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p-adic modular forms of non-integral weight over Shimura curves

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Abstract

In this work, we set up a theory of p-adic modular forms over Shimura curves over totally real fields which allows us to consider also non-integral weights. In particular, we define an analogue of the sheaves of k-th invariant differentials over the Shimura curves we are interested in, for any p-adic character. In this way, we are able to introduce the notion of overconvergent modular form of any p-adic weight. Moreover, our sheaves can be put in p-adic families over a suitable rigid-analytic space, that parametrizes the weights. Finally, we define Hecke operators. We focus on the U operator, showing that it is completely continuous on the space of overconvergent modular forms.

Contents

Introduction

The goal of this work is to develop a geometric theory of p-adic analytic families of modular forms over certain PEL Shimura curves. A similar problem, for the elliptic case, is the subject of the work in progress [**AIS11**]. Some of the basic ideas of this thesis are taken from [**AIS11**]. The elliptic case is addressed also in [**Pil09**], where slightly different techniques are used, from Hida theory. To motivate our work, we now briefly review some basic facts about p-adic modular forms.

Let $p > 5$ be a prime and let $N > 4$ be a fixed positive integer, with $(p, N) = 1$. Let R be a separated and complete $\mathbb{Z}_p$-algebra. The first precise definition of the concept of p-adic modular form, of level N, weight $k \in \mathbb{Z}$, with coefficients in R, was given by Serre in [**Ser73**]. He identified a modular form with its q-expansion, and he defined a p-adic modular form as a power series $f \in R[[q]]$ such that

$$f(q) = \varinjlim_n f_n(q),$$

where the f_n's are the q-expansions of a sequence of classical modular forms and the limit is calculated in the p-adic topology of $R[[q]]$. Let f be such a p-adic modular form and let $\{k_n\}$ be the sequence of weights of $\{f_n\}$. It turns out that the sequence $\{k_n\}$ can be not eventually constant, so it can be not convergent in $\mathbb{Z}$. This suggests that, in general, the weight of f is not an integer. However, Serre identified an integer k with the map

$$\begin{aligned} k\colon\ \mathbb{Z}_p^* &\to \mathbb{Z}_p^* \\ t &\mapsto t^k \end{aligned}$$

and he showed that the limit $\chi(t) := \varinjlim_n t^{k_n}$ exists for all $t \in \mathbb{Z}_p^*$. Moreover, the map

$$\chi\colon\ \mathbb{Z}_p^* \to \mathbb{Z}_p^*$$

is a continuous character. Since any continuous character occurs in this way, it is reasonable to believe that the weight of a p-adic modular form should be a continuous character $\mathbb{Z}_p^* \to \mathbb{Z}_p^*$. To motivate further his definition, Serre also introduced the notion of an analytic p-adic family of modular forms, parametrized by the weight, and he showed that the mere existence of the family of the p-adic Eisenstein series implies the analyticity of the p-adic zeta function.

There is a rigid analytic space $\mathcal{W}$ over $\mathbb{Q}_p$, called the *weight space*, such that its K-points, for any finite extension $K/\mathbb{Q}_p$, are the continuous characters $\mathbb{Z}_p^* \to K^*$. In particular, we see that the continuous characters $\mathbb{Z}_p^* \to \mathbb{Z}_p^*$ are the $\mathbb{Q}_p$-valued points of $\mathcal{W}$. To work with p-adic analytic families in general, it is then natural to look for modular forms whose weight is a continuous character $\mathbb{Z}_p^* \to K^*$. In this way, a p-adic analytic family of modular forms should be parametrized by the points of an admissible open subset of $\mathcal{W}$.

Serre's definition is very natural, but it has two drawbacks. First of all it is not 'modular', i.e. a p-adic modular form is, seemingly, unrelated to elliptic curves with level structure. Another problem is the fact that it relies essentially on the q-expansion of a modular form. To generalize the notion of p-adic modular forms to other Shimura curves, we cannot use cusps, so we do not have q-expansions in general. In this thesis we are going to consider certain *proper* Shimura curves, so we will need a different approach. The first really 'modular' definition of a p-adic modular form was given by Katz in [**Kat73**]. We are going to briefly recall his construction, see also [**Gou88**] for a concise presentation.

Let $Y_1(N)$ be the modular curve of level N, over $\mathbb{Q}_p$ ($\Gamma_1(N)$-level structure). Let $X_1(N)$ be the compactification of $Y(N)$, so we have a universal semi-abelian scheme $\pi\colon A \to X_1(N)$. Let $\underline{\omega} = \underline{\omega}_{X_1(N)}$ be the sheaf $e^*\Omega^1_{A/X_1(N)}$, where $e\colon X_1(N) \to A$ is the zero section. Classically, a modular form of level N and weight k could be defined as a global section of $\underline{\omega}^{\otimes k}$. In [**Kat73**], Katz gave a geometric interpretation of the notion of a p-adic modular form *of integral weight*. To understand Katz' definition, it is convenient to use rigid geometry. For any rational number $0 \leq w < 1$, let $X_1(N)(w)^{\mathrm{an}}$ be the affinoid subdomain of the analytification of $X_1(N)$ defined in [**Col97a**], relative to the Eisenstein series E_{p-1} (see Section 1.5 for more details about Coleman's construction). We can think about $X_1(N)(w)^{\mathrm{an}}$ as the subset of $X_1(N)^{\mathrm{an}}$ where E_{p-1} has valuation smaller than or equal to w. The complement of $X_1(N)(0)^{\mathrm{an}}$ is a finite union of discs, called the supersingular discs. Katz introduced the notion of p-adic modular form of level N, weight k, and growth condition w: it is a global section of $\underline{\omega}^{\otimes k}$ on $X_1(N)(w)^{\mathrm{an}}$. A modular form of growth condition 0 is called a *convergent* modular form, and one of growth condition $w > 0$ is called an *overconvergent* modular form. Katz defined also the usual Hecke operators acting on the space of p-adic modular forms, including the U operator, the analogue of the classical U_p operator of Atkin. Finally, Katz showed that his definition of a p-adic modular form generalizes Serre's. In particular, if f is a p-adic modular form in the sense of Serre, of integral weight k, then f can be identified with a convergent p-adic modular form, and conversely.

The goal of this work is to develop a similar theory for modular forms over certain PEL Shimura curves, that works for any weight and moreover allows us to consider analytic families.

To fully develop the theory of p-adic modular forms, an essential tool is Riesz theory for completely continuous operators on p-adic Banach modules. Let K be a finite extension of $\mathbb{Q}_p$. In [**Ser62**], Serre developed Riesz theory for completely continuous endomorphisms of orthonormizable Banach modules over K. An example of such a Banach module is provided by the space of p-adic modular forms over K, of growth condition w and weight any $\chi\colon \mathbb{Z}_p^* \to K^*$. It is a key fact that, if we consider only modular forms with growth condition $w > 0$, the U operator is completely continuous, so we have a good Riesz theory for it. In [**Col97b**], Coleman developed Riesz theory for a completely continuous operator on a family of orthonormizable Banach modules, generalizing Serre's work. Coleman defined p-adic families of modular forms, and using his own theory he was able to study the Riesz theory of the U operator (see below). In our work we will need a further generalization of Riesz theory. In [**Buz07**], Buzzard showed that Coleman's results remain true also for Banach modules that are direct summand of an orthonormizable Banach module (we will need this in Chapter 5).

In [**Col97b**], Coleman was able to prove the following theorem, that generalizes a previous result of Hida in [**Hid86**], where only the case $\mathrm{v}(a_p) = 0$ is studied.

THEOREM. *Let f be an overconvergent modular form of weight k that is an eigenform for the full Hecke algebra and let a_p be the* U*-eigenvalue. If*

$$\mathrm{v}(a_p) < k - 1,$$

then f is classical. Furthermore, any such modular form lies in a p-adic family of eigenforms over the weight space.

Since Katz' definition works only for integral weight, the first step needed to obtain Coleman's theorem is to define the notion of overconvergent modular form of any weight.

A natural approach is to generalize the sheaves $\underline{\omega}^{\otimes k}$ obtaining the sheaves $\underline{\omega}^{\otimes \chi}$ on $X_1(w)$, for any p-adic weight χ. Note that, since Coleman is interested in overconvergent modular forms, it is enough to work with arbitrarily small, but positive, w. If one has the sheaf $\underline{\omega}^{\otimes \chi}$, we can define a modular form of growth condition w and weight χ as a global section of $\underline{\omega}^{\otimes \chi}$. However, Coleman's approach is completely different. He made a heavy use of the Eisenstein series to compare modular forms of different levels. In this way he was able to define the notion of overconvergent modular form of weight χ through its q-expansion. He then proved that his definition makes sense, attaching a Galois representation to any p-adic modular form. Finally he was able to prove the above theorem. Coleman's approach is very interesting and powerful, but it seems difficult to generalize.

In [**AIS11**], Andreatta, Iovita, and Stevens proposed a geometric approach to this problem, as follows. Let $\chi \colon \mathbb{Z}_p^* \to K^*$ be a continuous character, where K is a finite extension of $\mathbb{Q}_p$ satisfying certain technical conditions. Then there is a rational number $w > 0$ and a locally free sheaf Ω_w^χ on $X_1(N)(w)^{\mathrm{an}}$, such that its global sections correspond naturally to p-adic modular forms of weight χ and growth condition w, with coefficients in K, as defined by Coleman. Furthermore we have Hecke operators and more importantly these sheaves can be put in p-adic families over the weight space.

In this work we study the case of modular forms over certain quaternionic Shimura curves. The notion of p-adic modular form in this context was introduced by Kassaei in [**Kas04**]. It is worth to note that Kassaei considered only *integral weight*. The definition is similar to Katz', but, since our Shimura curves are compact, we do not have any Eisenstein series, so passing from elliptic modular forms to quaternionic modular forms is really non trivial. Since Kassaei considered only integral weights, he has no families of modular forms. The goal of this work is to give a geometric definition of quaternionic modular forms of any weight and to prove that these modular forms can be put in families.

Here is a detailed description of the thesis. We will work with several curves, corresponding to different level structures. For the convenience of the reader and to stress the analogy between our curves (called quaternionic curves) and the classical ones, it is convenient to list now the curves we are interested in (without definition), with the corresponding classical modular curves. We consider here rigid analytic curves, but we will also need the integral and formal models.

Quaternionic curve	**Level**	**Classical curve**	**Classical level**
$\mathfrak{M}(H)^{\mathrm{rig}}$	$K(H)$	$X_1(N)^{\mathrm{an}}$	$\Gamma_1(N)$
$\mathfrak{M}(H, \varpi^n)^{\mathrm{rig}}$	$K(H, \varpi^n)$	$X_1(N; p^n)^{\mathrm{an}}$	$\Gamma_1(N) \cap \Gamma_0(p^n)$
$\mathfrak{M}(H\varpi^n)^{\mathrm{rig}}$	$K(H\varpi^n)$	$X_1(Np^n)^{\mathrm{an}}$	$\Gamma_1(Np^n)$

Chapter 1 contains only preliminary material. In Section 1.1 we briefly review the definition and basic properties of the Shimura varieties of PEL type. In Section 1.2, following [**Car86**], we define the Shimura curves we are interested in, over $\mathbb{C}$. These are Shimura varieties of PEL type defined as follows. Let $p \neq 2$ denote a fixed rational prime. Let F be a totally real field, with $[F : \mathbb{Q}] > 1$, and let B be a quaternion algebra over F that splits at exactly one infinite place of F and at $\mathcal{P}$, a prime of F above p. Attached to these data, there is an inverse system of Shimura varieties $\{M_K(\mathbb{C})\}$, parametrized by compact open subgroups of $G(\mathbb{A}^f)$, where G is a reductive algebraic group over $\mathbb{Q}$, defined using B. We prove (see Proposition 1.2.5) that each $M_K(\mathbb{C})$ is a compact Riemann surface. By general theory (Theorem 1.1.3), the $M_K(\mathbb{C})$ are moduli spaces of abelian varieties with additional structure. In Section 1.3, assuming that K is small enough, we define a canonical model M_K of $M_K(\mathbb{C})$ over a suitable number field E (Proposition 1.3.1). Let $F_\mathcal{P}$ be the completion of F at $\mathcal{P}$, and let $\mathcal{O}_\mathcal{P} := \mathcal{O}_{F_\mathcal{P}}$ be its ring of integers, with uniformizer ϖ. We have that $F_\mathcal{P}$ is an E-algebra. If K has some specific form, we can give a very explicit description of the moduli problem solved by M_K, over $F_\mathcal{P}$ (Section 1.3.2). Finally we define integral models of our curves (Theorem 1.3.8). In Section 1.4 we explain our assumption that $F \neq \mathbb{Q}$, and we show that the case $F = \mathbb{Q}$ is essentially a particular case of our work. In Section 1.5 we explain Coleman's construction of an admissible open subset of a rigid analytic curve associated with a section of a line bundle. Sections 1.6 and 1.7 contain a brief review of the theory of ϖ-divisible groups and of Faltings' theory of groups schemes with strict $\mathcal{O}_\mathcal{P}$-action. These theories will be essential for our work. In the particular case $\mathcal{O}_\mathcal{P} = \mathbb{Z}_p$ they are just the theory of p-divisible groups and of group schemes.

In Chapter 2 we define p-adic modular forms *of integral weight* over our Shimura curves. Section 2.1 is essentially due to Kassaei. We recall the definition of the analogue of the Hasse invariant in our situation. This allows us to define E_{q-1}, an analogue of the Eisenstein series. In this way using Coleman construction of Section 1.5, we are able to define the space of p-adic modular forms of level $K(H)$, weight $k \in \mathbb{Z}$, and growth condition $0 \leq w < 1$. In Section 2.2, we recall the theory of the canonical subgroup, as developed in [**Kas04**]. We also consider canonical subgroups of higher level (Proposition 2.2.4). We can decompose the p^n-torsion of the objects of our moduli problems, that are abelian schemes, to define a p-divisible group of dimension 1. In [**Kas04**], this p-divisible group is used to define the canonical subgroup. In order to obtain the results we want, we need the theory of ϖ-divisible group. We study the ϖ-divisible group attached to our abelian scheme and in Section 2.3 we define p-adic modular forms of level $K(H\varpi)$. Using the canonical subgroup, we are able to show (Proposition 2.3.5) the important new result that there is a modular form of level $K(H\varpi)$, called E_1, that satisfies

$$E_1^{q-1} = E_{q-1}.$$

Chapter 3 contains the most important technical results of the thesis. In Section 3.1 we define the map $\mathrm{d}\log$, that will be absolutely central in our theory. By Propositions 3.1.4 and 3.1.7 it is closely related with the canonical subgroup. Furthermore, the map $\mathrm{d}\log$ permits to link the modular form E_1 with the canonical subgroup. Indeed we have a canonical point γ of the dual of the canonical subgroup, and we have (Proposition 3.1.10)

$$\mathrm{d}\log(\gamma) \equiv E_1 \bmod \varpi^{1-w}.$$

These results are inspired by [**AIS11**], but are more complicated (see below). In Section 3.2 we use the previous results to construct the so called Hodge-Tate sequence. We prove that the homology of this sequence is killed by a certain power of ϖ (Theorem 3.2.11). This links the Tate module of our abelian schemes to the module of invariant differentials in a very precise way. Since an elliptic curve admits a canonical principal polarization, all the objects studied in [**AIS11**] are self-dual. This is not the case in our situation, in particular we need Proposition 3.2.3. This lack of self duality makes some of the arguments of Section 3.2 more delicate than those in [**AIS11**]. In the case $\mathcal{O}_\mathcal{P} = \mathbb{Z}_p$, we obtain some results of [**AIS11**] in a completely different way.

Chapter 4 is the heart of the thesis. We assume that $\mathcal{O}_\mathcal{P}$ is not too ramified over $\mathbb{Z}_p$ (but see Section 4.5, where we explain what can be done without this assumption). In Section 4.1 we study the continuous characters we are interested in (see Definition 4.1.1). In particular we define a suitable rigid analytic weight space $\mathcal{W}$ whose K points, for K a finite extension of $F_{\mathcal{P}}$, correspond to continuous characters $\mathcal{O}_{\mathcal{P}}^* \to K^*$. We also define an admissible covering $\{\mathcal{W}_r\}_{r\geq 0}$ of $\mathcal{W}$, made by affinoids that will be needed later on (see Lemma 4.1.7). In Section 4.2 we consider only a special class of characters, called accessible. We prove that we can generalize the definition of the sheaves $\underline{\omega}^{\otimes k}$ to any accessible character $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to K^*$. For any fixed χ, there is a rational $w > 0$ and a locally free sheaf Ω_w^χ on $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ (Corollary 4.2.14), such that (Lemma 4.2.21)

$$\Omega_w^\chi = \underline{\omega}^{\otimes k} \text{ if } \chi(t) = t^k.$$

In this way we are able to define the space of p-adic modular forms of weight χ. In order to define the sheaves Ω_w^χ, we need to consider the curve $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$. We start by defining a sheaf $\tilde{\Omega}_w^\chi$ on $\mathfrak{M}(H\varpi^n)(w)^{\mathrm{rig}}$. We then show that we have diamond operators acting on the push-forward of $\tilde{\Omega}_w^\chi$ to $\mathfrak{M}(H)(w)^{\mathrm{rig}}$. Taking invariants with respect to these operators, we obtain the sheaf Ω_w^χ. We also describe our modular forms using 'test object' (Section 4.2.2). In Section 4.3 we consider general characters. This requires working with curves of higher level. In this way we are able to define the notion of p-adic modular form of any weight. In Section 4.4 we consider analytic families over the weight space $\mathcal{W}$. To show that our definition of the sheaves Ω_w^χ makes sense, we prove that the Ω_w^χ's live in families (Proposition 4.4.4). More precisely, we prove that there are locally free sheaves $\Omega_{w,r}$ on $\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}$, such that Ω_w^χ is the pullback of $\Omega_{w,r}$ at the point defined by χ. Furthermore, the $\Omega_{w,r}$'s satisfy various compatibility conditions. This shows that our sheaves really 'interpolate' the sheaves $\underline{\omega}^{\otimes k}$, for various k. Furthermore, any modular form of weight χ lives in a p-adic family.

In Chapter 5 we consider Hecke operators. In Section 5.1 we prove (Corollary 5.1.3) that the space of p-adic modular forms of any weight is a Banach module that satisfies property (Pr) of [**Buz07**] (this is a slightly generalization of being orthonormizable). After that, we introduce the U operator, analogous the the classical U_p operator. We show that it is a completely continuous operator on the space of *overconvergent* modular forms (Proposition 5.1.7). In particular, using the work of Buzzard, we have Riesz theory for U. In Section 5.2 we construct the $\mathrm{T}_{\mathcal{L}}$ operators. These are analogous to the classical T_l operators. We also construct families of both U and $\mathrm{T}_{\mathcal{L}}$. The properties of the U operator imply formally (thanks to the machinery developed in [**Buz07**]) that a modular form that is an eigenvector for the U operator (with finite slope) lives in a p-adic analytic family of eigenforms (Proposition 5.1.11). This gives the analogue of the above theorem of Coleman (see also Remark 5.1.10).

In the Appendix we make a very detailed study of the canonical subgroup. In particular we give explicit formulas for the comultiplication and for the module of invariant differentials (Propositions A.4 and A.10), generalizing some results of [**Col05**]. Furthermore, we show that the trivial analogues of the results of [**AIS11**] are false in our situation. To be more precise, let $\mathcal{A}$ be an object of our moduli problem, of level $K(H\varpi)$. We have the canonical subgroup $\mathcal{C}$ of $\mathcal{A}[p]$. In [**AIS11**], it is shown that we have a canonical point γ' of $\mathcal{C}^{\mathrm{D}}$ (Cartier dual). One of the most important technical results of [**AIS11**] is that the image of γ' under the map $\mathrm{d}\log$ is congruent, modulo p^{1-w}, to E_1. Also in our situation we have the canonical point γ' (Proposition A.6), but, by Proposition A.8, we have

$$\mathrm{d}\log(\gamma') = 0$$

if $\mathcal{O}_{\mathcal{P}}$ is sufficiently ramified. This show that we need a different approach. The deep reason for this problem is that all the objects we are interested in are endowed with an action of $\mathcal{O}_{\mathcal{P}}$, and we really need to take this action into account. For example, Cartier duality does not work, since $\mathbb{G}_{\mathrm{m}}$ does not have a natural action of $\mathcal{O}_{\mathcal{P}}$. This is why we need the theory of group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action.

The whole thesis can be seen as the proof of the following two main technical results. Here K is a finite extension of $F_{\mathcal{P}}$ satisfying certain technical conditions.

THEOREM. *Let $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to K^*$ be a continuous character and assume that w is small enough. Then we have an invertible sheaf Ω_w^χ on $\mathfrak{M}(H)(w)_K^{\mathrm{rig}}$ and a completely continuous operator* U *on the space of global sections of Ω_w^χ. If $\chi(t) = t^k$ for all $t \in \mathcal{O}_{\mathcal{P}}$, then there is a natural isomorphism, commuting with the action of* U, *between $\mathrm{H}^0(\Omega_w^\chi, \mathfrak{M}(H)(w)_K^{\mathrm{rig}})$ and the space of modular forms of growth condition w as defined in* [**Kas04**].

THEOREM. *Let $r \geq 0$ be an integer. For any small enough w, we have an invertible sheaf $\Omega_{w,r}$ on $\mathcal{W}_r \times \mathfrak{M}(H)(w)_K^{\mathrm{rig}}$ such that its pullback to $\mathfrak{M}(H)(w)_K^{\mathrm{rig}}$ at any $\chi \in \mathcal{W}_r(K)$ is Ω_w^χ. We have Hecke operators on $\Omega_{w,r}$. Furthermore, any* U*-eigenform of finite slope can be analytically deformed.*

As an application of our work, we obtain the following theorem (see Theorem 5.3.1).

THEOREM. *There is a rigid space $\mathcal{C} \subseteq \mathcal{W} \times \mathbb{A}_K^{1,\mathrm{rig}}$, called the eigencurve, such that its L-points, where L is a finite extension of K, correspond naturally to systems of eigenvalues of overconvergent modular forms defined over L. If $x \in \mathcal{C}(L)$, let $\mathcal{M}_x$ be the set of overconvergent modular forms corresponding to x. Then all the elements of $\mathcal{M}_x$ have weight $\pi_1(x) \in \mathcal{W}(L)$ and the* U*-operator acts on $\mathcal{M}_x$ with eigenvalue $\pi_2(x)^{-1}$.*

Future developments

As in the classical case, we plan to give a cohomological interpretation of our modular forms, providing an isomorphism similar to the classical Eichler-Shimura isomorphism. This should allow us to use the powerful language of modular symbols, as for example in [**Bel09**].

Using the above mentioned cohomological interpretation, we believe we will also be able to apply the construction given in [**Urb10**]. We plan to compare this construction with the one given in Theorem 5.3.1, proving that they give the same eigencurve.

We finally hope that our approach to define p-adic families of overconvergent modular forms can be also used for algebraic groups different from G. We plan to develop such a theory in a future work.

Notations

We will try to always use standard notations.

All rings are assumed to have a unity element 1, and any morphism of rings sends 1 to 1. Unless explicitly stated, all rings are commutative.

If R is any ring and n is an integer, we will write $\mu_n(R)$ for the set of n-th roots of unity in R. We have $\mu_n(R) \subseteq R^*$, where R^* is the set of units of R.

If z is a complex number, we will write $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ to denote, respectively, the real and imaginary part of z.

We will use subscript to denote base-change over a fixed object, that will be clear from the context.

If R is any ring, then we will write $\mathrm{M}_n(R)$ for the (non commutative) ring of $n \times n$ matrices with coefficients in R. The transpose of a matrix M will be denoted with M^t.

If L/K is a finite extension of fields, we will write $\mathrm{Res}_{L/K}$ for the *Weil restriction functor*, i.e. the functor, from L-schemes to K-schemes, that is right adjoint to the base change $\cdot \otimes_K L$. We set $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$, so $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^*$ canonically.

We use the symbol $\prod'$ to denote the *restricted product*. For example $\mathbb{A}^f$, the ring of finite adele of $\mathbb{Q}$, is defined as $\mathbb{A}^f := \prod_p' \mathbb{Q}_p$.

Let A be an abelian scheme over a ring R. We will write $\mathrm{T}_p(A) := \varprojlim_n A[p^n]$ for the Tate module of A, viewed as an étale sheaf over $\mathrm{Spec}(R)$ (we will often identify this sheaf with its stalk at a geometric generic point). We set $\widehat{\mathrm{T}}(A) := \prod_p \mathrm{T}_p(A)$ and $\widehat{\mathrm{V}}(A) := \prod_p' V_p(A)$, where $\mathrm{V}_p(A) := \mathrm{T}_p(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If K is a number field, or a local field, we will write $\mathcal{O}_K$ for its ring of integers.

If K is a local field, with residue field κ, we will write

$$[\cdot]\colon \kappa^* \to \mu_q(\mathcal{O}_K),$$

where κ has q elements, for the Teichmüller character. We also set $[0] = 0$.

The Hamilton quaternions will be denoted with $\mathbb{H} = \langle 1, i, j, k \rangle$.

In Chapter 1, we will write $\mathbf{H}$ for the Poincaré half plane, i.e. the Riemann surface $\mathbf{H} := \{z \in C \text{ with } \Im(z) > 0\}$. Starting with Chapter 2, $\mathbf{H}$ will have a complete different meaning (it will be the *Hasse invariant*, see Section 2.1), but no confusion should arise.

Our conventions on Hodge structures are those of [**Mil05**]. In particular, if V is a real vector space and $h\colon \mathbb{S} \to \mathrm{GL}(V)$ is a morphism of algebraic groups, then the induced Hodge structure on $V_{\mathbb{C}}$ satisfies

$$h(z)v = z^{-p}\overline{z}^{-q}v,$$

for any $z \in \mathbb{C}^*$ and $v \in V^{p,q}$.

Let k be any field, and let R be any ring. Let V be a k-vector space of finite dimension, and suppose that R acts on V by k-linear endomorphisms. We define the map

$$\mathrm{Tr}_k(\cdot|V)\colon R \to A$$

that sends $x \in R$ to the trace of the k-linear map given by x.

We will write $\mathrm{Spm}(R)$ for the maximal spectrum of a ring R (we will use this notations only for rings that are admissible in the sense of [**BL93**]).

If I is any ideal in a ring R, we say that R is I-adically complete if it is *separated and complete* for the I-adic topology on R.

If S is a Noetherian scheme and $G \to S$ is a finite and flat commutative group scheme, we will write G^{D} for its Cartier dual.

CHAPTER 1

Preliminaries

In this chapter we review some preliminary material. In particular, we define the Shimura curves we are interested in. They are constructed starting with a quaternion algebra, and solve a suitable moduli problem, so we call them *quaternionic modular Shimura curves*. They are Shimura varieties of PEL type and have been introduced by Carayol in [**Car86**]. We then recall some results of rigid analytic geometry proved by Coleman in [**Col97a**]. We also review the theory of ϖ-divisible and of formal $\mathcal{O}_{\mathcal{P}}$-module as in [**Mes72**]. Finally we introduce the notion of group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action, following [**Fal02**].

1.1. Shimura varieties of PEL type

We briefly review the theory of Shimura varieties of PEL type. General references for Shimura varieties are [**Mil05**], [**Del71**], and [**Del79**]. We assume some knowledge about algebraic groups, the standard references are [**Wat79**] and [**Hum75**].

Let D be a finite dimensional simple $\mathbb{Q}$-algebra (not necessarily commutative). We let ${}^*\colon D \to D$ be an *involution*, i.e. a $\mathbb{Q}$-linear map that satisfies $(ab)^* = b^*a^*$ and $a^{**} = a$ for all a and b in D.

We assume that the decomposition into a product of simple algebras of $D_{\mathbb{C}}$ is the following:

$$D_{\mathbb{C}} = \prod_{i=1}^{N} (\mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})).$$

We make the further assumption that our involution respects this product and that on each factor we have, up to an inner automorphism of $\mathrm{M}_2(\mathbb{C})$,

$$(a,b)^* = (b^t, a^t).$$

In the language of [**Mil05**], Chapter 8, we are assuming that $(D, {}^*)$ is of *type A*.

Let E be the center of D and let F be the subalgebra of E fixed by *. We assume that E (and hence F) is a field. Let V be a free left D-module of finite rank. We make the assumption that the reduced dimension of V

$$\frac{\dim_E(V)}{[D:E]^{1/2}}$$

is even. In the language of [**Mil05**], Chapter 8, this means that $(D, {}^*)$ is of type *Aeven*, in particular the Shimura varieties we are going to define are moduli spaces of abelian varieties with additional structure.

Let $\Theta\colon V \times V \to \mathbb{Q}$ be an alternating non-degenerate $\mathbb{Q}$-bilinear form. We say that Θ is *symplectic* if

$$\Theta(du, v) = \Theta(u, d^*v)$$

for all $d \in D$ and all $u, v \in V$. We fix (V, Θ), with Θ symplectic.

Let g be in $\mathrm{Aut}_D(V)$. We say that g is a D-linear *symplectic similitude* of (V, Θ), if there exists $\mu(g) \in \mathbb{Q}^*$ such that

$$\Theta(g(u), g(v)) = \mu(g)\Theta(u, v)$$

for all $u, v \in V$. We extend this definition to (V_R, Θ_R), for any $\mathbb{Q}$-algebra R, in the obvious way. Let G be the algebraic group over $\mathbb{Q}$ such that its R-points are

$$G(R) = \{D\text{-linear symplectic similitudes of } (V_R, \Theta_R)\}.$$

Proposition 1.1.1. *The algebraic group G is* reductive. *Furthermore, the adjoint group G^{ad} is* simple.

Proof. Everything is proved in [**Mil05**], Chapter 8. See in particular Proposition 8.7. □

We write C for $\mathrm{End}_D(V)$. Note that $\cdot^*$ induces an involution ${}^*\colon C \to C$ given by

$$\Theta(c(u), v) = \Theta(u, c^*(v))$$

for all $u, v \in V$ and all $c \in C$. It follows that, for any $\mathbb{Q}$-algebra R, we have a functorial isomorphism

$$G(R) \cong \{x \in C \otimes_{\mathbb{Q}} R \text{ such that } xx^* \in R^*\}.$$

We assume that * is *positive*, i.e. that $\mathrm{Tr}_{V_{\mathbb{R}}/\mathbb{R}}(d^*d) > 0$ for all $d \in D_{\mathbb{R}} \setminus \{0\}$. By [**Mil06**], Chapter 1, Section 1, we have that F is totally real and that E is a CM-field.

Proposition 1.1.2. *There is a D-linear element $J \in C_{\mathbb{R}}$ such that, for all $u, v \in V_{\mathbb{R}}$, we have*

$$J^2 = -1, \ \Theta_{\mathbb{R}}(Ju, Jv) = \Theta_{\mathbb{R}}(u, v), \ \textit{and } \Theta_{\mathbb{R}}(u, Ju) > 0 \textit{ if } u \neq 0.$$

Furthermore, J is unique up to a conjugation by an element $c \in C$ that satisfies $cc^* = 1$.

Proof. This is [**Mil05**], Proposition 8.12. □

Each J as in the above proposition defines a complex structure on $V_{\mathbb{R}}$, hence it gives a morphism $h\colon \mathbb{S} \to \mathrm{GL}_{V_{\mathbb{R}}}$. Since $\Theta_{\mathbb{R}}(Ju, Jv) = \Theta_{\mathbb{R}}(u, v)$, we have that h factors through a morphism

$$h\colon \mathbb{S} \to G_{\mathbb{R}}.$$

We let X be the $G(\mathbb{R})$-conjugacy class of any h as above. By Proposition 1.1.2, we have that X is well defined.

Since (any) h as above is non trivial and G^{ad} is simple, we have that G^{ad} has no $\mathbb{Q}$-factors on which the projection of h is trivial. This is condition SV3 in [**Mil05**]. Using the notation of [**Del79**], this implies that $G^{\mathrm{ad}} = G_1^{\mathrm{ad}}$.

By [**Mil05**], Proposition 8.14, the couple (G, X) satisfies condition β (hence condition α) of [**Del79**]. Since $\Theta_{\mathbb{R}}(u, Ju) > 0$ for all $u, v \in V$, we have that J defines a Cartan involution of $G^{\mathrm{ad}} = G_1^{\mathrm{ad}}$, in particular (G, X) satisfies condition γ of [**Del79**] (see also [**Mil05**], Chapter 6).

For any h as above, we define

$$t\colon D \to \mathbb{C}$$
$$d \mapsto \mathrm{Tr}_{\mathbb{C}}(d | V_{\mathbb{C}} / F_h^0 V_{\mathbb{C}})$$

where $F_h^i V_{\mathbb{C}}$ is the Hodge filtration of $V_{\mathbb{C}}$ defined by h. Since condition α is satisfied, we have that t does not depend on h.

Let $K \subseteq G(\mathbb{A}^f)$ be a compact open subgroup. Recall the definition of the Shimura varieties attached to (G, X), of level K,

$$M_K(\mathbb{C}) := G(\mathbb{Q}) \backslash (G(\mathbb{A}^f) \times X)/K,$$

where $G(\mathbb{Q})$ acts by left multiplication on $G(\mathbb{A}^f)$ and by conjugation on X, while K acts by right multiplication on $G(\mathbb{A}^f)$ and trivially on X. By general theory, we have that $M_K(\mathbb{C})$ is a complex manifold.

THEOREM 1.1.3. *We have that $M_K(\mathbb{C})$ classifies the isomorphism classes of quadruples $(A, i, \bar{\theta}, \bar{\alpha})$, where*

(1) *A is a complex abelian variety, defined up to isogenies, with an action of D via $i\colon D \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that, for all $d \in D$, we have*
$$\mathrm{Tr}_{\mathbb{C}}(i(d)|\,\mathrm{Lie}(A)) = t(d);$$
(2) *$\bar{\theta}$ is a homogeneous polarization (i.e. a class of polarization up to multiplication by elements of $\mathbb{Q}^*$) such that the corresponding Rosati involution sends $i(d)$ to $i(d^*)$, for all $d \in D$;*
(3) *$\bar{\alpha}$ is a K-level structure, i.e. a class, modulo K, of a symplectic D-linear similitudes $\alpha\colon \widehat{\mathrm{V}}(A) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}^f$.*

In the last condition, the symplectic form on $\widehat{\mathrm{V}}(A)$ is given by the Weil pairing composed with the polarization.

PROOF. This is a restatement of **[Mil05]**, Theorem 8.17 and Proposition 8.19. □

The Shimura varieties obtained in this way are called *Shimura varieties of PEL type*, since by Theorem 1.1.3, they classify abelian varieties with polarization (given by $\bar{\theta}$), endomorphisms (given by D), and level structure (given by $\bar{\alpha}$).

1.2. Quaternion modular Shimura curves over $\mathbb{C}$

In this section we define the Shimura curve we are interested in, over $\mathbb{C}$. These varieties are particular cases of those consider in Section 1.1. We start with the field F and we define the algebra D, rather than obtaining F from D, but at the end the notations of this section will be completely compatible with those of Section 1.1. The main reference is **[Car86]** (with some results of **[Del71]**). We assume familiarity with basic theory of quaternion algebra, the standard reference is **[Vig80]**.

We fix F, a totally real field of degree $N > 1$ over $\mathbb{Q}$. The various embeddings of F into $\mathbb{R}$ will be denoted with $\tau_1, \dots, \tau_N$. We set $\tau := \tau_1$

Let B be a quaternion algebra over F. We assume that B is split at τ and that it is ramified at $\tau_2, \dots, \tau_N$ (these assumptions imply that the Shimura varieties we are going to consider have dimension 1). In particular we fix the identifications
$$B \otimes_{F,\tau} \mathbb{R} = \mathrm{M}_2(\mathbb{R}) \text{ and } B \otimes_{F,\tau_i} \mathbb{R} = \mathbb{H} \text{ for } i \neq 1.$$

Let $\lambda < 0$ be a rational number. Since F is totally real, we have $[E : F] = 2$, where $E := F(\sqrt{\lambda})$ (in other words E is a CM-field). We extend each τ_i to E by
$$\tau_i(x + \sqrt{\lambda}y) = \tau_i(x) + \sqrt{\lambda}\tau_i(y),$$
where $x, y \in F$. We embed E in the field of complex numbers via $\tau = \tau_1$.

