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Citation: Chaos **22**, 033124 (2012); doi: 10.1063/1.4745851
View online: http://dx.doi.org/10.1063/1.4745851
View Table of Contents: http://chaos.aip.org/resource/1/CHAOEH/v22/i3
Published by the American Institute of Physics.

Transition from order to chaos, and density limit, in magnetized plasmas

A. Carati,[1,a)] M. Zuin,[2,b)] A. Maiocchi,[1] M. Marino,[1] E. Martines,[2] and L. Galgani[1]
[1]*Università degli Studi di Milano, Milano, Italy*
[2]*Consorzio RFX, Associazione EURATOM-ENEA sulla Fusione, Padova, Italy*

(Received 8 February 2012; accepted 30 July 2012; published online 16 August 2012)

It is known that a plasma in a magnetic field, conceived microscopically as a system of point charges, can exist in a magnetized state, and thus remain confined, inasmuch as it is in an ordered state of motion, with the charged particles performing gyrational motions transverse to the field. Here, we give an estimate of a threshold, beyond which transverse motions become chaotic, the electrons being unable to perform even one gyration, so that a breakdown should occur, with complete loss of confinement. The estimate is obtained by the methods of perturbation theory, taking as perturbing force acting on each electron that due to the so–called microfield, i.e., the electric field produced by all the other charges. We first obtain a general relation for the threshold, which involves the fluctuations of the microfield. Then, taking for such fluctuations, the formula given by Iglesias, Lebowitz, and MacGowan for the model of a one component plasma with neutralizing background, we obtain a definite formula for the threshold, which corresponds to a density limit increasing as the square of the imposed magnetic field. Such a theoretical density limit is found to fit pretty well the empirical data for collapses of fusion machines. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4745851]

The existence of a transition from order to chaos in Hamiltonian systems, as a generic phenomenon occurring when a perturbation is added to an integrable system, is a well established fact. This fact involves deep mathematical features (see for example Ref. 1) and was made popular in the scientific community through the striking pictures of Hénon and Heiles[2] and the discovery, by Izrailev and Chirikov,[3] which the ordered motions found by Fermi, Pasta, and Ulam in their model[4] become chaotic above a certain threshold (see Figs. 4.3 and 4.5 of the review 5 or 6)). Transitions of this type were met also in the frame of plasma physics, in connection with the destruction of magnetic surfaces,[7,8] and also with the chaoticity thus induced on single particle motions.[9] See Refs. 10–12. On the other hand, in plasma physics, a phenomenon of great relevance exists that is yet unexplained, and for which we propose here an explanation just in terms of a transition from order to chaos. We refer to the loss of plasma confinement, a plasma collapse that is met when the plasma density is increased beyond a certain density limit (see Ref. 13, Fig. 3). Let us recall that confinement (i.e., keeping the charged particles away from the walls) is actually achieved by means of a suitable magnetic field, the form of which depends on the concrete machine (either just a field imposed from outside or a superposition of the imposed one with that due to a plasma current). So, when the phenomenon of destruction of magnetic surfaces was understood, people thought that it might play a role in explaining the breakdown occurring at the density limit (see for example Ref. 14). However, such considerations did not prove sufficient to explain the quick collapses of plasmas. Here, we propose a solution of a different character, completely unrelated to peculiarities of the field lines, up to the point of applying even in the extremely idealized case in which the field is uniform, so that the field lines are just straight parallel lines, covering the whole space (and the plasma is uniform too). We refer to the existence of a magnetic pressure, which is essential in keeping the particles away from the walls. The point is that such a pressure exists inasmuch as the plasma is diamagnetic, which means, in microscopic terms, that each electron is equivalent to a magnetic moment, just in virtue of its dynamical property of performing gyrational motions transverse to the field lines. This is the kind of ordered motions we are referring to. Indeed such ordered motions persist indefinitely in the unperturbed case, when one neglects the perturbation due to the so called microfield, i.e., the microscopic electric field acting on each charge and due to the Coulomb interactions with all the other ones (see the recent review Ref. 15). On the other hand, the intensity of such a perturbation clearly increases with the density, and so it seems natural to expect that when the perturbation is large enough, i.e., at a large enough density, a transition to a state of chaotic motions should occur, in which diamagnetism is lost, together with magnetic pressure. The proposal advanced here is that such a kind of transition may explain the loss of confinement, at least in its gross features, by providing a theoretical estimate of the density limit that should be compatible, as far as order of magnitude is concerned, with the observed ones. This is what we actually find out. Working with an extremely simple model, we predict a chaoticity threshold, which corresponds to a density limit that fits pretty well those observed in collapses of several kinds of fusion machines. To this end, we make use of quite recent results on perturbation theory holding in the thermodynamic limit,[16–18] and of an old result of of Iglesias,

a)Electronic mail: andrea.carati@unimi.it.
b)Electronic mail: matteo.zuin@igi.cnr.it.

