating the set of coordinates $X^M$ and $\xi^m$ respectively with $X^\cdot$ and $\xi^\cdot$, while the embedding function $\varphi^a$ is indicated with $\varphi^\cdot$. We assume here that the metric $\mathbf{g}$ is nondegenerate and pseudo-Riemannian.

In the following we will be mostly interested in codimension-1 brane, for which there is a fair amount of dedicated terminology and geometrical concepts to which we now turn.

### Codimension-1 braneworlds

We denote in general with a tilde $\tilde{}$ quantities pertaining to the codimension-1 brane. Taken a basis $\{\mathbf{b}_{(a)}\}_a$ $(a = 1, \ldots, D)$ of $T\Sigma \subset T\mathscr{M}$, we can define the vector $\mathbf{n}(\xi^\cdot)$ normal to the cod-1 brane in the following way

$$\mathbf{n}(\xi^\cdot) : \quad \begin{cases} \langle \mathbf{n}(\xi^\cdot) | \mathbf{b}_{(a)}(\xi^\cdot) \rangle_{\mathbf{g}} = 0 \\ |\langle \mathbf{n}(\xi^\cdot) | \mathbf{n}(\xi^\cdot) \rangle_{\mathbf{g}}| = 1 \end{cases}$$

where $\langle \ | \ \rangle_{\mathbf{g}}$ indicate the scalar product associated to the metric $\mathbf{g}$. There are two possibilities, depending on the sign of the squared modulus of $\mathbf{n}$: if the normal vector is spacelike $\|\mathbf{n}\| > 0$, the brane is said to be timelike, while if the normal vector is timelike $\|\mathbf{n}\| < 0$, the brane is said to be spacelike. We will consider only the case of a spacelike normal vector, which corresponds to having a "spatial" extra dimension. Even fixing the sign of $\|\mathbf{n}\|$, the system above does not define uniquely the normal vector since there are two possible choices which define the local orientation of the brane. Note that we can uniquely decompose a vector $\mathbf{w}$ into an orthogonal component $\mathbf{w}_\perp = w_\perp \mathbf{n}$ and a parallel component $\mathbf{w}_{\shortparallel}$ such that $\langle \mathbf{w}_{\shortparallel} | \mathbf{n} \rangle_{\mathbf{g}} = 0$.

Using the normal vector we can define the first fundamental form of the cod-1 brane

$$\mathbf{P}(\xi^\cdot) \equiv \mathbf{g} - \mathbf{g}(\mathbf{n}, \_) \otimes \mathbf{g}(\mathbf{n}, \_) \tag{2.6}$$

where $\mathbf{g}$ is evaluated in $X^\cdot = \varphi^\cdot(\xi^\cdot)$, and $\mathbf{n}$ is evaluated in $\xi^\cdot$. Acting on two vectors $\mathbf{c}$ and $\mathbf{d}$ the first fundamental form give as a result the scalar product computed with $\mathbf{g}$ between the parallel components of the two vectors

$$\mathbf{P}\big(\mathbf{c}, \mathbf{d}\big) = \mathbf{P}\big(\mathbf{c}_{\shortparallel}, \mathbf{d}_{\shortparallel}\big) = \mathbf{g}\big(\mathbf{c}_{\shortparallel}, \mathbf{d}_{\shortparallel}\big) \tag{2.7}$$

and therefore extracts the notion of metric on the brane from the bulk metric $\mathbf{g}$. To get an intrinsic object which defines metric concepts on the brane we can pull-back the first fundamental form to the brane using the embedding function, obtaining the (already introduced) induced metric

$$\tilde{\mathbf{g}} \equiv \varphi_\star(\mathbf{g}) = \varphi_\star(\mathbf{P}) \tag{2.8}$$

From the induced metric we can construct the associated symmetric and metric compatible connection, and the curvature tensors and scalar, which characterise the intrinsic geometry of the brane.

The second fundamental form of the cod-1 brane is defined as

$$\mathbf{K} \equiv -\frac{1}{2}\,\mathcal{L}_{\mathbf{n}}\,\mathbf{P} \tag{2.9}$$

and instead characterises the extrinsic geometry of the brane. Like the first fundamental form, it is a brane parallel object in the sense that it acts only on the parallel components of the vectors

$$\mathbf{K}\big(\mathbf{c},\mathbf{d}\big) = \mathbf{K}\big(\mathbf{c}_{\shortparallel},\mathbf{d}_{\shortparallel}\big) \tag{2.10}$$

To obtain from the second fundamental form an intrinsic object which describes the extrinsic geometry we can pull-back $\mathbf{K}$ to the brane, obtaining the extrinsic curvature $\tilde{\mathbf{K}}(\xi^{\cdot})$

$$\tilde{\mathbf{K}} \equiv \varphi_{\star}\big(\mathbf{K}\big) = -\frac{1}{2}\,\varphi_{\star}\big(\mathcal{L}_{\mathbf{n}}\,\mathbf{g}\big) \tag{2.11}$$

Using the expression (0.6) for the Lie derivative, and taking advantage of the fact that $\mathbf{n}$ and $\mathbf{v}_{(a)}$ are orthogonal for every $a$, we can express it as

$$\tilde{\mathbf{K}}(\xi^{\cdot}) = \tilde{\mathbf{K}}^{[og]}(\xi^{\cdot}) + \tilde{\mathbf{K}}^{[pg]}(\xi^{\cdot}) + \tilde{\mathbf{K}}^{[b]}(\xi^{\cdot}) \tag{2.12}$$

where we defined

$$\tilde{K}^{[og]}_{ab}(\xi^{\cdot}) \equiv -\frac{1}{2}\bigg(\big(\partial_{\mathbf{n}}\,\mathbf{g}\big)\big(\mathbf{v}_{(a)},\mathbf{v}_{(b)}\big)\bigg) \tag{2.13}$$

$$\tilde{K}^{[pg]}_{ab}(\xi^{\cdot}) \equiv \frac{1}{2}\bigg(\big(\partial_{\mathbf{v}_{(a)}}\,\mathbf{g}\big)\big(\mathbf{n},\mathbf{v}_{(b)}\big) + \big(\partial_{\mathbf{v}_{(b)}}\,\mathbf{g}\big)\big(\mathbf{n},\mathbf{v}_{(a)}\big)\bigg) \tag{2.14}$$

$$\tilde{K}^{[b]}_{ab}(\xi^{\cdot}) \equiv \frac{1}{2}\bigg(\mathbf{g}\big(\mathbf{n},\partial_{\xi^a}\mathbf{v}_{(b)}\big) + \mathbf{g}\big(\mathbf{n},\partial_{\xi^b}\mathbf{v}_{(a)}\big)\bigg) \tag{2.15}$$

where $\mathbf{g}$, $\partial_{\mathbf{v}_{(a)}}\,\mathbf{g}$ and $\partial_{\mathbf{n}}\,\mathbf{g}$ are evaluated in $X^{\cdot} = \varphi^{\cdot}(\xi^{\cdot})$. The first two pieces are named "orthogonal gradient" and "parallel gradient" as they are nonzero when the bulk metric has nonzero derivative respectively in the directions orthogonal and parallel to the cod-1 brane, even when the cod-1 brane is not bent. The third piece is instead due to the bending, since it is nonzero when the brane is bent even if the bulk metric is constant. The three contributions read in coordinates

$$\tilde{K}^{[og]}_{ab}(\xi^\cdot) \equiv -\frac{1}{2}\,\frac{\partial\varphi^A(\xi^\cdot)}{\partial\xi^a}\frac{\partial\varphi^B(\xi^\cdot)}{\partial\xi^b}\,n^L(\xi^\cdot)\frac{\partial\, g_{AB}}{\partial X^L}\Big|_{X^\cdot=\varphi^\cdot(\xi^\cdot)} \tag{2.16}$$

$$\tilde{K}^{[pg]}_{ab}(\xi^\cdot) \equiv \frac{1}{2}\,n^A(\xi^\cdot)\,\frac{\partial\varphi^B(\xi^\cdot)}{\partial\xi^{(a}}\,\frac{\partial\varphi^L(\xi^\cdot)}{\partial\xi^{b)}}\frac{\partial\, g_{AB}}{\partial X^L}\Big|_{X^\cdot=\varphi^\cdot(\xi^\cdot)} \tag{2.17}$$

$$\tilde{K}^{[b]}_{ab}(\xi^\cdot) \equiv n_L(\xi^\cdot)\,\frac{\partial^2\varphi^L(\xi^\cdot)}{\partial\xi^a\partial\xi^b} \tag{2.18}$$

where $n_M(\xi^\cdot) = g_{LM}(\varphi^\cdot(\xi^\cdot))\,n^M(\xi^\cdot)$.

Note that it is always possible, at least locally, to use (N-1) of the N bulk coordinates to parametrize the brane: for definiteness we can indicate the coordinates on the brane with $\xi^\cdot$, and the bulk coordinates as $X^\cdot = (\xi^\cdot, z)$, so essentially we recognize $z$ as the extra dimension. In this case all the components of the embedding function are trivial but $\varphi^z$, and (with a little abuse of notation) we call $\varphi$ the nontrivial component

$$\varphi^\cdot(\xi^\cdot) = (\xi^\cdot, \varphi(\xi^\cdot)) \tag{2.19}$$

Using this gauge fixing between the bulk coordinates and the brane coordinates, the system is now characterised by the bulk metric $g_{AB}(X^\cdot)$ and by one scalar function, the nontrivial component of the embedding $\varphi$. We can express the objects which define the geometrical properties of the brane using these quantities: the induced metric takes the simplified form

$$\tilde{g}_{ab}(\xi^\cdot) = \frac{\partial\varphi(\xi^\cdot)}{\partial\xi^a}\frac{\partial\varphi(\xi^\cdot)}{\partial\xi^b}\,g_{zz}\big(\varphi^\cdot(\xi^\cdot)\big) + \frac{\partial\varphi(\xi^\cdot)}{\partial\xi^{(a}}\,g_{z|b)}\big(\varphi^\cdot(\xi^\cdot)\big) + g_{ab}\big(\varphi^\cdot(\xi^\cdot)\big) \tag{2.20}$$

and a 1-form orthogonal to the brane can be found as

$$N_A(\xi^\cdot) \equiv \Big(-\frac{\partial\varphi}{\partial\xi^a}(\xi^\cdot), 1\Big) \tag{2.21}$$

Normalizing $N$ we obtain the normal form to the cod-1 brane

$$n_A(\xi^\cdot) \equiv \frac{1}{\sqrt{g^{LM}\,N_L\,N_M}}\;\Big(-\frac{\partial\varphi}{\partial\xi^a}(\xi^\cdot), 1\Big) \tag{2.22}$$

where $g^{LM}$ is evaluated in $X^\cdot = (\xi^\cdot, \varphi(\xi^\cdot))$ and $N_L$ is evaluated in $\xi^\cdot$. Using the results above we can express the extrinsic curvature in a simplified way as well, and for example the bending contribution to the extrinsic curvature reads

$$\tilde{K}^{[b]}_{ab}(\xi^{\cdot}) = \frac{1}{\sqrt{N_L\, N^L}}\ \frac{\partial^2 \varphi(\xi^{\cdot})}{\partial \xi^a \partial \xi^b} \tag{2.23}$$

## 2.2 The DGP model

The DGP model [40], in its original formulation, is a codimension-1 braneworld model in five dimensions. The complete spacetime $\mathscr{M} = \mathscr{B} \cup \Sigma$ is made up of a five dimensional bulk $\mathscr{B} = \mathscr{B}_- \cup \mathscr{B}_+$ constituted by the two disjoint pieces $\mathscr{B}_-$ and $\mathscr{B}_+$, which have in common a four dimensional boundary $\Sigma = \partial\mathscr{B}_- = \partial\mathscr{B}_+$. We assume that the topology of $\mathscr{B}_-$ and $\mathscr{B}_+$ is the same as $\mathbb{R}^4 \times \mathbb{R}$. The action of the model is

$$S = 2M_5^3 \int_{\mathscr{B}} d^5X \,\sqrt{-g}\, R + 2M_4^2 \int_{\Sigma} d^4x \,\sqrt{-\tilde{g}}\, \tilde{R} + \int_{\Sigma} d^4x \,\sqrt{-\tilde{g}}\, \mathscr{L}_M + \\ + S_{GH}(\Sigma_-) + S_{GH}(\Sigma_+) \tag{2.24}$$

where $S_{GH}(\Sigma_-)$ and $S_{GH}(\Sigma_+)$ are the Gibbons-Hawking terms[1] [63, 64] on the two sides of the brane, and $\mathscr{L}_M$ is the matter Lagrangian. Here $g$ is the determinant of the bulk metric and $R$ is the Ricci scalar constructed from it, while $\tilde{g}$ is the determinant of the induced metric on the brane and $\tilde{R}$ is the Ricci scalar constructed from it. We assume that the mass scales $M_5^3$ and $M_4^2$ obey the hierarchy $M_4^2/M_5^3 \gg 1$. The distinctive feature of this action is the *induced gravity term*

$$2M_4^2 \int_{\Sigma} d^4x \,\sqrt{-\tilde{g}}\, \tilde{R} \tag{2.25}$$

which as we shall see is responsible for the recovery of the correct 4D Newtonian behavior of gravity on the brane, for small and intermediate distances. This piece of the action can be introduced at classical level purely on phenomenological grounds, but can be also understood as contribution coming from loop corrections in the low energy action of a quantum description where matter is confined on the brane [40].

[1] $S_{GH} = -4M_5^3 \int d^4x \,\sqrt{-\tilde{g}}\, \tilde{K}$, where $K$ is the trace of the extrinsic curvature of the brane.

The equations of motion for this system are

$$\mathbf{G} = 0 \qquad \text{(bulk)} \tag{2.26}$$

$$M_5^3 \big[\tilde{\mathbf{K}} - \tilde{\mathbf{g}}\,\mathrm{tr}\,\tilde{\mathbf{K}}\big]_{\pm} + M_4^2\,\tilde{\mathbf{G}} = \tilde{\mathbf{T}} \qquad \text{(brane)} \tag{2.27}$$

where $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are the Einstein tensors constructed respectively from $\mathbf{g}$ and $\tilde{\mathbf{g}}$, $\tilde{\mathbf{K}}$ is the extrinsic curvature of the brane and $\tilde{\mathbf{T}}$ is the energy momentum tensor of the matter localized on the brane. Equation (2.26) is simply the vacuum Einstein equation in the bulk, while (2.27) is the Israel junction condition [65] on the brane. The notation $[\ ]_{\pm}$ indicates the jump across the brane of the quantity in square parenthesis, or equivalently $[\ ]_{\pm} = [\ ]_{\Sigma_+} - [\ ]_{\Sigma_-}$.

It is customary to assume that $\mathscr{B}_-$ and $\mathscr{B}_+$ are diffeomorphic and to impose a reflection symmetry across the brane ($\mathbb{Z}_2$ symmetry). In this case it is enough to solve the equations of motion in one of the two pieces to know the solution in all the bulk. Assuming that the $\mathbb{Z}_2$ symmetry holds, the equations of motion become

$$\mathbf{G} = 0 \qquad \text{(bulk)} \tag{2.28}$$

$$2M_5^3 \big(\tilde{\mathbf{K}} - \tilde{\mathbf{g}}\,\mathrm{tr}\,\tilde{\mathbf{K}}\big) + M_4^2\,\tilde{\mathbf{G}} = \tilde{\mathbf{T}} \qquad \text{(brane)} \tag{2.29}$$

where for definiteness the bulk equation is considered in $\mathscr{B}_+$ and the extrinsic curvature is evaluated in $\Sigma_+$. Note that assigning the energy-momentum tensor on the brane is not enough to fix univoquely the solution of the system above, so an additional condition is needed to render the model self-consistent. This is tipical of codimension-1 braneworld models: apart from the junction conditions, a condition on the behavior of the bulk metric at spatial infinity (in the normal direction to the brane) is to be imposed. It is standard to ask that, in the spatial infinity asymptotic region, the metric is a superposition of outgoing waves only, formalizing the idea that nothing can enter our universe from the extra dimension.

We will call $X^m = (x^\mu, y)$ the coordinates in the bulk. Although we could use a generic coordinate system on the brane, we will use four of the five bulk coordinates to parametrize the brane (for the sake of precision $x^\mu$), which is always possible (at least locally). Following the terminology of subsection (2.1.2), the embedding function reads

$$\varphi^m(x^{\cdot}) = \big(x^\mu, \varphi(x^{\cdot})\big) \tag{2.30}$$

The system is completely determined once we know the bulk metric $\mathbf{g}(X^{\cdot})$ and the brane embedding function $\varphi(x^{\cdot})$. Using the relations (2.28)-(2.29), it is straightforward to see that if the brane is empty ($\tilde{\mathbf{T}} = 0$), the configuration

$$\mathbf{g}(X^{\cdot}) = \bar{\mathbf{g}}(X^{\cdot}) = \boldsymbol{\eta} \tag{2.31}$$

$$\varphi(x^{\cdot}) = \bar{\varphi}(x^{\cdot}) = 0 \tag{2.32}$$

is a solution of the equations of motion, since all the curvature tensors (constructed from the bulk and from the induced metric) vanish. In fact using (2.20) it is easy to see that the induced metric is flat as well

$$\bar{\tilde{\mathbf{g}}}(x^{\cdot}) = \boldsymbol{\eta} \tag{2.33}$$

Therefore, a straight brane in a flat bulk is a vacuum solution of the theory. This vacuum solution is quite different from the warped solution of an empty (but of course tensionful) Randall-Sundrum brane, and it may seem surprising that gravity on a DGP brane can be very similar to 4D GR. We turn now to the analysis of weak gravity in the DGP model.

### 2.2.1 Weak gravity in the DGP model

Let's study perturbations around the flat-Minkowski solution. We indicate with $\pi(x^{\cdot})$ the perturbation of the embedding function and with $h_{ab}(x^{\cdot}, y)$ the perturbation of the bulk metric, explicitly

$$\varphi(x^{\cdot}) = \pi(x^{\cdot}) \tag{2.34}$$

$$g_{ab}(x^{\cdot}, y) = \eta_{ab} + h_{ab}(x^{\cdot}, y) \tag{2.35}$$

We define the perturbation in the induced metric as

$$\tilde{h}_{\mu\nu}(x^{\cdot}) \equiv \tilde{g}_{\mu\nu}(x^{\cdot}) - \eta_{\mu\nu} \tag{2.36}$$

and we indicate with $\tilde{\mathcal{T}}_{\mu\nu}$ the perturbation of the energy-momentum tensor localized on the brane.

While the above definitions do not assume that $\pi$, $h_{ab}$ and $\mathcal{T}_{\mu\nu}$ are small, we now focus on studying perturbative solutions to (2.28)-(2.29) at first order. It is very useful to choose a gauge which simplifies the expressions as much as we can: a common choice is to use Gaussian Normal Coordinates (GNC), where the brane is placed at $y = 0$ and the only nonzero components of the bulk metric perturbations are the 4D ones. This reference system is therefore defined by

$$\pi^{(GN)}(x^{\cdot}) = 0 \qquad\qquad h_{5a}^{(GN)}(X^{\cdot}) = 0 \tag{2.37}$$

and have the good property that the induced metric is exactly the bulk metric computed in $y = 0^+$

$$\tilde{g}_{\mu\nu}^{(GN)}(x^{\cdot}) = g_{\mu\nu}^{(GN)}\big|_{y=0^+}(x^{\cdot}) \tag{2.38}$$

We will use instead a different gauge choice, introduced by [66], where we do not fix the position of the brane, and so the bending becomes a physical perturbation mode. On one hand, this is mathematically useful since it simplifies the bulk equations. On the other hand, it is also physically useful because the bending mode has a direct geometrical interpretation and its dynamics turns out to be characterised by a different length scale compared to the bulk perturbations, which is important at nonlinear level. Without fixing the bending, it is possible to impose more gauge conditions on the bulk metric, and in fact it is possible to impose

$$h_{55}(X^{\cdot}) = h_{5\nu}(X^{\cdot}) = 0 \qquad\qquad \eta^{\mu\nu}h_{\mu\nu}(X^{\cdot}) = \partial_\mu\, h^{\mu}{}_{\nu}(X^{\cdot}) = 0 \tag{2.39}$$

where indices are raised with the background inverse metric $\eta^{\mu\nu}$. In this gauge, the only nonzero components of the bulk metric perturbations are the 4D ones and the bulk metric is transverse-traceless (TT-gauge), which is the 5D equivalent of what is usually done in GR to study gravitational waves [3]. Note that this gauge conditions can be imposed only in source-free regions, which is always true in our case since we consider an empty bulk.

We can now derive the the dynamical equations for the relevant degrees of freedom in this gauge. First, note that the trace of the junction conditions (2.29) gives

$$\Box_4\pi = -\frac{1}{6M_5^3}\,\tilde{\mathcal{T}} \tag{2.40}$$

where $\tilde{\mathcal{T}} = \eta^{\mu\nu}\tilde{\mathcal{T}}_{\mu\nu}$, and we use the notations $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\Box_4 = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\Box_5 = \Box_4 + \partial_y^2$. The latter equation confirms that $\pi$ is not a gauge mode but instead a physical perturbation mode which is sourced by the trace of the energy momentum tensor. The junction condition reads

$$-\frac{1}{2}\Big(2M_5^3\,\partial_y + M_4^2\,\Box_4\Big)\Big|_{y=0^+} h_{\mu\nu} = \tilde{\mathcal{T}}_{\mu\nu} + 2M_5^3\,\eta_{\mu\nu}\,\Box_4\pi - 2M_5^3\partial_\mu\partial_\nu\pi \qquad (2.41)$$

and we see that the bending mode acts as a source for the bulk metric $h_{\mu\nu}$ along with the energy momentum tensor. Using the trace equation (2.40) we can write the equations of motion for the bulk metric in a suggestive way: the bulk equation (2.28) reads

$$\Box_5 h_{\mu\nu} = 0 \qquad (2.42)$$

while the junction condition becomes

$$-\frac{1}{2}\Big(2M_5^3\,\partial_y + M_4^2\,\Box_4\Big)\Big|_{y=0^+} h_{\mu\nu} = \tilde{\mathcal{T}}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\,\tilde{\mathcal{T}} - 2M_5^3\partial_\mu\partial_\nu\pi \qquad (2.43)$$

**The DGP propagator**

A powerful way to study solutions to linear differential equations in presence of sources is to derive the *propagator*, which roughly speaking is the solution correspondent to a perfectly localized source (Green's function). More precisely, it can be defined as the object

$$\mathcal{D}_{\mu\nu}{}^{\alpha\beta}(x, y; x') \qquad (2.44)$$

such that the solution to the linear differential equation correspondent to a source configuration $\tilde{\mathcal{T}}_{\mu\nu}(x)$ is

$$h_{\mu\nu}(x, y) = \int d^4x'\,\mathcal{D}_{\mu\nu}{}^{\alpha\beta}(x, y; x')\,\tilde{\mathcal{T}}_{\alpha\beta}(x') \qquad (2.45)$$

Note that we can neglect the term $2M_5^3\partial_\mu\partial_\nu\pi$ in equation (2.43) since it produces in momentum space a contribution $\sim p_\mu p_\nu$, which have no effect at first order if we consider (as we do) test bodies whose energy-momentum tensor is conserved. Therefore the propagator for our system obeys

$$\Box_5\,\mathcal{D}_{\mu\nu}{}^{\alpha\beta}(x, y; x') = 0 \qquad (2.46)$$

$$-\frac{1}{2}\Big(2M_5^3\,\partial_y + M_4^2\,\Box_4\Big)\Big|_{y=0^+} \mathcal{D}_{\mu\nu}{}^{\alpha\beta}(x, y; x') = \bigg[\frac{1}{2}\Big(\delta_\mu{}^\alpha\delta_\nu{}^\beta + \delta_\nu{}^\alpha\delta_\mu{}^\beta\Big) - \frac{1}{3}\,\eta_{\mu\nu}\eta^{\alpha\beta}\bigg]\,\delta^{(4)}(x - x') \qquad (2.47)$$

To find a solution to this system, we can factorize a scalar part $\mathcal{D}_S(x-x',y)$ which depends on the coordinates (where we have made manifest that the propagator can depend only on the difference of the coordinates, due to the 4D translational inveriance of the model) and a purely numerical part which carries the tensor structure ${\mathcal{S}_{\mu\nu}}^{\alpha\beta}$

$$ {\mathcal{D}_{\mu\nu}}^{\alpha\beta}(x-x',y) = {\mathcal{S}_{\mu\nu}}^{\alpha\beta}\,\mathcal{D}_S(x-x',y) \tag{2.48} $$

Roughly speaking, the tensor part gives the relative weight between the different components of the resulting metric $h_{\mu\nu}$, while the scalar part fixes the dependence of the components from the distance. Substituting this expression into (2.46)-(2.47) one gets that the tensor structure is

$$ {\mathcal{S}_{\mu\nu}}^{\alpha\beta} = \frac{1}{2}\Big(\delta_\mu^{\ \alpha}\delta_\nu^{\ \beta} + \delta_\nu^{\ \alpha}\delta_\mu^{\ \beta}\Big) - \frac{1}{3}\,\eta_{\mu\nu}\eta^{\alpha\beta} \tag{2.49} $$

while the scalar propagator obeys

$$ \Box_5\,\mathcal{D}_S(x-x',y) = 0 \tag{2.50} $$

$$ -\frac{1}{2}\Big(2M_5^3\,\partial_y + M_4^2\,\Box_4\Big)\Big|_{y=0^+}\mathcal{D}_S(x-x',y) = \delta^{(4)}(x-x') \tag{2.51} $$

In the case where the source is static $\tilde{\mathcal{T}}_{\alpha\beta}(x') = \tilde{\mathcal{T}}_{\alpha\beta}(\vec{x}\,')$, the metric $h_{\mu\nu}$ evaluated on the brane (from equation (2.45)) takes the form

$$ h_{\mu\nu}(\vec{x},0) = {\mathcal{S}_{\mu\nu}}^{\alpha\beta}\int d^3\vec{x}\,'\;\tilde{\mathcal{T}}_{\alpha\beta}(\vec{x}\,')\,V(\vec{x}-\vec{x}\,') \tag{2.52} $$

where $V(\vec{x}-\vec{x}\,')$ is the (static) potential

$$ V(\vec{x}-\vec{x}\,') = \int dt'\;\mathcal{D}_S(\vec{x}-\vec{x}\,',t',0) \tag{2.53} $$

Note that the potential actually depends only on the module $r = \|\vec{x}-\vec{x}\,'\|$, due to the rotational symmetry of the system. The potential for the DGP model can be found exactly, and reads [40]

$$ V(r) = -\frac{1}{2\pi^2 M_4^2}\,\frac{1}{r}\left[\sin\Big(\frac{r}{r_c}\Big)\mathrm{Ci}\Big(\frac{r}{r_c}\Big) + \frac{1}{2}\cos\Big(\frac{r}{r_c}\Big)\Big(\pi - 2\,\mathrm{Si}\Big(\frac{r}{r_c}\Big)\Big)\right] \tag{2.54} $$

where $\mathrm{Ci}(z) \equiv \gamma + \ln(z) + \int_0^z(\cos(t)-1)dt/t$ and $\mathrm{Si}(z) \equiv \int_0^z \sin(t)dt/t$ are respectively the Cosine integral function and the Sine integral function, $\gamma \simeq 0.577$ is the

Euler-Mascheroni constant and the distance scale $r_c$ is defined as follows

$$r_c \equiv \frac{M_4^2}{2M_5^3} \tag{2.55}$$

It can be seen that $r_c$ is a "crossover" scale where the behavior of the gravitational potential changes from $4D$ to $5D$. In fact, at short distances $r \ll r_c$ the potential behaves as

$$V(r) \simeq -\frac{1}{2\pi^2 M_4^2}\frac{1}{r}\left[\frac{\pi}{2} + \Big(-1+\gamma+\ln\Big(\frac{r}{r_c}\Big)\Big)\Big(\frac{r}{r_c}\Big) + \mathcal{O}(r^2)\right] \tag{2.56}$$

and at leading order it has the 4D Newtonian $1/r$ scaling, while at large distances $r \gg r_c$ we obtain

$$V(r) \simeq -\frac{1}{2\pi^2 M_4^2}\frac{1}{r}\left[\frac{r_c}{r} + \mathcal{O}\Big(\frac{1}{r^2}\Big)\right] \tag{2.57}$$

so at leading order it has now the 5D behavior $1/r^2$. This results suggests that we may hope to reproduce GR results using the DGP model as long as we set $r_c$ to be much bigger than the length scales we are interested in, and tune

$$\frac{1}{M_4^2} \sim G \tag{2.58}$$

Note that this implies the following hierarchy of scales

$$r_g \equiv \frac{M}{M_4^2} \lll r_c \tag{2.59}$$

#### Weak GR gravity *vs.* weak DGP gravity

The story is however more complicate than that. Let's consider for definiteness a static and spherically symmetric point source of mass $M$: $\tilde{\mathcal{T}}_{\alpha\beta}(\vec{x}\,') = M\,\delta_\alpha^{\,0}\delta_\beta^{\,0}\,\delta^{(3)}(\vec{x}\,')$ (which may model a star or a planet). In this case the metric on the brane reads

$$h_{\mu\nu}(\|\vec{x}\|, 0) = \mathcal{S}_{\mu\nu}{}^{00}\, M\, V(\|\vec{x}\|) \tag{2.60}$$

and one can easily see from (2.49) that the off-diagonal components of $\mathcal{S}_{\mu\nu}{}^{00}$ are zero while $\mathcal{S}_{00}{}^{00} = 2/3 = 2\,\mathcal{S}_{ii}{}^{00}$. Note furthermore that at first order we have for the induced metric

$$\tilde{h}_{\mu\nu}(t,\vec{x}) \simeq h_{\mu\nu}(t,\vec{x}) \tag{2.61}$$

Therefore, indicating $r = \|\vec{x}\|$ and writing the induced metric in terms of the gravitational potentials $\Psi$ and $\Phi$

$$\tilde{h}_{00}(r) = -2\,\Phi(r) \tag{2.62}$$

$$\tilde{h}_{0i}(r) = 0 \tag{2.63}$$

$$\tilde{h}_{ij}(r) = -2\,\Psi(r)\,\delta_{ij} \tag{2.64}$$

we have that for $r \ll r_c$

$$\Phi(r) = -\frac{1}{3}\,\frac{M}{2\pi M_4^2}\,\frac{1}{r} \qquad\qquad \Psi(r) = \frac{1}{2}\,\Phi(r) \tag{2.65}$$

The situation is quite different from GR, where one has [1]

$$\Phi(r) = -GM/r \qquad\qquad \Psi(r) = \Phi(r) \tag{2.66}$$

Despite the fact that (for $r \ll r_c$) the two potentials in the DGP model scale as $1/r$, it is apparent that in DGP we can never reproduce the complete GR line element. In fact, suitably tuning the value of $M_4^2$ we can reproduce one of the two potentials, but never both of them. The fact is that, experimentally, we can test both the potentials independently: non-relativistic test bodies (realistically a planet orbiting around a star) are in fact influenced only by $\Phi(r)$, while the propagation of light is influenced by both of the potentials. Therefore if we put right the orbits of planets then the light deflection comes out wrong, and conversely if we reproduce the correct light deflection then the orbit of planets does not agree with observations anymore: the relative error we get is as big as 25% (see *e.g.* [67]). It seems then that the weak field gravity in the DGP model is irreparably different from the weak field gravity in GR. This difference can be traced back to the fact that in GR the tensor structure is

$$S_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\Big(\delta_\mu{}^\alpha\delta_\nu{}^\beta + \delta_\nu{}^\alpha\delta_\mu{}^\beta\Big) - \frac{1}{2}\,\eta_{\mu\nu}\eta^{\alpha\beta} \tag{2.67}$$

and, as a consequence of the coefficient of the last term being $\frac{1}{2}$ instead of $\frac{1}{3}$, one has

$$S_{00}{}^{00} = S_{ii}{}^{00} \tag{2.68}$$

Regarding the bending mode, as we already saw in the linear approximation it obeys equation (2.40). Considering the same form for the source term $\tilde{\mathcal{T}}_{\alpha\beta}(\vec{x}\,') =$

$M\,\delta_\alpha^{\ 0}\delta_\beta^{\ 0}\,\delta^{(3)}(\vec{x}\,')$ we used to find the gravitational potentials, we find the following profile for the bending mode in presence of a static, spherically symmetric and point-like source

$$\pi(r) = -\frac{M}{6M_5^3}\,\frac{1}{4\pi r} \tag{2.69}$$

where (the notation is not a happy one in this case) the $\pi$ in the denominator of the right hand side is the number $3.1415926\ldots$, while the $\pi$ in the left hand side is the bending mode.

### 2.2.2 Nonlinearities and the Vainshtein mechanism

From what we said above, it may seems that solar system observations rule out the DGP model for every choice of parameters. However, this conclusion relies on the implicit assumption that, since the motion of planets and light in the solar system are described by weak field (*i.e.* linearized) GR, in the DGP model it should be described by the weak field approximation of DGP. In GR, the scale at which nonlinearities become important around a spherically symmetric source is $r_s = GM$: we are then implicitly assuming that the scale at which nonlinearities become important in DGP is the scale $r_g \equiv M/M_4^2 \sim r_s$ correspondent to the scale at which nonlinearities become important in GR, or at least much smaller than the length scales we can probe in earth-solar system measurements. This is however not obvious.

To verify this, we should evaluate all the nonlinear terms when the dynamical variables take on their weak field value, and recognize at which length scales such nonlinear terms become comparable to the linear ones. Naively, we may in fact expect the presence of a different scale where nonlinearities become important in the DGP model: following [44], we notice that the profile for the bending mode in the linear approximation (2.69) becomes very large even for $r \gg r_g$, since

$$\pi(r) = -r_c\,r_g\,\frac{1}{12\pi r} \tag{2.70}$$

This can be traced back to the fact that, at linear level, $h_{\mu\nu}$ receives contributions both from the extrinsic curvature term (multiplied by $M_5^3$) and from the induced gravity term (multiplied by $M_4^2$): as a result of the competition between these two terms, there is a crossover scale $r_c$ above which $h_{\mu\nu}$ couples to $\tilde{\mathcal{T}}_{\mu\nu}$ with effective strength $G_5 = 1/M_5^3$, while below $r_c$ it couples with effective strength $G_4 = 1/M_4^2$. At "small" scales the behavior of $h_{\mu\nu}$ is then dictated by $M_4^2\tilde{\mathbf{G}}$, which sets the scale $r_g = M/M_4^2$ where nonlinear terms in $h_{\mu\nu}$ become important. The bending mode $\pi$, instead, at linear order receives contributions only from the extrinsic curvature term, and therefore couples to $\tilde{\mathcal{T}}$ with effective strength $G_5 = 1/M_5^3$ at all scales: as a result, the solution (2.69) contains only $M/M_5^3 \sim r_c r_g$. However, at quadratic order we have

$$\tilde{h}_{\mu\nu} = h_{\mu\nu}\Big|_{y=0^+} + \partial_\mu\pi\partial_\nu\pi + \mathcal{O}(h\pi) \tag{2.71}$$

so the equation of motion for the bending mode acquires a contribution from the induced gravity term as well: the competition between the linear term controlled by $M_5^3$ and the quadratic one controlled by $M_4^2$ may introduce a new scale where nonlinearities become important.

### The Vainshtein radius

It is actually not difficult to see that, for a static, spherically symmetric point-like source of mass $M$, the term $\partial_\mu\pi\partial_\nu\pi$ (evaluated with the linear profile (2.69)) becomes of the same order of $h_{\mu\nu}(y = 0^+)$ at the *Vainshtein radius*

$$r_V = \sqrt[3]{\frac{M M_4^2}{M_5^6}} \sim \sqrt[3]{r_g\, r_c^2} \tag{2.72}$$

and therefore below this radius the linear approximation cannot be trusted. The hierarchy between $r_g$ and $r_c$ implies that $r_g \ll r_V \ll r_c$: we conclude that the linear approximation for the DGP model breaks down at distances which are much bigger than the distance where the linear approximation breaks down in GR. To be quantitative, using $H_0 \sim 70\,\mathrm{km/s/Mpc}$ and $r_c \sim c/H_0$ we get[2] $r_c \sim 4.3 \times 10^3\,\mathrm{Mpc}$ and for the sun[3] we get $r_g^{sun} \sim 1.5\,\mathrm{km}$ and finally $r_V^{sun} \sim 3\times10^{15}\,\mathrm{km} \sim 10^2\,\mathrm{pc}$. Note that the average distance between Pluto and the sun is $\sim 6 \times 10^9\,\mathrm{km} \sim 10^{-6}\,r_V$: in practice, the light deflection experiments and the orbits of planet and satellites take place in the range $r_g < r < r_V$, so the analysis of the previous section does not

[2] 1 MegaParsec (Mpc) is approximately $1\,\mathrm{Mpc} \simeq 3.09 \times 10^{19}\,\mathrm{km}$

[3] $M_{sun} \sim 2 \times 10^{30}$ kg

apply. Note that we have not shown that above $r_V$ the linear approximation holds: in the complete perturbative expansion there will be interaction terms containing all powers of $\pi$, $h$ and mixed terms $\pi^n h^m$, each of which, when evaluated on the linear solutions, may become important at a different scale. In principle some nonlinear terms may become of the same order of the linear ones at scales which are even higher then $r_V$.

However, it has been shown [41, 42, 43, 44, 45, 46] that the approximation where $h_{\mu\nu}$ is treated at first order while we keep nonlinear terms in $\pi$ is consistent, and $r_V$ is indeed the highest of the scales where nonlinearities become important. To find out what happens below $r_V$ (*i.e.* for radii smaller than $r_V$ but bigger than the scales where other nonlinear terms become important), we can consider the approximated equations of motion where we keep the linear terms in $h$ and the *quadratic* terms in $\pi$. This is equivalent to postulate the following ordering of amplitudes

$$h_{\mu\nu} \sim \epsilon^2 \qquad\qquad \pi \sim \epsilon \tag{2.73}$$

and truncate the equations at the $\epsilon^2$ level. This does not change the extrinsic curvature part since corrections start at $\epsilon^3$ level ($h\,\pi$ terms), and changes just the induced gravity term which becomes

$$\begin{aligned}\tilde{G}_{\mu\nu} = -\frac{1}{2}\,\Box_4 h_{\mu\nu}\Big|_{y=0^+} + \Box_4\pi\,\partial_\mu\partial_\nu\pi - \partial_\mu\partial^\lambda\pi\,\partial_\nu\partial_\lambda\pi - \\ -\frac{1}{2}\,\eta_{\mu\nu}\Big(\Box_4\pi\,\Box_4\pi - \partial_\alpha\partial^\lambda\pi\,\partial^\alpha\partial_\lambda\pi\Big) + \mathcal{O}(\epsilon^3)\end{aligned} \tag{2.74}$$

Note that, despite the fact that calculating the Einstein tensor from $\partial_\mu\pi\partial_\nu\pi$ one would expect terms with three derivatives, all these terms cancel leaving out an expression which is of second order in derivatives. This property is highly nontrivial and very restrictive, and defines a very interesting class of Lagrangians of which the Lagrangian for the bending mode in the DGP model is just a particular case, as we will see in section (3.5.1). Taking the trace of the junction conditions, we obtain the nonlinear equation for the bending mode

$$\Box_4\pi + \frac{r_c}{3}\Big((\Box_4\pi)^2 - \partial_\alpha\partial^\lambda\pi\,\partial^\alpha\partial_\lambda\pi\Big) = -\frac{1}{6M_5^3}\,\tilde{\mathcal{T}} \tag{2.75}$$

which for a static, spherically symmetric, point-like source of mass $M$ can be exactly integrated [68] to give

$$\frac{\pi'}{r} + \frac{2r_c}{3}\left(\frac{\pi'}{r}\right)^2 = \frac{M}{6M_5^3}\,\frac{1}{4\pi r^3} \tag{2.76}$$

where we indicate derivatives with respect to $r$ with a prime. Inserting the linear profile (2.69) in the previous equation one recognizes that the nonlinear term becomes comparable to the linear term at the radius $r = r_V$: the Vainshtein radius is therefore not only the radius where nonlinearities in $\pi$ become comparable to $h_{\mu\nu}$ in the induced metric, but also the radius where nonlinearities become important in the equation of motion for $\pi$ itself. Equation (2.76) is an algebraic equation in $\pi'/r$, in fact a quadratic equation at fixed $r$: we can then solve it exactly obtaining

$$\left[\frac{\pi'(r)}{r}\right]_{\pm} = -\frac{3}{4r_c}\left(1 \pm \sqrt{1 + \frac{2}{9\pi}\,\frac{r_c^2 r_g}{r^3}}\right) \tag{2.77}$$

There are two branches of solutions, characterised by the sign $+$ or $-$: the $-$ solutions is decaying at infinity, while the $+$ one is not (we have $\pi \propto r^2$ for very large radii). The solution we are interested in here is the decaying one, since it has to reduce to (2.69) when $r \gg r_V$: from the previous equation we can obtain the asymptotic behaviors (note that $r_c^2 r_g = r_V^3/4$)

$$\pi'(r) \quad \begin{cases} = \dfrac{1}{12\pi}\,\dfrac{r_c r_g}{r^2} & \text{for}\, r \gg r_V \\[2ex] = \sqrt{\dfrac{1}{8\pi}}\,\sqrt{\dfrac{r_g}{r}} & \text{for}\, r \ll r_V \end{cases} \tag{2.78}$$

**The Vainshtein mechanism**

We can pictorially sum up the situation in the following way. The presence of a static point source on the brane has (in our language/gauge choice) two separate effects: it creates a nontrivial profile for the embedding $\pi$ of the brane, and it creates a nontrivial metric $h_{\mu\nu}$ in the 5D spacetime. The latter effect can in turn be split in the presence of a significant leaking of the gravitational force into the bulk (encoded in $\partial_y h_{\mu\nu}$ in the junction conditions) and the presence of a significant gravitational force on the brane (encoded in $\Box_4 h_{\mu\nu}$ in the junction conditions). The situation we described so far is then the following: there are two relevant length scales, the crossover radius $r_c$ and the Vainshtein radius $r_V \ll r_c$. Above the crossover scale, the leaking of the gravitational force into the bulk is non-negligible (the extra dimension “opens up”) so gravity on the brane has a 5D behavior. Below $r_c$ the gravitational leaking is instead negligible, and gravity on the brane is essentially 4D. Above the Vainshtein radius, the bending does not contribute appreciably to the induced gravity term, but acts as a source for $h_{\mu\nu}$ in such a way that the tensor structure of gravity on the brane is different from the one characteristic of GR. When we approach $r_V$, instead, nonlinearities in $\pi$ start becoming important and it starts contributing significantly to the induced gravity term.

