GROUPS WHOSE PRIME GRAPH ON CONJUGACY CLASS SIZES HAS FEW COMPLETE VERTICES

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Abstract. Let G be a finite group, and let $\Gamma(G)$ denote the prime graph built on the set of conjugacy class sizes of G. In this paper, we consider the situation when $\Gamma(G)$ has "few complete vertices", and our aim is to investigate the influence of this property on the group structure of G. More precisely, assuming that there exists at most one vertex of $\Gamma(G)$ that is adjacent to all the other vertices, we show that G is solvable with Fitting height at most 3 (the bound being the best possible). Moreover, if $\Gamma(G)$ has no complete vertices, then G is a semidirect product of two abelian groups having coprime orders. Finally, we completely characterize the case when $\Gamma(G)$ is a regular graph.

Introduction

A classical research field in the theory of finite groups is the analysis of the interplay between the structure of a (finite) group G and the set $\mathrm{cs}(G)$, whose elements are the sizes of the conjugacy classes of G. As a key tool in this kind of investigation, several authors considered the *prime graph* on $\mathrm{cs}(G)$, that we shall denote by $\Gamma(G)$ in this paper: the vertices of $\Gamma(G)$ are the prime numbers dividing some element of $\mathrm{cs}(G)$, and two (distinct) vertices p, q are adjacent in $\Gamma(G)$ if and only if there exists an element in $\mathrm{cs}(G)$ that is divisible by pq.

The graph $\Gamma(G)$ tends to have many edges, in the sense that the nonadjacency of two vertices p, q implies significant restrictions on the structure of G regarding the primes p and q (see [5, Theorem B]). For instance, as shown in Theorem 9 of [5], $\Gamma(G)$ is a complete graph provided the Fitting subgroup $\mathbf{F}(G)$ of G is trivial. In view of that, our aim in this paper is to describe the structure of finite groups G whose graph $\Gamma(G)$ has "many" pairs of nonadjacent vertices. More precisely, define a vertex of a graph to be *complete* if it is adjacent to all the other vertices of the graph: one of the main results is the following.

Theorem A. *Let G be a finite group. Assume that at most one vertex of $\Gamma(G)$ is complete. Then G is solvable, and the Fitting height of G is at most 3.*

In fact, under the assumptions of Theorem A, the factor group $G'\mathbf{F}(G)/\mathbf{F}(G)$ is nilpotent, whence G is a nilpotent-by-nilpotent-by-abelian group (Theorem 2.4); if in addition the prime 2 is not a complete vertex, then $G'\mathbf{F}(G)/\mathbf{F}(G)$ turns out to be abelian (see Remark 2.5), thus G is nilpotent-by-metabelian. Example 2.6 shows that the above description is from one point of view the best possible, in

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2000 *Mathematics Subject Classification.* 20E45 (primary), 20D60 (secondary).

The first three authors are partially supported by the MIUR project "Teoria dei gruppi e applicazioni". The research of the fourth author is partially supported by the Spanish MINECO proyecto MTM2010-15296, and Prometeo/2011/030 -Generalitat Valenciana.

the sense that the Fitting series of G can have length 3, and its first two factors (respectively, the first one in the latter situation) need not be abelian.

Following M.L. Lewis, who considered these problems in the context of irreducible character degrees instead of conjugacy class sizes (see [10]), there is a nice interpretation of Theorem A. As explained in the proof of Corollary B, if a graph Γ, occurring as $\Gamma(G)$ for some finite solvable group G, has more than one complete vertex, then it is possible to construct finite solvable groups H *of arbitrarily large Fitting height* such that $\Gamma(H) = \Gamma$. If this does not happen (i.e., if there is a bound for the Fitting height of finite solvable groups H such that $\Gamma(H) = \Gamma$), then we say that Γ is of *bounded Fitting height* for conjugacy class sizes. Given that, Theorem A yields the following immediate consequence.

Corollary B. *Let G be a finite solvable group such that $\Gamma(G)$ is of bounded Fitting height for conjugacy class sizes. Then the Fitting height of G is at most* 3.

We remark that several results of this kind have been established in the original context of character degrees, by Lewis himself and other authors. For instance, C.P. Morresi Zuccari recently obtained the bound 6 for the Fitting height of finite solvable groups whose prime graph on irreducible character degrees is of bounded Fitting height (see [11], [12]), thus getting very close to Lewis's conjecture that this bound should be 4.

If we strengthen the hypothesis of Theorem A assuming that $\Gamma(G)$ has no complete vertices, then we get a stronger conclusion.

Theorem C. *Let G be a finite group. Assume that no vertex of $\Gamma(G)$ is complete. Then, up to an abelian direct factor, $G = KH$ with $K \trianglelefteq G$, K and H abelian groups of coprime order. Moreover, $K = G'$, $K \cap \mathbf{Z}(G) = 1$ and the prime divisors of $|K|$ (resp. $|H|$) are pairwise adjacent vertices in $\Gamma(G)$.*

Given two graphs Γ_1 and Γ_2 with disjoint vertex sets $\mathrm{V}(\Gamma_1)$ and $\mathrm{V}(\Gamma_2)$ respectively, the join of Γ_1 and Γ_2 is the graph $\Gamma_1 * \Gamma_2$ whose vertex set is $\mathrm{V}(\Gamma_1) \cup \mathrm{V}(\Gamma_2)$, and two vertices are adjacent if and only if either one of them is in $\mathrm{V}(\Gamma_1)$ and the other one in $\mathrm{V}(\Gamma_2)$, or they are vertices of the same Γ_i and they are adjacent in Γ_i. It might be tempting to conjecture that a graph $\Gamma(G)$ as in the statement of Theorem C, and having diameter at most 2, is necessarily a join of disconnected graphs. Although, as shown by Example 4.1, this is in general not true, it does hold if $\Gamma(G)$ is assumed to be regular (and not complete).

A group G is called a *D-group* if $G = KH$ with $K \trianglelefteq G$, K and H are abelian groups of coprime order, $\mathbf{Z}(G) \leq H$ and $G/\mathbf{Z}(G)$ is a Frobenius group; in this case $\Gamma(G)$ is disconnected, and its connected components are complete graphs on the sets of prime divisors of $|K|$ and $|H/\mathbf{Z}(G)|$, respectively. We say that the D-group G is *m-balanced* if both the number of prime divisors of $|K|$ and of $|H/\mathbf{Z}(G)|$ are equal to m. We recall (see [1, Theorem 2] or [6, Theorem 4 and Remark 8]) that $\Gamma(G)$ is disconnected if and only if G is a direct product of a D-group and an abelian group. Finally, in the statement of the following Theorem D, Δ_m denotes a graph having two complete connected components of m vertices each.

Theorem D. *Let G be a finite group, and assume that $\Gamma(G)$ is a noncomplete regular graph of degree d with n vertices. Then $G \simeq A \times G_1 \times \cdots \times G_{n/2m}$, where $m = (n-1) - d$, the G_i are m-balanced D-groups of pairwise coprime orders, and A is an abelian group. Conversely, for a group G of this kind, $\Gamma(G)$ is the join of $n/2m$ copies of Δ_m.*

The above theorem provides a full characterization of finite groups G such that $\Gamma(G)$ is noncomplete and regular, but in fact it yields also a characterization of the noncomplete regular graphs arising as $\Gamma(G)$ for some finite group G.