Lebowitz and MacGowan[19] concerning the fluctuations of the microfield.

I. INTRODUCTION

The idea that the loss of confinement in magnetized plasmas corresponds to a transition from order to chaos is easily understood. Indeed, an essential point in guaranteeing confinement is the existence of a magnetic pressure. Now, in a macroscopic magnetohydrodynamic description of the problem, the existence of a magnetic pressure is derived from the constitutive equations of a plasma, i.e., from the assumption that the plasma be diamagnetic. However, from a microscopic point of view in which the plasma is modeled as a system of discrete charges, such an assumption has to be justified. In such a perspective, diamagnetism is a dynamical property that can be present or absent, according to the motions being ordered or chaotic. In fact, existence of magnetization corresponds to the prevailing of gyrational motions transverse to the field, whereas in the state of statistical equilibrium (i.e., with prevailing chaotic motions), magnetization vanishes (see for example Ref. 20). This breakdown is thus a global characteristic feature of magnetized plasmas, irrespective of the particular mechanism employed for obtaining confinement.

The conception that magnetization due to orbital motions can exist only in a nonequilibrium state, characterized by motions of ordered type, was apparently first proposed by Bohr (see Ref. 21, page 382). Now, Bohr took for granted that the relaxation time to equilibrium would be very short, as *"the collective motions of the electrons would disappear very rapidly."* On the other hand, we are well acquainted with the fact that the relaxation time from order to chaos can be very long, as occurs for example with glasses and with the Fermi, Pasta and Ulam (FPU) model, and was recently pointed out also in connection with orbital magnetization.[22] Thus, in order to establish up to which time is the magnetized state conserved, one should estimate a typical relaxation time after which the system becomes chaotic (see for example the *"characteristic time of mixing"* defined in Ref. 23, sec. 5).

It is well known that this is a quite hard task. However, estimates of the relaxation time from below are available through perturbation theory,[24–26] as we now recall. Indeed, in general such a theory allows one to construct adiabatic invariants $I^{(n)}$ at any order n, providing for their changes $I_t^{(n)} - I^{(n)}$ (where X_t denotes the time evolved at time t of any dynamical variable X) estimates which in their simplest form are of the type

$$\left| I_t^{(n)} - I^{(n)} \right| \le n! \epsilon^{n+1} \, \mathcal{I} \, \frac{t}{\tau}, \tag{1}$$

where ϵ is the perturbation parameter, while τ and $\mathcal{I}$ are a characteristic time and a characteristic value of I, of the system. Now, imposing $|I_t^{(n)} - I^{(n)}| \le \mathcal{I}\epsilon$, and recalling $n! \simeq (n/e)^n$, formula (1) gives $t \le \tau(e/n\epsilon)^n$ for all n, which, by taking the optimal value of n, $n(\epsilon) \simeq 1/\epsilon$, gives $t \le \tau \exp(1/\epsilon)$. Thus, a lower estimate to the relaxation time is obtained, which is exponentially long in $1/\epsilon$ as long as $\epsilon < 1$. It is thus clear that the condition $\epsilon = 1$ provides a natural chaoticity threshold, which should identify the relevant transition, at least as concerns the order of magnitude of the characteristic parameters of the problem. Indeed, for smaller ϵ, the motions keep an ordered character for practically infinite times, whereas for larger ϵ, the ordered character is not even guaranteed up to the microscopic time τ. As a matter of fact, the estimates for the changes of the adiabatic invariants are in general a little more complicated than Eq. (1), and the lower estimates for the relaxation time are found to increase as stretched (rather than pure) exponentials, but the conclusion for the chaoticity threshold to be drawn in a moment remains unaltered.