Nonlinearities in $\pi$ change the bending profile (as we saw) with respect to the linear case, and influence the induced metric since at order $\epsilon^2$ we have

$$\tilde{h}_{\mu\nu} = h_{\mu\nu}\Big|_{y=0^+} + \partial_\mu\pi\partial_\nu\pi \tag{2.79}$$

Furthermore, quadratic terms in $\pi$ are likely to modify the way the bending sources the metric $h_{\mu\nu}$, and so the behavior of the gravitational potentials may be significantly different from what we found in the context of the linear approximation. To study that, we focus on length scales around $r_V$, which in practice means (as we already said) length scales smaller than the crossover scale and larger than the scales where other nonlinear terms become important. This implies that we can work at order $\epsilon^2$ (in the sense of (2.73)), and at the same time safely neglect the $\partial_y h_{\mu\nu}$ term in the junction conditions. Therefore we have

$$M_4^2\tilde{G}_{\mu\nu} = \tilde{\mathcal{T}}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\tilde{\mathcal{T}} - 2M_5^3\partial_\mu\partial_\nu\pi - \frac{M_4^2}{3}\eta_{\mu\nu}\tilde{R} \tag{2.80}$$

where

$$\tilde{R}(r) = (\Box_4\pi)^2 - \partial_\alpha\partial^\lambda\pi\,\partial^\alpha\partial_\lambda\pi \tag{2.81}$$

Let’s consider as we did before a static, spherically symmetric, point-like source of mass $M$: the spherical symmetry allows us to write the induced metric in the same form (2.62)-(2.64) used at linear level, where now the gravitational potentials $\Phi$ and $\Psi$ contain a contribution from the bending mode as well as a contribution from $h_{\mu\nu}$, according to (2.79). Using the fact that $\tilde{R}_{00} = \triangle_3\Phi$ and $\tilde{G}_{00} = 2\triangle_3\Psi$, where $\triangle_3$ is the laplacian operator, we have

$$M_4^2 \triangle_3 \Phi = \tilde{\mathcal{T}}_{00} + \frac{1}{3}\,\tilde{\mathcal{T}} - \frac{1}{6}\,M_4^2\,\tilde{R} \tag{2.82}$$

$$M_4^2 \triangle_3 \Psi = \frac{1}{2}\Big(\tilde{\mathcal{T}}_{00} + \frac{1}{3}\,\tilde{\mathcal{T}}\Big) + \frac{1}{6}\,M_4^2\,\tilde{R} \tag{2.83}$$

where the induced curvature scalar takes the form

$$\tilde{R}(r) = \frac{2}{r^2}\frac{d}{dr}(r\pi'^2) \tag{2.84}$$

We then see that the two gravitational potentials couple differently with the energy-momentum tensor (which is the origin of the factor of two difference between the potentials in (2.65)), but at the same time the nonlinear contributions from the bending mode have opposite sign in the two cases. Integrating the equations above on a sphere of radius $r$ and centered on the point-like mass we get

$$\frac{\Phi'}{r} = \frac{2}{3}\,\frac{r_g}{4\pi r^3} - \frac{1}{3}\,\Big(\frac{\pi'}{r}\Big)^2 \tag{2.85}$$

$$\frac{\Psi'}{r} = \frac{1}{3}\,\frac{r_g}{4\pi r^3} + \frac{1}{3}\,\Big(\frac{\pi'}{r}\Big)^2 \tag{2.86}$$

and using the $r \gg r_V$ and $r \ll r_V$ behaviors (2.78) we arrive at

$$r \gg r_V \qquad \begin{cases} \dfrac{\Phi'}{r} = 2\,\dfrac{r_g}{12\pi r^3} \\[2ex] \dfrac{\Psi'}{r} = \dfrac{r_g}{12\pi r^3} \end{cases} \tag{2.87}$$

and

$$r \ll r_V \qquad \begin{cases} \dfrac{\Phi'}{r} = \dfrac{r_g}{8\pi r^3} \\[2ex] \dfrac{\Psi'}{r} = \dfrac{r_g}{8\pi r^3} \end{cases} \tag{2.88}$$

It is apparent that for $r \gg r_V$ the "linear" DGP behavior (2.65) is reproduced, with the factor two difference between the potentials, while well inside the Vainshtein radius the potentials are equal one to the other and therefore linear GR is reproduced. This is due to the fact that the quadratic contributions in $\pi'$ have opposite signs for the two potentials, and counterbalance the different way the two potentials couple with the energy-momentum tensor. We can conclude then that (quadratic) nonlinearities in the bending mode restore the agreement with GR on length scales where nonlinearities in GR are still negligible.

The fact that agreement with GR is restored *via* (derivative) self-coupling of a light degree of freedom is known as *Vainshtein mechanism*, and has been proposed for the first time by A. Vainshtein [69] in the context of massive gravity (which will be treated in the next two chapters).

### 2.2.3 Cosmology in the DGP model

Let's study now cosmological solutions in the DGP model. Following [47], we consider configurations where the 5D metric in the Gaussian normal coordinates reads

$$ds^2 = -N^2(\tau, y)d\tau^2 + A^2(\tau, y)\gamma_{ij}dx^i dx^j + B^2(\tau, y)dy^2 \tag{2.89}$$

where $\gamma_{ij}$ is a metric on a three dimensional space of constant curvature, and (as in section (1.1.1)) a parameter $k = +1, 0, -1$ identifies the three possible cases for the sign of the spatial curvature. The brane is located at $y = 0$, where $y$ is the extra dimension, and the induced metric reads

$$ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -n^2(\tau)d\tau^2 + a^2(\tau)\gamma_{ij}dx^i dx^j \tag{2.90}$$

where we denote the values of the bulk metric components on the brane with lower case letters

$$n(\tau) = N(\tau, 0) \qquad a(\tau) = A(\tau, 0) \qquad b(\tau) = B(\tau, 0) \tag{2.91}$$

We assume that the matter content of the brane have the usual cosmological form

$$\tilde{T}_\mu^{\ \nu}(\tau) = \mathrm{diag}\big(-\rho(\tau), p(\tau), p(\tau), p(\tau)\big) \tag{2.92}$$

Note that it is always possible to set $n(\tau) = 1$ using the gauge freedom and rescaling the time coordinate $\tau \to t$. Using this freedom the Hubble parameter on the brane takes the usual form

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{2.93}$$

We will make the further assumption that the bulk is flat, or equivalently that the bulk metric (2.89) can be transformed into the 5D Minkowski metric by a suitable change of coordinates.

### The modified Friedmann equations

These assumption imply that we can derive an evolution equation for $H(t)$ without having to solve the full equations and find the exact metric in the bulk. It can be shown that the Friedmann equations in this case take the form [47]

$$H^2 + \frac{k}{a^2} - \epsilon \frac{1}{r_c} \sqrt{H^2 + \frac{k}{a^2}} = \frac{1}{3M_4^2} \rho \tag{2.94}$$

where $r_c = M_P^2 / 2M_5^3$ is the DGP crossover scale, and also that the usual conservation equation holds for matter on the brane

$$\dot{\rho} + 3H(p + \rho) = 0 \tag{2.95}$$

Note that there are two branches of solutions, identified by the the value $\epsilon = \pm 1$ of the parameter $\epsilon$ in (2.94), which corresponds to the sign of the jump of $\partial_y A$ across the brane.

Inspecting the Friedmann equation, we can see that the usual 4D Friedmann equation is reproduced whenever the square root term in (2.94) is subdominant with respect to the other two terms. Explicitly this happens when

$$\sqrt{H^2 + \frac{k}{a^2}} \gg \frac{1}{r_c} \tag{2.96}$$

and, neglecting the curvature term, we find

$$H^{-1} \ll r_c \tag{2.97}$$

so the usual 4D cosmological evolution is reproduced when the Hubble radius is smaller than the crossover scale. Taking as initial condition at a certain $t = \bar{t}$ a configuration where the universe is expanding and satisfies (2.97), we want to study how the late time cosmology predicted by this model looks like. We assume that the 4D universe is filled with matter whose energy density is non-negative and goes to zero when $a \to +\infty$, or equivalently that the equation of state of matter is $p = w\rho$ with $w \geq -1$.

### Late time cosmology

It turns out that the late time cosmological evolution is quite different depending on which branch we consider. To see it more clearly, it is useful to recast the

Friedmann equation in the following form

$$\sqrt{H^2 + \frac{k}{a^2}} = \frac{1}{2r_c}\left(\epsilon + \sqrt{1 + \frac{4r_c}{3M_4^2}\,\rho}\right) \tag{2.98}$$

Let's start by considering the branch of solutions defined by $\epsilon = -1$. Considering just the cases $k = 0$ and $k = -1$, where the universe expands forever (*i.e.* $a(t) \to +\infty$ for $t \to +\infty$), we have that at late times the matter density goes to zero, so we can expand the square root in the right hand side of (2.98) to obtain

$$\sqrt{H^2 + \frac{k}{a^2}} = \frac{1}{6M_5^3}\,\rho \tag{2.99}$$

which is called the *5D regime*. In this branch, the universe continues expanding with $H \to 0$ for $t \to +\infty$, but at late times the expansion rate changes from $(H^2 + \frac{k}{a^2}) \propto \rho$ to $(H^2 + \frac{k}{a^2}) \propto \rho^2$. In practice, when the Hubble radius reaches the crossover scale $r_c$ the universe starts feeling the extra dimension, and there is a transition in the expansion rate. This branch is usually called the *conventional branch*.

Now consider the branch defined by $\epsilon = +1$. Also in this case we restrict the analysis to the cases $k = 0$ and $k = -1$, where $a(t) \to +\infty$ for $t \to +\infty$ and we have that at late times the matter density goes to zero. Differently from the conventional branch, in this case we have

$$\sqrt{H^2 + \frac{k}{a^2}} > H_{self} \equiv \frac{1}{r_c} \tag{2.100}$$

and we have that the Hubble parameter is bounded from below

$$H > H_{self} = \frac{1}{r_c} \tag{2.101}$$

This means that, for $t \to +\infty$, the energy density goes to zero and the scale factor goes to infinity, but the Hubble parameter asymptote the finite and nonzero value $H_{self}$. Therefore, when the Hubble radius reaches the crossover scale $r_c$ and the universe starts feeling the extra dimension, the universe enters an *accelerating* phase. This branch is usually called the *self-accelerating branch*.

It can be shown explicitly [47] that these solutions can be embedded in the 5D Minkowski spacetime, and therefore the treatment is self-consistent.

### Acceleration as self-acceleration and its problems

We have seen that in the DGP model there is a branch of cosmological solutions which displays a transition form the usual 4D cosmological evolution to an accelerated one. This happens without the need of introducing dark energy or a nonzero cosmological constant: it happens for geometric reasons. This results motivated the hope to explain the late time acceleration of the universe by geometrical means [48], where the transition to the accelerated phase is a consequence of the fact that the correct theory of gravity is not GR, and the difference starts to be felt when the Hubble radius reaches the critical scale $r_c$. Despite being a very appealing possibility, this does not solve the fine tuning problem which is present in the case of the cosmological constant, since to explain the cosmological observations we have to tune the 5D mass scale $M_5^3$ (and therefore $r_c$) such that the transition happens (in cosmological terms) very close to the matter-radiation equality.

Beside this unsatisfying aspect, there are much serious problems which cast doubts on the viability of this explanation of the late time acceleration. It has in fact been shown [52, 70, 71] that the predictions of the DGP cosmological models are in strong tension with the observational data, and so these models fits the data significantly worse than $\Lambda$CDM. Since DGP and $\Lambda$CDM have the same number of free parameters, and both of them need to be fine tuned, we can conclude that the observational data strongly disfavor DGP in comparison to $\Lambda$CDM as a description of the late time acceleration phenomenon. Furthermore, it has been shown [72, 73, 50, 45, 46, 51] that there is a ghost excitation in the self-accelerating branch: this implies that such a solution is unstable from a quantum point of view.

These issues are serious enough to force us to abandon the (original) DGP model as an explanation of the cosmic acceleration. However, there is still the possibility that some generalizations of the DGP model, involving higher codimensions for example, may be ghost-free and fit the data significantly better than the original version, thereby providing a geometrical explanation for the late time cosmic acceleration.

# Chapter 3

# dGRT massive gravity

We have seen in the previous chapters that a way to try to explain the apparent late time acceleration of the universe is to modify gravity in the infrared, *i.e.* at large distances. In particular, we have seen that the DGP model provides an interesting way to do that, and in that model gravitational potentials behave like $1/r$ below a crossover scale $r_c$ and like $1/r^2$ above it. However, in particle physics it is not unusual to have a theory which behaves like $1/r$ below a scale and decays much faster above it: Yukawa long ago proposed a model, which ought to describe the pion, in which a scalar field has exactly this property. This is linked with the idea that the mass of a particle fixes the range of the interaction it mediates: massive particles mediate finite range forces, while massless particles mediate infinite range forces. Considering a scalar field, the relativistic field equation for a massless field is the D'Alembert equation

$$\Box\phi = T \tag{3.1}$$

where $T$ is the source. Considering a static, spherically symmetric source, the solution outside the source is

$$\phi \propto \frac{1}{r} \tag{3.2}$$

However, giving a mass to the particle one obtains the equation of motion

$$(\Box - m^2)\phi = T \tag{3.3}$$

which is the Klein-Gordon equation, and admits a static, spherically symmetric vacuum solution

$$\phi \propto \frac{e^{-mr}}{r} \tag{3.4}$$

This is known as the *Yukawa potential*, and we can see that it behaves like $\sim 1/r$ for $r \ll r_c$ while it decays exponentially for $r \gg r_c$, where $r_c = 1/m$ is called the *Compton radius*. We then see that the interaction mediated by a massive scalar field has a finite length, set by the Compton radius or equivalently by the inverse mass.

It is quite natural to wonder if we could use this simple idea to modify gravity in the infrared, "giving a mass" to the graviton. This relies on the fact that GR can be considered as a theory of a massless field: we will see in fact that GR can be thought as an interacting theory of a massless helicity-2 field, which is consistent with the fact that gravitational interaction in GR have infinite range. More precisely we could try to formulate an interacting theory of a massive spin-2 field, and set its Compton radius of the order of the Hubble radius today $r_c \sim H_0^{-1}$. The hope is that we could construct in this way a theory which accurately reproduces GR below $r_c$, while behaves differently above that radius. Once done that, we could investigate if this modified gravity theory is able to explain the late time acceleration as an effect of the fact that gravity behaves differently when the Hubble's radius becomes comparable to the Compton radius.

The idea of formulating a theory of a massive spin-2 field which reduces to GR below the Compton radius is actually quite old, and can be traced back to the works of Fierz and Pauli (FP) in 1939 [74]. They formulated a theory of a free massive spin-2 field, whose action reduces to the one of linearized GR in the $m \to 0$ limit. However, the program we sketched above proved to be very difficult to implement. On one hand, it was argued that any nonlinear extension of the FP theory leads to the appearing of an additional "sixth" degree of freedom and the reintroduction of ghosts [75], and therefore the is no sensible way to formulate an interacting theory of a massive spin-2 graviton (apart from considering Lorentz violating theories [76]). On the other hand, it was shown that at linear level the FP theory does not reproduce GR, even below the Compton radius [77, 78, 79]. A possible way out of the latter problem has been suggested by Vainshtein [69], who proposed that nonlinearities could be crucial in restoring the agreement with GR, a mechanism which is known as *Vainshtein mechanism*. Recently, a class of nonlinear completions of the Fierz-Pauli theory which are Lorentz invariant and propagate exactly five degrees of freedom has been proposed [80, 81]. Even before considering cosmological solutions, it is crucial to establish if this class of theories reproduces GR in a suitable range of length scales, and therefore if the Vainshtein mechanism is effective or not.

The main aim of this thesis is to investigate the effectiveness of the Vainshtein mechanism in the class of theories known as dRGT Massive Gravity [80, 81]. In this chapter we therefore introduce the theory in its generality, while in the next chapter we focus on static, spherically symmetric solutions and on the Vainshtein mechanism. This chapter is largely based on the recent review [82].

## 3.1 GR as an interacting massless helicity-2 field

Let's consider the action of GR

$$S_{GR}[g_{\mu\nu}, \psi_{(i)}] = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, R + S_M[g_{\mu\nu}, \psi_{(i)}] \tag{3.5}$$

where the $\psi_{(i)}$ are matter fields while the matter action is

$$S_M = \int d^4x \sqrt{-g}\, \mathscr{L}_M \tag{3.6}$$

The energy momentum tensor is defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} S_M \tag{3.7}$$

so the equations of motion are the Einstein equations

$$G_{\mu\nu} = \frac{1}{M_P^2}\, T_{\mu\nu} \tag{3.8}$$

where $M_P^2 = 1/8\pi G$.

### 3.1.1 Linear GR as a free massless spin-2 field

Let's study perturbations around the Minkowski solution

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.9}$$

The linearized equations of motion can be deduced by expanding the equations of motions, or equivalently by varying the quadratic part of the action obtained by the expanding (3.5) in terms of $h_{\mu\nu}$, which reads

$$S^{(2)}_{GR} = \int d^4x \, \frac{M_P^2}{2}\left( -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \right) - h_{\mu\nu}T^{\mu\nu} \tag{3.10}$$

where indices has been raised using $\eta^{\mu\nu}$. To study the vacuum dynamics of perturbation from Minkowski spacetime, we can set to zero the energy momentum tensor in the action above: the vacuum equations of motion for $h_{\mu\nu}$ can be then deduced from the action

$$S^{(2)} = \int d^4x \left( -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \right) \tag{3.11}$$

We could pretend to forget for a moment where this action comes from, and just study its properties. In general, fields living in Minkowski spacetime can be categorized regarding their transformation properties with respect to Lorentz transformations: in particular, they can be decomposed in components of fixed mass and spin. It can be shown that action (3.11) describes exactly a massless helicity-2 field [82]. As a consistency check, we can show that a field whose dynamic is described by (3.11) propagates two degrees of freedom (d.o.f.): to do that, it is useful to use the Hamiltonian formalism.

#### Degrees of freedom count

Let's review some definitions of Lagrangian and Hamiltonian mechanics: for a dynamical system with $n$ degrees of freedom $\{q_i\}$ described by the Lagrangian $L(q, \dot{q}, t)$, the conjugate momenta are defined as

$$p_i(q, \dot{q}, t) \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{3.12}$$

If the matrix of second derivatives with respect to the coordinates is nonsingular

$$\det\left[\frac{\partial^2 L}{\partial q_i \partial q_j}\right] \neq 0 \tag{3.13}$$

then we may invert (3.12) to express $\dot{q}_i$ in terms of $q, p$ and $t$. In this case we can define the Hamiltonian of the system as the Legendre transform of the Lagrangian

$$H(q,p,t) = \sum_i p_i\, \dot{q}_i(q,p,t) - L\big(q, \dot{q}(q,p,t), t\big) \tag{3.14}$$

and the equations of motion take the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{3.15}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{3.16}$$

Given a function $f(q,p,t)$, its time evolution satisfies

$$\frac{df}{dt} = \{f, H\}_P + \frac{\partial f}{\partial t} \tag{3.17}$$

where the curly brackets are named Poisson brackets and are defined as

$$\{f, g\}_P = \sum_i \Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\Big) \tag{3.18}$$

We want to do a Hamiltonian analysis of the theory described be the action (3.11). In this case, the dynamical variables are the field components $h_{\mu\nu}$, but it can be seen that the condition (3.13) is not satisfied, since $\dot{h}_{00}$ and $\dot{h}_{0i}$ appear linearly. However, since total derivatives in the action do not change the physics of the system, it is possible to integrate by parts in the action: using this freedom, we end up with an action where $\dot{h}_{00}$ and $\dot{h}_{0i}$ do not appear, and instead $h_{00}$ and $h_{0i}$ appear linearly. We can do the Legendre transform of the new action with respect just to the spatial components: the conjugate momenta are then [82]

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - \partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \tag{3.19}$$

and we can invert this relation to get

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + \partial_{(i}h_{j)0} \tag{3.20}$$

Note that, since we are splitting space and time, it makes sense to perform purely spatial transformations and so the Kronecker delta $\delta_{ij}$ is indeed a tensor. Moreover, note that we are using the convention of implicit sum on repeated indices, but now the indices do not need to "up and down", so for example $\dot{h}_{kk}$ means $\sum_{k=1}^{3} \dot{h}_{kk}$. We can then write the Lagrangian as [82]

$$\mathcal{L}(h, \pi, h_{00}, h_{0i}) = \pi_{ij}\,\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\big(\partial_j \pi_{ij}\big) + h_{00}\big(\triangle h_{ii} - \partial_i \partial_j h_{ij}\big) \tag{3.21}$$

where $\mathcal{H}$ depends only on $h_{ij}$, $\pi_{ij}$ and their spatial derivatives. Note that $h_{00}$ and $h_{0i}$ indeed appear linearly, and they are multiplied by terms with no time derivatives: we can interpret $h_{00}$ and $h_{0i}$ as Lagrange multipliers which enforce the (primary) constraints

$$\partial_j \pi_{ij} = 0 \qquad\qquad \triangle h_{ii} - \partial_i \partial_j h_{ij} = 0 \tag{3.22}$$

and so consider the system described by (3.11) as a constrained Hamiltonian system. It can be checked that the matrix whose elements are the Poisson brackets of the constraints between themselves is vanishing when the fields satisfy the constraints, so each of the four constraints generate a gauge transformation, and that the Poisson bracket of the constraints with the Hamiltonian vanishes, so the constraints are conserved by the time evolution. To count the number of degrees of freedom, the $h_{ij}$ and $\pi_{ij}$ are $3 \times 3$ symmetric matrices, so have 6 independent components each. Of these 12 degrees of freedom, 4 can be eliminated using the constraints, and other 4 can be fixed using the gauge transformations. So in the end we are left with 4 phase space degrees of freedom, which correspond to 2 physical degrees of freedom.

**Massless helicity-2 and gauge invariance**

It is remarkable that, even if we didn't start from the complete GR action, we could have arrived at the action (3.11) following other paths. As we just said, the request that the action describes a massless, helicity-2 field singles out (apart from a multiplicative constant) the action above. Even if we just ask that the action describes a massless field which, upon decomposition in helicity-2, helicity-1 and helicity-0 components, contains a helicity-2 part, then the request of absence of ghost instabilities fixes the action to be (3.11) [83]. Therefore, if we started from a more field theoretical perspective, we would have singled out this action just asking that a massless helicity-2 field plays a role in the gravitational interaction. Note that there is yet another way of deriving this action, this time from the point of view of symmetries. The action (3.11) is invariant with respect to the (gauge) transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\xi(\eta)_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.23}$$

where $\xi_\mu(x)$ is an arbitrary 1-form field. From the perspective of GR, this is just a consequence of diffeomorphism invariance of the full theory, and the transformation above is the linearized form of an infinitesimal coordinate transformation. On the other hand, considering the most general quadratic, local and Lorentz invariant action for a symmetric field $h_{\mu\nu}$ on Minkowski spacetime, with no more than two derivatives, the request of invariance with respect to the transformation (3.23) fixes the action to be (3.11) [82, 84], again up to a multiplicative constant. Once again, we may have found the action above just asking reasonable physical properties plus gauge invariance, without knowing anything about GR. It is tempting to wonder if it is not just a chance that the action which describes linear GR has these properties, and if they may be considered instead the core of GR as a field theory of gravitation.

### 3.1.2 GR as an interacting massless spin-2 theory

It can in fact be seen that locality, Lorentz invariance, no higher derivatives and gauge invariance actually fix the theory also at nonlinear level. Let's start again from the complete action of GR (3.5): the theory is invariant with respect with general coordinate transformations, which for infinitesimal transformations read

$$X^\mu \to X^\mu - \xi^\mu(X) \tag{3.24}$$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{L}_\xi(h)_{\mu\nu} \tag{3.25}$$

Here, the full metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\xi^\mu$ is an infinitesimal vector field and indices are lowered/raised with the flat metric $\eta_{\mu\nu}/\eta^{\mu\nu}$. However, $h_{\mu\nu}$ is not necessarily small. Expanding around Minkowski space, we can write the full action in terms of powers of $h_{\mu\nu}$: the quadratic piece give the action (3.10), while higher powers of $h_{\mu\nu}$ can be interpreted as self-interaction pieces. The full action in vacuum schematically will be of the form

$$S = \int d^4x \left[\partial^2 h^2 + \partial^2 h^3 + \cdots + \partial^2 h^n + \cdots\right] + \tag{3.26}$$

where $\partial^2 h^n$ means that this piece contains two derivatives and $n$ factors coming from $h_{\mu\nu}$ (not that there is a second derivative of $h$ to the third power). The fact that this is an expansion of GR around Minkowski spacetime is encoded in the precise form of the terms which enter at every order, and in the values of the numerical coefficients which stand in front of each term.

### GR as a resummed theory

However, we may take the opposite perspective: we may start with the action (3.11) for a free massless helicity-2 graviton, and ask what higher power interaction terms can be added. The possible terms can be arranged in powers of the perturbations $h$ and their derivatives, so the general nonlinear extension of (3.11) will contain the type of terms present in (3.26) as well as many others. We may ask that the full action resulting from such an operation enjoys gauge invariance: the gauge transformations should reduce to (3.23) at linear order, but may have higher order corrections. It can be shown [82, 84] that these requirements are strong enough to force the interaction terms to be exactly the ones of full nonlinear GR. Therefore, we may equivalently see the full action of GR not as the starting point, but as the result of the summation of all the terms allowed by gauge invariance for an interacting theory of a massless helicity-2 field.

A note of caution is in order: this “bottom-up” construction which allows to see GR as an interacting theory of a massless helicity-2 field relies on the fact that we chose Minkowski space as the starting point. However, from this perspective a “miracle” happens when we add up all the interaction terms: despite the fact that we explicitly started from a definite background ($\eta_{\mu\nu}$), which is not dynamical (it is not determined by the theory itself), the field redefinition $h_{\mu\nu} \to g_{\mu\nu} - \eta_{\mu\nu}$ in the resummed theory completely eliminates the background metric from the action. Therefore, the fully interacting action turns out to be background independent, or in other words there is not a prior geometry in the theory.

### 3.1.3 Propagator and relevant scales

#### Propagator

Let's study the linear approximation of GR in presence of sources. As we already said, the theory is defined by the action (3.10) which gives the equations of motion

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = M_P^{-2} T_{\mu\nu} \tag{3.27}$$

where

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\Big[\, \delta^{\sigma}_{(\mu}\eta^{\lambda\rho}\partial_\lambda\partial_{\nu)} - \eta^{\rho\sigma}\partial_\mu\partial_\nu\, h - \delta_\mu^{\,\rho}\delta_\nu^{\,\sigma}\Box - \eta_{\mu\nu}\left(\eta^{\lambda\rho}\eta^{\alpha\sigma}\partial_\lambda\partial_\alpha - \eta^{\rho\sigma}\Box\right)\Big] \tag{3.28}$$

We would like to find the propagator of the (linear) theory, which roughly speaking is the solution of the equation above when the source is perfectly localized. However, the gauge invariance enjoyed by the theory implies that, for every configuration of the source term, there are an infinite number of solutions of the equation above and therefore the operator $\mathcal{E}_{\mu\nu}{}^{\rho\sigma}$ is not invertible. To find the propagator, we have to fix the gauge and render the differential operator invertible: once found the propagator in a particular gauge, the solution of (3.27) will be given by the sum of the gauge fixed solution and a pure gauge contribution. We choose to impose the harmonic gauge condition

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0 \tag{3.29}$$

and using this condition the equation of motion (3.27) can be simplified to give

$$\mathcal{O}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = M_P^{-2} T_{\mu\nu} \tag{3.30}$$

where

$$\mathcal{O}_{\mu\nu}{}^{\rho\sigma} = -\frac{1}{2}\Big[\delta_\mu^{\,\rho}\delta_\nu^{\,\sigma}\Box - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\Box\Big] \tag{3.31}$$

The propagator $D_{\mu\nu}{}^{\alpha\beta}(x;x')$ is then defined as the solution to the equation

$$\mathcal{O}_{\mu\nu}{}^{\rho\sigma} D_{\rho\sigma}{}^{\alpha\beta}(x;x') = \frac{1}{2}\Big(\delta_\mu^{\,\alpha}\delta_\nu^{\,\beta} + \delta_\nu^{\,\alpha}\delta_\mu^{\,\beta}\Big)\,\delta^{(4)}(x-x') \tag{3.32}$$

and as in the previous chapter we can factorize a scalar part $D_S(x;x')$ and a purely numerical part which carries the tensor structure $S_{\mu\nu}{}^{\alpha\beta}$ (note that the propagator depends only on the difference $(x-x')$ because of translational symmetry). One has then

$$S_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\left(\delta_\mu{}^\alpha \delta_\nu{}^\beta + \delta_\nu{}^\alpha \delta_\mu{}^\beta\right) - \frac{1}{2}\,\eta_{\mu\nu}\eta^{\alpha\beta} \tag{3.33}$$

$$-\frac{1}{2}\Box\, D_S(x-x') = \delta^{(4)}(x-x') \tag{3.34}$$

which confirms the formula (2.67) of the previous chapter.

### Static spherically symmetric solutions and nonlinearity scales

Considering now a static, spherically symmetric source point source of mass $M$: $T_{\alpha\beta}(\vec{x}\,') = M\,\delta_\alpha^{\,0}\delta_\beta^{\,0}\,\delta^{(3)}(\vec{x}\,')$, we get analogously to section (2.2.1)

$$h_{\mu\nu}(r) = S_{\mu\nu}{}^{00}\,\frac{M}{M_P^2}\,V_{GR}(r) \tag{3.35}$$

and so we have that $h_{\mu\nu}$ is diagonal and

$$h_{00}(r) = \frac{M}{M_P^2}\,\frac{1}{4\pi r} = \frac{2GM}{r} \tag{3.36}$$

$$h_{ii}(r) = \frac{M}{M_P^2}\,\frac{1}{4\pi r} = \frac{2GM}{r} \tag{3.37}$$

Remembering the definition of gravitational potentials

$$h_{00}(r) = -2\,\Phi(r) \tag{3.38}$$

$$h_{ii}(r) = -2\,\Psi(r)\,\delta_{ij} \tag{3.39}$$

we have that in GR

$$\Phi(r) = -\frac{M}{M_P^2}\,\frac{1}{8\pi r} = -\frac{GM}{r} \tag{3.40}$$

$$\Psi(r) = -\frac{M}{M_P^2}\,\frac{1}{8\pi r} = -\frac{GM}{r} \tag{3.41}$$

which gives (2.66). Note that this solution indeed satisfies the harmonic gauge condition (3.29).

To find the scale where nonlinearities become important in GR, we should insert the linear solution (3.40)-(3.41) in the full action (3.26), and see at what radius(es) the nonlinear terms become comparable with the linear ones. Due to the dependence $\propto 1/r$ of the components of $h_{\mu\nu}$, any term $\partial^2 h^n$ will be, apart from numerical factors, $\partial^2 h^n \sim h^n/r^2$ and so become comparable to $\partial^2 h^2 \sim h^2/r^2$ at $r \sim M/M_P^2$. We see that all the nonlinear terms become comparable to the linear ones at the same scale

$$r_g \sim GM \sim \frac{M}{M_P^2} \tag{3.42}$$

which is therefore the only scale where nonlinearities become important in presence of a spherical body of mass $M$.

## 3.2 The Fierz-Pauli theory

### 3.2.1 The Fierz-Pauli action

Having seen that GR can be considered in some sense as an interacting theory of a massless helicity-2 field on Minkowski spacetime, the first step in building a nonlinear theory of massive gravity is to find the action which describes the dynamics of a free *massive* spin-2 field on Minkowski spacetime. In the perturbative approach to construct the full theory, once found this free action we should add interaction terms which extend the theory at full nonlinear level. The problem of finding the action which describes a free massive spin-2 field on Minkowski spacetime has been solved already in 1939 by Fierz and Pauli [74] who proposed the following action for a symmetric tensor $h_{\mu\nu}$

$$S^{(2)}_{FP} = \int d^4x \Big[ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \\ + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{m^2}{2}\big(h_{\mu\nu}h^{\mu\nu} - h^2\big)\Big] \tag{3.43}$$

which is therefore called the *Fierz-Pauli action.* Analogously to the quadratic action for GR, there are several ways to look at it. We may notice in fact that it is a linear combination of all the possible contractions of two powers of $h_{\mu\nu}$ with up to two derivatives, which are the terms appearing in (3.11) plus two non-derivative terms. The coefficients of this linear combination are such that the derivative part exactly reproduces the quadratic GR action (3.11), while the relative coefficient between the two non-derivative terms is fixed to be $-1$: this is known as the *Fierz-Pauli tuning*. However, the most distinctive property of this action is seen from the point of view of the representations of the Lorentz group: this is exactly the action which describes the dynamics of one free massive spin-2 field. Any change in this action would either introduce other degrees of freedom along with the massive spin-2 field, or disrupt the fact that there is a massive spin-2 in the theory. The overall coefficient of the non-derivative terms plays the role of mass of the field, and the part $m^2(h_{\mu\nu}h^{\mu\nu} - h^2)/2$ is then called the mass term. As we did for the quadratic GR action, we can count the degrees of freedom as consistency check of the fact that the Fierz-Pauli action propagates the 5 degrees of freedom of a massive spin-2 field.

### Degrees of freedom count

Analogously to the case of linear GR, the fields $\dot{h}_{00}$ and $\dot{h}_{0i}$ appear linearly in the action, and the condition (3.13) is not satisfied. Also in this case we integrate by parts to have an action where $\dot{h}_{00}$ and $\dot{h}_{0i}$ do not appear at all. However, in this case the $h_{0i}$ do not appear linearly, since the mass term produces quadratic terms in $h_{0i}$, while $h_{00}$ still appears linearly, despite the mass term. We can do the Legendre transform of the (integrated by parts) action with respect just to the spatial components, and the conjugate momenta have the same form as in the $m = 0$ case [82]

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - \partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \tag{3.44}$$

and inverting this relation we get as in the $m = 0$ case

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + \partial_{(i}h_{j)0} \tag{3.45}$$

The contributions from the mass term show up in the Lagrangian, which can be written as [82]

$$\mathcal{L}(h, \pi, h_{00}, h_{0i}) = \pi_{ij}\,\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\big(\partial_j \pi_{ij}\big) + m^2 h_{0i}^2 + h_{00}\big(\triangle h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\big) \tag{3.46}$$

where again $\mathcal{H}$ depends only on $h_{ij}$, $\pi_{ij}$ and their spatial derivatives, and $h_{0i}^2$ is a shorthand for $\sum_i h_{0i}^2$. It is apparent that, as we said, $h_{00}$ still appears linearly, and still multiply a term with no time derivatives, but now the fields $h_{0i}$ appear quadratically. They can be interpreted as auxiliary variables: in this case they don't enforce any constraint, and their equations of motion give

$$h_{0i} = -\frac{1}{m^2}\,\partial_j \pi_{ij} \tag{3.47}$$

which can be plugged back into the action (3.46) to give [82]

$$S = \int d^4x \left[\pi_{ij}\dot{h}_{ij} - \mathscr{H} + h_{00}\Big(\triangle h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\Big)\right] \tag{3.48}$$

where

$$\mathscr{H} = \mathcal{H} + \frac{1}{m^2}\big(\partial_j \pi_{ij}\big)^2 \tag{3.49}$$

The field $h_{00}$ instead enforces the (primary) constraint $\mathcal{C}_1 = \triangle h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii} = 0$. However, this constraint is not automatically preserved by the time evolution of the system, since its Poisson bracket with the Hamiltonian $\mathcal{C}_2 \equiv \{\mathcal{C}_1, \mathscr{H}\}_P$ is neither zero nor proportional to $\mathcal{C}_1$. Therefore, we have to impose also the (secondary) constraint $\mathcal{C}_2 = 0$. The Poisson bracket of $\mathcal{C}_2$ with $H$ is instead linearly dependent with $\mathcal{C}_1$ and $\mathcal{C}_2$, so we don't need to impose any more constraints: in total the number of constraints we need to impose is therefore two. Since the Poisson bracket of the two constraints does not vanish, they don't generate any gauge symmetry. The degrees of freedom are then the $6 + 6 = 12$ of $h_{ij}$ and $\pi_{ij}$ minus one for each constraint: we have in total $12 - 2 = 10$ phase space degrees of freedom which correspond to 5 physical degrees of freedom.

### Massive spin-2 and absence of gauge invariance

Using the Hamiltonian formalism, it is actually quite easy to see why the Fierz-Pauli tuning is necessary: a generic mass term $a\,h_{\mu\nu}h^{\mu\nu} + b\,h^2$ contains $h_{00}^2$ in the form $(a+b)h_{00}^2$, so only if $a = -b$ we have that $h_{00}$ appears linearly. Explicitly

$$ah_{\mu\nu}h^{\mu\nu} + bh^2 = (a+b)h_{00}^2 - 2ah_{0i}^2 - 2bh_{00}h_{ii} + ah_{ij}h_{ij} + bh_{ii}^2 \tag{3.50}$$

We see that if $a = 0$ then $h_{0i}$ appear linearly in the action (due the derivative part), so there are at least 3 constraints and it is impossible to have 10 phase space degrees of freedom. Therefore, if $a = 0$ the action can never propagate the 5 physical degrees of freedom of a massive spin-2 graviton. However, if $a \neq 0$ the $h_{0i}$ become auxiliary variables: if $b \neq -a$ then $h_{00}^2$ appears in the action, and so it is a auxiliary variable as well meaning that there are no constraints at all. Therefore, in the latter case the number of physical degrees of freedom is 6. Only if $a \neq 0$ and $a = -b$ there can be 5 degrees of freedom, which describe the massive spin-2 field.

Note that this action is *not* invariant with respect to the gauge transformation (3.23): the gauge symmetry is broken by the mass term. Therefore, we cannot construct a nonlinear extension by enforcing a nonlinear version of the gauge symmetry, as can be done to construct (nonlinear) GR from the linear approximation. However, it can be shown that every modification at linearized level of (3.43) which still propagates a massive spin-2 field, have ghost instabilities [83]: necessarily the additional (sixth) degree of freedom turned on by the modification is a ghost. The Fierz-Pauli action is therefore the only quadratic action for a symmetric tensor on Minkowski spacetime which contains a massive spin-2 field and is ghost-free. This is a property we may hope to use as a criterion to build a nonlinear extension of the Fierz-Pauli action.

### 3.2.2 The VDVZ discontinuity and Vainshtein mechanism

We would like now to derive the weak field solution correspondent to a static, point-like mass in the Fierz-Pauli theory. The full Fierz-Pauli action including the source is

$$S_{FP}^{(2)} = \int d^4x \, \frac{M_P^2}{2} \Big[ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \\ + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{m^2}{2}\big(h_{\mu\nu}h^{\mu\nu} - h^2\big)\Big] - h_{\mu\nu}T^{\mu\nu} \quad (3.51)$$

and performing the variation with respect to $h^{\mu\nu}$ we obtain the equation of motion

$$-\frac{1}{2}\Big(\Box h_{\mu\nu} - \partial_\lambda \partial_{(\mu} h^\lambda{}_{\nu)} + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \\ - \eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h)\Big) = M_P^{-2}T_{\mu\nu} \quad (3.52)$$

We consider conserved sources, for which $\partial_\mu T^{\mu\nu} = 0$. Acting on the equations of motion (3.52) with $\partial^\mu$, we find

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0 \tag{3.53}$$

and, plugging this back into (3.52) and taking the trace, we find

$$-\frac{3}{2}\, m^2 h = \frac{1}{M_P^2}\, T \tag{3.54}$$

Using the last two relations, we can show that the equations of motion (3.52) are equivalent to the following system of differential equations

$$-\frac{1}{2}\left(\Box - m^2\right) h_{\mu\nu} = \frac{1}{M_P^2}\left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right) T\right] \tag{3.55}$$

$$\partial^\mu h_{\mu\nu} = -\frac{2}{3}\frac{1}{M_P^2 m^2}\, \partial_\nu T \tag{3.56}$$

$$h = -\frac{2}{3}\frac{1}{M_P^2 m^2}\, T \tag{3.57}$$

The general solution to (3.52) can be expressed in general as the sum of the general solution of the homogeneous equation plus a particular solution. The former is therefore the general solution of the system

$$\left(\Box - m^2\right) h_{\mu\nu} = 0 \tag{3.58}$$

$$\partial^\mu h_{\mu\nu} = 0 \tag{3.59}$$

$$h = 0 \tag{3.60}$$

and so is a transverse-traceless field. For the particular solution of the sourced equation, we impose boundary conditions which imply that the operator $\left(\Box - m^2\right)$ is invertible, and so the second and third equations (3.56) and (3.57) are implied by the first one (3.55). Therefore, in order to find a particular solution of the sourced field equations (3.52), it is sufficient to find a solution of

$$-\frac{1}{2}\left(\Box - m^2\right) h_{\mu\nu} = \frac{1}{M_P^2}\left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right) T\right] \tag{3.61}$$

To solve this equation, it is useful to go to momentum space. We express $h_{\mu\nu}(x)$ and $T_{\mu\nu}(x)$ *via* their Fourier transforms

$$h_{\mu\nu}(x) = \int d^4p \; e^{ip_\alpha x^\alpha} \, h_{\mu\nu}(p) \tag{3.62}$$

$$T_{\mu\nu}(x) = \int d^4p \; e^{ip_\alpha x^\alpha} \, T_{\mu\nu}(p) \tag{3.63}$$

and so we obtain

$$h_{\mu\nu}(p) = \frac{2}{M_P^2} \frac{1}{p_\alpha p^\alpha + m^2} \left[ \frac{1}{2} \, \delta_{(\mu}^{\;\;\rho} \, \delta_{\nu)}^{\;\;\sigma} - \frac{1}{3} \left( \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \eta^{\rho\sigma} \right] T_{\rho\sigma}(p) \tag{3.64}$$

Note that a static source $T_{\mu\nu}(x) = T_{\mu\nu}(\vec{x})$ has a Fourier transform of the form

$$T_{\mu\nu}(p) = \delta(p^0) \, T^{(3)}_{\mu\nu}(\vec{p}) \tag{3.65}$$

and in particular, for a point-like source of mass $M$ we have

$$T_{\mu\nu}(\vec{x}) = M \, \delta_\mu^{\;0} \, \delta_\nu^{\;0} \, \delta^{(3)}(\vec{x}) \qquad \longrightarrow \qquad T_{\mu\nu}(p) = \frac{\delta(p^0)}{(2\pi)^3} \, M \, \delta_\mu^{\;0} \, \delta_\nu^{\;0} \tag{3.66}$$

Indicating $r \equiv \sqrt{\vec{x}^{\,2}}$ and using the formulas

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \frac{1}{\vec{p}^{\,2} + m^2} = \frac{1}{4\pi} \, \frac{e^{-mr}}{r} \tag{3.67}$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \frac{p_i p_j}{\vec{p}^{\,2} + m^2} = -\partial_i \partial_j \int \frac{d^3\vec{p}}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \frac{1}{\vec{p}^{\,2} + m^2} \tag{3.68}$$

we have [82]

$$h_{00}(x) = \frac{4}{3} \, \frac{M}{M_P^2} \, \frac{e^{-mr}}{4\pi r} \tag{3.69}$$

$$h_{0i}(x) = 0 \tag{3.70}$$

$$h_{ij}(x) = \frac{2}{3} \, \frac{M}{M_P^2} \, \frac{e^{-mr}}{4\pi r} \left[ \frac{1 + mr + m^2 r^2}{m^2 r^2} \delta_{ij} - \frac{1}{m^2 r^4} (3 + 3mr + m^2 r^2) x_i x_j \right] \tag{3.71}$$

where $x_i = \delta_{ik} x^k$.