Corollary E. *Let Γ be a noncomplete regular graph. Then there exists a finite group G such that $\Gamma(G) = \Gamma$ if and only if Γ is the join of k copies of Δ_m, for some positive integers m and k.*

To close with, all the groups considered throughout the paper are assumed to be finite groups.

1. Preliminaries

In this brief section we recall some well-known facts, mostly concerning conjugacy class sizes. As mentioned in the Introduction, we denote by $\Gamma(G)$ the prime graph on the set of conjugacy class sizes of the group G. Note that $\Gamma(G \times A) = \Gamma(G)$ when A is an abelian group. The vertex set and the edge set of $\Gamma(G)$ will be denoted by $\mathrm{V}(G)$ and $\mathrm{E}(G)$ respectively, and we shall say that $p \in \mathrm{V}(G)$ is a complete vertex if, for every $q \in \mathrm{V}(G) \setminus \{p\}$, we have $\{p, q\} \in \mathrm{E}(G)$. Also, if x is an element of G, the symbol x^G will denote the conjugacy class of x in G. Finally, we shall use the notation $\pi(n)$ for the set of prime divisors of $n \in \mathbb{N}$, whereas $\pi(G)$ will stand for $\pi(|G|)$.

The next four lemmas will be often used with no reference.

Lemma 1.1. *Let G be a group, and let A, B, C be subgroups of G such that $A \cap B = 1$ and $C \leq \mathbf{N}_G(A) \cap \mathbf{N}_G(B)$. Let $a \in A$ and $b \in B$. Then $\mathbf{C}_C(ab) = \mathbf{C}_C(a) \cap \mathbf{C}_C(b)$.*

Lemma 1.2. *Let G be a group, and let $x, y \in G$ be elements of coprime order such that $xy = yx$. Then $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$.*

Lemma 1.3. *Let G be a group, N a normal subgroup of G, and x an element of G. Then $|(xN)^{G/N}|$ is a divisor of $|x^G|$ and, if $x \in N$, $|x^N|$ is a divisor of $|x^G|$. In particular, $\Gamma(N)$ and $\Gamma(G/N)$ are both subgraphs of $\Gamma(G)$.*

Lemma 1.4. *Let G be a group and p a prime number. Then*

- **(i):** *$p \notin \mathrm{V}(G)$ if and only if G has a central Sylow p-subgroup;*
- **(ii):** *if $p \notin \mathrm{V}(G/\mathbf{\Phi}(G))$, then a Sylow p-subgroup of G is a direct factor of G.*

Part (i) of the following theorem is Theorem B of [5], whereas part (ii) is a classical result by N. Ito (for a proof see, for instance, [5, Lemma 2]).

Theorem 1.5. *Let G be a group and p, q distinct primes. Assume $p, q \in \mathrm{V}(G)$, but $\{p, q\} \notin \mathrm{E}(G)$. Then*

- **(i):** *G is $\{p, q\}$-solvable, with abelian Sylow p-subgroups and Sylow q-subgroups.*
- **(ii):** *G is either p-nilpotent or q-nilpotent.*

The following result concerning coprime actions is [2, Lemma 2.4].

Lemma 1.6. *Let F be a nilpotent group, and A an abelian group acting faithfully on F. Assume that $|F|$ and $|A|$ are coprime. Then we have $F = \langle x \in F \ : \ \mathbf{C}_A(x) = 1 \rangle$.*

Finally, we recall that a group G is called an *A-group* if it is solvable and all of its Sylow subgroups are abelian.

2. Graphs with at most one complete vertex

In this section we shall prove Theorem A and Corollary B. The following two propositions deal with nonadjacent vertices in the prime graph on conjugacy class sizes, and, together with the subsequent Lemma 2.3, will be a key ingredient in our proof of Theorem A. Before stating Proposition 2.1, we recall that if the Frattini subgroup $\mathbf{\Phi}(G)$ of a group G is trivial, then every abelian normal subgroup of G has a complement in G and, by Gaschütz Theorem, $\mathbf{F}(G)$ is a direct product of minimal normal subgroups of G (see [8, III, 4.4 and 4.5]).

Proposition 2.1. *Let G be a group such that $\mathbf{\Phi}(G) = 1$ and let F be the Fitting subgroup of G. Let t be a prime number, and let T be a normal subgroup of G such that T/F is a nontrivial t-group. Assume that q is a vertex of $\Gamma(G)$, $q \neq t$, such that $\{q, t\} \notin \mathrm{E}(G)$. Let M be a minimal normal subgroup of G such that $M \leq [F, T]$, so that M is an elementary abelian r-group for a suitable prime r, and let H be a complement for M in G.*

- **(A):** *Assume that G has abelian Sylow r-subgroups.*
  - **(a):** *If $r \neq q$, then $\mathbf{Z}(\mathbf{C}_H(x))$ contains a Sylow q-subgroup of G for every nontrivial $x \in M$.*
  - **(b):** *If $r = q$, then G has an abelian normal Sylow q-subgroup Q contained in $\mathbf{Z}(\mathbf{C}_G(M))$, and $G/\mathbf{C}_G(M)$ acts fixed-point freely (by conjugation) on M.*
- **(B):** *Assume that G has nonabelian Sylow r-subgroups. Then $\mathbf{Z}(\mathbf{C}_H(\mu))$ contains a Sylow q-subgroup of G, for every nonprincipal $\mu \in \mathrm{Irr}(M)$.*

Proof. Observe that, since $\mathbf{O}_t(F)$ is a completely reducible module for the t-group $T/\mathbf{C}_T(F)$, we have $\mathbf{O}_t(F) \leq \mathbf{C}_F(T)$; moreover, $\mathbf{O}_{t'}(F) = [\mathbf{O}_{t'}(F), T/\mathbf{C}_T(F)] \times \mathbf{C}_{\mathbf{O}_{t'}(F)}(T/\mathbf{C}_T(F)) = [\mathbf{O}_{t'}(F), T] \times \mathbf{C}_{\mathbf{O}_{t'}(F)}(T)$, so that $F = [F, T] \times \mathbf{C}_F(T)$ and, in particular, $\mathbf{C}_M(T) = 1$. As a consequence, t divides $|x^G|$ for every $x \in M \setminus \{1\}$: in fact, if S is a Sylow t-subgroup of G contained in $\mathbf{C}_G(x)$, we get $T/F \leq SF/F$, whence $T \leq SF \leq \mathbf{C}_G(x)$, a contradiction.

Set $A = [F, T]$ and $B = A \cap H$, so that $A = M \times B$. As the first step, we show that *for all $y \in H$ and for all $x \in M \setminus [M, y]$, a Sylow q-subgroup of G is contained in $\mathbf{C}_G(xy)$.* In fact, for a proof by contradiction, assume that $\mathbf{C}_G(xy)$ does not contain any Sylow q-subgroup of G. Since $\{q, t\} \notin \mathrm{E}(G)$, we have that $\mathbf{C}_G(xy)$ contains a Sylow t-subgroup of G. In particular, there exists a Sylow t-subgroup T_0 of $T \cap H$ and an element $a \in G$ such that $\mathbf{C}_G(xy) \geq T_0^{a^{-1}}$. Note that a can be chosen in A, because T_0 is a Sylow t-subgroup of AT_0 and, since AT_0/A is a Sylow t-subgroup of the nilpotent group T/A, we have $AT_0 \trianglelefteq G$. Now, T_0 centralizes $(xy)^a = xy^a = x[a, y^{-1}]y$. Write $a = mb$ with $m \in M$ and $b \in B$, so that $x[a, y^{-1}] = x[m, y^{-1}][b, y^{-1}]$. So T_0 centralizes $x'y'$, where $x' = x[m, y^{-1}] \in M$ and $y' = [b, y^{-1}]y \in H$. Since T_0 normalizes both M and B, an application of Lemma 1.1 yields $x[m, y^{-1}] \in \mathbf{C}_M(T_0)$. On the other hand, we have $T = FT_0$, thus $\mathbf{C}_M(T_0) = \mathbf{C}_M(T) = 1$. It follows that $x = [m, y^{-1}]^{-1} \in [M, y^{-1}] = [M, y]$, a contradiction.