Let $z \mapsto \bar{z}$ be the non trivial element of $\mathrm{Gal}(E/F)$. On $D := B \otimes_F E$, we define an involution $\bar{\cdot}$, that sends $b \otimes_F z$ to $b' \otimes_F \bar{z}$, where $\cdot'\colon B \to B$ is the canonical involution of B. Let δ be any invertible element of D such that $\bar{\delta} = \delta$. We define an involution $\cdot^*\colon D \to D$ by
$$l^* = \delta^{-1}\bar{l}\delta.$$

LEMMA 1.2.1. *We have that D is a simple $\mathbb{Q}$-algebra and satisfies the assumptions of Section 1.1. Furthermore, the center of D is E, and F is the subset of E fixed by the involution *.*

PROOF. Since D is a quaternion algebra over E, we have that D is simple and E is its center. Note that, by definition, * restricted to E is the unique non trivial element of $\mathrm{Gal}(E/F)$, so F is the subset of E fixed by *. We have $D \otimes_{\mathbb{Q}} \mathbb{C} =$

$\prod_i D_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C}$, where $D_{\tau_i} := D \otimes_{F,\tau_i} \mathbb{R} = B \otimes_{F,\tau_i} \mathbb{R} \otimes_{F,\tau_i} E$. We now consider the cases $i = 1$ and $i \neq 1$.

If $i = 1$, we have $B \otimes_{F,\tau} \mathbb{R} = \mathrm{M}_2(\mathbb{R})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so $D_\tau = \mathrm{M}_2(\mathbb{C})$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \delta^{-1} \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix} \delta.$$

If $i \neq 1$, we have $B \otimes_{F,\tau} \mathbb{R} = \mathbb{H}$ and $(a + ib + jc + kd)' = a - ib - jc - kd$, so $D_{\tau_i} = \mathrm{M}_2(\mathbb{C})$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \delta^{-1} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \delta.$$

It follows that, for any i, we have $D_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$. By an easy computation, we have that, on each $D_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C}$,

$$(a, b)^* = (b^*, a^*),$$

for all $a, b \in \mathrm{M}_2(\mathbb{C})$. The lemma follows by **[Mil05]**, Proposition 8.3. □

Let V be the underlying $\mathbb{Q}$-vector space of D. Note that V is a left free (of rank 1) D-module. Clearly, the reduced dimension of V is 2. We now take any $\alpha \in E$ such that $\bar{\alpha} = -\alpha$, and we define the $\mathbb{Q}$-bilinear form

$$\Theta \colon V \times V \to \mathbb{Q}$$
$$(v, w) \mapsto \Theta(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\alpha \, \mathrm{Tr}_{D/E}(v \delta w^*))$$

LEMMA 1.2.2. *We have that Θ is a symplectic form on V.*

PROOF. Clearly Θ is $\mathbb{Q}$-bilinear. Since δ is invertible, the non degeneracy of Θ follows by the non degeneracy of $\mathrm{Tr}_{D/E}$ and of $\mathrm{Tr}_{E/F}$. We prove that Θ is alternating. Let x be in F, we have

$$\mathrm{Tr}_{E/\mathbb{Q}}(\alpha x) = \sum_{i=1}^N \tau_i(\alpha x) + \sum_{i=1}^N \tau_i(\overline{\alpha x}) = \sum_{i=1}^N \tau_i(\alpha x) - \sum_{i=1}^N \tau_i(\alpha x) = 0.$$

In particular, to prove that Θ is alternating, i.e. that $\Theta(v, v) = 0$ for all $v \in V$, we can prove that $\mathrm{Tr}_{D/E}(v\delta v^*) = \mathrm{Tr}_{D/E}(v\bar{v}\delta) \in F$ for all $v \in V$. Let $\cdot' \colon D \to D$ be the canonical involution of D. Since $\mathrm{Tr}_{D/E}(d) = d + d'$, we have that $\overline{\mathrm{Tr}_{D/E}(d)} = \mathrm{Tr}_{D/E}(\bar{d})$ for all $d \in D$. We have $\overline{v\bar{v}\delta} = \delta v \bar{v}$. In particular, the fact that $\mathrm{Tr}_{D/E}(v\delta v^*) \in F$ and that Θ is symplectic follow by $\mathrm{Tr}_{D/E}(d_1 d_2) = \mathrm{Tr}_{D/E}(d_2 d_1)$ for all $d_1, d_2 \in D$. □

Let V_{τ_i} be the $\mathbb{R}$-vector space underlying D_{τ_i}, where D_{τ_i} is defined in the proof of Lemma 1.2.1. By definition we have $V_{\mathbb{R}} \cong \oplus_i V_{\tau_i}$.

Thanks to Proposition 1.1.1, we have G, a reductive algebraic group over $\mathbb{Q}$ whose points are the symplectic similitudes of Θ.

LEMMA 1.2.3. *We can choose δ is such a way that the involution * is positive.*

PROOF. Since being positive is an open condition and V is dense in $V_{\mathbb{R}}$, it suffices to prove that we can choose, for all i, an element $\delta_{\tau_i} \in V_{\tau_i}$ such that the corresponding involution is positive. By the proof of Lemma 1.2.1, we have $V_{\tau_i} \cong \mathrm{M}_2(\mathbb{C})$ for all i, while the involution * depends on whether $i = 1$ or not. By an explicit computation we see that we can take $\delta_\tau = \mathrm{Im}(\tau(\alpha)) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $\delta_{\tau_i} = \mathrm{Im}(\tau_i(\alpha)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $i \neq 1$. □

From now on we assume that * is positive. It follows by the results of Section 1.1, that we have a morphism of algebraic groups

$$h\colon \mathbb{S} \to G_{\mathbb{R}}.$$

This morphism defines an Hodge structure of type $\{(-1,0),(0,-1)\}$ on $V_{\mathbb{R}}$, i.e. a complex structure on $V_{\mathbb{R}}$ (the multiplication by i is given by the action of $J = h(i)$). The fact that * is positive implies that Θ is a polarization for the Hodge structure on $V_{\mathbb{R}}$.

Let X be the conjugacy class of h. It is possible to describe X more concretely. We now sketch an equivalent, but more explicit, construction of X, see **[Car86]** for the details. Let G' be the reductive algebraic group over $\mathbb{Q}$ defined by $G' := \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ (note that in the literature, G and G' are often interchanged). By assumption, we can fix an identification

$$G'(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^*)^{N-1}.$$

The homomorphism $\mathbb{C}^* \to G'(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^*)^{N-1}$ that sends $x+iy$ to

$$\left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1}, 1, \ldots, 1\right),$$

comes from a unique morphism

$$h'\colon \mathbb{S} \to G'_{\mathbb{R}}.$$

We write X' for the conjugacy class of h'. We have that X' is the union of 2 copies of $\mathbf{H}$, the Poincaré half plane (in particular it has dimension 1). Let T be $\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{\mathrm{m},F})$, so T is isomorphic to the center of G'. In particular we have an exact sequence of algebraic groups

$$1 \longrightarrow G'_1 \longrightarrow G' \xrightarrow{v'} T \longrightarrow 1,$$

where G'_1 is the derived subgroup of G' (the morphism v' is induced by the norm of B). Let T_E be $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{\mathrm{m},E})$. We define G'' to be the colimit of the diagram $T_E \leftarrow T \to G'$, where $T \to G'$ is given by identifying T with the center of G'. Let U_E be the subgroup of T_E defined by the equation $z\bar{z} = 1$. We have a morphism $v\colon G'' \to T \times U_E$, given by $v(g,z) = (v'(g)z\bar{z}, z/\bar{z})$. Finally, let T' be the subtorus of $T \times U_E$ defined by $T' = \mathbb{G}_{\mathrm{m},\mathbb{Q}} \times U_E$.

LEMMA 1.2.4. *We have that G is isomorphic to the inverse image of T' under v. Furthermore, the derived subgroup of G is isomorphic to G'_1, the derived subgroup of G.*

PROOF. With our assumption, the ring $C = \mathrm{End}_D(V)$ is isomorphic to D. So, for any $\mathbb{Q}$-algebra R, we have that $G(R)$ can be identified with the set of $x \in D \otimes_{\mathbb{Q}} R$ such that $xx^* \in R^*$ (see Section 1.1). The lemma follows. $\square$

We have a natural isomorphism

$$T_E(\mathbb{C}) \cong (\mathbb{C}^*)^N,$$

so we can define a morphism $h_E\colon \mathbb{S} \to (T_E)_{\mathbb{R}}$ by the formula (on $\mathbb{R}$-points)

$$h_E(z) = (1, z^{-1}, \ldots, z^{-1}).$$

By universal property, we get a morphism $h' \times h_E\colon \mathbb{S} \to G''_{\mathbb{R}}$, that factors through $h\colon \mathbb{S} \to G_{\mathbb{R}}$. By the uniqueness part of Proposition 1.1.2, we have that the conjugacy class of h is our X defined above. Furthermore we see that X is isomorphic to $\mathbf{H}$. Using the decomposition of $V_{\mathbb{R}}$ given in the proof of Lemma 1.2.1, we can study more explicitly the morphism h. We have that J acts on each $V_{\tau_i} \cong \mathrm{M}_2(\mathbb{C})$

as multiplication (on the right) by a matrix in $\mathrm{GL}_2(\mathbb{C})$. If $i = 1$ then such a matrix is equal to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while if $\tau \neq 1$ then it is equal to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.

Recall that, for any compact open subgroup K of $G(\mathbb{A}^f)$, we have the Shimura varieties $M_K(\mathbb{C})$ associated to the couple (G, X).

Proposition 1.2.5. *We have that the $M_K(\mathbb{C})$'s are* compact Riemann surfaces. *If $K' \subseteq K$ are sufficiently small, the natural map*

$$M_{K'}(\mathbb{C}) \to M_K(\mathbb{C})$$

is étale.

Proof. By general theory of Shimura varieties we know that $M_K(\mathbb{C})$ is a quotient of X, so, since X has dimension (as complex variety) 1, we have that $M_K(\mathbb{C})$ is a Riemann surface. The compactness follows from the fact that D, being a division algebra, has no non-trivial nilpotent elements, so $G(\mathbb{R})$ has no non-trivial unipotent elements (see [**Mil05**], Theorem 3.3). The last assertion is a general fact about Shimura varieties, see [**Mil05**], Remark 5.29. □

1.3. Models of quaternionic modular Shimura curves

In this section we define a canonical model over E of the quaternionic modular curves. We use this model to base change our varieties to a p-adic field $F_{\mathcal{P}}$. Finally we work over $\mathcal{O}_{\mathcal{P}}$, the ring of integers of $F_{\mathcal{P}}$. Putting some condition on $K \subseteq G(\mathbb{A}^f)$, we also describe more explicitly our moduli problem. In particular, in Section 1.3.2, we prove that almost every object we are going to consider admits a particular decomposition, that will be used through the thesis.

1.3.1. The canonical model over the reflex field. For details about the reflex field of a Shimura variety, see [**Del79**] and [**Del71**].

Recall the morphism t, defined in Section 1.1,

$$t\colon D \to \mathbb{C}$$
$$d \mapsto \mathrm{Tr}_{\mathbb{C}}(d|V_{\mathbb{C}}/F_h^0 V_{\mathbb{C}})$$

Proposition 1.3.1. *For any sufficiently small $K \subseteq G(\mathbb{A}^f)$ compact open, the reflex field of $M_K(\mathbb{C})$ is E. In particular, for such a K, the Riemann surface $M_K(\mathbb{C})$ admits a canonical model, denoted M_K, over E. These curves solve the moduli problem of Theorem 1.1.3, but for E-algebras.*

Proof. By [**Del71**], Section 6, the reflex field of $M_K(\mathbb{C})$ is the subfield of $\mathbb{C}$ generated by the $t(d)$'s, for $d \in D$. We show that this field is E, embedded in $\mathbb{C}$ via τ. First of all note that $V_{\mathbb{C}}/F_h^0 V_{\mathbb{C}} = V_{\mathbb{C}}^{-1,0}$ (using the Hodge structure given by h). We have (see the proof of Lemma 1.2.1 for the notations)

$$V_{\mathbb{C}}^{-1,0} = \bigoplus_{i=1}^{N} (V_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C})^{-1,0},$$

furthermore the trace we are interested in is the sum of the various traces given by this decomposition.

If $i = 1$, then $V_\tau = \mathrm{M}_2(\mathbb{C})$, so $V_\tau \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$. Furthermore, $J = h(i)$ acts on V_τ as multiplication, on the right, by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on each $\mathrm{M}_2(\mathbb{C})$. In particular $(V_\tau \otimes_{\mathbb{R}} \mathbb{C})^{-1,0}$, is the set $(M_1, M_2) \in \mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$ such that $M_k J = iM_k$, for $k = 1, 2$. This last condition is equivalent to $M_k = \begin{pmatrix} a_k & a_k \\ ib_k & b_k \end{pmatrix}$, for some $a_k, b_k \in \mathbb{C}$. An element $d \in D_\tau$ acts on $\mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$ as multiplication on the left by d. This implies that the trace of the multiplication by d_τ on $V_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C}$ is equal to $\mathrm{Tr}(d_\tau) + \overline{\mathrm{Tr}(d_\tau)}$.

If $i \neq 1$ a similar argument works. We have $V_{\tau_i} = \mathrm{M}_2(\mathbb{C})$ and $V_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$. In this case J acts on V_{τ_i} as multiplication by i on the first factor, and by $-i$ on the second one. It follows that $(V_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C})^{-1,0}$ is equal to the first copy of $\mathrm{M}_2(\mathbb{C})$. In particular the trace of the multiplication by d_{τ_i} on $V_{\tau_i} \otimes_{\mathbb{R}} \mathbb{C}$ is equal to $2\,\mathrm{Tr}(d_{\tau_i})$.

We have proved that $t(d) = \sigma(\mathrm{Tr}_{D/E}(d))$, where

$$\sigma := \tau + \bar{\tau} + 2\tau_2 + \cdots + 2\tau_N.$$

Let $x, y \in F$. Since $E = F(\sqrt{\lambda})$, we have

$$\sigma(x + \sqrt{(}\lambda)y) = 2\,\mathrm{Tr}_{F/\mathbb{Q}}(x) + 2(\mathrm{Tr}_{F/\mathbb{Q}}(y) - y)\sqrt{\lambda}, \tag{1.3.1}$$

so the image of σ in $\mathbb{C}$ generates E. The proposition follows. □

From now on we assume that K is small enough to have the canonical model M_K. We conclude this section with a new version of Theorem 1.1.3.

THEOREM 1.3.2. *We have that M_K represents the functor (E-algebras)$^{\mathrm{op}} \to$* **set** *that sends R, an E-algebra, to the set of isomorphism classes of quadruples $(A, i, \bar{\theta}, \bar{\alpha})$, where*

(1) *A is an abelian scheme over R of relative dimension $4N$, defined up to isogenies, with an action of D via $i \colon D \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that, for all $d \in D$, we have*

$$\mathrm{Tr}_R(i(d) | \,\mathrm{Lie}(A)) = t(d);$$

(2) *$\bar{\theta}$ is a homogeneous polarization, such that the corresponding Rosati involution sends $i(d)$ to $i(d^*)$, for all $d \in D$;*

(3) *$\bar{\alpha}$ is a K-level structure.*

PROOF. Since V has the dimension $8N$ over $\mathbb{Q}$, the existence of the level structure implies that the dimension of A is $4N$. □

1.3.2. The canonical model over a local field. We now start working p-adically. We are going to define a local field $F_{\mathcal{P}}$ that contains E, so we can base-change our canonical model M_K to $F_{\mathcal{P}}$.

Let $p \neq 2$ be a prime number, fixed from now on. We assume that $\mathbb{Q}(\sqrt{\lambda})/\mathbb{Q}$ splits at p

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be the prime ideals of $\mathcal{O}_F$ lying over p. The completion of F at $\mathcal{P}_i$ will be denoted with $F_{\mathcal{P}_i}$. Let $d_i := [F_{\mathcal{P}_i} : \mathbb{Q}_p]$, so $N = d_1 + \cdots + d_m$. There is a canonical isomorphism of rings

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^{m} F_{P_i}.$$

We set

$$\mathcal{P} := \mathcal{P}_1,\ d := d_1, \text{ and } \mathcal{O}_{\mathcal{P}} := \mathcal{O}_{F_{\mathcal{P}}}.$$

We choose ϖ, a uniformizer of $\mathcal{O}_{\mathcal{P}}$. All our results do not depend on this choice. Let $\kappa := \mathcal{O}_{\mathcal{P}}/\varpi$ be the residue field of $\mathcal{O}_{\mathcal{P}}$. The ramification degree of $\mathcal{O}_{\mathcal{P}}$ will be denoted with e, so $\varpi^e = p$ in $\mathcal{O}_{\mathcal{P}}$. We write $q = p^f$ for the cardinality of κ. In particular we have $d = ef$. Let $\mathrm{v}(\cdot)$ be the valuation on $\mathcal{O}_{\mathcal{P}}$ normalized in such a way that $\mathrm{v}(\varpi) = 1$ (beware, $\mathrm{v}(p) = e \geq 1$). We choose the absolute value $|\cdot|$ on $F_{\mathcal{P}}$ that satisfies $|\varpi| = q^{-1}$. We fix $\overline{F}_{\mathcal{P}}$, an algebraic closure of $F_{\mathcal{P}}$ and we denote with $\mathbb{C}_p$ its completion (note that $\mathbb{C}_p$ is an algebraically closed field). Both $\mathrm{v}(\cdot)$ and $|\cdot|$ extend to the whole $\mathbb{C}_p$. The residue field of $\mathcal{O}_{\mathbb{C}_p}$ is $\bar{\kappa}$, an algebraic closure of κ.

We assume that B is split at $\mathcal{P}$. In particular, we fix the identification

$$B \otimes_F F_{\mathcal{P}} = \mathrm{M}_2(F_{\mathcal{P}}).$$

Since $\mathbb{Q}(\sqrt{\lambda})/\mathbb{Q}$ splits at p, there is $\mu \in \mathbb{Q}_p$ such that $\mu^2 = \lambda$, and we choose one such μ. There is an isomorphism of rings

$$\begin{aligned} E \otimes_{\mathbb{Q}} \mathbb{Q}_p &\to (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \\ x + y\sqrt{\lambda} \otimes 1 &\mapsto (x \otimes 1 + y \otimes \mu, x \otimes 1 - y \otimes \mu), \end{aligned} \tag{1.3.2}$$

for $x, y \in F$. In particular, the canonical isomorphism $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_i F_{\mathcal{P}_i}$ gives

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} (F_{\mathcal{P}_1} \times \cdots \times F_{\mathcal{P}_m}) \times (F_{\mathcal{P}_1} \times \cdots \times F_{\mathcal{P}_m}). \tag{1.3.3}$$

Composing twice the natural map $E \to E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with the projection on the first factor, we get a map $E \to F_{\mathcal{P}}$. We use this morphism to define a structure of E-algebra on $F_{\mathcal{P}}$.

From now on, we will consider the curve M_K as a smooth and proper scheme over $F_{\mathcal{P}}$, via $E \to F_{\mathcal{P}}$. We have that M_K solves the moduli problem of Theorem 1.3.2, but for $F_{\mathcal{P}}$-algebras. Using the properties of $F_{\mathcal{P}}$, we can be more explicit about the moduli problem.

The above decomposition of $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ gives

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_1^1 \times \cdots \times D_m^1 \times D_1^2 \times \cdots D_m^2. \tag{1.3.4}$$

Here $D_j^k \cong B \otimes_F F_{\mathcal{P}_j}$ as $F_{\mathcal{P}_j}$-algebra, so $D_1^1 \cong D_1^2 \cong \mathrm{M}_2(F_{\mathcal{P}})$ as $F_{\mathcal{P}}$-algebras. Note that $l \mapsto l^*$ switches D_j^1 and D_j^2.

Recall that a category $\mathcal{C}$ is called *preadditive* if it has a zero object (i.e. both initial and terminal), all the Hom-sets are abelian group, and composition is $\mathbb{Z}$-bilinear ('$\mathcal{C}$ is enriched over the monoidal category of abelian groups'). In particular, if X is an object of $\mathcal{C}$, then $\mathrm{End}(X)$ is a ring (with unit, but not necessarily commutative).

DEFINITION 1.3.3. A category $\mathcal{C}$ is called *pseudo-abelian* or *Karoubian*, if it is preadditive and the following holds. Let X be an object of $\mathcal{C}$ and let $f \in \mathrm{End}(X)$ be an *idempotent* endomorphism. Then $\ker(f)$ exists in $\mathcal{C}$.

LEMMA 1.3.4. *Let $\mathcal{C}$ be a pseudo-abelian category. Let X be an object of $\mathcal{C}$ and suppose that $f \in \mathrm{End}(X)$ is an idempotent endomorphism. Then $\mathrm{id} - f \in \mathrm{End}(X)$ is idempotent too and we have $X \cong \ker(f) \oplus \ker(\mathrm{id} - f)$ (existence of the direct sum is part of the lemma).*

PROOF. The only non trivial part is the one about the direct sum. We write $i_1\colon \ker(f) \to X$ for the natural morphism, and similarly for $i_2\colon \ker(\mathrm{id} - f) \to X$. We prove that (X, i_1, i_2) has the universal property of $\ker(f) \oplus \ker(\mathrm{id} - f)$. The composition $X \xrightarrow{\mathrm{id} - f} X \xrightarrow{f} X$ is 0, so, by universal property, we get a morphism $\pi_1\colon X \to \ker(f)$ such that $i_1 \circ \pi_1 = \mathrm{id} - f$. In an similar way, we have a morphism $\pi_2\colon X \to \ker(\mathrm{id} - f)$ such that $i_2 \circ \pi_2 = f$. Let Y be any object of $\mathcal{C}$, and suppose we have two morphisms $g_1\colon \ker(f) \to Y$ and $g_2\colon \ker(\mathrm{id} - f) \to Y$. We define a morphism $g\colon X \to Y$ by

$$g = g_1 \circ \pi_1 + g_2 \circ \pi_2.$$

By diagram chasing, it is easy to verify that

$$\pi_1 \circ i_1 = \mathrm{id}\,,\ \pi_2 \circ i_1 = 0,\ \pi_2 \circ i_2 = \mathrm{id}\,, \text{ and } \pi_1 \circ i_2 = 0.$$

This implies that $g \circ i_1 = g_1$ and $g \circ i_2 = g_2$. It is immediate to check that g is uniquely defined by g_1 and g_2. $\square$

As a particular case of the lemma, if $\mathcal{C}$ is a pseudo-abelian category and X is an object of $\mathcal{C}$ with an action of $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ (i.e. a map $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}(X)$ that is an homomorphism or an anti-homomorphism of rings), the isomorphism in (1.3.4) induces a decomposition (in $\mathcal{C}$)

$$X \cong X_1^1 \oplus \cdots \oplus X_m^1 \oplus X_1^2 \oplus \cdots X_m^2,$$

where each X_j^k has an action of D_j^k. We can use the idempotents $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of $D_1^2 \cong \mathrm{M}_2(F_{\mathcal{P}})$, to obtain a further decomposition:

$$X_1^2 \cong X_1^{2,1} \oplus X_1^{2,2}.$$

Note that $X_1^{2,1}$ and $X_1^{2,2}$ are isomorphic (take a matrix that switches the idempotents). The definitions of $X_1^{1,1}$ and of $X_1^{1,2}$ are similar.

REMARK 1.3.5. We will use the above notation for the rest of thesis. In particular we will consider the ${}^{2,1}_{1}$-part of various objects to reduce their dimension. The idea is that, taking this direct factor, we obtain an object of dimension 1, so we can work as if our abelian schemes were elliptic curves. Note that D does not act on $X_1^{2,1}$, in some sense we have already used its action to get the decomposition. The idea of taking such a decomposition goes back to Carayol in [**Car86**], and is absolutely central to our theory.

We want to state a moduli problem about abelian schemes rather than about isogeny classes of abelian schemes. To do this, we work with a maximal order in our quaternion algebra.

Let $\mathcal{O}_B$ be a maximal order of B. Since B is split at $\mathcal{P}$, we can fix an identification, of $\mathcal{O}_{\mathcal{P}}$-algebras,

$$\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathcal{P}} \cong \mathrm{M}_2(\mathcal{O}_{\mathcal{P}}).$$

The maximal order of D corresponding to $\mathcal{O}_B$ will be denoted with $\mathcal{O}_D$, or with $V_{\mathbb{Z}}$ if we want to see it as a $\mathbb{Z}$-lattice in the $\mathbb{Q}$-vector space V. The isomorphism in (1.3.3) implies that we have the following isomorphisms of rings

$$\begin{array}{ccccccccccccc} \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p & \cong & \mathcal{O}_{D_1^1} & \times & \cdots & \times & \mathcal{O}_{D_m^1} & \times & \mathcal{O}_{D_1^2} & \times & \cdots & \mathcal{O}_{D_m^2} \\ \bigcap & & \bigcap & & & & \bigcap & & \bigcap & & & \bigcap \\ D \otimes_{\mathbb{Q}} \mathbb{Q}_p & \cong & D_1^1 & \times & \cdots & \times & D_m^1 & \times & D_1^2 & \times & \cdots & D_m^2 \end{array} \tag{1.3.5}$$

Let X be an object of a pseudo-abelian category, and assume that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ acts on X. Using the decomposition given in (1.3.5), we extend, in the obvious way, the definition of X_j^k, $X_1^{1,i}$, and $X_1^{2,i}$.

We can choose $\mathcal{O}_B$, α, and δ in such a way that the following conditions are satisfied:

(1) $\mathcal{O}_D$ is stable under $l \mapsto l^*$;
(2) $\mathcal{O}_{D_j^k}$ is a maximal order in D_j^k and $\mathcal{O}_{D_1^2}$ is identified with $\mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$;
(3) Θ takes integer values on $V_{\mathbb{Z}}$;
(4) Θ induces a perfect pairing on $V_{\mathbb{Z}_p} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

From now on, we assume that these conditions are satisfied.

Let $V'_{\mathbb{Z}}$ be the dual lattice of $V_{\mathbb{Z}}$, with respect to Θ. From now on we assume that every compact open subgroup K is small enough to keep invariant the lattice $V_{\widehat{\mathbb{Z}}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Let R be an $F_{\mathcal{P}}$-algebra and let A be an abelian scheme over R. Suppose that $\mathcal{O}_D$ acts on A. Then $\mathrm{Lie}(A)$ is a $\mathcal{O}_D \otimes_{\mathbb{Q}} R$-module, hence a $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module. In particular, the R-module $\mathrm{Lie}(A)_i^j$ is defined. Since $\mathrm{Lie}(A)$ is a projective R-module and

$$\mathrm{Lie}(A) = \mathrm{Lie}(A)_1^1 \oplus \cdots \oplus \mathrm{Lie}(A)_m^1 \oplus \mathrm{Lie}(A)_1^2 \oplus \cdots \oplus \mathrm{Lie}(A)_m^2,$$

we have that $\mathrm{Lie}(A)_i^j$ is a projective R-module.

We are now ready to state our version the moduli problem over $F_{\mathcal{P}}$.

THEOREM 1.3.6. *We have that M_K represents the functor ($F_{\mathcal{P}}$-algebras)$^{\mathrm{op}} \to$ **set** that sends R, an $F_{\mathcal{P}}$-algebra, to the set of isomorphism classes of quadruples $(A, i, \theta, \bar{\alpha})$, where*

(1) *A is an abelian scheme over R of relative dimension $4N$ with an action of $\mathcal{O}_D$ via $i\colon \mathcal{O}_D \to \mathrm{End}_R(A)$ that satisfies:*
  (a) *the projective R-module $\mathrm{Lie}(A)_1^{2,1}$ has rank 1 and $\mathcal{O}_{\mathcal{P}}$ acts on it via the natural morphism $\mathcal{O}_{\mathcal{P}} \to R$;*
  (b) *for $j \geq 2$, we have $\mathrm{Lie}(A)_j^2 = 0$;*

(2) *θ is a polarization, of degree prime to p, such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$;*

(3) *$\bar{\alpha}$ is a K-level structure, i.e. a class modulo K of symplectic $\mathcal{O}_D$-linear isomorphisms $\alpha\colon \widehat{\mathrm{T}}(A) \xrightarrow{\sim} V_{\widehat{\mathbb{Z}}}$.*

PROOF. We outline the proof, that is contained in [**Car86**]. The starting point is Theorem 1.3.2. We write d_i^j for a generic element of D_i^j. Let $(A, i, \bar{\theta}, \bar{\alpha})$ be an object of the moduli problem of Theorem 1.3.2, with A defined over R. In Sections 2.4.1-2.4.3 of [**Car86**] it is shown that condition (1) of Theorem 1.3.2 is equivalent to the following conditions:

- $\mathrm{Tr}_R(d_1^1 | \mathrm{Lie}(A)_1^1) = 2\,\mathrm{Tr}_{F_{\mathcal{P}}/\mathbb{Q}_p}(\mathrm{Tr}_{D_1^1/F_{\mathcal{P}}}(d_1^1)) - \mathrm{Tr}_{D_1^1/F_{\mathcal{P}}}(d_1^1)$;
- $\mathrm{Tr}_R(d_i^1 | \mathrm{Lie}(A)_i^1) = 2\,\mathrm{Tr}_{F_{\mathcal{P}_i}/\mathbb{Q}_p}(\mathrm{Tr}_{D_i^1/F_{\mathcal{P}_i}}(d_i^1))$ if $i \neq 1$;
- $\mathrm{Tr}_R(d_1^2 | \mathrm{Lie}(A)_1^2) = \mathrm{Tr}_{D_1^2/F_{\mathcal{P}}}(d_1^2)$;
- $\mathrm{Lie}(A)_i^2 = 0$ if $i \neq 1$.

By [**Car86**], Sections 2.4.4 and 2.4.5, the last two conditions imply the others (since A has dimension $4N$). If follows that condition (1) of Theorem 1.3.2 is equivalent to condition (1) of the statement of the theorem we are proving.

We now want to define a unique abelian scheme A_0, rather than a class of abelian schemes up to isogenies. By our assumption on $V_{\widehat{\mathbb{Z}}}$, we have that $k^{-1}(V_{\widehat{\mathbb{Z}}}) \subseteq \widehat{V}(A)$ does not depend on the choice of $\alpha \in \bar{\alpha}$. There is a unique abelian scheme A_0 in the class of A such that $\widehat{\mathrm{T}}(A_0) = \alpha^{-1}(V_{\widehat{\mathbb{Z}}})$. Since $V_{\widehat{\mathbb{Z}}}$ is stable under $\mathcal{O}_D$, we have an action of $\mathcal{O}_D$ on A_0. We can also define a unique $\theta \in \bar{\theta}$ by requiring that $\alpha^{-1}(V_{\widehat{\mathbb{Z}}})$ is dual to $\alpha^{-1}(V'_{\widehat{\mathbb{Z}}})$, where $V'_{\widehat{\mathbb{Z}}} = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{A}^f$.

Decomposing $\mathrm{V}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, it is possible to prove (see [**Car86**], Section 2.5) that in order to define the symplectic D-linear similitude

$$\alpha\colon \widehat{V}(A) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}^f,$$

it is enough to specify:

- a symplectic D-linear similitude $\alpha^p\colon \prod'_{l \neq p} V_l(A) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}^f$;
- a similitude factor in $\mathbb{Q}_p$
- an $F_{\mathcal{P}}$-linear isomorphism $k_1^{2,1}\colon \mathrm{V}(A)_1^{2,1} \xrightarrow{\sim} V_1^{2,1}$.

In this way we can see $\bar{k}$ as a class of symplectic $\mathcal{O}_D$-linear isomorphisms

$$k\colon \widehat{\mathrm{T}}(A) \xrightarrow{\sim} V_{\widehat{\mathbb{Z}}}.$$

The theorem follows. □

Let $V_j^k := (V \otimes_{\mathbb{Q}} \mathbb{Q}_p)_j^k$. We have that V_j^k and V_i^h are orthogonal with respect to $\Theta \otimes_{\mathbb{Q}} \mathbb{Q}_p$ except if $j = i$ and $k \neq h$. Let f be in $G(\mathbb{Q}_p)$, so f is a symplectic similitude of $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We have that f is uniquely defined by its similitude factor and its restriction to the V_j^2's. This implies that

$$G(\mathbb{Q}_p) = \mathbb{Q}_p^* \times (B \otimes_F F_{\mathcal{P}_1})^* \cdots \times (B \otimes_F F_{\mathcal{P}_m})^*.$$

In particular we have

$$G(\mathbb{A}^f) \cong \mathbb{Q}_p^* \times \mathrm{GL}_2(F_{\mathcal{P}}) \times (B \otimes_F F_{\mathcal{P}_2})^* \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^* \times G(\mathbb{A}^{f,p}),$$

where $\mathbb{A}^{f,p}$ is the restricted product of the $\mathbb{Q}_l$'s (l prime), with $l \neq p$. We will simply write Γ for $(B \otimes_F F_{\mathcal{P}_2})^* \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^* \times G(\mathbb{A}^{f,p})$.

From now on, we assume that K is a compact open subgroup of $G(\mathbb{A}^f)$ of the form

$$K = \mathbb{Z}_p^* \times K_{\mathcal{P}} \times H,$$

where $K_{\mathcal{P}}$ is a subgroup of $\mathrm{GL}_2(F_{\mathcal{P}})$ and H is a subgroup of Γ.

Let $(A, i, \theta, \bar{\alpha})$ be an object of the moduli problem of Theorem 1.3.6. We write $\widehat{\mathrm{T}}^p(A)$ for $\prod_{l \neq p} \mathrm{T}_l(A)$ and $\widehat{\mathbb{Z}}^p$ for $\prod_{l \neq p} \mathbb{Z}_l$ (l prime). We denote $\mathrm{T}_p(A)_2^2 \oplus \cdots \oplus \mathrm{T}_p(A)_m^2$ with $\mathrm{T}_p^{\mathcal{P}}(A)$ and $\left(V_{\mathbb{Z}_p}\right)_2^2 \oplus \cdots \oplus \left(V_{\mathbb{Z}_p}\right)_m^2$ with $W_p^{\mathcal{P}}$. Let $\widehat{W}^p$ be $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$.

We can give a more explicit interpretation of $\bar{\alpha}$, that is a class of $\mathcal{O}_D$-linear isomorphisms $\widehat{\mathrm{T}}(A) \xrightarrow{\sim} V_{\widehat{\mathbb{Z}}}$. Using the decomposition $K = \mathbb{Z}_p^* \times K_{\mathcal{P}} \times H$, we have that (see the last part of the proof of Theorem 1.3.6) giving $\bar{\alpha}$ is equivalent to give $\bar{\alpha}^{\mathcal{P}}$ and $\bar{\alpha}_{\mathcal{P}}$, where

- $\bar{\alpha}^{\mathcal{P}}$ is a class, modulo H, of $\alpha^{\mathcal{P}} = \alpha_p^{\mathcal{P}} \oplus \alpha^p$, with $\alpha_p^{\mathcal{P}} \colon \mathrm{T}_p^{\mathcal{P}}(A) \xrightarrow{\sim} W_p^{\mathcal{P}}$ linear and $\alpha^p \colon \widehat{\mathrm{T}}^p(A) \xrightarrow{\sim} \widehat{W}^p$ symplectic;
- $\bar{\alpha}_{\mathcal{P}}$ is a class modulo $K_{\mathcal{P}}$ of isomorphisms of $\mathcal{O}_{\mathcal{P}}$-modules $\alpha_{\mathcal{P}} \colon \mathrm{T}_p(A)_1^{2,1} \xrightarrow{\sim} (V_{\mathbb{Z}_p})_1^{2,1} \cong \mathcal{O}_{\mathcal{P}}^2$.

In the case $K_{\mathcal{P}}$ has some specific form, we can be even more explicit. We write $A[\varpi]_1^{2,1}$ for the ϖ-torsion of $A[p]_1^{2,1}$ (see Section 1.6).