This classical scheme was implemented in a probabilistic frame in the paper.[27] Later, the scheme was shown to be applicable also for systems of macroscopic sizes, i.e., in the so called thermodynamic limit,[16–18] which is an essential point for our purposes. In such a probabilistic frame, one renounces to control the changes of the adiabatic invariant along all single trajectories and just controls mean properties with respect to a given invariant measure in phase space. For example, one can look at the time autocorrelation function $C_{I^{(n)}}(t)$ of the adiabatic invariant at order n, defined as usual by

$$C_{I^{(n)}}(t) = \langle I_t^{(n)} I^{(n)} \rangle - \langle I^{(n)} \rangle^2 ,$$

where $\langle \cdot \rangle$ denotes mean with respect to the given measure. In terms of the time autocorrelation function, the analogue of the classical estimate Eq. (1) then takes the form

$$\frac{C_{I^{(n)}}(t)}{\sigma^2_{I^{(n)}}} \ge 1 - \frac{1}{2} n! \epsilon^{n+1} \left(\frac{t}{\tau} \right)^2 ,$$

where $\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2$ is the variance of X. The latter provides a natural dimensional constant for the autocorrelation, since one has $C_X(0) = \sigma_X^2$. By optimization with respect to n, the time after which the adiabatic invariant may lose correlation is still found to be exponentially long in $1/\epsilon$, provided one has $\epsilon < 1$. So, $\epsilon = 1$ again turns out to be the perturbation estimate of the chaoticity threshold.

Our main task is thus to estimate the chaoticity threshold for a magnetized plasma, in the probabilistic frame just sketched. In Sec. II, we describe the model that will be studied, define the dynamical variable of interest (the component of the angular momentum of each electron along the field), and give the lowest order estimate for its time autocorrelation function. This leads to a natural conjecture for identifying the perturbation parameter ϵ, which then gives the chaoticity threshold by the condition $\epsilon = 1$. In Sec. III, we give a general formula for the threshold in terms of temperature and of the fluctuations of the microfield. Using the available analytical estimate of such fluctuations for the model of a one component plasma with a neutralizing background,[19] a definite formula for the theoretical density limit is then obtained. In Sec. IV, the theoretical density limit is compared to the empirical data for collapses in fusion machines. Some comments are finally added in the conclusions.

II. DEFINITION OF THE MODEL. CONJECTURE ON THE CHAOTICITY THRESHOLD

The model chosen for the magnetized plasma is the simplest one we could conceive in order to check the main idea of the present paper, namely, that the relevant feature concerning loss of confinement, regardless of the particular mechanism involved in each machine, is the occurring of a sharp transition from order to chaos as the perturbation due to the microfield is increased beyond a threshold. So, first of all, for what concerns the magnetic field $\mathbf{B}$, we consider the extremely idealized case in which it is uniform, say $\mathbf{B} = B\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector along the z axis. Concerning the plasma itself, the key point is that it should be conceived as a dynamical system of point charges and not as a continuum. Thus, any charge will be subject, in addition to the Lorentz confining force due to $\mathbf{B}$, also to the force of the microfield $\mathbf{E}$, defined as the vector sum of the Coulomb fields created by all the other charges. Mutual magnetic forces and retardation effects are neglected.

So, we have a dynamical system of several kinds of charges, and the Newton equation for the jth charge (in the nonrelativistic approximation) is then

$$m_j\ddot{\mathbf{x}}_j = e_j\mathbf{v}_j \wedge \mathbf{B} + e_j\mathbf{E}_j, \tag{2}$$

where m_j and e_j are the mass and the charge of the particle, $\mathbf{x}_j$ and $\mathbf{v}_j = \dot{\mathbf{x}}_j$ its position vector and velocity, and $\mathbf{E}_j$ the microfield evaluated at $\mathbf{x}_j$, i.e., the microscopic electric field acting on the jth particle and due to the Coulomb interactions with all the other ones; obviously $\mathbf{E}_j$ depends on the positions of all the charges. Finally, in order that the dynamical system be defined within the standard approach of ergodic theory, we consider as given also an invariant measure, a few minimal properties of which will be mentioned later.

If the microfield is neglected, the transverse motion of each particle is a uniform gyration about a field line with its characteristic cyclotron frequency $\omega_{cj} = |e_j|B/m_j$. So, the system is integrable, the z component of the angular momentum of each particle being a constant of motion. The microfield, acting as a perturbation, makes the system no more integrable.