Next, we consider the setting of (A)(a) (i.e., we assume that the Sylow r-subgroups of G are abelian, and that $r \neq q$). Let x be a nontrivial element of M, and let y be in $C := \mathbf{C}_H(x)$. Since the r-Sylow subgroups of G are abelian, the action of $\langle y \rangle / \mathbf{C}_{\langle y \rangle}(M)$ on M is coprime, thus we get $M = [M, y] \times \mathbf{C}_M(y)$. As $x \in \mathbf{C}_M(y) \setminus \{1\}$, we have $x \notin [M, y]$. Therefore, by the discussion in the paragraph above, there exists a Sylow q-subgroup Q of G with $Q \leq H$ and an element $w \in M$ such that $Q^{w^{-1}} \leq \mathbf{C}_G(xy)$. So Q centralizes $xy^w = x_0y$, where $x_0 = x[w, y^{-1}] \in M$. By Lemma 1.1, then $Q \leq \mathbf{C}_H(x_0) \cap \mathbf{C}_H(y)$. In particular, Q normalizes both $\mathbf{C}_M(y)$ and $[M, y]$, hence Q centralizes both x and $[w, y^{-1}]$. Thus $Q \leq C$ and hence $Q \leq \mathbf{C}_C(y)$. We conclude that every y in C centralizes a Sylow q-subgroup of C, therefore Q is in fact central in C, as desired.

Assume now the setting of (A)(b), so the Sylow r-subgroups of G are abelian and $r = q$. In this situation, M is contained in every Sylow q-subgroup of G. Let $x \in M \setminus \{1\}$, and consider $y \in \mathbf{C}_H(x)$. Then, as above, $M = [M, y] \times \mathbf{C}_M(y)$, so $x \notin [M, y]$. By the discussion in the second paragraph of this proof, $\mathbf{C}_G(xy)$ contains a Sylow q-subgroup Q of G. As $x \in M \leq Q$ and Q is abelian, it follows that $Q \leq \mathbf{C}_G(y)$. Thus, y centralizes M and we conclude that $\mathbf{C}_H(x) = \mathbf{C}_H(M)$. Now, $\mathbf{C}_G(M)$ contains all the Sylow q-subgroups of G, and every element of $\mathbf{C}_G(M)$ is centralized by one of them. We conclude that $\mathbf{C}_G(M)$ has a central Sylow q-subgroup, that is therefore normal in G. Finally, as $\mathbf{C}_H(x) = \mathbf{C}_H(M)$ for every nontrivial $x \in M$, we conclude that $H/\mathbf{C}_H(M)$ (and $G/\mathbf{C}_G(M)$) acts fixed-point freely (by conjugation) on M.

Finally we assume the setting of (B): G has nonabelian Sylow r-subgroups. Observe that, by Theorem 1.5, this implies $r \notin \{q, t\}$. In order to achieve the desired conclusion, we argue as in the proof of that theorem. Let $y \in H$ be such that $[M, y] < M$, and choose $x \in M \setminus [M, y]$. Then, by the discussion in the second paragraph, xy centralizes a Sylow q-subgroup $Q^{w^{-1}}$ of G, where $w \in M$ and $Q \leq H$. As above, Q centralizes $xy^w = x[w, y^{-1}]y$ and, by Lemma 1.1, $Q \leq \mathbf{C}_H(x[w^{-1}, y]) \cap \mathbf{C}_H(y)$. Therefore, for every $y \in H$ such that $[M, y] < M$, we have that $\mathbf{C}_H(y)$ contains a Sylow q-subgroup of H. Furthermore, considering the action of $\mathbf{C}_H(y)$ on $M/[M, y]$, every element of $M/[M, y]$ is centralized by a Sylow q-subgroup of $\mathbf{C}_H(y)$ (hence of H). By [9, Theorem A] it follows that, in the action of $\mathbf{C}_H(y)$ on the dual group $\widehat{M/[M, y]}$ (that is, the set $\mathrm{Irr}(M/[M, y])$ endowed with multiplication of characters), each irreducible character of $M/[M, y]$ is centralized by a Sylow q-subgroup of $\mathbf{C}_H(y)$ (i.e. of H). Consider now a nonprincipal μ in the dual group $\widehat{M}$, and let $y \in \mathbf{C}_H(\mu)$. Then $\mu \in \mathbf{C}_{\widehat{M}}(y)$, and this is equivalent to $[M, y] \leq \ker \mu < M$. Let μ_0 be the character of $M/[M, y]$ corresponding to μ. Then $\mathbf{C}_{\mathbf{C}_H(\mu)}(y) = \mathbf{C}_H(\mu) \cap \mathbf{C}_H(y) = \mathbf{C}_{\mathbf{C}_H(y)}(\mu_0)$ contains a Sylow q-subgroup of H. As this holds for every $y \in \mathbf{C}_H(\mu)$, we obtain that $\mathbf{C}_H(\mu)$ contains a Sylow q-subgroup of H, and hence of G, as a central subgroup, as claimed. □

Proposition 2.2. *Let G be a group such that $\mathbf{\Phi}(G) = 1$ and let F be the Fitting subgroup of G. Let t be a prime number, and let T be a normal subgroup of G such that T/F is a nontrivial t-group. Assume that q is a vertex of $\Gamma(G)$, $q \neq t$, such that $\{q, t\} \notin \mathrm{E}(G)$. Assume also that q divides $|G : F|$. Then $M = [F, T]$ is the unique minimal normal subgroup of G such that q divides $|G : \mathbf{C}_G(M)|$. Moreover, $G/\mathbf{C}_G(M)$ is isomorphic to a q-nilpotent subgroup of the semilinear group on M, and it is therefore metacyclic. Finally, q does not divide $|\mathbf{C}_G(M) : F|$.*

Proof. Set $A = [F, T]$, and observe that $A \neq 1$ because $T \not\leq F$ (see [8, III, 4.2b)]). Clearly, A is a normal subgroup of G, so we can consider a minimal normal subgroup M of G contained in A. This M is an elementary abelian r-group for a suitable prime r and, since by assumption G does not have a normal Sylow q-subgroup, Proposition 2.1 implies that $r \neq q$. Let H be a complement for M in G, and define

$\widetilde{M}$ to be M or the dual group $\widehat{M}$, depending on whether the Sylow r-subgroups of G are abelian or nonabelian respectively. Note that, by the Brauer Permutation Lemma, we have $\mathbf{C}_G(\widetilde{M}) = \mathbf{C}_G(M)$.