We define

$$K(H) := \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}),$$

$$K(H, \varpi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) \text{ s.t. } c \equiv 0 \bmod \varpi^n \right\},$$

and

$$K(H\varpi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) \text{ s.t. } a \equiv 1 \bmod \varpi^n \text{ and } c \equiv 0 \bmod \varpi^n \right\}.$$

In the case $K_{\mathcal{P}} = K(H)$, a choice of a level structure is equivalent to a choice of $\bar{\alpha}^{\mathcal{P}}$, where:

(1) $\bar{\alpha}^{\mathcal{P}}$ is a class, modulo H, of $\alpha^{\mathcal{P}} = \alpha_p^{\mathcal{P}} \oplus \alpha^p$, with $\alpha_p^{\mathcal{P}} \colon \mathrm{T}_p^{\mathcal{P}}(A) \xrightarrow{\sim} W_p^{\mathcal{P}}$ linear and $\alpha^p \colon \widehat{\mathrm{T}}^p(A) \xrightarrow{\sim} \widehat{W}^p$ symplectic.

In the case $K_{\mathcal{P}} = K(H, \varpi^n)$, a choice of a level structure is equivalent to a choice of $(C, \bar{\alpha}^{\mathcal{P}})$, where:

(1) C is a finite and flat subgroup scheme of rank q^n of $A[\varpi^n]_1^{2,1}$, stable under the action of $\mathcal{O}_{\mathcal{P}}$;
(2) $\bar{\alpha}^{\mathcal{P}}$ is as above.

In the case $K_{\mathcal{P}} = K(H\varpi^n)$, a choice of a level structure is equivalent to a choice of $(Q, \bar{\alpha}^{\mathcal{P}})$, where:

(1) Q is a point of exact $\mathcal{O}_{\mathcal{P}}$-order ϖ^n in $A[\varpi^n]_1^{2,1}$;
(2) $\bar{\alpha}^{\mathcal{P}}$ is as above.

In these cases, the curves M_K will be denoted, respectively, with $M(H)$, $M(H, \varpi^n)$, and $M(H\varpi^n)$. These are the curves we are mainly interested in.

REMARK 1.3.7. The curve $M(H)$ should be thought of as an analogue of the classical modular curve $X_1(M)$, where p does not divide M (our Shimura curves are already proper, so there is no need for compactification). In this philosophy, the curves $M(H,\varpi^n)$ and $M(H\varpi^n)$ are the analogue of the curves $X(M;p^n)$ ($\Gamma_1(M)\cap\Gamma_0(p^n)$-level structure) and $X_1(Mp^n)$.

1.3.3. Integral models. The last models we are interested in are over $\mathcal{O}_{\mathcal{P}}$.

THEOREM 1.3.8. *There are curves $\mathcal{M}(H)$, $\mathcal{M}(H,\varpi^n)$, and $\mathcal{M}(H\varpi^n)$ that are canonical $\mathcal{O}_{\mathcal{P}}$-models of $M(H)$, $M(H,\varpi^n)$, and $M(H\varpi^n)$. These curves solve the above moduli problems, but for $\mathcal{O}_{\mathcal{P}}$-algebras. We have that $\mathcal{M}(H)$ is smooth over $\mathcal{O}_{\mathcal{P}}$, while the other curves have semistable reduction.*

PROOF. This theorem is one of the main results of [**Car86**], see also [**Kas09**], Section 5. □

If we work over $\mathcal{O}_{\mathcal{P}}$, the level structure has the same description as above, but now Q is a point of exact $\mathcal{O}_{\mathcal{P}}$-order ϖ^n in the sense of Drinfel'd, i.e. a morphism of $\mathcal{O}_{\mathcal{P}}$-modules $\varphi\colon \mathcal{O}_{\mathcal{P}}/\varpi^n\mathcal{O}_{\mathcal{P}} \to \mathcal{A}[\varpi^n]_1^{2,1}(R)$ such that $\sum_{a\in\mathcal{O}_{\mathcal{P}}/\varpi^n\mathcal{O}_{\mathcal{P}}}[a]$ is a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule scheme of rank q^n of $\mathcal{A}[\varpi^n]_1^{2,1}$. Here $(\mathcal{A},i,\theta,\bar{\alpha})$ is an object of the moduli problem defined over R, $[a]$ is the closed subscheme of $\mathcal{A}[\varpi^n]_1^{2,1}$ corresponding to $\varphi(a)$, and the sum means product of the defining ideals. Note that $Q=\varphi(1)$ is a canonical R-point of $\mathcal{A}[\varpi^n]_1^{2,1}$ that generates a submodule of order q^n. If $n\geq m$, we have morphisms, given by the obvious natural transformations of functors

$$\begin{gathered}\mathcal{M}(H\varpi^n)\to\mathcal{M}(H\varpi^m),\\ \mathcal{M}(H,\varpi^n)\to\mathcal{M}(H,\varpi^m),\\ \mathcal{M}(H\varpi^n)\to\mathcal{M}(H,\varpi^n)\text{ and}\\ \mathcal{M}(H,\varpi^n)\to\mathcal{M}(H).\end{gathered}$$

The universal objects of the moduli problems of the curves $M(H)$, $M(H,\varpi^n)$, and $M(H\varpi^n)$ will be denoted with $A(H)$, $A(H,\varpi^n)$, and $A(H\varpi^n)$. They admit the canonical integral models $\mathcal{A}(H)$, $\mathcal{A}(H,\varpi^n)$, and $\mathcal{A}(H\varpi^n)$, that are universal objects for the moduli problems over $\mathcal{O}_{\mathcal{P}}$. The morphism $\mathcal{A}(H)\to\mathcal{M}(H)$ will be denoted with π, and its zero section with e, we use the same symbols for the other curves.

We will consider the ϖ-adic completion of these integral models, and after we take rigidification. Since our curves are proper, we could also analytify the generic fiber, but in Chapter 2 we will really need the formal model.

1.4. What if $F=\mathbb{Q}$?

One of our first assumptions is that N, the degree of F over $\mathbb{Q}$, is strictly greater than 1, so $F\neq\mathbb{Q}$. What happens if $F=\mathbb{Q}$? It turns out that all our results remain true also in this case, but some proofs must be adapted. But note that if $F=\mathbb{Q}$, then B is a quaternion algebra over $\mathbb{Q}$. In this case, our assumptions about B mean that B is a division algebra, that is split at p. In particular, the case $F=\mathbb{Q}$ *is not* the case of the classical modular curves, that is characterized by the fact that the Shimura curves need to be compactified. Nevertheless, all results of [**AIS11**] that involve the formal group of an elliptic curve (for example the property of the map $\mathrm{d}\log$ and of the Hodge-Tate sequence) can be deduced by our work.

The most important reason why we have chosen to exclude the case $F=\mathbb{Q}$ is the following. Recall the reductive algebraic group $G'=\mathrm{Res}_{F/\mathbb{Q}}(B^*)$ and the morphism $h'\colon\mathbb{S}\to G'_{\mathbb{R}}$ from Section 1.2. If $K'\subseteq G'(\mathbb{A}^f)$ is compact open, then we can define the Shimura curve $M'_{K'}(\mathbb{C})$ as usually, as double coset space. It is easy to prove that the weight morphism $w_X \colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to G'_{\mathbb{R}}$ is defined over $\mathbb{Q}$ if an only if $F = \mathbb{Q}$ (see [**Mil05**], page 87). So, if $F \neq \mathbb{Q}$, the Shimura curve $M'_{K'}(\mathbb{C})$ is not of PEL type (it is not even of abelian type). This lack of interpretation as moduli space is the main reason to introduce the group G. If $F = \mathbb{Q}$ the curve $M'_{K'}(\mathbb{C})$ is of PEL type, so there is no need to introduce the algebraic group G, and it is more natural to work with G'.

If $F = \mathbb{Q}$, the Riemann surface $M'_{K'}$ admits canonical smooth and proper models over $\mathbb{Q}$, denoted again $M'_{K'}$. The study of the curves $M'_{K'}$ is done in [**Kas99**]. It turns out (with some technical assumptions that we do not recall here) that $M'_{K'}$ classifies isomorphism class of *fake elliptic curves* with level structure. This are abelian varieties of dimension 2 (not 4, as one can suppose using $N = 1$ in our work). In this case the field $F_{\mathcal{P}}$ is equal to $\mathbb{Q}_p$, and the ring $\mathcal{O}_{\mathcal{P}}$ is $\mathbb{Z}_p$. There is nothing like the field E (so there is no D). If $\mathcal{C}$ is a pseudo-abelian category and X is an object of $\mathcal{C}$ with an action of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, then we have a decomposition

$$X = X^1 \oplus X^2.$$

Using X^1 instead of $X_1^{2,1}$ in the following chapters (and the curve $M'_{K'}$), we see that the case $F = \mathbb{Q}$ is essentially a particular case of our work. It is worth noting that the case $F \neq \mathbb{Q}$ is essentially more difficult than the case $F = \mathbb{Q}$. Indeed, since in general $\mathcal{O}_{\mathcal{P}} \neq \mathbb{Z}_p$ and $\varpi \neq p$, we are forced to use the full theory of group schemes with a strict action of $\mathcal{O}_{\mathcal{P}}$ and the theory of ϖ-divisible groups (see below). Of course, if $F = \mathbb{Q}$, these theories are just the theory of group schemes and of p-divisible groups. Another source of difficulties is the fact that, for the object we are interested in, X^1 and X^2 are isomorphic and also in duality, in particular X^1 is self-dual. If $F \neq \mathbb{Q}$, then $X_1^{2,1}$ and $X_1^{2,2}$ are isomorphic, as already stressed, but $X_1^{2,1}$ is dual to $X_1^{1,1}$ (See Section 1.7), so we do not know whether $X_1^{2,1}$ is self-dual. For example, Proposition 3.2.3 is trivial if $F = \mathbb{Q}$ (and also in the elliptic case).

1.5. The wide open subspace associated to a section of a line bundle

We assume some knowledge about rigid analytic geometry, the standard reference is [**BGR84**].

Let K be a finite extension of $F_{\mathcal{P}}$, with ring of integers V. Let X be an algebraic variety over K. By this we mean a reduced scheme X of finite type over $\operatorname{Spec}(K)$. Recall that we have a faithful functor

$$\begin{aligned} {}^{\mathrm{an}}\colon \text{ algebraic varieties}/K &\to \text{ rigid space}/K \\ X &\mapsto X^{\mathrm{an}} \end{aligned}$$

This functor extends to coherent sheaves. Furthermore, if X is a projective variety, then we have an analogue of the GAGA theorem.

Let us suppose that we have $\mathcal{X}$, a model of X over V, that is flat and of finite type. Associated to $\mathcal{X}$ we have $\mathfrak{X}$, the formal completion of $\mathcal{X}$ along the closed subscheme defined by $\varpi = 0$. We have that $\mathfrak{X}$ is an admissible formal scheme in the sense of [**BL93**], so we have the rigid space $\mathfrak{X}^{\mathrm{rig}}$. There is an open immersion

$$X^{\mathrm{an}} \hookrightarrow \mathfrak{X}^{\mathrm{rig}},$$

that is an isomorphism if $\mathcal{X}$ is proper.

The following definition is due to Coleman, see [**Col97b**].

Definition 1.5.1. Let X be a *smooth* and proper rigid analytic curve over $F_{\mathcal{P}}$. We say that a rigid space Y is a *wide open* subspace of X, if Y is isomorphic to an admissible open in X whose complement is (after a finite extension of K if necessary) isomorphic to a disjoint union of a non-zero number of closed affinoid disks.

Let $\mathcal{X} \to \mathrm{Spec}(V)$ be a reduced, flat, and proper scheme of finite type, of relative dimension 1. We denote with X the generic fiber $\mathcal{X}_K$. Suppose that $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$ and that s is a global section of $\mathcal{L}$. Let x be a point of X^{an}. The residue field K_x of X^{an} at x is a finite extension of K, and we denote with V_x the ring of integers of K_x. By properness, the morphism $\mathrm{Spm}(K_x) \to X^{\mathrm{an}}$ corresponding to x extends uniquely to a morphism $f_x \colon \mathrm{Spec}(V_x) \to \mathcal{X}$. Since $\mathcal{L}$ is locally free, we have that $f_x^*\mathcal{L}$ is free, generated by a section t. In particular, the pullback $f_x^* s$ can be written at, for some $a \in R_x$. We set

$$|s(x)| := |a|,$$

that does not depend on the various choices we made. Let w be a rational number such that $0 \leq w < 1$. We define

$$X^{\mathrm{an}}(w) = \left\{x \in X \text{ such that } |s(x)| \geq q^{-w}\right\}.$$

Proposition 1.5.2. *Using the above notation, suppose that $\mathcal{X} \to \mathrm{Spec}(V)$ is smooth. If $\mathcal{L} \neq s\mathcal{O}_{\mathcal{X}}$, then $X^{\mathrm{an}}(w)$ is an affinoid wide open subspace of X^{an}.*

Proof. This is done in [**Col97a**], Section 1. □

Remark 1.5.3. Let us suppose that there is an $a \in V$ such that $|a| = q^{-w}$. In this case it is possible to explicitly describe a formal model of $X^{\mathrm{an}}(w)$ over $\mathrm{Spf}(V)$. It is the ϖ-adic completion of

$$\mathrm{Spec}_{\mathcal{X}}(\mathrm{Sym}(\mathcal{L})/(s-a)).$$

Note that, by general theory, such a formal model is unique only up to a formal blow-up.

1.6. ϖ-divisible groups and formal $\mathcal{O}_{\mathcal{P}}$-modules

In this section we recall the basic properties of ϖ-divisible groups and of formal $\mathcal{O}_{\mathcal{P}}$-modules. A ϖ-divisible group is a generalization of a p-divisible group (see [**Tat67**]), and a formal $\mathcal{O}_{\mathcal{P}}$-module is a generalization of a formal group (see for example [**Sil09**], Chapter IV). Both these generalizations are needed when we want to take into account the action of $\mathcal{O}_{\mathcal{P}}$.

Let X be any $\mathcal{O}_{\mathcal{P}}$-scheme. Recall that a *$\varpi$-divisible group* $H \to X$ is a Barsotti-Tate group H over X, together with an embedding $\mathcal{O}_{\mathcal{P}} \hookrightarrow \mathrm{End}(H)$ such that the induced action of $\mathcal{O}_{\mathcal{P}}$ on $\mathrm{Lie}(H)$ is the one given by $H \to X \to \mathrm{Spec}(\mathcal{O}_{\mathcal{P}})$. If X is connected, there is a unique integer $\mathrm{ht}(H)$, called the *height* of H, such that $\mathrm{rk}(H[\varpi^n]) = q^{n\,\mathrm{ht}(H)}$ for all n. The basic properties of ϖ-divisible groups are very similar to those of p-divisible groups. Details can be found in [**HT01**], Chapter 2, in [**Mes72**], Chapters 1 and 2, and in the Appendix of [**Car86**]. In particular, if H is a ϖ-divisible group, its height, *as p-divisible group*, is $d\,\mathrm{ht}(H)$, where $d = [F_{\mathcal{P}} : \mathbb{Q}_p]$. We have also the notion of ϖ-divisible groups over an $\mathcal{O}_{\mathcal{P}}$-formal scheme, see [**HT01**], page 60.

Let $\mathfrak{X}$ be a ϖ-adic formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathcal{P}})$, and let $\mathfrak{G} \to \mathfrak{X}$ be a smooth formal group (i.e. a group object in the category of formal schemes). We say that $\mathfrak{G}$ is a *formal $\mathcal{O}_{\mathcal{P}}$-module* if we have an action of $\mathcal{O}_{\mathcal{P}}$ on $\mathfrak{G}$ such that the action of $\mathcal{O}_{\mathcal{P}}$ on $\mathrm{Lie}(\mathfrak{G})$ is given by $\mathfrak{G} \to \mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_{\mathcal{P}})$. Zariski locally on $\mathfrak{X}$, we have that $\mathfrak{G}$ is a formal group law (we are interested only in the dimension 1 case). If $\mathfrak{X}$ is a formal scheme over $\mathrm{Spf}(\kappa)$, and $\mathfrak{G} \to \mathfrak{X}$ is a formal $\mathcal{O}_{\mathcal{P}}$-module, then there are morphisms $\mathrm{F} \colon \mathfrak{G} \to \mathfrak{G}^{(q)}$ and $\mathrm{V} \colon \mathfrak{G}^{(q)} \to \mathfrak{G}$, that generalize the usual Frobenius and Verschiebung. Similar morphisms exist also for ϖ-divisible groups. Suppose that $\mathfrak{G}$ has dimension 1, with coordinate x (we use x also as coordinate of $\mathfrak{G}^{(q)}$). Let $[\varpi]_{\mathfrak{G}}(x)$ the power series given by the multiplication by ϖ on $\mathfrak{G}$ (we will simply write $[\varpi](x)$ if no confusion can arise). Then we have that $\mathrm{F}(x)$, the power series given by F, satisfies $\mathrm{F}(x) = x^q$ and, if $\mathrm{V}(x)$ is the power series given by V, we have $\mathrm{V}(\mathrm{F}(x)) = [\varpi](x)$ and $\mathrm{F}(\mathrm{V}(x)) = [\varpi]_{\mathfrak{G}^{(q)}}(x)$.

We now recall some facts proved in **[Mes72]**, Chapter 2. Let $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_{\mathcal{P}})$ be a ϖ-adic formal scheme, and let $\pi\colon \mathfrak{H} \to \mathfrak{X}$ be a ϖ-divisible group. Suppose we are given $e\colon \mathfrak{X} \to \mathfrak{H}$, a section of π. Attached to these data, there is a (smooth) formal $\mathcal{O}_{\mathcal{P}}$-module $\widehat{\mathfrak{H}} \to \mathfrak{X}$ called *the completion of* $\mathfrak{H}$ *along* e. We have that $\widehat{\mathfrak{H}}$ is of ϖ-torsion. Note that, to have this construction, one really needs to work with formal schemes, see **[Mes72]**. It is not always the case that $\widehat{\mathfrak{H}}$ is a Barsotti-Tate group (in this assertion, we consider all objects as f.p.p.f. sheaves). This happens if $\mathfrak{X}$ is the formal spectrum of an Artin ring (see also Proposition 4.2, Chapter 2 of **[Mes72]**). If $\mathfrak{X}$ is a $\mathrm{Spf}(\kappa)$-formal scheme and $\widehat{\mathfrak{H}}$ is a ϖ-divisible group of dimension 1, then we can recover the height of $\widehat{\mathfrak{H}}$ from its (local) structure of formal group law. Indeed, let x be a coordinate of $\widehat{\mathfrak{H}}$, then $\mathrm{ht}(\widehat{\mathfrak{H}})$ is the largest integer n such that $[\varpi](x)$ is a power series in x^{q^n} (compare this with **[Sil09]**, Section IV.7). In general we have $\mathrm{ht}(\widehat{\mathfrak{H}}) \leq \mathrm{ht}(\mathfrak{H})$.

We now attach a ϖ-divisible group to any object of the moduli problems considered in the previous sections. Let R be a ϖ-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra and let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem, with $\mathcal{A}$ defined over R. We have that $\mathcal{A}[p^n]$ has a natural action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$, for all n. In particular $\mathcal{A}[p^n]_j^k$ is defined, and we also have $\mathcal{A}[p^n]_1^{2,k}$, that has a natural action of $\mathcal{O}_{\mathcal{P}}$. We write $\mathcal{A}[\varpi^n]_1^{2,k}$ for its ϖ^n-torsion, and we set $\mathcal{A}[\varpi^n]_1^2 := A[\varpi^n]_1^{2,1} \oplus A[\varpi^n]_1^{2,2}$. We use a similar notation for $A[\varpi^n]_1^1 := A[\varpi^n]_1^{1,1} \oplus A[\varpi^n]_1^{1,2}$. We write $\mathcal{A}[\varpi^\infty]_1^{2,1}$ for $\varinjlim_n \mathcal{A}[\varpi^n]_1^{2,1}$ and we will call it the ϖ-divisible group associated to $\mathcal{A}$. The height of $\mathcal{A}[\varpi^\infty]_1^{2,1}$ is 2. We will use the same notations for $\mathfrak{A}$, the ϖ-adic completion of $\mathcal{A}$. Since the degree of the isogeny θ is prime to p, it induces an isomorphism $\mathcal{A}[p^n]_j^1 \xrightarrow{\sim} (\mathcal{A}[p^n]_j^2)^{\mathrm{D}}$. Since $\mathrm{Lie}(\mathcal{A})_j^2 = 0$, we have that $\mathcal{A}[p^n]_j^2$ is étale for $j \geq 2$.

Let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem, with $\mathcal{A}$ defined over a ϖ-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra R. We write $\mathfrak{A} \to \mathrm{Spf}(R)$ for the ϖ-adic completion of $\mathcal{A}$. Taking completion along the zero section we obtain $\widehat{\mathfrak{A}}$. Since $\widehat{\mathfrak{A}}$ has an action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we can consider $\widehat{\mathfrak{A}}_1^{2,1}$. It is a formal $\mathcal{O}_{\mathcal{P}}$-module of dimension 1. The formal $\mathcal{O}_{\mathcal{P}}$-module associated to $\mathfrak{A}[\varpi^\infty]_1^{2,1}$ is $\widehat{\mathfrak{A}}_1^{2,1}$. If $\widehat{\mathfrak{A}}_1^{2,1}$ is a ϖ-divisible group (this is the case if R is an Artin ring), its height h is either 1 or 2 and satisfies $q^h = \mathrm{rk}(\widehat{\mathfrak{A}}_1^{2,1}[\varpi]) \leq \mathrm{rk}(\mathcal{A}[\varpi]_1^{2,1})$. We will write $\widehat{\mathcal{A}}$ for $\widehat{\mathfrak{A}}$, but beware that this is the completion (along the zero section) of a ϖ-adic formal scheme. We will use the notation $\widehat{\mathcal{A}}[\varpi^n]_1^{2,1} := \widehat{\mathcal{A}}_1^{2,1}[\varpi^n]$.

1.7. Group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action

In this section we recall the theory of *group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action*, as developed in **[Fal02]**. This theory generalizes that of groups schemes. As in the case of ϖ-divisible groups, we need such a generalization to deal with the action of $\mathcal{O}_{\mathcal{P}}$. In particular, we want a good duality theory that takes into account the $\mathcal{O}_{\mathcal{P}}$-action: since $\mathbb{G}_{\mathrm{m}}$ has no action of $\mathcal{O}_{\mathcal{P}}$, the usual Cartier duality is not enough.

Throughout this section, R will be a ϖ-adically complete and ϖ-torsion free $\mathcal{O}_{\mathcal{P}}$-algebra.

Definition 1.7.1. Let G be a finite and flat group scheme over R. We say that G has a *strict $\mathcal{O}_{\mathcal{P}}$-action* if we have a ring homomorphism $\mathcal{O}_{\mathcal{P}} \to \mathrm{End}_R(G)$ such that the action on the Lie algebra of G is the natural one. Homomorphisms between group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action are homomorphisms that respect the action of $\mathcal{O}_{\mathcal{P}}$.

REMARK 1.7.2. Our definition of group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action is a particular case of the one given in [**Fal02**]. Indeed, by our assumptions on R, we have that the cotangent complex of $G \to \operatorname{Spec}(R)$ has non trivial cohomology only in degree 0 (since G is smooth over $R_{F_{\mathcal{P}}}$).

Let H be ϖ-divisible group over R. We have that, for any n, the ϖ^n-torsion $H[\varpi^n]$ is naturally a group scheme with a strict $\mathcal{O}_{\mathcal{P}}$-action.

On the power series ring $R[[x]]$ there is a unique action of $\mathcal{O}_{\mathcal{P}}$ such that the multiplication by ϖ has the form $[\varpi](x) = x^q + \varpi x$ and the action on the Lie algebra is the one induced by $\mathcal{O}_{\mathcal{P}} \to R$. This is the so called *Lubin-Tate ϖ-divisible group*, denoted $\mathcal{LT}$. It is clear that, for any n, the ϖ^n-torsion of $\mathcal{LT}$ is a group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action. The action of $\mathcal{O}_{\mathcal{P}}$ on $R[[x]]/(x^q + \varpi x)$ factors through κ, and $z \in \kappa$ sends x to $[z]x$. We now fix G, a finite and flat group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action.

LEMMA 1.7.3. *We have that G is killed by ϖ^n, for some n. In particular, any morphism $G \to \mathcal{LT}$ factors through $\mathcal{LT}[\varpi^n]$*

PROOF. This is [**Fal02**], Lemma 7. □

THEOREM 1.7.4. *The functor*

$$\varpi\textit{-adically complete and torsion free } R\textit{-algebras} \to \mathbf{groups}$$
$$S \mapsto \operatorname{Hom}(G_S, \mathcal{LT}_S)$$

is representable by a finite and flat group scheme over R, with strict $\mathcal{O}_{\mathcal{P}}$-action. We denote this group scheme with $G^\vee$.

PROOF. This is [**Fal02**], Theorem 8. □

REMARK 1.7.5. If $\mathcal{O}_{\mathcal{P}} = \mathbb{Z}_p$ and R contains a primitive p-th root of unity, then we have an isomorphism $\mathcal{LT} \cong \widehat{\mathbb{G}}_{\mathrm{m},R}$. In particular, the duality above coincides with the usual Cartier duality.

EXAMPLE 1.7.6. The following example will be very important for us, see [**Fal02**], Section 3 and [**Far07**], Section 1.1.2. Let E be an element in R such that $\frac{\varpi}{E} \in R$. By [**Far07**], page 6, on $G = \operatorname{Spec}(R[x]/(x^q + \frac{\varpi}{E}x))$ there is a unique structure of group scheme with strict action of $\mathcal{O}_{\mathcal{P}}$ such that $a \in \kappa$ acts on $R[x]/(x^q + \frac{\varpi}{E}x)$ as $x \mapsto [a]x$ (note that, since G has order q, an action of $\mathcal{O}_{\mathcal{P}}$ must factor through κ). With this assumptions, we have $G^\vee = \operatorname{Spec}(R[x]/(x^q - Ex))$. In particular, if there is a $q-1$-th root of E in R, we have an R-point of $G^\vee$. We will show that the canonical subgroup of the abelian schemes of our moduli problems (if it exists) is of this type. To be more precise, let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem of level $K(H)$, with $\mathcal{A}$ defined over R. With some assumptions (see Section 2.3), we have that $\mathcal{A}[\varpi]_1^{2,1}$ has a canonical subgroup $\mathcal{C}$ of order q. As a scheme, we have $\mathcal{C} = \operatorname{Spec}(R[x]/(x^q + \frac{\varpi}{E}x))$, where $E \in R$. Suppose that in R we have a fixed $q-1$-th root of $-\varpi$, denoted $(-\varpi)^{1/(q-1)}$. Suppose furthermore that we have a canonical non-trivial R-point of $\mathcal{A}[\varpi]_1^{2,1}$ that lies in $\mathcal{C}$ (this is what happens at level $K(H\varpi)$). We have a canonical solution of $x^{q-1} - \frac{\varpi}{E} = 0$ in R and so we have a canonical $q-1$-th root of E in R. In this way we obtain a canonical R-point of $\mathcal{C}^\vee$. Explicitly, this point is given by the morphism

$$\mathcal{C} = \operatorname{Spec}(R[x]/(x^q + \frac{\varpi}{E}x)) \to \mathcal{LT} = \operatorname{Spec}(R[[x]])$$
$$E^{1/(q-1)}x \leftarrow\!\shortmid x$$

CHAPTER 2

Quaternionic modular forms

We will use the notations of Chapter 1. In particular D is a quaternion algebra over a CM field E, obtained by base change of B, a quaternion algebra over F, the maximal totally real subfield of E. We also have a local field $F_{\mathcal{P}}$, with ring of integers $\mathcal{O}_{\mathcal{P}}$. Recall that in Section 1.3.3, we have defined several curves over $\mathcal{O}_{\mathcal{P}}$ that are fine moduli spaces (see Theorem 1.3.6). In particular we have the universal abelian schemes $\pi\colon \mathcal{A}(H) \to \mathcal{M}(H)$, $\pi\colon \mathcal{A}(H, \varpi^n) \to \mathcal{M}(H, \varpi^n)$, and $\pi\colon \mathcal{A}(H\varpi^n) \to \mathcal{M}(H\varpi^n)$. Recall that all object that are endowed with an action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ admit a decomposition as in Section 1.3.2, in particular the ${}_1^{2,1}$-part is defined. This allows us to work with one dimensional objects.

The goal of this chapter is to define ϖ-adic modular forms, with respect to D, *of integral weight*. We consider p-adic modular form of level $K(H)$, and also of higher levels. Furthermore, we define, and study, the canonical subgroup of our abelian schemes. This subgroup will be used in the following chapters to define p-adic modular forms of non integral weight.

Notation. From now on, we will use the following notation: objects defined over $\mathcal{O}_{\mathcal{P}}$ will be denoted using Italics letter, like $\mathcal{A}$. The completion along the subscheme defined by $\varpi = 0$ will be denoted using the corresponding Gothic letter, like $\mathfrak{A}$. For example the ϖ-adic completion of $\mathcal{M}(H)$ will be denoted with $\mathfrak{M}(H)$, and so on.

2.1. The Hasse invariant and ϖ-adic modular forms of level $K(H)$

In this section we study the abelian scheme $\pi\colon \mathcal{A}(H) \to \mathcal{M}(H)$. We consider also its ϖ-adic completion, this allows us to define ϖ-adic modular forms of level $K(H)$. Throughout this section, we will consider only the moduli problem of level $K(H)$, but later on we will consider higher levels.

First of all we define classical modular forms. Since the locally free sheaf $\pi_*\Omega^1_{\mathcal{A}(H)/\mathcal{M}(H)}$ has an action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we can define

$$\underline{\omega} := \underline{\omega}_{K(H)} := \left(\pi_*\Omega^1_{\mathcal{A}(H)/\mathcal{M}(H)}\right)_1^{2,1}.$$

The definitions of $\underline{\omega}_{K(H,\varpi^n)}$ and $\underline{\omega}_{K(H\varpi^n)}$ are analogous, we usually drop the subscript. By condition (1a) of Theorem 1.3.6, we have that $\underline{\omega}$ is a locally free sheaf of rank 1. If R is an $\mathcal{O}_{\mathcal{P}}$-algebra, the pullback of $\underline{\omega}$ to $\operatorname{Spec}(R)$ will be denoted $\underline{\omega}_R$ or $\underline{\omega}_{\mathcal{A}/R}$, where $\mathcal{A}$ is the pullback of the universal object to $\operatorname{Spec}(R)$.

Definition 2.1.1. Let R be an $\mathcal{O}_{\mathcal{P}}$-algebra and k an integer. The space of *modular forms with respect to D, level $K(H)$ and weight k, with coefficients in R*, is defined as

$$S^D(R, K(H), k) := \mathrm{H}^0(\mathcal{M}(H)_R, \underline{\omega}_R^{\otimes k}).$$

The definitions of $S^D(R, K(H, \varpi^n), k)$ and $S^D(R, K(H\varpi^n), k)$ are similar.

This definition is clearly inspired by the elliptic case. The next step is to define p-adic modular forms, that will have coefficients in R, a p-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra. We want to 'complete' (in quotes because we do not just take the p-adic completion) $S^D(R, K(H), k)$, using a modular definition. In other words, we want to define p-adic modular forms as section of $\underline{\omega}$ on a suitably defined (formal) scheme, that still parametrizes abelian schemes with additional structure, as $\mathcal{M}(H)$ does. This approach has been introduced by Katz in [**Kat73**], following ideas of Dwork (see [**Dwo73**]).

The following proposition is well known in the case of elliptic curves, see for example [**Sil09**], Chapter IV, Corollary 7.5.

PROPOSITION 2.1.2. *Let R be a κ-algebra and let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem, with $\mathcal{A}$ defined over R, and suppose that $\widehat{\mathcal{A}}_1^{2,1}$ is a ϖ-divisible group. The height of $\widehat{\mathcal{A}}_1^{2,1}$ is 1 if and only if $\mathcal{A}$ is ordinary.*

PROOF. We may assume that R is an algebraically closed field. We prove the claim counting the points of $\mathcal{A}[p]$. By definition of the moduli problem, $\mathcal{A}[p]_j^2$, for $j > 1$, has p^{4d_j} points. Suppose that the height of $\widehat{\mathcal{A}}_1^{2,1}$ is 1. It follows that its height, *as p-divisible group*, is d, so $\widehat{\mathcal{A}}_1^{2,1}[p]$, that is the connected component of $\mathcal{A}[p]_1^{2,1}$, has rank p^d. Again by the definition of the moduli problem, $\mathcal{A}[p]_1^{2,1}$ has rank p^{2d}, so $\mathcal{A}[p]_1^{2,1}$ has p^d points. Since $\mathcal{A}[p]_1^{2,1}$ and $\mathcal{A}[p]_1^{2,2}$ are isomorphic, $\mathcal{A}[p]_1^2$ has p^{2d} points. The Cartier duality between $\widehat{\mathcal{A}}_1^{2,1}$ and $\widehat{\mathcal{A}}_1^{1,1}$ implies that their heights are equal, so the same argument gives p^{4d} points from $\mathcal{A}[p]_1^1 \oplus \mathcal{A}[p]_1^2$. Since $N = d + d_2 + \cdots + d_m$, we have shown one implication. The converse follows by the same argument. $\square$

DEFINITION 2.1.3. Let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem, defined over a ϖ-adically complete $\mathcal{O}_\mathcal{P}$-algebra R, and assume that $\widehat{\mathcal{A}}_1^{2,1}$ is a ϖ-divisible group. We say that $(\mathcal{A}, i, \theta, \bar{\alpha})$, or simply $\mathcal{A}$, is *ordinary* if $\widehat{\mathcal{A}}_1^{2,1}$ has height 1. By Proposition 2.1.2, this is equivalent to the fact that its reduction is ordinary in the usual sense. Otherwise we say that $(\mathcal{A}, i, \theta, \bar{\alpha})$ is *supersingular.*

REMARK 2.1.4. Note that, using the ${}_1^{2,1}$-part, we are reduced to a one dimensional ϖ-divisible group, as in the case of elliptic curves. This is the reason why, even if we work with abelian schemes, we have 'ordinary if and only if not supersingular'.

With the above notations, suppose further that $\mathrm{Spf}(R)$ is so small that $\widehat{\mathcal{A}}_1^{2,1}$ is coordinatizable (this is the case if $\underline{\omega}_{\mathcal{A}/R}$ is a free R-module, see [**Mes72**], Chapter 2). It is proved in [**Kas04**], Proposition 4.3, that we can find a coordinate x_R on $\widehat{\mathcal{A}}_1^{2,1}$ such that the action of ϖ has the form

$$[\varpi](x_R) = \varpi x_R + a_R x_R^q + \sum_{j=2}^{\infty} c_j x_R^{j(q-1)+1}, \tag{2.1.1}$$

here a, c_j are in R and $c_j \in \varpi R$ unless $j \equiv 1 \bmod q$. If we assume that $\varpi = 0$ in R, the various a_R glue together to define $\mathbf{H}$, a (classical) modular form of level $K(H)$ and weight $q-1$, defined over κ, that is called the *Hasse invariant.* If $W = \mathrm{Spec}(R)$ is an open affine of $\mathcal{M}(H)_\kappa$ and we denote with ω the differential dual to the coordinate x_R defined above, we have

$$\mathbf{H}_{|W} = a_R \omega^{\otimes q-1}.$$

PROPOSITION 2.1.5. *We have the following Kodaira-Spencer (non canonical) isomorphism of sheaves*

$$(\underline{\omega}_{K(H)})^{\otimes 2} \cong \Omega^1_{\mathcal{M}(H)/\mathcal{O}_\mathcal{P}}.$$

PROOF. This is [**Kas04**], Proposition 4.1. $\square$

From now on, we assume that the compact open subgroup $K \subseteq G(\mathbb{A}^f)$ is so small that the curve $M(H)$ has genus g bigger than 1, this is possible by [**Shi94**], Proposition 1.40.

