Numerical analysis of a locking-free mixed finite element method for a bending moment formulation of Reissner-Mindlin plate model

LOURENÇO BEIRÃO DA VEIGA†
Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy.

DAVID MORA‡
Departamento de Matemática, Facultad de Ciencias, Universidad del Bío Bío, Casilla 5-C, Concepción, Chile.

AND

RODOLFO RODRÍGUEZ§
CI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

This paper deals with the approximation of the bending of a clamped plate modeled by Reissner-Mindlin equations. It is known that standard finite element methods applied to this model lead to wrong results when the thickness t is small. Here, we propose a new mixed formulation in terms of the bending moments, shear stress, rotations and transversal displacement. To prove that the resulting variational formulation is well posed, we use standard Babuška-Brezzi theory with appropriate t-dependent norms. The problem is discretized by standard finite elements and error estimates are proved with constants independent of the plate thickness. Moreover, these constants depend on norms of the solution that can be a priori bounded independently of the plate thickness, which leads to the conclusion that the method is locking-free. A local post-processing leading to H^1 approximations of transversal displacement and rotations is introduced. Finally, we report numerical experiments confirming the theoretical results.

Keywords: Reissner-Mindlin; bending moment formulation; locking-free finite elements; error analysis.

1. Introduction

The Reissner-Mindlin theory is the most used model to approximate the deformation of an elastic thin or moderately thick plate. It is very well understood that standard finite element methods applied to this model lead to wrong results when the thickness t is small with respect to the other dimensions of the plate, due to the so called locking phenomenon. Nevertheless, adopting for instance a reduced integration or a mixed interpolation technique, this phenomenon can be avoided. Indeed, nowadays several families of methods have been rigorously shown to be free from locking and optimally convergent. We mention the recent monograph by Falk (2008) for a thorough description of the state of the art and further references.

Among the existing methods, a large success has been shared by the mixed interpolation of tensorial components (MITC) methods introduced in Bathe & Dvorkin (1985) or variants of them (Durán

†Email: lourenco.beirao@unimi.it
‡Email: dmora@ubiobio.cl
§Email: rodolfo@ing-mat.udec.cl

& Liberman (1992)). Another solution is to write an equivalent formulation between the plate equations and an uncoupled system of two Poisson equations plus a rotated Stokes system, by means of a Helmholtz decomposition of the shear stress, as in Arnold & Falk (1989).

More recently, Amara *et al.* (2002) proposed and analyzed a conforming finite element method for the Reissner-Mindlin model satisfying various boundary conditions. In their analysis the bending moment is written in terms of three auxiliary variables belonging to classical Sobolev spaces. A mixed formulation in terms of these new variables is discretized by standard finite elements. Under regularity assumption on the exact solution, optimal error estimates were proved with constants independent of the plate thickness.

Another approach is presented by Behrens & Guzmán (2009). In this case the plate bending problem is posed in terms of six variables lying in L^2 and $H(\mathrm{div})$ spaces. A discretization in terms of discontinuous polynomials and enriched Raviart-Thomas elements is proposed. Error estimates with t-independent constants are proved. These estimates are quasi optimal in regularity, since they involve a norm of the shear stress which can not be a priori bounded independently of t.

In the present paper we consider a bending moment formulation for the plate problem. We introduce these moments (which in practice usually represents the quantity of true interest in applications) as a new unknown, together with the shear stress, the rotations and the transversal displacement. We obtain a mixed variational formulation (Elasticity-like system) which, thanks to Babuška-Brezzi theory, is shown to be well posed and stable in appropriate t-dependent norms. For the approximation of bending moment and rotations we employ PEERS finite elements introduced by Arnold *et al.* (1984), classical Raviart-Thomas elements are used for the shear stress and piecewise constants for the transversal displacement. We prove an uniform inf-sup condition with respect to the discretization parameter h and the thickness t. Moreover, the convergence rate is proved to be optimal $O(h)$. The obtained estimates only depend on norms of the quantities which are known to be bounded independent of t. Therefore, the method turns out thoroughly locking-free. In addition, we propose a local post-processing procedure which gives piecewise linear rotations and transversal displacement which converge to the exact solution in a stronger H^1 type discrete norm.

The outline of this paper is as follows: In Section 2, we first recall the Reissner-Mindlin equations and some regularity results. Then, we prove the unique solvability and stability properties of the proposed formulation. In Section 3, we present the finite element scheme, prove a stability result and show the (linear) convergence of the method. In addition, we introduce and analyze a local post-processing procedure for transversal displacements and rotations. Finally, in Section 4 we report numerical tests which allow us to assess the performance of the proposed method.

Throughout the paper we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes, or hats a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

From now on, we use the following notation for any tensor field $\boldsymbol{\tau} = (\tau_{ij})$ $i,j = 1,2$, any vector field $\eta = (\eta_i)$ $i = 1,2$ and any scalar field v:

$$\operatorname{div}\eta := \partial_1\eta_1 + \partial_2\eta_2, \qquad \operatorname{rot}\eta := \partial_1\eta_2 - \partial_2\eta_1, \qquad \nabla v := \begin{pmatrix}\partial_1 v\\ \partial_2 v\end{pmatrix}, \qquad \operatorname{curl} v := \begin{pmatrix}\partial_2 v\\ -\partial_1 v\end{pmatrix},$$

$$\mathbf{div}\,\boldsymbol{\tau} := \begin{pmatrix}\partial_1\tau_{11} + \partial_2\tau_{12}\\ \partial_1\tau_{21} + \partial_2\tau_{22}\end{pmatrix}, \qquad \mathsf{Curl}\,\eta := \begin{pmatrix}\partial_2\eta_1 & -\partial_1\eta_1\\ \partial_2\eta_2 & -\partial_1\eta_2\end{pmatrix}, \qquad \nabla\eta := \begin{pmatrix}\partial_1\eta_1 & \partial_2\eta_1\\ \partial_1\eta_2 & \partial_2\eta_2\end{pmatrix},$$

$$\boldsymbol{\tau}^{\mathbf{t}} := (\tau_{ji}), \qquad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \qquad \boldsymbol{\tau}^a : \boldsymbol{\tau}^b := \sum_{i,j=1}^{2} \tau_{ij}^a \tau_{ij}^b.$$

Moreover, we denote

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2. The plate model.

Consider an elastic plate of thickness t such that $0 < t \leqslant 1$, with reference configuration $\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, where Ω is a convex polygonal domain of $\mathbb{R}^2$ occupied by the midsection of the plate. The deformation of the plate is described by means of the Reissner-Mindlin model in terms of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers initially normal to the plate's midsurface, the scaled shear stress $\gamma = (\gamma_1, \gamma_2)$, and the transversal displacement w. Assuming that the plate is clamped on its whole boundary $\partial\Omega$, the following strong equations describe the plate's response to conveniently scaled transversal load $g \in L^2(\Omega)$:

$$-\,\mathbf{div}(\mathscr{C}(\varepsilon(\beta))) - \gamma = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$-\operatorname{div}\gamma = g \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\gamma = \frac{\kappa}{t^2}(\nabla w - \beta) \quad \text{in} \quad \Omega, \tag{2.3}$$

$$w = 0, \beta = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

where $\kappa := Ek/2(1+\nu)$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor usually taken as $5/6$ for clamped plates, $\varepsilon(\beta) := \frac{1}{2}(\nabla\beta + (\nabla\beta)^{\mathbf{t}})$ is the standard strain tensor, and $\mathscr{C}$ is the tensor of bending moduli, given by (for isotropic materials)

$$\mathscr{C}\boldsymbol{\tau} := \frac{E}{12(1-\nu^2)}\left((1-\nu)\boldsymbol{\tau} + \nu\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}\right) \quad \forall \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}.$$

The tensor $\mathscr{C}$ is invertible with its inverse given by

$$\mathscr{C}^{-1}\boldsymbol{\tau} := \frac{12(1-\nu^2)}{E}\left(\frac{1}{(1-\nu)}\boldsymbol{\tau} - \frac{\nu}{(1-\nu^2)}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}\right) \quad \forall \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}.$$

To write a variational formulation of the Reissner-Mindlin plate problem, we introduce the bending moment $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$, $i, j = 1, 2$ as a new unknown defined by

$$\boldsymbol{\sigma} := \mathscr{C}(\varepsilon(\beta)).$$

We can rewrite the equation above as:

$$\mathscr{C}^{-1}\boldsymbol{\sigma} = \nabla\beta + \left(\frac{1}{2}\operatorname{rot}\beta\right)\mathbf{J},$$

introducing the auxiliary unknown $r := -\frac{1}{2}\operatorname{rot}\beta$. Multiplying by test function and then integrating by parts, we get

$$\int_\Omega \mathscr{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \beta \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega r(\tau_{12} - \tau_{21}) = 0. \tag{2.5}$$

Now, by testing (2.1)-(2.3) with adequate functions, integrating by parts, using (2.5) and (2.4), and imposing weakly the symmetry of $\boldsymbol{\sigma}$, we obtain the following mixed variational formulation:

Find $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ *such that*

$$\begin{aligned} \int_\Omega \mathscr{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \frac{t^2}{\kappa}\int_\Omega \gamma\cdot\xi + \int_\Omega \beta\cdot(\mathbf{div}\,\boldsymbol{\tau}+\xi) + \int_\Omega r(\tau_{12}-\tau_{21}) + \int_\Omega w\,\mathrm{div}\,\xi &= 0,\\ \int_\Omega \eta\cdot(\mathbf{div}\,\boldsymbol{\sigma}+\gamma) + \int_\Omega s(\sigma_{12}-\sigma_{21}) + \int_\Omega v\,\mathrm{div}\,\gamma &= -\int_\Omega gv, \end{aligned} \tag{2.6}$$

for all $((\boldsymbol{\tau},\xi),(\eta,s,v)) \in \mathbf{H}\times\mathbf{Q}$, where

$$\mathbf{H} := H(\mathbf{div};\Omega)\times H(\mathrm{div};\Omega),$$

$$\mathbf{Q} := [L^2(\Omega)]^2 \times L^2(\Omega)\times L^2(\Omega),$$

with

$$H(\mathbf{div};\Omega) := \{\boldsymbol{\tau}\in[L^2(\Omega)]^{2\times 2} : \mathbf{div}\,\boldsymbol{\tau}\in[L^2(\Omega)]^2\},$$

and

$$H(\mathrm{div};\Omega) := \{\xi\in[L^2(\Omega)]^2 : \mathrm{div}\,\xi\in L^2(\Omega)\}.$$

In the analysis, we will utilize the following t-dependent norm for the space $\mathbf{H}$

$$\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\tau}+\xi\|_{0,\Omega} + \|\xi\|_{t,H(\mathrm{div};\Omega)},$$

where

$$\|\xi\|_{t,H(\mathrm{div};\Omega)} := t\|\xi\|_{0,\Omega} + \|\mathrm{div}\,\xi\|_{0,\Omega},$$

while for the space $\mathbf{Q}$, we will use

$$\|(\eta,s,v)\|_{\mathbf{Q}} := \|\eta\|_{0,\Omega} + \|s\|_{0,\Omega} + \|v\|_{0,\Omega}.$$

We note that for all $(\boldsymbol{\tau},\xi)\in H(\mathbf{div};\Omega)\times H(\mathrm{div};\Omega)$,

$$\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}} \leqslant C(\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} + \|\xi\|_{H(\mathrm{div};\Omega)}),$$

where C is independent of t and

$$\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} + \|\xi\|_{H(\mathrm{div};\Omega)} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} + \|\xi\|_{0,\Omega} + \|\mathrm{div}\,\xi\|_{0,\Omega}.$$

We rewrite problem (2.6) as follows:

Find $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ *such that*

$$\begin{aligned} a((\boldsymbol{\sigma},\gamma),(\boldsymbol{\tau},\xi)) + b((\boldsymbol{\tau},\xi),(\beta,r,w)) &= 0 \quad \forall(\boldsymbol{\tau},\xi)\in\mathbf{H},\\ b((\boldsymbol{\sigma},\gamma),(\eta,s,v)) &= F(\eta,s,v) \quad \forall(\eta,s,v)\in\mathbf{Q}, \end{aligned} \tag{2.7}$$

where the bilinear forms $a:\mathbf{H}\times\mathbf{H}\to\mathbb{R}$ and $b:\mathbf{H}\times\mathbf{Q}\to\mathbb{R}$, and the linear functional $F:\mathbf{Q}\to\mathbb{R}$, are defined by

$$\begin{aligned} a((\boldsymbol{\sigma},\gamma),(\boldsymbol{\tau},\xi)) &:= \int_\Omega \mathscr{C}^{-1}\boldsymbol{\sigma}:\boldsymbol{\tau} + \frac{t^2}{\kappa}\int_\Omega \gamma\cdot\xi\\ &= \frac{12(1-\nu^2)}{E}\left(\frac{1}{1-\nu}\int_\Omega \boldsymbol{\sigma}:\boldsymbol{\tau} - \frac{\nu}{1-\nu^2}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\tau})\right) + \frac{t^2}{\kappa}\int_\Omega \gamma\cdot\xi, \end{aligned} \tag{2.8}$$

$$b((\boldsymbol{\tau},\xi),(\eta,s,v)) := \int_\Omega \eta\cdot(\mathbf{div}\,\boldsymbol{\tau}+\xi)+\int_\Omega s(\tau_{12}-\tau_{21})+\int_\Omega v\,\mathrm{div}\,\xi, \tag{2.9}$$

and

$$F(\eta,s,v) := -\int_\Omega gv,$$

for all $(\boldsymbol{\sigma},\gamma),(\boldsymbol{\tau},\xi)\in\mathbf{H}$ and for all $(\eta,s,v)\in\mathbf{Q}$.

We recall the following regularity result for the solution of problem (2.7) (see Arnold & Falk (1989)).

PROPOSITION 2.1 Suppose that Ω is a convex polygon and $g\in L^2(\Omega)$. Let $((\boldsymbol{\sigma},\gamma),(\beta,r,w))$ be the solution of problem (2.7). Then, there exists a constant C, independent of t and g, such that

$$\|w\|_{2,\Omega}+\|\beta\|_{2,\Omega}+\|\gamma\|_{H(\mathrm{div};\Omega)}+t\|\gamma\|_{1,\Omega}+\|\boldsymbol{\sigma}\|_{1,\Omega}+t\|\,\mathbf{div}\,\boldsymbol{\sigma}\|_{1,\Omega}+\|r\|_{1,\Omega}\leqslant C\|g\|_{0,\Omega}.$$

Now, we will prove that problem (2.7) satisfies the hypotheses of the Babuška-Brezzi theory, which yields the unique solvability and continuous dependence of this variational formulation.

We first observe that the bilinear forms a, b, and the linear functional F are bounded with constants independent of plate thickness t.

Let $V := \{(\boldsymbol{\tau},\xi)\in\mathbf{H} : b((\boldsymbol{\tau},\xi),(\eta,s,v))=0 \quad \forall(\eta,s,v)\in\mathbf{Q}\}$ be the null space of the bilinear form b. From (2.9), we have that

$$V := \{(\boldsymbol{\tau},\xi)\in\mathbf{H} : \xi+\mathbf{div}\,\boldsymbol{\tau}=\mathbf{0},\ \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathbf{t}} \text{ and } \mathrm{div}\,\xi=0 \text{ in } \Omega\}.$$

The following lemma shows that the bilinear form a is V-elliptic, with a constant independent of the plate thickness t.

LEMMA 2.1 There exists $C>0$, independent of t, such that

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi))\geqslant C\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}^2 \quad \forall(\boldsymbol{\tau},\xi)\in V.$$

Proof. Given $(\boldsymbol{\tau},\xi)\in V$, using $\mathrm{tr}(\boldsymbol{\tau})^2\leqslant 2(\boldsymbol{\tau}:\boldsymbol{\tau}) \quad \forall\boldsymbol{\tau}\in[L^2(\Omega)]^{2\times 2}$, from (2.8) we obtain

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi))\geqslant\frac{12(1-\nu)}{E}\|\boldsymbol{\tau}\|_{0,\Omega}^2+\frac{t^2}{\kappa}\|\xi\|_{0,\Omega}^2.$$

Since $\|\,\mathbf{div}\,\boldsymbol{\tau}+\xi\|_{0,\Omega}=0$ and $\|\mathrm{div}\,\xi\|_{0,\Omega}=0$, we get

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi))\geqslant\min\left\{\frac{12(1-\nu)}{E},\frac{1}{\kappa}\right\}\left(\|\boldsymbol{\tau}\|_{0,\Omega}^2+\|\,\mathbf{div}\,\boldsymbol{\tau}+\xi\|_{0,\Omega}^2+t^2\|\xi\|_{0,\Omega}^2+\|\mathrm{div}\,\xi\|_{0,\Omega}^2\right).$$

Thus

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)) \quad\geqslant\quad C\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}^2,$$

where

$$C := \min\left\{\frac{6(1-\nu)}{E},\frac{1}{2\kappa}\right\}.$$

Therefore, a is V-elliptic, and we end the proof. □

In order to prove the corresponding inf-sup condition, we first prove the following lemma.

LEMMA 2.2 There exists $c>0$, independent of t, such that the following holds. For all $s\in L^2(\Omega)$, there exists $\boldsymbol{\tau}^s\in H(\mathbf{div};\Omega)$ such that $(\tau_{12}^s-\tau_{21}^s)=s$, $\mathbf{div}\,\boldsymbol{\tau}^s=\mathbf{0}$ in Ω, and $\|\boldsymbol{\tau}^s\|_{H(\mathbf{div};\Omega)}\leqslant c\|s\|_{0,\Omega}$.