Proposition 2.1 yields that, for every nontrivial $x \in \widetilde{M}$, a Sylow q-subgroup of G is contained in $\mathbf{C}_H(x)$ as a central subgroup. In particular, $\mathbf{C}_G(M)$ has a central Sylow q-subgroup, which is therefore normal in G: it is then clear that q does not divide $|\mathbf{C}_G(M) : F|$ and, as $q \mid |G : F|$, also $|G : \mathbf{C}_G(M)| = |H : \mathbf{C}_H(M)|$ is divisible by q. We are in a position to apply the Main Theorem of [3], obtaining that $H/\mathbf{C}_H(M)$ (thus $G/\mathbf{C}_G(M)$) is isomorphic to a q-nilpotent subgroup of the semilinear group on $\widetilde{M}$, which is a metacyclic group. Finally, set $F_0 = F \cap H$, so that $F = M \times F_0$. For a given nontrivial element $x \in \widetilde{M}$, denote by Q the Sylow q-subgroup of G lying in $\mathbf{Z}(\mathbf{C}_H(x))$. Since $F_0 \leq \mathbf{C}_H(x)$, we see that Q acts trivially (by conjugation) on F_0, and hence on F/M. It follows that Q centralizes every minimal normal subgroup of G other than M. Thus, M is the unique minimal normal subgroup of G such that q divides $|G : \mathbf{C}_G(M)|$, and therefore $M = A$. $\square$

Lemma 2.3. *Let ℓ, q be distinct vertices of $\Gamma(G)$ such that $\{\ell, q\} \notin \mathrm{E}(G)$. Let t be a prime number, and let D, T be normal subgroups of G such that $T/D \cap T$ is a nontrivial t-group. Suppose that G is ℓ-nilpotent, q divides $|g^G|$ for all $g \in DT \setminus D$, and a Sylow ℓ-subgroup L of G is contained in D. Then $[G, L] \leq \mathbf{O}^t(T)$.*

Proof. Let K be a normal ℓ-complement of G, and set $K_0 = D \cap K$, so that $D = K_0 L$. Since L is abelian by Theorem 1.5, we get $[G, L] = [K, L]$. Now, by coprimality, $[K, L] = [K, L, L]$ and, since L lies in $D \trianglelefteq G$, we get $[K, L, L] \leq [D \cap K, L] = [K_0, L] = [D, L]$. Thus we get $[G, L] = [D, L]$.

Setting $C = D \cap T$, we claim first that $[D, L] \leq C$. In fact, take $a \in D \setminus C$ and $b \in T \setminus C$; then ab lies in $DT \setminus D$, whence by assumption q divides $|(ab)^G|$. As $\{\ell, q\} \notin \mathrm{E}(G)$, it follows that $ab \in \mathbf{C}_G(L^x)$ for some x that can be chosen in D (recall that $L \leq D \trianglelefteq G$). We use now the bar convention for the factor group $\overline{G} = G/C$: we have $\overline{a}^{\overline{x}^{-1}} \overline{b} \in \mathbf{C}_{\overline{G}}(\overline{L})$, so $\overline{a}^{\overline{x}^{-1}} \in \mathbf{C}_{\overline{G}}(\overline{L})$ as $\overline{T}$ centralizes $\overline{D}$. Therefore we get

$$\overline{D} = \bigcup_{\overline{x} \in \overline{D}} \mathbf{C}_{\overline{D}}(\overline{L})^{\overline{x}}$$

and hence $\overline{L}$ is central in $\overline{D}$, so $[D, L] \leq C$. Note that, as a consequence, CL is a normal subgroup of G, whence $\mathrm{Syl}_\ell(G)$ is transitively permuted by the conjugation action of C.

Next, setting $N = \mathbf{O}^t(T)$, we claim that $[D, L] \leq N$. As q divides $|g^G|$ for all $g \in T \setminus C$, we have

$$T = C \cup \bigcup_{x \in C} \mathbf{C}_T(L)^x.$$

Thus we get $T = C \cup U$, where U is the normal closure of $\mathbf{C}_T(L)$ in T, but in fact $U = T$ because C is a proper subgroup of T. So, using the bar convention for the quotient group $\overline{T} = T/N$, we get $\overline{U} = \overline{T}$. Now, $\overline{U}$ is the normal closure of $\overline{\mathbf{C}_T(L)}$ in $\overline{T}$, and $\overline{T}$ is a t-group. Since the normal closure of a proper subgroup of a nilpotent group is a proper subgroup as well, it follows that $\overline{\mathbf{C}_T(L)} = \overline{T}$, i.e. $T = N\mathbf{C}_T(L)$. Therefore L acts trivially on $\overline{T}$ and, in particular, on $\overline{C}$. Since $\overline{D}$ is ℓ-nilpotent and L is abelian, we have $[\overline{D}, L] = [\overline{D}, L, L]$. But $[\overline{D}, L, L] \leq [\overline{C}, L] = 1$, whence $[D, L] \leq N$, as wanted. $\square$

Theorem A is a consequence of the following slightly stronger Theorem 2.4. We denote by $\mathbf{F}_2(G)$ the second term of the ascending Fitting series of G; i.e. $\mathbf{F}_2(G)/\mathbf{F}(G) = \mathbf{F}(G/\mathbf{F}(G))$.

Theorem 2.4. *Let G be a group such that $\Gamma(G)$ has at most one complete vertex. Then G is solvable and $G' \leq \mathbf{F}_2(G)$.*

Proof. Observe first that G is solvable, as by Theorem 1.5 it is p-solvable for every prime p, except perhaps one. Set $F = \mathbf{F}(G)$.

Arguing by induction on the order of the group, we can assume $\mathbf{\Phi}(G) = 1$. In fact, by Lemma 1.4 and Lemma 1.5(i) every noncomplete vertex of $\Gamma(G)$ is a vertex of $\Gamma(G/\mathbf{\Phi}(G))$, and hence also $\Gamma(G/\mathbf{\Phi}(G)))$ has at most one complete vertex. Moreover, the ascending Fitting series of $G/\mathbf{\Phi}(G)$ is the image under the natural homomorphism of the corresponding series of G.

Since $\mathbf{Z}(G)$ is now a direct factor of G, we can clearly assume also $\mathbf{Z}(G) = 1$. This implies, in particular, that $F \leq G'$. Set $K = \mathbf{F}_2(G')$; we have to show that $K = G'$.

For $t \in \pi(K/F)$, set $T/F = \mathbf{O}_t(K/F)$. Also, define $\mathcal{M}_t$ to be the set of minimal normal subgroups M of G such that $\mathbf{C}_T(M)$ is minimal, with respect to inclusion, in the set $\{\mathbf{C}_T(N) \mid N \text{ minimal normal in } G\}$.

Claim 1. For every $t \in \pi(K/F)$ and every $M \in \mathcal{M}_t$, the group $X_M := T/\mathbf{C}_T(M)$ is either cyclic or isomorphic to a generalized quaternion group.

Assume first that $t \in \pi(K/F)$ is not a complete vertex of $\Gamma(G)$. Then there exists a $q \in \mathrm{V}(G)$, $q \neq t$, such that $\{q, t\} \notin \mathrm{E}(G)$. So, by Theorem 1.5, both the Sylow q-subgroups and the Sylow t-subgroups of G are abelian, and G is either q-nilpotent or t-nilpotent. If G is t-nilpotent, then T/F is central in G/F and, as G/F acts irreducibly on M, $T/\mathbf{C}_T(M)$ is cyclic. So we can assume that G is q-nilpotent. Then q divides $|G : F|$, as otherwise G has a central Sylow q-subgroup, against the fact that q lies in $\mathrm{V}(G)$. By Proposition 2.2, $[F, T]$ is minimal normal in G. Hence, by the definition of $\mathcal{M}_t$, it follows that $M = [F, T]$. Thus, again by Proposition 2.2, $\overline{G} := G/\mathbf{C}_G(M)$ is metacyclic. So $T/\mathbf{C}_T(M) \cong \overline{T} \leq \overline{G}'$ is cyclic, and we are done.