By Proposition 2.1.5 and Serre's duality, we have that

$$\mathrm{H}^1(M(H)_{\overline{F}_{\mathcal{P}}}, \underline{\omega}^{\otimes q-1}) = \mathrm{H}^0(M(H)_{\overline{F}_{\mathcal{P}}}, \underline{\omega}^{\otimes 3-q})^*.$$

Again by Proposition 2.1.5, we see that $\deg(\underline{\omega}_{K(H),\overline{F}_{\mathcal{P}}}) = g-1$. If $q > 3$, this implies that $\deg(\underline{\omega}^{\otimes 3-q}_{K(H),\overline{F}_{\mathcal{P}}}) < 0$, hence

$$\mathrm{H}^1(M(H)_{\overline{F}_{\mathcal{P}}}, \underline{\omega}^{\otimes q-1}) = 0.$$

From now on we assume that $q > 3$.

By the above cohomological calculations and [**Har77**], Proposition 9.3, we see that $\mathrm{H}^1(M(H), \underline{\omega}^{\otimes q-1}) \otimes_{\mathcal{O}_{\mathcal{P}}} F_{\mathcal{P}} = 0$, hence

$$\varpi^n \, \mathrm{H}^1(\mathcal{M}(H), \underline{\omega}^{\otimes q-1}) = 0$$

for some n. Repeating what we have done above, we obtain $\mathrm{H}^1(\mathcal{M}(H)_{\overline{\kappa}}, \underline{\omega}^{\otimes q-1}) = 0$, that implies

$$\mathrm{H}^1(\mathcal{M}(H), \underline{\omega}^{\otimes q-1}) \otimes_{\mathcal{O}_{\mathcal{P}}} \kappa = 0.$$

This means that multiplication by ϖ is surjective on $\mathrm{H}^1(\mathcal{M}(H), \underline{\omega}^{\otimes q-1})$, but this $\mathcal{O}_{\mathcal{P}}$-module is killed by some power of ϖ, hence

$$\mathrm{H}^1(\mathcal{M}(H), \underline{\omega}^{\otimes q-1}) = 0.$$

Consider now the exact sequence of sheaves on $\mathcal{M}(H)$

$$0 \longrightarrow \underline{\omega}^{\otimes q-1} \longrightarrow \underline{\omega}^{\otimes q-1} \longrightarrow \underline{\omega}_{\kappa}^{\otimes q-1} \longrightarrow 0,$$

where the first map is multiplication by ϖ. Taking the associated long exact sequence we see, since $\mathrm{H}^1(\mathcal{M}(H), \underline{\omega}^{\otimes q-1}) = 0$, that the map

$$\mathrm{H}^0(\mathcal{M}(H), \underline{\omega}^{\otimes q-1}) \to \mathrm{H}^0(\mathcal{M}(H)_{\kappa}, \underline{\omega}^{\otimes q-1})$$

is surjective. In particular, we can lift the Hasse invariant $\mathbf{H}$ to a global section of $\underline{\omega}$ on $\mathcal{M}(H)$. We choose once and for all such a lifting, called E_{q-1}: in [**Kas04**], Corollary 13.2, it is shown that, even if our choice is not canonical, the whole theory does not depend on this choice. The independence of the theory on the choice of E_{q-1} is a consequence of the following results.

Proposition 2.1.6. *The zeros of* $\mathbf{H}$ *on* $\mathcal{M}(H)_{\overline{\kappa}}$ *are simple.*

Proof. This is [**Kas04**], Proposition 6.3. □

Proposition 2.1.7. *Let R be a κ-algebra and let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be an object of the moduli problem, with $\mathcal{A}$ defined over R and let z be a geometric point of* $\mathrm{Spec}(R)$. *Then the pullback of* $\mathbf{H}$ *to $\mathcal{M}(H)_R$ vanishes at z if and only if the pullback of $\mathcal{A}$ to z is supersingular. Furthermore, the set of supersingular geometric points of $\mathcal{M}(H)_{\kappa}$ is finite and not empty.*

Proof. We look at the formal $\mathcal{O}_{\mathcal{P}}$-module associated to $\mathcal{A}$, at z. If $\mathbf{H}$ vanishes at z, then multiplication by ϖ in this formal $\mathcal{O}_{\mathcal{P}}$-module is given by a power series $[\varpi](x)$ that satisfies (see Section 1.7) $[\varpi](x) = \mathrm{V}(\mathrm{F}(x)) = \mathrm{V}(x^q) = c_{q+1}x^{q^2} + c_{2q+1}x^{q(2q-2)} + \cdots$, where the last equality follows by (2.1.1). This shows that $\mathrm{V}(0) = \mathrm{V}'(0) = 0$, so $\mathrm{V}(x) = G(x^q)$ for some power series G (this is proved as [**Sil09**], Chapter IV, Proposition 7.2). We conclude that $[\varpi](x)$ is a power series in x^{q^2}, so $\mathcal{A}$ is supersingular. The converse is an immediate consequence of (2.1.1). Being $\mathbf{H}$ a global section of a non trivial line bundle, the set of geometric points where $\mathbf{H}$ vanishes is finite and non empty. □

By the results of Section 1.5, we know that the set of ordinary points of $\mathfrak{M}(H\varpi)^{\mathrm{rig}}$ is isomorphic to the complement of a finite number of discs. Propositions 2.1.7 and 2.1.6 show that E_{q-1} can be thought as a parameter on these discs. On a rigid analytic disc, it is possible to choose several different parameters, but we will work only in a region of the discs sufficiently near to the edge such that all parameters are equivalent (see [**Buz03**], page 36 for the details). This is the deep reason why our theory does not depend on the choice of the lifting E_{q-1}. A similar remark can be done also in the elliptic case, and it is interesting in the case the Eisenstein series are not defined (i.e. for small p).

The formal completion and the rigidification of E_{q-1} will be denoted with the same symbol.

REMARK 2.1.8. As the name suggests, $\mathbf{H}$ is the analogue of the classical Hasse invariant, while E_{q-1} is the analogue of the Eisenstein series: in our situation there are no cusps, so we cannot use the q-expansion to define it.

REMARK 2.1.9. Using the above notations, we have $E_{q-1|\,\mathrm{Spec}(R)} = E\omega^{\otimes q-1}$, for some $E \in R$. By [**Kas04**], Proposition 6.2, we have $a_R \equiv E \bmod \varpi$.

We now move on to ϖ-adic modular form. They are defined over ϖ-adically complete $\mathcal{O}_{\mathcal{P}}$-algebras, so it is convenient to work with formal schemes. Let V be a finite extension of $\mathcal{O}_{\mathcal{P}}$ and let $0 \leq w < 1$ be a rational number such that there is an element of V, denoted ϖ^w, of valuation w. We define

$$\mathcal{M}(H)(w)_V := \mathrm{Spec}_{\mathcal{M}(H)_V}(\mathrm{Sym}(\underline{\omega}_V^{\otimes q-1})/(E_{q-1} - \varpi^w)),$$

where the V in the subscript means only 'defined over V', it does not mean that it is the base change to V of a scheme over $\mathcal{O}_{\mathcal{P}}$.

PROPOSITION 2.1.10. *Let R by any V-algebra. We have that $\mathcal{M}(H)(w)_V(R)$ is naturally in bijection with the set of isomorphism classes of quintuples $(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$, where $(\mathcal{A}, i, \theta, \bar{\alpha})$ is as in Theorem 1.3.6 and Y is a global section of $\underline{\omega}_{\mathcal{A}/R}^{\otimes 1-q}$ that satisfies*

$$YE_{q-1} = \varpi^w.$$

PROOF. This follows from the moduli theoretic description of $\mathcal{M}(H)$. □

There is a natural morphism $\mathcal{M}(H)(w)_V \to \mathcal{M}(H)_V$, we write $\mathcal{A}(H)(w)_V$ for the pullback of $\mathcal{A}(H)_V$ with respect to this map.

DEFINITION 2.1.11. Let V and w as above. The space of ϖ-adic modular forms with respect to D, level $K(H)$, weight k and growth condition w, with coefficients in V, is defined as

$$S^D(V, w, K(H), k) := \mathrm{H}^0(\mathfrak{M}(H)(w)_V, \underline{\omega}^{\otimes k}).$$

Of course in this definition $\underline{\omega}$ is obtained by taking the formal completion of the base change of $\underline{\omega}_{K(H),V}$ to $\mathcal{M}(H)(w)_V$.

To better understand this definition, it is convenient to use rigid geometry. Let K be the fraction field of V. First of all note that, by properness, the natural morphism of K-rigid spaces

$$\mathfrak{M}(H)_V^{\mathrm{rig}} \hookrightarrow M(H)_K^{\mathrm{an}}$$

is an isomorphism.

The following proposition shows that $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ has a very nice description in terms of the modular form E_{q-1}.

PROPOSITION 2.1.12. *The rigidification of the map $\mathfrak{M}(H)(w)_V \to \mathfrak{M}(H)_V$ is the immersion $\mathfrak{M}(H)_V^{\mathrm{rig}}(w) \hookrightarrow \mathfrak{M}(H)_V^{\mathrm{rig}}$, where $\mathfrak{M}(H)_V^{\mathrm{rig}}(w)$ is the affinoid subdomain of $\mathfrak{M}(H)_V^{\mathrm{rig}}$ relative to E_{q-1} and w.*

PROOF. This is [**Kas04**], Proposition 9.7. □

We call $\mathfrak{M}(H)_V(0)^{\mathrm{rig}}$ the *ordinary locus*, it is an affinoid subdomain of $\mathfrak{M}(H)_V^{\mathrm{rig}}$: its complement is a finite union of discs, called the *supersingular discs*. Their points correspond to those objects of the moduli problem that are supersingular.

By rigid GAGA, elements of $S^D(V, K(H), k) \otimes_V K$ correspond to sections of (the rigidification of) $\underline{\omega}^{\otimes k}$ over $\mathfrak{M}(H)_V^{\mathrm{rig}}$ and elements of $S^D(V, w, K(H), k) \otimes_V K$ correspond to sections over $\mathfrak{M}(H)_V(w)^{\mathrm{rig}}$. Elements of $S^D(V, 0, K(H), k) \otimes_V K$ are called *convergent* modular forms over K, and elements of $S^D(V, w, K(H), k) \otimes_V K$, for $w > 0$, are called *overconvergent* modular forms over K, since they can be, partially, extended to the supersingular discs.

REMARK 2.1.13. Since $\mathcal{M}(H)(w)$ is a moduli space, we have a natural description of $\mathfrak{M}(H)(w)_V(R)$, where R is a ϖ-adically complete V-algebra, in terms of formal abelian schemes with level structure and a formal section of $\underline{\omega}^{\otimes 1-q}$, called Y, that satisfies $YE_{q-1} = \varpi^w$. We have a similar remark for $\mathfrak{M}(H)(w)_V^{\mathrm{rig}}$ and even for level $K(H\varpi^r)$ (but in this case there is no w).

PROPOSITION 2.1.14. *We have that* $\mathfrak{M}(H)(w)$ *is normal, with reduced special fiber.*

PROOF. This is [**Kas04**], Propositions 8.2 and 9.5. □

2.2. The canonical subgroup

In this section we review the theory of canonical subgroup of our abelian schemes, as developed in [**Kas04**]. This is similar to the theory of the canonical subgroup of elliptic curves, see [**Kat73**]. We also prove that the definition of [**Kas04**] coincide, at level of points, with the one of [**Far07**]. At the end of the section, we consider canonical subgroups of higher rank.

NOTATION. Let V, K, and w be as above. From now on, we will work over V, so we will consider the base change to V, or to K, of the various objects defined so far. For simplicity we will omit the subscripts $_V$ and $_K$. For example $\mathcal{M}(H)_V$ will be denoted with $\mathcal{M}(H)$. We fix π, a uniformizer of V. Beware that the valuation of V satisfies $\mathrm{v}(\varpi) = 1$, not $\mathrm{v}(\pi) = 1$. We assume that V contains a fixed primitive p-th root of unity, denoted ζ_p.

From now on we assume that
$$0 \leq w < \frac{q}{q+1}.$$

REMARK 2.2.1. In the following chapters, we will assume that $0 \leq w$ is smaller and smaller, but we never assume $w = 0$. Geometrically, this means that we will work in a smaller and smaller, but non trivial, annulus of the supersingular discs, near the edge. In particular all our objects will be 'overconvergent', in the sense that they can be, partially, extended to the supersingular discs.

Recall that $\mathfrak{A}(H)(w)$ is the completion of $\mathcal{A}(H)(w)$ along the closed subscheme defined by $\varpi = 0$. By Theorem 10.1 of [**Kas04**], the q-torsion of $\mathfrak{A}(H)(w)$ admits a canonical subgroup stable under $\mathcal{O}_{\mathcal{P}}$, that we call $\mathfrak{C}(\mathfrak{A}(H)(w))$ or simply $\mathfrak{C}$. Given $\mathrm{Spf}(R) \to \mathfrak{M}(H)(w)$, where R is a ϖ-adically complete V-algebra, let $(\mathcal{A}, i, \theta, \bar{\alpha})$ be the corresponding object of the moduli problem, so $\mathcal{A}$ is defined over R. By base change we obtain $\mathfrak{C}(\mathcal{A})$: it extends to a canonical subgroup, denoted with $\mathcal{C}(\mathcal{A})$, or simply $\mathcal{C}$, of the q-torsion of $\mathcal{A}$, and we have that $\mathcal{C}_1^{2,1}$ is killed by ϖ. Since we will mostly be interested in $\mathcal{C}_1^{2,1}$ rather than $\mathcal{C}$, we will say 'the canonical subgroup' meaning also its $_1^{2,1}$-part (this abuse of notation should not create any confusion).

Since $\mathcal{C}(\mathcal{A})_1^{2,1}$ has order q, we can use it to define a morphism $\mathcal{M}(H)(w) \to \mathcal{M}(H,\varpi)$, whose formal completion makes the diagram

$$\begin{tikzcd} \mathfrak{M}(H)(w) \arrow[r] \arrow[dr] & \mathfrak{M}(H) \\ & \mathfrak{M}(H,\varpi) \arrow[u] \end{tikzcd}$$

commutative. Its rigidification is a section, defined over $\mathfrak{M}(H)(w)^{\mathrm{rig}}$, of the morphism $\mathfrak{M}(H,\varpi)^{\mathrm{rig}} \to \mathfrak{M}(H)^{\mathrm{rig}}$. We define $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$ (at this point, the $^{\mathrm{rig}}$ here is just notation, we have not defined the formal scheme $\mathfrak{M}(H\varpi)(w)$ yet) to be the inverse image of (the image of) $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ with respect to the map $\mathfrak{M}(H\varpi)^{\mathrm{rig}} \to \mathfrak{M}(H,\varpi)^{\mathrm{rig}}$. It is an affinoid subdomain of $\mathfrak{M}(H\varpi)^{\mathrm{rig}}$ with a map to $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ that is finite and étale since we are in characteristic 0.

The canonical subgroup is defined starting with its ${}^{2,1}_{1}$-part, so let us briefly recall how $\mathfrak{C}_1^{2,1}$ is constructed, see **[Kas04]**, Lemma 10.2 for details. Let $\mathrm{Spf}(R)$ be an open affine of $\mathfrak{M}(H\varpi)(w)$, and let $\mathcal{A}$ be the abelian scheme associated to $\mathrm{Spf}(R)$. Suppose that $\underline{\omega}_R$ is free. We fix a coordinate x of the formal group associated to $\mathcal{A}$ such that $[\varpi](x) = \varpi x + ax^q + \sum_{j=2}^{\infty} c_j x^{j(q-1)+1}$, and let ω be the differential dual to x. Since $\underline{\omega}_R$ is generated by ω, we can write $E_{q-1|\,\mathrm{Spec}(R)} = E\omega^{\otimes q-1}$, with $E \in R$. Note that, by Remark 2.1.9, we have $E \cong a \bmod \varpi$. In particular there is $b \in R$ such that $E = a + b\varpi$, furthermore we can write $(a+\varpi b)y = \varpi^w$, with $y \in R$. We set $r_1 := -\varpi/\varpi^w \in V$ and $t_0 := r_1 y/(1+r_1 by) \in R$. We have that $\mathcal{C}_1^{2,1}$, as a scheme, is $\mathrm{Spec}(R[[x]]/(x^q - t_{\mathrm{can}}x))$, where $t_{\mathrm{can}} = t_0(1-t_\infty)$. Here t_∞ is an element of r_2R, where $r_2 \in V$ has positive valuation. Since t_{can} is topologically nilpotent, we have an isomorphism of schemes

$$\mathcal{C}_1^{2,1} \cong \mathrm{Spec}(R[x]/(x^q - t_{\mathrm{can}}x)).$$

Suppose furthermore that R is a discrete valuation ring, whose valuation extends the one of $\mathcal{O}_{\mathcal{P}}$. In this case we can give a more explicit description of the points of the canonical subgroup. Since $w < 1$ and $E \equiv a \bmod \varpi$, we have $\mathrm{v}(a) = \mathrm{v}(E)$, and, by the description of the moduli problem of $\mathfrak{M}(H)(w)$, we have $\mathrm{v}(E) \leq w$. We are going to show that $\mathcal{C}_1^{2,1}$ coincides, at level of points, with the canonical subgroup of $\widehat{\mathcal{A}}_1^{2,1}[\varpi]$ defined in **[Far07]**, Section 7.1 (of rank and level 1). If $\mathcal{A}$ is ordinary this is trivial since $\widehat{\mathcal{A}}_1^{2,1}[\varpi]$ has rank 1 and coincides with $\mathcal{C}_1^{2,1}$, so we assume that $\mathcal{A}$ is supersingular, i.e that $\widehat{\mathcal{A}}_1^{2,1}$ has height 2. We are going to study P, the Newton polygon of $[\varpi](x)$.

LEMMA 2.2.2. *We have that P is the convex hull of the points*

$$(0,+\infty),\ (1,1),\ (q,\mathrm{v}(a)),\ \textit{and}\ (q^2,0).$$

PROOF. By the explicit description of $[\varpi](x)$, the first two vertices of P are $(0,+\infty)$ and $(1,1)$. Furthermore $(q^2,0) \in P$ since $\widehat{\mathcal{A}}_1^{2,1}[\varpi]$ has rank 2. The points corresponding to x^i, with $q < i < q^2$, do not belong to P because $c_j \in \varpi R$ unless $j \equiv 1 \bmod q$. It remains only to show that $(q,\mathrm{v}(a)) \in P$. Let ℓ be the line through $(1,1)$ and $(q^2,0)$, so $(q,\frac{q}{q+1}) \in \ell$ and the fact that $(q,\mathrm{v}(a))$ is below ℓ is exactly our assumption on w. $\square$

By the lemma, we immediately see that the roots of $[\varpi](x)$ corresponding to points of $\mathcal{C}_1^{2,1}$ are those with the biggest valuation, so our canonical subgroup coincides with the one of **[Far07]**. In particular we see that, if $w < \frac{q}{q+1}$ and $\mathcal{A}$ has height 2, then $[\varpi](x)$ has q^2-1 non trivial roots in some extension of R (this was already clear since $\widehat{\mathcal{A}}_1^{2,1}[\varpi]$ has rank 2). There are $q-1$ roots with valuation

$\frac{1-\mathrm{v}(a)}{q-1}$, while the remaining q^2-q roots have valuation $\frac{\mathrm{v}(a)}{q^2-q} < \frac{1-\mathrm{v}(a)}{q-1}$. This shows clearly that there are $q-1$ 'distinguished' roots. If $w \geq \frac{q}{q+1}$, all non trivial roots have valuation $\frac{1}{q^2-1}$ and there is no way to define the canonical subgroup. This suggests that our bound $w < \frac{q}{q+1}$ is the best possible, and this is actually proved in [**Kas09**].

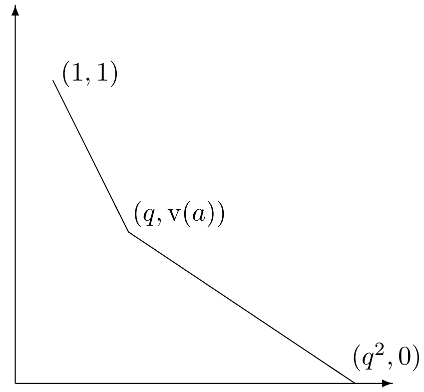

FIGURE 1. The Newton polygon of $[\varpi](x)$

REMARK 2.2.3. In [**AIS11**], Andreatta, Iovita, and Stevens, use the theory of canonical subgroup of abelian schemes developed in [**AG07a**]. For our approach it is convenient to work with formal $\mathcal{O}_{\mathcal{P}}$-modules, in order to take into account the action of $\mathcal{O}_{\mathcal{P}}$. One can prove that, if $\mathcal{D}$ is the canonical subgroup of [**AG07a**], we have

$$\mathcal{C}_1^{2,1} = \mathcal{D}_1^{2,1}[\varpi].$$

In the Appendix, we make a very detailed study of $\mathcal{C}_1^{2,1}$, obtaining an explicit formula for the comultiplication and some results about the module of invariant differentials.

2.2.1. Canonical subgroups of higher rank. In this section we fix an integer $r \geq 1$, the case of the canonical subgroup considered above corresponds to $r = 1$.

Let us suppose that

$$w < \frac{1}{q^{r-2}(q+1)}.$$

PROPOSITION 2.2.4. *We have that $\mathfrak{A}(H)(w)[\varpi^r]$ has a canonical subgroup $\mathfrak{C}_r$ stable under the action of D. Furthermore $(\mathfrak{C}_r)_1^{2,1}$ has order q^r and $\mathfrak{C}_1 = \mathfrak{C}$.*

PROOF. Let $\mathcal{A} \to \mathrm{Spec}(R)$ as above. We prove the proposition by induction on r, we already know the case $r = 1$. By assumption, $\mathcal{A}[\varpi]$ admits a canonical subgroup $\mathcal{C}$. In [**Kas04**], Section 4.4 and Theorem 10.1, it is proved that $\mathcal{A}/\mathcal{C}$ is another object of the moduli problem, and that the R-point corresponding to it lies in $\mathfrak{M}(H)(qw)$ (this can be proved also using an argument similar to that of the proof of Lemma 5.1.5). Since $qw \leq \frac{1}{q^{r-3}(q+1)}$, by induction hypothesis we have a canonical subgroup $\mathcal{C}'_{r-1} \subseteq (\mathcal{A}/)C[\varpi^{r-1}]$. We define $\mathcal{C}_r$ to be the kernel of the composite map

$$\mathcal{A} \to \mathcal{A}/\mathcal{C} \to (\mathcal{A}/\mathcal{C})/\mathcal{C}'_{r-1}.$$

The required properties of $\mathcal{C}_r$ follow from [**Kas04**]. $\square$

Note that, using the Newton polygon, one can obtain explicit formulas for the valuation of the roots of $[\varpi^r](x)$ as in the case $r = 1$. Again, the points of the canonical subgroup correspond to the roots with biggest valuation.

EXAMPLE 2.2.5. As an example, let us consider the case $r = 2$. We use the assumptions and the notations of Lemma 2.2.2. The roots of $[\varpi^2](x)$ are described by the polygons Q_z obtained taking the convex envelop of P and the points $(0, \mathrm{v}(z))$, where z is a root of $[\varpi](x)$. It is convenient to distinguish three cases

- if $z = 0$, then $Q_z = P$ and we have the trivial root, $q-1$ roots with valuation $\frac{1-\mathrm{v}(a)}{q-1}$ and q^2-q roots with valuation $\frac{\mathrm{v}(a)}{q^2-q}$;
- if z corresponds to a non trivial point of $\mathcal{C}_1^{2,1}$, then Q_z consists of two segments (the fact that there are two segments is equivalent to $\mathrm{v}(a) < \frac{1}{q+1}$), and there are q roots of valuation $\frac{1-q\,\mathrm{v}(a)}{q^2-q}$ and q^2-q roots of valuation $\frac{\mathrm{v}(a)}{q^2-q}$;
- if z corresponds to a point of $\mathcal{A}[\varpi]_1^{2,1} \setminus \mathcal{C}_1^{2,1}$, then Q_z has a single segment, and there are q^2 roots of valuation $\frac{1-\mathrm{v}(a)}{q^2(q-1)}$.

Since there are $q-1$ possible choices for points of the second type, and q^2-q possible choices for points of the third type, this gives a total of q^4 points as expected. In particular we see that the trivial root, the $q-1$ roots of valuation $\frac{1-\mathrm{v}(a)}{q-1}$ and the $(q-1)q$ roots of valuation $\frac{1-q\,\mathrm{v}(a)}{(q-1)q}$ are the roots with biggest valuation. They correspond to the q^2 points of $(\mathcal{C}_2)_1^{2,1}$.

Using the canonical subgroups of higher rank, everything we have done for level $K(H\varpi)$ can be repeated for level $K(H\varpi^r)$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{M}(H)(w) \arrow[r] \arrow[dr] & \mathfrak{M}(H) \\ & \mathfrak{M}(H,\varpi^r) \arrow[u] \end{tikzcd}$$

Its rigidification is a section, over $\mathfrak{M}(H)(w)^{\mathrm{rig}}$, of the morphism $\mathfrak{M}(H,\varpi^r)^{\mathrm{rig}} \to \mathfrak{M}(H)^{\mathrm{rig}}$. Following what we have done for level $K(H)$, we define the rigid space $\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$ as the inverse image of $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ under the map $\mathfrak{M}(H\varpi^r)^{\mathrm{rig}} \to \mathfrak{M}(H,\varpi^r)^{\mathrm{rig}}$.

2.3. Modular forms of level $K(H\varpi)$

In this section we introduce p-adic modular forms of higher level. We also continue the study of the canonical subgroup.

From now on we assume that V contains a fixed element whose $q-1$-th power is $-\varpi$, denoted $(-\varpi)^{1/(q-1)}$ (if $\varpi = p$ and $f = 1$ this is automatic since we have the p-th roots of unity). In the following chapters, we will make several assumptions on V similar to this one. In particular we will obtain a theory that works well over $\mathcal{O}_{\mathbb{C}_p}$ (but beware that V will always be a finite extension of $\mathcal{O}_{\mathcal{P}}$). We now move on to modular forms of level $K(H\varpi)$.

Let $\mathfrak{U} = \mathrm{Spf}(R)$ be an open affine of $\mathfrak{M}(H)(w)$. We write $\mathfrak{U}^{\mathrm{rig}} = \mathrm{Spm}(R_K)$ for its rigid analytic fiber. Since the morphism $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}} \to \mathfrak{M}(H)(w)^{\mathrm{rig}}$ is finite and étale, the inverse image of $\mathfrak{U}^{\mathrm{rig}}$ is an affinoid, $\mathfrak{V}^{\mathrm{rig}} = \mathrm{Spm}(S_K)$, with $R_K \to S_K$ finite and étale. Let S be the normalization of R in S_K, and write $\mathfrak{V}$ for $\mathrm{Spf}(S)$. Note that S is ϖ-torsion free, normal, since S_K is integrally closed by smoothness of $\mathfrak{M}(H\varpi)^{\mathrm{rig}} \cong M(H\varpi)^{\mathrm{an}}$, and finite over R. In particular S is

ϖ-adically complete. The various $\mathfrak{V}$'s glue together to define a formal scheme $\mathfrak{M}(H\varpi)(w)$, with a morphism to $\mathfrak{M}(H)(w)$. By construction we have the following

LEMMA 2.3.1. *The rigid analytic fiber of $\mathfrak{M}(H\varpi)(w)$ is $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$. Furthermore, the rigidification of $\mathfrak{M}(H\varpi)(w) \to \mathfrak{M}(H)(w)$ is the map $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}} \to \mathfrak{M}(H)(w)^{\mathrm{rig}}$ defined above.*

By definition $\mathfrak{M}(H\varpi)(w)$ is the normalization of $\mathfrak{M}(H)(w)$ in $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$, that is a finite extension of its generic fiber: this construction is well behaved, in particular it satisfies a suitable universal property, see **[FGL08]**, Annexe A, for more details. We write $\mathfrak{A}(H\varpi)(w)$ for the base change of $\mathfrak{A}(H)(w)$ to $\mathfrak{M}(H\varpi)(w)$. We do not know whether $\mathfrak{M}(H\varpi)(w)$ is a formal moduli space (indeed, since we take the normalization of $\mathfrak{M}(H)(w)$ in $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$, we do not have an integral model as in the case of $\mathfrak{M}(H)(w)$), but we have the following

PROPOSITION 2.3.2. *Let S be a normal and ϖ-adically complete V-algebra. There is a natural bijection between $\mathfrak{M}(H\varpi)(w)(S)$ and the set of isomorphism classes of quintuples $(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$, where:*

- *$(\mathcal{A}, i, \theta, \bar{\alpha})$ is an object of the moduli problem, with $\mathcal{A}$ defined over S, of $\mathcal{M}(H\varpi)$ and the canonical S-point of $\mathcal{A}[\varpi]_1^{2,1}$ generates, as $\mathcal{O}_{\mathcal{P}}$-module, the canonical subgroup of $\mathcal{A}[\varpi]$;*
- *Y is a section of $\underline{\omega}_{\mathcal{A}/S}^{\otimes 1-q}$ that satisfies $YE_{q-1} = \varpi^w$.*

PROOF. This is the analogue of **[AIS11]** Lemma 3.1. Let $x \in \mathfrak{M}(H\varpi)(w)(S)$. Taking the base change, via x, of $\mathfrak{A}(H\varpi)(w)$, we obtain a formal abelian scheme $\mathfrak{A} \to \mathrm{Spf}(S)$ endowed with a $K(H\varpi)$-level structure and $\mathfrak{Y}$, a formal section of $\underline{\omega}_{\mathfrak{A}/S}$ that satisfies $\mathfrak{Y}E_{q-1} = \varpi^w$. Using the natural polarization of $\mathfrak{A}$, coming from its level structure, we can embed $\mathfrak{A}$ in a formal projective space, in particular we obtain an integral model $\mathcal{A} \to \mathrm{Spec}(S)$ of it. By properness of $\mathfrak{A} \to \mathrm{Spf}(S)$ and GAGA, we see that also $\mathfrak{Y}$ comes from the required Y. The fact that the canonical S-point of $\mathcal{A}[\varpi]_1^{2,1}$ generates the canonical subgroup of $\mathcal{A}[\varpi]$ follows from the fact that the image of x^{rig} lies in $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$. This gives one direction of the correspondence.

For the converse, let $(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$ as in the statement of the proposition. By Remark 2.1.13, we obtain a morphism $g^{\mathrm{rig}} \colon \mathrm{Spm}(S_K) \to \mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}$. Indeed, forgetting the point of $\mathcal{A}[\varpi]_1^{2,1}$, we have a morphism $f \colon \mathrm{Spf}(S) \to \mathfrak{M}(H)(w)$. We need to prove that there is a unique formal model of g^{rig} that lifts f. Since the problem is local, we can assume that $f(\mathrm{Spf}(S))$ is contained in an affine of $\mathfrak{M}(H)(w)$, say $\mathrm{Spf}(W)$. Let $\mathrm{Spm}(T_K)$ be the inverse image of $\mathrm{Spm}(W_K)$ under the natural map $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}} \to \mathfrak{M}(H)(w)^{\mathrm{rig}}$. We write T for the normalization of W in T_K. By construction, $\mathrm{Spf}(T)$ is an open affine of $\mathfrak{M}(H\varpi)(w)$. Let $h \colon S \to T$ be the induced morphism, that is integral by definition, and let $g_K \colon T_K \to S_K$ be the morphism given by g^{rig}. Since $f_K = g_K \circ h_K$, it is enough to prove that $g_K(T)$ is contained in S. Let $t \in T$. We have that t is integral over W, so $g_K(t)$ is integral over S, hence, by normality, $g_K(t) \in S$ as required. □

DEFINITION 2.3.3. We define the space of ϖ-adic modular forms with respect to D, level $K(H\varpi)$, weight k and growth condition w, with coefficients in V, as

$$S^D(V, w, K(H\varpi), k) := \mathrm{H}^0(\mathfrak{M}(H\varpi)(w), \underline{\omega}^{\otimes k}),$$

where $\underline{\omega}$ has the obvious meaning. Note that we have

$$S^D(V, w, K(H\varpi), k)_K = \mathrm{H}^0(\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}}, \underline{\omega}^{\otimes k}).$$

We have a natural map $S^D(V, w, K(H), k) \to S^D(V, w, K(H\varpi), k)$. The image of E_{q-1} will be denoted with the same symbol. We are going to prove that, at level $K(H\varpi)$, this modular form admits a canonical $q-1$-th root, i.e. a modular form of weight 1 whose $q-1$-th power is E_{q-1}. We show this locally, so let us fix an open affine $\mathrm{Spf}(R)$ of $\mathfrak{M}(H)(w)$ with associated abelian scheme $\mathcal{A}$. We can assume that $\underline{\omega}_{\mathcal{A}/R}$ is a free R-module. Recall that there is a coordinate x on $\widehat{\mathcal{A}}_1^{2,1}$ such that the action of ϖ on the formal $\mathcal{O}_{\mathcal{P}}$-module has the form $[\varpi](x) = \varpi x + ax^q + \sum_{j=2}^{\infty} c_j x^{j(q-1)+1}$ (see Section 2.1). We fix such a coordinate and we denote with ω the differential dual to x, that is a generator of $\underline{\omega}_{\mathcal{A}/R}$. We write $E_{q-1|\,\mathrm{Spf}(R)} = E\omega^{\otimes q-1}$, with $E \in R$. Let $\mathrm{Spf}(S)$ be the base change of $\mathrm{Spf}(R)$ to $\mathfrak{M}(H\varpi)(w)$, we need to find a $q-1$-th root of E in S. This will be done using $\mathcal{C}$, the canonical subgroup of $\mathcal{A}$.

LEMMA 2.3.4. *There is a canonical non trivial point of $\mathcal{C}_1^{2,1}$, defined over S.*

PROOF. The pullback of $\mathcal{A}$ to S is given by an S-point $\mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$, so the lemma follows by Proposition 2.3.2. □

PROPOSITION 2.3.5. *There is $E_1 \in S^D(V, w, K(H\varpi), 1)$ such that*

$$E_1^{q-1} = E_{q-1}.$$

PROOF. Let $\alpha := \sum_{i=0}^{\infty} \binom{1/(q-1)}{i} (-t_\infty)^i$, so $\alpha^{q-1} = 1 - t_\infty$. The map induced by $x \mapsto \alpha x$ is an isomorphism between $R[x]/(x^q - t_{\mathrm{can}} x)$ and $R[x]/(x^q + \frac{\varpi}{a} x)$. Let us write β for $\sum_{i=0}^{\infty} \binom{1/(q-1)}{i} (b\frac{\varpi}{a})^i$, so $\beta^{q-1} = 1 + b\frac{\varpi}{a}$. We have an isomorphism $R[x]/(x^q + \frac{\varpi}{a}x) \cong R[x]/(x^q + \frac{\varpi}{E}x)$, induced by $x \mapsto \beta x$. By Lemma 2.3.4, the equation $x^{q-1} + \frac{\varpi}{E} = 0$ has a canonical solution $\alpha \in S$. Consider the element $E^{1/(q-1)} := \frac{(-\varpi)^{1/(q-1)}}{\alpha} \in S_K$: it is a canonical $q-1$-th root of E in S_K, that lies in S by normality. For the various R's, these roots glue together to define the required modular form. □

REMARK 2.3.6. In the proof of Proposition 2.3.5, we have shown that $\mathcal{C}_1^{2,1}$ is, as a scheme, $\mathrm{Spec}(R[x]/(x^q + \frac{\varpi}{E}x))$. By the Jacobian criterion, we have that $\mathcal{C}_1^{2,1}$ is étale over R_K. In particular, by Lemma 2.3.4, we have that the base change of $\mathcal{C}_1^{2,1}$ to S_K is a constant group scheme, with associated abstract group κ.

We can repeat what we have done in this section for level $K(H\varpi^r)$, for any integer $r \geq 0$. The formal scheme $\mathfrak{M}(H\varpi^r)(w)$ is defined as the normalization of $\mathfrak{M}(H)(w)$ in $\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$. In particular we have the following analogue of Proposition 2.3.2.