Proof. Let $s \in L^2(\Omega)$, let us put

$$P(s) := \frac{1}{|\Omega|}\int_\Omega s,$$

and consider $\lambda := (s - P(s))$. We have that $\lambda \in L_0^2(\Omega) := \{v \in L^2(\Omega) : \int_\Omega v = 0\}$ and moreover $\|\lambda\|_{0,\Omega} \leqslant c\|s\|_{0,\Omega}$. Then, there exists $\mathbf{v} \in [H_0^1(\Omega)]^2$ such that $\operatorname{div}\mathbf{v} = \lambda$ in Ω and $\|\mathbf{v}\|_{1,\Omega} \leqslant c\|\lambda\|_{0,\Omega}$.

We consider the following function

$$\varphi := \mathbf{v} + \frac{P(s)}{2}\begin{pmatrix} x \\ y \end{pmatrix},$$

which satisties $\operatorname{div}\varphi = s$ and $\|\varphi\|_{1,\Omega} \leqslant \|\mathbf{v}\|_{1,\Omega} + c\|s\|_{0,\Omega}$. Now, we define

$$\boldsymbol{\tau}^s := -\operatorname{Curl}\varphi = -\begin{pmatrix} \partial_2 v_1 & -\partial_1 v_1 - \frac{1}{2}P(s) \\ \partial_2 v_2 + \frac{1}{2}P(s) & -\partial_1 v_2 \end{pmatrix} \in [L^2(\Omega)]^{2\times 2}.$$

From this, we get that $\mathbf{div}\,\boldsymbol{\tau}^s = \mathbf{0}$, hence $\boldsymbol{\tau}^s \in H(\mathbf{div};\Omega)$. Moreover

$$(\tau_{12}^s - \tau_{21}^s) = \partial_1 v_1 + \frac{P(s)}{2} + \partial_2 v_2 + \frac{P(s)}{2} = \operatorname{div}\mathbf{v} + P(s) = \lambda + P(s) = s,$$

and it is easy to check that

$$\|\boldsymbol{\tau}^s\|_{H(\mathbf{div};\Omega)} \leqslant c\|s\|_{0,\Omega}.$$

Thus, we end the proof. □

We are ready to prove the inf-sup condition for the bilinear form b.

LEMMA 2.3 There exists $C > 0$, independent of t, such that

$$\sup_{(\boldsymbol{\tau},\xi)\in\mathbf{H}} \frac{|b((\boldsymbol{\tau},\xi),(\eta,s,v))|}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} \geqslant C\|(\eta,s,v)\|_{\mathbf{Q}} \quad \forall(\eta,s,v)\in\mathbf{Q}.$$

Proof. Let $(\eta,s,v) \in \mathbf{Q}$. From Lemma 2.2, there exists $\boldsymbol{\tau}^s \in H(\mathbf{div};\Omega)$ such that $\mathbf{div}\,\boldsymbol{\tau}^s = \mathbf{0}$ in Ω, $(\tau_{12}^s - \tau_{21}^s) = s$ in Ω and $\|\boldsymbol{\tau}^s\|_{H(\mathbf{div};\Omega)} \leqslant c\|s\|_{0,\Omega}$. Then,

$$\begin{aligned} \sup_{(\boldsymbol{\tau},\xi)\in\mathbf{H}} \frac{|b((\boldsymbol{\tau},\xi),(\eta,s,v))|}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} &\geqslant \frac{|b((\boldsymbol{\tau}^s,\mathbf{0}),(\eta,s,v))|}{\|\boldsymbol{\tau}^s\|_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}^s\|_{0,\Omega}} \\ &= \frac{1}{\|\boldsymbol{\tau}^s\|_{0,\Omega}}\left|\int_\Omega \eta\cdot\mathbf{div}\,\boldsymbol{\tau}^s + \int_\Omega s(\tau_{12}^s - \tau_{21}^s)\right| \\ &= \frac{\|s\|_{0,\Omega}^2}{\|\boldsymbol{\tau}^s\|_{0,\Omega}} \geqslant \frac{1}{c}\|s\|_{0,\Omega}. \end{aligned} \tag{2.10}$$

Now, let $\widetilde{\boldsymbol{\tau}} := -\varepsilon(z)$, where $z \in [H_0^1(\Omega)]^2$ is the unique weak solution (as a consequence of Korn's inequality and Lax-Milgram's lemma) of the following auxiliary problem

$$\begin{cases} -\mathbf{div}\,\varepsilon(z) = \eta & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We have that $\mathbf{div}\,\widetilde{\boldsymbol{\tau}} = \eta \in [L^2(\Omega)]^2$, hence $\widetilde{\boldsymbol{\tau}} \in H(\mathbf{div};\Omega)$ and $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\tau}}^{\mathbf{t}}$. Moreover, applying the continuous dependence result, there exists $\tilde{c} > 0$ depending only on Ω such that

$$\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \leqslant \tilde{c}\|\eta\|_{0,\Omega}. \tag{2.11}$$

Therefore,

$$\begin{aligned}\sup_{(\boldsymbol{\tau},\xi)\in\mathbf{H}} \frac{|b((\boldsymbol{\tau},\xi),(\eta,s,v))|}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} &\geqslant \frac{|b((\widetilde{\boldsymbol{\tau}},\mathbf{0}),(\eta,s,v))|}{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}} \\ &= \frac{1}{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}} \left|\int_\Omega \eta\cdot\mathbf{div}\,\widetilde{\boldsymbol{\tau}} + \int_\Omega s(\tilde{\tau}_{12} - \tilde{\tau}_{21})\right| \\ &= \frac{\|\eta\|_{0,\Omega}^2}{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}} \geqslant \frac{1}{\tilde{c}}\|\eta\|_{0,\Omega},\end{aligned} \tag{2.12}$$

where the last inequality follows from (2.11).

Finally, let $\widetilde{\xi} := -\nabla\tilde{z}$, where $\tilde{z} \in H_0^1(\Omega)$ is the unique weak solution (as a consequence of Poincaré inequality and Lax-Milgram's lemma) of the following auxiliary problem

$$\begin{cases} -\Delta\tilde{z} = v & \text{in} \quad \Omega, \\ \tilde{z} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

As before, there exists $\hat{c} > 0$ depending only on Ω such that

$$\|\widetilde{\xi}\|_{0,\Omega} + \|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega} \leqslant \hat{c}\|v\|_{0,\Omega}.$$

It follows that

$$\begin{aligned}\sup_{(\boldsymbol{\tau},\xi)\in\mathbf{H}} \frac{|b((\boldsymbol{\tau},\xi),(\eta,s,v))|}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} &\geqslant \frac{|b((\mathbf{0},\widetilde{\xi}),(\eta,s,v))|}{(1+t)\|\widetilde{\xi}\|_{0,\Omega} + \|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega}} \\ &\geqslant \frac{1}{2(\|\widetilde{\xi}\|_{0,\Omega} + \|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega})}\left|\int_\Omega \eta\cdot\widetilde{\xi} + \int_\Omega v\,\mathrm{div}\,\widetilde{\xi}\right| \\ &= \frac{1}{2(\|\widetilde{\xi}\|_{0,\Omega} + \|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega})}\left|\int_\Omega \eta\cdot\widetilde{\xi} + \|v\|_{0,\Omega}^2\right| \\ &\geqslant \frac{1}{2(\|\widetilde{\xi}\|_{0,\Omega} + \|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega})}\left(\|v\|_{0,\Omega}^2 - \left|\int_\Omega \eta\cdot\widetilde{\xi}\right|\right) \\ &\geqslant \frac{1}{2\hat{c}}\|v\|_{0,\Omega} - \|\eta\|_{0,\Omega}.\end{aligned}$$

Given

$$A := \sup_{(\boldsymbol{\tau},\xi)\in\mathbf{H}} \frac{|b((\boldsymbol{\tau},\xi),(\eta,s,v))|}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}},$$

we have proved that (cf. (2.12))

$$A \geqslant \frac{1}{\tilde{c}}\|\eta\|_{0,\Omega},$$

therefore,

$$A \geqslant \frac{1}{2\hat{c}(1+\tilde{c})}\|v\|_{0,\Omega}. \tag{2.13}$$

Thus, the proof follows from (2.10), (2.12) and (2.13). □

We are now in position to state the main result of this section which give the solvability of the continuous problem (2.7).

THEOREM 2.2 There exists a unique $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ solution of the mixed variational formulation (2.7), and the following continuous dependence result holds

$$\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))\|_{\mathbf{H}\times\mathbf{Q}} \leqslant C\|g\|_{0,\Omega},$$

where C is independent of t.

Proof. By virtue of Lemmas 2.1 and 2.3, the proof follows from a straightforward application of the abstract Theorem 1.1 in Chapter II of Brezzi & Fortin (1991). □

3. The Finite Element Scheme

Let $\mathscr{T}_h$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T with mesh size $h := \max\{h_T : T \in \mathscr{T}_h\}$, and such that there holds $\bar{\Omega} = \cup\{T : T \in \mathscr{T}_h\}$. In addition, given an integer $k \geqslant 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_k(S)$ the space of polynomials in two variables defined in S of total degree at most k, and for each $T \in \mathscr{T}_h$ we define the local Raviart-Thomas space of order zero

$$RT_0(T) := \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\},$$

where $\begin{pmatrix}x\\y\end{pmatrix}$ is a generic vector of $\mathbb{R}^2$.