### The VDVZ discontinuity

Note that, neglecting the term $\partial_\mu \partial_\nu T$ in (3.61) and therefore the term $p_\mu p_\nu T(p)$ in (3.64), we would obtain the following solution

$$h_{00}(x) = \frac{4}{3} \frac{M}{M_P^2} \frac{e^{-mr}}{4\pi r} \tag{3.72}$$

$$h_{0i}(x) = 0 \tag{3.73}$$

$$h_{ij}(x) = \frac{2}{3} \frac{M}{M_P^2} \frac{e^{-mr}}{4\pi r} \tag{3.74}$$

The term $p_\mu p_\nu T(p)$ produces a contribution to the metric field which has no observable consequences on a test body whose energy-momentum tensor obeys the conservation equation: in fact, the interaction amplitude $\int d^4x\, h_{\mu\nu} T_{tb}^{\mu\nu}$ between such a contribution to the metric and the conserved energy-momentum tensor of a test body vanishes. Therefore, regarding measurements like light deflection, planets orbits and so on, the metric (3.69)-(3.71) give the same predictions as the metric (3.72)-(3.74). Let's consider then the metric (3.72)-(3.74): the gravitational potentials reads

$$\Phi(r) = -\frac{2}{3} \frac{M}{M_P^2} \frac{e^{-mr}}{4\pi r} \qquad\qquad \Psi(r) = \frac{1}{2}\, \Phi(r) \tag{3.75}$$

For distances larger than the Compton length $r_c \equiv 1/m$, the potentials decay exponentially, with the tipical (Yukawa) behavior of massive fields $e^{-mr}/r$. On the other hand, for distances smaller than the Compton wavelength $r \ll r_c$, both of the gravitational potentials have the $1/r$ dependence of GR, but their ratio $\Phi(r)/\Psi(r)$ is twice the GR value. The situation is completely equivalent to the weak field solution of the DGP model inside the crossover scale: this mismatch is responsible for a 25% relative error in light deflection or planet orbits predictions compared to the GR ones. Note that this conclusion is not affected by taking $m$ as small as we like, since this will only make the Compton radius bigger and bigger without altering what happens well inside the Compton radius itself. However, if we set $m$ to be exactly zero, then the theory is exactly GR and trivially the predictions agree with the GR ones: therefore, there seems to be a discontinuity in the physical predictions of the theory when $m \to 0$. This has been noted and pointed out independently by Iwasaki [77], van Dam and Veltman [78] and Zakharov [79], and is known as the *VDVZ discontinuity.* This is *a priori* unexpected, since there seems to be no discontinuity in the $m \to 0$ limit in the action (3.51), and it usually assumed that if a theory is continuous in a parameter, then its physical predictions should be continuous in that parameter as well. However, the key point here is that taking the limit $m \to 0$ in the action is not the correct way to perform the $m \to 0$ limit in the theory: for example, the action (3.51) for every $m \neq 0$ propagates 5 degrees of freedom, as we saw, while the $m = 0$ action propagates only two degrees of freedom. Also, the $m = 0$ theory enjoys a gauge invariance which does not hold as soon as $m$ becomes different from zero. Therefore, the number of degrees of freedom and the symmetry properties of the action (3.51) are not continuous in the $m \to 0$ limit. We may conclude that the $m \to 0$ limit of the Fierz-Pauli theory is *not* described by the $m \to 0$ limit of the Fierz-Pauli action, and in particular the $m \to 0$ limit of the Fierz-Pauli theory is not GR. To elucidate this, it is useful to construct a different action which enjoys gauge invariance even in the $m \neq 0$ case and gives the same physical predictions of the FP one: this is achieved using the Stückelberg language, as we shall see in the next subsection.

**The Vainshtein mechanism**

The conclusion that the $m \to 0$ limit of the Fierz-Pauli theory is not GR seems to put an end to our hope to use a very small mass for the graviton as a way to explain the cosmological observations which indicate a late time acceleration: it seems that massive gravity is not a modified gravity theory in the sense of section (1.3.2). However, from the modified gravity perspective the FP theory is just the starting point: since the FP theory is linear (at the level of the field equations), it can never reproduce the strong field behavior of GR. The hope was that the FP theory reproduces the weak field limit of GR for distances smaller than the Compton length, and that a suitable nonlinear completion of the FP theory is able to reproduce also the strong field behavior of GR in the same range of length scales. Instead, we found that the FP theory does not reproduce GR either inside or outside the Compton radius. However, it has been proposed by Vainshtein [69] that interaction terms added to the FP action may be effective to restore agreement with GR also at length scales where the weak field approximation in GR is valid. This idea relies on the fact that nonlinear terms in the nonlinear extension of the FP theory may become relevant at a scale which is much larger than the scale $r_g = GM \sim M/M_P^2$ where nonlinear terms become relevant in GR, somewhat similarly to what happens in the DGP model.

Vainshtein considered a specific nonlinear extension of the Fierz-Pauli theory, namely the one obtained adding the mass term $m^2(h_{\mu\nu}h^{\mu\nu} - h^2)/2$ to the *full* nonlinear GR action expressed in terms of $\eta_{\mu\nu}$ and $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Considering a static and spherically symmetric source, he used the following *ansatz* for the metric

$$ds^2 = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2 \tag{3.76}$$

which at linear order (*i.e.* keeping only the quadratic terms in the action) have the vacuum solutions [82]

$$B_1(r) = -\frac{8GM}{3}\frac{e^{-mr}}{r} \tag{3.77}$$

$$C_1(r) = -\frac{8GM}{3}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2} \tag{3.78}$$

$$A_1(r) = \frac{4GM}{3}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2} \tag{3.79}$$

which are equivalent to (3.69)-(3.71). We can ask how these solutions are modified if we keep also the nonlinear terms in the equations of motion (or equivalently the interaction terms in the action). We can write

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots \\ A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots \end{aligned} \tag{3.80}$$

where $A_0 = B_0 = C_0 = 1$ and $\epsilon$ is a parameter that keeps track of which order in nonlinearities we are working at. Solving recursively the vacuum equations at each order in $\epsilon$ shows [69, 82] that the expansion in powers of nonlinearities shows up in the solutions for $A$, $B$ and $C$ as an expansion in the parameter $r_V/r$, where

$$r_V \equiv \sqrt[5]{\frac{GM}{m^4}} = \sqrt[5]{r_g r_c^4} \tag{3.81}$$

is called the *Vainshtein radius*. It follows that nonlinearities become important (*i.e.* comparable to the linear terms) when $r \approx r_V$, so the Vainshtein radius is the scale around a mass $M$ below which the linear approximation cannot be trusted. Note that, since we assume that $r_c/r_g \gg 1$, it follows that the Vainshtein radius is much bigger than the Schwarzschild radius $r_V/r_g \gg 1$ and so the scale where nonlinearities become important around a spherical object for the Fierz-Pauli theory is indeed much bigger than the scale where this happens in GR. In fact, setting $m = H_0$, for an object like the sun the Vainshtein radius (3.81) is $r_V \sim 10^5\,\mathrm{pc}$, which is bigger than the diameter of the Milky Way[1]: therefore the linear solution cannot be used to calculate the light bending and the planets' orbits in the solar system. Note also that the definition (3.81) for the Vainshtein radius in this nonlinear extension of Fierz-Pauli is different from the definition (2.72) of the Vainshtein radius in the DGP model: this is not strange, since the Vainshtein radius of a theory depends on the structure of the interaction terms, and theories which have different nonlinear structures are likely to have different Vainshtein radiuses.

To understand if this nonlinear extension of the Fierz-Pauli theory reproduces or not the GR predictions inside the Vainshtein radius, we should then solve the full equations (with all the nonlinear terms). Note that we have to solve necessarily for three unknown functions, we cannot reduce to just two unknown functions as we do in GR. In fact, reparametrising the radial coordinate according to

$$r \to \rho(r) = r\sqrt{A(r)} \tag{3.82}$$

we can eliminate the function $A$ from the metric and write the line element in terms of just two functions

$$ds^2 = -\tilde{B}(\rho)dt^2 + \tilde{C}(\rho)d\rho^2 + \rho^2 d\Omega^2 \tag{3.83}$$

In GR, performing this change of variables in the equations of motion results in the function $A$ disappearing also from them, as a consequence of the fact that the theory is invariant with respect of reparametrisations, and so indeed we can reduce the problem to solving for just two functions. However, the Fierz-Pauli theory is

[1]The diameter of the Milky Way is approximately $3 \times 10^4\,\mathrm{pc}$.

*not* invariant with respect to reparametrisations, and as a consequence the function $A \to \tilde{A}(\rho)$, despite disappearing from the metric, remains present in the equations of motion along with $\tilde{B}$ and $\tilde{C}$ when we reparametrise the radial coordinate. Of course, nothing prevents us from performing the change of coordinate and work with the unknown functions $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ instead of $A$, $B$ and $C$. In fact, Vainshtein suggests that this is a convenient thing to do to study the $m \to 0$ limit, since, regarding the functions $\tilde{B}$ and $\tilde{C}$, he suggests that the effects of nonlinearities inside the Vainshtein radius is just to rescale the numerical factors so that $\tilde{B}/\tilde{C} = 1$, while preserving the $\propto 1/r$ dependence. Instead, the effect of nonlinearities changes $\tilde{A}$ quite dramatically. He then concludes [69] that for $m \ll 1$ the functions $\tilde{B}$ and $\tilde{C}$ coincide to a very good approximation with their GR ($m = 0$) values inside the Vainshtein radius, and have a smooth $m \to 0$ limit. In other words, the nonlinear terms in the equation of motion modify all the three functions $A$, $B$ and $C$, but in such a way that, redefining the radial coordinate to get rid of $A$ in the metric, the nonlinear solutions for $\tilde{B}$ and $\tilde{C}$ inside the Vainshtein radius agree with the GR solutions, and so the nonlinear interaction terms restore the agreement with GR. Further studies on the recovery of GR results in the same nonlinear extension of the Fierz-Pauli theory considered by Vainshtein can be found in [85, 86, 87, 88, 89]. The mechanism of restoring agreement with GR *via* nonlinear interactions is named after Vainshtein and is known as the *Vainshtein mechanism*.

Note finally that, as the mass $m$ approaches 0, the Vainshtein radius grows and tends to infinity: in the limit $m \to 0$ the predictions of GR are recovered everywhere, and so the $m \to 0$ limit is indeed smooth for the theory. Therefore, while the linear Fierz-Pauli theory does not reduce to (linear) GR in the $m \to 0$ limit, it is possible that a nonlinear extension of the Fierz-Pauli theory does reduce to (nonlinear) GR in the same limit, and therefore that there is no VDVZ discontinuity at nonlinear level.

### 3.2.3 The Fierz-Pauli theory in the Stückelberg language

We have mentioned that the weak field predictions of the FP theory are significantly different from the ones of linearized GR, no matter how small is the mass of the FP graviton. Therefore, since the $m \to 0$ limit of the FP action is smooth and gives the GR action, the physical predictions of the FP theory seem not to be continuous in the $m \to 0$ limit. This is very surprising, since it usually assumed that if a theory is continuous in a parameter, then its physical predictions should be continuous in that parameter as well. However, as we already mentioned, a deeper look at the structure of the FP theory and of GR casts doubts on the fact that $m \to 0$ limit of the FP theory is given by the $m \to 0$ limit of the Fierz-Pauli action: in fact, the FP theory propagates five degrees of freedom, while GR propagates just two degrees of freedom; conversely, GR enjoys gauge invariance, which is instead broken in the Fierz-Pauli theory. Therefore, regarding the symmetry properties and the number of degrees of freedom, the $m \to 0$ limit of the Fierz-Pauli action is not continuous. It is tempting to conjecture that the VDVZ discontinuity and the discontinuity in symmetry properties and degrees of freedom are linked, and that the $m \to 0$ limit of the Fierz-Pauli theory is not described by the $m \to 0$ limit of the Fierz-Pauli action.

To understand the relation between the $m \to 0$ limit of the Fierz-Pauli theory and GR, thereby possibly sheding light on the origin of the VDVZ discontinuity, we would like to formulate a new theory which gives the same physical predictions of the Fierz-Pauli theory, but whose action in the $m \to 0$ limit still has the same symmetry properties and number of degrees of freedom of the $m \neq 0$ action. This task is achieved using the Stückelberg formalism.

### The Stückelberg formalism

Starting from the Fierz-Pauli action (3.51), we want to formulate a *different* theory which is invariant under gauge transformations, yet gives the same physical predictions of the FP action. This is achieved introducing auxiliary fields, called *Stückelberg fields*, whose transformation properties are defined exactly to render the action invariant. Let's in fact perform in the action (3.51) the substitution

$$h_{\mu\nu}(x) \to H_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_{(\mu} Z_{\nu)} \quad : \tag{3.84}$$

if we impose that the field $Z_\mu$ *shifts* under gauge transformations

$$x'^\mu = x^\mu - \xi^\mu(x) \tag{3.85}$$

$$Z'_\mu = Z_\mu - \xi^\mu \tag{3.86}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}\,\xi_{\nu)} \tag{3.87}$$

we have that the resulting action is invariant. Note that, given any field configuration ($h_{\mu\nu}$, $Z_\mu$) of the new theory, we can always perform a gauge transformation with parameter $\xi_\mu = Z_\mu$ in the new action and reobtain the original FP action. Therefore, despite the fact that the new action contains more fields that the original one, the physical prediction of the original action and of the "covariantized" one are precisely the same. On the other hand, the field $Z_\mu$ does *not* transform as a 1-form, but have an unusual transformation property.

Performing the substitution (3.84) inside the FP action, the kinetic part of the action does not change since the substitution (3.84) has the same form of a gauge transformation, and that part of the action is invariant (it is the action for linearized GR in fact). The only thing that changes is the mass term, and modulo total derivatives we get

$$\begin{aligned} S = \int d^4x\, M_P^2 \bigg[ & h^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}\big(h_{\mu\nu}h^{\mu\nu} - h^2\big) - \\ & - \frac{m^2}{2}F_{\mu\nu}F^{\mu\nu} - 2m^2\big(h_{\mu\nu}\partial^\mu Z^\nu - h\partial_\mu Z^\mu\big)\bigg] - h_{\mu\nu}T^{\mu\nu} \end{aligned} \tag{3.88}$$

where $F_{\mu\nu} = \partial_{[\mu}Z_{\nu]}$ and we raise/lower indices with the Minkowski metric $\eta^{\mu\nu}/\eta_{\mu\nu}$. We can redefine the field $Z_\mu \to \frac{1}{m}Z_\mu$ to render canonical its kinetic term: if we take the $m \to 0$ limit, we obtain an action for a massless graviton and a massless vector, which in total have four degrees of freedom. So at this point, we still lose one degree of freedom in the $m \to 0$ limit.

We can remedy to this problem by introducing an additional substructure in $Z_\mu$ by singling out explicitly a derivative part: we then write

$$Z_\mu = A_\mu + \partial_\mu\phi \tag{3.89}$$

and in terms of $A_\mu$ and $\phi$ the tensor $H_{\mu\nu}$ reads

$$H_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}A_{\nu)} + 2\,\partial_\mu\partial_\nu\phi \tag{3.90}$$

Note that the decomposition (3.89) is invariant with respect to the additional internal symmetry

$$\phi(x) \to \phi(x) - \Lambda(x) \tag{3.91}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{3.92}$$

and so there are now two gauge transformation under which the action is invariant

$$x'^\mu = x^\mu - \xi^\mu(x) \tag{3.93}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}\,\xi_{\nu)} \tag{3.94}$$

$$A'_\mu = A_\mu - \xi_\mu + \partial_\mu \Lambda \tag{3.95}$$

$$\phi' = \phi - \Lambda \tag{3.96}$$

In terms of the fields $A_\mu$ and $\phi$, the action (3.88) takes the form

$$\begin{aligned} S = \int d^4x\, M_P^2 \bigg[ & h^{\mu\nu} \mathcal{E}_{\mu\nu}^{\ \ \rho\sigma} h_{\rho\sigma} - \frac{m^2}{2}\big(h_{\mu\nu}h^{\mu\nu} - h^2\big) - \frac{m^2}{2} F_{\mu\nu}F^{\mu\nu} - \\ & - 2m^2\big(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu\big) - 2m^2\big(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial_\mu\phi\partial^\mu\phi\big)\bigg] - h_{\mu\nu}T^{\mu\nu} \end{aligned} \tag{3.97}$$

where now $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and again we have discarded total derivatives. Note that the quadratic piece in $\partial\partial f$ and the mixed term $\partial A \partial\partial f$ does not appear precisely for this reason: these terms rearrange in total derivatives, and therefore have no effect on the dynamic. As we will mention later, this is a consequence of the Fierz-Pauli tuning, since any other choice for the mass term of $h_{\mu\nu}$ in the starting action produces a quadratic piece in $\partial\partial f$ and a mixed piece $\partial A \partial\partial f$ which does not arrange themselves into total derivatives.

### The VDVZ discontinuity in the Stückelberg language

Note that, in the action (3.97), the field $\phi$ does not have a kinetic term on its own, but is kinetically mixed with $h_{\mu\nu}$. To be able to see more clearly the physical meaning of this action, it is useful to perform a field redefinition which demix kinetically the fields $h_{\mu\nu}$ and $\phi$, and at the same time creates a proper kinetic term for the latter field. The redefinition

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - m^2 \phi \, \eta_{\mu\nu} \tag{3.98}$$

$$\bar{A}_\mu = A_\mu \tag{3.99}$$

$$\bar{\phi} = \phi \tag{3.100}$$

has precisely this effect, and creates a coupling between $\bar{\phi}$ and the trace of the energy-momentum tensor as well. It is convenient to further redefine the fields to render the kinetic terms canonical

$$\hat{h}_{\mu\nu} = M_P \bar{h}_{\mu\nu} \tag{3.101}$$

$$\hat{A}_\mu = M_P m \bar{A}_\mu \tag{3.102}$$

$$\hat{\phi} = M_P m^2 \bar{\phi} \tag{3.103}$$

and in terms of the "hatted" fields the action (3.97) reads

$$S = \int d^4x \left[ \hat{h}^{\mu\nu} \mathcal{E}_{\mu\nu}{}^{\rho\sigma} \hat{h}_{\rho\sigma} - \frac{1}{2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - 3 \, \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{M_P} \hat{h}_{\mu\nu} T^{\mu\nu} - \frac{1}{M_P} \hat{\phi} T + \ldots \right] \tag{3.104}$$

where the dots stand for terms which are multiplied by $m$ or $m^2$. The $m \to 0$ limit of this action describes a theory of a massless graviton, a massless vector and a massless scalar, and so propagates five degrees of freedom exactly as the $m \neq 0$ theory.

Note that, in the $m \to 0$ limit, the action for the field $\hat{h}_{\mu\nu}$ is exactly the GR action (apart the $1/M_P$ rescaling); furthermore, the coupling of the field $\hat{\phi}$ with the trace of the energy-momentum tensor remains finite in the limit. Going back to the field $h_{\mu\nu}$ (whose dynamic is described by the action (3.97)) we can express it in terms of $\hat{h}_{\mu\nu}$ and $\hat{\phi}$ as

$$h_{\mu\nu} = \frac{\hat{h}_{\mu\nu}}{M_P} + \frac{\hat{\phi}}{M_P} \eta_{\mu\nu} \quad : \tag{3.105}$$

in the $m \to 0$ limit, it receives contributions both from a tensor field which satisfies the GR equations and a scalar field which couples with $T$ with finite strength. Since by construction the action (3.97) gives the same physical prediction of the Fierz-Pauli theory, we can conclude that indeed the $m \to 0$ limit of the FP theory is not equivalent to GR, but rather to a scalar-tensor theory.

Note finally that it is possible to impose gauge conditions which eliminate all the terms in the action (3.104) which are linear in $m$ [82]. This gauge transformation completely diagonalizes the action, and in the resulting action all the fields have a canonical kinetic term and a mass term, while only $\hat{h}_{\mu\nu}$ and $\hat{\phi}$ couple to the energy-momentum tensor. Therefore, if we consider a static and spherically symmetric source of mass $M$, the profile (in this gauge) for the fields $\hat{h}_{\mu\nu}$ and $\hat{\phi}$ inside the Compton wavelength $r_c = 1/m$ reads

$$\hat{h}_{\mu\nu} \sim \frac{M}{M_P}\frac{1}{r} \qquad\qquad \hat{\phi} \sim \frac{M}{M_P}\frac{1}{r} \tag{3.106}$$

apart from numerical factors.

## 3.3 Nonlinear extensions of the Fierz-Pauli theory

Having discussed the linear theory of a massive graviton, we would like to formulate now a nonlinear theory of massive gravity which reproduces the predictions of GR in a suitable range of length scales. To be more precise, we are looking for a theory which can be seen as an interacting theory of a massive graviton: therefore we ask that it reduces to the Fierz-Pauli theory in the weak field approximation, and that it propagates the same number of degrees of freedom (five) as the Fierz-Pauli theory. In the linear case, we can formulate the theory of a massive graviton by starting from the action of a massless graviton (linearized GR), and adding a suitable term (the mass term) which is weighted by a parameter which sets the range of the interaction, and does not contain derivatives of the field: we want to do the same also at nonlinear level. Therefore, we consider the full (nonlinear) GR Lagrangian and add a "mass" term, which in general we take to be nonlinear as well: this is to be a term which is weighted by a mass parameter, and contains no derivatives of the metric.

### 3.3.1 Generic nonlinear extension

In full generality, considering a Lorentz-invariant theory, such a mass term cannot be built from one metric tensor alone [75]: in fact, the identity $g_{\mu\lambda}g^{\lambda\nu} = \delta_{\mu}^{\ \nu}$ implies that it is impossible to construct a nontrivial scalar function out of $g_{\mu\nu}$ and $g^{\mu\nu}$

without using derivatives. Therefore, the theory will contain (at least) two metric tensors: there will be a *physical* metric $\mathbf{g}$, which is the metric test bodies feel and which determine in general the causal structure of the spacetime, and an *absolute* background metric $\mathbf{g}^{(0)}$, which is necessary to create nontrivial traces and contractions. To respect the equivalence principle, we postulate that matter fields couple only to the physical metric. Therefore the action will have the following structure

$$S = \int d^4x \, \sqrt{-g} \left[ \frac{M_P^2}{2} \Big( R[\mathbf{g}] - \frac{m^2}{2} \mathcal{U}[\mathbf{g}, \mathbf{g}^{(0)}] \Big) + \mathscr{L}_M[\mathbf{g}, \psi_{(i)}] \right] \tag{3.107}$$

where the $\psi_{(i)}$ are matter fields. Note that the mass term can equivalently be written as a function of the absolute metric $\mathbf{g}^{(0)}$ and of the physical metric $\mathbf{g}$, or as a function of the absolute metric $\mathbf{g}^{(0)}$ and of the difference between the two metrics $\mathbf{h} \equiv \mathbf{g} - \mathbf{g}^{(0)}$, or as a function of the physical metric $\mathbf{g}$ and of the difference $\mathbf{h}$. Despite the fact that we may use any absolute metric $\mathbf{g}^{(0)}$, a natural choice is to use the Minkowski metric as the absolute metric, and so in the following we assume $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. Assuming that the function $\mathcal{U}$ is analytic, we can therefore write the mass term as an (*a priori*) infinite sum of terms where each term contains a fixed number of powers of $h_{\mu\nu}$, and therefore we can write

$$\sqrt{-g}\,\mathcal{U}[\mathbf{g}, \mathbf{g}^{(0)}] = \sqrt{-det(\eta)} \sum_{k=2}^{+\infty} V_k[\eta, \mathbf{h}] \tag{3.108}$$

where each term $V_k[\eta, \mathbf{h}]$ is a linear combination of all the possible contractions of $k$ factors $h_{\mu\nu}$ with $k$ factors $\eta^{\alpha\beta}$

$$V_k[\eta, \mathbf{h}] = \sum_{p \in P_k} c_p^{(k)} \, \eta^{\mu_1 p(\nu_1)} \, \cdots \, \eta^{\mu_k p(\nu_k)} \, h_{\mu_1\nu_1} \cdots h_{\mu_k\nu_k} \tag{3.109}$$

where $P_k$ is the group of permutations of $k$ elements, and the sum runs on all the permutations $p$ belonging to $P_k$. Introducing the notation

$$\big[ h^n \big] \equiv \eta^{\mu\alpha_1} \, h_{\alpha_1\beta_1} \, \eta^{\beta_1\alpha_2} \, h_{\alpha_2\beta_2} \cdots \eta^{\beta_{n-1}\alpha_n} \, h_{\alpha_n\mu} \tag{3.110}$$

for the cyclic contraction of $n$ tensors $h_{\mu\nu}$, we can write the terms in the following more compact way

$$V_2[\eta, \mathbf{h}] = B_1 \big[h^2\big] + B_2 \big[h\big]^2 \tag{3.111}$$

$$V_3[\eta, \mathbf{h}] = C_1 \big[h^3\big] + C_2 \big[h^2\big] \big[h\big] + C_3 \big[h\big]^3 \tag{3.112}$$

$$V_4[\eta, \mathbf{h}] = D_1 \big[h^4\big] + D_2 \big[h^3\big] \big[h\big] + D_3 \big[h^2\big]^2 + D_4 \big[h^2\big] \big[h\big]^2 + D_5 \big[h\big]^4 \tag{3.113}$$

$$\begin{aligned} V_5[\eta, \mathbf{h}] = {} & F_1 \big[h^5\big] + F_2 \big[h^4\big] \big[h\big] + F_3 \big[h^3\big] \big[h\big]^2 + F_4 \big[h^3\big] \big[h^2\big] + F_5 \big[h^2\big]^2 \big[h\big] + \\ & + F_6 \big[h^2\big] \big[h\big]^3 + F_7 \big[h\big]^5 \end{aligned} \tag{3.114}$$

$$\vdots$$

and the requirement that the weak field limit should reproduce the Fierz-Pauli action implies that $B_2 = -B_1$. Inserting this expression in (3.107) and expanding also $\sqrt{-g}\, R[\eta, \mathbf{h}]$ in powers of $h_{\mu\nu}$, we can see that the resulting action is the one we would obtain in a perturbative approach adding interaction terms to the Fierz-Pauli action (3.43), with the condition that the derivative interaction terms are exactly the same as in (interacting) GR.

### Degrees of freedom and the Boulware-Deser ghost

The values of the numerical coefficients $C_i$, $D_i$, $F_i$, $\ldots$ (or at least consistency conditions on their values) are to be found imposing the condition that the theory be a viable theory of an interacting massive spin-2 field. This condition translates in several requirements, both of theoretical and phenomenological nature: from the theoretical point of view, we ask that the theory does not have ghost instabilities and that it propagates exactly 5 degrees of freedom, which match the degrees of freedom of the Fierz-Pauli theory. From the phenomenological point of view, we ask that GR predictions are reproduced in the range of length scales where GR is well tested. Note that, since the FP theory is not gauge invariant, we cannot use the requirement of gauge invariance as a guide to build the nonlinear theory: unlike in GR, whose nonlinear structure is completely fixed by this requirement, we have to implement directly the conditions relating to the absence of ghosts and the number of degrees of freedom. These are in fact quite strong requirements, and it has been actually claimed that any nonlinear extension of the Fierz-Pauli theory necessarily propagates six degrees of freedom and the Hamiltonian is not bounded from below [75], meaning that the "sixth" degree of freedom is a ghost (usually called the *Boulware-Deser ghost*). Although this conclusion is premature, it has been shown explicitly that any nonlinear completion of FP where the (nonlinear) mass term is of the form

$$\mathcal{U}[\mathbf{g}, \mathbf{g}^{(0)}] = \mathcal{U}\big(\big(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\big)h_{\mu\nu}h_{\alpha\beta}\big) \tag{3.115}$$

with[2] $\mathcal{U}'(0) = 1$, propagates six degrees of freedom and has an Hamiltonian which is unbounded from below. The nonlinear completion originally considered by Vainshtein in [69] (see section (3.2.2)) falls in this category, and is therefore plagued by ghost instabilities.

We could try to tackle the problem in full generality using the Hamiltonian formalism, and try to find consistency relations between the numerical coefficients $C_i$, $D_i$, $F_i$, ... above imposing that the theory does not propagate a sixth degree of freedom. However, this approach turns out to be very difficult to implement. Another approach is to first use appropriate limits and approximations of the theory to try to guess what a reasonable nonlinear extension could be, and restrict the domain of possible values for the coefficients $C_i$, $D_i$, $F_i$, ...: only in a second moment would we use the Hamiltonian formalism, with the hope that the analysis of the selected class of actions turns out to be less cumbersome than the general analysis. We follow the latter approach: the tools we use to simplify the analysis of the nonlinear massive actions are provided by the Stückelberg language in its full nonlinear form, and the use of a "decoupling" limit which select relevant subsets of nonlinear operators and focus on specific aspects/scales of the nonlinear dynamics. To apply the Stückelberg formalism to interacting massive gravity, it will be more useful to write the action (3.107) in terms of the physical metric $\mathbf{g}$ and the difference between the physical and absolute metric $\mathbf{h} \equiv \mathbf{g} - \mathbf{g}^{(0)}$. In complete analogy with what has been done above, we can write

$$\sqrt{-g}\,\mathcal{U}[\mathbf{g}, \mathbf{g}^{(0)}] = \sqrt{-g}\sum_{k=2}^{+\infty} U_k[\mathbf{g}, \mathbf{h}] \tag{3.116}$$

[2] This condition enforces the fact that the weak field limit is the Fierz-Pauli theory.

where each term $U_k[\mathbf{g}, \mathbf{h}]$ has exactly the same structure of (3.109) with the only difference that each index raised factor $\eta^{\alpha\beta}$ is now substituted with $g^{\alpha\beta}$. Also, introducing the notation

$$\langle h^n \rangle \equiv g^{\mu\alpha_1}\, h_{\alpha_1\beta_1}\, g^{\beta_1\alpha_2}\, h_{\alpha_2\beta_2} \cdots g^{\beta_{n-1}\alpha_n}\, h_{\alpha_n\mu} \tag{3.117}$$

we can write the terms $U_k$ in the more compact way

$$U_2[\mathbf{g}, \mathbf{h}] = b_1\langle h^2\rangle + b_2\langle h\rangle^2 \tag{3.118}$$

$$U_3[\mathbf{g}, \mathbf{h}] = c_1\langle h^3\rangle + c_2\langle h^2\rangle\langle h\rangle + c_3\langle h\rangle^3 \tag{3.119}$$

$$U_4[\mathbf{g}, \mathbf{h}] = d_1\langle h^4\rangle + d_2\langle h^3\rangle\langle h\rangle + d_3\langle h^2\rangle^2 + d_4\langle h^2\rangle\langle h\rangle^2 + d_5\langle h\rangle^4 \tag{3.120}$$

$$\begin{aligned} U_5[\mathbf{g}, \mathbf{h}] = {} & f_1\langle h^5\rangle + f_2\langle h^4\rangle\langle h\rangle + f_3\langle h^3\rangle\langle h\rangle^2 + f_4\langle h^3\rangle\langle h^2\rangle + f_5\langle h^2\rangle^2\langle h\rangle + \\ & + f_6\langle h^2\rangle\langle h\rangle^3 + f_7\langle h\rangle^5 \end{aligned} \tag{3.121}$$

$$\vdots$$

where again the requirement that the weak field limit should reproduce the Fierz-Pauli action implies that $b_2 = -b_1$. These two formulations (*i.e.* in terms of $\eta$ and $\mathbf{h}$ or $\mathbf{g}$ and $\mathbf{h}$) are completely equivalent, and the upper case numerical coefficients $C_i$, $D_i$, $F_i$, ... are biunivocally related to the lower case numerical coefficients $c_i$, $d_i$, $f_i$, ...: it is possible to see it explicitly expressing the inverse and the determinant of the full metric in terms of the inverse and determinant of the absolute metric

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}\Big(h_{\alpha\beta} - \eta^{\lambda\rho}h_{\alpha\lambda}h_{\rho\beta} + \eta^{\lambda\rho}\eta^{\sigma\tau}h_{\alpha\lambda}h_{\rho\sigma}h_{\tau\beta} + \cdots\Big) \tag{3.122}$$

$$\sqrt{-g} = 1 + \frac{1}{2}\,\eta^{\mu\nu}h_{\mu\nu} - \frac{1}{4}\Big(\eta^{\mu\nu}\eta^{\alpha\beta} - \frac{1}{2}\,\eta^{\mu\alpha}\eta^{\nu\beta}\Big)h_{\mu\alpha}h_{\nu\beta} + \cdots \tag{3.123}$$

and substituting in (3.116)-(3.121) and finally comparing with (3.108)-(3.114).

### 3.3.2 The nonlinear Stückelberg formalism

We have seen in section (3.2.3) that the introduction of auxiliary fields which restore gauge invariance is a powerful tools in studying the Fierz-Pauli theory, since it elucidates the origin of the VDVZ discontinuity and allows to perform the $m \to 0$ limit of the theory without losing degrees of freedom. We would like to apply the same formalism to the full nonlinear massive gravity, as first proposed by [90]. As we already mentioned, the theory contains two metrics, the physical metric $\mathbf{g}$ which transforms *covariantly* with respect to general coordinate transformations

$$x'^{\mu} = (f^{-1})^{\mu}(x) \tag{3.124}$$

$$g'_{\mu\nu}(x') = \frac{\partial f^{\alpha}(x')}{\partial x'^{\mu}} \frac{\partial f^{\beta}(x')}{\partial x'^{\nu}} g_{\mu\nu}(f(x')) \tag{3.125}$$

and the absolute metric $\mathbf{g}^{(0)}$ (which we choose to be the Minkowski metric) which transform *invariantly*

$$g^{(0)\,\prime}_{\mu\nu} = g^{(0)}_{\mu\nu} = \eta_{\mu\nu} \tag{3.126}$$

To construct a new action which is physically equivalent to (3.107) and enjoys invariance with respect to general coordinate transformations, we first promote the absolute metric to a covariant tensor

$$\eta_{\mu\nu} \to \Sigma_{\mu\nu}(x) \equiv \eta_{\alpha\beta} \frac{\partial \phi^{\alpha}(x)}{\partial x^{\mu}} \frac{\partial \phi^{\beta}(x)}{\partial x^{\nu}} \tag{3.127}$$

using four *scalar* fields $\phi^{\alpha}(x)$ which are called the *Stückelberg fields*. It can be checked that the chain rule for the derivative of composite functions gives the correct tensorial transformation law for $\Sigma_{\mu\nu}(x)$. We then define the covariantisation of the difference between the physical and the absolute metric $h_{\mu\nu}(x) = g_{\mu\nu}(x) - \eta_{\mu\nu}$ as

$$H_{\mu\nu}(x) \equiv g_{\mu\nu}(x) - \Sigma_{\mu\nu}(x) \tag{3.128}$$

Now, remembering the expression

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} \Big( R[\mathbf{g}] - \frac{m^2}{2} \mathcal{U}[\mathbf{g}, h] \Big) + \mathscr{L}_M[\mathbf{g}, \psi_{(i)}] \right] \tag{3.129}$$

where $\mathcal{U}[\mathbf{g}, h] = \sum_{k=2}^{+\infty} U_k[\mathbf{g}, \mathbf{h}]$ has the structure (3.118)-(3.121), we can construct a theory which is diffeomorphism invariant by replacing

$$h_{\mu\nu}(x) \to H_{\mu\nu}(x) \tag{3.130}$$

By construction, for every configuration of the Stückelberg fields $\phi^{\alpha}$ we can perform a suitable coordinate change such that the covariantized absolute metric $\Sigma_{\mu\nu}(x)$ becomes the Minkowski metric: in this reference system, the covariantized theory and the original theory are equal, and so the two descriptions are physically equivalent.

**Perturbative expansion**

In order to perform a perturbative analysis, it is useful to define a new object $Z^\alpha$ which can be considered the perturbation in the Stückelberg fields

$$\phi^\alpha = x^\alpha - Z^\alpha \tag{3.131}$$

and so we can express $H_{\mu\nu}$ in terms of $h_{\mu\nu}$ and $Z^\mu$ (we raise/lower indices with the Minkowski metric, so $Z_\nu = \eta_{\nu\alpha} Z^\alpha$)

$$H_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu} Z_{\nu)} - \eta_{\alpha\beta} \partial_\mu Z^\alpha \partial_\nu Z^\beta \tag{3.132}$$

Note that $Z^\mu$ does *not* transform as a vector with respect to general coordinate transformation: under infinitesimal coordinate transformations with gauge parameter $\xi^\alpha$ we have

$$x'^\mu = x^\mu - \xi^\mu(x) \tag{3.133}$$

$$Z'^\mu = Z^\mu - \xi^\mu + \xi^\lambda \, \partial_\lambda Z^\mu \tag{3.134}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu} \, \xi_{\nu)} + \mathcal{L}_\xi(h)_{\mu\nu} \tag{3.135}$$

and we can see that at linear order $Z^\mu$ simply shifts. As we did in the linear case, it is useful to introduce an additional substructure in $Z^\mu$ singling out explicitly a derivative part and writing

$$Z_\mu = A_\mu + \partial_\mu \phi \tag{3.136}$$

and in terms of $A^\alpha$ and $\phi$ the tensor $H_{\mu\nu}$ reads

$$\begin{aligned} H_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu} A_{\nu)} + 2\, \partial_\mu \partial_\nu \phi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \\ - \partial_{(\mu} A^\alpha \partial_{\nu)} \partial_\alpha \phi - \partial_\mu \partial^\alpha \phi \, \partial_\nu \partial_\alpha \phi \end{aligned} \tag{3.137}$$

Note that the decomposition (3.136) is invariant with respect to the internal symmetry

$$\phi(x) \to \phi(x) - \Lambda(x) \tag{3.138}$$

$$A_\alpha(x) \to A_\alpha(x) + \partial_\alpha \Lambda(x) \tag{3.139}$$

and so the fields transform under the joint action of the two symmetries in the following way

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x) \tag{3.140}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}\,\xi_{\nu)} + \mathcal{L}_{\xi}(h)_{\mu\nu} \tag{3.141}$$

$$A'_{\mu} = A_{\mu} - \xi_{\mu} + \xi^{\lambda}\,\partial_{\lambda}A_{\mu} + \partial_{\mu}\Lambda \tag{3.142}$$

$$\phi' = \phi + \xi^{\lambda}\,\partial_{\lambda}\phi - \Lambda \tag{3.143}$$

At linear order, the relations (3.132)-(3.143) reduce to the analogous relations introduced in section (3.2.3) to study the Fierz-Pauli theory with the Stückelberg language. Note finally that $A_{\mu}$ and $\phi$ does *not* transform respectively as a vector and as a scalar with respect to general coordinate transformation, as a consequence of the fact that $Z^{\mu}$ does not transform as a vector. We will use in the following the notation

$$\Pi_{\mu\nu} \equiv \partial_{\mu}\partial_{\nu}\phi \tag{3.144}$$

## 3.4 Stückelberg analysis of nonlinear massive gravity

We want now to study the theory defined by the action (3.107)

$$S = \int d^4x\,\sqrt{-g}\,\left[\,\frac{M_P^2}{2}\Big(R[\mathbf{g}] - \frac{m^2}{2}\mathcal{U}[\mathbf{g},\mathbf{g}^{(0)}]\Big) + \mathscr{L}_M[\mathbf{g},\psi_{(i)}]\,\right] \tag{3.145}$$

from a perturbative point of view, similarly to what we did in section (3.1.2) when we interpreted the full theory of GR as a resummation of an infinite expansion in powers of perturbations of the metric around Minkowski spacetime. Expanding the action (3.145) around the vacuum solution $g_{\mu\nu} = g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, we would indeed obtain an interacting theory of the field $h_{\mu\nu}$. However, since we want to work with a gauge invariant formulation, we first introduce the Stückelberg fields by expressing the potential part $\sqrt{-g}\,\mathcal{U}[\mathbf{g},\mathbf{g}^{(0)}]$ as in (3.116) and performing the replacement (3.130). Expanding also the inverse physical metric $g^{\mu\nu}$ in terms of $h^{\mu\nu}$, we then obtain an interacting action expressed in terms of the fields $h_{\mu\nu}$, $A_{\mu}$, $\phi$, where the interaction terms are expressed as linear combinations of powers of $h_{\mu\nu}$, $A_\mu$, $\phi$ and their derivatives. In the following, we raise/lower indices on perturbation fields with the Minkowski metric.

### 3.4.1 Interaction terms

Note first of all that the introduction of the Stückelberg fields have no effect on the "Einstein-Hilbert" part of the action, since it has the same form of a gauge transformation and the Einstein-Hilbert term is gauge invariant. Therefore, the nonlinear terms coming from this piece of the action do not contain the Stückelberg fields $A$ and $\phi$ and are exactly the same as in GR

$$\frac{M_P^2}{2}\sqrt{-g}\,R[\mathbf{g}] \sim M_P^2 \sum_{k=2}^{+\infty} \partial^2 h^k \sim \sum_{k=2}^{+\infty} M_P^{2-k}\,\partial^2 \tilde{h}^k \tag{3.146}$$

where $\tilde{h}_{\mu\nu} = M_P h_{\mu\nu}$. On the other hand, the mass term is not gauge invariant: since $A_\mu$ appears always derivated once in the Stückelberg formalism and $\phi$ appears always derivated twice, the interaction terms coming from the mass term will be of the form

$$\frac{M_P^2\,m^2}{4}\sqrt{-g}\,\mathcal{U} \supset M_P^2\,m^2\,h^i\,(\partial A)^j\,(\partial\partial\phi)^r \sim M_P^{2-i-j-r}\,m^{2-j-2r}\,\tilde{h}^i\,(\partial\tilde{A})^j\,(\partial\partial\tilde{\phi})^r \tag{3.147}$$

with $i, j, r \geq 2$ and the tilde fields are defined as follows

$$\tilde{h}_{\mu\nu} = M_P\,h_{\mu\nu} \tag{3.148}$$

$$\tilde{A}_\mu = M_P\,m\,A_\mu \tag{3.149}$$

$$\tilde{\phi} = M_P\,m^2\,\phi \tag{3.150}$$

To be more precise, note that every $U_k$ for $k \geq 2$ contains a piece $\left[H^k\right]$ which contains all the combinations of the form $h^i\,(\partial A)^j\,(\partial\partial\phi)^r$ with $i + j + r = k$. Therefore, if we don't assume the Fierz-Pauli tuning, the most general mass term actually contains *all* the possible combinations of terms of the type (3.147) with $i + j + r \geq 2$ and $i$, $j$, $r$ non-negative. If we assume the Fierz-Pauli tuning, the quadratic part have a special form, while the interaction part (terms which are cubic or higher in the fields) contains all the possible combinations of terms of the type (3.147) with $i + j + r \geq 3$ and $i$, $j$, $r$ non-negative.

**Quadratic term**

Let us look at the quadratic terms first, assuming the Fierz-Pauli tuning. They can be obtained using only the part of $H_{\mu\nu}$ which is linear in $h_{\mu\nu}$, $\partial A_\mu$, $\partial\partial\phi$ (which we indicate with $\bar{H}_{\mu\nu}$) and replacing $g^{\mu\nu}$ with $\eta_{\mu\nu}$ so it reads

$$-\frac{M_P^2\, m^2}{4}\Big(\big[\bar{H}^2\big] - \big[\bar{H}\big]^2\Big) \tag{3.151}$$

and is therefore equivalent to the mass term obtained in the Stückelberg analysis of the Fierz-Pauli action. Using the tilde fields, it contains (modulo total derivatives) a canonic kinetic term for $\tilde{A}_\mu$, the FP mass term for $\tilde{h}_{\mu\nu}$, a mixing term $m\,\tilde{h}\partial\tilde{A}$ and a kinetic mixing between $\tilde{h}$ and $\tilde{\phi}$. Note that the quadratic terms in $\tilde{\phi}$ appear in the combination

$$\big[\tilde{\Pi}^2\big] - \big[\tilde{\Pi}\big]^2 \tag{3.152}$$

which is indeed a total derivative, however if we don't assume the Fierz-Pauli tuning we would get the term

$$b_1\,\big[\tilde{\Pi}^2\big] + b_2\,\big[\tilde{\Pi}\big]^2 \tag{3.153}$$

instead. This term is not a total derivative if $b_1 \neq -b_2$, and would give rise to higher derivative terms (*i.e.* terms with derivatives of order three or higher) in the equation of motion for $\tilde{\phi}$. Higher derivative terms in the equation of motion are usually associated with ghost instabilities, by Ostrogradski theorem [91, 92]. This is consistent with the already mentioned result that any violation of the Fierz-Pauli tuning imply that the theory propagates also a sixth degree of freedom, which is a ghost [83]. The Fierz-Pauli mass term can therefore be uniquely identified in the Stückelberg language at quadratic order by the request that the scalar mode $\phi$ does not have higher derivative terms in the equations of motion.