PROPOSITION 2.3.7. *Let S be a normal and ϖ-adically complete V-algebra. There is a natural bijection between $\mathfrak{M}(H\varpi^{r+1})(w)(S)$ and the set of isomorphism classes of quintuples $(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$, where:*

- *$(\mathcal{A}, i, \theta, \bar{\alpha})$ is an object of the moduli problem, with $\mathcal{A}$ defined over S, of $\mathcal{M}(H\varpi^{r+1})$ and the canonical S-point of $\mathcal{A}[\varpi^{r+1}]_1^{2,1}$ generates, as $\mathcal{O}_{\mathcal{P}}$-module, the canonical subgroup of $\mathcal{A}[\varpi^{r+1}]$;*
- *Y is a section of $\underline{\omega}_{\mathcal{A}/S}^{\otimes 1-q}$ that satisfies $YE_{q-1} = \varpi^w$.*

CHAPTER 3

The Hodge-Tate sequence

In this chapter we obtain some very important technical results. We introduce the map $\mathrm{d}\log$, that will be absolutely central for our work. We use this map to define the so-called Hodge-Tate sequence. We prove that this sequence is a complex, whose homology is killed by a power of ϖ. This allows us to define the notion of overconvergent modular form of non integral weight in the next chapter.

3.1. The map $\mathrm{d}\log$

DEFINITION 3.1.1. Let R be a V-algebra. We say that R is *small* if:

- $\mathrm{Spec}(R)$ is connected and R is ϖ-adically complete;
- there is a morphism $V\{T_1, \ldots, T_s\}/(T_1 \cdots T_j - \pi^a) \to R$ that is topologically of finite type and ϖ-adically formally étale, where $a \geq 0$ is an integer.

A *small affine* is a scheme of the form $\mathrm{Spf}(R)$, with R small.

PROPOSITION 3.1.2. *There is an open covering of $\mathfrak{M}(H)(w)$ by small affines.*

PROOF. By smoothness, we can find $\{\mathrm{Spec}(S_i)\}_{i \in I}$, an open covering of $\mathcal{M}(H)$, such that, for all i, there is an étale map $V[X] \to S_i$. We can assume that each $\mathrm{Spec}(S_i)$ intersects the special fiber, and, up to taking a refinement, that the pullback of $\underline{\omega}$ to $\mathrm{Spec}(S_i)$ is trivial, generated by t_i. Over $\mathrm{Spec}(S_i)$, we can write $E_{q-1} = a_i t_i$, with $a_i \notin \pi S_i$ (since the set of supersingular geometric points of the reduction of $\mathrm{Spec}(S_i)$ is finite). By construction, the pullback of $\mathcal{M}(H)(w)$ to $\mathrm{Spec}(S_i)$ is $\mathrm{Spec}(T_i)$, where $T_i := S_i[t_i]/(a_i t_i - \varpi^w)$. If $\mathbf{H}$ does not vanish on $\mathrm{Spec}(T_i)$, the proposition is clear, so let z be a geometric point of $\mathrm{Spec}(T_i)_{V/\pi V}$ that is a zero of $\mathbf{H}$ (in particular we can suppose $w > 0$). Enlarging V, we can assume that z is defined over $V/\pi V$. Let us consider the morphism $V[X,Y]/(XY - \varpi^w) \to T_i$, that sends X to a_i and Y to t_i. It is enough to prove that its reduction modulo ϖ is étale at z (since we want ϖ-adic formal étalness). The proposition follows by looking at the completed local ring of $\mathrm{Spec}(T_i)_{V/\pi V}$ at z, since a is a generator of the maximal ideal of the completed local ring of S_i at the image of z by **[Kas04]**, Proposition 6.3. □

In this section we mainly work over $\mathfrak{M}(H)(w)$ and over $\mathfrak{M}(H\varpi)(w)$. Let $\{\mathrm{Spf}(R_i)_{i \in I}\}$ be a covering of $\mathfrak{M}(H)(w)$ by affine irreducible formal schemes (sometimes we will assume that each R_i is small, this is possible by Proposition 3.1.2). Our local situation will be the following: we choose one of the R_i's, called simply R. Note that, by Proposition 2.1.14, we have that R is a normal ring. Its pullback to $\mathfrak{M}(H\varpi)(w)$ will be denoted with $\mathrm{Spf}(S)$. By construction, $\mathrm{Spf}(R)$ is equipped with $\mathfrak{A}$, a formal abelian scheme of dimension $4N$. We have that $\mathfrak{A}$ is the ϖ-adic completion of an abelian scheme $\pi\colon \mathcal{A} \to \mathcal{U}$, where $\mathcal{U} := \mathrm{Spec}(R)$. Here $\mathcal{A}$ is part of an object of the moduli problem of Theorem 1.3.6. Up to taking a refinement, we can assume that $\underline{\omega}_{\mathcal{A}/R} = \left(\pi_* \Omega^1_{\mathcal{A}/R}\right)_1^{2,1}$ is a free $\mathcal{O}_{\mathcal{U}}$-module, generated by ω, and we write $E_{q-1|\,\mathrm{Spf}(R)} = E\omega^{\otimes q-1}$. Let $\eta = \mathrm{Spec}(\mathbb{K})$ be a generic geometric point of

$\mathcal{U}$, we write $\mathcal{G}$ for $\pi_1(\mathcal{U}_K, \eta)$. We denote with $\overline{R}$ the direct limit of all R-algebras $T \subseteq \mathbb{K}$ which are normal and such that T_K is finite and étale over R_K. Note that $\mathcal{G} = \mathrm{Gal}(\overline{R}_K/R_K)$, so it acts continuously on $\widehat{\overline{R}}$, the ϖ-adic completion of $\overline{R}$.

We now give the definition of the map $\mathrm{d}\log$ in the generality we need. We are going to use the Lubin-Tate ϖ-divisible group $\mathcal{LT} = R[[x]]$, defined in Section 1.7.

Let G be an abelian group with an action of $\mathcal{O}_\mathcal{P}$. We define the *Tate module* of G as $\mathrm{T}_\varpi(G) := \varprojlim_n G[\varpi^n]$. If G is a ϖ-divisible group, we define $\mathrm{T}_\varpi(G) := \mathrm{T}_\varpi(G(\overline{R}_K))$. Note that, if we define the (constant) ϖ-divisible group

$$F_\mathcal{P}/\mathcal{O}_\mathcal{P} := \varinjlim_{n\geq 0} \varpi^{-n}\mathcal{O}_\mathcal{P}/\mathcal{O}_\mathcal{P},$$

then we have

$$\mathrm{T}_\varpi(G) = \mathrm{Hom}(F_\mathcal{P}/\mathcal{O}_\mathcal{P}, G).$$

Let G be a finite and flat group scheme over R with an action of $\mathcal{O}_\mathcal{P}$ (we will always assume the condition on the action on the Lie algebra), and let us suppose that G is killed by ϖ^n for some integer n. Recall that, by Theorem 1.7.4, the functor $(R\text{-algebras})^{\mathrm{op}} \to \mathbf{set}$ that sends T to $\hom_{\mathcal{O}_\mathcal{P}}(G_T, \mathcal{LT}_T)$ is representable by $G^\vee$. We have a similar results for ϖ-divisible groups.

Let G be a ϖ-divisible group and let H be a sub $\mathcal{O}_\mathcal{P}$-module of $\mathrm{T}_\varpi(G^\vee)$. By duality between G and $G^\vee$, we obtain $H^\perp$, the orthogonal of H, that is a sub $\mathcal{O}_\mathcal{P}$-module of $\mathrm{T}_\varpi(G)$.

If $D \subseteq G[\varpi^n](\overline{R}_K)$ is a sub $\mathcal{O}_\mathcal{P}$-module, we write D^{cl} for the schematic closure of D in $G[\varpi^n]$. If R is a discrete valuation ring, whose valuation extends the one of $\mathcal{O}_\mathcal{P}$, we have that D^{cl} and $(D^\perp)^{\mathrm{cl}}$ are group schemes. In this case, by **[Far07]**, Proposition 1, we have

$$(D^{\mathrm{cl}})^\vee \cong G[\varpi^n]^\vee/(D^\perp)^{\mathrm{cl}}.$$

Let W be a normal Noetherian R-algebra, without ϖ-torsion. Let G be a group scheme with an action of $\mathcal{O}_\mathcal{P}$, and let $\underline{\omega}_{G/R}$ be the module of invariant differential of G. If G is killed by ϖ^n, we define a map

$$\mathrm{d}\log_G := \mathrm{d}\log_{G,W} \colon G^\vee(W_K) \to \underline{\omega}_{G/R} \otimes_R W/\varpi^n W$$

in the following way: given x, a W_K-valued point of $G^\vee$, it extends, by normality, to a W-valued point of $G^\vee$, called again x. It gives a group scheme homomorphism (that respects the action of $\mathcal{O}_\mathcal{P}$) $f_x \colon G_W \to \mathcal{LT}_W$. We define

$$\mathrm{d}\log_{G,W}(x) := f_x^* \,\mathrm{d}(T).$$

LEMMA 3.1.3. *Let G be as above.*

- *Let R' be an R-algebra, and let G' be the base change of G to R'. If W is a normal, Noetherian, ϖ-torsion free R'-algebra, then the following diagram commutes.*

$$\begin{tikzcd} G^\vee(W_K) \arrow[r, "\mathrm{d}\log_{G,W}"] \arrow[d] & \underline{\omega}_{G/R} \otimes_R W/\varpi^n W \arrow[d] \\ (G')^\vee(W_K) \arrow[r, "\mathrm{d}\log_{G',W}"] & \underline{\omega}_{G'/R'} \otimes_{R'} W/\varpi^n W \end{tikzcd}$$

- *Let G' be a finite and flat group scheme over R, with an action of $\mathcal{O}_\mathcal{P}$. Assume that G' is killed by ϖ^n and suppose we are given a morphism $G' \to G$ that respects the action of $\mathcal{O}_\mathcal{P}$. If W is as above, then the following* *diagram commutes.*

$$\begin{tikzcd} G^\vee(W_K) \arrow[r, "\mathrm{d}\log_{G,W}"] \arrow[d] & \underline{\omega}_{G/R} \otimes_R W/\varpi^n W \arrow[d] \\ (G')^\vee(W_K) \arrow[r, "\mathrm{d}\log_{G',W}"] & \underline{\omega}_{G'/R} \otimes_R W/\varpi^n W \end{tikzcd}$$

- *Let W be as above and let W' be a normal, Noetherian, and ϖ-torsion free W-algebra. Then the following diagram commutes.*

$$\begin{tikzcd} G^\vee(W_K) \arrow[r, "\mathrm{d}\log_{G,W}"] \arrow[d] & \underline{\omega}_{G/R} \otimes_R W/\varpi^n W \arrow[d] \\ G^\vee(W'_K) \arrow[r, "\mathrm{d}\log_{G,W'}"] & \underline{\omega}_{G/R} \otimes_R W'/\varpi^n W' \end{tikzcd}$$

PROOF. We prove the first point of the lemma, the other ones are similar. We write $i\colon R \to R'$, and we use the same notation for the map $G' \to G$. Let x be a point in $G^\vee(W_K)$ and let $f_x\colon G_W \to \mathcal{LT}_W$ be the corresponding morphism. We have $\mathrm{d}\log_{G,W}(x) = f_x^* \,\mathrm{d}(T)$, so its image in $\underline{\omega}_{G'/R'} \otimes_{R'} W/\varpi^n W$ is $i^* f_x^* \,\mathrm{d}(T)$. By definition we have that $i^\vee(x)$ corresponds to the morphism $f_x \circ i\colon G'_W \to \mathcal{LT}_W$. It follows that

$$\mathrm{d}\log_{G',W}(i^\vee(x)) = (f_x \circ i)^* \,\mathrm{d}(T) = i^* f_x^* \,\mathrm{d}(T)$$

as required. $\square$

In particular, we can take $G = \mathcal{A}[\varpi^n]_1^{2,1}$, and we obtain the map

$$\mathrm{d}\log_{n,W}\colon (\mathcal{A}[\varpi^n]_1^{2,1})^\vee(W_K) \to \underline{\omega}_{\mathcal{A}[\varpi^n]_1^{2,1}} \otimes_R W/\varpi^n W.$$

Taking the direct limit over all W as above, we get the map

$$\mathrm{d}\log_{n,\mathcal{A}}\colon (\mathcal{A}[\varpi^n]_1^{2,1})^\vee(\overline{R}_K) \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi^n \overline{R}.$$

Finally, taking the projective limit, we obtain the map

$$\mathrm{d}\log_{\mathcal{A}}\colon \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}.$$

Suppose that R is a discrete valuation ring, whose valuation extends the one of $\mathcal{O}_\mathcal{P}$. In particular we have that $\widehat{\mathcal{A}}_1^{2,1}$ is a ϖ-divisible group, and so is the formal $\mathcal{O}_\mathcal{P}$-module associated to $\mathcal{A}[\varpi^\infty]_1^{2,1}$. From $\mathrm{d}\log_{\mathcal{A}}$, we obtain the maps $\mathrm{d}\log_{n,\widehat{\mathcal{A}}}$ and the map

$$\mathrm{d}\log_{\widehat{\mathcal{A}}}\colon \mathrm{T}_\varpi((\widehat{\mathcal{A}}_1^{2,1})^\vee) \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}},$$

that is denoted $\alpha^{\overline{\mathcal{O}}_\mathcal{P}}_{(\widehat{\mathcal{A}}_1^{2,1})^\vee}$ in **[Far07]**, Section 1. We extend the notation introduced above for $\mathcal{A}[\varpi^\infty]_1^{2,1}$ to any ϖ-divisible group in the obvious way.

We return now to the case of a general ϖ-adically complete R.

PROPOSITION 3.1.4. *Let us suppose that that R is a discrete valuation ring and that $\mathcal{A}$ is supersingular. Then $\mathrm{d}\log_{1,\widehat{\mathcal{A}}}$ has non trivial kernel if and only if $w \leq \frac{1}{q}$. In this case we have*

$$\ker(\mathrm{d}\log_{1,\widehat{\mathcal{A}}}) = (\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}(\overline{R}_K),$$

where the orthogonal is taken with respect to $\widehat{\mathcal{A}}[\varpi]_1^{2,1}$.

PROOF. Since R is a discrete valuation ring, we have that $\widehat{\mathcal{A}}_1^{2,1}$ is a formal ϖ-divisible group. We re-write part of Section 1.2 of **[Far07]**, adapted to our case. Let $y \in \widehat{\mathcal{A}}_1^{2,1}[\varpi]^\vee(\overline{R}_K)$ and let $\mathcal{D} \subseteq \widehat{\mathcal{A}}_1^{2,1}[\varpi]^\vee(\overline{R}_K)$ be the $\mathcal{O}_\mathcal{P}$-module generated by y. We have $\underline{\omega}_{(\mathcal{D}^{\mathrm{cl}})^\vee} \cong R/\gamma R$, with $\mathrm{v}(\gamma) = 1 - \sum \mathrm{v}(z)$, where the sum is over $\mathcal{D}^\perp \setminus \{0\}$ (we consider everything as a subscheme of $R[[x]]$, so the valuation of a point makes sense), see **[Far07]** for details. Since the map

$$\overline{R}/\gamma\overline{R} \cong \underline{\omega}_{(\mathcal{D}^{\mathrm{cl}})^\vee/R} \otimes_R \overline{R}/\varpi\overline{R} \hookrightarrow \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R} \cong \overline{R}/\varpi\overline{R}$$

is the multiplication by $\frac{\varpi}{\gamma}$, it is injective. In particular, we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{A}}_1^{2,1}[\varpi]^\vee(\overline{R}_K) \arrow[r] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R} \\
\mathcal{D}^{\mathrm{cl}}(\overline{R}_K) \arrow[r] \arrow[u, hook] & \underline{\omega}_{(\mathcal{D}^{\mathrm{cl}})^\vee/R} \otimes_R \overline{R}/\varpi\overline{R} \arrow[u, hook]
\end{tikzcd}$$

so we can study the map $\mathrm{d}\log_{(D^{\mathrm{cl}})^\vee,\overline{R}}$. We now have that $\mathrm{d}\log_{(D^{\mathrm{cl}})^\vee,\overline{R}}(y) \equiv \beta \bmod \gamma$, with $\mathrm{v}(\beta) = \frac{1-\mathrm{v}(\gamma)}{q-1}$, so $\mathrm{d}\log_{(D^{\mathrm{cl}})^\vee,\overline{R}}(y) = 0$ if and only if $\mathrm{v}(\gamma) \leq \frac{1-\mathrm{v}(\gamma)}{q-1}$, i.e. if and only if

$$\mathrm{v}(\gamma) \leq \frac{1}{q}.$$

If $y \in \mathcal{C}_1^{2,1}(\overline{R}_K)^\perp \setminus \{0\}$, we have that $\mathcal{D}^\perp = \mathcal{C}_1^{2,1}(\overline{R}_K)$ and $\mathrm{v}(\gamma) = \mathrm{v}(E) \leq w$. It follows that $\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp$ is contained in the kernel of $\mathrm{d}\log_{1,\mathcal{A}}$ if and only if $w \leq \frac{1}{q}$. We now prove that if $y \notin (\mathcal{C}_1^{2,1})(\overline{R}_K)^\perp$ then $y \notin \ker(\mathrm{d}\log_{1,\widehat{\mathcal{A}}})$. If $y \notin (\mathcal{C}_1^{2,1})(\overline{R}_K)^\perp$, the valuation of the points of $\mathcal{D}^\perp$ is $\frac{\mathrm{v}(E)}{q(q-1)}$ (by Lemma 2.2.2), so $\mathrm{v}(\gamma) = 1 - \frac{\mathrm{v}(E)}{q} > \frac{1}{q}$ (since $w < \frac{q}{q+1}$). Since we have $\widehat{\mathcal{A}}_1^{2,1}[\varpi]^\vee/(\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}} \cong (\mathcal{C}_1^{2,1})^\vee$ we obtain $\ker(\mathrm{d}\log_{1,\widehat{\mathcal{A}}}) = (\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}(\overline{R}_K)$. □

From now on we assume that $w \leq \frac{1}{q}$, so, if R is a discrete valuation ring, then $\ker(\mathrm{d}\log_{1,\widehat{\mathcal{A}}}) = (\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}(\overline{R}_K)$.

REMARK 3.1.5. Let R be a complete discrete valuation ring. If $\mathcal{A}$ is supersingular, then the above proposition shows that $\ker(\mathrm{d}\log_{1,\mathcal{A}}) = (\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}(\overline{R}_K)$. Indeed, we have $\widehat{\mathcal{A}}[\varpi]_1^{2,1} = \mathcal{A}[\varpi]_1^{2,1}$. In general we have that $\widehat{\mathcal{A}}[\varpi]_1^{2,1}$ and $\mathcal{A}[\varpi]_1^{2,1}$ have the same module of invariant differentials. It follows by Lemma 3.1.3 and **[Far07]**, Lemme 1, that the remark is true also in the case $\mathcal{A}$ is ordinary. In particular we have

$$\ker(\mathrm{d}\log_{1,\mathcal{A}}) = (\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}(\overline{R}_K),$$

where the orthogonal is taken in $\mathcal{A}[\varpi]_1^{2,1}$.

REMARK 3.1.6. Let $\mathcal{D}$ be a subgroup of $\mathcal{A}[\varpi^n]_1^{2,1}$. Suppose we want to prove that $\mathcal{D} \subseteq \ker(\mathrm{d}\log_{n,\mathcal{A}})$. We need to prove that $\mathcal{D}(W_K) \subseteq \ker(\mathrm{d}\log_{n,W})$, for all big enough normal R-algebras $W \subseteq \overline{R}$. Let $(\varpi) \subseteq \mathfrak{p}$ be a prime ideal of W of height 1. If $x \in \mathcal{D}(W)$, be Lemma 3.1.3, we can prove that $\mathrm{d}\log_{n,W}(x) = 0$ after localizing at $\mathfrak{p}$ and taking $\mathfrak{p}$-adic completion. It follows that we may assume that W and R are complete discrete valuation rings, whose valuation extends the one of $\mathcal{O}_\mathcal{P}$. In particular we can use Remark 3.1.5, and we obtain a map, denoted again $\mathrm{d}\log_{1,\mathcal{A}}$,

$$\mathrm{d}\log_{1,\mathcal{A}} \colon (\mathcal{C}_1^{2,1})^\vee(\overline{R}_K) \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R}.$$

From now on we will use the notation

$$v := \frac{w}{q-1}.$$

PROPOSITION 3.1.7. *By base change we obtain the map*

$$\mathrm{d}\log_{1,\mathcal{A}} \colon (\mathcal{C}_1^{2,1})^\vee(\overline{R}_K) \otimes_\kappa \overline{R}/\varpi\overline{R} \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R}.$$

We have that its cokernel is killed by ϖ^v, in particular we have

$$\ker(\mathrm{d}\log_{1,\mathcal{A}}) = \mathcal{C}_1^{2,1}(\overline{R}_K)^\perp.$$

PROOF. Fix an isomorphism $\underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R} \cong \overline{R}/\varpi\overline{R}$ and let us denote with $a \in \overline{R}/\varpi\overline{R}$ a generator of the image of $\mathrm{d}\log_{1,\mathcal{A}}$. We need to prove that ϖ^v is a multiple of a. To prove this, we can replace $\overline{R}$ with a normal R-algebra $W \subseteq \overline{R}$ such that W_K is finite and étale over R_K. By normality, we have $W = \cap W_\mathfrak{p}$, where the intersection is over the set of prime ideals of W of height 1. So wee can assume that R is a discrete valuation ring, whose valuation extends the one of $\mathcal{O}_\mathcal{P}$ (if $(\varpi) \nsubseteq \mathfrak{p}$ the proposition becomes trivial over $W_\mathfrak{p}$). Taking completion, we may even assume that R is complete.

We can work with $\widehat{\mathcal{A}}_1^{2,1}$. We start by proving the proposition in the case $\mathcal{A}$ is supersingular. We continue the calculations made in the proof of Proposition 3.1.4, using the same notations. Looking at the map $\underline{\omega}_{(\mathcal{D}^{\mathrm{cl}})^\vee/R} \otimes_R \overline{R}/\varpi\overline{R} \hookrightarrow \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R}$, we see that, if $y \notin (\mathcal{C}_1^{2,1})(\overline{R}_K)^\perp$, then $\mathrm{d}\log_{1,\mathcal{A}}(y) \equiv \beta \bmod \varpi$, with

$$\mathrm{v}(\beta) = \frac{q}{q-1}(1-\mathrm{v}(\gamma)) = \frac{\mathrm{v}(E)}{q-1} \leq v$$

as required. The ordinary case is similar. □

REMARK 3.1.8. Looking at the proof of Proposition 3.1.7, we see that the map $\mathrm{d}\log_{1,\mathcal{A}}$ is surjective if and only if $\mathcal{A}$ is ordinary.

REMARK 3.1.9. In [**AIS11**], the condition $w < \frac{1}{p}$ is used from the very beginning to define the canonical subgroup and it is not directly related to the map $\mathrm{d}\log$. In our situation, before Proposition 3.1.7, it was enough to assume $w < \frac{q}{q+1}$. Looking at the proof of the above propositions, we see that if we want to relate the map $\mathrm{d}\log$ with the canonical subgroup, the assumption $w \leq \frac{1}{q}$ is really essential.

Recall that $\mathrm{Spf}(S)$ is the inverse image of $\mathrm{Spf}(R)$ under the map $\mathfrak{M}(H\varpi)(w) \to \mathfrak{M}(H)(w)$. Furthermore we have $E^{1/(q-1)}$, a canonical $q-1$-th root of E in S, and we know that $\mathcal{C}_1^{2,1}$ becomes constant over S_K. We now consider the morphism (see Example 1.7.6)

$$\begin{aligned}(\mathcal{C}_1^{2,1})_S = \mathrm{Spec}(S[x]/(x^q + \frac{\varpi}{E}x)) &\to \mathcal{LT}_S = \mathrm{Spec}(S[[x]]) \\ E^{1/(q-1)}x &\leftarrow\!\shortmid x\end{aligned}$$

By Lemma A.3, it respects the action of $\mathcal{O}_\mathcal{P}$, so it gives a canonical non trivial point $\gamma \in (\mathcal{C}_1^{2,1})^\vee(S_K)$ that, for dimension reasons, is a generator of the κ-vector space $(\mathcal{C}_1^{2,1})^\vee(S_K)$. We are now ready to relate the modular form E_1 with the map $\mathrm{d}\log$.

PROPOSITION 3.1.10. *We have the equality*

$$\mathrm{d}\log_{1,S}(\gamma) \equiv E_{1|\,\mathrm{Spf}(S)} \bmod \varpi^{1-w}.$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd} (\mathcal{A}[\varpi]_1^{2,1})^\vee(S_K) \arrow[r] \arrow[d] & \underline{\omega}_{\mathcal{A}/R} \otimes_R S/\varpi^{1-w}S \arrow[d] \\ (\mathcal{C}_1^{2,1})^\vee(S_K) \arrow[r] \arrow[ur, dotted] & \underline{\omega}_{\mathcal{C}_1^{2,1}/R} \otimes_R S/\varpi^{1-w}S \end{tikzcd}$$

where the horizontal arrows are the corresponding $\mathrm{d}\log$ maps (base-changed to $S/\varpi^{1-w}S$) and the existence of the dotted arrow follows by Remark 3.1.5. Being $h\colon \mathcal{C}_1^{2,1} \to \mathcal{A}[\varpi]_1^{2,1}$ a closed immersion, the right vertical map is surjective, but both its domain and codomain are free $S/\varpi^{1-w}S$-module of rank 1: indeed $\underline{\omega}_{\mathcal{A}/R}$ is free of rank 1, furthermore $\underline{\omega}_{\mathcal{C}_1^{2,1}/R} \cong R/\frac{\varpi}{E}R$ by Lemma A.3 and $\frac{\varpi}{E} = 0$ in $S/\varpi^{1-w}S$ by our assumption on E. It follows that the right vertical map is an isomorphism, so we can prove that

$$\mathrm{d}\log_{\mathcal{C}_1^{2,1},S}(\gamma) \equiv h^*(E_{1\,|\,\mathrm{Spf}(S)}) \bmod \varpi^{1-w}$$

By the explicit description of γ, we have $\mathrm{d}\log_{\mathcal{C}_1^{2,1},S}(\gamma) = E^{1/(q-1)}\,\mathrm{d}(x)$, so we can conclude by Proposition A.10. $\square$

3.2. The Hodge-Tate sequence

We now use the results of the previous section to relate the module of invariant differential of $\mathcal{A}$ with its Tate module.

We continue to work locally as in the previous section, using the same notations. In this section we will work with the further assumption that R *is small*, so we can use the results of [**Bri08**].

By definition we have an isomorphism of $\mathrm{Gal}(\overline{F}_{\mathcal{P}}/F_{\mathcal{P}})$-modules

$$\mathrm{T}_\varpi(\mathcal{LT}) \cong \mathcal{O}_{\mathcal{P}}(1),$$

where $\mathcal{O}_{\mathcal{P}}(1)$ means that the action of $\mathrm{Gal}(\overline{F}_{\mathcal{P}}/F_{\mathcal{P}})$ is twisted by the Lubin-Tate character. We have a map

$$\mathrm{d}\log\colon \mathcal{LT}(\overline{\mathcal{O}}_{\mathcal{P}})[\varpi^\infty] \to \Omega^1_{\overline{V}/V}$$
$$x \mapsto x^*\,\mathrm{d}(T),$$

where $\overline{V}$ is the normalization of V in $\overline{F}_{\mathcal{P}}$. Taking Tate modules, we get a map

$$\mathrm{d}\log\colon \mathcal{O}_{\mathcal{P}}(1) \to \mathrm{T}_\varpi(\Omega^1_{\overline{V}/V}).$$

By [**Fon82**], Theorem 1, we have a canonical isomorphism $\Omega^1_{\overline{V}/V} \cong \overline{F}_{\mathcal{P}}/\varpi^{-\rho}\overline{V}(1)$, where $\varpi^\rho \in \overline{\mathcal{O}}_{\mathcal{O}_{\mathcal{P}}}$ (the rational number ρ depends on q and on the different of K over $W(k)[\varpi]$). This implies that

$$\mathrm{T}_\varpi(\Omega^1_{\overline{V}/V}) \cong \varpi^{-\rho}\widehat{\overline{V}}(1).$$

It is immediate to check that, under this isomorphism, the map

$$\mathrm{d}\log\colon \mathcal{O}_{\mathcal{P}}(1) \to \varpi^{-\rho}\widehat{\overline{V}}(1)$$

is the natural immersion.

Let G be a ϖ-divisible group over R. By pullback of differentials, we get a bilinear map

$$\mathrm{T}_\varpi(G) \times \underline{\omega}_{G/R} \otimes_R \overline{R} \to \mathrm{T}_\varpi(\Omega^1_{\overline{R}/R})$$
$$(x,y) \mapsto \langle x, y\rangle$$

We also have the perfect pairing, given by duality

$$\begin{aligned} \mathrm{T}_\varpi(G) \times \mathrm{T}_\varpi(G^\vee) &\to \mathcal{O}_{\mathcal{P}}(1) \\ (x,y) &\mapsto \langle x,y\rangle \end{aligned}$$

LEMMA 3.2.1. *Let $x \in \mathrm{T}_\varpi(G)$ and $y \in \mathrm{T}_\varpi(G^\vee)$. Via the map $\Omega^1_{\overline{V}/V} \to \Omega^1_{\overline{R}/R}$, we have*

$$\mathrm{d}\log(\langle x,y\rangle) = \langle x, \mathrm{d}\log_G(y)\rangle + \langle y, \mathrm{d}\log_{G^\vee}(x)\rangle.$$

PROOF. This is proved in exactly the same way as **[Fal87]**, Lemma 1. $\square$

REMARK 3.2.2. Let x and y be as in the above lemma. Suppose further that x and y are in the kernel of the corresponding $\mathrm{d}\log$ maps. By the lemma, we see that $\mathrm{d}\log(\langle x,y\rangle) = 0$. But the map $\mathrm{d}\log\colon \mathcal{O}_{\mathcal{P}}(1) \to \varpi^{-\rho}\widehat{\overline{V}}(1)$ is injective, so we have that x and y are orthogonal.

We now need some results about $\widehat{\mathcal{A}}_1^{2,1}[\varpi]^\vee \cong \widehat{\mathcal{A}}_1^{1,1}[\varpi]$ (the isomorphism comes from the polarization of $\mathcal{A}$). Everything we have said until now, in the whole thesis, can be proved, in exactly the same way, using the ${}_1^{1,1}$-part instead of the ${}_1^{2,1}$-part of the various objects. For example we have the modular form $\mathbf{H}'$, the analogue of $\mathbf{H}$, and in general we will use the $'$ to denote that we are 'in the ${}_1^{1,1}$ case'. We have

$$\underline{\omega}'_{\mathcal{A}/R} \cong \underline{\omega}_{\mathcal{A}[\varpi^\infty]_1^{1,1}/R},$$

and we write $\underline{\omega}_{\mathcal{A}^\vee/R}$ for this R-module. We set $E'_{q-1|\operatorname{Spf}(S)} = E'\omega'$, with $E' \in S$. Using E'_{q-1}, that is a global section of an invertible sheaf on $\mathfrak{M}(H)^{\mathrm{rig}}$, we can define the affinoid subdomain $\mathfrak{M}(H)'(w)^{\mathrm{rig}}$, for any rational number w. We are going to prove that $\mathfrak{M}(H)(w)^{\mathrm{rig}}$ and $\mathfrak{M}(H)'(w)^{\mathrm{rig}}$ are the same subset of $\mathfrak{M}(H)^{\mathrm{rig}}$. Compare the following proposition with **[Far10]**, Proposition 2.

PROPOSITION 3.2.3. *Let us suppose that R is a discrete valuation ring, whose valuation extends the one of $\mathcal{O}_{\mathcal{P}}$. Then the valuation of E' is the same as the valuation of E. Furthermore $(\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}$ is the canonical subgroup of $\widehat{\mathcal{A}}[\varpi]_1^{1,1}$.*

PROOF. If $\widehat{\mathcal{A}}_1^{2,1}$ has height 1, also its dual must have height 1, so both E and E' are units of R, hence we can assume that $\widehat{\mathcal{A}}_1^{2,1}$ has height 2. We claim that the map

$$\mathrm{d}\log'_{1,\mathcal{A}}\colon \mathcal{A}[\varpi]_1^{2,1}(\overline{R}_K) \to \underline{\omega}'_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R}$$

has $\mathcal{C}_1^{2,1}(\overline{R}_K)$ as kernel. Indeed, let $y \in \mathcal{C}_1^{2,1}(\overline{R}_K)$, since we have a commutative diagram

$$\begin{tikzcd} \mathcal{A}_1^{2,1}[\varpi](\overline{R}_K) \arrow[r] & \underline{\omega}'_{\mathcal{A}/R} \otimes_R \overline{R}/\varpi\overline{R} \\ \mathcal{C}_1^{2,1}(\overline{R}_K) \arrow[r] \arrow[u, hook] & \underline{\omega}_{(\mathcal{C}_1^{2,1})^\vee/R} \otimes_R \overline{R}/\varpi\overline{R} \arrow[u, hook] \end{tikzcd}$$

to prove that $\mathrm{d}\log'_{1,\mathcal{A}}(y) = 0$ it suffices to show that $\mathrm{d}\log_{(\mathcal{C}_1^{2,1})^\vee,\overline{R}}(y) = 0$. But by **[Fal02]**, Section 3, we have $(\mathcal{C}_1^{2,1})^\vee \cong \operatorname{Spec}(R[x]/(x^q - Ex))$, so $\underline{\omega}_{(\mathcal{C}_1^{2,1})^\vee/R} \cong R/ER$. With this isomorphism, we have $\mathrm{d}\log_{(\mathcal{C}_1^{2,1})^\vee,\overline{R}}(y) = \gamma$, with $\mathrm{v}(\gamma) = \frac{1-\mathrm{v}(E)}{q-1} \geq \mathrm{v}(E)$ since $\mathrm{v}(E) \leq \frac{1}{q}$. The claim follows since, by Remark 3.2.2 and Proposition 3.1.4, the kernel of $\mathrm{d}\log'_{1,\mathcal{A}}$ is orthogonal to $\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp$ and hence has κ-dimension at most 1. By the analogue of Proposition 3.1.4, we see that the fact that $\mathrm{d}\log'_{1,\mathcal{A}}$ has a non trivial kernel implies that $\mathrm{v}(E') \leq \frac{1}{q}$. The statement about $(\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}$ follows. It remains to bound the valuation of E', or equivalently, to bound the valuation of the points of $\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp$, that is $\frac{1-\mathrm{v}(E')}{q-1}$. Let us consider the isogeny $\widehat{\mathcal{A}}_1^{1,1} \twoheadrightarrow \widehat{\mathcal{A}}_1^{1,1}/(\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}}$, by **[Far07]**, Remarque 2, it is given, after a suitable choice of coordinates, by the map

$$\begin{aligned} R[[x]] &\to R[[x]] \\ x &\mapsto \prod_{\lambda \in \mathcal{C}_1^{2,1}(\overline{R}_K)^\perp} (x-\lambda) \end{aligned}$$

Since the valuation of the points of $\mathcal{A}[\varpi]_1^{1,1}$ that are not in $\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp$ is $\frac{\mathrm{v}(E')}{q(q-1)}$, that is smaller than $\frac{1-\mathrm{v}(E')}{q-1}$, we have that the valuation of the image of these points under the isogeny is $\frac{\mathrm{v}(E')}{q-1}$. But $\mathcal{A}[\varpi]_1^{1,1}/(\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp)^{\mathrm{cl}} \cong (\mathcal{C}_1^{2,1})^\vee$, whose points have valuation $\frac{\mathrm{v}(E)}{q-1}$, so $\mathrm{v}(E) = \mathrm{v}(E')$ as required. □

REMARK 3.2.4. The above proposition implies that our results about $\mathcal{A}[\varpi^\infty]_1^{2,1}$ have an analogue for $\mathcal{A}[\varpi^\infty]_1^{1,1}$, *with the same constant* w. For example, we have an analogue of Proposition 3.1.7 and so on.