On the other hand, for each triangle $T \in \mathscr{T}_h$ we take the unique polynomial $b_T \in \mathbb{P}_3(T)$ that vanishes on ∂T and is normalized by $\int_T b_T = 1$. This cubic bubble function is extended by zero onto the region $\Omega - T$ and therefore it becomes an element of $H_0^1(\Omega)$. Hence, we define

$$B(\mathscr{T}_h) := \{\boldsymbol{\tau}_h \in H(\mathbf{div},\Omega) : (\tau_{i1h},\tau_{i2h}) \in Z(T), i = 1,2, \forall T \in \mathscr{T}_h\},$$

where $Z(T) := \operatorname{span}\{\operatorname{curl}(b_T), T \in \mathscr{T}_h\}$.

Hence, we define the following finite element subspaces:

$$H_h^{\boldsymbol{\sigma}} := X_h \oplus B(\mathscr{T}_h),$$

where X_h is the global Raviart-Thomas space of lowest order,

$$X_h := \{\boldsymbol{\tau}_h \in H(\mathbf{div},\Omega) : \boldsymbol{\tau}_h|_T \in [RT_0(T)^{\mathbf{t}}]^2,\ \forall T \in \mathscr{T}_h\},$$

$$H_h^{\gamma} := \{\xi_h \in H(\operatorname{div},\Omega) : \xi_h|_T \in RT_0(T),\ \forall T \in \mathscr{T}_h\},$$

$$Q_h^{w} := \{v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}_0(T),\ \forall T \in \mathscr{T}_h\},$$

$$Q_h^{\beta} := \{\eta_h \in [L^2(\Omega)]^2 : \eta_h|_T \in [\mathbb{P}_0(T)]^2,\ \forall T \in \mathscr{T}_h\},$$

$$Q_h^r := \left\{s_h \in H^1(\Omega) : s_h|_T \in \mathbb{P}_1(T),\ \forall T \in \mathscr{T}_h\right\}.$$

At this point we recall that $H_h^{\boldsymbol{\sigma}} \times Q_h^{\beta} \times Q_h^r$ correspond to the PEERS-space given by Arnold, Brezzi and Douglas in Arnold *et al.* (1984).

Defining $\mathbf{H}_h := H_h^{\boldsymbol{\sigma}} \times H_h^{\gamma}$ and $\mathbf{Q}_h := Q_h^{\beta} \times Q_h^r \times Q_h^w$ our mixed finite element scheme associated with the continuous formulation (2.7) reads as follows:

Find $((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *such that*

$$\begin{aligned} a((\boldsymbol{\sigma}_h,\gamma_h),(\boldsymbol{\tau}_h,\xi_h)) + b((\boldsymbol{\tau}_h,\xi_h),(\beta_h,r_h,w_h)) = 0 \quad \forall(\boldsymbol{\tau}_h,\xi_h) \in \mathbf{H}_h,\\ b((\boldsymbol{\sigma}_h,\gamma_h),(\eta_h,s_h,v_h)) = F(\eta_h,s_h,v_h) \quad \forall(\eta_h,s_h,v_h) \in \mathbf{Q}_h. \end{aligned} \tag{3.1}$$

Our next goal is to show the corresponding discrete versions of Lemmas 2.1 and 2.3 to have the solvability and stability of problem (3.1). With this aim, we note that the discrete null space of the bilinear form b reduces to:

$$V_h := \left\{(\boldsymbol{\tau}_h,\xi_h) \in \mathbf{H}_h : \int_\Omega \eta_h \cdot (\mathbf{div}\,\boldsymbol{\tau}_h + \xi_h) + \int_\Omega s_h(\tau_{12h} - \tau_{21h}) + \int_\Omega v_h \operatorname{div}\xi_h = 0 \quad \forall(\eta_h,s_h,v_h) \in \mathbf{Q}_h\right\}.$$

Let $(\boldsymbol{\tau}_h,\xi_h) \in V_h$, taking $(\mathbf{0},0,v_h) \in \mathbf{Q}_h$ and using that $(\operatorname{div}\xi_h)|_T$ is a constant, since $v_h|_T$ is also a constant, we conclude that $\operatorname{div}\xi_h = 0$ in Ω.

Now, taking $(\eta_h,0,0) \in \mathbf{Q}_h$, since $\mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0}$ in Ω $\forall\boldsymbol{\tau}_h \in B(\mathscr{T}_h)$, we have that $(\mathbf{div}\,\boldsymbol{\tau}_h)|_T$ is a constant vector. Moreover, since $\operatorname{div}\xi_h = 0$, we have that $\xi_h|_T$ is also a constant vector. Therefore, since $\eta_h|_T$ is also a constant vector, we conclude that $(\mathbf{div}\,\boldsymbol{\tau}_h + \xi_h) = \mathbf{0}$ in Ω. Thus, we obtain

$$V_h = \left\{(\boldsymbol{\tau}_h,\xi_h) \in \mathbf{H}_h \ :\ \xi_h + \mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0},\ \int_\Omega s_h(\tau_{12h} - \tau_{21h}) = 0 \quad \forall s_h \in Q_h^r \text{ and } \operatorname{div}\xi_h = 0 \text{ in } \Omega\right\}.$$

Note that the second condition in the above definition does not guarantee the symmetry of the tensors $H_h^{\boldsymbol{\sigma}}$, as it was the case for the continuous kernel of b.

Hence, we have that V_h is not included in V, however the proof of Lemma 2.1 can be repeated (because in this proof we never used that $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathbf{t}}$), to get the following result

LEMMA 3.1 There exists $C > 0$ such that

$$a((\boldsymbol{\tau}_h,\xi_h),(\boldsymbol{\tau}_h,\xi_h)) \geqslant C\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}^2 \quad \forall(\boldsymbol{\tau}_h,\xi_h) \in V_h,$$

where the constant C is independent of h and t.

We introduce the Raviart-Thomas interpolation operator $\mathscr{R} : [H^1(\Omega)]^2 \to H_h^{\gamma}$. Let us review some properties of this operator that we will use in the sequel:

- Let $\mathscr{P}$ be the orthogonal projection from $L^2(\Omega)$ onto the finite element subspace Q_h^w. Then for all $\xi \in [H^1(\Omega)]^2$, we have
$$\operatorname{div}\mathscr{R}\xi = \mathscr{P}(\operatorname{div}\xi). \tag{3.2}$$

- There exists $c > 0$, independent of h, such that

$$\|\xi - \mathscr{R}\xi\|_{0,\Omega} \leqslant ch\|\xi\|_{1,\Omega} \quad \forall \xi \in [H^1(\Omega)]^2. \tag{3.3}$$

Now, let $\Pi_h : [H^1(\Omega)]^{2\times 2} \to X_h$ be the usual interpolation operator defined as the cartesian product of Raviart-Thomas interpolation operator $\mathscr{R}$, which satisfies (see Arnold *et al.* (1984)):

- Let $\widetilde{\mathscr{P}}$ be the orthogonal projection from $[L^2(\Omega)]^2$ onto the finite element subspace Q_h^{β}. Then for all $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$, we have
$$\mathbf{div}\,\Pi_h \boldsymbol{\tau} = \widetilde{\mathscr{P}}(\mathbf{div}\,\boldsymbol{\tau}).$$

- There exists $c > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}\|_{0,\Omega} \leqslant ch\|\boldsymbol{\tau}\|_{1,\Omega} \quad \forall \boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}.$$

Since $X_h \subset H_h^{\boldsymbol{\sigma}}$, the operator Π_h can be considered from $[H^1(\Omega)]^{2\times 2}$ into $H_h^{\boldsymbol{\sigma}}$, with the same properties given above.

Moreover, we let $\mathscr{P}^1 : L^2(\Omega) \to Q_h^r$ the ortogonal projection. Then, we have

$$\|s - \mathscr{P}^1(s)\|_{0,\Omega} \leqslant Ch\|s\|_{1,\Omega}. \tag{3.4}$$

We continue with the following lemma establishing the discrete analogue of Lemma 2.3.