## 3.4.2 Strong coupling scales and decoupling limit

Let us now turn to the interaction terms. As we already mentioned, a general nonlinear extension of the Fierz-Pauli theory contains all the possible combinations of terms

$$M_P^{2-i-j-r}\, m^{2-j-2r}\, \tilde{h}^i\, (\partial\tilde{A})^j\, (\partial\partial\tilde{\phi})^r \tag{3.154}$$

with $i$, $j$, $r$ non-negative and $i + j + r \geq 3$. Note that each of the terms $\tilde{h}^i (\partial \tilde{A})^j (\partial\partial\tilde{\phi})^r$ is suppressed by a dimensionful factor

$$M_P^{i+j+r-2}\, m^{j+2r-2} \tag{3.155}$$

where $M_P$ appears with positive power since $i + j + r \geq 3$. This factor sets a (mass) scale $\Lambda_{(ijr)}$

$$\Lambda_{(ijr)}^{i+2j+3r-4} = M_P^{i+j+r-2}\, m^{j+2r-2} \tag{3.156}$$

and, since the kinetic terms are in canonical form, the lowest of this mass scales is the strong coupling scale of the system, which is the scale where quantum corrections become non-negligible and need to be taken into account. Note that, despite the fact that $M_P$ appears always with positive power in the suppressing factor, $m$ appears with negative or zero power if $0 \leq j + 2r \leq 2$: in these cases (which comprise the non-derivative self interaction of $\tilde{h}$ for example) the associated scale $\Lambda$ is bigger than $M_P$. For the other cases (for which $j + 2r > 2$) the associated scale $\Lambda$ is smaller than $M_P$, and to see more clearly which is the lowest of these mass scales it is useful to write them in the following way

$$\Lambda_\lambda = \sqrt[\lambda]{M_P\, m^{\lambda-1}} \tag{3.157}$$

where (as it follows from (3.156)) we have

$$\lambda = \lambda(i, j, r) = \frac{i + 2j + 3r - 4}{i + j + r - 2} \tag{3.158}$$

Since we assume $m \ll M_P$, we have that the bigger $\lambda$ the lower the scale $\Lambda_\lambda$. Note that in general $\lambda$ is a rational number: $\lambda \in \mathbb{Q}$. The strong coupling scale of the system is therefore set by the biggest allowed $\lambda$, which we call $\lambda_{max}$: once found $\lambda_{max}$, we can immediately read the strong coupling scale $\Lambda_{sc} = \Lambda_{\lambda_{max}}$ from (3.157).

#### Strong coupling scales

To see which are the allowed values for $\lambda$, we note that at fixed $i, j$ the function $\lambda(i, j, r)$ becomes a function of $r$ only which is a hyperbola

$$\lambda_{i,j}(r) = \frac{3r - (4 - i - 2j)}{r - (2 - i - j)} \quad , \tag{3.159}$$

apart from the cases $(i,j) = (1,0)$ and $(i,j) = (0,2)$ where $\lambda_{i,j}(r) = 3$ and is independent of $r$. For the other cases, the hyperbola $\lambda_{i,j}(r)$ has the horizontal asymptote $\lambda = 3$ and the vertical asymptote $\lambda = 2 - i - j$. Since we have $i + j + r \geq 3$, at fixed $(i,j)$ (which must be positive) only the values $r \geq 3 - i - j$ are allowed, and since they are bigger that the position of the vertical asymptote, it follows that the allowed points $(r, \lambda(r))$ lie on the branch of the hyperbola which extends to $r \to +\infty$. It is easy to see that this branch is a decreasing function for the cases $(i,j) = (0,0)$ and $(i,j) = (0,1)$, while is an increasing function in the other cases (apart the particular cases $(i,j) = (1,0)$, $(i,j) = (0,2)$ as mentioned above). Furthermore, in the cases $(i,j) = (0,0)$, $(i,j) = (0,1)$ for which $\lambda_{i,j}(r)$ is a decreasing function, the biggest value for $\lambda$ is set by the lowest possible value for $r$, which is respectively $r = 3$ and $r = 2$. Therefore we conclude that the allowed values for $\lambda$ in the case $(i,j) = (0,0)$ lie in the range

$$(i,j) = (0,0) \qquad \Rightarrow \qquad 3 < \lambda_{0,0}(r) \leq 5 \quad , \quad r \geq 3 \tag{3.160}$$

and in particular we have

| $(i,j) = (0,0)$ | $r$ | $\to$ | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | $\lambda(r)$ | $\to$ | 5 | 4 | 11/3 | 7/2 | $\cdots$ |

while for $(i,j) = (0,1)$ the allowed values for $\lambda$ lie in the range

$$(i,j) = (0,1) \qquad \Rightarrow \qquad 3 < \lambda_{0,1}(r) \leq 4 \quad , \quad r \geq 2 \tag{3.161}$$

and in particular we have

| $(i,j) = (0,1)$ | $r$ | $\to$ | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | $\lambda(r)$ | $\to$ | 4 | 7/2 | 10/3 | 13/4 | $\cdots$ |

As already mentioned, for the cases $(i,j) = (1,0)$ $(i,j) = (0,2)$ we have

$$(i,j) = (1,0) \text{ or } (0,2) \qquad \Rightarrow \qquad \lambda_{i,j}(r) = 3 = \text{constant} \tag{3.162}$$

while for the other cases we have

$$(i,j) \neq \{(0,0), (0,1), (1,0), (0,2)\} \qquad \Rightarrow \qquad \lambda_{i,j}(r) < 3 \tag{3.163}$$

since in the latter cases the relevant branch of the hyperbola is a monotonically increasing function and asymptotes the value $\lambda = 3$.

We then conclude that for a generic nonlinear mass term (or equivalently for a generic choice of the coefficients $c_i$, $d_i$, $f_i$, $\ldots$) we have $\lambda_{max} = 5$ and the strong coupling scale of the system is

$$\Lambda_5 = \sqrt[5]{M_P\, m^4} \tag{3.164}$$

which is carried only by the cubic self-interaction term of $\tilde{\phi}$

$$\frac{1}{\Lambda_5^5} \left(\partial^2 \tilde{\phi}\right)^3 \tag{3.165}$$

The second lowest scale is instead

$$\Lambda_4 = \sqrt[4]{M_P\, m^3} \tag{3.166}$$

which is carried by the quartic self-interaction term of $\tilde{\phi}$ and by the interaction term which is quadratic in $\tilde{\phi}$ and linear in $\tilde{A}$

$$\frac{1}{\Lambda_4^8} \left(\partial^2 \tilde{\phi}\right)^4 \qquad\qquad \frac{1}{\Lambda_4^4}\, \partial \tilde{A} \left(\partial^2 \tilde{\phi}\right)^2 \tag{3.167}$$

We then have the higher order self-interaction terms of $\tilde{\phi}$ with or without a term which linear in $\tilde{A}$

$$\propto \left(\partial^2 \tilde{\phi}\right)^n \qquad\qquad \propto \partial \tilde{A} \left(\partial^2 \tilde{\phi}\right)^l \tag{3.168}$$

with $n \geq 5$ and $l \geq 3$, which carry scales $\Lambda_\lambda$ such that $3 < \lambda < 4$, and finally terms of the type

$$\frac{1}{\Lambda_3^{3(s-1)}}\, \tilde{h}\, (\partial\partial\tilde{\phi})^s \qquad\qquad \frac{1}{\Lambda_3^{3p}}\, (\partial \tilde{A})^2\, (\partial\partial\tilde{\phi})^p \tag{3.169}$$

with $s \geq 2$ and $p \geq 1$, which carry the scale

$$\Lambda_3 = \sqrt[3]{M_P\, m^2} \tag{3.170}$$

All the remaining terms carry scales $\Lambda_\lambda$ such that $\lambda < 3$.

### The Vainshtein radius

Having found the scale where quantum correction become important, we turn now to the scale where classical nonlinearities become important. Let's consider a static spherically symmetric source of mass $M$: as we saw in section (3.2.3), in terms of the redefined fields $\hat{h}_{\mu\nu}$, $\hat{A}_\mu$ and $\hat{\phi}$ the kinetic terms are in canonical form, and a gauge can be chosen so that there are no mixed terms at quadratic order. Therefore, the fields profile at linear order are ($\sim$ here means "apart from dimensionless factors")

$$\hat{h}_{\mu\nu} \sim \frac{M}{M_P}\frac{1}{r} \qquad\qquad \hat{\phi} \sim \frac{M}{M_P}\frac{1}{r} \tag{3.171}$$

which is to be expected since the "hatted" fields, as well as the tilded "fields", has dimension (length)$^{-1}$. In particular this implies that also the tilded fields have the same behavior, modulo a gauge mode which has no effect since the theory is now gauge invariant. Therefore it is quite simple to see at which radius each interaction term become comparable to the quadratic terms in the action. An interaction term of the form

$$M_P^{2-i-r}\, m^{2-2r}\, \tilde{h}^i\, (\partial\partial\tilde{\phi})^r \tag{3.172}$$

gives a contribution

$$\sim M_P^{2-i-r}\, m^{2-2r} \left(\frac{M}{M_P}\right)^{i+r} \left(\frac{1}{r}\right)^{i+3r} \tag{3.173}$$

while the quadratic terms give a contribution

$$\sim \left(\frac{M}{M_P}\right)^2 \left(\frac{1}{r}\right)^4 \tag{3.174}$$

so the interaction term (3.172) become comparable to the quadratic ones at the radius

$$r_{(ir)} \sim \left[ m^{2-2r} \left(\frac{M}{M_P^2}\right)^{i+r-2} \right]^{1/(i+3r-4)} \tag{3.175}$$

The largest of these radiuses is the one where the linear theory breaks down (at a classical level), and is therefore the Vainshtein radius of the theory. The interaction terms which correspond to this radius are the ones which first go nonlinear when from spatial infinity we move towards the source, and this happens at the Vainshtein radius: they are the only relevant interaction terms when we consider scales close to the Vainshtein radius. To see more clearly which is the biggest radius $r_{(ijr)}$ defined by (3.175) when $i+j+r \geq 3$, we write it in the following form

$$r_{(ijr)} = r_\mu = \sqrt[\mu]{r_g r_c^{\mu-1}} \tag{3.176}$$

where we have introduced the Compton radius of the theory $r_c = 1/m$ and the gravitational radius $r_g = M/M_P^2$ (which depends on the mass of the source). The hierarchy $M_P \gg m$ implies $r_g \ll r_c$, and so the bigger $\mu$ the bigger $r_\mu$: the Vainshtein radius is set by the maximum allowed value for $\mu$, which we indicate with $\mu_{max}$. Comparing (3.175) with (3.176) we find

$$\mu = \mu(i,j,r) = \frac{i+2j+3r-4}{i+j+r-2} = \lambda \tag{3.177}$$

and so $\mu$ is precisely equal to the number $\lambda$ associated to the interaction term individuated by $(ijr)$ which we have introduced when studying the strong coupling scales. In particular, it follows that $\mu_{max} = \lambda_{max}$. Therefore, the interaction terms which set the strong coupling scale are also the terms which set the Vainshtein radius: for the most general mass term the strong coupling scale is $\Lambda_5$ and the Vainshtein radius is

$$r_V = \sqrt[5]{r_g r_c^4} \tag{3.178}$$

where the only term which goes nonlinear at this scale is the cubic self-interaction term for $\tilde{\phi}$

$$\frac{1}{\Lambda_5^5} \left(\partial^2 \tilde{\phi}\right)^3 \tag{3.179}$$

We recover then the result (3.81) obtained in a somewhat different way in section (3.2.2). In that case we were considering the particular nonlinear extension of the Fierz-Pauli theory obtained adding the quadratic Fierz-Pauli term to the full nonlinear GR action: this action in fact contains the cubic self-interaction term for $\tilde{\phi}$, and so the Vainshtein radius is indeed (3.178).

### The decoupling limit

We have seen that there exists in the theory a special subclass of interaction terms which set both the strong coupling scale and the Vainshtein radius. We would like to define a formal limit of the theory which kills all the other interaction terms, and leaves us with a theory which contains only the kinetic terms and this special class of interaction terms.

We notice that, if we formally send $m \to 0$ and $M_P \to +\infty$ while keeping $\Lambda_{sc}$ fixed, all the scales $\Lambda$ bigger than $\Lambda_{sc}$ diverge. Therefore, taking this formal limit in the action, we have that all the interaction terms suppressed by scales larger than the strong coupling scale disappear. However, also the source term disappears since it is suppressed by $M_P$. If we want to construct a theory which contains only the desired interaction terms, but where the fields are still sourced by the energy and momentum of matter fields, we have to ask that also the energy-momentum tensor scales in some way in the limit, in order to compensate the fact that $M_P$ diverges. Therefore, we define the so called *decoupling limit* (first introduced by [45] in the context of the DGP model) as

$$m \to 0 \ , \quad M_P \to +\infty \ , \quad T_{\mu\nu} \to +\infty \ , \quad \Lambda_{sc} \text{ and } \frac{T_{\mu\nu}}{M_P} \text{ fixed} \tag{3.180}$$

By construction, this limit does not change the strong coupling scale of the theory, but it is easy to check that it leaves untouched the Vainshtein radius as well. Therefore, we could see this formal limit as a way to focus on the behavior of the complete theory at the scales correspondent to the strong coupling and the Vainshtein radius: it seems likey that the decoupling limit should be appropriate to study the effectiveness of the Vainshtein mechanism.

## 3.5 dGRT massive gravity

We have so far introduced a very general class of actions (3.107) which can be seen as nonlinear extensions of the Fierz-Pauli theory. We have then restored gauge invariance using the Stückelberg language, and identified the scales where quantum corrections and nonlinearities become important. In this section and in the next chapter, we want to select a subset of actions which ought to describe a phenomenologically viable theory of an interacting massive spin-2 field. As we already mentioned, to be viable these actions have to be meet several requirements: they must propagate exactly five degrees of freedom (as many as the free theory of a massive spin-2 field), they have to be free of ghost instabilities, and they have to reproduce GR in the range of scales where GR is well tested, which practically translates to the requirement that there have to be an efficient screening mechanism at work (the Vainshtein mechanism in this case). In this section we deal with the first two requirements, namely the number of degrees of freedom and absence of ghosts, which are anyway closely related [75]. We will select a two-parameters class of actions, which are shown to propagate the correct number of degrees of freedom. We dedicate the next chapter, instead, to the study of the effectiveness of the Vainshtein mechanism in this restricted class of theories, with the aim to select the range of parameters for which the correspondent theory is phenomenologically viable.

### 3.5.1 The $\Lambda_3$ theory

As we have already mentioned, to impose the condition of having just five degrees of freedom in principle we could perform a Hamiltonian analysis of the general action (3.107), and see for which values of the parameters $c_i$, $d_i$, $f_i$, ... there are constraints which kill one degree of freedom. This idea turns out to be very difficult to implement, so we instead try to reach the goal in two steps: first we select a subclass of actions which we expect to be good candidates for propagating five degrees of freedom, and only after that we apply the Hamiltonian formalism to properly count the numer of degrees of freedom.

#### Arranging self-interactions in total derivatives

We saw that, at quadratic level, the Fierz-Pauli action is the only action (apart an overall numerical factor) which has no ghosts and propagates exactly five degrees of freedom. We have also seen, using the Stückelberg language, that this requirement is precisely equivalent to the requirement that the scalar component $\phi$ of the Stückelberg fields have no higher derivative terms in the equations of motion (which is in turn linked to the absence of ghosts by Ostrogradski theorem [91, 92]), which implies that quadratic terms in $\partial\partial\phi$ in the action rearrange themselves to produce a total derivative term. We decide to follow this guideline also at full nonlinear level, and therefore we look for actions of the form (3.107) where, *at* *every order*, self-interaction terms in $\partial\partial\phi$ rearrange themselves to produce total derivative terms. This is also consistent with the indications in [90, 93, 94] that the nonlinear interactions of the scalar mode are related to sixth degree of freedom at full nonlinear level.

Since we are (for the time being) only interested in self-interacting terms in $\phi$, we may set

$$h_{\mu\nu} = 0 \qquad\qquad A_\mu = 0 \qquad\qquad H_{\mu\nu} = 2\,\Pi_{\mu\nu} - \Pi_\mu^{\ \alpha}\,\Pi_{\alpha\nu} \tag{3.181}$$

where indices are raised/lowered with $\eta^{\mu\nu}/\eta_{\mu\nu}$ and $\Pi_{\mu\nu}$ is defined in (3.144). The only terms which survive are the ones belonging to the nonlinear mass term, and the action takes the form

$$S = -\frac{M_P^2 m^2}{4} \int d^4x \sum_{k=2}^{+\infty} U_k[\Pi] \tag{3.182}$$

where

$$U_2[\Pi] = \big[H^2\big] - \big[H\big]^2 \tag{3.183}$$

$$U_3[\Pi] = c_1\big[H^3\big] + c_2\big[H^2\big]\big[H\big] + c_3\big[H\big]^3 \tag{3.184}$$

$$U_4[\Pi] = d_1\big[H^4\big] + d_2\big[H^3\big]\big[H\big] + d_3\big[H^2\big]^2 + d_4\big[H^2\big]\big[H\big]^2 + d_5\big[H\big]^4 \tag{3.185}$$

$$\begin{aligned} U_5[\Pi] = {} & f_1\big[H^5\big] + f_2\big[H^4\big]\big[H\big] + f_3\big[H^3\big]\big[H\big]^2 + f_4\big[H^3\big]\big[H^2\big] + \\ & + f_5\big[H^2\big]^2\big[H\big] + f_6\big[H^2\big]\big[H\big]^3 + f_7\big[H\big]^5 \end{aligned} \tag{3.186}$$

$$\vdots$$

and here $H_{\mu\nu}$ contains only the $\Pi$ tensor as explicitly said in (3.181).

The idea is now to work perturbatively order by order, starting at order 3 and choosing (if possible) the coefficients $c_1$, $c_2$, $c_3$ such that the cubic piece in $\Pi$ contained in $U_2[\Pi] + U_3[\Pi]$ is a total derivative, then going to order 4 and choosing (if possible) the coefficients $d_1$, $d_2$, $d_3$, $d_4$, $d_5$ such that the quartic piece in $\Pi$ contained in $U_2[\Pi] + U_3[\Pi] + U_4[\Pi]$ is a total derivative, and so on. The first attempt to realize this program has been done in [93], where it is was mistakenly concluded that there is no way to tune the free coefficients in (3.183)-(3.186) in order to produce total derivatives at fourth order and above. Later, it has been proved in [80] (building on previous works [95, 96, 97]) that it is indeed possible to carry on successfully this procedure at every order. It can be shown [68] that, at every order in $\Pi$, there is essentially only one linear combination of contractions of $\Pi$ which is a total derivative, which at order $n$ is explicitly

$$\mathcal{L}_n^{TD}(\Pi) = \sum_{p\in P_n} (-1)^p \, \eta^{\mu_1 p(\nu_1)} \cdots \eta^{\mu_n p(\nu_n)} \, \Pi_{\mu_1\nu_1} \cdots \Pi_{\mu_n\nu_n} \tag{3.187}$$

where the sum runs on all the permutations $p$ of $n$ elements. "Essentially" means that all the other linear combination of contractions of $\Pi$ at order $n$ which are total derivatives, are actually proportional to $\mathcal{L}_n^{TD}(\Pi)$. Note that, for $n \geq 5$, the sum in (3.187) vanishes identically by symmetry reasons: therefore, at each order $n$ there is a one-dimensional variety of total derivative terms if $n = 2, 3$ and 4, while for $n \geq 5$ the variety is zero-dimensional: the total derivative structures have in total three free parameters.

It is actually not difficult to see that the system is compatible, meaning that it is always possible to tune the coefficients in (3.183)-(3.186) to rearrange the terms in total derivatives at all orders: if we fix $n$ and insert in $U_n(\Pi)$ only the part of $H_{\mu\nu}$ which is *linear* in $\Pi$, we generate the most general linear combination of contraction of $n$ tensors $\Pi_{\mu\nu}$ with $n$ inverse metrics $\eta^{\alpha\beta}$. Therefore we can always use the free coefficients in $U_n(\Pi)$ to compensate exactly for the terms of order $n$ in $\Pi$ which come from the lower orders of the potential, and create the total derivative combination (3.187) at each order. Furthermore, since there are three free parameters in the total derivatives combinations correspondents to the orders $n = 2, 3$ and 4, there will be a three-parameters class of Lagrangians where the $\phi$ self-interactions are removed at all orders: the parameter coming from order two is reabsorbed in the overall mass parameter $m$ in the action, so we end up with a genuinely two-parameters class of actions. Explicitly, the values of the tuned coefficients in (3.183)-(3.186) are [80] to fourth order

$$c_1 = 2c_3 + \frac{1}{2} \qquad c_2 = -3c_3 - \frac{1}{2} \tag{3.188}$$

$$d_1 = -6d_5 + \frac{1}{16}(24c_3 + 5) \qquad d_2 = 8d_5 - \frac{1}{4}(6c_3 + 1) \tag{3.189}$$

$$d_3 = 3d_5 - \frac{1}{16}(12c_3 + 1) \qquad d_4 = -6d_5 + \frac{3}{4}c_3 \tag{3.190}$$

**The effect on the strong coupling scale**

Considering now the strong coupling scale of the theory, from what said in section (3.4.2) we can immediately conclude that the removal of all $\phi$ self-interaction terms raises the strong coupling scale to $\Lambda_4$, which is carried by the term

$$\frac{1}{\Lambda_4^4}\,\partial\tilde{A}\,\big(\partial^2\tilde{\phi}\big)^2 \tag{3.191}$$

However, it can be shown [90, 80] that the choice of coefficients in the nonlinear mass term which remove the self-interaction terms in $\phi$, automatically remove also the terms of the form

$$M_P^{1-l}\,m^{1-2l}\,\partial\tilde{A}\,(\partial\partial\tilde{\phi})^l \tag{3.192}$$

with $l \geq 2$, which carry the strong coupling scales $\Lambda_\lambda$ with $4 \geq \lambda > 3$. Therefore, removing the scalar self-interactions actually raises the strong coupling scale to

$$\Lambda_3 = \sqrt[3]{M_P\,m^2} \tag{3.193}$$

which is carried by terms of the form

$$\frac{1}{\Lambda_3^{3(s-1)}}\,\tilde{h}\,(\partial\partial\tilde{\phi})^s \qquad\qquad \frac{1}{\Lambda_3^{3p}}\,(\partial\tilde{A})^2\,(\partial\partial\tilde{\phi})^p \tag{3.194}$$

with $s \geq 2$ and $p \geq 1$. Note that these terms are the only terms which survive in the decoupling limit, since we proved in section (3.4.2) that all the other interaction terms are suppressed by scales $\Lambda_\lambda$ with $\lambda < 3$. The two-parameter theory defined by tuning the interaction terms so to remove the $\phi$ self-interactions is usually called the $\Lambda_3$ *theory*.

Note that the vector field $A_\mu$ does not couple directly to $T^{\mu\nu}$, and therefore setting it to zero and solving for $h_{\mu\nu}$ and $\phi$ always give a consistent solution of the theory. This however does not means that $A_\mu$ does not play any role, since the most general solution of the theory contains also the $A_\mu$ field since it couples to $h_{\mu\nu}$ and $\phi$, and in fact the $A_\mu$ sector may contain ghost instabilities (at least around some backgrounds) [98]. Setting anyway $A_\mu$ to zero for the time being, the decoupling limit Lagrangian up to total derivatives is given by the kinetic term for $\tilde{h}_{\mu\nu}$ plus the part of the mass term which is linear in $\tilde{h}_{\mu\nu}$. As shown in [80], it has at most quartic couplings in $\tilde{h}_{\mu\nu}$ and $\tilde{\phi}$ and explicitly reads

$$S = \int d^4x \left[ \tilde{h}^{\mu\nu} \mathcal{E}_{\mu\nu}{}^{\rho\sigma} \tilde{h}_{\rho\sigma} - \frac{1}{2}\,\tilde{h}^{\mu\nu} \left( -4\tilde{X}^{(1)}_{\mu\nu}(\tilde{\phi}) + \frac{4(6c_3-1)}{\Lambda_3^3}\,\tilde{X}^{(2)}_{\mu\nu}(\tilde{\phi}) + \right.\right.$$
$$\left.\left. + \frac{16(8d_5+c_3)}{\Lambda_3^6}\,\tilde{X}^{(3)}_{\mu\nu}(\tilde{\phi}) \right) - \frac{1}{M_P}\,\tilde{h}_{\mu\nu} T^{\mu\nu} \right] \tag{3.195}$$

where the operator $\mathcal{E}_{\mu\nu}{}^{\rho\sigma}$ has been defined in (3.28) and the tensors $\tilde{X}^{(n)}_{\mu\nu}$ are of order $n$ in $\tilde{\Pi}$ and are defined in the Appendix (A). Note finally that, in the decoupling limit, the Lagrangian has a finite number of interaction terms between $\tilde{h}_{\mu\nu}$ and $\tilde{\phi}$, while it has an infinite number of interaction terms between $\tilde{h}_{\mu\nu}$ and $\tilde{A}_{\mu}$.

**Demixing in the decoupling limit and galileons**

In the decoupling limit Lagrangian (3.195), the scalar mode $\tilde{\phi}$ does not have a kinetic term on its own but is kinetically mixed to $\tilde{h}_{\mu\nu}$: furthermore, all the interaction terms are in mixed form. To make more transparent the physical meaning of this action, we would like to disentangle as much as we can the dynamics of $\tilde{h}_{\mu\nu}$ and that of $\tilde{\phi}$.

First of all, we kinetically demix $\tilde{h}_{\mu\nu}$ and $\tilde{\phi}$ by redefining the fields, as we did in section (3.2.3), and going to the "hatted" fields: this transformation creates a canonical kinetic term for $\hat{\phi}$, as well as coupling $\hat{\phi}$ to the trace of the energy-momentum tensor $T$. At this point there are still couplings $\hat{h}\,\hat{\Pi}^2$ and $\hat{h}\,\hat{\Pi}^3$ between $\hat{h}$ and $\hat{\phi}$, while derivative self-interaction terms for $\hat{\phi}$ has appeared. It is possible to further demix the action and remove the cubic $\hat{h}\,\hat{\Pi}^2$ coupling, performing the field redefinition

$$\check{h}_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{2(6c_3-1)}{\Lambda_3^3}\,\partial_\mu\hat{\phi}\,\partial_\nu\hat{\phi} \tag{3.196}$$

After this operation the Lagrangian reads

$$\begin{aligned} S = \int d^4x \Bigg[ & \check{h}^{\mu\nu} \mathcal{E}_{\mu\nu}{}^{\rho\sigma} \check{h}_{\rho\sigma} + \frac{\mathcal{C}_1}{\Lambda_3^6} \check{h}^{\mu\nu} \check{X}^{(3)}_{\mu\nu} + \frac{1}{M_P} \check{h}_{\mu\nu} T^{\mu\nu} - \\ & - 3 \, (\partial\check{\phi} \cdot \partial\check{\phi}) + \frac{\mathcal{C}_2}{\Lambda_3^3} (\partial\check{\phi} \cdot \partial\check{\phi}) \, \Box\check{\phi} + \frac{\mathcal{C}_3}{\Lambda_3^6} (\partial\check{\phi} \cdot \partial\check{\phi}) \Big( [\check{\Pi}]^2 - [\check{\Pi}^2] \Big) + \\ & + \frac{\mathcal{C}_4}{\Lambda_3^9} (\partial\check{\phi} \cdot \partial\check{\phi}) \Big( [\check{\Pi}]^3 - 3[\check{\Pi}^2][\check{\Pi}] + 2[\check{\Pi}^3] \Big) + \\ & + \frac{1}{M_P} \check{\phi} \, T + \frac{\mathcal{C}_5}{\Lambda_3^3 M_P} \partial_\mu\check{\phi} \, \partial_\nu\check{\phi} \, T^{\mu\nu} \Bigg] \end{aligned} \tag{3.197}$$

while it is instead not possible to demix further the action and remove the quartic mixing $\check{h} \, \check{\Pi}^3$ keeping the action local, since only a nonlocal field redefinition could remove that mixing term. The notation $\big(\partial\check{\phi} \cdot \partial\check{\phi}\big)$ here stands for $\big(\partial_\alpha\check{\phi} \, \partial^\alpha\check{\phi}\big)$, while the numerical coefficients $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ depend only on $c_3$ and $d_5$, and their explicit expression can be found for example in [82]

$$\mathcal{C}_1 = -8(8d_5 + c_3) \tag{3.198}$$

$$\mathcal{C}_2 = 6(6c_3 - 1) \tag{3.199}$$

$$\mathcal{C}_3 = -4\big((6c_3 - 1)^2 - 4(8d_5 + c_3)\big) \tag{3.200}$$

$$\mathcal{C}_4 = -40(6c_3 - 1)(8d_5 + c_3) \tag{3.201}$$

$$\mathcal{C}_5 = 2(6c_3 - 1) \quad . \tag{3.202}$$

Note that they are all written in terms of the combinations $6c_3 - 1$ and $8d_5 + c_3$, so they all disappear from the action when both these combinations vanish. Furthermore, the coupling $\check{h}^{\mu\nu} \check{X}^{(3)}_{\mu\nu}$ disappears when $8d_5 + c_3 = 0$, irrespectively of whether $6c_3 - 1 = 0$ vanishes or not, while the coupling $\partial_\mu\check{\phi} \, \partial_\nu\check{\phi} \, T^{\mu\nu}$ disappear when $6c_3 - 1 = 0$, irrespectively of the value of $d_5$.

The action (3.197) has several interesting features. First, note that, beside the coupling $\check{\phi} \, T$ of the scalar mode with the trace $T$ of the energy-momentum tensor, there is a new form of coupling between $\check{\phi}$ and the energy-momentum tensor which involves the derivatives $\partial\check{\phi}$ and *not* the trace $T$. This implies in particular that the scalar mode $\check{\phi}$ couples also to the electromagnetic field, whose energy-momentum tensor is traceless. Second, turning to the interaction terms, apart from the mixed term $\check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu} \sim \check{h}\,\check{\Pi}^3$ (which disappear from the action when $8d_5 + c_3 = 0$), the scalar mode has now three self-interaction terms, respectively at order 3, 4 and 5. Dropping the symbol $\check{}$ for clarity, the kinetic and the self-interaction terms have the structure

$$\mathcal{L}_2 = -\frac{1}{2}\big(\partial\phi\cdot\partial\phi\big) \tag{3.203}$$

$$\mathcal{L}_3 = -\frac{1}{2}\,(\partial\phi\cdot\partial\phi)\big[\Pi\big] \tag{3.204}$$

$$\mathcal{L}_4 = -\frac{1}{2}\,(\partial\phi\cdot\partial\phi)\big(\big[\Pi\big]^2 - \big[\Pi^2\big]\big) \tag{3.205}$$

$$\mathcal{L}_5 = -\frac{1}{2}\,(\partial\phi\cdot\partial\phi)\Big(\big[\Pi\big]^3 - 3\big[\Pi\big]\big[\Pi^2\big] + 2\big[\Pi^3\big]\Big) \tag{3.206}$$

These terms are known as *Galileon terms* [68], and have the defining property that they give rise to equations of motion where the field appears only derivated twice, and that they are invariant with respect to the “galilean” transformation

$$\phi \to \phi + b_\mu x^\mu + c \tag{3.207}$$

(for the sake of precision, the Lagrangians are not invariant themselves but the galilean transformation produce a total derivative, therefore the action is invariant). It can be shown [68] that at each order in $\phi$ they are the only terms with these properties, up to total derivatives. Historically, apart from the quadratic term, the first of these terms to be studied was the cubic galileon term, which describes the dynamic of the brane bending mode in the decoupling limit of the DGP model (see section 2.2.2). It has later been recognized that, in general, an action which produces nonlinear equations of motion in which the field appears only through its second derivatives, can be used to modify gravity at large distances since the field may shield itself around a spherical source *via* the Vainshtein mechanism [68].

Note that the scalar mode of the Stückelberg fields trivially enjoys the galilean symmetry, since by construction it appears only derivated twice. Instead, the absence of higher derivatives in the equations of motion (despite the Lagrangian containing second derivatives already) is highly nontrivial. The fact that the decoupling limit of the $\Lambda_3$ theory produces only self-interactions of galileon type, which are ghost free, is a promising signal that the full theory may be indeed free of the BD ghost. Even more, it has been argued in [80] that the complete decoupling limit Lagrangian (containing also $\check{h}_{\mu\nu}$ and its coupling with $\check{\phi}$) is indeed free of ghosts. Note finally that the galileon interaction terms arise in the decoupling limit only when we demix the fields $\tilde{h}_{\mu\nu}$ and $\tilde{\phi}$: in particular, the first transformation $(\tilde{h}, \tilde{\phi}) \to (\hat{h}, \hat{\phi})$ (which demixes the kinetic terms) create the cubic and quartic galileon terms, and the second trasformation $(\hat{h}, \hat{\phi}) \to (\check{h}, \check{\phi})$ (which eliminates the $\hat{h}\,\hat{\Pi}^2$ coupling) creates also the fifth galileon term. The demixing procedure is on the other hand responsible for the coupling of $\phi$ to matter: initially, the field $\tilde{\phi}$ in fact does not couple with $T_{\mu\nu}$; the first redefinition (which removes the kinetic $\tilde{h}\,\Pi$ term) creates the "trace" coupling $\hat{\phi}\,T$, while the second redefinition (which removes the $\hat{h}\,\hat{\Pi}^2$ term) creates the "derivative" coupling $\partial_\mu\check{\phi}\,\partial_\nu\check{\phi}\,T^{\mu\nu}$.

### 3.5.2 Resummation of $\Lambda_3$ massive gravity

In the previous sections we saw that there is a way to tune order by order the coefficients of a generic nonlinear extension of the Fierz-Pauli action, in order to avoid the appearence of higher derivatives in the equations of motion for the scalar mode of the Stückelberg fields. Although the theory is uniquely defined (once we specify the values of the free parameters), and we could be just satisfied with this perturbative formulation, we may like to reformulate it in a more compact and manageable form.

Let's consider for example the case of GR. As we already mentioned, we can formulate GR as a theory of a massless helicity-2 field, specifying the value of all the coefficients which enter the infinite expansion in powers of the difference $h_{\mu\nu}$ between the physical metric $g_{\mu\nu}$ and Minkowski metric $\eta_{\mu\nu}$. This fixes the theory in a unique way. Expressing the action in this form is indeed suitable and very useful if we want to study perturbatively the metric field produced by a source, for example if we want to focus on the weak field regime, at first or even second order. However, suppose we want to find the exact metric produced by a source. If we express the theory *via* an infinite perturbative expansion, we have to solve iteratively the coupled equation at each order, obtaining the full solution as an infinite expansion: in the case of a static, spherically symmetric source, we are luckily able to sum the series and express the solution as a unique nonlinear function, obtaining the Schwarzschild metric. However, if the source is less symmetric, it is unlikely that we are able to sum the series, and we have to work with a metric expressed as an infinite sum.

On the other hand, we know we can express the action of GR in terms of two quantities, the scalar curvature and the square root of the determinant of the metric, which themselves contain all powers of $h_{\mu\nu}$: in this "resummed" form, the action is made of one term only, $\sqrt{-g}R$, instead of an infinite sum. If we want to find the exact metric correspondent to a source configuration, varying the action we obtain just a finite number of equations, corresponding to the different components, which are however intrinsically nonlinear. We could say that the expanded form and the resummed form are both useful, depending on what we want to use them for. However, in general it is easier to perform a Taylor expansion of an object than to resum a perturbative expansion.

### The square root formulation

We would like then to provide a resummed form of the theory of nonlinear massive gravity we defined so far. To do that, we should identify an object which make it possible to express the full action as the sum of a finite number of terms. Looking back to the problem of rearranging the $\phi$ self-interaction terms in total derivatives, we notice that the reason why the tuning of coefficients goes on to an infinite number of orders is that, in the Stückelberg language, the generic nonlinear mass term is expressed as a power series of $H_{\mu\nu}$, which is *quadratic* in $\Pi_{\mu\nu}$. As a consequence, every order $n$ of the potential generates terms in $\Pi_{\mu\nu}$ which are of order $m > n$, and, as we construct the total derivative at order $n$, we are generating higher order terms which will need to be taken care of. We could try instead to express the generic mass term (3.116)-(3.121) of a nonlinear extension of FP in terms of an object which is *linear* in $\Pi$, at least when $h_{\mu\nu}$ and $A_\mu$ are vanishing since the condition we want to impose involves $\phi$ self-interactions only.

In fact, this is possible if we define the object [81]

$$\mathcal{K}^\mu_{\ \nu}(g,H) \equiv \delta^\mu_{\ \nu} - \sqrt{\delta^\mu_{\ \nu} - H^\mu_{\ \nu}} \tag{3.208}$$

where $H^\mu_{\ \nu} = g^{\mu\lambda}H_{\lambda\nu}$ and the square root of a matrix $\mathcal{A}^\mu_{\ \nu}$ is defined as the matrix $\mathcal{R}^\mu_{\ \nu}$ such that $\mathcal{A}^\mu_{\ \nu} = \mathcal{R}^\mu_{\ \alpha}\mathcal{R}^\alpha_{\ \nu}$. Since $\mathcal{K}^\mu_{\ \nu}$ can be expressed (at least perturbatively, when its components are small) as power series of $H^{\mu}_{\ \nu}$

$$\mathcal{K}^{\mu}_{\ \nu} = \sum_{n=1}^{\infty} \tilde{\beta}_n \, (H^n)^{\mu}_{\ \nu} \qquad\qquad \tilde{\beta}_n = -\frac{(2n)!}{(1-2n)(n!)^2 4^n} \quad , \tag{3.209}$$

the most general nonlinear extension of the Fierz-Pauli theory (3.107) can be expressed as an expansion in powers of the tensor $\mathcal{K}^{\mu}_{\ \nu}$

$$\sqrt{-g}\,\mathcal{U}[\mathbf{g}, \mathbf{g}^{(0)}] = \sqrt{-g}\,\sum_{k=2}^{+\infty} W_k[\mathcal{K}] \tag{3.210}$$

where

$$W_2[\mathcal{K}] = \langle \mathcal{K}^2 \rangle - \langle \mathcal{K} \rangle^2 \tag{3.211}$$

$$W_3[\mathcal{K}] = \tilde{c}_1 \langle \mathcal{K}^3 \rangle + \tilde{c}_2 \langle \mathcal{K}^2 \rangle \langle \mathcal{K} \rangle + \tilde{c}_3 \langle \mathcal{K} \rangle^3 \tag{3.212}$$

$$W_4[\mathcal{K}] = \tilde{d}_1 \langle \mathcal{K}^4 \rangle + \tilde{d}_2 \langle \mathcal{K}^3 \rangle \langle \mathcal{K} \rangle + \tilde{d}_3 \langle \mathcal{K}^2 \rangle^2 + \tilde{d}_4 \langle \mathcal{K}^2 \rangle \langle \mathcal{K} \rangle^2 + \tilde{d}_5 \langle \mathcal{K} \rangle^4 \tag{3.213}$$

$$W_5[\mathcal{K}] = \tilde{f}_1 \langle \mathcal{K}^5 \rangle + \ldots \tag{3.214}$$

$$\vdots$$

and where the angled brackets here mean

$$\langle \mathcal{K}^n \rangle = \mathcal{K}^{\mu}_{\ \alpha_2} \, \mathcal{K}^{\alpha_2}_{\ \alpha_3} \cdots \mathcal{K}^{\alpha_n}_{\ \mu} \tag{3.215}$$

On the other hand, if we set $h_{\mu\nu} = 0$ and $A_\mu = 0$, remarkably the powers of the linear and the quadratic pieces in $\Pi$ which constitute $H_{\mu\nu}$ nearly cancel out, when the power expansion of the square root (3.209) is performed, leaving only the linear term

$$\mathcal{K}^{\mu}_{\ \nu}\Big|_{h=0,A=0} = \delta^{\mu}_{\ \nu} - \sqrt{\delta^{\mu}_{\ \nu} - \left( \Pi^{\mu}_{\ \nu} - \Pi^{\mu}_{\ \alpha} \, \Pi^{\alpha}_{\ \nu} \right)} = \Pi^{\mu}_{\ \nu} \tag{3.216}$$

and so $\mathcal{K}^{\mu}_{\ \nu}$ is precisely equal to $\Pi^{\mu}_{\ \nu}$ when $h_{\mu\nu} = 0$ and $A_\mu = 0$. Therefore, it is much simpler to impose the condition that the self-interaction terms of $\phi$ rearrange in total derivatives when we express the nonlinear mass term in terms of $\mathcal{K}^{\mu}_{\ \nu}$, since it reduces to the conditions

$$W_3[\Pi] = \alpha_3\,\mathcal{L}_3^{\mathrm{TD}}(\Pi) \tag{3.217}$$

$$W_4[\Pi] = \alpha_4\,\mathcal{L}_4^{\mathrm{TD}}(\Pi) \tag{3.218}$$

$$W_5[\Pi] = 0 \tag{3.219}$$

$$W_6[\Pi] = 0 \tag{3.220}$$

$$\vdots$$

without any higher order tuning. Comparing with (A.6)-(A.7), we deduce

$$\tilde{c}_1 = 2\,\alpha_3 \qquad \tilde{c}_2 = -3\,\alpha_3 \qquad \tilde{c}_3 = \alpha_3 \tag{3.221}$$

$$\tilde{d}_1 = -6\,\alpha_4 \qquad \tilde{d}_2 = 8\,\alpha_4 \qquad \tilde{d}_3 = 3\,\alpha_4 \qquad \tilde{d}_4 = -6\,\alpha_4 \qquad \tilde{d}_5 = \alpha_4 \tag{3.222}$$

while $\tilde{f}_i$ and all the coefficients of the orders of $W_k$ higher than four are vanishing. The coefficients $\alpha_3$ and $\alpha_4$ are free parameters, and correspond to the free parameters $c_3$ and $d_5$ in the other formulation.

