.

EXTENDED DERDZINSKI-SHEN THEOREM FOR CURVATURE TENSORS

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Abstract. We extend a remarkable theorem of Derdzinski and Shen, on the restrictions imposed on the Riemann tensor by the existence of a nontrivial Codazzi tensor. We show that the Codazzi equation can be replaced by a more general algebraic condition. The ensuing extension applies both to the Riemann and to generalized curvature tensors.

1. Introduction

On a Riemannian manifold with metric g_{ij} and Riemann connection ∇_i, a *Codazzi tensor* is a symmetric tensor that satisfies the Codazzi equation:

$$\nabla_j b_{kl} - \nabla_k b_{jl} = 0. \tag{1.1}$$

In terms of differential forms (1.1) is the condition for closedness of the 1-form $b_{jk}dx^k$ [3, 11]. Codazzi tensors are of great interest in geometry and have been studied by several authors, such as Berger and Ebin [1], Bourguignon [3], Derdzinski [5, 6], Derdzinski and Shen [7], Ferus [8], Simon [15, 16]; a compendium of results is reported in Besse's book [2] (pp. 436-440). They occur in the study of Riemannian manifolds with harmonic curvature. For example, the Ricci tensor is a Codazzi tensor if and only if $\nabla_m R_{jkl}{}^m = 0$, i.e. the manifold has harmonic Riemann curvature [2] (page 435). The Weyl 1-form $\left[R_{kj} - \frac{R}{2(n-1)} g_{kj}\right] dx^k$ is closed if and only if $\nabla_m C_{jkl}{}^m = 0$, i.e. the manifold has harmonic conformal curvature [2] (page 435).

In ref. [3] important geometric and topological consequences of the existence of a non-trivial Codazzi tensor are examined, in particular the restrictions imposed on the structure of the curvature operator. Derdzinski and Shen improved them and stated the remarkable theorem (the theorem is reported in Besse's book [2], page 438):

Theorem 1.1 (Derdzinski-Shen, [7]). *Let b_{ij} be a Codazzi tensor on a Riemannian manifold M, x a point of M, λ and μ two eigenvalues of the*

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operator $b_i{}^j(x)$, with eigenspaces V_λ and V_μ in T_xM. Then, the subspace $V_\lambda \wedge V_\mu$ is invariant under the action of the curvature operator R_x.

Then, they obtained the following result: in a n-dimensional Riemannian manifold with a Codazzi tensor having n distinct eigenvalues almost everywhere, all real Pontryagin classes vanish.

We point out that the Codazzi equation is a *sufficient* condition for the theorem to hold. A more general one is suggested by the following lemma:

Lemma 1.2. *Any symmetric Codazzi tensor b_{kl} satisfies the algebraic identity*

$$R_{jkl}{}^m b_{im} + R_{kil}{}^m b_{jm} + R_{ijl}{}^m b_{km} \;=\; 0, \tag{1.2}$$

Proof. The following identity among commutators is true for a Codazzi tensor:

$$[\nabla_j, \nabla_k] b_{il} + [\nabla_k, \nabla_i] b_{lj} + [\nabla_i, \nabla_j] b_{kl} = 0 \tag{1.3}$$

Each commutator is evaluated: $[\nabla_i, \nabla_j] b_{kl} = R_{ijk}{}^m b_{ml} + R_{ijl}{}^m b_{km}$. Cancellations occur by the first Bianchi identity and the result is obtained. □

Our first extension of the theorem states that if a symmetric tensor b_{kl} satisfies the algebraic condition (1.2), then the same conclusions of the Derdzinski and Shen's theorem are valid for the Riemann tensor. It turns out that the proof of the extended theorem is much simpler than Derdzinski and Shen's proof.
The replacement of the Codazzi equation by an algebraic condition allows for a further natural extension of the theorem to generalized curvature tensors. It includes well known tensors such as the conformal, the concircular and the conharmonic tensors [14, 9, 12].

2. Extension of D-S theorem for the Riemann tensor.

As in the original theorem, we need an auxiliary tensor and a lemma to prove that it is a generalized curvature tensor, a concept introduced by Kobayashi and Nomizu [10] (page 198). The algebraic condition (1.2) is here used rather than Codazzi's equation to prove both the lemma and the extended theorem.

Definition 2.1. A tensor K_{ijlm} is a *generalized curvature tensor* if it has the symmetries of the Riemann curvature tensor:
a) $K_{ijkl} = -K_{jikl} = -K_{ijlk}$,

b) $K_{ijkl} = K_{klij}$,
c) $K_{ijkl} + K_{jkil} + K_{kijl} = 0$ (first Bianchi identity).

Lemma 2.2. *If a symmetric tensor b_{kl} satisfies eq.(1.2), then $K_{ijkl} = R_{ijrs}{b_k}^r{b_l}^s$ is a generalized curvature tensor.*

Proof. Properties a) are shown easily. For example: $K_{ijlk} = R_{ijrs}{b_l}^r{b_k}^s = R_{ijsr}{b_l}^s{b_k}^r = -R_{ijrs}{b_l}^s{b_k}^r = -K_{ijkl}$.
Property c) follows from (1.2): $K_{ijkl}+K_{jkil}+K_{kijl} = R_{ijrs}{b_k}^r{b_l}^s+R_{jkrs}{b_i}^r{b_l}^s+R_{kirs}{b_j}^r{b_l}^s = ({R_{jis}}^r b_{kr} + {R_{kjs}}^r b_{ir} + {R_{iks}}^r b_{jr}){b_l}^s = 0$.
Property b) follows from c): $K_{ijkl} + K_{jkil} + K_{kijl} = 0$. Sum the identity over cyclic permutations of all indices i, j, k, l and use the symmetries a) (this fact was pointed out in [10]).
It is easy to see that a first Bianchi identity holds also for the last three indices: $K_{ijkl} + K_{iklj} + K_{iljk} = 0$. □