LEMMA 3.2 There exists $C > 0$, independent of h and t, such that

$$\sup_{(\boldsymbol{\tau}_h,\xi_h)\in \mathbf{H}_h} \frac{|b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))|}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} \geqslant C\|(\eta_h,s_h,v_h)\|_{\mathbf{Q}} \quad \forall (\eta_h,s_h,v_h) \in \mathbf{Q}_h.$$

Proof. Let $(\eta_h,s_h,v_h) \in \mathbf{Q}_h$. From Lemma 4.4 in Arnold *et al.* (1984), we have that there exists $\widetilde{\boldsymbol{\tau}}_h \in H_h^{\boldsymbol{\sigma}}$ and $\tilde{c} > 0$ such that,

$$\frac{\int_\Omega \eta_h \cdot \mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h + \int_\Omega s_h(\tilde{\tau}_{12h} - \tilde{\tau}_{21h})}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}} \geqslant \tilde{c}\left\{\|\eta_h\|_{0,\Omega} + \|s_h\|_{0,\Omega}\right\}. \tag{3.5}$$

Thus, using (3.5), we have

$$\sup_{(\boldsymbol{\tau}_h,\xi_h)\in \mathbf{H}_h} \frac{|b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h)|}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} \geqslant \frac{|b((\widetilde{\boldsymbol{\tau}}_h,\mathbf{0}),(\eta_h,s_h,v_h)|}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}}$$
$$= \frac{\int_\Omega \eta_h \cdot \mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h + \int_\Omega s_h(\tilde{\tau}_{12h} - \tilde{\tau}_{21h})}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}} \geqslant \tilde{c}\left\{\|\eta_h\|_{0,\Omega} + \|s_h\|_{0,\Omega}\right\},$$

which yields

$$\sup_{(\boldsymbol{\tau}_h,\xi_h)\in \mathbf{H}_h} \frac{|b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))|}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} \geqslant \tilde{c}\left\{\|\eta_h\|_{0,\Omega} + \|s_h\|_{0,\Omega}\right\} \quad \forall (\eta_h,s_h,v_h) \in \mathbf{Q}_h. \tag{3.6}$$

Let now z be the unique weak solution (as a consequence of Poincaré inequality and Lax-Milgram's lemma) of the following problem:

$$\begin{cases} -\Delta z = v_h & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $v_h \in L^2(\Omega)$ and Ω is a convex domain, a classical elliptic regularity result guarantees that $z \in H^2(\Omega) \cap H_0^1(\Omega)$ and there exists $\bar{c} > 0$ such that $\|z\|_{2,\Omega} \leqslant \bar{c}\|v_h\|_{0,\Omega}$.

Now, we define $\widetilde{\xi} := -\nabla z \in [H^1(\Omega)]^2$. We note that $\operatorname{div}\widetilde{\xi} = v_h$ in Ω, and

$$\|\widetilde{\xi}\|_{1,\Omega} = \|\nabla z\|_{1,\Omega} \leqslant \|z\|_{2,\Omega} \leqslant \bar{c}\|v_h\|_{0,\Omega}. \tag{3.7}$$

Let $\widetilde{\xi}_h := \mathscr{R}\widetilde{\xi}$. From (3.2) and the fact that $\operatorname{div}\widetilde{\xi} = v_h$, we have that $\operatorname{div}\widetilde{\xi}_h = v_h$ in Ω. Hence, using the approximation property (3.3), we deduce that

$$\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega} \leqslant \|\widetilde{\xi}_h - \widetilde{\xi}\|_{0,\Omega} + \|\widetilde{\xi}\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}\|_{0,\Omega} \leqslant ch\|\widetilde{\xi}\|_{1,\Omega} + \|\widetilde{\xi}\|_{1,\Omega}.$$

From estimates (3.7), we obtain

$$(1+t)\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega} \leqslant 2(\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega}) \leqslant \hat{c}\|v_h\|_{0,\Omega}.$$

It follows that

$$\begin{aligned} \sup_{(\boldsymbol{\tau}_h,\xi_h)\in\mathbf{H}_h} \frac{|b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))|}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} &\geqslant \frac{|b((\mathbf{0},\widetilde{\xi}_h),(\eta_h,s_h,v_h))|}{(1+t)\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega}} \\ &\geqslant \frac{1}{2(\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega})}\left|\int_\Omega \eta_h\cdot\widetilde{\xi}_h + \int_\Omega v_h \operatorname{div}\widetilde{\xi}_h\right| \\ &\geqslant \frac{1}{2(\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega})}\left|\int_\Omega \eta_h\cdot\widetilde{\xi}_h + \|v_h\|_{0,\Omega}^2\right| \\ &\geqslant \frac{1}{2(\|\widetilde{\xi}_h\|_{0,\Omega} + \|\operatorname{div}\widetilde{\xi}_h\|_{0,\Omega})}\left(\|v_h\|_{0,\Omega}^2 - \left|\int_\Omega \eta_h\cdot\widetilde{\xi}_h\right|\right) \\ &\geqslant \frac{1}{\hat{c}}\|v_h\|_{0,\Omega} - \|\eta_h\|_{0,\Omega}. \end{aligned}$$

Given

$$A_h := \sup_{(\boldsymbol{\tau}_h,\xi_h)\in\mathbf{H}_h} \frac{|b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))|}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}},$$

we have proved that (cf. (3.6))

$$A_h \geqslant \tilde{c}\|\eta_h\|_{0,\Omega},$$

and therefore we have that

$$A_h \geqslant \frac{\tilde{c}}{(1+\tilde{c})\hat{c}}\|v_h\|_{0,\Omega}.$$

This allows us to conclude the proof. □

We are now in a position to establish the unique solvability, the stability, and the convergence properties of the discrete problem (3.1).

THEOREM 3.1 There exists a unique $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution of the discrete problem (3.1). Moreover, there exist $\widetilde{C}, C > 0$, independent of h and t, such that

$$\|((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \widetilde{C}\|g\|_{0,\Omega},$$

and

$$\begin{aligned}\|((\boldsymbol{\sigma},\gamma),(\beta,r,w)) - ((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\|_{\mathbf{H}\times\mathbf{Q}} \\ \leqslant C \inf_{((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))\in\mathbf{H}_h\times\mathbf{Q}_h} \|((\boldsymbol{\sigma},\gamma),(\beta,r,w)) - ((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))\|_{\mathbf{H}\times\mathbf{Q}},\end{aligned} \tag{3.8}$$

where $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ is the unique solution of the mixed variational formulation (2.7).

Proof. Is a direct application of the Theorem 2.1 in Chapter II of Brezzi & Fortin (1991). □

The following theorem provides the rate of convergence of our mixed finite element scheme (3.1).

THEOREM 3.2 Let $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solutions of the continuous and discrete problems (2.7) and (3.1), respectively. If $g \in H^1(\Omega)$, then,

$$\|((\boldsymbol{\sigma},\gamma),(\beta,r,w)) - ((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\|_{\mathbf{H}\times\mathbf{Q}} \leqslant Ch\|g\|_{1,\Omega}.$$

Proof. First, we note that (see (2.1))

$$\|(\mathbf{div}\,\boldsymbol{\sigma}+\gamma) - (\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h)\|_{0,\Omega} = \|\,\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h\|_{0,\Omega}.$$

From the second equation of (3.1), we have that

$$\int_\Omega \eta_h\cdot(\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h) = 0 \quad \forall \eta_h \in Q_h^\beta,$$

hence

$$-(\mathbf{div}\,\boldsymbol{\sigma}_h)|_T = \frac{1}{|T|}\int_T \gamma_h = \widetilde{\mathscr{P}}(\gamma_h).$$

Thus, also recalling (2.2),

$$\begin{aligned}\|\,\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h\|_{0,\Omega}^2 &= \sum_{T\in\mathscr{T}_h}\|\,\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h\|_{0,T}^2 = \sum_{T\in\mathscr{T}_h}\|\gamma_h-\widetilde{\mathscr{P}}(\gamma_h)\|_{0,T}^2 \leqslant C_1\sum_{T\in\mathscr{T}_h}h_T^2|\gamma_h|_{1,T}^2 \\ &\leqslant \widetilde{C}_1\sum_{T\in\mathscr{T}_h}h_T^2\|\mathrm{div}\,\gamma_h\|_{0,T}^2 = \widetilde{C}_1\sum_{T\in\mathscr{T}_h}h_T^2\|\mathscr{P}(\mathrm{div}\,\gamma)\|_{0,T}^2 \leqslant Ch^2\|g\|_{0,\Omega}^2.\end{aligned}$$

Therefore

$$\|\,\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h\|_{0,\Omega} \leqslant Ch\|g\|_{0,\Omega}. \tag{3.9}$$

On the other hand, we have

$$\|\mathrm{div}\,\gamma-\mathrm{div}\,\gamma_h\|_{0,\Omega}^2 = \sum_{T\in\mathscr{T}_h}\|\mathrm{div}\,\gamma-\mathrm{div}\,\gamma_h\|_{0,T}^2 = \sum_{T\in\mathscr{T}_h}\|\mathrm{div}\,\gamma-\mathscr{P}(\mathrm{div}\,\gamma)\|_{0,T}^2 \leqslant C\sum_{T\in\mathscr{T}_h}h_T^2|\mathrm{div}\,\gamma|_{1,T}^2,$$

which, using (2.2), yields

$$\|\mathrm{div}\,\gamma-\mathrm{div}\,\gamma_h\|_{0,\Omega} \leqslant Ch|g|_{1,\Omega}. \tag{3.10}$$

Now, it is easy to check that (see Lemma 2.1)

$$\left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + t^2\|\xi\|_{0,\Omega}^2\right) \leqslant Ca((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)) \quad \forall(\boldsymbol{\tau},\xi)\in \mathbf{H}.$$

In particular, taking $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\gamma-\gamma_h)\in\mathbf{H}$, we get

$$\left(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2 + t^2\|\gamma-\gamma_h\|_{0,\Omega}^2\right) \leqslant Ca((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\gamma-\gamma_h),(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\gamma-\gamma_h)),$$

and, using the first equation of (2.7), we obtain

$$\left(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2 + t^2\|\gamma-\gamma_h\|_{0,\Omega}^2\right) \leqslant Cb((\boldsymbol{\sigma}_h-\boldsymbol{\sigma},\gamma_h-\gamma),(\beta,r,w)). \tag{3.11}$$