For what concerns the adiabatic invariant to be investigated, in principle, we should look at the magnetization of the system, to which each charge contributes through the z component of its angular momentum. However, it is well known that only the electrons are relevant, the ions contribution to magnetization being negligible. So, we will consider the contribution to magnetization due to any single electron, i.e., the z component of its angular momentum. Since now on, the index j referring to a chosen electron will be left understood. Thus, as zeroth order approximation for the adiabatic invariant, we take the quantity

$$L = \frac{m^2}{eB}\, v_\perp^2; \tag{3}$$

($\mathbf{v}_\perp$ denoting transverse velocity of the chosen electron), which is proportional to the transverse kinetic energy of the electron. One immediately checks (see page 16 of the book of Alfvén,[28] or any plasma physics textbook) that L is the z component of the angular momentum of the chosen electron, referred to its instantaneous gyration center (or guiding center), the latter being calculated in the approximation in which the perturbing force is neglected.

We also add here the formula for the time derivative $\dot{L}$ of L, as we will need it in a moment. As L is a multiple of $\mathbf{v}_\perp \cdot \mathbf{v}_\perp$, $\dot{L}$ is immediately obtained through dot multiplication of Newton's Eq. (2) by $2\mathbf{v}_\perp$, which gives

$$\dot{L} = \frac{2m}{B}\,\mathbf{v}_\perp \cdot \mathbf{E}_\perp, \tag{4}$$

where $\mathbf{E}_\perp$ denotes the transverse component of the microfield.

We come now to the main point: to find the dimensionless perturbation parameter ϵ, which determines the chaoticity threshold corresponding to the destruction of the chosen adiabatic invariant (and of all the adiabatic invariants corresponding to each electron). This would require performing the corresponding perturbation estimates at all orders, which at the moment, we are unable to do. What we can easily do is to perform the zeroth order estimate for the time autocorrelation function of L, which turns out to be

$$\frac{C_L(t)}{\sigma_L^2} \geq 1 - \frac{1}{2}\frac{\sigma\dot{L}^2}{\omega_c^2\sigma_L^2}(\omega_c t)^2. \tag{5}$$

Notice that as characteristic microscopic time τ of the unperturbed electron's motion, we have naturally taken $1/\omega_c$.

The proof of Eq. (5) is rather simple. One starts from the elementary identity

$$C_L(t) = \sigma_L^2 - \frac{1}{2}\langle(L_t - L)^2\rangle$$

and uses the inequality $\langle(L_t - L)^2\rangle \leq \langle\dot{L}^2\rangle t^2$, which is just a function theoretic analogue of the Lagrange finite increment formula of elementary calculus, and basically follows from unitarity of the time evolution of the dynamical variables (see Ref. 16, Theorem 1, and Ref. 17, Sec. 7). This already gives inequality Eq. (5), with $\langle\dot{L}^2\rangle$ in place of $\sigma\dot{L}2$. Relation (5) then immediately follows by noting that, due to the time–invariance of the measure, for any dynamical variable X, one has $\langle\dot{X}\rangle = 0$, so that $\langle\dot{X}^2\rangle = \sigma_{\dot{X}}^2$.

As explained in the Introduction, from Eq. (5), we are led to conjecture that the relevant dimensionless parameter of the problem is

$$\epsilon = \frac{\sigma\dot{L}}{\omega_c\sigma_L}; \tag{6}$$

($\sigma_X = \sqrt{\sigma_X^2}$ denoting standard deviation), and this leads to a chaoticity threshold given by

$$\frac{\sigma\dot{L}}{\omega_c\sigma_L} = 1. \tag{7}$$

We add now a comment of a general character, which concerns the way in which a relaxation time for L (or analogously for any variable X) turns out to be identified in the present approach, which combines perturbation and statistical mechanics methods. From Eq. (5), one sees that the relaxation time t_L^{rel} of L is given by $t_L^{rel} = \sigma_L/\sigma_{\dot L}$, a formula which involves standard deviations. Now, compare such a formula with the one generally met in textbooks, i.e., $t_L^{rel} = L/\dot L$, where it should be understood that "typical values" are to be taken for the numerator and the denominator. But, this requires a great ingenuity from the part of the reader, especially when variables are involved which have vanishing mean. So, one may say that the identification of the relaxation time provided by perturbation theory in a probabilistic frame, namely, $t_L^{rel} = \sigma_L/\sigma_{\dot L}$, appears to be some definite quantitative implementation of the intuitive idea underlying the familiar informal definition and amounts to the prescription that the informal qualification "typical values" should be understood in the sense of "standard deviations."