On the other hand, assume that t is the (unique) complete vertex. Let $\{q\} = \pi(M)$. We can assume $q \neq t$, as otherwise $\mathbf{C}_T(M) = T$ and we are done. Then there exists a vertex $\ell \neq q$ such that $\{\ell, q\} \notin \mathrm{E}(G)$. Note that G is ℓ-nilpotent, as otherwise we have the contradiction $M \leq \mathbf{Z}(G) = 1$. Write $D = \mathbf{C}_G(M)$, $C = \mathbf{C}_T(M)$, and let L be a Sylow ℓ-subgroup of G. Observe that $[G, L] \leq D$: in fact, for all $g \in G \setminus D$, q divides $|g^G|$, so ℓ does not. This implies that G/D has a central Sylow ℓ-subgroup, and then $[G, DL] \leq D$. In particular, DL is normal in G, and $[M, L] = [M, DL]$ is normal in G as well.

Assume first $L \not\leq D$. Then $[M, L] \neq 1$, whence $[M, L] = M$. By coprimality, we have $\mathbf{C}_M(L) = 1$ and hence $\ell \in \pi(|x^G|)$ for all nontrivial $x \in M$ (recall that the G-conjugates of L are in fact D-conjugates of L, as $DL \trianglelefteq G$). Let $yC \in T/C = X_M$, where y can be chosen to be a t-element. Assume that there exists a nontrivial $x \in \mathbf{C}_M(y)$. Thus $\pi(|(xy)^G|) \supseteq \pi(|x^G|) \cup \pi(|y^G|)$, and hence $q \notin \pi(|y^G|)$. So y centralizes M, i.e. yC is the trivial element of X_M. We conclude that X_M acts fixed-point freely on M. Since X_M is a t-group, it is then cyclic or a generalized quaternion group, and we are done.

We consider now the case $L \leq D$. As observed above, q divides $|g^G|$ for all $g \in G \setminus D$. As we can clearly assume that T/C is nontrivial, Lemma 2.3 applies, yielding $[G, L] \leq \mathbf{O}^t(T)$ (the latter subgroup being the Hall t'-subgroup of F). Let N be a minimal normal subgroup of G such that $N \leq [G, L] = [G, L, L] = [F, L]$ (recall that G is ℓ-nilpotent and L is abelian). Observe that, since $FL \trianglelefteq G$ and F acts transitively on $\mathrm{Syl}_\ell(G)$, we get $\ell \in \pi(|x^G|)$ for all nontrivial $x \in N$. Let yC be a nontrivial element of X_M, where again y can be chosen to be a t-element, and let $x \in \mathbf{C}_N(y)$. Then (as $q \in \pi(|y^G|)$ and N is a t'-group) $q \in \pi(|x^G|) \cup \pi(|y^G|) \subseteq \pi(|(xy)^G|)$, so $\ell \notin \pi(|x^G|)$ and hence $x = 1$. Therefore $\mathbf{C}_T(N) \leq C$, so by the minimality of C we have $\mathbf{C}_T(N) = C$. We conclude that X_M acts fixed-point freely on N, whence it is either cyclic or generalized quaternion, as wanted.

Claim 2. For every $t \in \pi(K/F)$ and every $M \in \mathcal{M}_t$, we have $G' \leq T\mathbf{C}_G(X_M)$.

We have to show that the group of outer automorphisms $G/T\mathbf{C}_G(X_M)$ induced by G on X_M is abelian. This follows from Claim 1, except when X_M is isomorphic to the quaternion group of order 8. In this case, $\mathrm{Out}(X_M)$ is isomorphic to the symmetric group on three objects S_3. Set $D = \mathbf{C}_G(X_M)$: as we want to show that G/DT is abelian, we shall assume $G/DT \simeq S_3$, aiming to a contradiction.

Note that $G/DT \simeq S_3$ may hold only if $t = 2$ is the unique complete vertex. Therefore, $3 \in \mathrm{V}(G)$ is not complete, so there exists a vertex $\ell \neq 3$ such that $\{3, \ell\} \notin \mathrm{E}(G)$. Since G has a quotient isomorphic to S_3, G is not 3-nilpotent, therefore it must be ℓ-nilpotent. Let L be a Sylow ℓ-subgroup of G. Then $L \leq D$ (because $\ell \notin \{2, 3\}$ and $G/D \simeq S_4$). Moreover, 3 divides $|g^G|$ for all $g \in DT \setminus D$, because DT/D is the Klein subgroup of G/D. We can hence apply Lemma 2.3 and get $[G, L] \leq \mathbf{O}^t(T) \leq F$.

Write $R = FL$. Then $R \trianglelefteq G$, R/F is a nontrivial ℓ-group (otherwise L would be central in G), and 3 divides $|G : F|$. Writing $A = [F, R]$, an application of Proposition 2.2 gives that A is a minimal normal subgroup of G, and if $B \leq F$ is such that $F = A \times B$, then B is centralized by all the Sylow 3-subgroups of G. Setting $E = \mathbf{C}_G(A)$, Proposition 2.2 yields also that G/E is 3-nilpotent. Thus 3 does not divide $|G : EDT|$, as otherwise we would have $EDT = DT$ (recall that $G/DT \simeq S_3$), and now G/DT would be 3-nilpotent, a contradiction. As a consequence, 3 divides $|E : E \cap DT|$. Finally, let Q_0 be a Sylow 3-subgroup of E. Then Q_0 centralizes $A \times B = F$ and so $Q_0 \leq F \leq DT$, against the fact that $3 \mid |E : E \cap DT|$. This final contradiction completes the proof of Claim 2.

In order to finish the proof, write $H = G'$, $\overline{H} = H/F$ and define $\mathcal{Y}$ to be the set of all $\overline{H}$-chief factors $Y = \overline{U}/\overline{V}$, with $\overline{U}, \overline{V} \leq \overline{K}$ and $\overline{U}, \overline{V}$ normal in $\overline{H}$. Observe that $\overline{K}$ is the intersection of the centralizers in $\overline{H}$ of the groups $Y \in \mathcal{Y}$. Now, every $Y \in \mathcal{Y}$ is isomorphic (as $\overline{H}$-group) to a section of $\overline{T}$, for some $t \in \pi(\overline{K})$ (since $\overline{K} = \prod_{t \in \pi(\overline{K})} \overline{T}$). As $\bigcap_{M \in \mathcal{M}_t} \mathbf{C}_T(M) = F$, it follows that Y is isomorphic to a section of some $\overline{X_M}$, for a suitable $M \in \mathcal{M}_t$. Hence, by Claim 2 we deduce that

$$\overline{H} \leq \bigcap_{Y \in \mathcal{Y}} \mathbf{C}_{\overline{H}}(Y) = \overline{K}\ .$$

Therefore, $H = K$, as wanted. □

Remark 2.5. *Let us consider the statement of Theorem A with the extra assumption that the prime 2 is not a complete vertex of $\Gamma(G)$. In this setting, we get $G'' \leq \mathbf{F}(G)$. In fact, adopting the notation of the above proof, in Claim 1 one* *can now show that every X_M is cyclic and consequently that $G' \leq \mathbf{C}_G(X_M)$, so $[G',T] \leq \mathbf{C}_T(M)$, for every $M \in \mathcal{M}_t$. It follows that $[G',T] \leq F$, for every $t \in \pi(K/F)$ and hence $G'' \leq [G',K] \leq F$.*

By Theorem A and the above remark, we see that a group G whose graph $\Gamma(G)$ has at most one complete vertex is a nilpotent-by-nilpotent-by-abelian group, and if in addition the prime 2 is not a complete vertex, then G is in fact nilpotent-by-metabelian. Example 2.6 shows that this description is, in a sense, sharp.