We have the map

$$\mathrm{d}\log_{\mathcal{A}} \colon \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}},$$

and also its analogue for $(\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee \cong \mathcal{A}[\varpi^\infty]_1^{1,1}$,

$$\mathrm{d}\log_{\mathcal{A}^\vee} \colon \mathrm{T}_\varpi(\mathcal{A}[\varpi^\infty]_1^{2,1}) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \to \underline{\omega}_{\mathcal{A}^\vee/R} \otimes_R \widehat{\overline{R}}.$$

Let $\cdot^*$ mean 'dual module', then we have an isomorphism of $\mathcal{G}$-modules (recall that $\mathcal{G} = \mathrm{Gal}(\overline{R}_K/R_K)$)

$$\mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \cong \mathrm{T}_\varpi(\mathcal{A}[\varpi^\infty]_1^{2,1})^*(1),$$

where $(\cdot)(1)$ means that the action of the whole $\mathcal{G}$ is twisted by the Lubin-Tate character. We define

$$a_{\mathcal{A}} := \mathrm{d}\log^*_{\mathcal{A}^\vee}(1).$$

DEFINITION 3.2.5. The *Hodge-Tate sequence of* $\mathcal{A}$ is the following sequence of $\widehat{\overline{R}}$-modules with semilinear action of $\mathcal{G}$:

$$0 \to \underline{\omega}^*_{\mathcal{A}^\vee/R} \otimes_R \widehat{\overline{R}}(1) \xrightarrow{a_{\mathcal{A}}} \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \xrightarrow{\mathrm{d}\log_{\mathcal{A}}} \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}} \to 0,$$

where the action of $\mathcal{G}$ on $\underline{\omega}_{\mathcal{A}^\vee/R}$ and $\underline{\omega}_{\mathcal{A}\vee/R}$ is trivial, and on the other objects it is the natural one.

PROPOSITION 3.2.6. *The Hodge-Tate sequence of* $\mathcal{A}$ *is a complex.*

PROOF. We need to prove that $a_{\mathcal{A}} \circ \mathrm{d}\log_{\mathcal{A}} = 0$, so it is enough to show that $\mathrm{H}^0(\widehat{\overline{R}}(-1), \mathcal{G}) = 0$. This is well known in the case of twist by the cyclotomic character (**[Bri08]**, Proposition 3.1.8), but all the theory of almost étale extensions works also in our case, see **[Fal02]**, Section 9. □

In general the Hodge-Tate sequence is not exact, we are going to prove that its homology is killed by ϖ^v.

LEMMA 3.2.7. *We have that* ϖ *is not a* 0*-divisor in* $\widehat{\overline{R}}$ *and that the natural map* $\overline{R} \to \widehat{\overline{R}}$ *is injective.*

PROOF. This is **[Bri08]**, Proposition 2.0.3. □

PROPOSITION 3.2.8. *We have that the cokernel of the map* $\mathrm{d}\log_{\mathcal{A}}$ *is killed by* ϖ^v *and* $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})$ *is a free* $\widehat{\overline{R}}$*-module of rank* 1. *Furthermore,* $\ker(\mathrm{d}\log_{\mathcal{A}})$ *is a projective* $\widehat{\overline{R}}$*-module of rank* 1.

PROOF. We rewrite the proof of [**AIS11**], Lemma 2.5, adapted to our situation. Since $\mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee)$ is a free $\mathcal{O}_{\mathcal{P}}$-module, the statement about $\ker(\mathrm{d}\log_{\mathcal{A}})$ follows from the one about $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})$. We write ω for $\omega \otimes_R 1$, it is a basis of $\underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}$. Note that

$$\mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee)/\varpi\, \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \cong (\mathcal{A}[\varpi]_1^{2,1})^\vee(\overline{R}_K)$$

and that the reduction modϖ of $\mathrm{d}\log_{\mathcal{A}}$ is the map $\mathrm{d}\log_{1,\mathcal{A}}$ extended to $\overline{R}$, that factors through $(\mathcal{C}_1^{2,1})^\vee(\overline{R}_K) \otimes_\kappa \overline{R}/\varpi\overline{R}$. By Proposition 3.1.7, the cokernel of this last map is killed by ϖ^v. Recall that γ is a generator of $(\mathcal{A}[\varpi]_1^{2,1})^\vee(\overline{R}_K)$ as κ-vector space, we write γ also for $\gamma \otimes_\kappa 1$. Let $\delta \in \overline{R}/\varpi\overline{R}$ be such that $\mathrm{d}\log_{1,\mathcal{A}}(\gamma) = \delta\omega$ and let $\tilde{\delta} \in \widehat{\overline{R}}$ be a lifting of δ. We can write

$$\varpi^v\omega = \varpi A\omega + B\delta\omega$$

for some A and B in $\widehat{\overline{R}}$, so, being $1 - \varpi^{1-v}A$ invertible, we find $s \in \widehat{\overline{R}}$ such that $s\tilde{\delta} = \varpi^v$. Let G by the $\widehat{\overline{R}}$-module generated by $\tilde{\delta}\omega$. It is free of rank 1: indeed, after inverting ϖ, we have just proved that $\tilde{\delta}$ is invertible, so any $a \in \widehat{\overline{R}}$ that kills G becomes 0 in $\widehat{\overline{R}}[\varpi^{-1}]$, hence it is 0 already in $\widehat{\overline{R}}$ again by Lemma 3.2.7. Since $\mathrm{d}\log_{\mathcal{A}}(s\tilde{\gamma}) = \varpi^v\omega$, where $\tilde{\gamma} \in \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\overline{R}}$ is any lifting of γ, we see that G contains $\varpi^v\underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}$, so the proposition follows if we show that $G = \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})$. Let $x \in \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\overline{R}}$, by definition of δ, there are C and D in $\widehat{\overline{R}}$ such that $\mathrm{d}\log_{\mathcal{A}}(x) = C\tilde{\delta}\omega + \varpi D\omega$, but this element is in G since $\varpi^v\underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}} \subseteq G$, so $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}}) \subseteq G$. Since by definition $G = \varpi G + \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})$, the conclusion follows by Nakayama's lemma ((ϖ) is contained in the Jacbson radical of $\widehat{\overline{R}}$ by completeness). $\square$

LEMMA 3.2.9. *The map* $a_{\mathcal{A}}$ *is injective.*

PROOF. By Remark 3.2.4 and Proposition 3.2.8, we know that the cokernel of $\mathrm{d}\log_{\mathcal{A}^\vee}$ is killed by ϖ^v, so the same must be true for the kernel of $a_{\mathcal{A}}$, but by Lemma 3.2.7 this implies that $\ker(a_{\mathcal{A}}) = 0$. $\square$

NOTATION. Let $\mathcal{D}_1^{2,1}$ be $\mathcal{C}_1^{2,1}(\overline{R}_K)^\perp$. From now on we will omit $(\overline{R}_K)$ in the notation, it should be clear from the context whether we are talking about the group scheme or about the group of points. We also write $\overline{R}_z$ for $\overline{R}/\varpi^z\overline{R}$ (and similarly for other objects). Note that, by functoriality of $\mathrm{d}\log_{\mathcal{A}}$, we have that $\ker(\mathrm{d}\log_{\mathcal{A}})$ and $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})$ are $\mathcal{G}$-modules, and similarly for $\mathrm{d}\log_{\mathcal{A}^\vee}$.

LEMMA 3.2.10. *We have a commutative diagram, with exact bottom row:*

$$\begin{tikzcd}
& \underline{\omega}^*_{\mathcal{A}^\vee/R} \otimes_R \overline{R}_1(1) \arrow[r] \arrow[d] & \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_\kappa \overline{R}_1 \arrow[r] \arrow[d, equal] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_1 & \\
0 \arrow[r] & \mathcal{D}_1^{2,1} \otimes_\kappa \overline{R}_1 \arrow[r] & (\mathcal{A}[\varpi]_1^{2,1})^\vee \otimes_\kappa \overline{R}_1 \arrow[r] & (\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_1 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Furthermore we have an isomorphism $\ker(\mathrm{d}\log_{\mathcal{A}}) \cong \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee})^*(1)$.

PROOF. This is the second step of the proof of Proposition 2.4 in [**AIS11**]. By Remark 3.2.4 and Proposition 3.2.8, we have that $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee})$ is a free $\widehat{\overline{R}}$-module of rank 1, and $\ker(\mathrm{d}\log_{\mathcal{A}^\vee})$ is a projective $\widehat{\overline{R}}$-module of rank 1. The exactness of the bottom row is clear, and the right vertical map is given by Remark 3.1.6. Let h be the isomorphism

$$h\colon \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \to \mathrm{T}_\varpi(\mathcal{A}[\varpi^\infty]_1^{2,1})^* \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}}(1).$$

By Remark 3.2.2, we have that

$$h(\ker(\mathrm{d}\log_\mathcal{A})) \subseteq \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee})^*(1),$$

in particular we obtain a morphism $h'\colon \mathrm{Im}(\mathrm{d}\log_\mathcal{A}) \to \ker(\mathrm{d}\log_{\mathcal{A}^\vee})^*(1)$. The natural map

$$\mathrm{T}_\varpi(\mathcal{A}[\varpi^\infty]_1^{2,1})^* \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}}(1) \to \ker(\mathrm{d}\log_{\mathcal{A}^\vee})^*(1)$$

is surjective, because $\ker(\mathrm{d}\log_{\mathcal{A}^\vee})$ is a projective module, so h' is surjective, hence it is an isomorphism. It follows that h gives the required isomorphism $\ker(\mathrm{d}\log_\mathcal{A}) \cong \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee})^*(1)$. Let i be the natural map $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee}) \to \underline{\omega}_{\mathcal{A}^\vee} \otimes_R \widehat{\overline{R}}$, we have

$$i^* \otimes_{\widehat{\overline{R}}} \overline{R}_1(1)\colon \underline{\omega}^*_{\mathcal{A}^\vee/R} \otimes_R \overline{R}_1(1) \to \ker(\mathrm{d}\log_\mathcal{A}) \otimes_\kappa \overline{R}_1.$$

Since by Proposition 3.1.7, we have a map

$$\ker(\mathrm{d}\log_\mathcal{A})/\varpi\ker(\mathrm{d}\log_\mathcal{A}) \to \mathcal{D}_1^{2,1} \otimes_\kappa \overline{R}_1,$$

we obtain, by composition, the left vertical map. The lemma follows. □

THEOREM 3.2.11. *The homology of the Hodge-Tate sequence is killed by ϖ^v, and we have a commutative diagram of $\mathcal{G}$-modules, with exact rows and vertical isomorphisms,*

$$\begin{tikzcd}
0 \arrow[r] & \ker(\mathrm{d}\log_\mathcal{A})_{1-v} \arrow[r] \arrow[d, "\wr"] & \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{1-v} \arrow[r] \arrow[d, equal] & \mathrm{Im}(\mathrm{d}\log_\mathcal{A})_{1-v} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{D}_1^{2,1} \otimes_\kappa \overline{R}_{1-v} \arrow[r] & (\mathcal{A}[\varpi]_1^{2,1})^\vee \otimes_\kappa \overline{R}_{1-v} \arrow[r] & (\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_{1-v} \arrow[r] & 0
\end{tikzcd}$$

Furthermore, $\ker(\mathrm{d}\log_\mathcal{A})$ *is a free* $\widehat{\overline{R}}$*-module of rank* 1.

PROOF. This is the third step of the proof of Proposition 2.4 in [**AIS11**]. By Remark 3.2.4 and Proposition 3.2.8, we have $\varpi^v(\underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}) \subseteq \mathrm{Im}(\mathrm{d}\log_{\mathcal{A}^\vee})$, so, by Lemma 3.2.10, we have $\varpi^v \ker(\mathrm{d}\log_\mathcal{A}) \subseteq \underline{\omega}^*_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}(1) \subseteq \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}}$, where the last inclusion is Lemma 3.2.9. By Lemma 3.2.10, we have that $\underline{\omega}^*_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}(1)$ goes to 0 in $(\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_1$, so the same must be true for $\varpi^v \ker(\mathrm{d}\log_\mathcal{A})$. But $(\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_1$ is a free $\overline{R}_1$-module and the kernel of the multiplication by ϖ^v in $\overline{R}_1$ is $\varpi^{1-v}\overline{R}/\varpi\overline{R}$ by Lemma 3.2.7, so $\ker(\mathrm{d}\log_\mathcal{A})$ goes to $\varpi^{1-v}(\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_1$, and hence is 0 modulo ϖ^{1-v}. This gives the map

$$\mathrm{Im}(\mathrm{d}\log_\mathcal{A})_{1-v} \to (\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_{1-v},$$

that is a surjective morphism between free modules of the same rank, so it is an isomorphism. Being $\mathrm{Im}(\mathrm{d}\log_\mathcal{A})$ free, we have, non canonically, $\mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \cong \ker(\mathrm{d}\log_\mathcal{A}) \oplus \mathrm{Im}(\mathrm{d}\log_\mathcal{A})$ (this decomposition is not $\mathcal{G}$-equivariant), so the map $\ker(\mathrm{d}\log_\mathcal{A})_{1-v} \to \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{1-v}$ is injective and its image in $(\mathcal{A}[\varpi]_1^{2,1})^\vee \otimes_\kappa \overline{R}_{1-v}$ must be $\mathcal{D}_1^{2,1} \otimes_\kappa \overline{R}_{1-v}$, this gives the required diagram. Since $\ker(\mathrm{d}\log_\mathcal{A})$ is a projective $\widehat{\overline{R}}$-module by Proposition 3.2.8 and its reduction modulo ϖ^{1-v} is free, we obtain that it must be free. It remains only to show that

$\ker(\mathrm{d}\log_{\mathcal{A}})/\operatorname{Im}(a_{\mathcal{A}})$ is killed by ϖ^v. This quotient is isomorphic, as $\widehat{\overline{R}}$-module, to $\widehat{\overline{R}}/a\widehat{\overline{R}}$, for some $a \in \widehat{\overline{R}}$ and we need to prove that ϖ^v is a multiple of a. But, by the part of the theorem proved so far, we have $\varpi^v + x\varpi^{1-v} = ya$ for some x and y in $\widehat{\overline{R}}$, the conclusion follows. □

PROPOSITION 3.2.12. *The Hodge-Tate sequence is exact if and only if $\mathcal{A}$ is ordinary.*

PROOF. This follows by Remark 3.1.8 (see also the calculations made in the proof of Proposition 3.1.7). □

We now work over S. Let us briefly recall the notations introduced in the proof of Proposition 3.2.8: δ is an element of $\overline{R}_1$ that satisfies $\mathrm{d}\log_{1,\mathcal{A}}(\gamma) = \delta\omega$ and $\tilde{\delta} \in \widehat{\overline{R}}$ is a lifting of δ. Since γ is defined over S, we see that we can assume $\delta \in S/\varpi S$ and $\tilde{\delta} \in S$. We write $\mathcal{H}$ for $\mathrm{Gal}(\overline{R}_K/S_K)$: it is a subgroup of $\mathcal{G}$.

PROPOSITION 3.2.13. *Let $\mathcal{F}(S) \subseteq \underline{\omega}_{\mathcal{A}/R} \otimes_R S$ be the submodule generated by $\tilde{\delta}\omega \otimes 1$.*

(1) *We have that $\mathcal{F}(S)$ is a free S-module of rank 1, with basis $\tilde{\delta}\omega$ and $\mathcal{F}(S)\otimes_S \widehat{\overline{R}} \cong \operatorname{Im}(\mathrm{d}\log_A)$;*
(2) *the S-module $\operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})^{\mathcal{H}}$ is equal to $\mathcal{F}(S)$;*
(3) *there is an isomorphism $\mathcal{F}(S)_{1-v} \cong (\mathcal{C}_1^{2,1})^\vee \otimes_\kappa S_{1-v}$, its base change to $\widehat{\overline{R}}$ gives, via $\mathcal{F}(S) \otimes_S \widehat{\overline{R}} \cong \operatorname{Im}(\mathrm{d}\log_A)$, the isomorphism of Theorem 3.2.11;*
(4) *there is an isomorphism $\mathcal{F}(S)^*(1) \otimes_S \widehat{\overline{R}} \cong \ker(\mathrm{d}\log_{\mathcal{A}})$.*

Furthermore, all the above isomorphisms are $\mathcal{H}$-equivariant.

PROOF. This is the analogue of Proposition 2.6 of [**AIS11**].

(1) The fact that $\mathcal{F}(S)$ is free with basis $\tilde{\delta}\omega$ is proved in exactly the same way of Proposition 3.2.8, where it is also shown that $\operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})$ is the $\widehat{\overline{R}}$-submodule of $\underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}$ generated by $\tilde{\delta}\omega$ as required;
(2) by part (1), we have that $\mathcal{F}(S) \subseteq \operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})^{\mathcal{H}}$. Any $x \in \operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})$ can be written as $x = a\delta\omega$, for some $a \in \widehat{\overline{R}}$. Since $\delta \in S$, we see that $\delta\omega$ is invariant under $\mathcal{H}$, so if x is invariant under the action of $\mathcal{H}$, we have that $a \in \widehat{\overline{R}}^{\mathcal{H}} = S$, where the last equality follows from [**Bri08**];
(3) this follows from the fact that the map $\operatorname{Im}(\mathrm{d}\log_{\mathcal{A}}) \to (\mathcal{C}_1^{2,1})^\vee \otimes_\kappa \overline{R}_{1-v}$ sends $\tilde{\delta}\omega$ to the reduction of γ;
(4) by taking the $\mathcal{H}$-invariants of $\ker(\mathrm{d}\log_{\mathcal{A}}) \cong \operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})^*(1)$ we obtain the isomorphism $\ker(\mathrm{d}\log_{\mathcal{A}})^{\mathcal{H}} \cong \mathcal{F}(S)^*(1)$, since $\ker(\mathrm{d}\log_{\mathcal{A}})$ is a free $\widehat{\overline{R}}$-module, the conclusion follows.

□

The following lemma will be used in the next chapter.

LEMMA 3.2.14. *Let $\mathrm{Spf}(R')$ be a small affine of $\mathfrak{M}(H)(w)$ and suppose that R' is an R-algebra. We write $\mathcal{A}'$ for the base change of $\mathcal{A}$ to R'. Let $\mathrm{Spf}(S')$ be the inverse image of $\mathrm{Spf}(R')$ under the map $\mathfrak{M}(H\varpi)(w) \to \mathfrak{M}(H)(w)$, then we have a natural isomorphism $\mathcal{F}(S) \otimes_S S' \cong \mathcal{F}(S')$, compatible with $\underline{\omega}_{\mathcal{A}/R} \otimes_R R' \cong \underline{\omega}_{\mathcal{A}'/R'}$.*

PROOF. By functoriality of $\mathrm{d}\log$, we have a natural morphism

$$\operatorname{Im}(\mathrm{d}\log_{\mathcal{A}}) \otimes_{\widehat{\overline{R}}} \widehat{\overline{R'}} \to \operatorname{Im}(\mathrm{d}\log_{\mathcal{A}'}),$$

that is compatible with the isomorphism $\underline{\omega}_{\mathcal{A}/R} \otimes_R R' \cong \underline{\omega}_{\mathcal{A}'/R'}$. Taking Galois invariants we obtain, by Proposition 3.2.13, a morphism

$$\mathcal{F}(S) \otimes_S S' \to \mathcal{F}(S'),$$

that is an isomorphism modulo ϖ^{1-v} by Theorem 3.2.11. The lemma follows since both $\mathcal{F}(S) \otimes_S S'$ and $\mathcal{F}(S')$ are free S'-modules of rank 1. $\square$

CHAPTER 4

Modular forms of non-integral weight

This chapter, together with the next one, is the hearth of the thesis. For any continuous morphism $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to K^*$, we define a sheaf Ω_w^χ on $\mathfrak{M}(H)(w)$. Using Theorem 3.2.11, we prove that Ω_w^χ is locally free of rank 1. Furthermore, if $\chi(t) = t^k$ for some integer k, we have that the rigidifications of Ω_w^χ and of $\underline{\omega}^{\otimes k}$ are naturally isomorphic. This allows us to the define the notion of overconvergent modular form of any weight. We prove that the Ω_w^χ's, for various χ, can be put in p-adic analytic families.

4.1. Generalities about continuous characters

In order to define the sheaf Ω_w^χ, we need some technical results about the space of continuous characters we are interested in. Since $\mathcal{O}_{\mathcal{P}}$ can be ramified, this space is slightly more complicated then the classical one, where $\mathcal{O}_{\mathcal{P}} = \mathbb{Z}_p$ (see in particular Definition 4.1.1).

We assume that $e \leq p-1$ (but see Section 4.5 for an explanation of what can be done without this assumption). Note that in this case the p-adic logarithm gives an isometric isomorphism

$$(1+\varpi\mathcal{O}_{\mathcal{P}}) \cong (1+\varpi)^{\mathcal{O}_{\mathcal{P}}},$$

where the first group is denoted multiplicatively and the latter additively.

Recall that $\varpi^e = p$ and that the valuation $\mathrm{v}(\cdot)$ is normalized in such a way that $\mathrm{v}(\varpi) = 1$. We have an isomorphism of topological groups

$$\mathcal{O}_{\mathcal{P}}^* \cong \mu_{q-1} \times (1+\varpi\mathcal{O}_{\mathcal{P}}).$$

Notation. Let $t \in \mathcal{O}_{\mathcal{P}}^*$. We will use the following notation:

- $[t]$ means $[\cdot]$ applied to the reduction of t modulo ϖ (recall that $[\cdot]\colon$ is the Teichmüller character);
- $\langle t\rangle := t/[t]$.

Let A be any affinoid $\mathcal{O}_{\mathcal{P}}$-algebra and let χ be a continuous morphism of multiplicative groups $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to A^*$. Note that $\chi(1+\varpi) = 1+a$, where $a \in A$ is topologically nilpotent (so $\log(1+a) \in A$ makes sense). Being $\mathcal{O}_{\mathcal{P}}$ a free $\mathbb{Z}_p$-module of rank d, it is not always the case that χ is uniquely defined by $\chi_{|\mu_{q-1}}$ and $\chi(1+\varpi)$. To avoid these problems, we will consider only characters as in the following

Definition 4.1.1. We say that any χ as above is a *continuous character* if the $\mathbb{Z}_p$-linear map

$$\mathcal{O}_{\mathcal{P}} \to A$$
$$t \mapsto \log(\chi((1+p)^t))$$

is also $\mathcal{O}_{\mathcal{P}}$-linear.

Remark 4.1.2. By definition, if χ is a continuous character, then $\chi_{|1+\varpi\mathcal{O}_{\mathcal{P}}}$ is completely defined by $\chi(1+\varpi)$. If $\mathcal{O}_{\mathcal{P}} = \mathbb{Z}_p$ then our notion of continuous character coincides with the usual one. We use this notion in order to take into account the action of $\mathcal{O}_\mathcal{P}$ that we have on $1+\varpi\mathcal{O}_\mathcal{P}$.

DEFINITION 4.1.3. If $s \in \mathbb{C}_p$ satisfies

$$\mathrm{v}(s) > \frac{e}{p-1} - 1$$

we say that s is *good*. This is equivalent to require that $(1+\varpi x)^s := \exp(s\log(1+\varpi x))$ is defined for any $x \in \mathbb{C}_p$. More generally, for any integer $r \geq 1$, we say that s is r-good if it satisfies

$$\mathrm{v}(s) > \frac{e}{p-1} - r.$$

This is equivalent to require that $(1+\varpi^r x)^s$ is defined for any $x \in \mathbb{C}_p$.

DEFINITION 4.1.4. Following **[CM98]**, Section 1.4, we say that a continuous character $\chi\colon \mathcal{O}_\mathcal{P}^* \to K^*$ is *accessible* if it is of the form

$$\begin{aligned} \chi\colon \mathcal{O}_\mathcal{P}^* &\to K^* \\ t &\mapsto [t]^i \langle t\rangle^s \end{aligned}$$

where:

- $i \in \mathbb{Z}/(q-1)\mathbb{Z}$;
- $s \in K$ is good.

In this case we write $\chi = (s,i)$. Given an integer k, we view it as the almost accessible character $t \mapsto t^k$, that is (k,k) in the above notation.

DEFINITION 4.1.5. Let $\chi\colon \mathcal{O}_\mathcal{P}^* \to K^*$ be any continuous character, that in general will not be almost accessible. We say that χ is *r-accessible*, where $r \geq 1$ is an integer, if $\mathrm{v}(\langle t\rangle - 1) \geq r$ implies that $\chi(t) = [t]^i\langle t\rangle^s$ for some $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ and some r-good $s \in K$.

REMARK 4.1.6. Note that accessible means 1-accessible and that any continuous character is r-accessible for some r.

Let $\mathcal{W}$ be the *weight space* for our continuous characters: it is an $F_\mathcal{P}$-rigid analytic space whose A-points, for any $F_\mathcal{P}$-affinoid algebra A, are

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_\mathcal{P}^*, A^*).$$

It is possible to give a more concrete description of $\mathcal{W}$: by the above discussion, we see that there is a natural bijection between the set of connected components of $\mathcal{W}$ and $\mathrm{Hom}(\mu_{q-1}(K), K^*) = \mathbb{Z}/(q-1)\mathbb{Z}$. Let $\mathcal{B}$ be the component corresponding to the identity. It follows that we have an isomorphism of rigid spaces

$$\mathcal{W} = \coprod_{\mathbb{Z}/(q-1)\mathbb{Z}} \mathcal{B}.$$

Let χ be in $\mathcal{B}(\mathbb{C}_p)$. By our definition of continuous character, we have that the map $\chi \mapsto \chi(1+\varpi) - 1$ gives an isomorphism of rigid analytic spaces

$$\mathcal{B} \cong \mathring{\mathbb{D}}(0,1),$$

where $\mathring{\mathbb{D}}(0,1)$ is the open disc of radius 1 centered in the origin.

Let t_1 be $|\varpi|^{\frac{e}{p-1}}$, and, given an integer $r \geq 2$, we define t_r by the following condition: for $x \in \mathbb{C}_p$, we have $|x| < t_r$ if and only if $|y| < t_1$, where y is any element of $\mathbb{C}_p$ that satisfies $|\log(y)| = \left|\frac{\log(1+\varpi^r)}{\log(1+\varpi)}\log(x)\right|$. We have $t_r \to 1$ as $r \to \infty$. Let $\mathcal{B}_r$ be the open ball of radius t_r. For $r \geq 1$, we fix $\mathcal{D}_r$, a closed (hence affinoid) ball such that $\mathcal{B}_{r-1} \subset \mathcal{D}_r \subset \mathcal{B}_r$. We write $\mathcal{W}_r$ for $\coprod_{\mathbb{Z}/(q-1)\mathbb{Z}} \mathcal{D}_r$. Note that each $\mathcal{W}_r$ is an affinoid subdomain of $\mathcal{W}$ and that $\{\mathcal{W}_r\}_{r\geq 1}$ gives an admissible covering of $\mathcal{W}$.

LEMMA 4.1.7. *Any $\chi \in \mathcal{W}_r(K)$ is an r-admissible character.*

PROOF. We may assume $\chi \in \mathcal{D}_r(K) \subseteq \mathcal{B}_r(K)$. In this case it is enough to take

$$s := \frac{\log(\chi(1+\varpi^r))}{\log(1+\varpi^r)}.$$

□

REMARK 4.1.8. The definition of the radius of $\mathcal{B}_r$ is quite complicated, and not totally explicit. This is not important for us, the only thing to keep in mind is that $\mathcal{W}_r$ is an affinoid subdomain of $\mathcal{W}$ and that $\{\mathcal{W}_r\}_{r\geq 0}$ is an admissible covering of $\mathcal{W}$. In particular, any character $\chi \in \mathcal{W}(K)$ lies in some $\mathcal{W}_r(K)$. Furthermore we know that any $\chi \in \mathcal{W}_r(K)$ is r-admissible.

4.2. The sheaves Ω_w^χ for accessible characters

We start by defining the sheaves Ω_w^χ assuming that χ is accessible. This allows us to define p-adic modular forms of weight χ. We compare modular forms of level $K(H\varpi)$ with modular forms of level $K(H)$. Finally, we prove that the definition of p-adic modular forms given in the previous chapters is a particular case of the one given below, and we also reformulate the definition using 'test objects'.

We start working over $\mathfrak{M}(H\varpi)(w)$, but later on we will need to consider curves of higher level. We write ϑ for the natural morphism

$$\vartheta\colon \mathfrak{M}(H\varpi)(w) \to \mathfrak{M}(H)(w).$$

If $\mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$ is as in the previous chapters, we have $\mathcal{F}(S)$, that is a free S-module of rank 1, contained in $\underline{\omega}_{\mathcal{A}/R} \otimes_R S$ (see Proposition 3.2.13).

DEFINITION 4.2.1. We write $\mathcal{F}$ for the unique locally free $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$-module of rank 1 that satisfies $\mathcal{F}(\mathrm{Spf}(S)) = \mathcal{F}(S)$, for $\mathrm{Spf}(S)$ an open affine of $\mathfrak{M}(H\varpi)(w)$ as above.

Recall that at the end of Section 3.1 we have defined γ, that is a non trivial canonical S_K-point of $(\mathcal{C}_1^{2,1})^\vee$. Let $(\mathfrak{C}_1^{2,1})^\vee$ be the sheaf on $\mathfrak{M}(H\varpi)(w)$ associated to the various groups $(\mathcal{C}_1^{2,1})^\vee$ (here $(\mathcal{C}_1^{2,1})^\vee$ should be thought as group of points). The various γ glue together, to define a global section of $(\mathfrak{C}_1^{2,1})^\vee$, denoted again by γ. By Theorem 3.2.11, we have an isomorphism of sheaves

$$\mathcal{F}/\varpi^{1-v}\mathcal{F} \cong (\mathfrak{C}_1^{2,1})^\vee \otimes_\kappa \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}/\varpi^{1-v}\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}.$$

DEFINITION 4.2.2. Using the above isomorphism, we define $\mathcal{F}'_v$ as the inverse image of the constant sheaf of sets $(\mathfrak{C}_1^{2,1})^\vee \setminus \{0\}$ under the natural map $\mathcal{F} \to \mathcal{F}/\varpi^{1-v}\mathcal{F}$. We will write $\mathcal{F}''_v$ for the inverse image of the sheaf $\{\gamma \otimes 1\}$.

We can give a more concrete description of this sheaf using Proposition 3.1.10.

LEMMA 4.2.3. *Let $\mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$ be an open affine, with associated abelian scheme $\mathcal{A} \to \mathrm{Spec}(S)$, and assume that $\underline{\omega}_{\mathcal{A}/S}$ is free, generated by ω. Then we have that $\mathcal{F}(\mathrm{Spf}(S))$ is free, and $\omega^{\mathrm{std}} := E_{1|\mathrm{Spf}(S)}$ gives a basis.*

PROOF. We can write $E_{1|\mathrm{Spf}(S)} = E^{1/(q-1)}\omega$, for some $E \in S$ (we use this notation to remain coherent with the previous chapters). To prove that $E_{1|\mathrm{Spf}(S)}$ is a basis of $\mathcal{F}(\mathrm{Spf}(S))$, we can work locally, so we can assume that $\mathrm{Spf}(S)$ is the inverse image of $\mathrm{Spf}(R) \to \mathfrak{M}(H)(w)$, with R small. We write $\mathcal{A}$ also for the abelian scheme over $\mathrm{Spec}(R)$. We use the notations of the proof of Proposition 3.2.8. In that proof, we have shown that $\mathcal{F}(S)$ is generated by $\tilde{\delta}\omega$. By Proposition 3.1.10, we have $\tilde{\delta} \equiv E^{1/(q-1)} \bmod \varpi^{1-w}$, so there is A in $\widehat{\overline{R}}$ such that $E^{1/(q-1)} = \tilde{\delta} + \varpi^{1-w}A$. Since $\mathcal{F}(S)$ contains $\varpi^v(\underline{\omega}_{\mathcal{A}/R}\otimes_R\widehat{\overline{R}})$ and $1-w \geq v$, we have that $E^{1/(q-1)}\omega \in \mathcal{F}(S)$.

Let G be the submodule of $\underline{\omega}_{\mathcal{A}/S}$ generated by $E^{1/(q-1)}\omega$. It remains to show that $\tilde{\delta}\omega \in G$. Since $E^{1/(q-1)} = \tilde{\delta} + \varpi^{1-w}A$, we have $\mathcal{F}(S) = \varpi^{1-v}\mathcal{F}(S) + G$, the claim follows by Nakayama's lemma. □

COROLLARY 4.2.4. *We have that $\mathcal{F}$ is a* free *$\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$-module of rank 1, with ω^{std} as basis.*

DEFINITION 4.2.5. Let $\mathcal{S}_v$ be the sheaf of abelian groups, on $\mathfrak{M}(H\varpi)(w)$, defined by

$$\mathcal{S}_v := \mathcal{O}_{\mathcal{P}}^*(1 + \varpi^{1-v}\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}).$$

Similarly, we define the sheaf $\mathcal{S}''_v$ as

$$\mathcal{S}''_v := 1 + \varpi^{1-v}\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}.$$

PROPOSITION 4.2.6. *We have that $\mathcal{F}'_v$ is a Zariski $\mathcal{S}_v$-torsor, and that $\mathcal{F}''_v$ is a Zariski $\mathcal{S}''_v$-torsor. Furthermore, both these torsors are generated by ω^{std}. Furthermore, via the inclusion $\mathcal{S}''_v \subseteq \mathcal{S}_v$, we have that $\mathcal{F}'_v$ is the pushed-out torsor $\mathcal{F}''_v \times^{\mathcal{S}''_v} \mathcal{S}_v$.*

PROOF. This follows from part (3) of Proposition 3.2.13 and Lemma 4.2.3. □

We are now ready to consider more general weights.

The morphism $\vartheta^{\mathrm{rig}} \colon \mathfrak{M}(H\varpi)(w)^{\mathrm{rig}} \to \mathfrak{M}(H)(w)^{\mathrm{rig}}$ is finite and étale, and its Galois group is canonically identified with κ^*. Furthermore, the action of κ^* on ϑ^{rig} extends to an action on ϑ (this follows from the moduli theoretic description of $\mathfrak{M}(H\varpi)(w)$ and $\mathfrak{M}(H)(w)$). If $c \in \mathcal{O}_{\mathcal{P}}^*$ we will write $\bar{c}$ for the image of c in κ^*, and we view $\mathcal{O}_{\mathcal{P}}^*$ acting on ϑ via the projection.

Throughout this section, we fix an accessible character $\chi = (s, i)$. We will assume that

$$w < (q-1)\left(\mathrm{v}(s) + \mathrm{v}(\log(1+\varpi)) - \frac{e}{p-1}\right).$$

Let x be a local section of $\mathcal{S}_v$ over $\mathfrak{V} = \mathrm{Spf}(S)$. We can write $x = ub$, where u is a section of $\mathcal{O}_{\mathcal{P}}^*$ and b is a section of $1 + \varpi^{1-v}\mathcal{O}_{\mathfrak{V}}$. Note that $b^s = \exp(s\log(b))$ makes sense thanks to our assumption on w. We set $x^\chi := \chi(u)b^s$, that is another section of $\mathcal{S}_v$. Note that, if $t \in 1 + p\mathcal{O}_{\mathcal{P}}$, we have $\chi(t) = t^s$, so x^χ is well defined, in the sense that it depends only on the product ub. We will write $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi)}$ for $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$ with the action of $\mathcal{S}_v$ by multiplication, twisted by χ.

We have a natural action, by multiplication, of $\mathcal{S}_v$ on $\mathcal{F}'_v$. In particular we can consider the sheaf

$$\tilde{\Omega}_w^\chi := \mathscr{H}om_{\mathcal{S}_v}(\mathcal{F}'_v, \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}),$$

where $\mathscr{H}om_{\mathcal{S}_v}(\cdot,\cdot)$ means homomorphisms of sheaves with an action of $\mathcal{S}_v$. By Proposition 4.2.6, we have that $\tilde{\Omega}_w^\chi$ is an invertible sheaf of $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$-modules.