We add now a final remark in which the previous comment is used in order to read in a quite transparent way the condition (7), which defines the chaoticity threshold. Indeed, through formula (9) of Sec. III, it will be seen that the condition for the threshold can be expressed in the form $(\sigma_{E_\perp}/B\sigma_v) = 1$. Thus, just in virtue of the previous comment relating standard deviations and typical values, one sees that the threshold occurs when the typical value of the perturbing force due to the microfield equals the typical value of the Lorentz force, which characterizes the unperturbed motions. So, the condition $\epsilon = 1$, which we have assumed as a definition of the threshold within a rather abstract point of view, is just what one would immediately guess, as the naivest implementation of the idea that a threshold occurs when the perturbing force equals the unperturbed one.

III. THE CHAOTICITY THRESHOLD IN TERMS OF MACROSCOPIC PARAMETERS: THE THEORETICAL DENSITY LIMIT

Our aim is now to express the chaoticity threshold (7) in terms of the macroscopic parameters T, n, B, temperature, electron number density, and field strength. Recalling the expressions (3) and (4) of L and $\dot L$, and the definition $\omega_c = |e|B/m$, the threshold (7) takes the form

$$\frac{2}{B}\,\frac{\sigma_{\mathbf{v}_\perp\cdot\mathbf{E}_\perp}}{\sigma_{v_\perp^2}} = 1. \tag{8}$$

It is clear that the standard deviations appearing in Eq. (8) depend on the model of plasma adopted, which determines the microfield, as well as on the chosen invariant measure. The choice of the invariant measure is a quite delicate problem, particularly in a nonequilibrium situation as the one we are discussing here. A general introduction may be found in the book.[29] For example, it is obvious that $\sigma_{\mathbf{v}_\perp}$ and $\sigma_{v_\perp^2}$ should be expressed in terms of temperature, *albeit* with coefficients, which depend on the assumptions made for the velocity distribution. Analogously, the statistical properties of the microfield may be different for a system composed by electrons plus a neutralizing background, rather than for a system of electrons and ions.

Quite natural assumptions on the measure are: (i) that velocities and positions are independent variables; (ii) that the distribution of the transverse velocities is Maxwellian at a temperature T; (iii) that the distribution of positions is isotropic. Under these natural assumptions, the Eq. (8) for the threshold is seen to take the form

$$\sqrt{\frac{2}{3}}\,\frac{\sigma_{\mathbf{E}}}{B\sqrt{k_BT/m}} = 1. \tag{9}$$

Indeed, from (i) and (ii), one gets

$$\sigma^2_{\mathbf{v}_\perp\cdot\mathbf{E}_\perp} = \frac{1}{2}\,\sigma^2_{\mathbf{v}_\perp}\,\sigma^2_{\mathbf{E}_\perp},$$

the variance of a vector $\mathbf{F}$ being defined by $\sigma^2_{\mathbf{F}} = \sigma^2_{F_x} + \sigma^2_{F_y} + \sigma^2_{F_z}$. One also gets

$$\frac{1}{2}\,\sigma^2_{\mathbf{v}_\perp} = \frac{k_BT}{m}$$

(k_B being the Boltzmann constant) and furthermore, as one easily checks,

$$\sigma^2_{v_\perp^2} = 4\left(\frac{k_BT}{m}\right)^2.$$

Finally, from (iii), one gets $\sigma^2_{\mathbf{E}_\perp} = (2/3)\,\sigma^2_{\mathbf{E}}$

The form (9) of the equation for the threshold already constitutes in our opinion a significant result. Indeed, the fluctuation $\sigma^2_{\mathbf{E}}$ of the microfield should in principle be itself a measurable quantity, which depends on the macroscopic state of the plasma, namely, electron number density n and the temperatures of the several constituents. So, the previous relation provides in principle the density limit as a function of the macroscopic state of the plasma.