### The resummed action

To get the complete action of nonlinear massive gravity, we have to reintroduce in some way the fields $h_{\mu\nu}$ and $A_\mu$. Since the tensor $\mathcal{K}^\mu_{\ \nu}$ naturally contains them, we can *define* the complete action of nonlinear massive gravity to be expressed in terms of $\mathcal{K}^\mu_{\ \nu}$ precisely in the same way as it is in the case $h_{\mu\nu} = 0$ and $A_\mu = 0$: the action in the resummed form then reads

$$S = \int d^4x\,\sqrt{-g}\,\bigg[\,\frac{M_P^2}{2}\Big(R[\mathbf{g}] - \frac{m^2}{2}\mathcal{U}[\mathcal{K}]\Big) + \mathscr{L}_M[\mathbf{g},\psi_{(i)}]\bigg] \tag{3.223}$$

where

$$\mathcal{U}[\mathbf{g},\mathcal{K}] = \mathcal{U}_2[\mathcal{K}] + \alpha_3\,\mathcal{U}_3[\mathcal{K}] + \alpha_4\,\mathcal{U}_4[\mathcal{K}] \tag{3.224}$$

and

$$\mathcal{U}_2 = (\mathrm{tr}\mathcal{K})^2 - \mathrm{tr}(\mathcal{K}^2) \tag{3.225}$$

$$\mathcal{U}_3 = (\mathrm{tr}\mathcal{K})^3 - 3(\mathrm{tr}\mathcal{K})(\mathrm{tr}\mathcal{K}^2) + 2\,\mathrm{tr}\mathcal{K}^3 \tag{3.226}$$

$$\mathcal{U}_4 = (\mathrm{tr}\mathcal{K})^4 - 6(\mathrm{tr}\mathcal{K})^2(\mathrm{tr}\mathcal{K}^2) + 8(\mathrm{tr}\mathcal{K})(\mathrm{tr}\mathcal{K}^3) + 3(\mathrm{tr}\mathcal{K}^2)^2 - 6\,\mathrm{tr}\mathcal{K}^4 \tag{3.227}$$

The infinite series of terms which made up the mass term in the previous formulation is expressed, in the resummed form, with just three terms. Note that in (3.208) we have defined the tensor $\mathcal{K}$ in terms of $H^{\mu}_{\ \nu} = g^{\mu\alpha} H_{\alpha\nu}$, where $H_{\mu\nu}$ is the "covariantization" of the difference $h_{\mu\nu}$ between the physical metric $g_{\mu\nu}$ and the absolute metric $g^{(0)}_{\mu\nu}$. To construct the theory, we found more convenient to express the theory in terms of $h_{\mu\nu}$ and $g_{\mu\nu}$, but now we want to express the full resummed action in terms of the absolute and physical metrics themselves. Remembering that $H_{\mu\nu}$ is defined as

$$H_{\mu\nu} = g_{\mu\nu} - \Sigma_{\mu\nu} \quad , \tag{3.228}$$

where the $\mathbf{\Sigma}$ tensor is the "covariantization" of the absolute metric $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and is defined as

$$\Sigma_{\mu\nu}(x) = g^{(0)}_{\alpha\beta} \, \frac{\partial \phi^{\alpha}(x)}{\partial x^{\mu}} \, \frac{\partial \phi^{\beta}(x)}{\partial x^{\nu}} \quad , \tag{3.229}$$

we have that

$$\delta^{\mu}_{\ \nu} - H^{\mu}_{\ \nu} = g^{\mu\alpha}\Sigma_{\alpha\nu} \quad . \tag{3.230}$$

We can therefore express the $\mathcal{K}$ tensor in terms of the physical metric $\mathbf{g}$, the absolute metric $\mathbf{g}^{(0)}$ and the Stückelberg fields $\phi^{\alpha}$ as

$$\mathcal{K}^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} - \left[ \sqrt{\mathbf{g^{-1}} \cdot \mathbf{\Sigma}} \, \right]^{\mu}_{\ \nu} \tag{3.231}$$

where the dot stands for the matrix multiplication operation.

The last expression, together with (3.223) - (3.227), defines the theory in the resummed form. Note that, by construction, the theory is reparametrization-invariant, by means of the Stückelberg fields $\phi^{\alpha}$. The introduction of the Stückelberg fields and the restoration of gauge invariance proved in fact to be very helpful in clarifying the analysis of a general non-linear extension of the Fierz-Pauli theory. However, as we stressed above, a theory with gauge invariance restored by means of Stückelberg fields is completely equivalent from a physical point of view to a theory without Stückelberg fields where gauge invariance is broken. Without using the Stückelberg formalism, the non-linear theory of massive gravity we obtained is described by the action

$$S = \int d^4x \, \sqrt{-g} \, \left[ \, \frac{M_P^2}{2} \Big( R[\mathbf{g}] - \frac{m^2}{2} \mathcal{U}\big[\mathbf{g}, \mathbf{g}^{(0)}\big] \Big) + \mathscr{L}_M[\mathbf{g}, \psi_{(i)}] \right] \tag{3.232}$$

where

$$\mathcal{U}\left[\mathbf{g},\mathbf{g}^{(0)}\right] = \mathcal{U}_2\left[\sqrt{\mathbf{g^{-1}}\cdot\mathbf{g}^{(0)}}\,\right] + \alpha_3\,\mathcal{U}_3\left[\sqrt{\mathbf{g^{-1}}\cdot\mathbf{g}^{(0)}}\,\right] + \alpha_4\,\mathcal{U}_4\left[\sqrt{\mathbf{g^{-1}}\cdot\mathbf{g}^{(0)}}\,\right] \tag{3.233}$$

and the explicit form of the potentials can be obtained plugging in (3.225)-(3.227) the expression

$$\mathcal{K}^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} - \left[\sqrt{\mathbf{g^{-1}}\cdot\mathbf{g}^{(0)}}\,\right]^{\mu}_{\ \nu} \tag{3.234}$$

**Absence of the Boulware-Deser mode and prior geometry**

We go back now to the problem of the number of degrees of freedom. As we already mentioned, a legitimate interacting theory of a massive graviton has to propagate five degrees of freedom, as many as a massive spin-2 field propagates. The absence of a sixth degree of freedom is also important from the point of view of the stability of the theory, since the additional degree of freedom is usually associated with ghost instabilities (Boulware-Deser ghost). The number of degrees of freedom can in principle be established recasting the theory in Hamiltonian form, however (as we said above) performing a full Hamiltonian analysis on the most general nonlinear extension of Fierz-Pauli action is very hard. By restoring gauge invariance and asking that the scalar component of the Stückelberg fields does not have higher derivatives in the equations of motion, it has been possible to single out a two-parameters class of non-linear extensions of the Fierz-Pauli theory. The hope is that the Hamiltonian analysis of this restricted class of theories turns out to be easier to perform.

A full Hamiltonian analysis on this restricted class of actions has indeed been performed in [99, 100, 101, 102], with the result that it has been confirmed that these actions propagate exactly five degrees of freedom. Therefore, the theories defined by (3.225) - (3.227) and (3.232) - (3.234) are legitimate interacting theories of a massive graviton, and are known as *dRGT Massive Gravity* (from the name of the authors de Rham, Gabadadze and Tolley) or also *Ghost-Free Massive Gravity*. The latter denomination is due to the fact that in these theories the Boulware-Deser ghost is absent. However, it is fair to say that the absence of the BD ghost does not imply that the theory is ghost-free, since some of the five degrees of freedom may still be a ghost, at least on some backgrounds [98]. Leaving aside this issue, a necessary condition for these theories to be phenomenologically viable is that they reproduce GR results on length scales/configurations where these results are experimentally tested. This implies that they have to admit static spherically symmetric solutions where the Vainshtein mechanism is effective. In the next chapter, we will systematically study static spherically symmetric solutions in the dGRT massive gravity theories, to characterise in which part of the phase space of theories spanned by $(\alpha_3, \alpha_4)$ we can find solutions which display the Vainshtein mechanism. This is a crucial step in establishing the phenomenological viability of non-linear massive gravity.

Note that the absolute metric $\mathbf{g}^{(0)}$ is explicitly present in the resummed action (3.232) - (3.234), therefore the dRGT Massive Gravity has a prior geometry, which is set by the absolute metric. This is in stark constrast with GR, where the absolute metric disappears from the resummed action when we substitute $h_{\mu\nu}$ with $g_{\mu\nu} - g^{(0)}_{\mu\nu}$, and so there is no prior geometry. It follows in particular that each choice for the absolute geometry generates a different theory of non-linear massive gravity. On the other hand, we can see that the theory really depends on the absolute *geometry*, and not on the coordinates chosen to express the absolute metric. In fact, let's consider two absolute metrics $g^{(0)}_{\mu\nu}$ and $g^{(0)\prime}_{\mu\nu}$ which describe the same absolute geometry, and so are linked by a change of coordinates: we may introduce an absolute metric manifold $\mathscr{M}_{(0)}$, and two system of references $y^\mu$ and $y'^\mu$ on $\mathscr{M}_{(0)}$, so that

$$g^{(0)\prime}_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, g^{(0)}_{\alpha\beta} \tag{3.235}$$

The physical metric in general is determined by the absolute metric and the energy-momentum tensor. Let's consider on one side the theory associated with the absolute metric $g^{(0)}_{\mu\nu}$, and consider a source term $T_{\mu\nu}$ in this theory, and on the other side the theory associated with the absolute metric $g^{(0)\prime}_{\mu\nu}$, and consider in this second theory a source term $T'_{\mu\nu}$ which is linked to $T_{\mu\nu}$ by the same relation which links $g^{(0)}_{\mu\nu}$ and $g^{(0)\prime}_{\mu\nu}$

$$T'_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, T_{\alpha\beta} \quad . \tag{3.236}$$

Let's call $g_{\mu\nu}$ the solution for the physical metric in the first theory and $g'_{\mu\nu}$ the solution for the physical metric in the second theory. If $g_{\mu\nu}$ and $g'_{\mu\nu}$ are *not* linked by the same relation which links the absolute metrics and the source terms, then we may say that the dRGT massive gravity depends not only on the absolute geometry, but also on the coordinate system chosen to express the absolute metric. Conversely, if $g_{\mu\nu}$ and $g'_{\mu\nu}$ are indeed linked by the relation

$$g'_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, g_{\alpha\beta} \quad , \tag{3.237}$$

then we may say that the dRGT massive gravity depends only on the absolute geometry, and not on the coordinate system chosen to express the absolute metric.

It is in fact not difficult to see that the latter case is the correct one. In fact, despite the fact that the action (3.232) is not invariant with respect to coordinate changes (which change the physical metric and the energy-momentum tensor but leaves untouched the absolute metric), the action is invariant with respect to the *formal* transformation

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, g_{\alpha\beta} \quad g^{(0)}_{\mu\nu} \to g^{(0)\prime}_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, g^{(0)}_{\alpha\beta} \quad T_{\mu\nu} \to T'_{\mu\nu} = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial y^\alpha}{\partial y'^\nu}\, T_{\alpha\beta} \tag{3.238}$$

as a consequence of the structure $\sqrt{\mathbf{g^{-1}} \cdot \mathbf{g}^{(0)}}$ in the potential. This is more in general a consequence of the fact that we started from the general action (3.107) whose potential term is written in terms of contractions of the inverse of the physical metric $g^{\mu\nu}$ and of the difference between the physical and absolute metric $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$.

# Chapter 4

# The Vainshtein mechanism in dRGT massive gravity

In the previous chapter we introduced a class of non-linear completions of the Fierz-Pauli action, known as dRGT massive gravity, which are free of the Boulware-Deser ghost and so seem to be potentially phenomenologically viable. To provide a reliable description of the gravitational interaction, they necessarily have to pass stringent experimental constraints, and agree with the predictions of GR which have been tested to a very high accuracy. A necessary condition for this to happen is that the vDVZ discontinuity is cured by non-linear interactions, or in other words that the Vainshtein mechanism is effective. In particular, since this class of non-linear completions of the Fierz-Pauli action has two free parameters (the Fierz-Pauli action has already a free parameter, the mass), it is crucial to understand for which values of the free parameters the Vainshtein mechanism works, and so to identify the regions in the phase space of free parameters which correspond to phenomenologically viable theories. The aim of this chapter is to find a precise answer to this problem. Therefore, we study static, spherically symmetric vacuum solutions in the dGRT massive gravity model with flat absolute geometry, and classify the types of solutions that the theory admits. We then determine in which regions of the two parameters phase space the Vainshtein mechanism is effective.

## 4.1 Spherically symmetric solutions

We consider the theory defined by equations (3.225) - (3.227) and (3.232) - (3.234) in the case where the absolute geometry is flat. To study static and spherically symmetric solutions in this case, we start by expressing the absolute metric $\mathbf{g}^{(0)}$ in spherical coordinates, which are more suited to the symmetry of the problem

$$ds^2 = g^{(0)}_{\mu\nu} dy^\mu dy^\nu = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{4.1}$$

where $y^\mu = (t, r, \theta, \varphi)$ indicates collectively the spherical coordinates. The most general form for the physical metric allowed by the condition of staticity and spherical symmetry is

$$ds^2 \;=\; -C(r)\, dt^2 + A(r)\, dr^2 + 2D(r)\, dtdr + B(r) d\Omega^2 \tag{4.2}$$

and, varying the action (3.223) and considering vacuum regions, we obtain the following equations of motion

$$G_{\mu\nu} = \frac{m^2}{2} T^{\mathcal{U}}_{\mu\nu} \tag{4.3}$$

where we have defined

$$T^{\mathcal{U}}_{\mu\nu} = \frac{1}{\sqrt{-g}} \, \frac{\delta \sqrt{-g}\; \mathcal{U}}{\delta g^{\mu\nu}} \tag{4.4}$$

### 4.1.1 The two branches

For metrics of the form (4.2), the Einstein tensor $G_{\mu\nu}$ satisfies the identity

$$D(r)\, G_{tt} + C(r)\, G_{tr} \;=\; 0 \tag{4.5}$$

which implies the following algebraic constraint on $T^{\mathcal{U}}_{\mu\nu}$

$$D(r)\, T^{\mathcal{U}}_{tt} + C(r)\, T^{\mathcal{U}}_{tr} = 0 \tag{4.6}$$

This last equation reduces to

$$D(r) \left( b_0 r - \sqrt{B(r)} \right) = 0 \tag{4.7}$$

where $b_0$ is a function of $\alpha_3$ and $\alpha_4$ only [98]. This constraint is solved in two possible ways, defining two class of solutions: either the metric is diagonal $D = 0$, which defines the *diagonal branch*, or $B = b_0^2 r^2$, which defines the *non-diagonal branch*. Note that it is possible to map a physical metric belonging to the diagonal branch into one of the non-diagonal branch *via* a change of coordinates, and *viceversa*: however, in dRGT massive gravity these two branches are physically distinct. To see it, it is convenient to restore gauge invariance by using the Stückelberg formalism. Consider, before introducing the Stückelberg fields, a configuration where the absolute metric has the form (4.1) and indicate with $\bar{g}_{\mu\nu}$ a solution of the equations of motion belonging to the diagonal branch, while indicate with $\bar{\bar{g}}_{\mu\nu}$ a solution of the equations of motion belonging to the non-diagonal branch. We then introduce the Stückelberg fields $\phi^\mu$ and form the "covariantized" version of the absolute metric

$$\Sigma_{\mu\nu}(x) = g^{(0)}_{\alpha\beta} \, \frac{\partial \phi^\alpha(y)}{\partial y^\mu} \, \frac{\partial \phi^\beta(y)}{\partial y^\nu} \tag{4.8}$$

where (analogously to section 3.3.2) we decompose the Stückelberg fields $\phi^\mu$ in the following way

$$\phi^t = t - \tilde{Z}^t \tag{4.9}$$

$$\phi^r = r - \tilde{Z}^r \tag{4.10}$$

$$\phi^\theta = \theta - \tilde{Z}^\theta \tag{4.11}$$

$$\phi^\varphi = \varphi - \tilde{Z}^\varphi \tag{4.12}$$

Substituting the absolute metric $\mathbf{g}^{(0)}$ with $\mathbf{\Sigma}$ in the action restores gauge invariance in the theory, and it is customary to call *unitary gauge* the situation when $\tilde{Z}^\mu = 0$. Therefore, the configurations $\big(\bar{g}_{\mu\nu}, g^{(0)}_{\mu\nu}\big)$ and $\big(\bar{\bar{g}}_{\mu\nu}, g^{(0)}_{\mu\nu}\big)$ we introduced above correspond, upon introducing the Stückelberg fields, to a situation where the physical metric is respectively $\bar{g}_{\mu\nu}$ and $\bar{\bar{g}}_{\mu\nu}$ in the unitary gauge. Suppose we now change coordinates and map $\bar{g}_{\mu\nu}$ into a metric $\bar{g}'_{\mu\nu}$ which belongs to the non-diagonal branch: the change of coordinates excites some components of the Stückelberg fields. Both $\bar{g}'_{\mu\nu}$ and $\bar{\bar{g}}_{\mu\nu}$ are non-diagonal metrics, but in the first case the Stückelberg fields are non-zero, while in the second case they vanish. Since the Stückelberg fields explicitly appear in the equations of motion, we conclude that $\bar{g}'_{\mu\nu}$ and $\bar{\bar{g}}_{\mu\nu}$ obey *different* equations of motion, and therefore are different. This implies that there are indeed two physically distinct branches of static and spherically symmetric solutions. This is in stark contrast with the GR case, where the theory is gauge invariant without the need to introduce the Stückelberg fields. In that case, $\bar{g}'_{\mu\nu}$ and $\bar{\bar{g}}_{\mu\nu}$ obey *the same* equations of motion, and so the two branches are physically identical.

As we shall see shortly, the Vainshtein mechanism in the diagonal branch is related to the role of non-linearities for the radial component of the Stückelberg fields. However, it has been shown [103] that, in the non-diagonal branch, the scalar mode of the Stückelberg fields does not couple directly to the energy-momentum tensor in the decoupling limit. In fact, the results of GR in this branch are reproduced without the need of the Vainshtein mechanism: the non-diagonal branch is very interesting and it can be shown that in this branch static, spherically symmetric solutions leads to Schwarzschild or Schwarzschild-de Sitter solutions [104, 105, 106, 107, 108, 109, 110]. Other interesting discussions on the non-diagonal branch can be found for example in [111, 112, 98].

Anyway, we conclude that the only branch which is relevant for the Vainshtein mechanism is the diagonal one: therefore, from now on we will consider only the diagonal branch.

### 4.1.2 The diagonal branch

To study the diagonal branch, let's start from the following ansatz for the physical metric

$$ds^2 = -\tilde{N}(r)^2 dt^2 + \tilde{F}(r)^{-1} dr^2 + r^2 \tilde{H}(r)^{-2} d\Omega^2 \quad , \tag{4.13}$$

and the form (4.1) for the absolute metric. To derive the equations of motion, we have to compute the form of the potential $\mathcal{U}\big(\mathbf{g}, \mathbf{g}^{(0)}\big)$ in terms of $\tilde{N}(r)$, $\tilde{F}(r)$ and $\tilde{H}(r)$: this amounts to evaluate the trace of $\sqrt{\mathcal{M}}$, $\mathcal{M}$, $\sqrt{\mathcal{M}}^3$ and $\mathcal{M}^2$, where $\mathcal{M} = \mathbf{g}^{-1}\mathbf{g}^{(0)}$. Note that, if a matrix $\mathcal{D}$ is diagonal, we have

$$tr\sqrt{\mathcal{D}}^{\,k} = \sum_i \sqrt{\lambda_i}^{\,k} \tag{4.14}$$

where $\lambda_i$, $i = 1, \cdots, 4$ are the eigenvalues of $\mathcal{D}$ and $k$ is a natural number. Furthermore, if a matrix $\mathcal{M}$ is diagonalizable (*i.e.* $\mathcal{M} = \mathcal{A}\mathcal{D}\mathcal{A}^{-1}$, for some invertible matrix $\mathcal{A}$), then we have

$$tr\mathcal{M} = tr\big(\mathcal{A}\mathcal{D}\mathcal{A}^{-1}\big) = tr\mathcal{D} \tag{4.15}$$

and using these relations we find

$$tr\sqrt{\mathcal{M}}^{\,k}=tr\Big(\mathcal{A}\sqrt{\mathcal{D}}\mathcal{A}^{-1}\cdots\mathcal{A}\sqrt{\mathcal{D}}\mathcal{A}^{-1}\Big)=tr\Big(\mathcal{A}\,\sqrt{\mathcal{D}}^{\,k}\mathcal{A}^{-1}\Big)=\sum_i\sqrt{\lambda_i}^{\,k} \tag{4.16}$$

Therefore, to compute $\mathcal{U}\big(\mathbf{g},\mathbf{g}^{(0)}\big)$ one has to find the eigenvalues of the matrix $\mathbf{g}^{-1}\mathbf{g}^{(0)}$ and plug them in (3.225)-(3.227) : this has been done in [105], where it was found that

$$\begin{aligned}\sqrt{-g}\,\mathcal{U}\big(\mathbf{g},\mathbf{g}^{(0)}\big)=-\frac{r^2}{\sqrt{\tilde{F}}\tilde{H}^2}\Bigg[&2\Big[\sqrt{\tilde{F}}\big((2\tilde{H}-3)\tilde{N}+1\big)+\tilde{H}^2\tilde{N}+\tilde{H}(2-6\tilde{N})+6\tilde{N}-3\Big]-\\&-6\alpha_3(\tilde{H}-1)\Big[\sqrt{\tilde{F}}\big((\tilde{H}-3)\tilde{N}+2\big)-2\tilde{H}\tilde{N}+\tilde{H}+4\tilde{N}-3\Big]-\\&-24\alpha_4(1-\sqrt{\tilde{F}})(1-\tilde{N})(1-\tilde{H})^2\Bigg]\end{aligned} \tag{4.17}$$

Varying the action with respect to $\tilde{N}(r)$, $\tilde{F}(r)$ and $\tilde{H}(r)$, one obtains the exact equations of motion for static, spherically symmetric solutions in the diagonal branch [105]. These equations are however very complicated, and to solve them it will be convenient to do some approximations.

Note that, in order to study the Vainshtein mechanism, we need to compare the solutions of this theory with the ones of GR: it may turn out to be convenient to rescale the radial coordinate $r\to\rho$ to recast the physical metric in a form where the angular components of the metric are just the square of a radial coordinate, since the linearized Schwarzschild solution has this form. It is crucial to notice, however, that it is impossible to eliminate completely the field $\tilde{H}$ from the equations. In fact, if we don't use the Stückelberg formalism the theory is not invariant with respect to reparametrizations, and if we perform the coordinate change the field $\tilde{H}$ disappears from the line element but does not disappear from the equations of motion. Using the Stückelberg formalism, instead, the theory is invariant with respect to reparametrizations and the field $\tilde{H}$ itself disappears when we rescale the radius; however, the transformation excites a component of the Stückelberg fields, which is related to $\tilde{H}$ and appears explicitly in the equations of motion. This is analogous to what happens in the non-linear extension of the Fierz-Pauli action considered by Vainshtein in [69], as explained in section (3.2.2).

Vainshtein [69] in fact suggested that the behavior of the system below the Vainshtein radius is in some sense more transparent with the second coordinate choice, in which the angular components of the metric are just the square of the radial coordinate. In particular, he suggested that, inside the Vainshtein radius, the effect of non-linearities on the two remaining components of the physical metric is just to rescale them by a numerical factor, so that they remain small even around and inside the Vainshtein radius. Instead, the Stückelberg field is strongly affected by the non-linearities. Therefore, we perform a coordinate change in the radial coordinate $r \to \rho$ so that in the new coordinate system we have

$$ds^2 = -N(\rho)^2 dt^2 + F(\rho)^{-1} d\rho^2 + \rho^2 d\Omega^2 \quad , \tag{4.18}$$

and we define $\tilde{H}\big(r(\rho)\big) = 1 + h(\rho)$. We also write

$$N(\rho) = 1 + \frac{n(\rho)}{2} \qquad F(\rho) = 1 + f(\rho) \quad , \tag{4.19}$$

which for the time being is just a field redefinition.

As we said above, this change of coordinates excites the perturbations of the Stückelberg fields $Z^\mu$. Since the Stückelberg fields $\phi^\mu$ transform as scalars, after changing coordinates we have[1]

$$y'^\mu(y) - Z^\mu\big(y'(y)\big) = y^\mu - \tilde{Z}^\mu(y) \tag{4.20}$$

and since, before changing coordinates, we were in the unitary gauge, we have $\tilde{Z}^\mu = 0$. The fact that only the the radial coordinate is involved in the transformation implies then

$$Z^t = 0 \tag{4.21}$$

$$Z^\rho(\rho) = \rho - r(\rho) \tag{4.22}$$

$$Z^\theta = 0 \tag{4.23}$$

$$Z^\varphi = 0 \tag{4.24}$$

[1] We indicate with $y^\mu$ and $\tilde{Z}^\mu$ the coordinates and Stückelberg fields in the $(t, r, \theta, \varphi)$ coordinate system, while we indicate with $y'^\mu$ and $Z^\mu$ the coordinates and Stückelberg fields in the $(t, \rho, \theta, \varphi)$ coordinate system.

and, remembering the internal decomposition $Z_\mu = A_\mu + \partial_\mu \phi$ and the fact that $\rho^2 = r^2/\tilde{H}^2$, we have that $A_\mu = 0$ and the only non-zero component of $\partial_\mu \phi$ is

$$\partial_\rho \phi = -\rho\, h(\rho) \quad . \tag{4.25}$$

We conclude that the field $h$ and the scalar component of the Stückelberg fields $\phi$ play exactly the same role in this case: we can then work equivalently with the fields $n$, $f$ and $h$, or with $n$, $f$ and $\dot{\phi} \equiv \partial_\rho \phi$. It will turn out to be more convenient to work with $h$ instead of $\dot{\phi}$, so from now on we will work with the fields $n$, $f$ and $h$.

### 4.1.3 Focusing on the Vainshtein mechanism

Let's first study the behavior around and above the Compton radius $r_c = 1/m$ of solutions which decay at infinity. At linear order in the fields $n$, $f$ and $h$, the physical line element reads

$$ds^2 = -(1+n)\, dt^2 + (1-f)\, d\rho^2 + \rho^2 d\Omega^2 \tag{4.26}$$

and the equations of motion read [105]

$$0 = \left(m^2\rho^2 + 2\right) f + 2\rho \left(\dot{f} + m^2\rho^2 \dot{h} + 3\, m^2 \rho h\right) \tag{4.27}$$

$$0 = \frac{1}{2}\, m^2 \rho^2 (n - 4h) - \rho\, \dot{n} - f \tag{4.28}$$

$$0 = f + \frac{1}{2}\, \rho\, \dot{n} \tag{4.29}$$

where we have indicated derivatives with respect to $\rho$ with an overdot $\dot{}$. The solutions for $n$ and $f$ are

$$n = -\frac{8GM}{3\rho}\, e^{-m\rho} \tag{4.30}$$

$$f = -\frac{4GM}{3\rho}\, (1 + m\rho)\, e^{-m\rho} \tag{4.31}$$

where we fixed the integration constant so that $M$ is the mass of a point particle at the origin, and $8\pi G = M_{pl}^{-2}$. It is apparent that the solutions display the

Yukawa exponential suppression for scales larger than the Compton radius, and for scales smaller than the Compton radius exhibit the vDVZ discontinuity, since the ratio between $n$ and $f$ is 2 in the massless limit $m \to 0$. This results agree with the spherically symmetric solutions in the Fierz-Pauli model we found in section (3.2.2), and are exactly what we expected: since the dRGT massive gravity is a non-linear completion of the Fierz-Pauli theory, the linearized solution of the equations of motion in the former theory should reproduce the solutions of the latter.

We now want to focus on the Vainshtein mechanism. As we already mentioned, the findings of Vainshtein [69] suggest that, when we focus on scales around and below the Vainshtein radius $r_v$, the effects of non-linearities show up mostly in the Stückelberg field, while the gravitational potentials $n$ and $f$ remain small. Therefore, to study the Vainshtein mechanism we decide to treat the gravitational potentials at first order in the equations of motion, and instead keep all the non-linearities in the field $h$. It can be shown [105] that in this approximation the equations of motion reduce to the following system of equations

$$f = -\frac{2GM}{\rho} - (m\rho)^2 \Big[ h - (1 + 3\alpha_3)h^2 + (\alpha_3 + 4\alpha_4)h^3 \Big] \tag{4.32}$$

$$\dot{n} = \frac{2GM}{\rho^2} - m^2\rho \Big[ h - (\alpha_3 + 4\alpha_4)h^3 \Big] \tag{4.33}$$

$$\begin{aligned} \frac{GM}{\rho}\Big[1 - 3(\alpha_3 + 4\alpha_4)h^2\Big] = -(m\rho)^2 \Bigg[ \frac{3}{2}h - 3(1+3\alpha_3)h^2 + \\ + \Big( (1+3\alpha_3)^2 + 2(\alpha_3 + 4\alpha_4) \Big) h^3 - \frac{3}{2}(\alpha_3 + 4\alpha_4)^2 h^5 \Bigg] \end{aligned} \tag{4.34}$$

Note that the field $h$ obeys a decoupled equation, since the gravitational potentials are not present in (4.34): this equation is in fact an *algebraic* equation, and for the sake of precision is a polynomial of fifth degree in $h$. In the following, we will refer to this equation as *the quintic equation*.

There is another way to derive the system of equations above, starting from the decoupling limit Lagrangian (3.195) [105]. As we mentioned in the previous chapter, the decoupling limit leaves the Vainshtein radius fixed and sends the

Compton radius to infinity, while sends the gravitational radius to zero: in some sense, this limit focuses on the scales above the gravitational length and below the Compton wavelength. Also, the decoupling limit selects a subclass of the interaction terms which appear in the action, and sends to zero all the others: these terms can be thought to be the ones which are more relevant regarding the effect of non-linear interactions on the linearized solutions when we focus on scales comparable to the Vainshtein radius. We then expect that there should be a connection between the equations for static, spherically symmetric solutions obtained from the decoupling limit Lagrangian and the equations obtained above.

To see it, it is actually more convenient to work with the fields $\check{h}_{\mu\nu}$ and $\check{\phi}$ , because their dynamics is coupled by just one interaction term, as is apparent in the Lagrangian (3.197). Apart from the interaction term $\propto \check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu}$, the dynamics of the field $\check{\phi}$ is described by a Galileon Lagrangian: as shown in [68], for static and spherically symmetric configurations the equations of motion for a Galileon field can be integrated exactly, obtaining an algebraic equation for $\partial_\rho\check{\phi}/\rho$

$$a_1\left(\frac{\partial_\rho\check{\phi}}{\rho}\right)+a_2\left(\frac{\partial_\rho\check{\phi}}{\rho}\right)^2+a_3\left(\frac{\partial_\rho\check{\phi}}{\rho}\right)^3\propto\frac{M}{4\pi r^3}\quad . \tag{4.35}$$

The coefficients $a_1$, $a_2$ and $a_3$ depend on the coefficients of the Galileon terms in the Lagrangian (3.197): therefore, if we neglect the interaction term $\propto \check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu}$, the equation for $\check{\phi}$ is polynomial in $\partial_\rho\check{\phi}/\rho$ and it is at most a cubic. As shown in [105], the effect of the interaction term $\propto \check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu}$ is to add to the left hand side of the cubic equation above a contribution proportional to

$$\big(8d_5+c_3\big)\left(\frac{\partial_\rho\check{n}}{\rho}\right)\left(\frac{\partial_\rho\check{\phi}}{\rho}\right)^2 \tag{4.36}$$

where $\check{n}=\check{h}_{tt}$, and $8d_5+c_3$ is proportional to $\alpha_3+4\alpha_4$. Varying the action with respect to $\check{h}_{\mu\nu}$, instead, one obtains that the equations of motion for $\check{n}$ and $\check{f}$: these equations imply that $\partial_\rho\check{n}/\rho$ can be expressed as a linear combination of a Newtonian term $GM/\rho^3$ and of a term $\propto \big(\alpha_3+4\alpha_4\big)\big(\partial_\rho\check{\phi}/\rho\big)^3$, which again comes from the interaction term $\check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu}$ in the Lagrangian. Substituting this expression for $\partial_\rho\check{n}/\rho$ in the equation for $\check{\phi}$, one obtains the quintic equation (4.34) for $h=\partial_\rho\check{\phi}/\rho$: in particular, the $h^5$ term in the quintic is generated by substituting this expression for $\partial_\rho\check{n}/\rho$ in (4.36). Therefore, the interaction term $\check{h}^{\mu\nu}\check{X}^{(3)}_{\mu\nu}$ (which is the only one which cannot be removed from the action by a local field redefinition) is responsible for the fact that the degree of the polynomial equation which $\partial_\rho \check{\phi}/\rho$ obey changes from three (as it is in a general Galileon theory) to five. Note however that when $(8d_5 + c_3) \propto (\alpha_3 + 4\alpha_4) = 0$ this coupling vanishes, and the polynomial equation becomes a cubic as in a Galileon theory. It is possible to verify [105] that also the equations (4.32) - (4.33) can be derived from the decoupling limit Lagrangian: this strongly supports the idea that the system of equations (4.32) - (4.34) is a good description of the full theory when we focus on scales comparable to the Vainshtein radius, and therefore this system is the starting point for our analysis of the Vainshtein mechanism in dRGT massive gravity.

## 4.2 The quintic equation

For notational convenience, it is useful to define the parameters $\alpha \equiv 1 + 3\,\alpha_3$ and $\beta \equiv \alpha_3 + 4\,\alpha_4$: in terms of these new parameters, the system (4.32)-(4.34) takes the form

$$f = -2\,\frac{GM}{\rho} - (m\rho)^2 \Big( h - \alpha h^2 + \beta h^3 \Big) \tag{4.37}$$

$$\dot{n} = 2\,\frac{GM}{\rho^2} - m^2 \rho \,\Big( h - \beta h^3 \Big) \tag{4.38}$$

$$\frac{3}{2}\,\beta^2\, h^5(\rho) - \Big(\alpha^2 + 2\beta\Big)\, h^3(\rho) + 3\,\Big(\alpha + \beta A(\rho)\Big)\, h^2(\rho) - \frac{3}{2}\, h(\rho) - A(\rho) = 0 \tag{4.39}$$

where $A(\rho) = \big(\rho_v/\rho\big)^3$ and $\rho_v$ is the Vainshtein radius defined as $\rho_v \equiv \big(GM/m^2\big)^{1/3}$. The new parameters have a clear physical interpretation: in fact, the two combinations of the parameters $c_3$ and $d_5$ which appear in the decoupling limit action (3.197) are easily expressed in terms of $\alpha$ and $\beta$

$$\alpha \propto 6c_3 - 1 \qquad\qquad \beta \propto 8d_5 + c_3 \quad . \tag{4.40}$$

In particular, the case $\beta = 0$ corresponds to a situation where the coupling $\check{h}^{\mu\nu} \check{X}^{(3)}_{\mu\nu}$ is absent and so the field $\check{\phi}$ is exactly a Galileon, while the case $\alpha = 0$ corresponds to a situation where the derivative coupling $\partial_\mu \check{\phi}\, \partial_\nu \check{\phi}\, T^{\mu\nu}$ is absent and so the field

$\check{\phi}$ does not couple to the electromagnetic field. In the case $\alpha = \beta = 0$ all the Galileon self-interaction terms vanish, and in the decoupling limit we are left with a Lagrangian for a free tensor field $\check{h}_{\mu\nu}$ and a free scalar $\check{\phi}$ both of which interact with the energy-momentum tensor *via* non-derivative couplings.

As we already mentioned, the equation (4.39) does not contain the gravitational potentials $n$ and $f$, so $h$ obeys a decoupled equation: furthermore, if we know the solution for $h$, the fields $f$, $n$ are uniquely determined (up to an integration constant) by the other two equations (4.37) and (4.38) in terms of $h$. Therefore, our aim has been to study all the solutions which the equation (4.39) admits, for every value of the parameters $\alpha$ and $\beta$, and characterize their geometrical properties using the equations (4.37)-(4.38). Note that in the particular case of $\beta = 0$, the equation for $h$ becomes a cubic equation and it is possible to obtain solutions for $h$ and the metric perturbations exactly. These solutions were studied in [104, 105] and it was shown that the solutions exhibit the Vainshtein mechanism. Therefore, in what follows, we assume $\beta \neq 0$. Note that a systematic approach to Vainshtein effects in theories which have connections with massive gravity have been performed in [113], regarding covariant Galileon theory, and in [114, 115], regarding general scalar-tensor theories.

### 4.2.1 The quintic equation

The equation of motion for $h$, which we rewrite here

$$\frac{3}{2}\,\beta^2\, h^5(\rho) - \Big(\alpha^2 + 2\beta\Big)\, h^3(\rho) + 3\,\Big(\alpha + \beta A(\rho)\Big)\, h^2(\rho) - \frac{3}{2}\, h(\rho) - A(\rho) = 0 \quad (4.41)$$

is an algebraic equation for $h$, $A$, $\alpha$ and $\beta$; at fixed $\rho$, $\alpha$ and $\beta$ it is, in fact, a polynomial equation of fifth degree in $h$ (except, as we already mentioned, in the special case $\beta = 0$). In the following, we will refer to it as *the quintic equation.* To study the Vainshtein mechanism in this theory, the most convenient thing to do would be to find exact solutions of the quintic equation, derive their physical predictions inside the Vainshtein radius, and determine if they agree with the ones of GR. However, finding exact solutions of this equation is almost impossible: a general theorem of algebra, the *Abel-Ruffini theorem* (see, for example, [116]), states that is impossible to express the general solution of a polynomial equation of degree five or higher in terms of radicals (while it possible for quadratic, cubic and quartic equations). Even if the quintic equation (4.41) lacks of the $h^4$ term, and so it is not the most general quintic equation, it seems arduous to find explicit solutions as a function of $\rho$.

However, it is indeed possible to find explicitly the number and properties of solutions which the quintic equation admits in a neighborhood of $\rho \to +\infty$, which we call the *asymptotic solutions*, and the number and properties of solutions which the quintic equation admits in a neighborhood of $\rho \to 0^+$, which we call the *inner solutions*. This fact offers the possibility to study the Vainshtein mechanism without finding the complete solutions of (4.41). In fact, suppose for example that we are able to show that (for some $\alpha$ and $\beta$) there exists a global solution of (4.41) (*i.e.* a solution which is defined on the domain $\rho \in (0, +\infty)$) which interpolates between an inner solution which reproduces GR results, and an asymptotic solution which displays the vDVZ discontinuity. We can then conclude that the Vainshtein mechanism is working for the theory defined by this choice of parameters. More in general, we can make precise statement on the effectiveness of the Vainshtein mechanism just by characterizing the properties of asymptotic and inner solutions in all the phase space of parameters, and by determining if there are global solutions which interpolates between each couple of asymptotic/inner solutions. In the following, when there is a global solution which interpolates between an inner and an asymptotic solution, we say that there is matching between the two solutions.

This is precisely the approach we take in studying the Vainshtein mechanism in dRGT massive gravity: in sections 4.3 and 4.4 we find exactly the number and properties of asymptotic and inner solutions in every point of the phase space, and in the section 4.5 we discuss the details of the matching between asymptotic and inner solutions. We will not restrict ourselves to asymptotically decaying solutions and to inner solutions which reproduce GR, but we will study the matching properties of all kinds of asymptotic and inner solutions.

It is worthwhile to point out that our starting equations (4.37)-(4.39) were constructed assuming $GM < \rho < 1/m$, but in the following analysis we use the whole radial domain $0 < \rho < +\infty$ . On one hand, this allows us to characterize exactly the number and properties of solutions on large and small scales. On the other hand, the picture we have in mind is that the Compton wavelength of the gravitational field $\rho_c = 1/m$ is of the same order of the Hubble radius today, and that there is a huge hierarchy between $\rho_c$ and the gravitational radius[2] $\rho_g = GM$, *i.e.* $\rho_c/\rho_g \ggg 1$. Therefore, we expect that extending the analysis to the whole radial domain captures the correct physical results.

### 4.2.2 Symmetry of the quintic and dual formulation

#### Symmetry of the quintic

To be able to describe how the matching works in all the phase space, in principle we should study separately every point $(\alpha, \beta)$. However, this is not necessary since equation (4.41) obeys a remarkable symmetry: defining the quintic function as

$$q\left(h, A; \alpha, \beta\right) \equiv \frac{3}{2}\,\beta^2\,h^5 - \left(\alpha^2 + 2\beta\right)h^3 + 3\left(\alpha + \beta A\right)h^2 - \frac{3}{2}\,h - A \tag{4.42}$$

it is simple to see that

$$q\left(\frac{h}{k}, \frac{A}{k}; k\,\alpha, k^2\beta\right) = \frac{1}{k}\,q\left(h, A; \alpha, \beta\right) \tag{4.43}$$

Therefore if a local solution of (4.41) exists for a given $(\alpha, \beta)$ within a certain radial interval, it would also be present for $(k\alpha, k^2\beta)$, for $k > 0$, with $h$ being replaced by $h/k$ and the radial interval rescaled by $1/\sqrt[3]{k}$. As a result, each point belonging to the $\alpha > 0$ part of the parabola $\beta = c\,\alpha^2$ of the phase space (with $c$ any non-vanishing constant) shares the same physics, hence having the same number of global solutions and matching properties. The same is true for the points belonging the $\alpha < 0$ part of the parabola. So, to understand the global structure of the phase space, it is sufficient to analyze one point for each of the half parabolas present in the phase space.

#### Dual formulation

In order to find the asymptotic and the inner solutions, we need to study the quintic equation in the limits $\rho \to +\infty$ and $\rho \to 0^+$. In particular, we will consider both

[2] We are using units where the speed of light speed has unitary value.

decaying and diverging solutions. To do this, it is very useful to formulate the theory in terms of quantities which remain finite in the limit.

Note that the radial coordinate $\rho$ is defined for $\rho \in (0, +\infty)$: this implies that the function $A(\rho)$ is always non-zero, and the map $\rho \to A(\rho)$ is a diffeomorphism[3] of $(0, +\infty)$ into itself. In particular, this means that we can use equivalently $\rho$ and $A$ as radial coordinates: the latter choice is more convenient to study asymptotic solutions, since the limit $\rho \to +\infty$ is expressed as the limit $A \to 0^+$. Furthermore, it will be useful to work with dimensionless radial coordinates, at least as far as only the solutions of the quintic are concerned, so instead of $\rho$ we will often use the coordinate $x \equiv \rho/\rho_v$ and, as we mentioned, $A = 1/x^3$.

The fact that $A$ is always different from zero implies that a solution $h$ of (4.41) never vanishes in the domain of definition, since the quintic function (4.42) for $h = 0$ is equal to $A$. Therefore, we can divide the quintic equation by $h^5$ obtaining the following quintic equation for $v \equiv 1/h$

$$d\big(v, A; \alpha, \beta\big) \equiv A\, v^5 + \frac{3}{2}\, v^4 - 3\,\big(\alpha + \beta A\big)\, v^3 + \big(\alpha^2 + 2\beta\big)\, v^2 - \frac{3}{2}\, \beta^2 = 0 \quad : \tag{4.44}$$

since we are considering the $\beta \neq 0$ case, every solution to the new quintic (4.44) is again never vanishing. It follows that, if we find a solution $h$ of the "original" quintic equation (4.41), then its reciprocal $1/h$ is a solution of the "new" quintic (4.44), and conversely the reciprocal of every solution of (4.44) is a solution of (4.41). This implies that it is completely equivalent to work with the field $h$ or with the field $v$: the quintic equation (4.44), together with the equations which we obtain substituting $h = 1/v$ in the equations (4.37)-(4.38), provides a completely equivalent formulation of the (decoupling limit) theory defined by the equations (4.37)-(4.39). We will refer to the formulation in terms of $v$ as the dual formulation.