Now, from the definition of the bilinear form $b(\cdot,\cdot)$ we get

$$\begin{aligned} b((\boldsymbol{\sigma}_h-\boldsymbol{\sigma},\gamma_h-\gamma),(\beta,r,w)) = &\int_\Omega \beta\cdot(\mathbf{div}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\gamma_h-\gamma)) \\ &+\int_\Omega r\,((\boldsymbol{\sigma}_{12h}-\boldsymbol{\sigma}_{21h})-(\boldsymbol{\sigma}_{12}-\boldsymbol{\sigma}_{21}))+\int_\Omega w\mathrm{div}\,(\gamma_h-\gamma). \end{aligned} \tag{3.12}$$

Subtracting the second equation of (3.1) from the second equation (2.7), we have that

$$\int_\Omega \eta_h\cdot(\mathbf{div}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\gamma_h-\gamma))=0 \quad \forall \eta_h\in Q_h^\beta, \tag{3.13}$$

$$\int_\Omega s_h\,((\boldsymbol{\sigma}_{12h}-\boldsymbol{\sigma}_{21h})-(\boldsymbol{\sigma}_{12}-\boldsymbol{\sigma}_{21}))=0 \quad \forall s_h\in Q_h^r, \tag{3.14}$$

$$\int_\Omega v_h\mathrm{div}\,(\gamma_h-\gamma)=0 \quad \forall v_h\in Q_h^w. \tag{3.15}$$

Considering $\widetilde{\mathscr{P}}(\beta)\in Q_h^\beta$, $\mathscr{P}^1(r)\in Q_h^r$ and $\mathscr{P}(w)\in Q_h^w$, using (3.13), (3.14) and (3.15), we rewrite (3.12) as follow

$$\begin{aligned} b((\boldsymbol{\sigma}_h-\boldsymbol{\sigma},\gamma_h-\gamma),(\beta,r,w)) = &\int_\Omega (\beta-\widetilde{\mathscr{P}}(\beta))\cdot(\mathbf{div}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\gamma_h-\gamma)) \\ &+\int_\Omega (r-\mathscr{P}^1(r))\,((\boldsymbol{\sigma}_{12h}-\boldsymbol{\sigma}_{21h})-(\boldsymbol{\sigma}_{12}-\boldsymbol{\sigma}_{21})) \\ &+\int_\Omega (w-\mathscr{P}(w))\mathrm{div}\,(\gamma_h-\gamma). \end{aligned}$$

From (3.11), the above equation and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2 + t^2\|\gamma-\gamma_h\|_{0,\Omega}^2 \leqslant &\, C\|\beta-\widetilde{\mathscr{P}}(\beta)\|_{0,\Omega}\|\mathbf{div}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\gamma_h-\gamma)\|_{0,\Omega} \\ &+C\|r-\mathscr{P}^1(r)\|_{0,\Omega}\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,\Omega} \\ &+C\|w-\mathscr{P}(w)\|_{0,\Omega}\|\mathrm{div}\,(\gamma_h-\gamma)\|_{0,\Omega}. \end{aligned}$$

Applying the inequality $pq\leqslant \frac{1}{2}p^2+\frac{1}{2}q^2$, from the above bound it follows

$$\begin{aligned} \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2 + t^2\|\gamma-\gamma_h\|_{0,\Omega}^2 \leqslant &\, C\|\beta-\widetilde{\mathscr{P}}(\beta)\|_{0,\Omega}\|\mathbf{div}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\gamma_h-\gamma)\|_{0,\Omega} \\ &+\frac{1}{2}C^2\|r-\mathscr{P}^1(r)\|_{0,\Omega}^2+\frac{1}{2}\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,\Omega}^2 \\ &+C\|w-\mathscr{P}(w)\|_{0,\Omega}\|\mathrm{div}\,(\gamma_h-\gamma)\|_{0,\Omega}. \end{aligned}$$

Using standard error estimates arguments, (3.4), (3.9) and (3.10), yields

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2+t^2\|\gamma-\gamma_h\|_{0,\Omega}^2 \leqslant \hat{C}\big(ch^2\|\beta\|_{1,\Omega}\|g\|_{0,\Omega}+ch^2\|r\|_{1,\Omega}+ch^2\|w\|_{1,\Omega}|g|_{1,\Omega}\big),$$

and thus, from Proposition 2.1, we get

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}+t\|\gamma-\gamma_h\|_{0,\Omega} \leqslant Ch\|g\|_{1,\Omega}. \tag{3.16}$$

Finally, using (3.8) we obtain

$$\begin{aligned}\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))-((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\|_{\mathbf{H}\times\mathbf{Q}} & \\ \leqslant C\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))-((\boldsymbol{\sigma}_h,\gamma_h),(\widetilde{\mathscr{P}}(\beta),\mathscr{P}^1(r),\mathscr{P}(w)))\|_{\mathbf{H}\times\mathbf{Q}} & \\ \leqslant Ch\|g\|_{1,\Omega}, &\end{aligned}$$

where in the last inequality we have used standard error estimates arguments, (3.4), (3.9), (3.10) and (3.16). We conclude the proof. □

3.1 *A post processing of transversal displacement and rotations*

In this section we present an element-wise post processing procedure which allows to build piecewise linear transversal displacement and rotations with improved approximation properties. In the following, we indicate with e a general edge of the triangulation and with $\mathscr{E}_h$ the set of all such edges. Moreover, we indicate with h_e the length of $e \in \mathscr{E}_h$ and associate to each edge a unit normal vector $\mathbf{n}_e$, chosen once and for all. For each internal edge e of $\mathscr{E}_h$, we indicate with T^+ and T^- the two triangles of the mesh which have the edge e in common, where $\mathbf{n}_e$ corresponds to the outward normal for T^+ and the opposite for T^-. Then, given any piecewise regular (scalar or vector) function v on Ω, for each $e \in \mathscr{E}_h$ we define the jump on internal edges

$$[\![v]\!] = v^+|_e - v^-|_e,$$

where $v^\pm$ is the restriction of v to $T^\pm$. On boundary edges, the jump is simply given by the value of v on the edge. We introduce the following H^1 type discrete norm

$$\|v\|_{1,h}^2 = \sum_{T\in\mathscr{T}_h}\|\nabla v\|_{0,T}^2 + \sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2,$$

for all sufficiently regular (scalar or vector) functions v.

Given, the discrete solution $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))$, we define a post-processed transversal displacement $w_h^\star \in L^2(\Omega)$ as follows. For all $T \in \mathscr{T}_h$ let $w_h^\star \in \mathbb{P}_1(T)$ such that

$$\begin{aligned}\mathscr{P}w_h^\star &= w_h,\\ \nabla w_h^\star &= \widetilde{\mathscr{P}}(\beta_h + t^2\kappa^{-1}\gamma_h).\end{aligned} \tag{3.17}$$

It is immediate to check that $w_h^\star$ is well defined and unique.

Let $w_I = \mathscr{P}w$, we start proving the following preliminary result.

LEMMA 3.3 There holds

$$\|w_I - w_h\|_{1,h} \leqslant Ch\|g\|_{1,\Omega}.$$

Proof. To prove the result, we will apply the following inf-sup condition: For all $v_h \in Q_h^w$, there exists $\xi_h \in H_h^\gamma$ such that

$$\int_\Omega v_h \operatorname{div} \xi_h = \|v_h\|_{1,h}^2 , \qquad \|\xi_h\|_{0,\Omega} \leqslant C\|v_h\|_{1,h}. \tag{3.18}$$

The simple proof of the above inf-sup condition will be shown briefly. Defining the degrees of freedom of ξ_h by $\xi_h \cdot \mathbf{n}_e := h_e^{-1}[\![v_h]\!]$ for all $e \in \mathscr{E}_h$, an element-wise integration by parts and the definition of the jump operator yield

$$\int_\Omega v_h \operatorname{div} \xi_h = \sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![v_h]\!]\|_{0,e}^2 = \|v_h\|_{1,h}^2.$$

The second bound in (3.18) follows easily by a scaling argument.

Applying (3.18) to $v_h = (w_h - w_I)$, noting that $\operatorname{div}\xi_h$ is piecewise constant and finally using the discrete equations (3.1), we obtain

$$\|w_I - w_h\|_{1,h}^2 = \int_\Omega (w_I - w_h)\operatorname{div}\xi_h = \int_\Omega (w - w_h)\operatorname{div}\xi_h = \int_\Omega (\beta - \beta_h)\xi_h + t^2\kappa^{-1}\int_\Omega (\gamma - \gamma_h)\xi_h.$$

The proof then follows from the above equation using a Cauchy-Schwarz inequality, recalling Theorem 3.2 and using (3.18). □

We have the following improved convergence result for the post-processed transversal displacement.