Theorem 2.3. *Let M be an n-dimensional Riemannian manifold with a symmetric tensor b_{kl} that satisfies the algebraic equation*

$$b_{im}{R_{jkl}}^m + b_{jm}{R_{kil}}^m + b_{km}{R_{ijl}}^m = 0$$

If X, Y and Z are three eigenvectors of the matrix ${b_r}^s$ at a point x of the manifold, with eigenvalues λ, μ and ν, then

$$X^i\, Y^j\, Z^k\, R_{ijkl} = 0 \qquad (2.1)$$

provided that λ and μ are different from ν.

Proof. Consider the first Bianchi identity for the Riemann tensor, the condition eq.(1.2) and the first Bianchi identity for the curvature $K_{lijk} = R_{lirs}{b_j}^r{b_k}^s$, and apply them to the three eigenvectors. The three algebraic relations can be put in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \mu\nu & \lambda\nu & \lambda\mu \end{bmatrix} \begin{bmatrix} R_{lijk}X^iY^jZ^k \\ R_{ljki}X^iY^jZ^k \\ R_{lkij}X^iY^jZ^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the matrix is $(\lambda-\mu)(\lambda-\nu)(\nu-\mu)$. If the eigenvalues are all different then $R_{lijk}X^iY^jZ^k = 0$; by the symmetries of the Riemann tensor the statement is true for the contraction of any three indices.
Suppose now that $\lambda = \mu \neq \nu$, i.e. X and Y belong to the same eigenspace; the system of equations implies that $R_{lkij}X^iY^jZ^k = 0$. □

This completes the first extension of the theorem, involving the Riemann tensor. The question arises about non-Codazzi symmetric tensor fields that fulfill (1.2), and make the theorem applicable. We give some examples.

Definition 2.4. A Riemannian manifold with symmetric tensor b_{ij} is "weakly b-symmetric" if

$$\nabla_i b_{kl} = A_i b_{kl} + B_k b_{il} + D_l b_{ik} \tag{2.2}$$

where A, B and D are 1-forms.

Weakly Ricci-symmetric manifolds [17] (see [4] and [12] for a compendium): $\nabla_i R_{kl} = A_i R_{kl} + B_k R_{il} + D_l R_{ik}$, and the more general Weakly Z-symmetric manifolds [13] are of this sort.
If $A - B$ is closed, the evaluation of the three commutators in the left hand side of (1.3) yields zero, then b_{ij} is Riemann compatible.

3. Extension of D-S theorem for generalized curvature tensors.

The theorem can be generalized further, and continues to hold if the Riemann tensor R is replaced by a generalized curvature tensor K, and a symmetric tensor field exists such that the condition (1.2) is valid with R replaced by K.

Definition 3.1. Let K be a generalized curvature tensor; a symmetric tensor b_{ij} is *K-compatible* if

$$K_{jkl}{}^m b_{im} + K_{kil}{}^m b_{jm} + K_{ijl}{}^m b_{km} = 0. \tag{3.1}$$

The metric tensor is trivially K-compatible, by the first Bianchi identity for K.
The following lemma is needed. Its proof is identical to that of lemma 2.2, with eq.(3.1) being used:

Lemma 3.2. *If K is a generalized curvature tensor and b_{kl} is K-compatible, then $\hat{K}_{ijkl} = K_{ijrs} b_k{}^r b_l{}^s$ is a generalized curvature tensor.*

The following fact can be proven by exactly the same argument as Theorem 2.3.

Theorem 3.3. *Let M be a n-dimensional Riemannian manifold with a generalized curvature tensor K and a K-compatible tensor b. If X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point x of the manifold, with eigenvalues λ, μ and ν, then*

$$X^i\, Y^j\, Z^k\, K_{ijkl} = 0 \tag{3.2}$$

provided that λ and μ are different from ν.

Consider the following family of curvature tensors K:

$$
\begin{aligned}
K_{jkl}{}^m = & R_{jkl}{}^m + \varphi(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j{}^m g_{kl} - R_k{}^m g_{jl}) \\
& + \chi(\delta_j^m g_{kl} - \delta_k^m g_{jl}),
\end{aligned} \tag{3.3}
$$

where φ, χ are scalar functions. Appropriate choice of the scalars give the conformal, concircular or conharmonic curvature tensors.
For this family, eq.(1.2) is a sufficient condition for the extended theorem to apply:

Proposition 3.4. *Let K be a tensor of the form* (3.3). *If a symmetric tensor b_{kl} is Riemann compatible, i.e.* (1.2) *holds, then it is K-compatible, and Theorem 3.3 applies.*

Proof. The proof is based on this identity, that holds for curvature tensors of the form (3.3):

$$
b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m
$$
$$
+\varphi\,[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)]
$$

K-compatibility requires the right hand side to be zero. If b_{kl} is Riemann compatible, then also $b_{im}R_j^m - b_{jm}R_i^m = 0$ (this is obtained by transvecting (1.2)), and all terms in the right hand side vanish. □

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