REMARK 4.2.7. Similarly to the case of $\mathcal{S}_v$, we have that $\mathcal{S}''_v$ acts both on $\mathcal{F}''_v$ and on $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}$. Note that the action of $\mathcal{S}''_v$ on $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}$ depends only on s and not on the whole $\chi = (s, i)$. Since $\mathcal{F}'_v \cong \mathcal{F}''_v \times^{\mathcal{S}''_v} \mathcal{S}_v$, we have a natural isomorphism of sheaves

$$\tilde{\Omega}_w^\chi \cong \mathscr{H}om_{\mathcal{S}''_v}(\mathcal{F}''_v, \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}).$$

It follows that, to specify f, a global section of $\tilde{\Omega}_w^\chi$, it is enough to give $f(\omega^{\mathrm{std}})$.

Since κ^* acts on $(\mathfrak{C}_1^{2,1})^\vee\backslash\{0\}$, we have an action of κ^* on $\mathcal{F}'_v$ compatible with the action of κ^*, by pullbacks, on $\underline{\omega}_{K(H\varpi)}$. Let a be in κ^*. We can write $a=(a,a^\sharp)$, viewing it as the morphism of ringed spaces $a\colon \mathfrak{M}(H\varpi)(w)\to\mathfrak{M}(H\varpi)(w)$. We have that $a^\sharp$ is a morphism

$$a^\sharp\colon \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)}\to a_*\mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)}.$$

Taking ϑ_* on both sides, we obtain, since $\vartheta=\vartheta\circ a$, an action of κ^* on $\vartheta_*\mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)}$. All these actions extend to actions of $\mathcal{O}_{\mathcal{P}}^*$, via the projection.

Since ϑ is finite, we have that $\vartheta_*\tilde{\Omega}_w^\chi$ is a coherent sheaf of $\mathcal{O}_{\mathfrak{M}(H)(w)}$-modules. The action of κ^* on $\mathcal{F}'_v$ and on $\vartheta_*\mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)}$ gives an action of κ^* on $\vartheta_*\tilde{\Omega}_w^\chi$. Explicitly, suppose that $a\in\kappa^*$ and that $f\colon \mathcal{F}'_{v|\mathfrak{V}}\to\mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)_{|\mathfrak{V}}}$, where $\mathfrak{V}=\vartheta^{-1}(\mathfrak{U})$, for some open $\mathfrak{U}\subseteq\mathfrak{M}(H)(w)$. Then af is the map given by the following diagram

$$\begin{tikzcd}
\mathcal{F}'_{v|\mathfrak{V}} \arrow[d, "a^{-1}"] & \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)_{|\mathfrak{V}}} \\
\mathcal{F}'_{v|\mathfrak{V}} \arrow[r, "f"] & \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi)(w)_{|\mathfrak{V}}} \arrow[u, "a"']
\end{tikzcd}$$

Note that κ^* acts on $\vartheta_*\tilde{\Omega}_w^\chi$ by automorphisms of $\mathcal{O}_{\mathfrak{M}(H)(w)}$-modules. In particular, we have an action of κ^* on the global section of $\tilde{\Omega}_w^\chi$. We will write this action by $f\mapsto f_{|\langle a\rangle}$, for $a\in\kappa^*$. These operators will be called *diamond operators*. As above, we let $\mathcal{O}_{\mathcal{P}}^*$ act via the natural projection.

Definition 4.2.8. We define the sheaf $\Omega_w^\chi=\Omega_w^{(s,i)}$ on $\mathfrak{M}(H)(w)$ as

$$\Omega_w^\chi := \left(\vartheta_*\tilde{\Omega}_w^\chi\right)^{\kappa^*}.$$

Let $\mathfrak{V}=\mathrm{Spf}(S)\to\mathfrak{M}(H\varpi)(w)$ be an open affine. We will write $X_{\chi,v}$ for the unique element of $\tilde{\Omega}_w^\chi(\mathfrak{V})$ (see Remark 4.2.7) that satisfies

$$X_{\chi,v}(b\omega^{\mathrm{std}})=b^{-s},$$

for all $b\in\mathcal{S}''_v(\mathfrak{V})=1+\varpi^{1-v}S$. For various $\mathfrak{V}$'s, the $X_{\chi,v}$'s glue together, so we obtain a global section of $\tilde{\Omega}_w^\chi$, denoted again $X_{\chi,v}$.

Lemma 4.2.9. *We have that $\tilde{\Omega}_w^\chi$ is a* free *$\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$-module of rank* 1*, with $X_{\chi,v}$ as basis.*

Proof. This follows from Lemma 4.2.3 and Proposition 4.2.6. □

Remark 4.2.10. Let $\chi'=(s,j)$ be another accessible character (note that we have the same s for χ and χ'). We have a canonical isomorphism

$$\beta_{\chi,\chi'}\colon \tilde{\Omega}_w^\chi \xrightarrow{\sim} \tilde{\Omega}_w^{\chi'},$$

that sends $X_{\chi,v}$ to $X_{\chi',v}$. This isomorphism does not respect the action of κ^*, but we have that $\beta_{\chi,\chi'}$ induces an isomorphism $\tilde{\Omega}_w^\chi\cong\tilde{\Omega}_w^{\chi'}[j-i]$. Here, by $\tilde{\Omega}_w^{\chi'}[j-i]$ we mean $\tilde{\Omega}_w^{\chi'}$ with the action of κ^* twisted by $[\cdot]^{j-i}$.

We can now make the definition of p-adic modular forms of weight χ, of level $K(H\varpi)$.

DEFINITION 4.2.11. We define the space of ϖ-adic modular forms with respect to D, level $K(H\varpi)$, weight χ and growth condition w, with coefficients in K, as

$$S^D(K, w, K(H\varpi), \chi) := \mathrm{H}^0(\mathfrak{M}(H\varpi)(w), \tilde{\Omega}_w^\chi)_K.$$

Note that, by Lemma 4.2.21 below, we have

$$S^D(K, w, K(H\varpi), \chi) = S^D(V, w, K(H\varpi), \chi)_K$$

in the case χ is an integer.

We now consider modular forms of level $K(H)$.

PROPOSITION 4.2.12. *There is a canonical and κ^*-equivariant isomorphism of $\mathcal{O}_{\mathfrak{M}(H)(w)}$-modules*

$$\vartheta_* \tilde{\Omega}_w^\chi = \bigoplus_{j \in \mathbb{Z}/(q-1)\mathbb{Z}} \Omega_w^{(s,j)},$$

such that $\Omega_w^{(s,j)}$ is identified with the submodule of $\vartheta_* \tilde{\Omega}_w^\chi$ on which κ^* acts via multiplication by $[\cdot]^{j-i}$.

PROOF. This is the analogue of Lemma 3.3 of [**AIS11**]. Recall that, by Remark 4.2.10, we have a canonical and κ^*-equivariant isomorphism $\tilde{\Omega}_w^{(s,j)} \cong \tilde{\Omega}_w^\chi[i-j]$. Hence $\Omega_w^{(s,j)}$ consists of the κ^*-invariants of $\vartheta_* \tilde{\Omega}_w^\chi[i-j]$, so it is the submodule of $\vartheta_* \tilde{\Omega}_w^\chi$ where κ^* acts via $[\cdot]^{j-i}$. The order of κ^* is $q-1$, that is invertible in all our rings, so $\vartheta_* \tilde{\Omega}_w^\chi$ can be decomposed, locally on $\mathfrak{M}(H)(w)$, as the direct sum of eigenspace of κ^*. Since the only characters $\kappa^* \to \mathcal{O}_{\mathcal{P}}^*$ are of the form $[\cdot]^j$, for some $j \in \mathbb{Z}/(q-1)\mathbb{Z}$, the action is the one described in the proposition. This concludes the proof. $\square$

REMARK 4.2.13. From now on we will use the above proposition to tacitly identify $\Omega_w^{(s,j)}$ with the submodule of $\vartheta_* \tilde{\Omega}_w^\chi$ on which κ^* acts via $[\cdot]^{j-i}$.

COROLLARY 4.2.14. *The rigidification of $\Omega_w^{(s,i)}$ is an invertible sheaf. Furthermore we have a decomposition*

$$S^D(K, w, K(H\varpi), \chi) = \bigoplus_{j \in \mathbb{Z}/(q-1)\mathbb{Z}} \mathrm{H}^0(\mathfrak{M}(H)(w), \Omega_w^{(s,j)})_K.$$

PROOF. The first statement is a consequence of the fact that ϑ^{rig} is finite and étale with Galois group κ^*. The corollary follows from Proposition 4.2.12. $\square$

REMARK 4.2.15. Since $\tilde{\Omega}_w^\chi$ is locally free, we have that $\vartheta_* \tilde{\Omega}_w^\chi$ is a reflexive sheaf of modules (see [**Har80**] for the basic properties of reflexive sheaves). It follows that Ω_w^χ is also reflexive. Suppose now that ϖ^w is a uniformizer of V. Since $\mathfrak{M}(H)(w)$ is, locally, of the form $\mathrm{Spf}(R\langle X, Y\rangle/(XY - \varpi^w))$, we have that $\mathfrak{M}(H)(w)$ is regular. It follows by [**Har80**], Corollary 1.4, that Ω_w^χ is locally free.

By Proposition 4.2.12, we see that any modular form of level $K(H\varpi)$ and weight χ has components that can be identified with global sections of $\Omega_w^{(s,j)}$, for various $j \in \mathbb{Z}/(q-1)\mathbb{Z}$.

DEFINITION 4.2.16. We define the space of ϖ-adic modular forms with respect to D, level $K(H)$, weight χ and growth condition w, with coefficients in K, as

$$S^D(K, w, K(H), \chi) := \mathrm{H}^0(\mathfrak{M}(H)(w), \Omega_w^{(s,i)})_K.$$

Note that, by Proposition 4.2.24 below, we have

$$S^D(K, w, K(H), \chi) = S^D(V, w, K(H), \chi)_K$$

in the case χ is an integer.

Let w' be a rational number that satisfies the same conditions of w. We set $v' := \frac{w'}{q-1}$ and we assume that $w' \geq w$. Note that we have a natural morphism $f_{w,w'} \colon \mathfrak{M}(H)(w) \to \mathfrak{M}(H)(w')$ induced by the inclusion $\mathfrak{M}(H)(w)^{\mathrm{rig}} \hookrightarrow \mathfrak{M}(H)(w')^{\mathrm{rig}}$. We also have the morphism $g_{w,w'} \colon \mathfrak{M}(H\varpi)(w) \to \mathfrak{M}(H\varpi)(w')$, that comes from $\mathfrak{M}(H\varpi)(w)^{\mathrm{rig}} \hookrightarrow \mathfrak{M}(H\varpi)(w')^{\mathrm{rig}}$.

LEMMA 4.2.17. *We have a natural isomorphism of $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}$-modules*

$$\tilde{\rho}_{v,v'} \colon g_{w,w'}^*(\tilde{\Omega}_{w'}^\chi) \cong \tilde{\Omega}_w^\chi.$$

We have that $\tilde{\rho}_{v,v} = \mathrm{id}$ and, if $w'' \geq w'$ satisfies the same conditions of w, we have $\tilde{\rho}_{v,v''} = \tilde{\rho}_{v,v'} \circ g_{w,w'}^*(\tilde{\rho}_{v',v''})$, where $v'' := \frac{w''}{q-1}$. Furthermore, we obtain a canonical morphism

$$\rho_{v,v'} \colon f_{w,w'}^*(\Omega_{w'}^\chi) \to \Omega_w^\chi,$$

that is an isomorphism after rigidification. The $\rho_{v,v'}$'s satisfy the same conditions as the $\tilde{\rho}_{v,v'}$'s do.

PROOF. This is Lemma 3.5 of [**AIS11**]. Let $\mathcal{T}_{v'}$ be the sheaf on $\mathfrak{M}(H\varpi)(w)$ defined by

$$\mathcal{T}_{v'} := \mathcal{O}_{\mathcal{P}}^*(1 + \varpi^{1-v'}\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}).$$

Note that $\mathcal{S}_v$ is a subsheaf of $\mathcal{T}_{v'}$, and this latter sheaf acts in a natural way on $\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}$ (the twisting by χ makes sense, since we are assuming that w' is small enough). We define $\mathcal{G}_{v'}'$ to be the sheaf of sets on $\mathfrak{M}(H\varpi)(w)$ given by inverse image of $(\mathfrak{C}_1^{2,1})^\vee \setminus \{0\}$ under the natural map

$$\mathcal{F} \to (\mathfrak{C}_1^{2,1})^\vee \otimes_\kappa \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}/\varpi^{1-v'}\mathcal{O}_{\mathfrak{M}(H\varpi)(w)}.$$

Since $1-v' \leq 1-v$, we have that $\mathcal{F}_v'$ is a subsheaf of $\mathcal{G}_{v'}'$, that is a $\mathcal{T}_{v'}$-torsor (this is proved in exactly the same way as for $\mathcal{F}_v'$). The two inclusions of sheaves just defined are compatible and we have ω^{std}, a canonical generator of both torsors. In particular we have an isomorphism $\mathcal{G}_{v'}' \cong \mathcal{F}_v' \times^{\mathcal{S}_v} \mathcal{T}_{v'}$. By universal property of push-out, we have an isomorphism

$$\tilde{\Omega}_w^\chi = \mathscr{H}\!om_{\mathcal{S}_v}(\mathcal{F}_v', \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}) \cong \mathscr{H}\!om_{\mathcal{T}_{v'}}(\mathcal{G}_{v'}', \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}).$$

Using $g_{w,w'}^*$, we have a natural morphism $\mathcal{S}_{v'} \to (g_{w,w'})_*\mathcal{T}_{v'}$. By adjunction, we have a natural morphism $g_{w,w'}^{-1}(\mathcal{S}_{v'}) \to \mathcal{T}_{v'}$, that is compatible with $g_{w,w'}^{-1}(\mathcal{F}_{v'}') \to \mathcal{G}_{v'}'$ (given by adjunction too). Using the generator ω^{std}, we obtain an isomorphism

$$\mathcal{G}_{v'}' \cong g_{w,w'}^{-1}(\mathcal{F}_{v'}') \times^{g_{w,w'}^{-1}(\mathcal{S}_{v'})} \mathcal{T}_{v'},$$

so, by universal property, we have an isomorphism

$$\mathscr{H}\!om_{\mathcal{T}_{v'}}(\mathcal{G}_{v'}', \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}) \cong \mathscr{H}\!om_{g_{w,w'}^{-1}(\mathcal{S}_{v'})}(g_{w,w'}^{-1}(\mathcal{F}_{v'}'), \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})}).$$

The natural morphism

$$g_{w,w'}^*(\tilde{\Omega}_{w'}^\chi) \to \mathscr{H}\!om_{g_{w,w'}^{-1}(\mathcal{S}_{v'}')}(g_{w,w'}^{-1}(\mathcal{F}_{v'}'), \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(\chi^{-1})})$$

is an isomorphism. Indeed, looking at $X_{\chi,v}$ we obtain the surjectivity, and both the domain and the codomain are free sheaves of rank 1. In particular we have obtained the isomorphism $\tilde{\rho}_{v,v'} \colon g_{w,w'}^*(\tilde{\Omega}_{w'}^\chi) \cong \tilde{\Omega}_w^\chi$ of the statement of the lemma. It easy to check that $\rho_{v,v} = (\mathrm{id})$, and that $\rho_{v,v''} = \rho_{v,v'} \circ f_{w,w'}^*(\rho_{v',v''})$. One can check that this isomorphism, when push-forwarded to $\mathfrak{M}(H)(w)$, is compatible with the action of κ^*. We have the commutative diagram

$$\begin{tikzcd} \mathfrak{M}(H\varpi)(w) \arrow[r, "g_{w,w'}"] \arrow[d, "\vartheta_w"'] & \mathfrak{M}(H\varpi)(w') \arrow[d, "\vartheta_{w'}"] \\ \mathfrak{M}(H)(w) \arrow[r, "f_{w,w'}"] & \mathfrak{M}(H)(w') \end{tikzcd}$$

The natural morphism $\Psi_{w,w'}\colon f^*_{w,w'}\circ\vartheta_{w',*}\tilde{\Omega}^\chi_{w'} \to \vartheta_{w,*}\circ g^*_{w,w'}\tilde{\Omega}^\chi_{w'}$ respects the action of κ^*, and gives the required $\rho_{v,v'}$ taking κ^*-invariants. Since the above diagram becomes Cartesian after passing to rigid analytic fiber, and the vertical arrows becomes finite and étale, we have that $\Psi_{w,w'}\otimes_V K$ is an isomorphism, the lemma follows. $\square$

Remark 4.2.18. Note that in the above lemma we have proved that

$$\rho_{v,v'}(X_{\chi,v'}) = X_{\chi,v}.$$

Definition 4.2.19. Using Lemma 4.2.17, we can define the space of *overconvergent* modular forms with respect to D, level $K(H)$, weight χ and growth condition w, with coefficients in K, as

$$S^D_\dagger(K,K(H),\chi) := \varinjlim_{w>0} S^D(K,w,K(H),\chi).$$

Note that the direct limit is taken over strictly positive w's (that satisfy all our assumptions).

Remark 4.2.20. In this remark we prove a sort of functoriality property of the sheaf Ω^χ_w. Suppose we are given $i_{\mathcal{A}}, i_{\mathcal{B}}\colon \mathrm{Spf}(S)\to\mathfrak{M}(H\varpi)(w)$, two affine points of $\mathfrak{M}(H\varpi)(w)$. We write $\mathcal{A}$ and $\mathcal{B}$ for the abelian schemes corresponding to $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$, respectively. Suppose we are given a morphism $f\colon\mathcal{B}\to\mathcal{A}$ over S. We obtain, by functoriality of $\mathrm{d}\log$, a morphism $\mathrm{Im}(\mathrm{d}\log_{\mathcal{A}})\to\mathrm{Im}(\mathrm{d}\log_{\mathcal{B}})$ compatible with the natural pullback $\underline{\omega}_{\mathcal{A}/S}\to\underline{\omega}_{\mathcal{B}/S}$. Taking Galois invariants we obtain, by Proposition 3.2.13, a morphism $f^*\colon\mathcal{F}(i_{\mathcal{A}}(\mathrm{Spf}(S)))\to\mathcal{F}(i_{\mathcal{B}}(\mathrm{Spf}(S)))$. Let us now suppose that $f\colon\mathcal{B}\to\mathcal{A}$ is an isogeny, and that its kernel intersects trivially the canonical subgroup of $\mathcal{B}$. In this case we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(i_{\mathcal{A}}(\mathrm{Spf}(S)))_{1-v} \arrow[r, "f^*_{1-v}"] \arrow[d, "\wr"'] & \mathcal{F}(i_{\mathcal{B}}(\mathrm{Spf}(S)))_{1-v} \arrow[d, "\wr"] \\ (\mathcal{C}(\mathcal{A})^{2,1}_1)^\vee\otimes_\kappa S_{1-v} \arrow[r, "(f^{2,1}_1)^\vee"] & \mathcal{C}((\mathcal{B})^{2,1}_1)^\vee\otimes_\kappa S_{1-v} \end{tikzcd}$$

By assumption, $(f^{2,1}_1)^\vee$ is an isomorphism, so f^* is an isomorphism, modulo ϖ^{1-v}. By Nakayama's lemma, we have that f^* is surjective. But both its domain and its codomain are free modules of rank 1, so f^* is an isomorphism. By definition of $\mathcal{F}'_v$, this implies that we also have an isomorphism $\mathcal{F}'_v(i_{\mathcal{A}}(\mathrm{Spf}(S)))\cong\mathcal{F}'_v(i_{\mathcal{B}}(\mathrm{Spf}(S)))$. In particular we obtain an isomorphism of sheaves

$$\mathscr{H}om_{\mathcal{S}_{v|i_{\mathcal{B}}(\mathrm{Spf}(S))}}(\mathcal{F}'_{v|i_{\mathcal{B}}(\mathrm{Spf}(S))},\mathcal{O}^{(\chi^{-1})}_{\mathrm{Spf}(S)}) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{S}_{v|i_{\mathcal{A}}(\mathrm{Spf}(S))}}(\mathcal{F}'_{v|i_{\mathcal{A}}(\mathrm{Spf}(S))},\mathcal{O}^{(\chi^{-1})}_{\mathrm{Spf}(S)}).$$

As in the proof of Lemma 4.2.17, we can show that

$$\mathscr{H}om_{\mathcal{S}_{v|i_{\mathcal{A}}(\mathrm{Spf}(S))}}(\mathcal{F}'_{v|i_{\mathcal{A}}(\mathrm{Spf}(S))},\mathcal{O}^{(\chi^{-1})}_{\mathrm{Spf}(S)}) \cong i^*_{\mathcal{A}}\tilde{\Omega}^\chi_w,$$

and similarly for $\mathcal{B}$. This gives an isomorphism $i^*_{\mathcal{A}}\tilde{\Omega}^\chi_w\to i^*_{\mathcal{B}}\tilde{\Omega}^\chi_w$. We will be more interested in its inverse

$$\tilde{f}^\chi\colon i^*_{\mathcal{B}}\tilde{\Omega}^\chi_w\to i^*_{\mathcal{A}}\tilde{\Omega}^\chi_w.$$

Note that if we drop the assumption that the kernel of f intersects trivially the canonical subgroup of $\mathcal{B}$, we still obtain a morphism $i_{\mathcal{A}}^*\tilde{\Omega}_w^\chi \to i_{\mathcal{B}}^*\tilde{\Omega}_w^\chi$. But in general we cannot consider its inverse, so we do not have the morphism $\tilde{f}^\chi$, that goes in the expected direction (i.e. the direction of the natural pullback of differentials).

All the various maps we have defined, when push-forwarded to $\mathfrak{M}(H)$, respect the action of κ^*, so the same must be true for $\tilde{f}^\chi$. Suppose that $\mathrm{Spf}(S)$ is the base change of $\mathrm{Spf}(R) \to \mathfrak{M}(H)$. We write $\vartheta\colon \mathrm{Spf}(S) \to \mathrm{Spf}(R)$. Since the Galois group of ϑ^{rig} is κ^*, we have a canonical isomorphism $\vartheta^{\mathrm{rig},*}\Omega_w^\chi \cong \tilde{\Omega}_w^\chi$ obtained inverting ϖ. In particular, we have the morphism

$$f^\chi\colon (\vartheta \circ i_{\mathcal{B}})^*\Omega_w^\chi \otimes_V K \to (\vartheta \circ i_{\mathcal{A}})^*\Omega_w^\chi \otimes_V K.$$

Note that f^χ exists only after having inverted ϖ. In the case $\chi = (k,k)$ is an integer, via the isomorphism of Lemma 4.2.21, f^k is the pullback of the k-th power of the invariant differentials with respect to the isogeny.

4.2.1. Modular forms of integral weight. We now prove that, for integral weights, our new definition of modular forms agrees with the one given in Sections 2.1 and 2.3.

Let k be an integer. If $\mathfrak{V} \subseteq \mathfrak{M}(H\varpi)(w)$, let $\phi_{k,\mathfrak{V}}\colon (\tilde{\Omega}_w^k)(\mathfrak{V}) \to (\underline{\omega}_{K(H\varpi)}^{\otimes k})(\mathfrak{V})$ be the map given by

$$\phi_{k,\mathfrak{V}}(f) = f(\omega^{\mathrm{std}})(\omega^{\mathrm{std}})^{\otimes k},$$

for $f \in \tilde{\Omega}_w^k(\mathfrak{V})$. These maps glue to a morphism of sheaves

$$\phi_k\colon \tilde{\Omega}_w^k \to \underline{\omega}_{K(H\varpi)}^{\otimes k}.$$

LEMMA 4.2.21. *For all integer k, we have that $\phi_k \otimes_V K$ is an isomorphism. In particular we have the identification*

$$S^D(V,w,K(H\varpi),k)_K = \mathrm{H}^0(\mathfrak{M}(H\varpi)(w), \tilde{\Omega}_w^{(k,k)})_K.$$

PROOF. The lemma follows since $\omega^{\mathrm{std}} \otimes 1$ is a generator of $\underline{\omega}_{K(H\varpi)} \otimes_V K$ by Theorem 3.2.11. □

REMARK 4.2.22. By Proposition 3.2.12, we see that ϕ_k is an isomorphism if and only if $w = 0$. In general, by Theorem 3.2.11, we have that $\mathrm{coker}(\phi_k)$ is killed by ϖ^{kv}.

Recall that we have the modular form E_1, of level $K(H\varpi)$ and weight 1, that is given by ω^{std}.

LEMMA 4.2.23. *We have that κ^* acts on $E_1 \in S^D(K,w,K(H\varpi),(1,1))$ via $[\cdot]^{-1}$, so E_1 is identified with a global section of $\Omega_w^{(1,0)}$.*

PROOF. By Remark 4.2.13, it is enough to prove the first part of the proposition. Let $\mathrm{Spf}(S) \subseteq \mathfrak{M}(H\varpi)(w)$ be an open affine. Let f be the element of $\tilde{\Omega}_w^{(1,1)}(\mathrm{Spf}(S))$ corresponding to $E_{1|\mathrm{Spf}(S)}$, in particular it gives the morphism

$$\begin{aligned} f\colon \mathcal{F}_v''(\mathrm{Spf}(S)) = (1+\varpi^{1-v}S)E_{1|\mathrm{Spf}(S)} &\to \mathcal{O}_{\mathfrak{M}(H\varpi)(w)}^{(-1,-1)}(\mathrm{Spf}(S)) = S \\ E_{1|\mathrm{Spf}(S)} &\mapsto 1 \end{aligned}$$

Let $a \in \kappa^*$. Recall that (see Proposition 2.3.5) E_1 is, locally, defined as $\frac{(-\varpi)^{1/(q-1)}}{\alpha}\omega$, where $\alpha \in S$ comes from the canonical S point of $\mathcal{C}_1^{2,1}$. We have (see the Appendix) $a^\sharp(\alpha) = [a]\alpha$, so $a^\sharp(\frac{(-\varpi)^{1/(q-1)}}{\alpha}) = [a]^{-1}\frac{(-\varpi)^{1/(q-1)}}{\alpha}$. Since ω is a section of $\underline{\omega}_{K(H)}$, we have $a^*(\omega) = \omega$. In particular, a^{-1} sends ω^{std} to $[a]\omega^{\mathrm{std}}$. It follows that $f_{|\langle a\rangle}$ is the map that sends ω^{std} to $a^\sharp(f([a]\omega^{\mathrm{std}}))$. But $f \in \tilde{\Omega}_w^{(1,1)}(\mathrm{Spf}(S))$, so

$f([a]\omega^{\mathrm{std}}) = [a]^{-1}f(\omega^{\mathrm{std}}) = [a]^{-1}$. Since $a^\sharp$ is an automorphism of $\mathcal{O}_{\mathcal{P}}$-algebras, we see that $f_{|\langle a\rangle}(\omega^{\mathrm{std}}) = [a]^{-1}$, so

$$f_{|\langle a\rangle} = [a]^{-1}f.$$

□

Since $\tilde{\Omega}^k_w$ is locally free, it has no ϖ-torsion, so ϕ_k is a monomorphism of sheaves. In particular we will consider $\tilde{\Omega}^k_w$ as a subsheaf of $\underline{\omega}^{\otimes k}_{K(H\varpi)}$ via ϕ_k. We have $\underline{\omega}^{\otimes k}_{K(H)} \otimes_V K \subseteq \vartheta_*\underline{\omega}^{\otimes k}_{K(H\varpi)} \otimes_V K = \vartheta_*\tilde{\Omega}^k_w \otimes_V K = \oplus_j \Omega^{(k,j)}_w \otimes_V K$.

Proposition 4.2.24. *We have that $\underline{\omega}^{\otimes k}_{K(H)} \otimes_V K = \Omega^{(k,k)}_w \otimes_V K$, so, inside $\oplus_j \Omega^{(k,j),\mathrm{rig}}_w$, we have the equality*

$$\underline{\omega}^{\otimes k,\mathrm{rig}}_{K(H)} = \Omega^{(k,k),\mathrm{rig}}_w.$$

In particular

$$S^D(V,w,K(H),k)_K = \mathrm{H}^0(\mathfrak{M}(H)(w), \Omega^{(k,k)}_w)_K.$$

Proof. We need to study the action of κ^* on $\underline{\omega}^{\otimes k}_{K(H)} \otimes_V K \subseteq \vartheta_*\tilde{\Omega}^k_w \otimes_V K$. We work locally, as in the proof of Lemma 4.2.23, using the same notations. Let $f \otimes 1 \in \tilde{\Omega}^k_w(\mathrm{Spf}(S)) \otimes_V K$ be the element corresponding to $\omega^{\otimes k} \otimes 1 \in \underline{\omega}^{\otimes k}_R \otimes_V K$, where ω is a generator of $\underline{\omega}_R$, that we can assume to be free. We can write $E_{1\,|\,\mathrm{Spf}(S)} = \frac{(-\varpi)^{1/(q-1)}}{\alpha}\omega$. It follows that f gives the map

$$f\colon \mathcal{F}''_v(\mathrm{Spf}(S)) = (1+\varpi^{1-v}S)E_{1\,|\,\mathrm{Spf}(S)} \to \mathcal{O}^{(-k,-k)}_{\mathfrak{M}(H\varpi)(w)}(\mathrm{Spf}(S)) = S$$
$$E_{1\,|\,\mathrm{Spf}(S)} \mapsto \left(\frac{\alpha}{(-\varpi)^{1/(q-1)}}\right)^k$$

As in the proof of Lemma 4.2.23, we have that

$$f_{|\langle a\rangle}(\omega^{\mathrm{std}}) = a^\sharp(f([a]\omega^{\mathrm{std}})) = [a]^{-k}a^\sharp\left(\frac{\alpha}{(-\varpi)^{1/(q-1)}}\right)^k.$$

Since $a^\sharp$ is an automorphism of V-algebras, it fixes $(-\varpi)^{1/(q-1)}$. Furthermore, we have $a^\sharp(\alpha) = [a]\alpha$, so $f_{|\langle a\rangle}(\omega^{\mathrm{std}}) = 1$, hence $f_{|\langle a\rangle} = f$. This shows that $\underline{\omega}^{\otimes k}_{K(H\varpi)} \otimes_V K \subseteq \Omega^{(k,k)}_w \otimes_V K$. For the other inclusion, note that any element f of $\tilde{\Omega}^k_w(\mathrm{Spf}(S)) \otimes_V K$ can be written as $f = s\omega$, for some $s \in S_K$. A calculation similar to the one above shows that, if the diamond operators act trivially on f, then $s \in R_K$ as required. □

Remark 4.2.25. By Corollary 4.2.14, we have a decomposition

$$S^D(K,w,K(H\varpi),\chi) = \bigoplus_{j \in \mathbb{Z}/(q-1)\mathbb{Z}} S^D(K,w,K(H),(s,j)).$$

In other words, any modular form of level $K(H\varpi)$ has components that are modular forms of level $K(H)$. Note that if f is of level $K(H\varpi)$ and has integral weight, say k, we cannot identify it with a modular form of integral weight k and level $K(H)$. Instead, f will have components that are modular forms of level $K(H)$ and weight $x \mapsto \langle x\rangle^k[x]^j$, for various $j \in \mathbb{Z}/(q-1)\mathbb{Z}$. We have that f can be identified with a modular form of weight k and level $K(H)$ if and only if there is only one non trivial component, corresponding to $i = k$. This is very similar to the case of elliptic modular forms (see **[Gou88]**, Sections I.3.4-7).

REMARK 4.2.26. Fix an open affine $\mathfrak{U} = \mathrm{Spf}(R) \subseteq \mathfrak{M}(H)(w)$, and let $\mathfrak{V} = \mathrm{Spf}(S)$ be the inverse image of $\mathfrak{U}$ under ϑ. We write $\mathcal{A} \to \mathrm{Spec}(S)$ for the corresponding abelian scheme. We let κ^* act on $\underline{\omega}_S$ by pullbacks. It follows from the above discussion that we have κ^*-equivariant isomorphisms

$$\Omega_w^1 \cong \mathcal{F} \text{ and } \Omega_w^{-1} \cong \mathcal{F}^*.$$

In particular, we see that there is a 'corrected' Hodge-Tate sequence

$$0 \to \Omega_w^{-1}(\mathfrak{V}) \otimes_R \widehat{\overline{R}}(1) \to \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_\kappa \widehat{\overline{R}} \to \Omega_w^1(\mathfrak{V}) \otimes_R \widehat{\overline{R}} \to 0$$

that is exact.

REMARK 4.2.27. We have $X_{1,v} = E_1$. Working as in the proof of Lemma 4.2.23, we see that $X_{\chi,v}$ is a global section of $\Omega_w^{(s,0)}$. It follows that $X_{\chi,v}$, when considered as a modular form of level $K(H)$, has integral weight if and only if s is an integer congruent to 0 modulo $q-1$. For example we have that $X_{q-1,v} = E_{q-1}$ has integral weight $q-1$ as one expects.

4.2.2. Katz' modular forms. We would like to describe our modular forms in a more familiar way, using 'test objects'. See [**Kat73**], for this description in the case of elliptic modular forms.

DEFINITION 4.2.28. A test object is a sextuple $(\mathcal{A}/S, i, \theta, \bar{\alpha}, Y, \eta)$, where:

- $\mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$ is an affine point, with S a normal and ϖ-adically complete V-algebra;
- $(\mathcal{A}, i, \theta, \bar{\alpha})$ is an object of the moduli problem of level $K(H\varpi)$, with $\mathcal{A}$ defined over S;
- Y is a section of $\underline{\omega}_{\mathcal{A}/S}^{\otimes 1-q}$ that satisfies $YE_{q-1} = \varpi^w$;
- η is a global section of the pullback of $\mathcal{F}'$ to $\mathrm{Spf}(S)$.

PROPOSITION 4.2.29. *To give an element f of $S^D(K, w, K(H\varpi), \chi)$ is equivalent to give a rule that assigns to every test object $T = (\mathcal{A}/S, i, \theta, \bar{\alpha}, Y, \eta)$ an element $\tilde{f}(T) \in S_K$ such that:*

- *$\tilde{f}(T)$ depends only on the isomorphism class of T;*
- *if $\varphi\colon S \to S'$ is a morphism of normal and ϖ-adically complete V-algebras, and we denote with T' the base change of T to S', we have $\tilde{f}(T') = \varphi(\tilde{f}(T))$.*

PROOF. This Lemma 3.10 of [**AIS11**]. Let us start with a modular form f and take a test object T as in the statement of the proposition. By the moduli description of $\mathfrak{M}(H\varpi)(w)$ we know that T comes from a unique morphism $i\colon \mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$. It follows that $\eta = ri^*(\omega^{\mathrm{std}})$, for some $r = ub \in \mathcal{O}_{\mathcal{P}}^*(1 + \varpi^{1-v}S)$. Furthermore we can write $f = cX_{\chi,v}$, where $c \in \mathrm{H}^0(\mathfrak{M}(H\varpi)(w), \mathcal{O}_{\mathfrak{M}(H)(w)})_K$. We set

$$\tilde{f}(T) := \chi(u^{-1})b^{-s}i^*(c).$$

By the moduli interpretation of $\mathfrak{M}(H)(w)$, we have that $\tilde{f}$ satisfies the conditions of the proposition. Conversely, suppose we are given $\tilde{f}$ as above. Let $\{\mathfrak{V}_i\}_{i\in I}$ be an affine covering of $\mathfrak{M}(H\varpi)(w)$, with $\mathfrak{V}_i = \mathrm{Spf}(S_i)$. We assume that each S_i is normal and ϖ-adically complete. The elements $c_i \in S_{i,K}$ given by $\tilde{f}$, glue together to define $c \in \mathrm{H}^0(\mathfrak{M}(H\varpi)(w), \mathcal{O}_{\mathfrak{M}(H)(w)})_K$. We set $f := cX_{\chi,v}$. It is easy to verify that the two procedures just described are one the inverse of the other, as required. $\square$

COROLLARY 4.2.30. *Let f be an element of $S^D(K, w, K(H\varpi), \chi)$. We have that $f \in S^D(K, w, K(H), (s, j))$ if and only if, for any test object $T = (\mathcal{A}/S, i, \theta, \bar{\alpha}, Y, \eta)$, we have*

$$\tilde{f}_{|\langle a \rangle}(T) = [a]^{j-i} \tilde{f}(T)$$

for any $a \in \kappa^*$.