PROPOSITION 3.3 There holds

$$\|w - w_h^\star\|_{1,h} \leqslant Ch\|g\|_{1,\Omega}.$$

Proof. We note that we can split

$$w_h^\star = w_h + \widetilde{w}_h , \qquad w = w_I + \widetilde{w}, \tag{3.19}$$

where w_h and w_I, already defined above, are piecewise constant while $\widetilde{w}_h$ and $\widetilde{w}$ have zero average on each element.

Applying a scaled trace inequality on each triangle T and using that $\widetilde{w}_h$ and $\widetilde{w}$ have zero average on each element, we have that

$$\sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![\widetilde{w}_h - \widetilde{w}]\!]\|_{0,e}^2 \leqslant C\sum_{T\in\mathscr{T}_h}(h_T^{-2}\|\widetilde{w}_h - \widetilde{w}\|_{0,T}^2 + |\widetilde{w}_h - \widetilde{w}|_{1,T}^2) \leqslant C\sum_{T\in\mathscr{T}_h}\|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2. \tag{3.20}$$

We now observe that, due to (3.19), there hold $\nabla w_h^\star|_T = \nabla\widetilde{w}_h|_T$ and $\nabla w|_T = \nabla\widetilde{w}|_T$ for all $T \in \mathscr{T}_h$. Therefore, first due to definition (3.17) and (2.3), then using standard properties of the projector $\widetilde{\mathscr{P}}$, for all $T \in \mathscr{T}_h$ we obtain

$$\begin{aligned}
\|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2 &= \|\widetilde{\mathscr{P}}(\beta_h + t^2\kappa^{-1}\gamma_h) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 \\
&\leqslant \|\widetilde{\mathscr{P}}(\beta_h + t^2\kappa^{-1}\gamma_h) - \widetilde{\mathscr{P}}(\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 + \|\widetilde{\mathscr{P}}(\beta + t^2\kappa^{-1}\gamma) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 \\
&\leqslant \|(\beta_h + t^2\kappa^{-1}\gamma_h) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 + Ch_T^2|(\beta + t^2\kappa^{-1}\gamma)|_{1,T}^2 \\
&\leqslant C(\|\beta_h - \beta\|_{0,T}^2 + t^4\|\gamma_h - \gamma\|_{0,T}^2 + h_T^2|\beta|_{1,T}^2 + h_T^2 t^4|\gamma|_{1,T}^2).
\end{aligned}$$

The above estimate, combined with Theorem 3.2 and Proposition 2.1, immediately yields

$$\sum_{T\in\mathscr{T}_h}\|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2 \leqslant Ch^2\|g\|_{1,\Omega}^2. \tag{3.21}$$

From (3.20), (3.21) and the definition of $\|\cdot\|_{1,h}$ norm, we finally obtain

$$\|\widetilde{w}_h - \widetilde{w}\|_{1,h} \leqslant Ch\|g\|_{1,\Omega}.$$

The above estimate, combined with Lemma 3.3 and a triangle inequality, finally gives the proof of the Proposition:

$$\|w - w_h^\star\|_{1,h} \leqslant \|w_I - w_h\|_{1,h} + \|\widetilde{w}_h - \widetilde{w}\|_{1,h} \leqslant Ch\|g\|_{1,\Omega}.$$

□

We define also a post-processed rotation field $\beta_h^\star \in [L^2(\Omega)]^2$ as follows: For all $T \in \mathscr{T}_h$ let $\beta_h^\star \in [\mathbb{P}_1(T)]^2$ such that

$$\begin{aligned} \widetilde{\mathscr{P}}\beta_h^\star &= \beta_h, \\ \nabla\beta_h^\star &= \widehat{\mathscr{P}}(\mathscr{C}^{-1}\boldsymbol{\sigma}_h + r_h\mathbf{J}), \end{aligned} \tag{3.22}$$

where $\widehat{\mathscr{P}}$ is the L^2 projection onto the space of piecewise constant $\mathbb{R}^{2\times 2}$ tensor fields. It is immediate to check that $\beta_h^\star$ is well defined and unique.

Moreover, the following results can be proved following the same lines shown above.

PROPOSITION 3.4 There holds

$$\|\beta - \beta_h^\star\|_{1,h} \leqslant Ch\|g\|_{1,\Omega}.$$

Finally note that both post-processing procedures are fully local and therefore have a negligible computational cost.

REMARK 3.1 Although the main purpose of this scheme is to compute a better approximation of the stresses, using this post-processing a piecewise linear approximation of transversal displacement and rotations, converging in a H^1 type norm can be recovered. Note in particular that, from the definition of the norm $\|\cdot\|_{1,h}$ and the fact that the jumps of w and β are null, it follows that at the limit for $h \to 0$ the post-processed discrete functions will also be continuous.

4. Numerical results

We report in this section some numerical experiments which confirm the theoretical results proved above. The numerical method analyzed has been implemented in a MATLAB code.

As a *test problem* we have taken an isotropic and homogeneous plate $\Omega := (0,1)\times(0,1)$ clamped on the whole boundary for which the analytical solution is explicitly known (see Chinosi *et al.* (2006)). We analyze the convergence properties of the elements proposed here by considering different uniform decompositions as shown in Figure 1, and keeping the thickness $t = 0.001$.

Choosing the transversal load g as:

$$\begin{aligned} g(x,y) = \frac{E}{12(1-\nu^2)}\Big[&12y(y-1)(5x^2-5x+1)\left(2y^2(y-1)^2 + x(x-1)(5y^2-5y+1)\right) \\ &+12x(x-1)(5y^2-5y+1)\left(2x^2(x-1)^2 + y(y-1)(5x^2-5x+1)\right)\Big], \end{aligned}$$

the exact solution of problem (2.7) is given by

$$\begin{aligned} w(x,y) =& \frac{1}{3}x^3(x-1)^3y^3(y-1)^3 \\ &- \frac{2t^2}{5(1-\nu)}\Big[y^3(y-1)^3x(x-1)(5x^2-5x+1) + x^3(x-1)^3y(y-1)(5y^2-5y+1)\Big], \end{aligned}$$

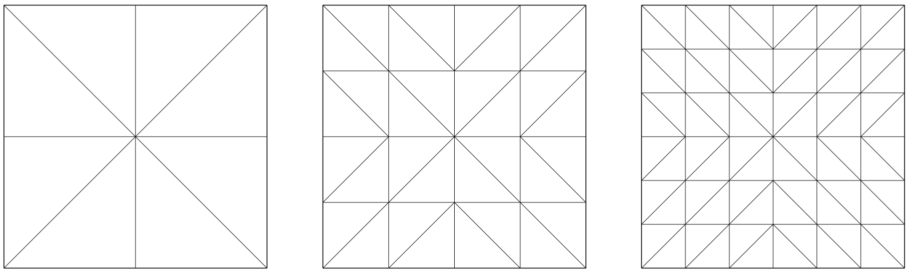

FIG. 1. Square plate: uniform meshes.

$$\beta_1(x,y) = y^3(y-1)^3x^2(x-1)^2(2x-1),$$

$$\beta_2(x,y) = x^3(x-1)^3y^2(y-1)^2(2y-1).$$

The material constants have been chosen: $E = 1$, $\nu = 0.30$ and the shear correction factor has been taken $k = 5/6$.

In what follows, N denotes the number of degrees of freedom, namely, $N := \dim(\mathbf{H}_h \times \mathbf{Q}_h)$. Moreover, we define the individual errors by:

$$\mathbf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}, \quad \mathbf{e}(\boldsymbol{\sigma},\gamma) := \|(\mathbf{div}\,\boldsymbol{\sigma} + \gamma) - (\mathbf{div}\,\boldsymbol{\sigma}_h + \gamma_h)\|_{0,\Omega}, \quad \mathbf{e}(\gamma) := \|\gamma - \gamma_h\|_{t,H(\mathrm{div};\Omega)},$$

$$\mathbf{e}(r) := \|r - r_h\|_{0,\Omega}, \qquad \mathbf{e}(\beta) := \|\beta - \beta_h\|_{0,\Omega}, \qquad \mathbf{e}(w) := \|w - w_h\|_{0,\Omega},$$

where $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H} \times \mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ are the unique solutions of problems (2.7) and (3.1), respectively.

Also, we define the experimental rates of convergence (rc) for the errors $\mathbf{e}(\boldsymbol{\sigma})$, $\mathbf{e}(\boldsymbol{\sigma},\gamma)$, $\mathbf{e}(\gamma)$, $\mathbf{e}(r)$, $\mathbf{e}(\beta)$ and $\mathbf{e}(w)$ by

$$rc(\cdot) := -2\frac{\log(\mathbf{e}(\cdot)/\mathbf{e}'(\cdot))}{\log(N/N')},$$

where N and N' denote the degrees of freedom of two consecutive triangulations with errors $\mathbf{e}$ and $\mathbf{e}'$.

Tables 1, 2 and 3 show the convergence history of the mixed finite element scheme (3.1) applied to our *test problem*.

Table 1. Errors and experimental rates of convergence for variables $\boldsymbol{\sigma}$ and $(\mathbf{div}\,\boldsymbol{\sigma} + \gamma)$, computed on uniform meshes.