It will be useful, especially when studying inner solutions, to work with the $x$ coordinate: to derive the quintic equations in terms of $x$, we can divide the quintic equation (4.41) by $A$ obtaining the following quintic equation

$$b\big(h, x; \alpha, \beta\big) \equiv x^3 \left(\frac{3}{2}\, \beta^2\, h^5 - \big(\alpha^2 + 2\beta\big)\, h^3 + 3\,\alpha\, h^2 - \frac{3}{2}\, h\right) + 3\,\beta\, h^2 - 1 = 0 \tag{4.45}$$

Furthermore, dividing the equation above by $h^5$ we obtain the quintic in the dual

[3] By diffeomorphism we mean a smooth and invertible function whose inverse is smooth.

formulation in terms of the radial coordinate $x$

$$g\big(v,x;\alpha,\beta\big) \equiv v^5 + \frac{3}{2}\,x^3\,v^4 - 3\,\big(\beta + \alpha\,x^3\big)\,v^3 + \big(\alpha^2 + 2\beta\big)\,x^3\,v^2 - \frac{3}{2}\,\beta^2\,x^3 = 0 \quad (4.46)$$

This four quintic equations provide equivalent descriptions of the same problem, when $\beta \neq 0$. Note that the dual formulation is more suited to discuss the $\beta \to 0$ limit of our results and the connection with exact results of the $\beta = 0$ case [105], since the quintic equations in the dual formulation remain of degree five even in the $\beta \to 0$ limit.

# 4.3 Asymptotic and inner solutions

We turn now to the study of asymptotic and inner solutions of the quintic equation (4.41), in the $\beta \neq 0$ case. Note that interesting results about asymptotic and inner solutions of the quintic equation have been obtained in [105] and [117], however the existence of the solution was not proved there. Furthermore, an exact characterization of the number of asymptotic and inner solutions in the phase space is missing in these papers. See also [118] for related studies on the phenomenology of solutions in this branch of massive gravity.

## 4.3.1 Asymptotic solutions

Let's suppose that a solution $h(\rho)$ of the quintic equation (4.41) exists in a neighborhood of $\rho = +\infty$, and that it has a well defined limit as $\rho \to +\infty$. We can immediately conclude that this solution cannot be divergent. In fact, suppose that indeed the solution is divergent $|\lim_{\rho\to+\infty} h(\rho)| = +\infty$: in the dual formulation, this corresponds to the case $\lim_{A\to 0} v(A) = 0$. Performing the limit $A \to 0$ in the quintic (4.44) one obtains $\beta = 0$, which is precisely against our initial assumption. Therefore, asymptotic solutions of the quintic equation (4.41) have to be finite.

Suppose now that $\lim_{\rho\to+\infty} h(\rho)$ is finite, and let's call it $C$. Then both of the sides of the quintic equation (4.41) have a finite limit when $\rho \to +\infty$, and taking this limit one gets

$$\frac{3}{2}\,\beta^2\,C^5 - \big(\alpha^2 + 2\beta\big)\,C^3 + 3\,\alpha\,C^2 - \frac{3}{2}\,C = 0 \quad . \qquad (4.47)$$

It follows then that the allowed asymptotic values at infinity for $h(\rho)$ are the roots of the following equation, which we call the *asymptotic equation*

$$\mathscr{A}(y) \equiv \frac{3}{2}\,\beta^2\,y^5 - \left(\alpha^2 + 2\beta\right) y^3 + 3\,\alpha\,y^2 - \frac{3}{2}\,y = 0 \quad . \tag{4.48}$$

Note that $y = 0$ is always a root of this equation, and in fact a simple root (*i.e.* a root of multiplicity one) since $\frac{d}{dy}\mathscr{A}(0) = -3/2 \neq 0$. Dividing by $y$, one obtains that the other asymptotic values for $h(\rho)$ are the roots of the *reduced asymptotic equation*

$$\mathscr{A}_r(y) \equiv \frac{3}{2}\,\beta^2\,y^4 - \left(\alpha^2 + 2\beta\right) y^2 + 3\,\alpha\,y - \frac{3}{2} = 0 \quad . \tag{4.49}$$

This last equation is a quartic, so it can have up to 4 (real) roots, depending on the specific values of $\alpha$ and $\beta$. Since

$$\lim_{y\to-\infty} \mathscr{A}_r(y) = +\infty \qquad \mathscr{A}_r(0) = -\frac{3}{2} < 0 \qquad \lim_{y\to+\infty} \mathscr{A}_r(y) = +\infty \quad , \tag{4.50}$$

we have, by the intermediate value theorem (see, for example, [119]), that the reduced asymptotic equation has always at least two roots, one positive and one negative. For the same reason, it cannot have two positive and two negative roots, since at each simple root the quartic function changes sign.

As we show in the appendix E, in the regions of the phase space below the parabola $\beta = c_-\,\alpha^2$ and above the parabola $\beta = c_+\,\alpha^2$ the asymptotic equation has three real roots, which are simple roots, while in the regions $c_-\,\alpha^2 < \beta < 0$ and $0 < \beta < c_+\,\alpha^2$ the asymptotic equation has five real roots, which are again simple roots. Note that $c_+ = 1/4$ and $c_-$ is the only real root of the equation $8 + 48\,y - 435\,y^2 + 676\,y^3 = 0$. On the two parabolas $\beta = c_\pm\,\alpha^2$ (which we call the five-roots-at-infinity parabolas) there are four roots, one of which is a root of multiplicity two. This is summarized in figure 4.1.

We name the roots in the following way: the $y = 0$ root is denoted as $\mathbf{L}$. For the phase space points where there are just three roots, the positive root is denoted as $\mathbf{C}_+$ and the negative one as $\mathbf{C}_-$ . For points in the five-roots regions, we adopt the following convention. Be $(\alpha_5, \beta_5)$ a point where there are five roots. In the same quadrant of the phase space, take another point $(\alpha_3, \beta_3)$ where there are three roots, and a path $\mathscr{C}$ which connects the two points. Following the path $\mathscr{C}$, two of the four non-zero roots of $(\alpha_5, \beta_5)$ smoothly flow to the non-zero roots of

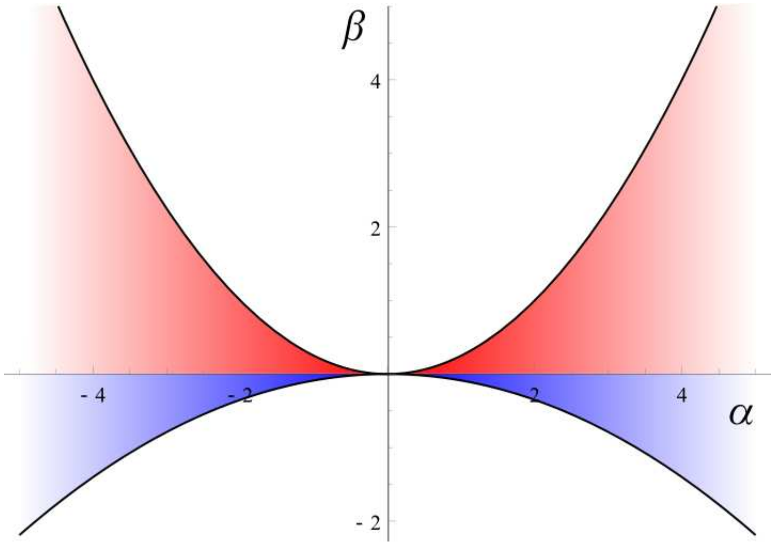


Figure 4.1: phase space diagram for the number of asymptotic solutions

$(\alpha_3, \beta_3)$, and are denoted as $\mathbf{C}_+$ and $\mathbf{C}_-$ themselves. The other two non-zero roots of $(\alpha_5, \beta_5)$, instead, disappear when (following $\mathscr{C}$) the boundary of the five-roots region is crossed, and are denoted as $\mathbf{P}_1$ and $\mathbf{P}_2$. We adopt the convention that $|\mathbf{P}_1| \leq |\mathbf{P}_2|$. The definition is independent of the particular choice of the point $(\alpha_3, \beta_3)$ and of the path $\mathscr{C}$ used. A careful study of the asymptotic equation and of its derivatives permits to show that we have $\mathbf{C}_- < \mathbf{C}_+ < \mathbf{P}_1 < \mathbf{P}_2$ for $\alpha > 0$ and $\mathbf{P}_2 < \mathbf{P}_1 < \mathbf{C}_- < \mathbf{C}_+$ for $\alpha < 0$. On the boundaries $\beta = c_\pm \alpha^2$ we have $\mathbf{P}_1 = \mathbf{P}_2 \equiv \mathbf{P}$.

### 4.3.2 Inner solutions

Suppose now that a solution of the quintic equation exists in a neighborhood of $\rho = 0^+$ (possibly not defined in $\rho = 0$), and that it has a well defined limit when $\rho \to 0^+$. We can immediately see that such a solution cannot tend to zero as $\rho \to 0^+$. In fact, suppose that indeed the solution tends to zero $\lim_{x \to 0^+} h(x) = 0$: taking the limit in the quintic equation (4.45), we get $-1 = 0$ which contradicts our assumption. Therefore, if $h(\rho)$ is an inner solution then $\lim_{\rho \to 0^+} h(\rho) \neq 0$.

This means that, in the dual formulation, all the inner solutions $v(x)$ have a finite limit for $x \to 0^+$. Considering the quintic in the dual formulation (4.46), the permitted limiting values for a inner solution $v$ are then the roots of the equation obtained performing the limit $x \to 0^+$ in the quintic (4.46), namely

$$v^5 - 3\,\beta\, v^3 = 0 \tag{4.51}$$

For $\beta > 0$ there are three roots, namely $v_0 = 0$, $v_+ = +\sqrt{3\,\beta}$ and $v_- = -\sqrt{3\,\beta}$ ; for $\beta < 0$, instead, there is only the root $v = 0$. Therefore, the permitted limiting behaviors for $h$ when $\rho \to 0^+$ are

$$|h(\rho)| \to +\infty \tag{4.52}$$

for $\beta \neq 0$, and

$$h \to \mathbf{F}_\pm \equiv \pm\sqrt{\frac{1}{3\,\beta}} \tag{4.53}$$

only for $\beta > 0$.

### 4.3.3 Existence of the asymptotic and inner solutions

Note that so far we have not proved that inner and asymptotic solutions exist, but just found the values that have to be the limit of these solutions if they exist. The existence and uniqueness of solutions can be proved applying the *implicit function theorem* (known also as Dini's theorem) which we enunciate in appendix B. Regarding asymptotic solutions, to apply the implicit function theorem we can artificially extend the domain of definition of the equation (4.41) to $A < 0$ as well: apart from the five-roots-at-infinity boundaries, all the asymptotic roots are simple roots. Therefore we can apply the implicit function theorem, which tells us that there exist a local solution of (4.41) associated to every root of the asymptotic equation: restricting now the domain of definition of these local solutions to $A > 0$, we obtain the desired asymptotic solutions to the quintic equation. It follows that to each of the asymptotic roots $\mathbf{L}$, $\mathbf{C}_+$, $\mathbf{C}_-$, $\mathbf{P}_1$ and $\mathbf{P}_2$ we can associate a local solution of the quintic equation in a neighborhood of $\rho \to +\infty$, and we indicate the root and the associate local solution with the same letter.

On the five-roots-at-infinity boundaries, a separate analysis is needed for the double root $\mathbf{P}_1 = \mathbf{P}_2 \equiv \mathbf{P}$. It can be shown that for $\alpha > 0$ and $\beta = c_+\,\alpha^2$ there are no local solutions of (4.41) which tend to $\mathbf{P}$ when $\rho \to +\infty$, and the same holds for $\alpha < 0$ and $\beta = c_-\,\alpha^2$. On the other hand, for $\alpha > 0$ and $\beta = c_-\,\alpha^2$ there are *two* different local solutions of (4.41) which tend to $\mathbf{P}$ when $\rho \to +\infty$, and the same holds for $\alpha < 0$ and $\beta = c_+\,\alpha^2$. Despite having the same limit for $\rho \to +\infty$, these two local solutions are different when $A \neq 0$: we then call $\mathbf{P}_1$ the solution which in absolute value is smaller, and $\mathbf{P}_2$ the solution which in absolute value is bigger. Therefore, on the boundaries between the three-roots-at-infinity regions and the five-roots-at-infinity regions, for $\alpha \gtrless 0$, $\beta = c_\pm\,\alpha^2$ there are three asymptotic solutions of (4.41), while for $\alpha \gtrless 0$, $\beta = c_\mp\,\alpha^2$ there are five asymptotic solutions of (4.41).

Regarding inner solutions, the existence of local solutions in a neighborhood of $\rho = 0^+$ associated to the limiting values $\mathbf{F}_+$ and $\mathbf{F}_-$ can be proved extending the validity of (4.46) to $x < 0$ and applying the implicit function theorem at $(v = \pm\sqrt{3\beta}, x = 0)$. Restricting then to $x > 0$ the domain of definition of the solutions obtained this way, we get two local solutions $v_\pm(x)$ of (4.46) which tend to $\pm\sqrt{3\beta}$ as $\rho \to 0^+$: the reciprocal $h_\pm(\rho) = 1/v_\pm(x(\rho))$ of these solutions are local solutions of the quintic (4.41) in a neighborhood of $\rho \to 0^+$, and are the inner solutions associated to $\mathbf{F}_\pm$. We will use $\mathbf{F}_\pm$ to denote both the limiting values and the inner solutions associated to the limiting values. For the solution associated to the limiting value $v = 0$, we cannot apply the implicit function theorem straightaway, because the function $g\big(v, x; \alpha, \beta\big)$ is such that $\frac{\partial g}{\partial v} = 0$ in $(v, x) = (0, 0)$. However, using the results of appendix C, it can be shown that, for $\beta > 0$, there always exists a neighborhood of $A \to +\infty$ where there is a simple root of the quintic (4.41) which is $< \mathbf{F}_-$ and decreases when $A$ increases. Applying the implicit function theorem to (4.41) in this neighborhood of $A \to +\infty$, we obtain a local solution of (4.41) which corresponds to the limiting value $v = 0$, which will be denoted by $\mathbf{D}$. For $\beta < 0$, instead, there always exists a neighborhood of $A \to +\infty$ where there is a simple root of the quintic (4.41) which is $> \mathbf{F}_+$ and increases when $A$ increases. Analogously to the $\beta > 0$ case, applying the implicit function theorem to (4.41) in this neighborhood we obtain a local solution of (4.41) which corresponds to the limiting value $v = 0$, which will be denoted as well by $\mathbf{D}$.

## 4.4 Characterization of the asymptotic and inner solutions

We sum up here the results obtained in the previous section on the existence and properties of asymptotic and inner solutions of eq. (4.41), together with their leading behaviors and geometrical meaning. We refer to the appendix F for the derivation of the leading behaviors.

### 4.4.1 Asymptotic solutions

In a neighborhood of $\rho \to +\infty$ there are, depending on the value of $(\alpha, \beta)$, three or five solutions to eq. (4.41). In particular:

- There is always a decaying solution, which we indicate with **L**. Its asymptotic behavior is

$$h(\rho) = -\frac{2}{3}\left(\frac{\rho_v}{\rho}\right)^3 + R(\rho) \tag{4.54}$$

where $\lim_{\rho\to+\infty} \rho^3 R(\rho) = 0$. This solution corresponds to a spacetime which is asymptotically flat, as one can see from eqs. (4.37)-(4.38).

- Additionally, there are two or four solutions to eq. (4.41) which tend to a finite, nonzero value as $\rho \to +\infty$. We name these solutions with $\mathbf{C}_+$, $\mathbf{C}_-$, $\mathbf{P}_1$ and $\mathbf{P}_2$. Their asymptotic behavior is

$$h(\rho) = C + R(\rho) \tag{4.55}$$

where $\lim_{\rho\to+\infty} R(\rho) = 0$ and $C$ is a root of the reduced asymptotic equation (4.49). From eqs. (4.37)-(4.38), one can get convinced that these solutions correspond to spacetimes which are asymptotically non-flat. Interestingly, the leading term in the gravitational potentials scales as $\rho^2$ for large radii, the same scaling which we find in a de Sitter spacetime. It is worthwhile to point out that, since we are working on scales below the Compton wavelength of the gravitational field, "asymptotically non-flat" really means that (from the point of view of the full and non-approximated theory) the spacetime correspondent to this solution tends to a non-flat spacetime when the Compton wavelength is approached. To understand the "true" asymptotic behavior of this solution, one should use the non-approximated equations. Note that, even if $C$ (and so $h$) is much smaller than one, the gravitational potentials $n$ and $f$ can be very large (as they behave like $\propto \rho^2$ far from the origin in this case): therefore, the linear approximation (for the gravitational potentials) we used to obtain eqs. (4.30)-(4.31) is not valid. Instead, the asymptotic fate of the solution is dictated by the nonlinear behavior of the non-approximated equations. This seems not easy to predict without a separate analysis, and we don't attempt to address this interesting problem.

### 4.4.2 Inner solutions

In a neighborhood of $\rho \to 0^+$ there are either one or three solutions to eq. (4.41). For $\beta > 0$ there are exactly three inner solutions, while for $\beta < 0$ there is only one inner solution. In particular:

- There is always a diverging solution, which we denote by **D**. Its leading behavior is
$$h(\rho) = -\sqrt[3]{\frac{2}{\beta}}\,\frac{\rho_v}{\rho} + R(\rho) \tag{4.56}$$
where $\lim_{\rho\to 0^+}(R(\rho)/\rho)$ is finite. This solution exists for both $\beta > 0$ and $\beta < 0$, with opposite signs for each case. Using this solution in eqs. (4.37)-(4.38), one realizes that the $h^3$ term cancels the $GM/\rho$ term, so the gravitational field is self-shielded and does not diverge as $\rho \to 0^+$. This solution is in strong disagreement with gravitational observations.

- For $\beta > 0$, there are two additional solutions to eq. (4.41), which tend to a finite, non-zero value as $\rho \to 0^+$. We indicate these solutions by $\mathbf{F}_+$ and $\mathbf{F}_-$. Their leading behavior is
$$h(\rho) = \pm\sqrt{\frac{1}{3\,\beta}} + R(\rho) \tag{4.57}$$
where $\lim_{\rho\to 0^+} R = 0$. Notice that for $\beta < 0$ there are no solutions to eq. (4.41) which tend to a finite value as $\rho \to 0^+$.

The expressions (4.37)-(4.38) for the gravitational potentials imply that the metric associated to these solutions ($\mathbf{F}_+$ and $\mathbf{F}_-$) approximate the linearized Schwarzschild metric as $\rho \to 0^+$.

From the behavior of the inner solutions, one concludes that only in the $\beta > 0$ part of the phase space solutions may exhibit the Vainshtein mechanism, but not necessarily for all values of $\alpha$. In the next subsection we see more in detail how this mechanism works.

### 4.4.3 Vainshtein mechanism and solutions matching

In order to study where in the phase space the Vainshtein mechanism works, it is useful to compare the gravitational potentials $f$ and $n$ with their counterparts in the GR case. In the weak field limit, the Schwarzschild solution of GR reads

$$ds^2 = -\left(1 - \frac{2GM}{\rho}\right) dt^2 + \left(1 + \frac{2GM}{\rho}\right) d\rho^2 + \rho^2\, d\Omega^2 \tag{4.58}$$

so by calling $f_{GR} = n_{GR} = -2GM/\rho$ we obtain

$$\frac{f}{f_{GR}} = 1 + \frac{1}{2}\left(\frac{\rho}{\rho_v}\right)^3 \left(h - \alpha h^2 + \beta h^3\right) \tag{4.59}$$

$$\frac{n'}{n'_{GR}} = 1 - \frac{1}{2}\left(\frac{\rho}{\rho_v}\right)^3 \left(h - \beta h^3\right) \tag{4.60}$$

Let us now first discuss the asymptotic solutions. For the decaying solution $\mathbf{L}$, we have that the linear contribution in $h$ rescales the coefficients of the Schwarzschild-like terms, so we obtain $f/f_{GR} \to 2/3$ and $n'/n'_{GR} \to 4/3$ for $\rho \to +\infty$. For the non-decaying solutions $\mathbf{C}_\pm$ and $\mathbf{P}_{1,2}$, the leading behavior for $f/f_{GR}$ and $n'/n'_{GR}$ is proportional to $(\rho/\rho_v)^3$ in both cases, however the proportionality coefficients generally differ since they have a different functional dependence on $\alpha$ and $\beta$. There are some special cases for $(\alpha, \beta)$ where these asymptotic solutions lead to $f/n \to 1$ as $\rho \to +\infty$, and therefore have the same behavior as in a de Sitter spacetime.

Consider instead the inner solutions. For the finite solutions $\mathbf{F}_\pm$ we obtain $(f/f_{GR}) \to 1$ and $(n'/n'_{GR}) \to 1$ as $\rho \to 0^+$, where the corrections scale like $\rho^3$.

On the contrary, for the diverging solution $\mathbf{D}$, the cubic terms in $h$ cancel out the contribution coming from the Schwarzschild-like terms, as explained above, and so $(f/f_{GR}) \to 0$ and $(n'/n'_{GR}) \to 0$ when $\rho \to 0^+$. In this case, corrections are linear in $\rho$.

Therefore, any global solution of equation (4.41) which interpolates between $\mathbf{L}$ and $\mathbf{F}_\pm$ provides a realization of the Vainshtein mechanism in an asymptotically flat spacetime, whereas an interpolation between $\mathbf{C}_\pm$ or $\mathbf{P}_{1,2}$ with $\mathbf{F}_\pm$ exhibits the Vainshtein mechanism in an asymptotically non-flat spacetime. Furthermore, notice that any asymptotic solution which interpolates with the inner solution $\mathbf{D}$ does no lead to the Vainshtein mechanism. These matchings will be explicitly exposed in the next section.

## 4.5 Phase space diagram for solutions matching

In the previous section, we characterized the number and properties of asymptotic and inner solutions in all the phase space. As we mentioned in section (4.2), to make precise statements about the effectiveness of the Vainshtein mechanism it is enough to establish (for every point of the phase space) which asymptotic solution is connected to which inner solution by a global solution which interpolates between them. The aim of this section is to study the matching of asymptotic and inner solutions in all the phase space.

### 4.5.1 Local solutions and the shape of the quintic

Since finding exact solutions of the quintic equation is extremely difficult, we need another method to determine, given a fixed asymptotic solution and a fixed inner solution, if there exists a global solution interpolating between them. To explain how this can be done, let's first of all note that we may see the quintic function (4.42), which is a function of two variables (when we keep $\alpha$ and $\beta$ fixed), as a collection of functions of $h$ whose shape depend continuously on a parameter $A$. This idea can be formalized introducing the *shape function* $q_A\left(h;\alpha,\beta\right)$ which is defined as

$$q_A\left(h;\alpha,\beta\right) = q\left(h,A;\alpha,\beta\right) \quad : \tag{4.61}$$

the shape function is a function of $h$ only, and essentially, given a value of $A$, it is the quintic in $h$ which one obtains keeping fixed $A$ in the quintic function (4.42). At every $A$, the shape function has a certain set of roots $\{r_i(A)\}_i$, which change continuously when $A$ changes: if $h(A)$ is a solution of the quintic equation, by definition $h(A)$ describes the continuous flow with $A$ of a particular zero of the shape function. Note that we study the flow with $A$ at $\alpha$ and $\beta$ fixed, so for simplicity from now on we will omit to write the dependence from $\alpha$ and $\beta$.

We would like to follow the opposite path, and infer the existence of a solution of the quintic equation from the study of the flow of the zeros of the shape function. This is indeed possible thanks to the implicit function theorem (see appendix B). In fact, if we start from a fixed $\bar{A}$ and find a simple zero $\bar{h}$ of the shape function, the implicit function theorem tells us that there exists a (local) solution $\bar{h}(A)$ of the quintic equation, which is defined in a neighborhood of $\bar{A}$, and which describes the flow with $A$ of the zero $\bar{h}$ we started with. Moreover, as we explain in the appendix B, there is a criterion which permits to infer the existence of global solutions of the quintic equation: if the flow of zero $\bar{h}$ is such that the zero remains simple[4] for every value of $A$, then the local solution $\bar{h}(A)$ can be extended maximally to a global solution. Therefore, we are in principle able to find global solutions to the quintic equation just by studying how the shape of $q_A\left(h\right)$ evolves with $A$.

### 4.5.2 Creation and annihilation of local solutions

Let's consider instead what happens when, extending a local solution $h(A)$, we reach a point $\tilde{A}$ when $dq_A/dh = 0$ and so the zero of the shape function is not simple. This situation geometrically means that the shape function has a stationary point on the $h$ axis. Consider for example the case where the shape function has a local minimum below the $h$ axis, and there are two zeros around the minimum. If this minimum translates upwards when $A$ increases and eventually crosses the $h$ axis at a certain $A = \tilde{A}$, the two zeros join together and disappear at the axis crossing: it follows that the two local solutions $h_{12}(A)$ associated to the zeros stop existing at $A = \tilde{A}$. When this happens, by (B.3) the derivative $dh_{12}/dA$ diverges

[4]We say that a zero $\bar{h}$ of the shape function $q_A\left(h\right)$ is simple if $\bar{h}$ is a simple root of the equation $q_A\left(h\right) = 0$.

at $A = \tilde{A}$, but the functions $h_{12}(A)$ remain bounded. The same happens when a local maximum of the shape function crosses the $h$ axis translating downwards. We will say in these cases that two local solution "annihilate" at $A = \tilde{A}$. If instead a local minimum of the shape function translates downwards when $A$ increases and crosses the $h$ axis at a certain $A = \tilde{A}$, two new zeros appear at $A = \tilde{A}$ and therefore two local solutions $h_{12}(A)$ of the quintic equation start existing at $A = \tilde{A}$: again, by (B.3) the derivative $dh_{12}/dA$ diverges at the point $A = \tilde{A}$, but the values of the functions remain bounded. The same happens if a local maximum of the shape function translates upwards and crosses the $h$ axis. We will say in these cases that two local solution "are created" at a certain $A = \tilde{A}$. The creation and annihilation of local solutions and its relation with local maxima and minima of the shape function is well illustrated in figure 4.7 and in figure 4.8.

The phenomenon of creation and annihilation of local solutions is found to be a general feature of the phenomenology of equation (4.41). In fact, in most part of the phase space the number of asymptotic solution is different from the number of inner solutions: the reason why some of these solutions cannot be continued to all the radial domain $0 < \rho < +\infty$ is always that they annihilate with some other local solution. Note that, in general, the solutions are created and annihilated in pairs, and the pairs of solutions have infinite slope when they are created or they annihilate. Anyway, a note of caution is in order: the fact that a stationary point appears on the $h$ axis does not necessarily means that a solution disappears or is created. For example, if a horizontal inflection point of the shape function crosses the $h$ axis, then there is a value $A = \tilde{A}$ where there is a stationary point on the $h$ axis, and the implicit function theorem cannot be applied. Nevertheless, in this case the solution continues existing, even if at $A = \tilde{A}$ it has an infinite first derivative.

It is crucial to point out that, since the first derivative of a local solution of the quintic equation diverges at a creation/annihilation point, the gravitational potentials associated with this solution have diverging derivatives themselves at this point. This implies that, when a creation/annihilation point is approached, the approximations we used to derive the system of equations (4.37)-(4.39) does not hold anymore (*i.e.* the linear approximation on the gravitational potentials), and to understand what happens to the spacetime described by this solutions we should study the full theory. We don't attempt to do this, and therefore we cannot say anything about what happens to the spacetimes described by local solutions of the quintic equation which in our analysis cannot be extended to the complete radial domain.

### 4.5.3 Analysis strategy

Our analysis strategy is therefore the following: for every point of the phase space, we start from the zeros of the shape function at infinity $A = 0$ (*i.e.* from the roots of the asymptotic equation), and we follow the evolution of the shape function when $A$ goes form zero to $+\infty$. In this way, we determine which asymptotic solutions flow into an inner solution, and we determine which asymptotic solutions matches which inner solution. The study is done in three different ways.

On one hand, we study analytically the evolution of the shape function, in particular focusing on the evolution of the number and position of its inflection points. In many cases, the study of the position of the inflection points is enough to establish that in a certain interval of values for $h$ there always (*i.e.* for every value of $A$) exists one simple zero of the shape function, thereby proving analytically the existence of the global solution of the quintic equation which corresponds to this zero. For this study it is necessary to characterize precisely the properties of the shape function at infinity, and the evolution of its properties when $A$ goes from zero to $+\infty$: the details of the study of these properties are given in the appendices C and D.

On the other hand, we plot numerically the shape function and continuously change the value of $A$ (of course, since it is a numerical procedure the modulation is not really continuous but procedes by small finite steps). Despite being less rigorous than the former procedure, this allows to visualize in a very efficient way the evolution of the shape function. Note that, as we explain in the appendix C, there is no need to follow the evolution till $A \to +\infty$ because for every $\alpha$ and $\beta$ there is a critical value $A_{crit}$ (which depends on $\alpha$ and $\beta$) such that for $A > A_{crit}$ there are no more creations/annihilations of solutions, and so from the shape function at $A = A_{crit}$ one can infer unambiguously the matching of the solutions. Note that, since $h$ is defined on $(-\infty, +\infty)$, we don't plot the shape function $q_A(h)$ itself but its composition with the tangent function $q_A\big(tg(h)\big)$: this has the effect of compactifying the real axis into the interval $(-\pi/2, +\pi/2)$, and at the same time does not change the number and the relative order of the zeros.

Finally, we check the results of these two (somehow complementary) methods by solving with the software Mathematica© for symbolic and numeric calculations[5] the condition of the presence of a stationary point on the $h$ axis. More precisely, we impose the condition that there exist a couple of values $(h,A)$ where both the shape function $q_A(h;\alpha,\beta)$ and its first derivative $dq_A/dh$ vanish: solving this condition gives constraints on the values for $\alpha$ and $\beta$, and identifies the regions of the phase space where solution can annihilate/be created.

These three different approaches permit us to characterize the solution matching in a detailed way, and in the next section we present our results.

### 4.5.4 Phase space diagram

The phase space diagram which displays our results about solution matching is given in figure 4.2. We discuss separately the $\beta > 0$ and $\beta < 0$ part of the phase space, and refer to the figure for the numbering of the regions. The notation $\mathbf{I} \leftrightarrow \mathbf{A}$ means that there is matching between the inner solution **I** and the asymptotic solution **A**.

$\beta < 0$

In this part of the phase space, there is only one inner solution, **D**, so there can be at most one global solution to (4.41). There are three distinct regions which differ in the way the matching works:

- region 1: $\mathbf{D} \leftrightarrow \mathbf{C}_+$. In this region, there are three or five asymptotic solutions, and only one of them, $\mathbf{C}_+$, is positive. This solution is the one which connects with the inner solution **D**, which is also positive, leading to the only global solution of eq. (4.41). The boundaries of this region are the line $\beta = 0$ for $\alpha < 0$ and the parabola $\beta = c_{12}\,\alpha^2$ for $\alpha > 0$, where $c_{12}$ is the negative[6]

[5] `http://www.wolfram.com/mathematica/`

[6] The equation $-4 - 8\,y + 88\,y^2 - 1076\,y^3 + 2883\,y^4 = 0$ has only two real roots, one positive and one negative.

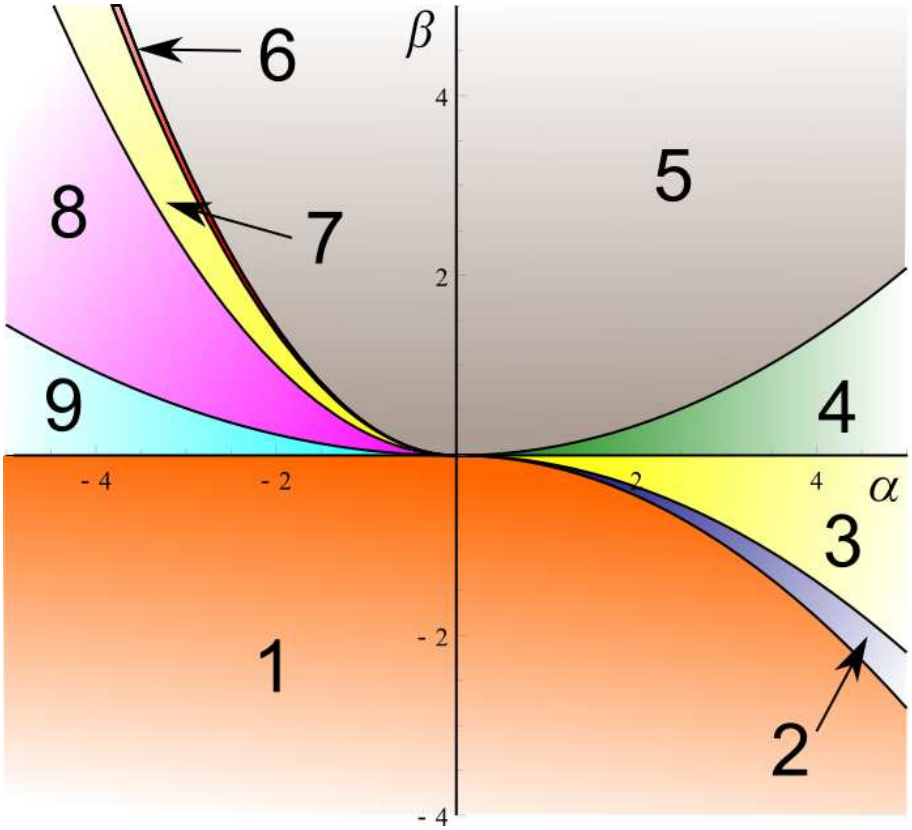


Figure 4.2: Phase space diagram in $(\alpha, \beta)$ for the solutions to the quintic equation (4.41) in $h$, where the different regions show different matching of inner solutions to asymptotic ones. The lines splitting the regions are half parabolas ($\beta \propto \alpha^2$, with $\alpha > 0$ or $\alpha < 0$) due to rescaling symmetry of eq. (4.41).

root of the equation $-4-8\,y+88\,y^2-1076\,y^3+2883\,y^4=0$ (approximately, $c_{12}\simeq -0.1124$). On the boundary $\beta = c_{12}\,\alpha^2$ the matching $\mathbf{D}\leftrightarrow\mathbf{C}_+$ still holds, however the solution $h(\rho)$ displays an inflection point with vertical tangent.

- region 2: **No matching**. In this region there are three asymptotic solutions. However, none of them can be extended all the way to $\rho\to 0^+$, and so, despite the fact that local solutions exist both at infinity and near the origin, equation (4.41) does not admit any global solution. The boundaries of this region are the parabola $\beta = c_{12}\,\alpha^2$ and the (negative) five-roots-at-infinity parabola $\beta = c_-\,\alpha^2$, where $c_-$ is the only real root of the equation $8+48\,y-435\,y^2+676\,y^3=0$ (approximately, $c_-\simeq -0.0876$).
- region 3: $\mathbf{D}\leftrightarrow\mathbf{P}_2$. This region coincides with the $\alpha>0$, $\beta<0$ part of the five roots at infinity region of the phase space (see fig. 4.1). The largest positive asymptotic solution, $\mathbf{P}_2$, is the one which connects to $\mathbf{D}$, leading to the only global solution of eq. (4.41). On the boundary $\beta = c_-\,\alpha^2$ the matching $\mathbf{D}\leftrightarrow\mathbf{P}_2$ still holds, but the solution $h$ seen as a function of $A$ has infinite derivative in $A=0$.

$\beta>0$

In this part of the phase space, there are three inner solutions, $\mathbf{D}$, $\mathbf{F}_+$ and $\mathbf{F}_-$, so there can be at most three global solutions to eq. (4.41). There are six distinct regions with different matching properties:

- region 4: $\mathbf{F}_-\leftrightarrow\mathbf{L}$ , $\mathbf{D}\leftrightarrow\mathbf{C}_-$. This region lies inside the $\alpha>0$, $\beta>0$ part of the five roots at infinity region of the phase space (see fig. 4.1), so there are five asymptotic solutions. Of the five asymptotic solution, $\mathbf{C}_-$ and $\mathbf{L}$ can always be extended to $\rho\to 0^+$, while $\mathbf{C}_+$, $\mathbf{P}_1$ and $\mathbf{P}_2$ cannot. So there are just two global solutions to eq. (4.41). The boundaries of this region are the parabola $\beta = c_{45}\,\alpha^2$, where $c_{45}=1/12\simeq 0.0833$, and the line $\beta=0$. On the boundary $\beta = c_{45}\,\alpha^2$ there is the additional matching $\mathbf{F}_+\leftrightarrow\mathbf{C}_+$, and the correspondent solution is $h(\rho)=const=+\sqrt{1/3\,\beta}\,$.

- region 5: $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$ , $\mathbf{F}_- \leftrightarrow \mathbf{L}$, $\mathbf{D} \leftrightarrow \mathbf{C}_-$. In this region there are three or five asymptotic solutions; $\mathbf{C}_-$ , $\mathbf{C}_+$ and $\mathbf{L}$ can always be extended to $\rho \to 0^+$, while $\mathbf{P}_1$ and $\mathbf{P}_2$ , where present, cannot. So there are three global solutions to (4.41). The boundaries of this region are the parabola $\beta = c_{45}\,\alpha^2$ for $\alpha > 0$ and the parabola $\beta = c_{56}\,\alpha^2$ for $\alpha < 0$, where $c_{56} = (5+\sqrt{13})/24 \simeq 0.3586$. On the $\alpha < 0$ boundary $\beta = c_{56}\,\alpha^2$ the matching works as in the rest of the region, but the solution $\mathbf{F}_- \leftrightarrow \mathbf{L}$ has an inflection point with vertical tangent.

- region 6: $\mathbf{D} \leftrightarrow \mathbf{C}_-$ , $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$. In this region there are three asymptotic solutions, however only two of them can be extended to $\rho \to 0^+$, while $\mathbf{L}$ cannot. Therefore, there are just two global solutions to eq. (4.41). The boundaries of this region are the parabolas $\beta = c_{56}\,\alpha^2$ and $\beta = c_{67}\,\alpha^2$, where $c_{67}$ is the positive root of the equation $-4-8\,y+88\,y^2-1076\,y^3+2883\,y^4 = 0$ (approximately, $c_{67} \simeq 0.3423$). On the boundary $\beta = c_{67}\,\alpha^2$ the matching works as in the rest of the region, but the solution $\mathbf{D} \leftrightarrow \mathbf{C}_-$ has an inflection point with vertical tangent.

- region 7: $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$. In this region there are three asymptotic solutions, however only one of them can be extended to $\rho \to 0^+$, while $\mathbf{L}$ and $\mathbf{C}_-$ cannot. The boundaries of this region are the parabola $\beta = c_{67}\,\alpha^2$ and the (positive) five-roots-at-infinity parabola $\beta = c_+\,\alpha^2$, where $c_+ = 1/4$. Note that on the $(\alpha < 0)$ part of the parabola $\beta = 1/3\,\alpha^2$ there is the additional matching $\mathbf{F}_- \leftrightarrow \mathbf{C}_-$, so for these points there are two global solutions to eq. (4.41). On the boundary $\beta = c_+\,\alpha^2$ there are the additional matchings $\mathbf{F}_- \leftrightarrow \mathbf{P}_1$ , $\mathbf{D} \leftrightarrow \mathbf{P}_2$, and the solutions corresponding to both of these additional matchings, seen as functions of $A$, display an infinite derivative in $A = 0$.

- region 8: $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$ , $\mathbf{F}_- \leftrightarrow \mathbf{P}_1$ , $\mathbf{D} \leftrightarrow \mathbf{P}_2$. This region lies inside the $\alpha < 0$, $\beta > 0$ part of the five roots at infinity region of the phase space (see fig. 4.1), so there are five asymptotic solutions. Only three of them can be extended to $\rho \to 0^+$, while $\mathbf{C}_-$ and $\mathbf{L}$ cannot. The boundaries of this region are the parabolas $\beta = c_+\,\alpha^2$ and $\beta = c_{89}\,\alpha^2$, where $c_{89} = (5-\sqrt{13})/24 \simeq 0.0581$.

On the boundary $\beta = c_{89}\,\alpha^2$ the matchings are the same as in the rest of the region, but the solution $h(\rho)$ correspondent to the matching $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$ has an inflection point with vertical tangent.

- region 9: $\mathbf{F}_- \leftrightarrow \mathbf{P}_1$ , $\mathbf{D} \leftrightarrow \mathbf{P}_2$. This region lies inside the $\alpha < 0$, $\beta > 0$ part of the five roots at infinity region of the phase space (see fig. 4.1), so there are again five asymptotic solutions. The matching is similar to that of region 8, apart from the fact that $\mathbf{C}_+$ cannot be extended to $\rho \to 0^+$ anymore; hence there are just two global solutions to eq. (4.41). The boundaries of this region are the parabola $\beta = c_{89}\,\alpha^2$ and line $\beta = 0$.

We note that the decaying solution $\mathbf{L}$ never connects to the diverging one $\mathbf{D}$, so we cannot have a spacetime which is asymptotically flat and exhibit the self-shielding of the gravitational field at the origin. On the other hand, finite non-zero asymptotic solutions ($\mathbf{C}_\pm$ or $\mathbf{P}_{1,2}$) can connect to both finite and diverging inner solutions. Therefore, one can have an asymptotically non-flat spacetime which presents self-shielding at the origin, or an asymptotically non-flat spacetime which tends to Schwarzschild spacetime for small radii. More precisely, for $\beta < 0$ there are only solutions displaying the self-shielding of the gravitational field, apart from region 2 where there are no global solutions. Therefore the Vainshtein mechanism never works for $\beta < 0$. In contrast, for $\beta > 0$ all three kinds of global solutions are present. Solutions with asymptotic flatness and the Vainshtein mechanism are present in regions 4 and 5, while solutions which are asymptotically non-flat and exhibit the Vainshtein mechanism do exist in all ($\beta > 0$) regions but region 4. Finally, solutions which display the self-shielding of the gravitational field are present in all ($\beta > 0$) regions but region 7.

## 4.6 Numerical solutions

We said in the previous sections that, having characterized geometrically the asymptotic and inner solutions, to study the Vainshtein mechanism it is enough to know how the matching between asymptotic and inner solutions works. To verify this assertion and corroborate the validity of our results, we solved numerically the system of equations $(4.37) - (4.39)$ in several points of the phase space and for each of the three different types of matching. Therefore, we present here the numerical solutions for the $h$ field and the gravitational potentials in some representative cases. We choose a specific realization for each of the three physically distinct cases, namely asymptotic flatness with Vainshtein mechanism, asymptotically non-flat spacetime with Vainshtein mechanism, and asymptotically non-flat spacetime with self-shielded gravitational field at the origin. In addition, we consider the case in which there are no global solutions to eq. (4.41). This provides an illustration of what happens, in general, to local solutions of eq. (4.41) which cannot be extended to the whole radial domain, and give an insight on the phenomenology of the equation (4.41).

### 4.6.1 Asymptotic flatness with Vainshtein mechanism

Let's consider the case in which the solution of eq. (4.41) connects to the decaying solution at infinity $\mathbf{L}$ and to a finite inner solution (in this case $\mathbf{F}_-$). In figure 4.3, the numerical solutions for $h$ (dashed line), $f/f_{GR}$ (bottom continuous line) and $n'/n'_{GR}$ (top continuous line) are plotted as functions of the dimensionless radial coordinate $x \equiv \rho/\rho_v$. These solutions correspond to the point $(\alpha, \beta) = (0\,, 0.1)$ of the phase space.