Example 2.6. *(1) Let p, q be distinct primes and assume that q does not divide $p^q - 1$. Then, for every $1 \leq k < q$ there exist p-groups P of class k with $|P| = p^{qk}$, as explained in* [13, Section 2]. *Next, one can construct groups G (see* [13, Section 5]*) such that $P = \mathbf{F}(G)$, $K = \mathbf{F}_2(G)$ is a Frobenius group with kernel P and cyclic complement of order $c = (p^q - 1)/(p - 1)$, and G/P is a Frobenius group with complement of order q. Moreover, every element of P centralizes a Sylow q-subgroup of G. Then, it is not hard to check that p is the only complete vertex of $\Gamma(G)$ (in fact, $\pi(c) \cup \{p\}$ induces a complete subgraph of $\Gamma(G)$, whereas q is only adjacent to p). Here, the Fitting height of G is 3, and the class of $\mathbf{F}(G)$ grows with k.*

(2) Let $H = \mathrm{SL}(2,3) \times C_5$ and $F = A \times C_7$, with $A = C_{11} \times C_{11}$ (where we denote by C_n a cyclic group of order n). Let H act fixed-point freely on A and as a group of automorphisms of order 3 on C_7. Define $G = FH$ as the corresponding semidirect product. Then

$$\mathrm{cs}(G) = \{1, 3, 120, 121, 363, 726, 3388\}$$

so 2 is the only complete vertex of $\Gamma(G)$. Observe that G has Fitting height 3, $G' \leq \mathbf{F}_2(G)$ and $G'/\mathbf{F}(G) \cong Q_8$ is nonabelian.

(3) Consider a direct product G of groups G_2 as in (2) and G_1 as in (1), with $k > 1$, $p = 2$ and $\pi(G_1) \cap \pi(G_2) = \{2\}$. Then $\Gamma(G)$ has only 2 as a complete vertex and both $\mathbf{F}(G)$ and $\mathbf{F}_2(G)/\mathbf{F}(G)$ are nonabelian.

We conclude the section with a proof of Corollary B.

Proof of Corollary B. Let Γ be a graph such that there exists a solvable group G with $\Gamma(G) = \Gamma$, and assume that Γ has two vertices p, q that are both complete. For any fixed positive integer n, take a $\{p, q\}$-group H whose Fitting height is larger than n (for instance, an iterated wreath product whose factors are alternately p-groups and q-groups), and consider the group $G \times H$. Then we clearly have $\Gamma(G \times H) = \Gamma(G) = \Gamma$, and the Fitting height of $G \times H$ is larger than n.

In other words, if $\Gamma = \Gamma(G)$ is of bounded Fitting height, then it has at most one complete vertex. Now Theorem A applies, and yields the conclusion. □

3. Graphs with no complete vertices

This section is devoted to the proof of Theorem C.

Proof of Theorem C. Observe first that we can clearly assume that G has no abelian direct factors and that $\mathrm{V}(G) = \pi(G) \neq \emptyset$. By (i) of Theorem 1.5, G is an A-group. Let $\delta = \delta(G)$ be the set of primes p in $\pi(G)$ such that there exists a normal p-complement in G. We set K to be the intersection of all the normal p-complements of G as p runs over δ. Our assumptions and (ii) of Theorem 1.5 imply that δ is not empty and that $\pi(K)$ induces a complete subgraph of $\Gamma(G)$.

Now, K is a normal Hall δ'-subgroup of G, hence there exists a complement H for K in G; note that $H \simeq G/K$ is abelian.

Let $F = \mathbf{F}(G) \cap K = \mathbf{F}(K)$; observe that $\mathbf{F}(G) = F \times (\mathbf{Z}(G) \cap H)$.

Our main goal is to show that K is abelian. To this end, we can assume that $\mathbf{\Phi}(G) = 1$. In fact, writing $\overline{G} = G/\mathbf{\Phi}(G)$, $\mathrm{V}(\overline{G}) = \mathrm{V}(G)$ by Lemma 1.4 and $\delta(\overline{G}) = \delta(G)$. So, from $\overline{K} \leq \mathbf{F}(\overline{G})$ it follows that $K \leq \mathbf{F}(G)$ is abelian.

Assume, working by contradiction, that K is nonabelian. Then $N := \mathbf{F}_2(K) > F$. Let t be a prime divisor of $|N/F|$ and let T/F be a Sylow t-subgroup of N/F. By assumption, there exists a vertex q of $\Gamma(G)$ which is not adjacent to t. Note that then $q \in \pi(H)$. Recalling that $\mathbf{F}(G)$ is the product of F and a central subgroup of G, by Proposition 2.2 we deduce that $F = M \times A$, where $M = [F, T]$ is minimal normal in G, q divides $|G : \mathbf{C}_G(M)|$ and q does not divide $|G : \mathbf{C}_G(A)|$. Now, when Q is the Sylow q-subgroup of H, we have $\mathbf{C}_M(Q) \neq 1$ and thus $1 < [F, Q] < M$.

Set $L = \mathbf{C}_H(K/F)$ and let $\{p\} = \pi(M)$. We prove that p is adjacent in $\Gamma(G)$ to every prime in $\pi(H/L)$. First we observe that $\mathbf{F}(G/F) = N/F \times LF/F$, hence $\mathbf{C}_H(N/F) = L$. Then H/L acts faithfully and coprimely on the abelian group N/F, so by Lemma 1.6, N/F is generated by the elements lying in regular orbits for this action. Since $F \leq \mathbf{C}_N(M) < N$, there exists an element $x \in N \setminus \mathbf{C}_N(M)$ such that xF lies in a regular H/L-orbit. Then $p \in \pi(|x^G|)$ and, writing $\overline{G} = G/LF$, we have $\pi(|\overline{x}^{\overline{NH}}|) \supseteq \pi(\overline{H}) = \pi(H/L)$. Hence $\pi(|x^G|) \supseteq \pi(H/L)$, as $NH \trianglelefteq G$.

Since $\pi(K)$ induces a complete subgraph of $\Gamma(G)$, we conclude that p is not adjacent to some prime $r \in \pi(L) \setminus \pi(H/L)$. Let R be the Sylow r-subgroup of H. As $R \leq L$, $[K, R] \leq F$ and so $FR \trianglelefteq G$. Let now $a \in \mathbf{C}_F(Q)$ and choose an element $x \in Q \setminus \mathbf{C}_Q(M)$. Then $\mathbf{C}_G(ax) = \mathbf{C}_G(a) \cap \mathbf{C}_G(x)$, because a and x have coprime order. Since p divides $|x^G|$ (as x does not centralize M), it follows that ax is centralized by some conjugate R^b of R, with $b \in F$ (remember that $FR \trianglelefteq G$). Thus, R centralizes $(ax)^{b^{-1}} = ax^{b^{-1}}$ and hence R centralizes a. We conclude that $\mathbf{C}_F(Q) \leq \mathbf{C}_F(R)$ and therefore, as R normalizes Q,

$$[F, FR] = [F, R] \leq [F, Q] < M \ .$$

As M is a minimal normal subgroup, this yields $[F, R] = 1$, so $[K, R] = 1$ and hence $r \notin \mathrm{V}(G)$, a contradiction.