However, we were unable to find in the literature sufficient experimental information on the fluctuation $\sigma^2_{\mathbf{E}}$ of the microfield. So, in order to have a definite theoretical formula to be compared with the experimental data, we limit ourselves to the consideration of a particular model for which an estimate of $\sigma^2_{\mathbf{E}}$ is available. In fact a formula for $\sigma^2_{\mathbf{E}}$ at equilibrium with respect to the Gibbs distribution was given by Iglesias, Lebowitz, and MacGowan[19] for the model of a one component plasma with neutralizing background, namely,

$$\sigma^2_{\mathbf{E}} = \frac{n\,k_BT}{\varepsilon_0}, \tag{10}$$

where ε_0 is the vacuum dielectric constant and n the electron number density (see Ref. 19, formula (2.5), substituting n for ρ and $1/\varepsilon_0$ for 4π).

So, for a one component plasma with neutralizing background at temperature T, the chaoticity threshold (9) takes the form

$$n = \frac{3}{2}\,\frac{\varepsilon_0}{m}\,B^2, \tag{11}$$

in which temperature disappeared, so that the threshold only involves density and field strength. Notice however that this might not be true with a more realistic model of a plasma, in which the temperature appearing in Eq. (10), which refers to the plasma as a whole, may be different from the electron transverse temperature which enters the previous formulas.

Formula (11) for the limit density (holding for a one component plasma with neutralizing background, at temperature T) is the type of result we were looking for, inasmuch as it provides a definite theoretical formula for the density limit that can be compared to the available empirical data for collapses in fusion machines, as will be done in Sec. IV.

Notice that formula (11) for the chaoticity threshold can be written in the enlightening form

$$\frac{\omega_c}{\omega_p} \simeq 1, \quad \text{or equivalently,} \quad \frac{\lambda_D}{r_L} \simeq 1, \tag{12}$$

where ω_c and ω_p are the cyclotron and plasma frequencies, while λ_D and r_L are the Debye length and the Larmor radius, with their usual meanings.

IV. COMPARISON WITH THE EMPIRICAL DATA FOR PLASMA COLLAPSES IN FUSION MACHINES

We now check whether the transition from order to chaos discussed here has anything to do with the empirical data for collapses in fusion machines. We recall that a proportionality of the density limit to the square of the magnetic field in tokamaks was suggested by Granetz[30] on the basis of empirical data, but apparently was not confirmed by later observations.[13,47] It is well known that, while at first, a proportionality to the magnetic field (through B/R, where R is the major radius of the torus) had been proposed on an empirical basis for tokamaks by Murakami,[31] in the plasma physics community, the common opinion is rather that the density limit for tokamaks should be proportional to the Greenwald parameter I_p/r_a^2, where I_p is the plasma current and r_a the minor radius of the torus (see Ref. 13).

We do not enter here a discussion of this point, and only content ourselves with plotting in Figure 1, a collection of available data of the density limit for several fusion machines versus their operating magnetic field B in log–log scale, comparing the data to the theoretical formula (11). The first thing that comes out from the figure is that the order of magnitude of the theoretical threshold is correct, and this without having introduced any phenomenological parameter. There is no adjustable parameter in the theory, and no fitting at all. One is thus tempted to say that the essence of the phenomenon has perhaps been captured, especially in consideration of the extreme simplicity of the model (see Ref. 55 for a discussion of the complexity of the problem), with respect to the variety of machines and of operational conditions to which the experimental data refer.

Entering now in some more details, one sees that the theoretical law appears to correspond not so badly to the data for the high field machines (tokamak and stellarators), whereas a sensible discrepancy is met for the low field machines (spherical tokamaks), for which the experimental data are larger by even an order of magnitude. Perhaps this discrepancy might be attributed to the fact that we are discussing here a model describing an isolated, non sustained, system (i.e., with no input heating power), whereas the low field machines considered in the figure are just the ones characterized, in general, by lower confinement time and thus by larger sustainment. Indeed (see the empirical Sudo limit for stellarators[51]) larger densities are expected to be accessible as the input power is increased (although this is not so clear for tokamaks[13]). This is illustrated, in the figure, by the three points reported for the same device (the stellarator WS-A7 (Ref. 48)) at essentially the same applied field, which however corresponds to three different (increasing) input heatings.

V. CONCLUSIONS

In view of the lack of any first principles rationale for the existence of a density limit in fusion machines, the comparison between theory and experiments exhibited in Figure 1 appears encouraging. Particularly so, if one considers the extreme simplicity of the model (uniform plasma in a uniform field) with respect to the variety of machines and of operational conditions to which the experimental data refer.