N	$\mathbf{e}(\boldsymbol{\sigma})$	$rc(\boldsymbol{\sigma})$	$\mathbf{e}(\boldsymbol{\sigma},\gamma)$	$rc(\boldsymbol{\sigma},\gamma)$
1345	0.40270e-04	–	0.29609e-03	–
5249	0.19649e-04	1.054	0.14805e-03	1.018
20737	0.09760e-04	1.019	0.07404e-03	1.009
82433	0.04868e-04	1.008	0.03702e-03	1.004
328705	0.02431e-04	1.004	0.01851e-03	1.002

Table 2. Errors and experimental rates of convergence for variables γ and r, computed on uniform meshes.

N	$\mathbf{e}(\gamma)$	$rc(\gamma)$	$\mathbf{e}(r)$	$rc(r)$
1345	0.31715e-02	–	0.87462e-04	–
5249	0.15876e-02	1.016	0.39217e-04	1.178
20737	0.07942e-02	1.008	0.15009e-04	1.398
82433	0.03971e-02	1.004	0.05491e-04	1.457
328705	0.01986e-02	1.002	0.01991e-04	1.466

Table 3. Errors and experimental rates of convergence for variables β and w, computed on uniform meshes.

N	$\mathbf{e}(\beta)$	$rc(\beta)$	$\mathbf{e}(w)$	$rc(w)$
1345	0.39713e-04	–	0.66226e-05	–
5249	0.18189e-04	1.147	0.27707e-05	1.280
20737	0.08884e-04	1.043	0.13136e-05	1.086
82433	0.04416e-04	1.013	0.06478e-05	1.025
328705	0.02205e-04	1.004	0.03228e-05	1.007

We observe there that the rate of convergence $O(h)$ predicted by Theorem 3.2 is attained for all variables.

Figure 2 shows the profiles of the discrete transversal displacement w_h (left) and the first component of the discrete rotation vector β_{1h} (right) for $t = 0.001$, and the finest mesh.

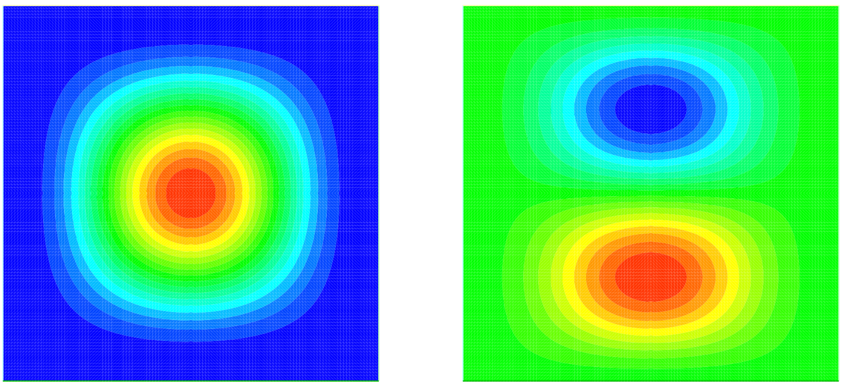

FIG. 2. Approximate transversal displacement (left) and the first component of the rotation vector (right).

Figure 3 shows the profiles of the discrete shear stress γ_h for $t = 0.001$, and the finest mesh.

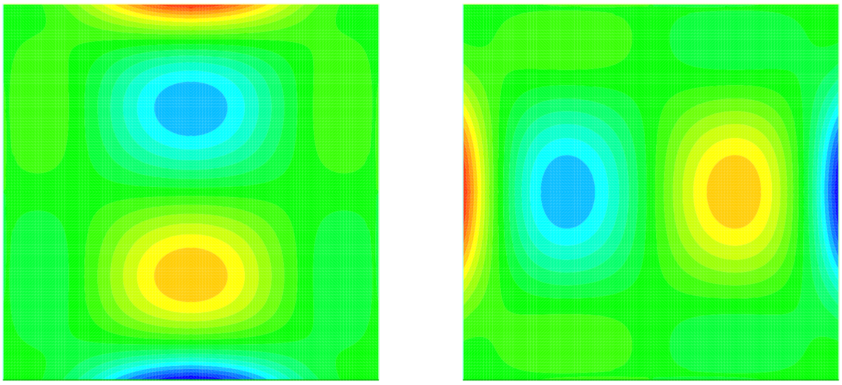

FIG. 3. Approximate shear stress: first component (left) and second component (right).

Figures 4 and 5 show the profiles of the bending moment tensor $\boldsymbol{\sigma}_h = (\boldsymbol{\sigma}_{ijh}), i, j = 1, 2$, for $t = 0.001$, and the finest mesh.

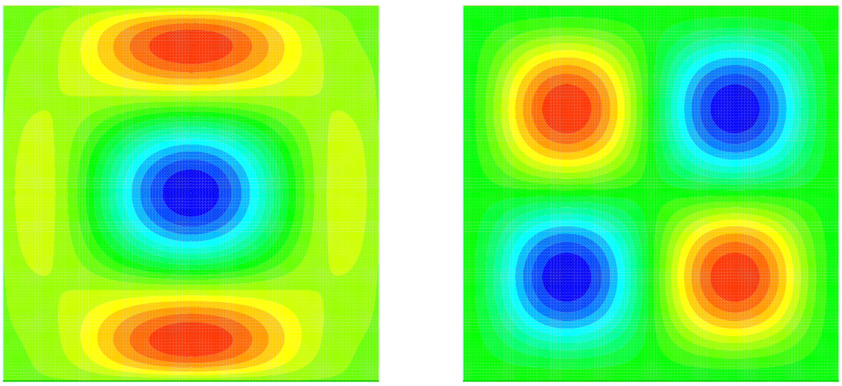

FIG. 4. Approximate bending moment: $\boldsymbol{\sigma}_{11h}$ (left) and $\boldsymbol{\sigma}_{12h}$ (right).

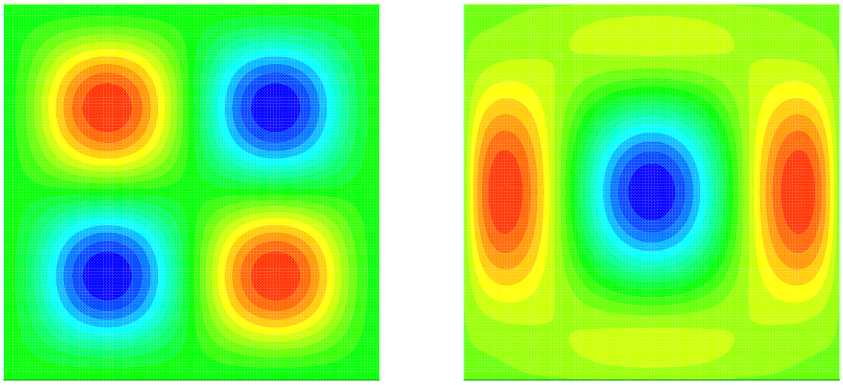

FIG. 5. Approximate bending moment: $\boldsymbol{\sigma}_{21h}$ (left) and $\boldsymbol{\sigma}_{22h}$ (right).

Funding

FONDAP and BASAL projects CMM, Universidad de Chile (Chile) to L.B. and R.R.; CONICYT (Chile) to D.M.

REFERENCES

AMARA, M., CAPATINA-PAPAGHIUC, D. & CHATTI, A. (2002) New locking-free mixed method for the Reissner-Mindlin thin plate model, *SIAM J. Numer. Anal.*, **40**, 1561–1582.

ARNOLD, D. N., BREZZI, F. & DOUGLAS, J. (1984) PEERS: A new mixed finite element for the plane elasticity, *Japan J. Appl. Math.*, **1**, 347–367.

ARNOLD, D. N. & FALK, R. S. (1989) A uniformly accurate finite element method for the Reissner-Mindlin plate, *SIAM J. Numer. Anal.*, **26**, 1276–1290.

BATHE, K. J. & DVORKIN, E. N. (1985) A four-node plate bending element based on Mindlin-Reissner plate theory and a mixed interpolation, *Int. J. Numer. Methods Engrg.* **21**, 367–383.

BEHRENS, E. M. & GUZMÁN, J. (2009) A new family of mixed methods for the Reissner-Mindlin plate model based on a system of first-order equations, Scientific Computing Group Reports (2009-45), Brown University.

BREZZI, F. & FORTIN, M. (1991) *Mixed and Hybrid Finite Element Methods.* New York: Springer.

CHINOSI, C., LOVADINA, C. & MARINI, L. D. (2006) Nonconforming locking-free finite elements for Reissner-Mindlin plates, *Comput. Methods Appl. Mech. Engrg.*, **195**, 3448–3460.

DURÁN, R. & LIBERMAN, E. (1992) On mixed finite elements methods for the Reissner-Mindlin plate model, *Math. Comp.*, **58**, 561–573.

FALK, R. (2008) Finite Elements for the Reissner-Mindlin Plate, *in Mixed Finite Elements, Compatibility Conditions, and Applications, D. Boffi and L. Gastaldi, eds.*, Berlin: Springer, 195–230.