Therefore, we have proved that K is abelian. By coprimality, we get $K = [K, H] \times \mathbf{C}_K(H)$ and, as we are assuming that G has no abelian direct factors, we deduce that $K = [K, H] = G'$ and $K \cap \mathbf{Z}(G) = 1$.

Finally, by Lemma 1.6 there exists an element $x \in K$ such that $\mathbf{C}_H(x) = H \cap \mathbf{Z}(G)$. Thus $\pi(|x^G|) = \pi(H/H \cap \mathbf{Z}(G)) = \pi(H)$ and hence $\pi(H)$ induces a complete subgraph of $\Gamma(G)$. The proof is complete. □

4. Regular graphs

There are two classes of groups that obviously satisfy the assumptions of Theorem C. One of them consists of the groups G such that $\Gamma(G)$ has diameter 3 (we recall that for every group the diameter of $\Gamma(G)$ is at most 3, and the groups which meet the bound are described in [4]). The other one consists of direct products of groups G such that $\Gamma(G)$ is disconnected, where the direct factors have pairwise coprime orders (see Proposition 4.2).

As mentioned in the Introduction, it might be reasonable to ask whether these two classes of groups contain essentially all the groups satisfying the hypotheses of

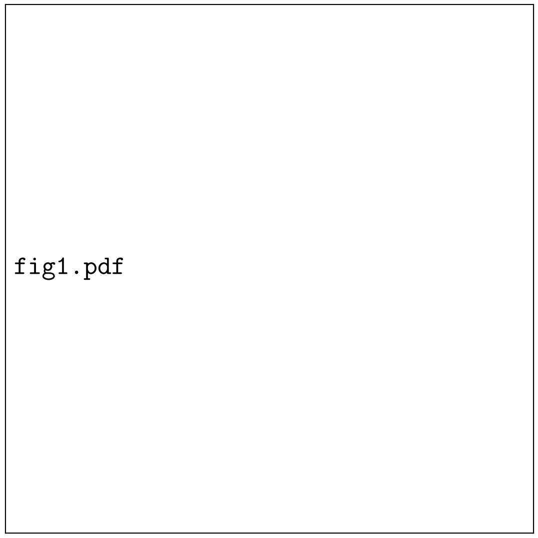

FIGURE 1. The graph $\Gamma(G)$ of Example 4.1

Theorem C. The following Example 4.1 shows that this is false and, in our opinion, not much can be said on this kind of groups besides the conclusions of Theorem C (it is probably worth pointing out that, by means of the construction described in [4, Section 4], it is easy to produce in a uniform way both groups satisfying the assumptions of Theorem C and groups not satisfying them). However, the situation clears up if the hypotheses are strengthened and $\Gamma(G)$ is assumed to be a regular graph. This leads to Theorem D, that we prove after Example 4.1 and Proposition 4.2 (which is only stated, as it is trivial).

Example 4.1. *Let p, p_1, p_2 and q, q_1, q_2 be distinct primes such that $2pp_1p_2$ divides $q-1$, q_1-1 and q_2-1 (such primes certainly exist by Dirichlet's Theorem on primes in arithmetic progression). Denoting by C_n a cyclic group of order n, set $A = C_q \times C_{q_1} \times C_{q_2}$, and $H = C_p \times C_{2p_1} \times C_{2p_2}$. Define also an action of H on A as follows: choosing generators x, y_1, y_2 of C_p, C_{2p_1}, C_{2p_2} respectively, let x act fixed-point freely on A, let y_1 act fixed-point freely on C_{q_1} and trivially on $C_q \times C_{q_2}$, and let y_2 act fixed-point freely on C_{q_2} and trivially on $C_q \times C_{q_1}$.*

Considering the semidirect product $G = A \rtimes H$ formed accordingly, it is easy to check (see Figure 1) that $\Gamma(G)$ is a connected graph with no complete vertices, it has diameter 2, and it is not the join of two (both nonempty) graphs. Also, G is not a direct product of proper subgroups.

Proposition 4.2. *Let G_1, G_2 be groups having coprime orders. Then $\Gamma(G_1 \times G_2)$ is the join $\Gamma(G_1) * \Gamma(G_2)$.*

We define, for a graph Γ and a vertex $p \in \mathrm{V}(\Gamma)$

$$\Lambda_p(\Gamma) = \{q \in \mathrm{V}(\Gamma) \setminus \{p\} \mid q \text{ is not adjacent to } p \text{ in } \Gamma\}\ .$$

Theorem D will be derived by the following more general result.

Theorem 4.3. *Let G be a group and let $\Gamma = \Gamma(G)$. Then the following are equivalent.*

(i): *$G \simeq A \times G_1 \times \cdots \times G_k$, where A is abelian, $G_1, \ldots, G_k$ are D-groups of pairwise coprime orders and $k \geq 1$;*
(ii): *$\Gamma = \Gamma_1 * \cdots * \Gamma_k$ is a join of disconnected graphs and $k \geq 1$;*
(iii): *Γ is not a complete graph and, for every choice of vertices p and q of Γ, $\Lambda_p(\Gamma)$ is not a proper subset of $\Lambda_q(\Gamma)$.*

Proof. Clearly, (i) implies (ii) by Proposition 4.2, as each graph $\Gamma(G_i)$ is disconnected.

Assume (ii): $\Gamma = \Gamma_1 * \cdots * \Gamma_k$, where $k \geq 1$ and the graphs Γ_i are disconnected. Note that, by the definition of join, two vertices of the same graph Γ_i are adjacent in Γ_i if and only if they are adjacent in Γ. Now, the independence (or stability) number of $\Gamma = \Gamma(G)$ is two (see Theorem A of [7]); that is, given any three vertices of Γ, at least two of them are adjacent in Γ. For every $i \in \{1, \ldots, k\}$, the same is hence true in Γ_i and this forces that Γ_i consists of two connected components Θ_i and Ξ_i, which are complete graphs. Thus, for every vertex p of Θ_i, we have $\Lambda_p(\Gamma) = \Lambda_p(\Gamma_i) = \Xi_i$ and, conversely, for $q \in \mathrm{V}(\Xi_i)$, we have $\Lambda_q(\Gamma) = \Lambda_q(\Gamma_i) = \Theta_i$. Therefore, (iii) follows.

Finally, we assume (iii) and prove (i). Note that Γ has no complete vertices, as otherwise $\Lambda_p(\Gamma)$ would be empty for every $p \in \mathrm{V}(\Gamma)$, contradicting the assumption that Γ is not complete. Thus Theorem C gives that, up to an abelian direct factor, $G = KH$, where $K \trianglelefteq G$, K and H are abelian, $K \cap \mathbf{Z}(G) = 1$ (so $\mathbf{Z}(G) \leq H$) and $(|K|, |H|) = 1$. Moreover, $\mathrm{V}(G) = \pi(K) \cup \pi(H)$, and both $\pi(K)$ and $\pi(H)$ induce complete subgraphs of $\Gamma(G)$.