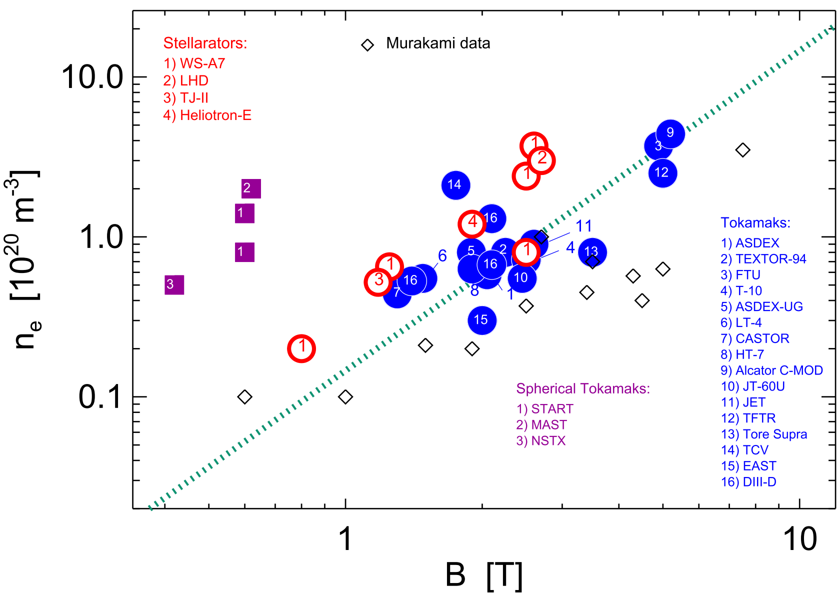

FIG. 1. Density limit values *vs* B for various machines: conventional tokamaks, for which recent data are shown (see Refs. 32–47) along with the original ones of Murakami (see Ref. 31), stellarator machines,[48–51] and spherical tokamaks.[52–54] Dotted line is the theoretical density limit (11).

The essence of the phenomenon seems to have been captured.

Actually, one might conceive that the chaoticity threshold discussed here may be of some interest even outside the domain of fusion machines, for example for astrophysical plasmas, although this is not at all clear. See for example Sec. 3.3 of Alfvén's book.[28] So, we leave this subject for possible future investigations.

So, one might consider as plausible the main proposal advanced in the present paper, namely, that the density limit characterizing the empirical collapses of fusion machines corresponds to a transition from order to chaos in the following sense. At low densities, ordered motions due to the imposed magnetic field prevail, with the electrons performing transverse gyrational motions, and thus with a magnetic pressure. Then, as density is increased, the perturbations caused by the fluctuations of the microfield (which increase as the density) introduce some chaotization, until a chaoticity limit (and so a density limit) is attained, beyond which ordered motions are lost, together with magnetic pressure and confinement.

A key feature of the present approach, with respect to treatments involving the continuum approximation, such as magnetohydrodynamics, is that we are dealing here with the plasma as a discrete system of charges. Indeed in our treatment, an essential role is played by the microfield acting on a single electron, and so it is not clear how the instability found here could find place within the continuum approximation, or any other approximation involving high–frequency cutoffs. For an analogous role of discreteness of matter in cosmology, see Refs. 56 and 57.

Actually, even in plasma physics theory, there exists a huge literature in which the discrete nature of matter is taken into account, following the approach of kinetic theory (see for example Ref. 58). A comparison with the results obtained here within the approach of dynamical systems theory would thus be in order. We hope to come back to this problem in the future.

A further remark is that the existence of a density limit proportional to the square of the magnetic field is well known in the frame of nonneutral plasmas (see Ref. 59), under the name of Brillouin limit. The physical context is however rather different, because the density limit in the latter case refers to the existence of a particular motion, in which the plasma, dealt with as a continuum, performs a rigid rotation about the z axis. Actually, it is clear that a magnetization threshold in the sense discussed here should exist for nonneutral plasmas too. The only problem is that we are unaware of any estimate of the standard deviation of the microfield in such a case. We hope to come back to this problem in the future.

We finally point out that the proportionality of the density limit to the square of the magnetic field predicted by the theoretical law (11), if confirmed, might have relevant implications for future tokamaks.

ACKNOWLEDGMENTS

This paper is dedicated to Giovanni Gallavotti, on the occasion of his seventieth birthday. We also thank N. Vianello for fruitful discussions.

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