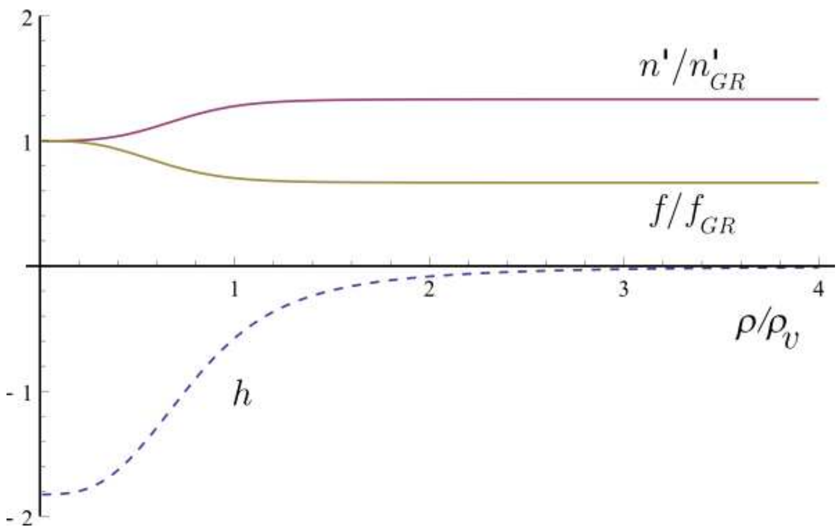


Figure 4.3: Numerical solutions for the case $\mathbf{F}_- \leftrightarrow \mathbf{L}$.

This plot displays very clearly the presence of the vDVZ discontinuity and its resolution *via* the Vainshtein mechanism. For large scales, $h$ is small and the gravitational potentials behave like the Schwarzschild one, however their ratio is different from one, unlike the massless case. Note that the ratio of the two potentials for $\rho \gg \rho_v$ is independent of $m$, so does not approach one as $m \to 0$ (vDVZ discontinuity). However, on small scales $h$ is strongly coupled, and well inside the Vainshtein radius the two potentials scale again as the Schwarzschild one, but their ratio is now one even if $m \neq 0$. So, the strong coupling of the $h$ field on small scales restores the agreement with GR (Vainshtein mechanism).

### 4.6.2 Asymptotically non-flat spacetime with Vainshtein mechanism

Let's consider now the case in which the solution of eq. (4.41) connects to a finite solution at infinity and to a finite inner solution. We consider for definiteness the phase space point $(\alpha, \beta) = (0\,, 0.1)$. In figure 4.4, we plot the numerical results for the gravitational potentials (normalised to their GR values) and the global solution of eq. (4.41) which interpolates between the inner solution $\mathbf{F}_+$ and the asymptotic solution $\mathbf{C}_+$.

We can see that, on large scales, the gravitational potentials are not only different one from the other but also behave very differently compared to the GR case. However, on small scales there is a macroscopic region where the two potentials agree, and their ratio with the Schwarzschild potential stays nearly constant and equal to one. Therefore, also in this case the small scale behavior of $h$ guarantees that GR results are recovered, even if the spacetime is not asymptotically flat. This behavior provide then, in a more general sense, a realization of the Vainshtein mechanism.

### 4.6.3 Asymptotically non-flat spacetime with self-shielding

We turn now to the case where the solution of eq. (4.41) connects to a finite solution at infinity and to the diverging inner solution. In figure 4.5, we plot the global solution $h$ and the associated gravitational potentials, normalized to their

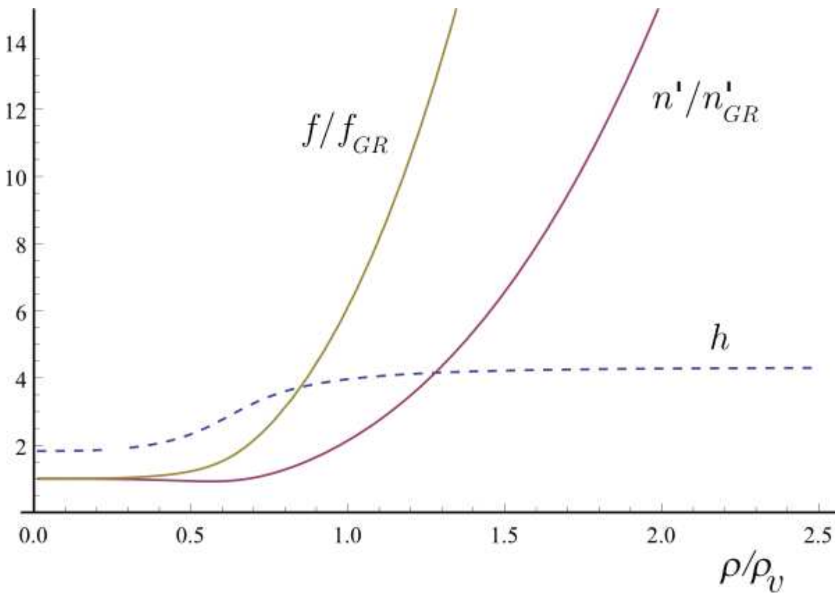


Figure 4.4: Numerical solutions for the case $\mathbf{F}_+ \leftrightarrow \mathbf{C}_+$.

GR values, correspondent to the phase space point $(\alpha, \beta) = (-1\,, -0.5)$. It is apparent that there are no regions where the solutions behave like in the GR case.

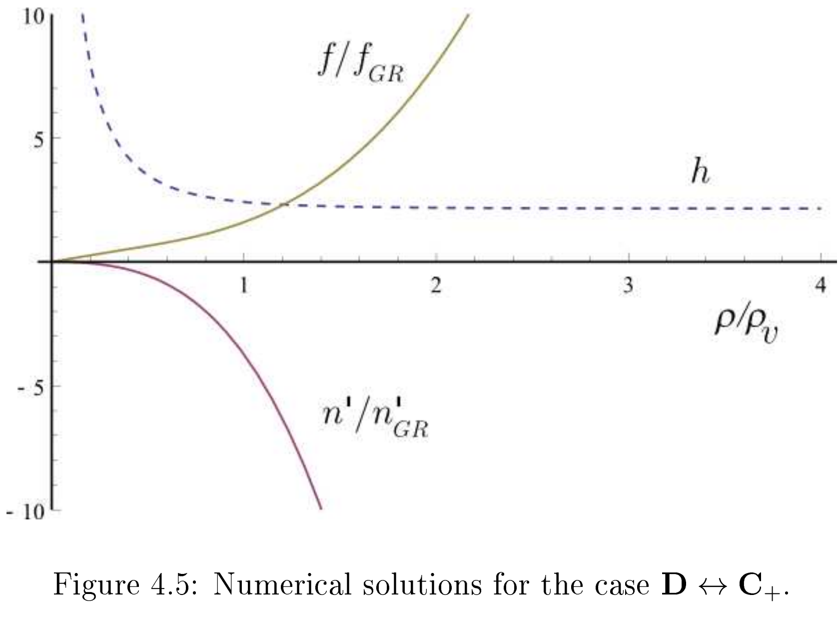


Figure 4.5: Numerical solutions for the case $\mathbf{D} \leftrightarrow \mathbf{C}_+$.

To see that the gravitational potentials are indeed finite at the origin, we plot in figure 4.6 the potentials $f$ and $n'$ themselves, as functions of $\rho/\rho_v$. We choose for definiteness the following ratio between the Compton wavelength and the gravitational radius $\rho_c/\rho_g = 10^6$, and plot the potentials for $0.01 < \rho/\rho_v < 2$. Note that, since in this case $\rho_c/\rho_v = \sqrt[3]{\rho_c/\rho_g} = 10^2$, the range where the functions are plotted is well inside the range of validity of our approximations. We can see that the potentials approach a finite value as $\rho \to 0^+$, and so indeed the gravitational field does not diverge at the origin.

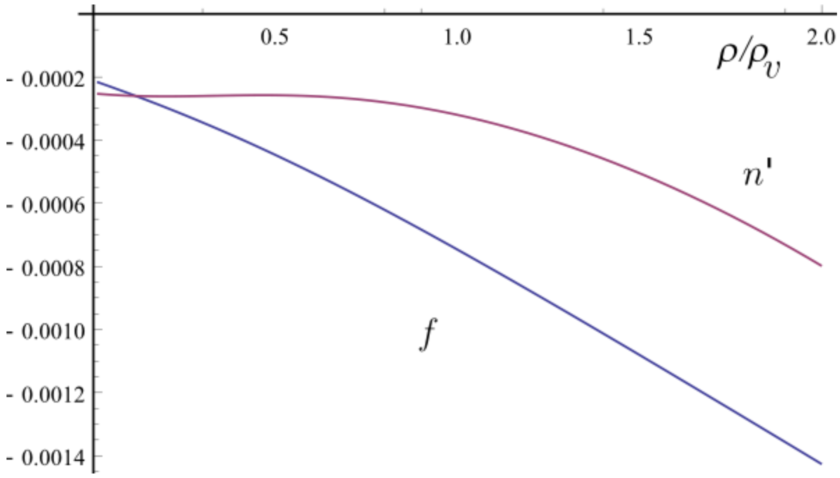


Figure 4.6: Numerical solutions for the gravitational potentials, for the case $\mathbf{D} \leftrightarrow \mathbf{C}_+$.

### 4.6.4 No matching

Finally, we consider the case in which equations (4.37) $-$ (4.39) do not admit global solutions. We consider for definiteness the phase space point $(\alpha, \beta) = (1\,, -0.092)$. In figure 4.7 we plot all the local solutions of the quintic equation (4.41) as functions of the dimensionless radial coordinate $x \equiv \rho/\rho_v$.

For $0 < x < 0.38$, there is only one local solution (the top continuous curve), which connects to the diverging inner solution $\mathbf{D}$. At $x \simeq 0.38$ a pair of solutions is created (dashed and continuous negative valued curves), and at $x \simeq 0.9$, another pair of solutions is created (positive valued dashed curve and positive valued

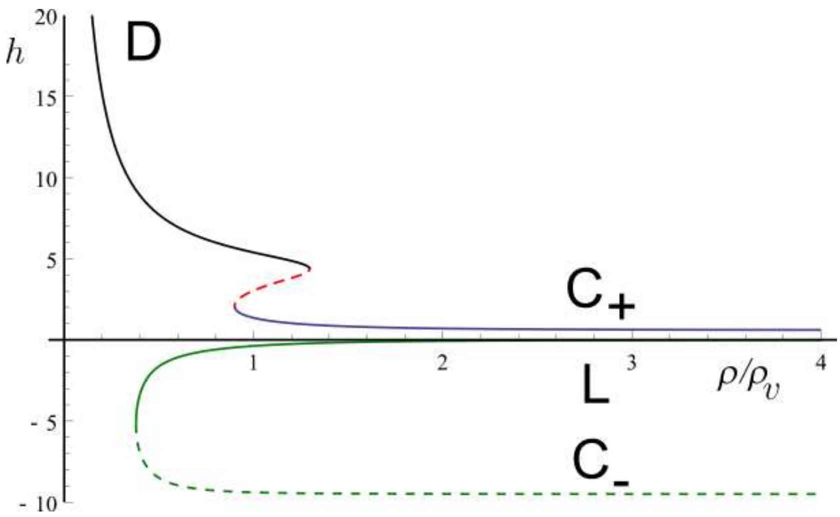


Figure 4.7: Numerical results for all local solutions of eq. (4.41) in the case where there is no matching.

bottom continuous curve). However, at $x \simeq 1.3$ one of the newly created functions (the positive valued dashed curve) annihilates with the solution which connects to the inner solution, so for $x > 1.3$ there are three local solutions, which finally connect with the asymptotic solutions $\mathbf{C}_-$ , $\mathbf{L}$ and $\mathbf{C}_+$. Therefore, the number of existing local solutions is one for $0 < x < 0.38$, three for $0.38 < x < 0.9$, five for $0.9 < x < 1.3$ and three for $x > 1.3$. We can see that, despite the fact that for every $\rho$ there is at least one local solution, there does not exist a solution which extends over the whole radial domain.

To clarify the meaning of figure 4.7, we plot in figure 4.8 several snapshots of the quintic function at different values of $A$, for the same phase space point $(\alpha, \beta) = (1\,, -0.092)$. Figure 4.8 shows the creation and annihilation of solutions from the point of view of the quintic instead of from the point of view of the implicitly defined functions: note that the quintic is plotted for increasing values of $A = 1/x^3$, while in figure 4.7 the local solutions are plotted as functions of $x$. The plots of the quintic correspond to the following values of $A$: $A = 0$, $A = 0.456 \leftrightarrow x = 1.3$, $A = 0.716$, $A = 1.356 \leftrightarrow x = 0.9$, $A = 2$, $A = 6.93$, $A = 17.9$, $A = 18.35 \leftrightarrow x = 0.38$ and $A = 18.68$.

At $A = 0$ there are three roots, one negative, one positive and the zero root,

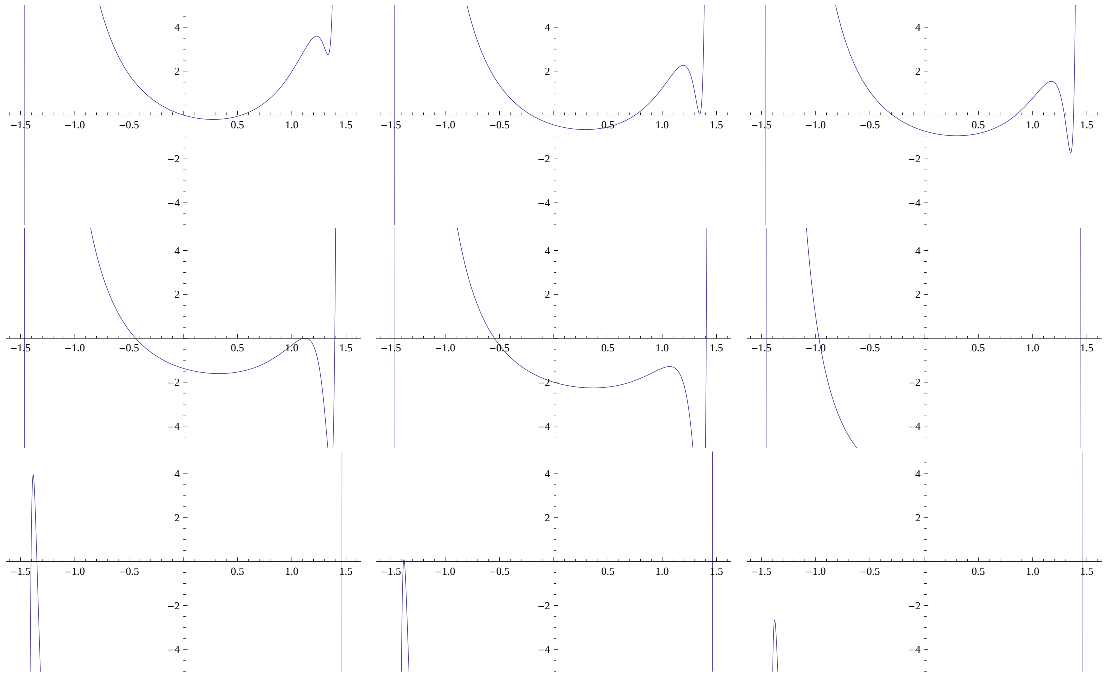


Figure 4.8: Quintic function for increasing values of $A$, no-matching case.

which correspond to the three asymptotic solutions $\mathbf{C}_-$ , $\mathbf{C}_+$ and $\mathbf{L}$. At $A = 0.456 \leftrightarrow x = 1.3$ a new double root appears, and two local solutions are created: these are the top continuous and dashed curve of figure 4.7. As is apparent in the $A = 0.716$ plot, for $0.456 < A < 1.356$ there are five roots and so five local solutions. At $A = 1.356 \leftrightarrow x = 0.9$ one of the newly created solutions (the top dashed curve of figure 4.7) annihilates with the asymptotic solution $\mathbf{C}_+$, which ceases existing: for $1.356 < A < 18.35$ there are three roots and therefore three local solutions. At $A = 18.35 \leftrightarrow x = 0.38$ the asymptotic solution $\mathbf{C}_-$ annihilates with the asymptotic solution $\mathbf{L}$, and for $A > 18.35$ only one local solution survives, the one created at $A = 0.456 \leftrightarrow x = 1.3$ which correspond to the top continuous curve in figure 4.7. This solution is the one which connects to the inner solution $\mathbf{D}$ when $A \to +\infty \leftrightarrow x \to 0^+$.

Note that, as we discussed in general in section 4.5.2 and in appendix B, the solutions are created and annihilated in pairs. Furthermore, the pairs of solutions have infinite slope when they are created and when they annihilate, while their values remain bounded.

# Conclusions

Recent cosmological observations seem to suggest that the universe is currently undergoing a period of accelerated expansion. Despite being unexpected, this result can be explained assuming the presence of a nonzero and fine-tuned cosmological constant, or the existence of an exotic source of energy which is usually termed dark energy. However, from another point of view, these observations may indicate the fact that General Relativity is not a good description for gravity at very large scales. To test this idea, we consider theories whose predictions differ from the ones of General Relativity only at very large scales, and see if they can fit the data satisfactorily. In general, theories which modify gravity at large distances involve more degrees of freedom than General Relativity, and for these theories to be phenomenologically viable it is necessary that the extra degrees of freedom are screened at terrestrial and astrophysical scales. A well known screening mechanism is the Vainshtein mechanism, where derivatives self-interactions of a field are responsible for its screening.

In this thesis, we have considered a specific class of theories which modify gravity at large distance, and investigated the effectiveness of the Vainshtein mechanism. In particular, we considered a class of nonlinear massive gravity theories, known as dGRT Massive Gravity: these theories contain a mass parameter, which sets the Compton radius of the theory, and two additional free parameters. We established for which values of the two free parameters the Vainshtein mechanism is working. More in general, we classified the existence and asymptotic properties of static, spherically symmetric solutions in the branch of configurations where the Vainshtein mechanism can occur.

In chapter 1 we introduced the Standard Cosmological Model, emphasizing in particular the hypothesis which lie at its foundations. We presented the problem of the late time acceleration and the possible ways out of it, suggesting that a possible way to explain the recent cosmological observations is to abandon the assumption that General Relativity is a good description of gravity at extremely large scales.

In chapter 2 we introduced the braneworld models, which provide an appealing way to construct theories which modify gravity at short and/or large distances, and focused on the DGP model, discussing both the realization of the Vainshtein mechanism and the cosmological solutions of the model.

In chapter 3 we introduced the concept of a massive theory of gravity, and explained why such a theory can be interesting from the point of view of the late time acceleration problem of cosmology. We then considered generic nonlinear extensions of the Fierz-Pauli theory, and described how, keeping the mass fixed, it is possible to select a two-parameters subclass of extensions which propagate five degrees of freedom. These subclass of nonlinear extensions are named dGRT Massive Gravity theories.

In chapter 4, we studied static, spherically symmetric solutions in the dGRT Massive Gravity theories. There are two branches of solutions which satisfy this symmetry requirement, and we considered only the branch where the Vainshtein mechanism can be effective. We focused on scales smaller than the Compton radius of the gravitational field, and considered the weak field limit for the gravitational potentials, while keeping all non-linearities of the scalar mode which is involved in the screening. For every point of the two free parameter phase space, we characterised completely the number and properties of asymptotic solutions on large scales and also of inner solutions on small scales. In particular, there are two kinds of asymptotic solutions, where one of them is asymptotically flat and the other one is not. There are also two kinds of inner solutions, one which displays the Vainshtein mechanism and the other which exhibits the self-shielding of the gravitational field near the origin.

We described under which circumstances the theory admits global solutions interpolating between the asymptotic and inner solutions, and found that the asymptotically flat solution connects only to inner solutions displaying the Vainshtein mechanism, while solutions which diverge asymptotically can connect to both kinds of inner solutions. Furthermore, we showed that there are some regions in the parameter space where global solutions do not exist, and characterised precisely in which regions of the phase space the Vainshtein mechanism is working. We showed that there is a significant part of the phase space where the Vainshtein mechanism is effective, which correspond to theories which are phenomenologically viable.

Our study embraces all of the phase space spanned by the two parameters of the theory. Notably, we found that, within our approximations, the asymptotic and inner solutions cannot in general be extended to the whole radial domain. In particular, we exhibited extreme cases in which global solutions do not exist at all. This happens because at a finite radius the derivatives of the metric components diverge, while the metric components themselves remain bounded. When the derivatives of the metric cease to be small, the approximations we used to derive the equations under study break down. It would be interesting to study what happens at this radius in the full theory.

We conclude that the dGRT Massive Gravity class of theories are very promising candidates for consistently modifying gravity at very large distances. On one hand, they are very appealing from a cosmological point of view, since they may provide a way to explain the late time acceleration problem without introducing dark energy or a nonzero cosmological constant. On the other hand, they are very interesting also from a purely theoretical point of view, since they may provide a realization of the idea that it should be possible to deform a theory (in this case General Relativity) by a continuous parameter (in this case the mass), and obtain theories whose predictions are continuous in the same parameter. It would be very interesting to see if it is possible to realize the phenomenon of self-acceleration in this framework.

# Appendix A

# Total derivative combinations

We review here the main definitions and properties of total derivative combinations of the field $\phi$ and related objects.

## A.1 Total derivative combinations of $\Pi_{\mu\nu}$

Let's remind the definition of the object $\Pi$ constructed from the second derivatives of the field $\phi$

$$\Pi_{\mu\nu} = \partial_\mu \partial_\nu \phi \quad . \tag{A.1}$$

As already mentioned in the main text, at every order in $\Pi$ (or equivalently in $\phi$) there is a unique (up to an overall constant) contraction of $\Pi$ factors (we raise/lower indices with the Minkowski metric $\eta^{\mu\nu}/\eta_{\mu\nu}$ ) which is in the form of a total derivative. Explicitly, at order $n$ it takes the form [68]

$$\mathcal{L}_n^{TD}(\Pi) = \sum_p (-1)^p \, \eta^{\mu_1 p(\nu_1)} \, \cdots \, \eta^{\mu_n p(\nu_n)} \, \Pi_{\mu_1 \nu_1} \, \cdots \, \Pi_{\mu_n \nu_n} \quad , \tag{A.2}$$

where the sum runs on all the permutations $p$ of $n$ elements. To facilitate the comparison with the $\Pi$ structures coming from the nonlinear mass term, we can group together some of the contractions in (A.2) using the fact that $\eta^{\mu\nu}$ and $\Pi_{\mu\nu}$ are symmetric, and using the notation

$$\big[\Pi^n\big] \equiv \eta^{\mu\alpha_1} \, \Pi_{\alpha_1\beta_1} \, \eta^{\beta_1\alpha_2} \, \Pi_{\alpha_2\beta_2} \cdots \eta^{\beta_{n-1}\alpha_n} \, \Pi_{\alpha_n\mu} \tag{A.3}$$

we obtain

$$\mathcal{L}_1^{\rm TD}(\Pi) = [\Pi] \tag{A.4}$$

$$\mathcal{L}_2^{\rm TD}(\Pi) = [\Pi]^2 - [\Pi^2] \tag{A.5}$$

$$\mathcal{L}_3^{\rm TD}(\Pi) = [\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3] \tag{A.6}$$

$$\mathcal{L}_4^{\rm TD}(\Pi) = [\Pi]^4 - 6[\Pi^2][\Pi]^2 + 8[\Pi^3][\Pi] + 3[\Pi^2]^2 - 6[\Pi^4] \quad . \tag{A.7}$$

Note that the terms $\mathcal{L}_n^{\rm TD}(\Pi)$ vanish identically for $n \geq 5$ (in general, they vanish for $n > D$, where $D$ is the spacetime dimension), and $\mathcal{L}_2^{\rm TD}(h)$ is the Fierz-Pauli term. Furthermore, they satisfy a recursion relation

$$\mathcal{L}_n^{\rm TD}(\Pi) = -\sum_{m=1}^{n} (-1)^m \frac{(n-1)!}{(n-m)!} \, [\Pi^m] \, \mathcal{L}_{n-m}^{\rm TD}(\Pi) \tag{A.8}$$

with $\mathcal{L}_0^{\rm TD}(\Pi) = 1$.

## A.2 The $X_{\mu\nu}^{(n)}$ tensors

From the total derivative Lagrangians $\mathcal{L}_n^{\rm TD}(\Pi)$, we can construct the tensors $X_{\mu\nu}^{(n)}$ by deriving with respect to $\Pi^{\mu\nu}$

$$X_{\mu\nu}^{(n)} = \frac{1}{n+1} \frac{\partial}{\partial \Pi^{\mu\nu}} \, \mathcal{L}_{n+1}^{\rm TD}(\Pi) \tag{A.9}$$

obtaining in general

$$X_{\mu\nu}^{(n)} = \sum_{m=0}^{n} (-1)^m \frac{n!}{(n-m)!} \, \Pi_{\mu\nu}^m \, \mathcal{L}_{n-m}^{\rm TD}(\Pi) \tag{A.10}$$

The tensors $X_{\mu\nu}^{(n)}$ satisfy the recursion relation

$$X_{\mu\nu}^{(n)} = -n \, \Pi_\mu^{\ \alpha} X_{\alpha\nu}^{(n-1)} + \Pi^{\alpha\beta} X_{\alpha\beta}^{(n-1)} \eta_{\mu\nu} \tag{A.11}$$

and, since $\mathcal{L}_n^{\mathrm{TD}}(\Pi)$ vanishes for $n > 4$, they vanish for $n \geq 4$ ($n \geq D$ in a spacetime of dimension $D$). Explicitly they read

$$
\begin{aligned}
X^{(0)}_{\mu\nu} &= \eta_{\mu\nu} \\
X^{(1)}_{\mu\nu} &= [\Pi]\,\eta_{\mu\nu} - \Pi_{\mu\nu} \\
X^{(2)}_{\mu\nu} &= \left([\Pi]^2 - \left[\Pi^2\right]\right)\eta_{\mu\nu} - 2\,[\Pi]\,\Pi_{\mu\nu} + 2\Pi^2_{\mu\nu} \\
X^{(3)}_{\mu\nu} &= \left([\Pi]^3 - 3\,[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right]\right)\eta_{\mu\nu} - 3\left([\Pi]^2 - \left[\Pi^2\right]\right)\Pi_{\mu\nu} + 6\,[\Pi]\,\Pi^2_{\mu\nu} - 6\Pi^3_{\mu\nu}
\end{aligned}
$$

The following relations involving the massless kinetic operator (3.28) make clear which is the form of transformations we can perform on $h_{\mu\nu}$ to remove the mixing terms $h^{\mu\nu}X^{(j)}_{\mu\nu}$ from the $\Lambda_3$ action in the decoupling limit

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\left(\phi\,\eta_{\alpha\beta}\right) = -(D-2)\,X^{(1)}_{\mu\nu} \tag{A.12}$$

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\left(\partial_\alpha\phi\,\partial_\beta\phi\right) = X^{(2)}_{\mu\nu} \tag{A.13}$$

Finally, it can be shown that the $X^{(n)}_{\mu\nu}$ tensors are symmetric and identically conserved

$$X^{(n)}_{\mu\nu} = X^{(n)}_{\nu\mu} \tag{A.14}$$

$$\partial^\mu X^{(n)}_{\mu\nu} = 0 \qquad . \tag{A.15}$$

# Appendix B

# The implicit function theorem

The implicit function theorem, also known as Dini's theorem, is used repeatedly throughout the text. Although the theorem is more general, we give here its formulation in the specific case of a function of two (real) variables. For the proof, see [120] for the general case and [121] for the particular case treated here.

## B.1 Formulation of the theorem

**Theorem 1 (Implicit function theorem, or Dini's theorem)** *Let $F(x,y)$ be a function defined in an open set $A \subset \mathbb{R}^2$, and let $F$ be derivable with continuous partial derivatives. Be $(x_0, y_0) \in A$ such that*

$$F(x_0, y_0) = 0 \qquad , \qquad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \quad . \tag{B.1}$$

*Then there exist:*

- *An open neighborhood $U$ of $x_0$ and an open neighborhood $V$ of $y_0$, such that $U \times V \subset A$ ;*

- *A function $f : U \to V$ such that, for all $(x,y) \in U \times V$, we have*

$$F(x,y) = 0 \qquad \Leftrightarrow \qquad y = f(x) \quad . \tag{B.2}$$

*Furthermore, the function $x \to f(x)$ is derivable with continuous derivative, and we have*

$$f'(x) = -\frac{\partial_x F\big(x, f(x)\big)}{\partial_y F\big(x, f(x)\big)} \tag{B.3}$$

Roughly speaking, the implicit function theorem states that, provided the conditions (B.1) are satisfied, a zero of a function of two real variables defines implicitly a functional relation between the two variables, at least locally. Furthermore, it says that this functional relation is regular, and gives an expression for the derivative of the function which links the two variables. Note that the conditions (B.1) are sufficient but not necessary for the existence of the "implicit" solution.

### B.1.1 The quintic equation and implicit functions

The implicit function theorem is crucial for our analysis of the Vainshtein mechanism in massive gravity, since (at $\alpha$, $\beta$ fixed) the equation which the field $h(\rho)$ obeys (the quintic equation) is of the form $F(h(\rho), \rho) = 0$. Note that it is equivalent to work with $\rho$ as a radial coordinate or with $x = \rho/\rho_v$, or $A = 1/x^3$, since all these coordinates are related by diffeomorphisms. If we work with the coordinate $A$, the solutions for the field $h(A)$ are then implicitly defined by the equation $q\left(h(A), A; \alpha, \beta\right) = 0$, where the quintic function $q$ is defined in (4.42).

At $\alpha$ and $\beta$ fixed, the function $q\left(h, A\right)$ is defined on $\mathbb{R} \times (0, +\infty)$ and is derivable an arbitrary number of times with continuous partial derivatives. Suppose that we find, at a certain $A = \bar{A}$ (*i.e.* at a certain radius $\bar{\rho} = \rho_v / \sqrt[3]{\bar{A}}$ ), a root $\bar{h}$ of the equation $q_{\bar{A}}\left(h\right) = 0$, where $q_A\left(h\right)$ is the shape function (4.61): the condition $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ translates in this case to the fact that $\bar{h}$ is a *simple root* of the equation $q_{\bar{A}}\left(h\right) = 0$. Therefore, if we find at a certain $A = \bar{A}$ a simple root $\bar{h}$ of the equation $q_{\bar{A}}\left(h\right) = 0$, then the conditions (B.1) are satisfied, and the implicit function theorem assures us that there exist a neighborhood of $\bar{A}$ (*i.e.* a neighborhood of $\bar{\rho}$) where there exists a solution $h(A)$ of the quintic equation such that $h(\bar{A}) = \bar{h}$.

### B.1.2 Maximal extension of implicitly defined solutions

Our aim in the end is to find *global* solutions of the quintic equation, that is solutions $h(A)$ of the quintic equation which are defined for $A \in (0, +\infty)$. Therefore, it is important to establish when a local solution can be extended to the whole radial domain. Suppose we have a local solution $h(A)$ of the quintic equation defined on $(A_i, A_f) \subset (0, +\infty)$. If the conditions (B.1) are satisfied also at $A = A_i$ and $A = A_f$, we can extend the solution to an interval $(A_i^{(2)}, A_f^{(2)}) \supset (A_i, A_f)$, and we can iterate this procedure. Therefore, we can extend the local solution until we reach a point $\tilde{A}$ where the conditions (B.1) are not both satisfied: this can happen only if one of the following conditions are true

1. $\lim_{A \to \tilde{A}} |h(A)| = +\infty$
2. $\dfrac{\partial q}{\partial h}(\tilde{h}, \tilde{A}) = 0$

where in the second case $\tilde{h} \equiv \lim_{A \to \tilde{A}} h(A)$. However, it is possible to see that the first case cannot happen. In fact, suppose hypothetically that there exists a solution $h(A)$ of the quintic equation such that $\lim_{A \to \tilde{A}} |h(A)| = +\infty$ with $\tilde{A}$ finite and non-zero. This means that, in the dual formulation, there is a solution $v(A)$ of the equation (4.44) such that $\lim_{A \to \tilde{A}} v(A) = 0$, with $\tilde{A}$ finite and non-zero: this implies that $\lim_{A \to \tilde{A}} d\big(v(A), A; \alpha, \beta\big) = \frac{3}{2}\beta^2 \neq 0$, since we are considering the $\beta \neq 0$ case. But, by the continuity of the function $d\big(v, A; \alpha, \beta\big)$ and the fact that $d\big(v(A), A; \alpha, \beta\big) = 0$ identically since $v(A)$ is a solution of (4.44), we have that $\lim_{A \to \tilde{A}} d\big(v, A; \alpha, \beta\big) = 0$. The hypothesis led us to a contradiction, so it follows that there cannot exist solutions $h(A)$ of the quintic equation such that $\lim_{A \to \tilde{A}} |h(A)| = +\infty$ with $\tilde{A}$ finite and non-zero.

Therefore, a local solution $h(A)$ of the quintic equations can be extended until we meet a finite and non-zero $\tilde{A}$ where $\frac{\partial q}{\partial h}(\tilde{h}, \tilde{A}) = 0$ (with $\tilde{h} \equiv \lim_{A \to \tilde{A}} h(A)$), or equivalently until we meet a finite and non-zero $\tilde{A}$ where the function $q_A\big(h; \alpha, \beta\big)$ has a stationary point on the horizontal axis. Note that, when this happens, the derivative $h'(A)$ of the solution diverges as $A \to \tilde{A}$, as can be deduced from (B.3), while the solution $h(A)$ itself remains bounded.

# Appendix C

# Useful properties of the quintic function

We discuss here some important properties of the quintic function, which are useful for the analytic study of the solutions matching. In this thesis, we study the quintic equation in the domain of definition $h \in (-\infty, +\infty)$, $A \in (0, +\infty)$, $\alpha \in (-\infty, +\infty)$ and $\beta \in (-\infty, +\infty)$, $\beta \neq 0$. However, it is very useful to extend the domain of definition of $A$ to $A = 0$ as well, which corresponds to the asymptotic limit $\rho \to +\infty$.

## C.1 General properties

The quintic function and its derivatives reads explicitly

$$q\left(h, A; \alpha, \beta\right) = \frac{3}{2}\,\beta^2\,h^5 - \left(\alpha^2 + 2\beta\right) h^3 + 3\left(\alpha + \beta A\right) h^2 - \frac{3}{2}\,h - A \qquad \text{(C.1)}$$

$$q'\left(h, A; \alpha, \beta\right) = \frac{15}{2}\,\beta^2\,h^4 - 3\left(\alpha^2 + 2\beta\right) h^2 + 6\left(\alpha + \beta A\right) h - \frac{3}{2} \qquad \text{(C.2)}$$

$$q''\left(h, A; \alpha, \beta\right) = 30\,\beta^2\,h^3 - 6\left(\alpha^2 + 2\beta\right) h + 6\left(\alpha + \beta A\right) \qquad \text{(C.3)}$$

$$q'''\left(h, A; \alpha, \beta\right) = 90\,\beta^2\,h^2 - 6\left(\alpha^2 + 2\beta\right) \qquad \text{(C.4)}$$

where we indicated the derivatives with respect to $h$ with a prime $'$. Note first of all that

$$\lim_{h\to+\infty} q\left(h, A; \alpha, \beta\right) = +\infty \qquad , \qquad \lim_{h\to-\infty} q\left(h, A; \alpha, \beta\right) = -\infty \tag{C.5}$$

and that

$$q\left(0, A; \alpha, \beta\right) = -A \leq 0 \tag{C.6}$$

$$q'\left(0, A; \alpha, \beta\right) = -\frac{3}{2} < 0 \tag{C.7}$$

$$q''\left(0, A; \alpha, \beta\right) = 6\left(\alpha + \beta A\right) \tag{C.8}$$

Therefore, for the intermediate value theorem, there is always (for every value of $A$) a root of the quintic for $h \in (0, +\infty)$. In particular, if we take into account the multiplicity of the roots, there is always an odd number of real roots. Note that $q'\left(0, A; \alpha, \beta\right)$ is indipendent of $A$, while $q\left(0, A; \alpha, \beta\right)$ is linear and decreasing with respect to $A$. We may see the evolution with $A$ of the quintic as the sum of an overall rigid translation due to the constant term of the polynomial, and of a change of shape due to the contribution $3\beta A\, h^2$ to the quadratic piece of the polynomial.

## C.2 Evolution with $A$

### C.2.1 The quintic function

To study how the quintic function evolves with $A$, let's consider its partial derivative with respect to $A$. It is easy to verify that

$$\frac{\partial q}{\partial A}\left(h, A; \alpha, \beta\right) = 3\,\beta h^2 - 1 \tag{C.9}$$

and this relation implies that, if $\beta < 0$, we have

$$\beta < 0 \qquad \Rightarrow \qquad \frac{\partial q}{\partial A}\left(h, A; \alpha, \beta\right) < 0 \tag{C.10}$$

for every $h$, $A$, $\alpha$ and $\beta < 0$. Therefore, at every $h$ the value of the quintic function decreases monotonically when $A$ goes from 0 to $+\infty$. On the other hand, if $\beta > 0$ we have

$$\beta > 0 \qquad \Rightarrow \qquad \begin{cases} \dfrac{\partial q}{\partial A}\big(h, A; \alpha, \beta\big) < 0 \quad \text{for} \quad |h| < \dfrac{1}{\sqrt{3\beta}} \\ \dfrac{\partial q}{\partial A}\big(h, A; \alpha, \beta\big) > 0 \quad \text{for} \quad |h| > \dfrac{1}{\sqrt{3\beta}} \\ \dfrac{\partial q}{\partial A}\big(h, A; \alpha, \beta\big) = 0 \quad \text{for} \quad |h| = \dfrac{1}{\sqrt{3\beta}} \end{cases} \tag{C.11}$$

and we conclude that, at every $h$ such that $-1/\sqrt{3\beta} < h < 1/\sqrt{3\beta}$, the value of the quintic function decreases monotonically when $A$ goes from 0 to $+\infty$, while it increases monotonically at every $h$ such that $h < -1/\sqrt{3\beta}$ or $h > 1/\sqrt{3\beta}$. Finally, there are two fixed points of the evolution of the quintic with $A$, which correspond to the following values for $h$

$$h = \pm\frac{1}{\sqrt{3\beta}} = \mathbf{F}_{\pm} \tag{C.12}$$

which (as already indicated above) are precisely the limiting values of the finite inner solutions $\mathbf{F}_{\pm}$.

### C.2.2 The first derivative

Consider now the first derivative of the quintic $q'\big(h, A; \alpha, \beta\big)$. We have

$$\frac{\partial q'}{\partial A}\big(h, A; \alpha, \beta\big) = 6\,\beta h \quad , \tag{C.13}$$

which implies that the only fixed point of the evolution of $q'$ corresponds to the value $h = 0$, and (as already mentioned) we have

$$q'\big(0, A; \alpha, \beta\big) = -\frac{3}{2} \tag{C.14}$$

independently of $\alpha$ and $\beta$. Furthermore, we have that

$$\beta < 0 \qquad \Rightarrow \qquad \begin{cases} \dfrac{\partial q'}{\partial A}\big(h, A; \alpha, \beta\big) < 0 \quad \text{for} \quad h > 0 \\ \dfrac{\partial q'}{\partial A}\big(h, A; \alpha, \beta\big) > 0 \quad \text{for} \quad h < 0 \end{cases} \tag{C.15}$$

so, for $\beta < 0$, at every fixed $h > 0$ the first derivative of the quintic decreases when $A$ goes from 0 to $+\infty$, while it increases at every fixed $h < 0$. Conversely, we have that

$$\beta > 0 \qquad \Rightarrow \qquad \begin{cases} \dfrac{\partial q'}{\partial A}\big(h, A; \alpha, \beta\big) < 0 \quad \text{for} \quad h < 0 \\ \dfrac{\partial q'}{\partial A}\big(h, A; \alpha, \beta\big) > 0 \quad \text{for} \quad h > 0 \end{cases} \tag{C.16}$$

and so, for $\beta < 0$, at every fixed $h > 0$ the first derivative of the quintic increases when $A$ goes from 0 to $+\infty$, while it decreases at every fixed $h < 0$.

### C.2.3 The second derivative

For what concerns the second derivative of the quintic $q''\left(h, A; \alpha, \beta\right)$, we have

$$\frac{\partial q''}{\partial A}\big(h, A; \alpha, \beta\big) = 6\,\beta \tag{C.17}$$

and this implies that there are no fixed points in the evolution with $A$ of $q''$. In fact, from (C.3) it is evident that $q''\left(h, A; \alpha, \beta\right)$ translates rigidly when $A$ changes, and in particular translates towards $h \to +\infty$ when $\beta > 0$ while translates towards $h \to -\infty$ when $\beta < 0$. Note that the value of $\alpha$ sets the value of the second derivative in $h = 0$ at $A = 0$

$$q''\left(0, 0; \alpha, \beta\right) = 6\,\alpha \tag{C.18}$$

and that

$$\lim_{h\to+\infty} q''\left(h, A; \alpha, \beta\right) = +\infty \qquad , \qquad \lim_{h\to-\infty} q''\left(h, A; \alpha, \beta\right) = -\infty \quad . \tag{C.19}$$

This implies that, for every value of $\alpha$ and $\beta$ (still with $\beta \neq 0$), there is always a critical value $A_{crit}(\alpha, \beta)$ such that the second derivative $q''\left(h, A; \alpha, \beta\right)$ has one and only one root for $A > A_{crit}(\alpha, \beta)$. This root is negative when $\beta$ is positive, and conversely is positive when $\beta$ is negative. Therefore, for $A > A_{crit}(\alpha, \beta)$, the quintic has zero inflection points for $h > 0$ and one inflection point for $h < 0$ in the case $\beta > 0$, while has one inflection point for $h > 0$ and zero inflection points for $h < 0$ in the case $\beta < 0$. Roughly speaking, this critical value for $A$ can be regarded as the value after which there cannot be anymore creations and annihilations of local solutions.

Note that, since the second derivative $q''\left(h, A; \alpha, \beta\right)$ translates rigidly when $A$ changes, it is very useful to characterize completely its shape at infinity (*i.e.* at $A = 0$) for every value of $\alpha$ and $\beta$ in the phase space.

# Appendix D

# Asymptotic structure of the quintic function

In this and in the next appendix, we summarize the main properties of the quintic function (4.42) when $A = 0$, which corresponds to the asymptotic limit $\rho \to +\infty$. In these appendices, when we say that a function has some property at infinity we mean at radial infinity, *i.e.* at $A = 0$.

As we mentioned above, for $A = 0$ the quintic function reduces to the asymptotic function

$$\mathscr{A}\big(h;\alpha,\beta\big) = \frac{3}{2}\,\beta^2\,h^5 - \big(\alpha^2 + 2\beta\big)\,h^3 + 3\,\alpha\,h^2 - \frac{3}{2}\,h \tag{D.1}$$

which can be factorized as

$$\mathscr{A}\big(h;\alpha,\beta\big) = h\;\mathscr{A}_r\big(h;\alpha,\beta\big) \tag{D.2}$$

where the function $\mathscr{A}_r\big(h;\alpha,\beta\big)$ is called the reduced asymptotic function and reads

$$\mathscr{A}_r\big(h;\alpha,\beta\big) = \frac{3}{2}\,\beta^2\,h^4 - \big(\alpha^2 + 2\beta\big)\,h^2 + 3\,\alpha\,h - \frac{3}{2} \quad . \tag{D.3}$$

Note that, as a consequence of the symmetry (4.43) of the quintic function, the asymptotic function has the following symmetry

$$\mathscr{A}\left(\frac{h}{k};k\,\alpha,k^2\beta\right) = \frac{1}{k}\,\mathscr{A}\left(h;\alpha,\beta\right) \tag{D.4}$$

which, differently from the symmetry (4.43), holds also for $k < 0$. Therefore, we may restrict the study of the asymptotic function only to the semi-plane $\alpha > 0$.