Given a vertex t of Γ, we write Λ_t for $\Lambda_t(\Gamma)$. We observe that $\Lambda_p \subseteq \pi(H)$ if $p \in \pi(K)$ and, conversely, $\Lambda_q \subseteq \pi(K)$ if $q \in \pi(H)$. For $q \in \pi(H)$ (respectively, $q \in \pi(K)$) we denote by H_q (respectively, K_q) the Sylow q-subgroup of H (respectively, of K).

Let $p \in \pi(H)$ be such that $A = [K, H_p]$ is minimal with respect to inclusion (observe that, for every $q \in \pi(H)$, the subgroup H_q is noncentral, hence $[K, H_q]$ is never trivial). Consider $q \in \pi(A)$ and $t \in \Lambda_q$. By coprimality we have $\mathbf{C}_A(H_p) = 1$, whence there exists an element $x \in H_p$ such that the Sylow q-subgroup of A is not centralized by x, so q is a divisor of $|x^G|$. Again by coprimality, we have $K = \mathbf{C}_K(x) \times [K, x]$ and, for every $b \in \mathbf{C}_K(x)$, t does not divide $|b^G|$ (for otherwise $qt \mid |(xb)^G|$). Hence $\mathbf{C}_K(x) \subseteq \mathbf{C}_K(H_t)$, and consequently

$$[K, H_t] = [\mathbf{C}_K(x), H_t] \times [K, x, H_t] \leq [K, x] \leq [K, H_p] = A.$$

By our choice of p, we get $[K, H_t] = A$. So, we proved that for every $q \in \pi(A)$ and every $t \in \Lambda_q$, we have $[K, H_t] = A$ (thus $\mathbf{C}_A(H_t) = 1$).

Our next claim is that, setting $C = \mathbf{C}_H(A)$, the factor group H/C acts as a group of fixed-point free automorphisms on A. In fact, let $y \in H \setminus C$, so that $|y^G|$ is divisible by some prime $q \in \pi(A)$. Taking $t \in \Lambda_q$, we see as before that t does not divide $|b^G|$ for every $b \in \mathbf{C}_K(y)$, yielding $\mathbf{C}_K(y) \leq \mathbf{C}_K(H_t)$. By the discussion in the paragraph above, we get $\mathbf{C}_K(y) \cap A = 1$, as wanted.

Define now σ to be the set $\{t \in \pi(H) \mid [K, H_t] = A\}$, and let L be the Hall σ-subgroup of G. Observe that $[K, L] = A$, and so

$$[K, L \cap C] = [K, L \cap C, L \cap C] \leq [A, C] = 1.$$

Therefore we get $L \cap C \leq \mathbf{Z}(G)$.

Finally, we show that C contains the Hall σ'-subgroup M of H. For a proof by contradiction, suppose that there exists a prime divisor ℓ of $|H/C|$ that does not belong to σ. Let x be an element of $\mathbf{C}_A(H_\ell)$; since ℓ is a divisor of H/C, there exists an element of $H_\ell \setminus C$ that centralizes x. But H/C acts fixed-point freely on A, and this forces x to be trivial. In other words we have $\mathbf{C}_A(H_\ell) = 1$ and, by coprimality, $A = [A, H_\ell]$. As a consequence, we get $A \leq [K, H_\ell]$ (but the inclusion must be proper, because $\ell \notin \sigma$), so that $\mathbf{C}_K(H_\ell) \leq \mathbf{C}_K(H_p)$ for every $p \in \sigma$. Also, we have $\Lambda_\ell \cap \pi(A) = \emptyset$: in fact, if $r \in \Lambda_\ell \cap \pi(A)$, then $\ell \in \Lambda_r$, and we showed that in this situation $[K, H_\ell] = A$, again contradicting $\ell \notin \sigma$. Take now $q \in \pi(A)$ and $p \in \Lambda_q$ (hence $p \in \sigma$); then $q \in \Lambda_p \setminus \Lambda_\ell$. By our assumption (iii), there exists a prime $t \in \Lambda_\ell \setminus \Lambda_p$. Then $\{t, p\} \in E(G)$, and there is an element $g = ax$ with $a \in K$, $x \in H$ and $ax = xa$, such that t divides $|x^G|$ and p divides $|a^G|$. But then $a \notin \mathbf{C}_K(H_p)$ and so, by what we observed above, $a \notin \mathbf{C}_K(H_\ell)$, thus yielding the contradiction $\ell t || g^G|$.

In conclusion, defining G_1 as the metabelian group AL, and setting $R = \mathbf{C}_K(L)M$, we get $G = G_1 \times R$. Since $\mathrm{V}(G) = \pi(G)$ and $\Gamma(G)$ has no complete vertices, one immediately sees that $(|G_1|, |R|) = 1$. Also, $G_1/L \cap C = G_1/\mathbf{Z}(G_1)$ is a Frobenius group, with kernel $A\mathbf{Z}(G_1)/\mathbf{Z}(G_1) \simeq A$ and complement $L/\mathbf{Z}(G_1)$. Hence, G_1 is a D-group.

Finally, Proposition 4.2 yields $\Gamma(G) = \Gamma(G_1) * \Gamma(R)$. Thus the condition (iii) is inherited by the graph $\Gamma(R)$, and an obvious inductive argument gives the desired result concerning the structure of G. □

Proof of Theorem D. If $\Gamma = \Gamma(G)$ is a noncomplete regular graph, then condition (iii) of Theorem 4.3 holds for Γ. We conclude that $G \simeq A \times G_1 \times \cdots \times G_k$, where A is abelian, $G_1, \ldots, G_k$ are D-groups of pairwise coprime orders and $k \geq 1$. Writing $G_i = K_i H_i$ with $K_i \trianglelefteq G_i$, K_i, H_i abelian and $Z_i := \mathbf{Z}(G_i) \leq H_i$, then $\Gamma(G_i)$ has two complete connected components with vertices $\pi(K_i)$ and $\pi(H_i/Z_i)$. Thus, elementary considerations on the graph force $|\pi(K_i)| = |\pi(H_i/Z_i)| = m$, where $m = |\Lambda_p(\Gamma)|$, for any $p \in \mathrm{V}(\Gamma)$.

Finally, if G is (up to an abelian direct factor) a direct product of m-balanced D-groups of pairwise coprime orders, then using again Theorem 4.3 we deduce that $\Gamma(G)$ is a join of copies of the graph Δ_m (the complement of the complete bipartite graph $K_{m,m}$). □

Finally, we prove Corollary E.

Proof of Corollary E. In view of Theorem D, we only need to prove the "if part" of the statement.

For every positive integer m, there exists an infinite family $\mathcal{H}$ of groups having pairwise coprime orders, such that $\Gamma(H) = \Delta_m$ for every $H \in \mathcal{H}$. In fact, for every choice of m distinct primes $p_1, ..., p_m$, by Dirichlet's theorem on primes in arithmetic progression, there are infinitely many choices of m distinct primes $q_1, ..., q_m$ such that $\prod_{j=1}^m q_j \equiv 1 \pmod{\prod_{j=1}^m p_j}$. Denoting by C_1 a cyclic group of order $\prod_{j=1}^m q_j$, and by C_2 a cyclic group of order $\prod_{j=1}^m p_j$, consider the fixed-point free action of C_2 on C_1 and define H to be the semidirect product $C_1 \rtimes C_2$ formed accordingly. Then H is a m-balanced D-group and hence $\Gamma(H) = \Delta_m$.

Now, taking into account Proposition 4.2, if G is the direct product of k distinct groups in $\mathcal{H}$, then $\Gamma(G)$ is the join of k copies of Δ_m. □

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