## D.1 Study of the second derivative

In order to study analytically the matching of solutions, it is very important to establish how many inflection points the quintic function has at infinity, and where they are located in relation to the fixed points of the quintic.

### D.1.1 Inflection points at infinity

The second derivative of the quintic at $A = 0$ is equal to the second derivative of the asymptotic function which reads

$$\mathscr{A}''\big(h;\alpha,\beta\big) = 30\,\beta^2\,h^3 - 6\left(\alpha^2+2\beta\right)h + 6\,\alpha \tag{D.5}$$

To find the number of roots of $\mathscr{A}''$, it is enough to study just the case $\alpha > 0$, since the symmetry (D.4) implies that the number of roots at $(-\alpha,\beta)$ and at $(\alpha,\beta)$ are equal. Considering then the case $\alpha > 0$, the function $\mathscr{A}''\big(h;\alpha,\beta\big)$ has the following properties

$$\lim_{h\to-\infty}\mathscr{A}''\big(h;\alpha,\beta\big) = -\infty \qquad \mathscr{A}''\big(0;\alpha,\beta\big) > 0 \qquad \lim_{h\to+\infty}\mathscr{A}''\big(h;\alpha,\beta\big) = +\infty \tag{D.6}$$

so for the intermediate value theorem there is always a root of $\mathscr{A}''\big(h;\alpha,\beta\big)$ for $h < 0$, which we call $r_0$. To understand if there are other roots, it is useful to study its first derivative

$$\mathscr{A}'''\big(h;\alpha,\beta\big) = 90\,\beta^2\,h^2 - 6\left(\alpha^2+2\beta\right) \quad : \tag{D.7}$$

it is easy to check that the quadratic equation $\mathscr{A}'''\big(h;\alpha,\beta\big) = 0$ admits solutions only if

$$\beta \geq -\frac{1}{2}\,\alpha^2 \tag{D.8}$$

in which case the roots are

$$h_\pm = \pm\frac{\sqrt{\alpha^2+2\beta}}{\sqrt{15}\,|\beta|} \quad . \tag{D.9}$$

Therefore, for $\beta \leq -(1/2)\,\alpha^2$ the function $\mathscr{A}'''\big(h;\alpha,\beta\big)$ is positive for all values of $h$, and the function $\mathscr{A}''\big(h;\alpha,\beta\big)$ is monotonically increasing. On the other hand, for $\beta > -(1/2)\alpha^2$ the function $\mathscr{A}''\big(h;\alpha,\beta\big)$ has a relative minimum at $h = h_+$ and a relative maximum at $h = h_-$. The number of roots of the equation $\mathscr{A}''\big(h;\alpha,\beta\big) = 0$ is determined by the fact that $\mathscr{A}''\big(h_+;\alpha,\beta\big)$ is positive or negative: if it is positive, then the equation $\mathscr{A}'' = 0$ has only one root (which has negative value), while if it is negative the equation $\mathscr{A}'' = 0$ has three roots (one root which has negative value and two roots, $r_1$ and $r_2$, which have positive values). The phase space boundaries between the regions where $\mathscr{A}'' = 0$ has three roots and the regions where $\mathscr{A}'' = 0$ has one root are defined by the condition $\mathscr{A}''\big(h_+;\alpha,\beta\big) = 0$: in this case, the equation $\mathscr{A}'' = 0$ has two roots, one simple root and one double root. The condition $\mathscr{A}''\big(h_+;\alpha,\beta\big) = 0$ is equivalent to the following condition on $y = \beta/\alpha^2$

$$8\,y^3 - \frac{87}{4}\,y^2 + 6\,y + 1 = 0 \quad : \tag{D.10}$$

this equation is a cubic and has positive discriminant, therefore has three real roots whose approximated values are $y_1 = in_1 \simeq -0.115898$, $y_2 = in_2 \simeq 0.452816$ and $y_3 = in_3 \simeq 2.38183$. It can be checked that for $-0.5\,\alpha^2 < \beta < in_1\,\alpha^2$ and for $in_2\,\alpha^2 < \beta < in_3\,\alpha^2$ we have $\mathscr{A}''\big(h_+;\alpha,\beta\big) > 0$, while for $in_1\,\alpha^2 < \beta < 0$, $0 < \beta < in_2\,\alpha^2$ and $\beta > in_3\,\alpha^2$ we have $\mathscr{A}''\big(h_+;\alpha,\beta\big) < 0$.

Therefore, for $\beta < \alpha^2$ the function $\mathscr{A}''\big(h;\alpha,\beta\big)$ is monotonic and the quintic function has one inflection point at infinity. For $\beta > \alpha^2$ the function $\mathscr{A}''\big(h;\alpha,\beta\big)$ is not monotonic, and:

- for $-0.5\,\alpha^2 < \beta < in_1\,\alpha^2$ the quintic function has one inflection point at infinity;
- for $in_1\,\alpha^2 < \beta < 0$ and for $0 < \beta < in_2\,\alpha^2$ the quintic function has three inflection points at infinity;
- for $in_2\,\alpha^2 < \beta < in_3\,\alpha^2$ the quintic function has one inflection point at infinity;
- for $\beta > in_3\,\alpha^2$ the quintic function has three inflection points at infinity.

This is summarized in figure D.1, where the parabolas $\beta = -0.5\,\alpha^2$, $\beta = in_1\,\alpha^2$, $\beta = in_2\,\alpha^2$ and $\beta = in_3\,\alpha^2$ are displayed together with the five-roots-at-infinity parabolas (which are the dashed curves).

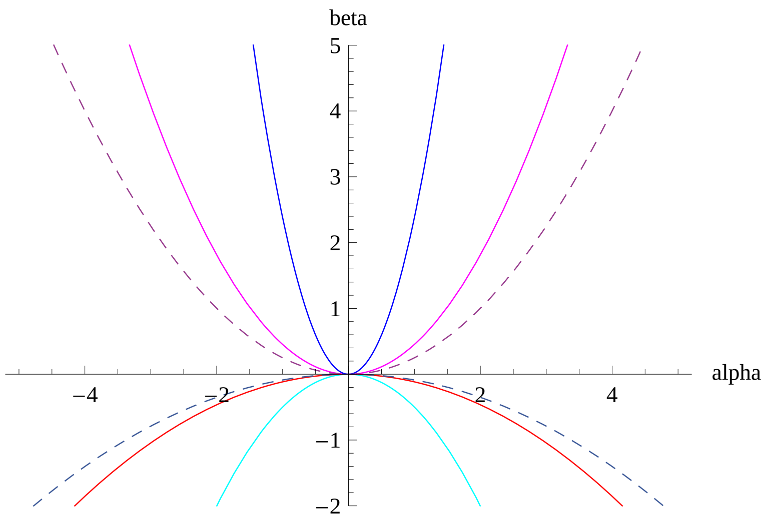


Figure D.1: Inflection points at infinity and five roots parabolas

### D.1.2 Inflection points and fixed points

To study analytically the matching of solutions, it is useful to know if the inflection points of the asymptotic function are located at a value of $h$ which is larger or smaller than the fixed points $h = \mathbf{F}_{\pm}$. We consider here only the case $\alpha > 0$ and $\beta > 0$, since $\mathbf{F}_{\pm}$ are defined only for $\beta$ positive.

Let's consider first the negative root $r_0$. The properties (D.6) imply that $\mathscr{A}''$ is negative for $\beta < r_0$, while is positive for $r_0 < \beta < 0$: therefore, we have that if $\mathscr{A}''\big(\mathbf{F}_-;\alpha,\beta\big) < 0$, then we have $\mathbf{F}_- < r_0$, while if $\mathscr{A}''\big(\mathbf{F}_-;\alpha,\beta\big) > 0$ we have $\mathbf{F}_- > r_0$. Indicating $x = \sqrt{\beta}/\alpha$, we have explicitly

$$\mathscr{A}''\big(\mathbf{F}_-;\alpha,\beta\big) = 2\frac{\alpha^2}{\sqrt{3\beta}}\left(x^2 + 3\sqrt{3}x + 3\right) \quad , \tag{D.11}$$

and the roots of the quadratic equation $x^2 + 3\sqrt{3}x + 3 = 0$ are both negative. Therefore, for $\alpha > 0$ we have $\mathscr{A}''\big(\mathbf{F}_-;\alpha,\beta\big) > 0$, which implies that $r_0 < \mathbf{F}_-$.

Let's consider now the positive roots $r_1$ and $r_2$, and let's introduce the convention $r_1 < r_2$. The properties (D.6) imply that $\mathscr{A}''$ is positive for $0 < \beta < r_1$ and $\beta > r_2$, while is negative for $r_1 < \beta < r_2$: therefore, we have that if $\mathscr{A}''\big(\mathbf{F}_+;\alpha,\beta\big) < 0$, then $r_1 < \mathbf{F}_+ < r_2$. On the other hand, if $\mathscr{A}''\big(\mathbf{F}_+;\alpha,\beta\big) > 0$ it follows that either $\mathbf{F}_+ < r_1 < r_2$ or $r_1 < r_2 < \mathbf{F}_+$: in particular, we have that if $\mathscr{A}'''\big(\mathbf{F}_+;\alpha,\beta\big) < 0$ then $\mathbf{F}_+ < r_1 < r_2$, while if $\mathscr{A}'''\big(\mathbf{F}_+;\alpha,\beta\big) > 0$ we have $r_1 < r_2 < \mathbf{F}_+$. Indicating $x = \sqrt{\beta}/\alpha$, we have explicitly

$$\mathscr{A}''\big(\mathbf{F}_+;\alpha,\beta\big) = -2\frac{\alpha^2}{\sqrt{3\beta}}\left(x^2 - 3\sqrt{3}x + 3\right) \quad , \tag{D.12}$$

and the roots of the quadratic equation $x^2 - 3\sqrt{3}x + 3 = 0$ are

$$x_{12} = \frac{\sqrt{3}}{2}\left(3 \pm \sqrt{5}\right) \quad . \tag{D.13}$$

Defining $k_1 = (3/4)\left(3 - \sqrt{5}\right)^2$ and $k_2 = (3/4)\left(3 + \sqrt{5}\right)^2$, we have

$$\mathscr{A}''\big(\mathbf{F}_+;\alpha,\beta\big) \ : \qquad \begin{cases} < 0 \quad \text{for} \quad 0 < \beta < k_1\alpha^2 \ \text{ and } \ \beta > k_2\alpha^2 \\ > 0 \quad \text{for} \quad k_1\alpha^2 < \beta < k_2\alpha^2 \end{cases} \tag{D.14}$$

where $k_1$ and $k_2$ have the approximate values $k_1 \simeq 0.437694$ and $k_2 \simeq 20.5623$. Furthermore, we have

$$\mathscr{A}'''\big(\mathbf{F}_+;\alpha,\beta\big) = 6\left(3\beta - \alpha^2\right) \tag{D.15}$$

and so

$$\mathscr{A}'''\big(\mathbf{F}_+;\alpha,\beta\big) \qquad : \qquad \begin{cases} < 0 \quad \text{for} \quad 0 < \beta < \dfrac{1}{3}\,\alpha^2 \\ > 0 \quad \text{for} \quad \beta > \dfrac{1}{3}\,\alpha^2 \end{cases} \tag{D.16}$$

We can then conclude that

- for $0 < \beta < k_1\,\alpha^2$ we have the ordering $r_1 < \mathbf{F}_+ < r_2$;
- for $k_1\,\alpha^2 < \beta < in_2\,\alpha^2$ we have the ordering $r_1 < r_2 < \mathbf{F}_+$;
- for $in_2\,\alpha^2 < \beta < in_3\,\alpha^2$ there are no inflection points for $h > 0$;
- for $in_3\,\alpha^2 < \beta < k_2\,\alpha^2$ we have the ordering $r_1 < r_2 < \mathbf{F}_+$;
- for $\beta > k_2\,\alpha^2$ we have the ordering $r_1 < \mathbf{F}_+ < r_2$.

# Appendix E

# Roots at infinity

We continue the summary started in the previous appendix about the main properties of the quintic function (4.42) when $A = 0$, which corresponds to the asymptotic limit $\rho \to +\infty$. We want to study here how many zeros the asymptotic function has, in relation to the value of $\alpha$ and $\beta$.

## E.1 Zeros of the asymptotic function

The asymptotic function (D.1) is a quintic, and therefore can have at most five real zeros. As we explained in section 4.3, $h = 0$ is always a zero, and in fact a simple one[1]. From the factorization (D.2) it follows that, to find the other zeros of the asymptotic function, we can study the zeros of the reduced asymptotic function $\mathscr{A}_r\big(h; \alpha, \beta\big)$

$$\mathscr{A}_r\big(h; \alpha, \beta\big) = \frac{3}{2}\,\beta^2\,h^4 - \big(\alpha^2 + 2\beta\big)\,h^2 + 3\,\alpha\,h - \frac{3}{2} \quad . \tag{E.1}$$

This function (see section 4.3) has always two zeros, one positive and one negative, and can have up to 4 real zeros, depending on the specific values of $\alpha$ and $\beta$.

[1] As we mentioned in section 4.5.1, we say that $y$ is a simple/double zero of a function $f$ if $y$ is a simple/double root of the equation $f = 0$

### E.1.1 Five-roots-at-infinity boundaries

The regions where the asymptotic functions has five zeros, if they exist, have to be inside the regions where there are three inflection points at infinity, since it is impossible to have five zeros and just one or two inflection points. Since the function $\mathscr{A}\big(h;\alpha,\beta\big)$ changes smoothly with $\alpha$ and $\beta$, the boundaries between regions where there are five zeros and regions where there are three zeros are found enforcing that $\mathscr{A}\big(h;\alpha,\beta\big) = 0$ has a multiple root. In this case the asymptotic function has to have a stationary point on the horizontal axis, and so if $h$ is the multiple root then we have $\mathscr{A}\big(h;\alpha,\beta\big) = \mathscr{A}'\big(h;\alpha,\beta\big) = 0$. Asking that this condition is satisfied for some $h$ and solving this condition with the software Mathematica, we get that the asymptotic function has a stationary point on the horizontal axis only if $\beta = c_+\,\alpha^2$ and $\beta = c_-\,\alpha^2$, where $c_+ = 1/4$ and $c_-$ is the only real root of the equation $8 + 48\,y - 435\,y^2 + 676\,y^3 = 0$ which has the approximate value $c_- \simeq -0.0876193$. The regions above the positive parabola and below the negative one have only three zeros, which are simple zeros, while the regions between the two parabolas (except $\beta = 0$) have five zeros, which are again simple zeros. On the boundaries $\beta = c_\pm\,\alpha^2$ between the three-zeros regions and the five-zeros regions there are four zeros, one of which is a zero of multiplicity two. Note that this result implies that for $\beta > in_3\alpha^2$, where in principle there could be five zeros (since there are three inflection points), there are only three zeros. This is summarized in figure 4.1.

These findings have been verified plotting the asymptotic function for many values of $\alpha$ and $\beta$. Note that, because of the symmetry (D.4), we can set $\alpha = 1$ and vary only the parameter $\beta$. In figure (E.1.1) we plot the asymptotic function for $\alpha = 1$ and increasing values of this parameter: because of space constraints, we plot the function only for fifteen values of $\beta$, and precisely for $\beta = -5\,,\ \beta = -1\,,\ \beta = -0.5\,,\ \beta = -0.2\,,\ \beta = -0.1\,,\ \beta = -0.09\,,\ \beta = c_-\,,\ \beta = -0.08\,,\ \beta = 0.19\,,\ \beta = c_+\,,\ \beta = 0.38\,,\ \beta = 0.5\,,\ \beta = 2\,,\ \beta = 5\,,\ \beta = 10\,.$ For the sake of precision, as already mentioned we don't plot the function itself but its composition with the tangent function, since this compactifies the real axis into the interval $(-\pi/2, +\pi/2)$ and at the same time does not change the number and the relative position of the zeros.

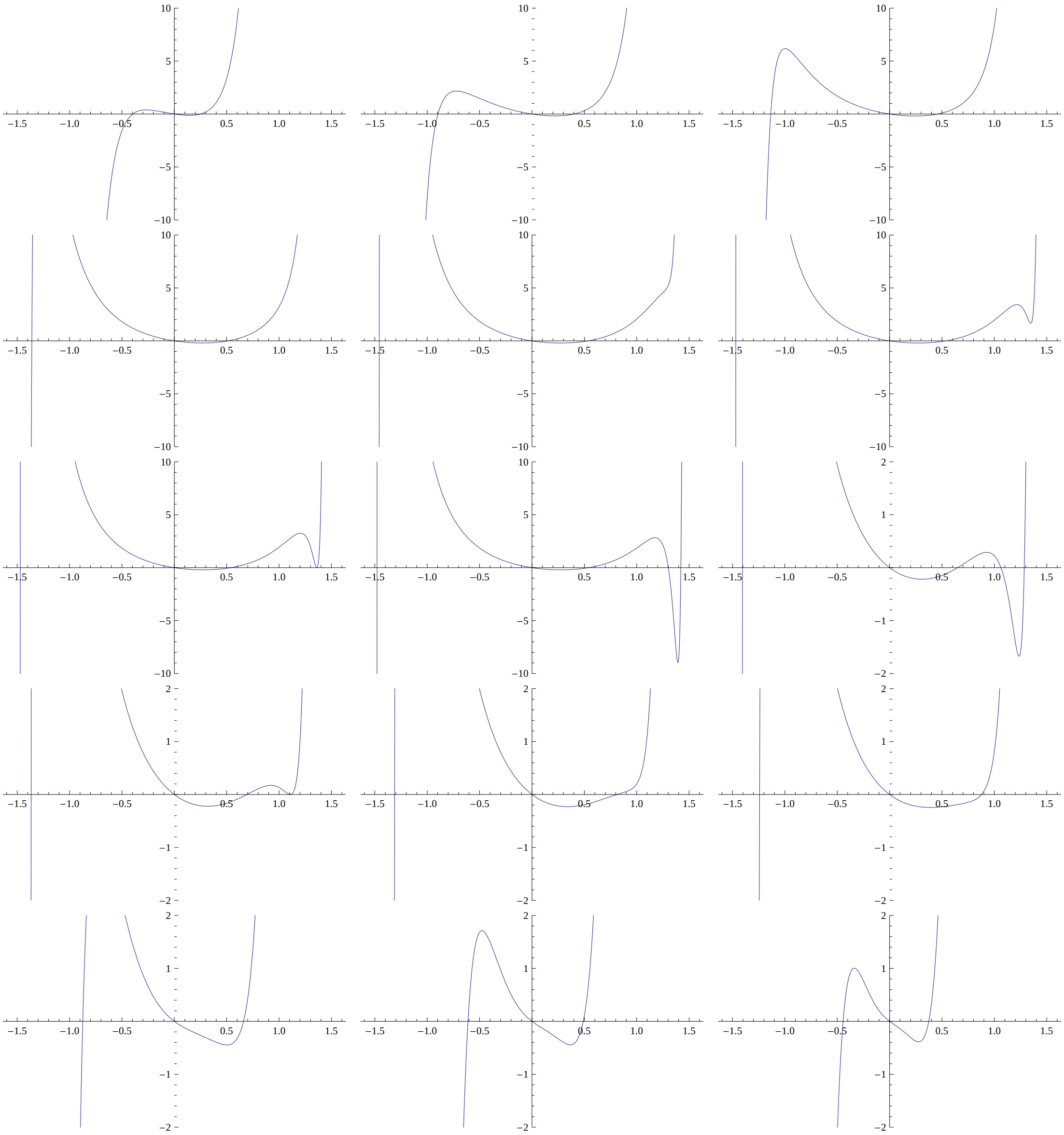

Figure E.1: Asymptotic function at $\alpha = 1$ for increasing values of $\beta$.

# Appendix F

# Leading behaviors

In this appendix we study the leading behaviors of the inner and asymptotic solutions. As previously mentioned we consider only the $\beta \neq 0$ case.

## F.1 Finite asymptotic and inner solutions

For the finite inner solutions $\mathbf{F}_{\pm}$ and finite non-zero asymptotic solutions $\mathbf{C}_{\pm}$ and $\mathbf{P}_{1,2}$, the behavior is

$$h(\rho) = C + R(\rho) \tag{F.1}$$

where $C \neq 0$ is their limiting value, and $R$ is respectively such that $\lim_{\rho \to 0^+} R = 0$ (inner solutions) and $\lim_{\rho \to +\infty} R = 0$ (asymptotic solutions).

## F.2 Asymptotic decaying solution L

Let's consider the solution $\mathbf{L}$, which satisfies $\lim_{\rho \to +\infty} h(\rho) = 0$. Dividing the quintic equation (4.41) by $h$, we get

$$\frac{3}{2}\beta^2 h^4 - \left(\alpha^2 + 2\beta\right) h^2 + 3\,\alpha\, h - \frac{3}{2} = \left(\frac{\rho_v}{\rho}\right)^3 \left(\frac{1}{h} - 3\,\beta\, h\right) \tag{F.2}$$

The left hand side has a finite limit when $\rho \to +\infty$, so the same has to hold for the right hand side: taking this limit in the equation above gives

$$\lim_{\rho \to +\infty} \left(\frac{\rho_v}{\rho}\right)^3 \frac{1}{h} = -\frac{3}{2} \tag{F.3}$$

which implies that

$$h(\rho) = -\frac{2}{3}\left(\frac{\rho_v}{\rho}\right)^3 + R(\rho) \tag{F.4}$$

with $\lim_{\rho\to+\infty} \rho^3 R(\rho) = 0$.

## F.3 Inner diverging solution D

Let's consider now the solution **D**, which satisfies $\lim_{\rho\to 0^+}|h(\rho)| = +\infty$. Dividing the equation (4.46) by $v^3$, one finds that

$$v^2 - 3\,\beta = \left(\frac{\rho}{\rho_v}\right)^3 \frac{1}{v^3}\Big( -\frac{3}{2}\,v^4 + 3\,\alpha\, v^3 - \big(\alpha^2 + 2\beta\big)\, v^2 + \frac{3}{2}\,\beta^2\Big) \tag{F.5}$$

One more time, the left hand side has a finite limit when $\rho \to 0^+$, so the same should hold for the right hand side. Therefore, the $\rho \to 0^+$ limit in the equation above gives

$$\lim_{\rho\to 0^+}\left(\frac{\rho}{\rho_v}\right)^3 \frac{1}{v^3} = -\frac{2}{\beta} \tag{F.6}$$

and so

$$v(\rho) = -\sqrt[3]{\frac{\beta}{2}}\,\frac{\rho}{\rho_v} + \mathrm{R}(\rho) \tag{F.7}$$

with $\lim_{\rho\to 0^+} \mathrm{R}(\rho)/\rho = 0$. To understand the behavior of the gravitational potentials (4.37)-(4.38) in this case, it is useful to calculate the next to leading order behavior. In fact, it turns out that, after going back to $h = 1/v$, the leading behavior precisely cancels the Schwarzschild-like contribution, so to understand if the gravitational potentials are finite at the origin it is essential to know how R behaves for very small radii. Inserting (F.7) into (4.46) and dividing by $x^5$, one obtains taking the limit $\rho \to 0^+$ that

$$\lim_{\rho\to 0^+}\frac{\mathrm{R}}{x^3} = \frac{1}{9\,\beta}\left(\alpha^2 + \frac{3}{2}\,\beta\right) \tag{F.8}$$

where $x = \rho/\rho_v$. We have then

$$v(\rho) = -\sqrt[3]{\frac{\beta}{2}}\,\frac{\rho}{\rho_v} + \mathcal{N}\left(\frac{\rho}{\rho_v}\right)^3 + \mathcal{R}(\rho) \tag{F.9}$$

where

$$\mathcal{N} = \frac{1}{9\,\beta}\left(\alpha^2 + \frac{3}{2}\,\beta\right) \tag{F.10}$$

and $\lim_{\rho\to 0^+}(\mathcal{R}(\rho)/\rho^3) = 0$. Finally, going back to the function $h$ we get

$$h(\rho) = -\sqrt[3]{\frac{2}{\beta}}\,\frac{\rho_v}{\rho} - \mathcal{M}\,\frac{\rho}{\rho_v} + \mathscr{R}(\rho) \tag{F.11}$$

where

$$\mathcal{M} = \frac{1}{9}\,\sqrt[3]{\frac{4}{\beta^5}}\left(\alpha^2 + \frac{3}{2}\,\beta\right) \tag{F.12}$$

and $\lim_{\rho\to 0^+}(\mathscr{R}(\rho)/\rho) = 0$. It can be shown that in the special case $\alpha^2 + 3\,\beta/2 = 0$, the next to leading order term scales as $\rho^2$ instead of $\rho$, and that $\lim_{\rho\to 0^+}(\mathscr{R}(\rho)/\rho^2) = 0$.

Therefore, we can conclude that in general the diverging inner solution **D** is such that

$$h(\rho) = -\sqrt[3]{\frac{2}{\beta}}\,\frac{\rho_v}{\rho} + R(\rho) \tag{F.13}$$

where $\lim_{\rho\to 0^+}(R(\rho)/\rho)$ is finite.

# Bibliography


[1] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation*. W. H. Freeman, San Francisco, 1973.

[2] Edward W. Kolb and Michael Turner. *The Early Universe*. Westview Press, 1990.

[3] Robert M. Wald. *General Relativity*. The University of Chicago Press, Chicago and London, 1984.

[4] Albert Einstein. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen der Physik*, 49:769822, 1916.

[5] Albert Einstein. *Über die spezielle und die allgemeine Relativitätstheorie*. F. Vieweg and Sohn, Braunschweig, 1920.

[6] Arno A. Penzias and Robert W. Wilson. A Measurement of Excess Antenna Temperature at 4080 Mc/s. *Astrophysical Journal*, 142:419421, 1965.

[7] Antonio Riotto. Inflation and the theory of cosmological perturbations. 2002.

[8] Mark Trodden and Sean M. Carroll. TASI lectures: Introduction to cosmology. 2004.

[9] Scott Dodelson. *Modern Cosmology*. Academic Press, San Diego, California, 2003.

[10] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Addison-Wesley, San Francisco, 2004.

[11] Adam G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.*, 116:1009–1038, 1998.

[12] S. Perlmutter et al. Measurements of Omega and Lambda from 42 high redshift supernovae. *Astrophys. J.*, 517:565–586, 1999.

[13] Adam G. Riess, Lucas Macri, Stefano Casertano, Megan Sosey, Hubert Lampeitl, et al. A Redetermination of the Hubble Constant with the Hubble Space Telescope from a Differential Distance Ladder. *Astrophys. J.*, 699:539–563, 2009.

[14] Will J. Percival et al. Baryon Acoustic Oscillations in the Sloan Digital Sky Survey Data Release 7 Galaxy Sample. *Mon. Not. Roy. Astron. Soc.*, 401:2148–2168, 2010.

[15] E. Komatsu et al. Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. *Astrophys.J.Suppl.*, 192:18, 2011.

[16] T. Padmanabhan. Why Does Gravity Ignore the Vacuum Energy? *Int.J.Mod.Phys.*, D15:2029–2058, 2006.

[17] Ruth Durrer and Roy Maartens. Dark Energy and Modified Gravity. 2008.

[18] Steven Weinberg. The Cosmological Constant Problem. *Reviews of Modern Physics*, 61:122, 1989.

[19] Leonard Susskind. The Anthropic landscape of string theory. 2003.

[20] Kari Enqvist and Teppo Mattsson. The effect of inhomogeneous expansion on the supernova observations. *JCAP*, 0702:019, 2007.

[21] R. Ali Vanderveld, Eanna E. Flanagan, and Ira Wasserman. Mimicking dark energy with Lemaitre-Tolman-Bondi models: Weak central singularities and critical points. *Phys.Rev.*, D74:023506, 2006.

[22] George Ellis, Hermann Nicolai, Ruth Durrer, and Roy Maartens. Editorial on the GRG special issue on dark energy. *Gen.Rel.Grav.*, 40:219–220, 2008.

[23] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified Gravity and Cosmology. *Phys.Rept.*, 513:1–189, 2012.

[24] Theodor Kaluza. Zum Unitätsproblem in der Physik. *Sitzungsber.Preuss.Akad.Wiss.Phys.Math*, 96:6972, 1921.

[25] T. Appelquist, A. Chodos, and P.G.O. Freund. *Modern Kaluza-Klein Theories.* Addison-Wesley, Menlo Park, 1987.

[26] O. Klein. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift fur Physik*, 37:895–906, 1926.

[27] Christian Fronsdal. Completion and Embedding of the Schwarzschild Solution. *Phys.Rev.*, 116:778, 1959.

[28] D.W. Joseph. Coordinate covariance and the particle spectrum. *Phys.Rev.*, 126:319, 1962.

[29] K. Akama. An Early Proposal of 'Brane World'. *Lect.Notes Phys.*, 176:267–271, 1982.

[30] V.A. Rubakov and Shaposhnikov M.E. Do we live inside a domain wall? *Phys. Lett.*, B125:136, 1983.

[31] Matt Visser. An Exotic Class of Kaluza-Klein Models. *Phys.Lett.*, B159:22, 1985.

[32] Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. The Hierarchy problem and new dimensions at a millimeter. *Phys.Lett.*, B429:263–272, 1998.

[33] Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity. *Phys.Rev.*, D59:086004, 1999.

[34] Ignatios Antoniadis, Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. New dimensions at a millimeter to a Fermi and superstrings at a TeV. *Phys.Lett.*, B436:257–263, 1998.

[35] Lisa Randall and Raman Sundrum. A Large mass hierarchy from a small extra dimension. *Phys.Rev.Lett.*, 83:3370–3373, 1999.

[36] Lisa Randall and Raman Sundrum. An Alternative to compactification. *Phys.Rev.Lett.*, 83:4690–4693, 1999.

[37] Ruth Gregory, V.A. Rubakov, and Sergei M. Sibiryakov. Opening up extra dimensions at ultra large scales. *Phys.Rev.Lett.*, 84:5928–5931, 2000.

[38] G.R. Dvali, G. Gabadadze, and M. Porrati. Metastable gravitons and infinite volume extra dimensions. *Phys.Lett.*, B484:112–118, 2000.

[39] Csaba Csaki, Joshua Erlich, and Timothy J. Hollowood. Quasilocalization of gravity by resonant modes. *Phys.Rev.Lett.*, 84:5932–5935, 2000.

[40] G.R. Dvali, Gregory Gabadadze, and Massimo Porrati. 4-D gravity on a brane in 5-D Minkowski space. *Phys.Lett.*, B485:208–214, 2000.

[41] Cedric Deffayet, G.R. Dvali, Gregory Gabadadze, and Arkady I. Vainshtein. Nonperturbative continuity in graviton mass versus perturbative discontinuity. *Phys.Rev.*, D65:044026, 2002.

[42] Takahiro Tanaka. Weak gravity in DGP brane world model. *Phys.Rev.*, D69:024001, 2004.

[43] Andrei Gruzinov. On the graviton mass. *New Astron.*, 10:311–314, 2005.

[44] M. Porrati. Fully covariant van Dam-Veltman-Zakharov discontinuity, and absence thereof. *Phys.Lett.*, B534:209–215, 2002.

[45] Markus A. Luty, Massimo Porrati, and Riccardo Rattazzi. Strong interactions and stability in the DGP model. *JHEP*, 0309:029, 2003.

[46] Alberto Nicolis and Riccardo Rattazzi. Classical and quantum consistency of the DGP model. *JHEP*, 0406:059, 2004.

[47] Cedric Deffayet. Cosmology on a brane in Minkowski bulk. *Phys.Lett.*, B502:199–208, 2001.

[48] Cedric Deffayet, G.R. Dvali, and Gregory Gabadadze. Accelerated universe from gravity leaking to extra dimensions. *Phys.Rev.*, D65:044023, 2002.

[49] Gia Dvali, Gregory Gabadadze, and M. Shifman. Diluting cosmological constant in infinite volume extra dimensions. *Phys.Rev.*, D67:044020, 2003.

[50] Dmitry Gorbunov, Kazuya Koyama, and Sergei Sibiryakov. More on ghosts in DGP model. *Phys.Rev.*, D73:044016, 2006.

[51] Christos Charmousis, Ruth Gregory, Nemanja Kaloper, and Antonio Padilla. DGP Specteroscopy. *JHEP*, 0610:066, 2006.

[52] Roy Maartens and Elisabetta Majerotto. Observational constraints on self-accelerating cosmology. *Phys.Rev.*, D74:023004, 2006.

[53] Gia Dvali, Stefan Hofmann, and Justin Khoury. Degravitation of the cosmological constant and graviton width. *Phys.Rev.*, D76:084006, 2007.

[54] R. Geroch and J.H. Traschen. Strings and Other Distributional Sources in General Relativity. *Phys.Rev.*, D36:1017, 1987.

[55] James M. Cline, Julie Descheneau, Massimo Giovannini, and Jeremie Vinet. Cosmology of codimension two brane worlds. *JHEP*, 0306:048, 2003.

[56] S.L. Dubovsky and V.A. Rubakov. Brane induced gravity in more than one extra dimensions: Violation of equivalence principle and ghost. *Phys.Rev.*, D67:104014, 2003.

[57] Marko Kolanovic, Massimo Porrati, and Jan-Willem Rombouts. Regularization of brane induced gravity. *Phys.Rev.*, D68:064018, 2003.

[58] Claudia de Rham, Gia Dvali, Stefan Hofmann, Justin Khoury, Oriol Pujolas, et al. Cascading gravity: Extending the Dvali-Gabadadze-Porrati model to higher dimension. *Phys.Rev.Lett.*, 100:251603, 2008.

[59] Masato Minamitsuji. Self-accelerating solutions in cascading DGP braneworld. *Phys.Lett.*, B684:92–95, 2010.

[60] Claudia de Rham, Stefan Hofmann, Justin Khoury, and Andrew J. Tolley. Cascading Gravity and Degravitation. *JCAP*, 0802:011, 2008.

[61] Parvin Moyassari and Masato Minamitsuji. Degravitation Features in the Cascading Gravity Model. 2011.

[62] Claudia de Rham, Justin Khoury, and Andrew J. Tolley. Cascading Gravity is Ghost Free. *Phys.Rev.*, D81:124027, 2010.

[63] G.W. Gibbons and S.W. Hawking. Action Integrals And Partition Functions In Quantum Gravity. *Phys.Rev.*, D15:2752, 1977.

[64] S.W. Hawking and G.T. Horowitz. The Gravitational Hamiltonian, action, entropy and surface terms. *Class.Quant.Grav.*, 13:1487, 1996.

[65] W. Israel. Singular Hypersurfaces and Thin Shells in General Relativity. *Nuovo Cim.*, 44:4349, 1966.

[66] Jaume Garriga and Takahiro Tanaka. Gravity in the brane world. *Phys.Rev.Lett.*, 84:2778–2781, 2000.

[67] Arthur Lue. The phenomenology of dvali-gabadadze-porrati cosmologies. *Phys.Rept.*, 423:1–48, 2006.

[68] Alberto Nicolis, Riccardo Rattazzi, and Enrico Trincherini. The Galileon as a local modification of gravity. *Phys.Rev.*, D79:064036, 2009.

[69] A.I. Vainshtein. To the problem of nonvanishing gravitation mass. *Phys.Lett.*, B39:393, 1972.

[70] Sara Rydbeck, Malcolm Fairbairn, and Ariel Goobar. Testing the DGP model with ESSENCE. *JCAP*, 0705:003, 2007.

[71] Wenjuan Fang, Sheng Wang, Wayne Hu, Zoltan Haiman, Lam Hui, et al. Challenges to the DGP Model from Horizon-Scale Growth and Geometry. *Phys.Rev.*, D78:103509, 2008.

[72] Kazuya Koyama. Are there ghosts in the self-accelerating brane universe? *Phys.Rev.*, D72:123511, 2005.

[73] Kazuya Koyama. Ghosts in the self-accelerating universe. *Class.Quant.Grav.*, 24:R231–R253, 2007.

[74] M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc.Roy.Soc.Lond.*, A173:211–232, 1939.

[75] D.G. Boulware and S. Deser. Can gravitation have a finite range? *Phys.Rev.*, D6:3368–3382, 1972.

[76] V.A. Rubakov. Lorentz-violating graviton masses: Getting around ghosts, low strong coupling scale and VDVZ discontinuity. 2004.

[77] Y. Iwasaki. Consistency condition for propagators. *Phys.Rev.*, D2:2255–2256, 1970.

[78] H. van Dam and M.J.G. Veltman. Massive and massless Yang-Mills and gravitational fields. *Nucl.Phys.*, B22:397–411, 1970.

[79] V.I. Zakharov. Massive and massless Yang-Mills and gravitational fields. *JETP Letters (Sov.Phys.)*, 12:312, 1970.

[80] Claudia de Rham and Gregory Gabadadze. Generalization of the Fierz-Pauli Action. *Phys.Rev.*, D82:044020, 2010.

[81] Claudia de Rham, Gregory Gabadadze, and Andrew J. Tolley. Resummation of Massive Gravity. *Phys.Rev.Lett.*, 106:231101, 2011.

[82] Kurt Hinterbichler. Theoretical Aspects of Massive Gravity. *Rev.Mod.Phys.*, 84:671–710, 2012.

[83] P. Van Nieuwenhuizen. On ghost-free tensor lagrangians and linearized gravitation. *Nucl.Phys.*, B60:478–492, 1973.

[84] Hans C. Ohanian and Remo Ruffini. *Gravitation and Spacetime (Second Edition)*. W. W. Norton & Company, 1994.

[85] Thibault Damour, Ian I. Kogan, and Antonios Papazoglou. Spherically symmetric space-times in massive gravity. *Phys.Rev.*, D67:064009, 2003.

[86] E. Babichev, C. Deffayet, and R. Ziour. The Vainshtein mechanism in the Decoupling Limit of massive gravity. *JHEP*, 0905:098, 2009.

[87] Cedric Deffayet. Spherically symmetric solutions of massive gravity. *Class.Quant.Grav.*, 25:154007, 2008.

[88] E. Babichev, C. Deffayet, and R. Ziour. Recovering General Relativity from massive gravity. *Phys.Rev.Lett.*, 103:201102, 2009.

[89] E. Babichev, C. Deffayet, and R. Ziour. The Recovery of General Relativity in massive gravity via the Vainshtein mechanism. *Phys.Rev.*, D82:104008, 2010.

[90] Nima Arkani-Hamed, Howard Georgi, and Matthew D. Schwartz. Effective field theory for massive gravitons and gravity in theory space. *Annals Phys.*, 305:96–118, 2003.

[91] M. Ostrogradski. . *Mem.Ac.St.Petersbourg VI*, 4:385, 1850.

[92] Richard P. Woodard. Avoiding dark energy with 1/r modifications of gravity. *Lect.Notes Phys.*, 720:403–433, 2007.

[93] Paolo Creminelli, Alberto Nicolis, Michele Papucci, and Enrico Trincherini. Ghosts in massive gravity. *JHEP*, 0509:003, 2005.

[94] Cedric Deffayet and Jan-Willem Rombouts. Ghosts, strong coupling and accidental symmetries in massive gravity. *Phys.Rev.*, D72:044003, 2005.

[95] Gregory Gabadadze. General Relativity With An Auxiliary Dimension. *Phys.Lett.*, B681:89–95, 2009.

[96] Claudia de Rham. Massive gravity from Dirichlet boundary conditions. *Phys.Lett.*, B688:137–141, 2010.

[97] Claudia de Rham and Gregory Gabadadze. Selftuned Massive Spin-2. *Phys.Lett.*, B693:334–338, 2010.

[98] Kazuya Koyama, Gustavo Niz, and Gianmassimo Tasinato. The Self-Accelerating Universe with Vectors in Massive Gravity. *JHEP*, 1112:065, 2011.

[99] S.F. Hassan and Rachel A. Rosen. Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity. *JHEP*, 1204:123, 2012.

[100] Mehrdad Mirbabayi. A Proof Of Ghost Freedom In de Rham-Gabadadze-Tolley Massive Gravity. 2011.

[101] J. Kluson. Non-Linear Massive Gravity with Additional Primary Constraint and Absence of Ghosts. *Phys.Rev.*, D86:044024, 2012.

[102] Kurt Hinterbichler and Rachel A. Rosen. Interacting Spin-2 Fields. *JHEP*, 1207:047, 2012.

[103] Claudia de Rham, Gregory Gabadadze, Lavinia Heisenberg, and David Pirtskhalava. Cosmic Acceleration and the Helicity-0 Graviton. *Phys.Rev.*, D83:103516, 2011.

[104] Kazuya Koyama, Gustavo Niz, and Gianmassimo Tasinato. Analytic solutions in non-linear massive gravity. *Phys.Rev.Lett.*, 107:131101, 2011.

[105] Kazuya Koyama, Gustavo Niz, and Gianmassimo Tasinato. Strong interactions and exact solutions in non-linear massive gravity. *Phys.Rev.*, D84:064033, 2011.

[106] Th.M. Nieuwenhuizen. Exact Schwarzschild-de Sitter black holes in a family of massive gravity models. *Phys.Rev.*, D84:024038, 2011.

[107] Andrei Gruzinov and Mehrdad Mirbabayi. Stars and Black Holes in Massive Gravity. *Phys.Rev.*, D84:124019, 2011.

[108] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, and A.J. Tolley. On Black Holes in Massive Gravity. *Phys.Rev.*, D85:044024, 2012.

[109] Mikhail S. Volkov. Hairy black holes in the ghost-free bigravity theory. *Phys.Rev.*, D85:124043, 2012.

[110] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo. Spherically Symmetric Solutions in Ghost-Free Massive Gravity. *Phys.Rev.*, D85:024044, 2012.

[111] A. Emir Gumrukcuoglu, Chunshan Lin, and Shinji Mukohyama. Open FRW universes and self-acceleration from nonlinear massive gravity. *JCAP*, 1111:030, 2011.

[112] A. Emir Gumrukcuoglu, Chunshan Lin, and Shinji Mukohyama. Cosmological perturbations of self-accelerating universe in nonlinear massive gravity. *JCAP*, 1203:006, 2012.

[113] Nemanja Kaloper, Antonio Padilla, and Norihiro Tanahashi. Galileon Hairs of Dyson Spheres, Vainshtein's Coiffure and Hirsute Bubbles. *JHEP*, 1110:148, 2011.

[114] Rampei Kimura, Tsutomu Kobayashi, and Kazuhiro Yamamoto. Vainshtein screening in a cosmological background in the most general second-order scalar-tensor theory. *Phys.Rev.*, D85:024023, 2012.

[115] Antonio De Felice, Ryotaro Kase, and Shinji Tsujikawa. Vainshtein mechanism in second-order scalar-tensor theories. *Phys.Rev.*, D85:044059, 2012.

[116] Nathan Jacobson. *Basic Algebra 1 (2nd edition).* Dover, 2009.

[117] Giga Chkareuli and David Pirtskhalava. Vainshtein Mechanism In $\Lambda_3$ - Theories. *Phys.Lett.*, B713:99–103, 2012.

[118] Stefan Sjors and Edvard Mortsell. Spherically Symmetric Solutions in Massive Gravity and Constraints from Galaxies. 2011.

[119] Douglas A. Clarke. *Foundations of Analysis.* Appleton-Century-Crofts, 1971.

[120] Walter Rudin. *Principles of Mathematical Analysis, 3rd edition.* McGraw-Hill, 1976.

[121] Enrico Giusti. *Analisi Matematica 2 (in Italian).* Bollati Boringhieri, 1983.