PROOF. This follows from Proposition 4.2.12. □

4.3. The sheaves Ω_w^χ for general characters

We now want to define the sheaf Ω_w^χ for any continuous character $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to K^*$. To do this, we use the theory of canonical subgroup of higher rank developed in Section 2.2.1.

In this section we fix an integer $r \geq 1$, the case of an accessible character corresponds to $r = 1$. Let us assume that

$$w < \frac{1}{q^{r-2}(q+1)},$$

and let $\mathrm{Spf}(R) \to \mathfrak{M}(H)(w)$ be as above. We write $\mathcal{A} \to \mathrm{Spec}(R)$ for the corresponding abelian scheme.

PROPOSITION 4.3.1. *The kernel of the map $\mathrm{d}\log_{r,\mathcal{A}}\colon (\mathcal{A}[\varpi^r]_1^{2,1})^\vee \to \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_r$ is $(\mathcal{D}_r)_1^{2,1} := ((\mathcal{C}_r(\overline{R}_K))_1^{2,1})^\perp$, where $\mathcal{C}_r$ is the canonical subgroup of $\mathcal{A}[\varpi^r]$.*

PROOF. We prove the proposition by induction, the case $r = 1$ follows by Proposition 3.1.7. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{A}[\varpi]_1^{2,1})^\vee \arrow[r] \arrow[d] & (\mathcal{A}[\varpi^r]_1^{2,1})^\vee \arrow[r] \arrow[d] & (\mathcal{A}[\varpi^{r-1}]_1^{2,1})^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{C}_1^{2,1})^\vee \arrow[r] & ((\mathcal{C}_r)_1^{2,1})^\vee \arrow[r] & ((\mathcal{C}_{r-1})_1^{2,1})^\vee \arrow[r] & 0
\end{tikzcd}$$

This implies, for dimension reasons, that we have an exact sequence

$$0 \to \mathcal{D}_1^{2,1} \to (\mathcal{D}_r)_1^{2,1} \to (\mathcal{D}_{r-1})_1^{2,1} \to 0$$

By functoriality of $\mathrm{d}\log$ (see Lemma 3.1.3) we obtain a commutative diagram, with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}_1^{2,1} \arrow[r] \arrow[d] & (\mathcal{D}_r)_1^{2,1} \arrow[r] \arrow[d] & (\mathcal{D}_{r-1})_1^{2,1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{A}[\varpi]_1^{2,1})^\vee \arrow[r] \arrow[d] & (\mathcal{A}[\varpi^r]_1^{2,1})^\vee \arrow[r] \arrow[d] & (\mathcal{A}[\varpi^{r-1}]_1^{2,1})^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_1 \arrow[r] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_r \arrow[r] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_{r-1} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact by **[Far07]**, Corollaire 1. We know that $\mathcal{D}_1^{2,1}$ is the kernel of $\mathrm{d}\log_{1,\mathcal{A}}$, and by induction hypothesis we have that $(\mathcal{D}_{r-1})_1^{2,1}$ is the kernel of $\mathrm{d}\log_{r-1,\mathcal{A}}$. The proposition follows. □

LEMMA 4.3.2. *We have a commutative diagram, with exact bottom row:*

$$\begin{tikzcd}
\underline{\omega}^*_{\mathcal{A}^\vee/R} \otimes_R \overline{R}_r(1) \arrow[r] \arrow[d] & \mathrm{T}_\varpi((\mathcal{A}[\varpi^\infty]_1^{2,1})^\vee) \otimes_{\mathcal{O}_{\mathcal{P}}} \overline{R}_r \arrow[r] \arrow[d, equal] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_r \\
(\mathcal{D}_r)_1^{2,1} \otimes_{\mathcal{O}_{\mathcal{P}}} \overline{R}_r \arrow[r, hook] & (\mathcal{A}[\varpi^r]_1^{2,1})^\vee \otimes_{\mathcal{O}_{\mathcal{P}}} \overline{R}_r \arrow[r, two heads] & ((\mathcal{C}_r)_1^{2,1})^\vee \otimes_{\mathcal{O}_{\mathcal{P}}} \overline{R}_r \arrow[u]
\end{tikzcd}$$

PROOF. This is proved exactly in the same way as Lemma 3.2.10, using Proposition 4.3.1. □

PROPOSITION 4.3.3. *We have a natural $\mathcal{G}$-equivariant isomorphism*

$$\operatorname{Im}(\mathrm{d}\log_{\mathcal{A}})_{r-v} \cong ((\mathcal{C}_r)_1^{2,1})^\vee \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{r-v}.$$

PROOF. Using Lemma 4.3.2, this is proved exactly in the same way as Theorem 3.2.11. □

We now work over $\mathfrak{M}(H\varpi^r)(w)$. As in the case $r = 1$, we have a natural morphism

$$\vartheta_r \colon \mathfrak{M}(H\varpi^r)(w) \to \mathfrak{M}(H)(w).$$

Its rigidification is finite and étale, and its Galois group is canonically identified with $G_r := (\mathcal{O}_\mathcal{P}/\varpi^r\mathcal{O}_\mathcal{P})^*$. As above, we have that G_r acts on ϑ_r too.

Let $\mathfrak{U} = \operatorname{Spf}(R) \subseteq \mathfrak{M}(H)(w)$ be an open affine. We will write $\mathfrak{V}_r = \operatorname{Spf}(S_r)$ for the inverse image of $\mathfrak{U}$ under ϑ_r. Since $(\mathcal{C}_r)_1^{2,1}$ is an extension of $(\mathcal{C}_{r-1})_1^{2,1}$ we see, by induction, that $(\mathcal{C}_r)_1^{2,1}$ becomes constant over $S_{r,K}$. Furthermore, by the moduli interpretation of $\mathfrak{M}(H\varpi^r)(w)$ given in Proposition 2.3.7, we have a canonical point of $(\mathcal{C}_r)_1^{2,1}$, defined over S_r (this point is a generator of $(\mathcal{C}_r)_1^{2,1}$ as $\mathcal{O}_\mathcal{P}/\varpi^r$-module). In particular we can repeat what we have done for $\mathcal{C}_1^{2,1}$, and we obtain an isomorphism of sheaves of $\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}$-modules

$$\mathcal{F}/\varpi^{r-v}\mathcal{F} \cong ((\mathfrak{C}_r)_1^{2,1})^\vee \otimes_{\mathcal{O}_\mathcal{P}} \mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}/\varpi^{r-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}.$$

Recall that $\mathcal{L}T$ is the group scheme associated to $R[[x]]$. It has an action of $\mathcal{O}_\mathcal{P}$ for which the multiplication by ϖ has the form $[\varpi](x) = x^q + \varpi x$. We now fix $\{\zeta_n\}_{n\geq 1}$, a sequence of $\mathbb{C}_p$-points of $\mathcal{LT}$ such that the order of ζ_n is exactly ϖ^n. We assume that $\varpi\zeta_{n+1} = \zeta_n$ for each r, and that ζ_1 is our fixed $(-\varpi)^{1/(q-1)}$. This is the analogue of fixing a coherent sequence of primitive p^n-roots of unity. As usual in out theory, we need to use $\mathcal{L}T$ instead of $\mathbb{G}_{\mathrm{m}}$.

If $\zeta_r \in V$, we can use it and the canonical S_r-point of $(\mathcal{C}_r)_1^{2,1}$ to obtain γ_r, a canonical S_r-point of $((\mathcal{C}_r)_1^{2,1})^\vee$. The various γ_r's glue together to define a canonical global section of $((\mathfrak{C}_r)_1^{2,1})^\vee$, denoted again by γ_r. Note that $\gamma_1 = \gamma$.

If w is smaller than $1/(q^{r-2}(q+1))$, we define the sheaf $\mathcal{F}'_{r,v}$ on $\mathfrak{M}(H\varpi^r)(w)$ as the inverse image of the constant sheaf of sets given by the subset of $((\mathfrak{C}_r)_1^{2,1})^\vee$ of points of order exactly ϖ^r. As in the case of $\mathcal{F}'_v$, we can prove that $\mathcal{F}'_{r,v}$ is a Zariski $\mathcal{S}_{r,v}$-torsor, where

$$\mathcal{S}_{r,v} := \mathcal{O}_\mathcal{P}^*(1 + \varpi^{r-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}).$$

We also have the sheaves $\mathcal{S}''_{r,v} = 1 + \varpi^{r-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}$ and $\mathcal{F}''_{r,v}$, defined using γ_r. It follows that $\mathcal{F}''_{r,v}$ is a Zariski $\mathcal{S}''_{r,v}$-torsor.

We now fix χ, an r-accessible character. We assume that $\zeta_r \in V$. Let s be the r-good element of $\mathbb{C}_p$ associated to χ (see Definitions 4.1.3 and 4.1.5). We assume that w is smaller than $1/(q^{r-2}(q+1))$, so we have the canonical subgroup of level r. Since we are also assuming that $w < (q-1)\left(\mathrm{v}(s) + \mathrm{v}(\log(1+\varpi)) - \frac{e}{p-1}\right)$, we have in particular

$$w < (q-1)\left(\mathrm{v}(s) + \mathrm{v}(\log(1+\varpi^r)) - \frac{e}{p-1}\right),$$

so the p-adic exponential below is well defined. Let x be a local section of $\mathcal{S}_{r,v}$. We can write $x = ub$, where u is a section of $\mathcal{O}_\mathcal{P}^*$ and b is a section of $1 + \varpi^{r-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}$. We have that $b^s := \exp(s\log(b))$ makes sense, so we can write $x^\chi := \chi(u)b^s$, that is another section of $S_{r,v}$. Note that x^χ depends only on the product ub, as in the case $r = 1$. We write $\mathcal{O}^{(\chi)}_{\mathfrak{M}(H\varpi^r)(w)}$ for the sheaf $\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}$ with the action of $\mathcal{S}_{r,v}$ twisted by χ. As in the case of accessible characters, we define

$$\tilde{\Omega}^{\chi}_w := \mathscr{H}om_{\mathcal{S}_{r,v}}(\mathcal{F}'_{r,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)}).$$

Since $\mathcal{F}'_{r,v}$ is an $\mathcal{S}_{r,v}$-torsor, we have that $\tilde{\Omega}^{\chi}_w$ is a locally free sheaf of rank 1.

There is a natural isomorphism of sheaves

$$\tilde{\Omega}^{\chi}_w \cong \mathscr{H}om_{\mathcal{S}''_{r,v}}(\mathcal{F}''_{r,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)}).$$

Let $\mathfrak{V}_r = \mathrm{Spf}(S_r) \to \mathfrak{M}(H\varpi^r)$ be an open affine. We can define $X_{\chi,v}$, a canonical section of $\tilde{\Omega}^{\chi}_w$ on $\mathfrak{V}_r$, by

$$X_{\chi,v}(b\omega^{\mathrm{std}}) = b^{-s},$$

for all $b \in \mathcal{S}''_{r,v}(\mathfrak{V}_r) = 1 + \varpi^{r-v}S_r$. In this way we obtain a canonical generator of the global sections of $\tilde{\Omega}^{\chi}_w$, called $X_{\chi,v}$ too.

Since G_r acts on $((\mathfrak{C}_r)^{2,1}_1)^\vee \setminus \{0\}$, we have an action of G_r on $\mathcal{F}'_{r,v}$. Similarly to the case of accessible characters, G_r acts also on $\vartheta_{r,*}\mathcal{O}^{\chi^{-1}}_{\mathfrak{M}(H\varpi^r)(w)}$. Repeating what we have done in the case $r = 1$, we obtain a coherent sheaves of $\mathcal{O}_{\mathfrak{M}(H)(w)}$-modules

$$\vartheta_{r,*}\tilde{\Omega}^{\chi}_w$$

that is endowed with an action of G_r.

Definition 4.3.4. We define the sheaf Ω^{χ}_w on $\mathfrak{M}(H)(w)$ as

$$\Omega^{\chi}_w := \left(\vartheta_{r,*}\tilde{\Omega}^{\chi}_w\right)^{G_r}.$$

Since $\vartheta^{\mathrm{rig}}_r$ is finite and étale with Galois group G_r, we have that $\Omega^{\chi}_w \otimes_V K$ is a locally free sheaf of rank 1.

Definition 4.3.5. We define the space of ϖ-adic modular forms with respect to D, level $K(H)$, weight χ and growth condition w, with coefficients in K, as

$$S^D(K, w, K(H), \chi) := \mathrm{H}^0(\mathfrak{M}(H)(w), \Omega^{\chi}_w)_K.$$

The space $S^D(K, w, K(H\varpi^r), \chi)$ is defined using the sheaf $\tilde{\Omega}^{\chi}_w$.

Everything we have done in the case of an accessible character can be repeated for χ. In particular we have the analogue of Lemma 4.2.17 and the analogue of Remark 4.2.20.

Definition 4.3.6. We define the space of overconvergent modular forms with respect to D, level $K(H)$, weight χ, with coefficients in K, as

$$S^D_\dagger(K, K(H), \chi) := \varinjlim_{w>0} S^D(K, w, K(H), \chi).$$

Let h be an integer with $r \geq h$. Suppose that χ is also h-accessible. In this case we can repeat the above construction starting with $\mathfrak{M}(H\varpi^h)(w)$, obtaining another sheaf on $\mathfrak{M}(H\varpi)(w)$ (note that the conditions on w, relative to h, are automatically satisfied). We are going to prove that, if we invert ϖ, this latter sheaf is naturally isomorphic to the Ω^{χ}_w defined above.

For $r \geq h$, let $\vartheta_{r,h}$ be the natural morphism

$$\vartheta_{r,h} \colon \mathfrak{M}(H\varpi^r)(w) \to \mathfrak{M}(H\varpi^h)(w),$$

defined using the canonical generator of $(\mathfrak{C}_r)^{2,1}_1$. The rigidification of $\vartheta_{r,h}$ is finite and étale. Let $G_{r,h} \subseteq G_r$ be the image of $1 + \varpi^h\mathcal{O}_{\mathcal{P}}$. It follows that the Galois group of $\vartheta^{\mathrm{rig}}_{r,h}$ is $G_{r,h}$, and the latter group also acts on $\vartheta_{r,h}$.

PROPOSITION 4.3.7. *Let $h \leq r$ be integers, and suppose that χ is an h-accessible character. Let $w \leq 1/(q^{r-2}(q+1))$ be a rational number. We have a natural G_h-equivariant morphism*

$$\mathscr{H}om_{\mathcal{S}_{h,v}}(\mathcal{F}'_{h,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^h)(w)}) \to \\ \to \left(\vartheta_{r,h,*}\,\mathscr{H}om_{\mathcal{S}_{r,v}}(\mathcal{F}'_{r,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^{r+1})(w)})\right)^{G_{r,h}},$$

that is an isomorphism if we invert ϖ. After the push-forward via ϑ_h and taking G_h-invariants, we get an isomorphism of $\mathcal{O}_{\mathfrak{M}(H)(w)} \otimes_V K$-modules

$$\sigma_{r,h}\colon \left(\vartheta_{h,*}\,\mathscr{H}om_{\mathcal{S}_{h,v}}(\mathcal{F}'_{h,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^h)(w)}) \otimes_V K\right)^{G_h} \cong \\ \cong \left(\vartheta_{r,*}\,\mathscr{H}om_{\mathcal{S}_{r,v}}(\mathcal{F}'_{r,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)}) \otimes_V K\right)^{G_r}.$$

Furthermore $\sigma_{r,r} = \mathrm{id}$, and, if $t \leq h$ is an integer, we have $\sigma_{r,t} = \sigma_{h,t} \circ \sigma_{r,h}$.

PROOF. This is Lemma 3.20 of **[AIS11]**. The proof is similar to the one of Lemma 4.2.17. We write $\mathcal{S}_{r,h,v}$ for the sheaf on $\mathfrak{M}(H\varpi^r)(w)$ defined by

$$\mathcal{S}_{r,h,v} := \mathcal{O}^*_{\mathcal{P}}(1 + \varpi^{h-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}).$$

Since $h \leq r$, we have $\mathcal{S}_{r,h,v} \subseteq \mathcal{S}_{r,v}$. By Proposition 4.3.3, we have an isomorphism

$$\mathcal{F}/\varpi^{h-v}\mathcal{F} \cong ((\mathfrak{C}_r)_1^{2,1})^\vee \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}/\varpi^{h-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}.$$

Via this isomorphism, we define $\mathcal{F}'_{r,h,v}$ to be the inverse image of $((\mathfrak{C}_r)_1^{2,1})^\vee \setminus \{0\}$ under the natural map $\mathcal{F} \to \mathcal{F}/\varpi^{h-v}\mathcal{F}$. Since $\mathcal{F}'_{r,h,v}$ is the pushed-out torsor $\mathcal{F}'_{r,v} \times^{\mathcal{S}_{r,v}} \mathcal{S}_{r,h,v}$, we have, by universal property, that the natural map

$$\mathscr{H}om_{\mathcal{S}_{r,v}}(\mathcal{F}'_{r,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)}) \to \mathscr{H}om_{\mathcal{S}_{r,h,v}}(\mathcal{F}'_{r,h,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)})$$

is an isomorphism. We have a natural isomorphism

$$\mathcal{F}'_{r,h,v} \cong \vartheta_{r,h}^{-1}(\mathcal{F}'_{h,v}) \times^{\vartheta_{r,h}^{-1}(\mathcal{S}_{h,v})} \mathcal{S}_{r,h,v},$$

so we obtain φ, a natural map

$$\vartheta^*_{r,h}\,\mathscr{H}om_{\mathcal{S}_{h,v}}(\mathcal{F}'_{h,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^h)(w)}) \to \mathscr{H}om_{\mathcal{S}_{r,h,v}}(\mathcal{F}'_{r,h,v}, \mathcal{O}^{(\chi^{-1})}_{\mathfrak{M}(H\varpi^r)(w)}).$$

Looking at ω^{std} we deduce that φ is surjective. But both the domain and the codomain of φ are locally free of rank 1, so φ is an isomorphism. All the map we have defined respects the action of $\mathcal{O}^*_{\mathcal{P}}$ (and of all of its quotients). Since $\vartheta^{\mathrm{rig}}_{r,h}$ is finite and étale with Galois group $G_{r,h}$, the first part of the proposition follows by taking the push forward via $\vartheta_{r,h}$ and then taking $G_{r,h}$-invariants of φ. The proposition follows since $\vartheta_h \circ \vartheta_{r,h} = \vartheta_r$ and $G_r/G_h = G_{r,h}$. □

Let $\chi\colon \mathcal{O}^*_{\mathcal{P}} \to K^*$ be a continuous character. The rigidification of Ω^χ_w will be denoted with the same symbol. We have proved that this sheaf does not depend on r, if χ is r-accessible and w satisfies the usual conditions. By definition we have the equality

$$S^D(K, w, K(H), \chi) = \mathrm{H}^0(\mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega^\chi_w).$$

4.4. The sheaves $\Omega_{r,w}$

We would like to put the sheaves Ω_w^χ in families, defining the sheaves $\Omega_{r,w}$, for any integer $r \geq 0$ and any rational $w \leq 1/(q^{r-2}(q+1))$. The existence of these sheaves is a strong argument to guarantee that our definition of Ω_w^χ makes sense.

Let π_i, for $i = 1, 2$, be the natural projection from $\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$ to the i-th factor. We write $\mathcal{S}_{r,v}$ also for $\pi_2^{-1}(\mathcal{S}_{r,v})$ and $\mathcal{F}'_{r,w}$ also for $\pi_2^{-1}(\mathcal{F}'_{r,w})$.

There is a natural action of $\mathcal{S}_{r,v}$ on $\mathcal{O}_{\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}}$ that 'glues' the actions of $\mathcal{S}_{r,v}$ on $\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}}$ twisted by various points of $\mathcal{W}_r$. Let $x = ub$ be a section of $\mathcal{S}_{r,v}$, where u is a section of $\mathcal{O}_{\mathcal{P}}^*$ and b is a section of $1 + \varpi^{r-v}\mathcal{O}_{\mathfrak{M}(H\varpi^r)(w)}$. If $A \otimes B$ is a local section of $\mathcal{O}_{\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}}$, we define $x(A \otimes B)$ to be the local section of $\mathcal{O}_{\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}}$ that corresponds to the function

$$(\chi, z) \mapsto \chi(a)A(\chi)b^\chi B(z),$$

for $\chi \in \mathcal{W}_r(T)$ and $z \in \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}(T)$, where T is any affinoid K-algebra. This is well defined by Lemma 4.1.7, and in particular it is an analytic function, so $x(A \otimes B)$ is really a section of $\mathcal{O}_{\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}}$. We define the sheaf

$$\tilde{\Omega}_{r,w} := \mathscr{H}om_{\mathcal{S}_{r,v}}(\mathcal{F}'_{r,v}, \mathcal{O}_{\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)}).$$

REMARK 4.4.1. It is possible to put also the $X_{\chi,v}$ in families. Let $\mathfrak{V}_r = \mathrm{Spf}(S_r)$ be an open affine of $\mathfrak{M}(H\varpi^r)(w)$ as always. We write $X_{r,v}$ for the unique element of $\tilde{\Omega}_{r,w}(\mathcal{W}_r \times \mathfrak{V}_r^{\mathrm{rig}})$ that satisfies

$$X_{r,v}(\omega^{\mathrm{std}}) = 1.$$

In this way we obtain a canonical generator of the global sections of $\tilde{\Omega}_{r,w}$.

PROPOSITION 4.4.2. *The sheaves $\tilde{\Omega}_{r,w}$ are invertible sheaves of modules on $\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$. For any $\chi \in \mathcal{W}_r(K)$, we have a natural isomorphism*

$$(\chi, \mathrm{id})^*(\tilde{\Omega}_{r,w}) \cong \tilde{\Omega}_w^\chi.$$

Furthermore, if $w' \leq w$ is a rational number, then the restriction of $\Omega_{r,w}$ to $\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w')^{\mathrm{rig}}$ coincides with $\Omega_{r,w'}$.

PROOF. As for $\tilde{\Omega}_w^\chi$, the first statement follows from the fact that $\mathcal{F}'_{r,w}$ is a Zariski $\mathcal{S}_{r,v}$-torsor. Since $\Omega_{r,w}$ is locally free, its construction commutes with base change, so we obtain the isomorphism $(\chi, 1)^*(\tilde{\Omega}_{r,w}) \cong \tilde{\Omega}_w^\chi$. The proposition follows by Lemma 4.2.17 and its analogue for r-accessible characters. $\square$

As in the case of a single character, we have that G_r acts on $(\mathrm{id} \times \vartheta_r)_*\tilde{\Omega}_{r,w}$.

DEFINITION 4.4.3. Let $r \geq 0$ be an integer, and let $w \leq 1/(q^{r-2}(q+1))$ be a rational number. On $\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}$, we define the sheaf

$$\Omega_{r,w} := ((\mathrm{id} \times \vartheta_r)_*\tilde{\Omega}_{r,w})^{G_r}.$$

PROPOSITION 4.4.4. *The sheaves $\Omega_{r,w}$ are invertible sheaves of modules on $\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}$. For any $\chi \in \mathcal{W}_r(K)$, we have a natural isomorphism*

$$(\chi, \mathrm{id})^*(\Omega_{r,w}) \cong \Omega_w^\chi.$$

Furthermore, if r_1 and r_2 are integers greater than 0 and $w_i \leq 1/(q^{r_i-2}(q+1))$, for $i = 1, 2$, are rational numbers, then the restrictions of Ω_{r_1,w_1} and Ω_{r_2,w_2} to $\mathcal{W}_{r_1} \cap \mathcal{W}_{r_2} \times \mathfrak{M}(H)(w_1)^{\mathrm{rig}} \cap \mathfrak{M}(H)(w_2)^{\mathrm{rig}}$ coincide.

PROOF. As for Ω_w^χ, the first statement follows by the fact that $(\mathrm{id} \times \vartheta_r)^{\mathrm{rig}}$ is finite and étale, with Galois group G_r. By construction and Proposition 4.4.2 we have that $(\chi, 1)^*(\Omega_{r,w}) \cong \Omega_w^\chi$. The last statement follows by Proposition 4.4.2 and Proposition 4.3.7. $\square$

Any local section f of $\Omega_{r,w}$ should be thought as a p-adic analytic family of modular forms. By Proposition 4.4.4, we see that the pullback of f to any $\chi \in \mathcal{W}_r(K)$ is a modular form of weight χ. We have that the natural morphism of sheaves on $\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}$, given by adjunction,

$$\Omega_{r,w} \to (\chi, \mathrm{id})_* \Omega_w^\chi$$

is an epimorphism. Since $\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}$ is an affinoid, Tate's acyclicity Theorem (see [**Tat71**]) implies that the natural specialization map

$$\mathrm{H}^0(\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega_{r,w}) \to S^D(K, w, K(H), \chi)$$

is surjective. In particular, any p-adic modular form of weight χ lives in an analytic family.

Remark 4.4.5. We have the analogue of Remark 4.2.20 for the sheaves $\tilde{\Omega}_{r,w}$ and $\Omega_{r,w}$. Assume we are given an isogeny $f\colon \mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian schemes corresponding to $i_{\mathcal{A}}, i_{\mathcal{B}}\colon \mathrm{Spf}(S_r) \to \mathfrak{M}(H\varpi^r)(w)$. Suppose that the kernel of f intersects trivially the canonical subgroup of $\mathcal{B}$. We write $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$ also for the corresponding maps

$$\mathcal{W}_r \times \mathrm{Spf}(S_r)^{\mathrm{rig}} \to \mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}.$$

The morphisms $\tilde{f}^\chi \otimes_V K$, for various χ, can be put in families, obtaining a map

$$\tilde{f}_r\colon i_{\mathcal{A}}^* \tilde{\Omega}_{r,w} \to i_{\mathcal{B}}^* \tilde{\Omega}_{r,w}.$$

A similar remark applies to the maps f^χ, obtaining

$$f_r\colon ((\mathrm{id} \times \vartheta_r^{\mathrm{rig}}) \circ i_{\mathcal{A}})^* \Omega_{r,w} \to (((\mathrm{id} \times \vartheta_r^{\mathrm{rig}}) \circ i_{\mathcal{B}})^* \Omega_{r,w}.$$

4.5. The deeply ramified case

We now briefly explain what can be done without assuming that $e \leq p-1$. Let n be the largest integer such that μ_{p^n} is contained in $\mathcal{O}_{\mathcal{P}}$. By [**Rob00**], Chapter 5, we have an exact sequence

$$0 \to \mu_{p^n} \to \mathcal{O}_{\mathcal{P}}^* \to \mathcal{O}_{\mathcal{P}} \to 0,$$

where the last group is denoted additively. By choosing a continuation of the exponential, we obtain a splitting of the above exact sequence and an isomorphism

$$\mathcal{O}_{\mathcal{P}}^* \cong \mu_{q-1} \times \mu_{p^n} \times \mathcal{O}_{\mathcal{P}}.$$

We can assume that $1+\varpi$ maps to 1 under the maps $\mathcal{O}_{\mathcal{P}}^* \to \mathcal{O}_{\mathcal{P}}$ given by the above decomposition, so, with a little abuse of notation, we can write $\mathcal{O}_{\mathcal{P}} \cong (1+\varpi)^{\mathcal{O}_{\mathcal{P}}}$ (but note that the logarithm is not injective on $1+\varpi\mathcal{O}_{\mathcal{P}}$). We consider characters χ such that the map $t \mapsto \log(\chi((1+\varpi)^t))$ is $\mathcal{O}_{\mathcal{P}}$-linear. In this way $\mathcal{W}$ becomes isomorphic to the disjoint union of $(q-1)p^n$ open disks of radius 1. We define the notion of r-accessible character as above, but only in the case $r \geq \frac{e}{p-1}$. In this way the definition of $\mathcal{W}_r$ can be adapted without problems. More importantly, if χ is r-accessible and x is a local section of $\mathcal{S}_{r,v}$, we have that x^s is a well defined section of $\mathcal{S}_{r,v}$. The rest of the theory goes smoothly. Thus, the real difference is that we do not have an integral structure for the space of modular forms of level $K(H\varpi^r)$ and weight χ for any r, but only for r big enough. However, if we invert ϖ (i.e. if we take rigidification), the maps ϑ_r and $\vartheta_{r,h}$ are étale, furthermore we have a residual action of G_r and $G_{r,h}$ on our sheaves, so there are no problems in this case.

CHAPTER 5

Hecke operators

In this chapter we define the analogue of the usual Hecke operators acting on the space of p-adic modular forms. First of all we define the U operator, showing that it is a completely continuous operator on the space of overconvergent modular forms. We finally define the $\mathrm{T}_{\mathcal{L}}$ operators, that are analogues to the usual T_l operators.

5.1. The U operator

Let $\chi\colon \mathcal{O}_{\mathcal{P}}^* \to K^*$ be a character in $\mathcal{W}_r$, where $r \geq 1$ is an integer, and let $w \leq 1/(q^{r-2}(q+1))$ be a rational number. In Chapter 4 we have constructed the sheaf $\tilde{\Omega}_w^\chi$ on $\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$ and in particular the spaces $S^D(K, w, K(H\varpi^r), \chi)$ and $S^D(K, w, K(H), \chi)$ are defined. We are going to prove that $S^D(K, w, K(H), \chi)$ is naturally a K-Banach module that satisfies property (Pr) (see [**Buz07**], Part I) and to define U, a completely continuous operator on it. As in the classical case, to obtain that U is completely continuous, we need to restrict to overconvergent modular forms, so in this section we will assume that w is *positive*

Let z be a point of $\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$, and let L be its residue field (it is a finite extension of K), so z comes from a morphism $\gamma_z\colon \mathrm{Spm}(L) \to \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$. We write $\tilde{\gamma}_z\colon \mathrm{Spf}(\mathcal{O}_L) \to \mathfrak{M}(H\varpi^r)(w)$ for the rigid point associated to z. We have

$$\mathrm{H}^0(\mathrm{Spm}(L), \gamma_z^*\tilde{\Omega}_w^\chi) = \mathrm{H}^0(\mathrm{Spf}(\mathcal{O}_L), \tilde{\gamma}_z^*\tilde{\Omega}_w^\chi) \otimes_{\mathcal{O}_L} L.$$

We fix an identification $\mathrm{H}^0(\mathrm{Spf}(\mathcal{O}_L), \tilde{\gamma}_z^*\tilde{\Omega}_w^\chi) \cong \mathcal{O}_L$ and, if f is an element of $\mathrm{H}^0(\mathrm{Spm}(L), \gamma_z^*\tilde{\Omega}_w^\chi)$, we define $|f|_z$ using the natural absolute value on $\mathcal{O}_L$. This definition is independent of all the choices we made. Let now f be an element of $\mathrm{H}^0(\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}, \tilde{\Omega}_w^\chi)$, we define

$$|f(z)| := |\gamma_z^* f|_z,$$

and we set

$$|f| := \sup_{z \in \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}} \{|f(z)|\},$$

where a priori this sup could be infinite.

Definition 5.1.1. Let M be a Banach A-module, where A is an affinoid K-algebra. Following [**Buz07**], we say that M satisfies the *property (Pr)*, if there is a Banach A-module N such that $M \oplus N$ is potentially orthonormizable. By [**Buz07**], we have the notion of a completely continuous operator on such an M, and we also have spectral theory.

Proposition 5.1.2. *The* sup *defined above is always finite, and it is a norm that makes $S^D(K, w, K(H\varpi^r), \chi)$ a potentially orthonormizable K-Banach module.*

Proof. Since we have that $\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}$ is an affinoid, the proposition follows by [**Kas09**], Lemma 2.14. □

Corollary 5.1.3. *We have that $S^D(K, w, K(H), \chi)$ is a K-Banach module that satisfies the property (Pr).*

PROOF. This follows by Proposition 5.1.2, since $S^D(K, w, K(H), \chi)$ is the G_r-invariant subspace of $S^D(K, w, K(H\varpi^r), \chi)$, and G_r is a finite group. $\square$

To define the U operator we need to introduce another type of curves. We use the notations of Section 1.3. We define

$$K(H\varpi^r, q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(H\varpi^r) \text{ s.t. } b \equiv 0 \bmod \varpi \right\}.$$

By [**Kas09**], Section 5, in the case $K_{\mathcal{P}} = K(H\varpi^r, q)$, a choice of a level structure is equivalent to a choice of $(Q, D, \bar{\alpha}^{\mathcal{P}})$, where (here (A, i, θ, α) is an object of the moduli problem for $F_{\mathcal{P}}$-algebras):

(1) Q is an R-point of exact $\mathcal{O}_{\mathcal{P}}$-order ϖ^r in $A[\varpi^r]_1^{2,1}$;
(2) D is a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule of $A[\varpi^r]_1^{2,1}$ of order q which intersects the $\mathcal{O}_{\mathcal{P}}$-submodule scheme generated by Q trivially;
(3) $\bar{\alpha}^{\mathcal{P}}$ is as always.

In this case, the curve M_K will be denoted $M(H\varpi^r, q)$, it is a proper and smooth scheme over K. There is a natural morphism

$$\pi_1 \colon M(H\varpi^r, q) \to M(H\varpi^r),$$

defined by the natural transformation of functors that forgets D. We have that π_1 is flat, and, since $M(H\varpi^r) \to \operatorname{Spec}(K)$ is proper, also π_1 must be proper. It follows that π_1 is finite, being proper and quasi finite.

Given C, a subgroup scheme of $A[q]$ of rank q^{4N}, stable under the action of $\mathcal{O}_D$, we say, following [**Kas04**], Section 4.4, that it is of 'type 2' if

$$C_2^2 \oplus \cdots \oplus C_m^2 = A[q]_2^2 \oplus \cdots \oplus A[q]_m^2$$

and the isomorphism $\theta \colon A[q] \xrightarrow{\sim} A[q]^{\mathrm{D}}$ sends $C \hookrightarrow A[q]$ to $(A[q]/C)^{\mathrm{D}} \hookrightarrow A[q]^{\mathrm{D}}$. Note that C, if it is of type 2, it is uniquely determined by $C_1^{2,1}$. Given D, a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule of $A[\varpi]_1^{2,1}$, we write $t_2(D)$ for the unique subgroup scheme of $A[q]$, of type 2, such that $t_2(D)_1^{2,1} = D$. We can now define another morphism $\pi_2 \colon M(H\varpi^r, q) \to M(H\varpi^r)$. At level of points, it is defined by taking the quotient over $t_2(D)$: in [**Kas04**], Section 4.4, it is shown how to put a level structure on $A/t_2(D)$, except for the point of exact $\mathcal{O}_{\mathcal{P}}$-order ϖ^r, but, since D intersects trivially the $\mathcal{O}_{\mathcal{P}}$-submodule scheme generated by Q, we can take for it the image of Q under the natural map $A \to A/t_2(D)$. We are interested in the analytification of π_1 and π_2, denoted respectively $\pi_{1,\mathrm{rig}}$ and $\pi_{2,\mathrm{rig}}$.

The rigid space associated to $M(H\varpi^r, q)$ will be denoted $\mathfrak{M}(H\varpi^r, q)^{\mathrm{rig}}$, and we write $\mathfrak{M}(H\varpi^r, q)(w)^{\mathrm{rig}}$ for $\pi_{1,\mathrm{rig}}^{-1}(\mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}})$. To define the formal model $\mathfrak{M}(H\varpi^r, q)(w)$, we can proceed as for the definition of $\mathfrak{M}(H\varpi^r)(w)$, taking the normalization, via $\pi_{1,\mathrm{rig}}$, of $\mathfrak{M}(H\varpi^r)(w)$ in $\mathfrak{M}(H\varpi^r, q)(w)^{\mathrm{rig}}$. In this way we obtain $\mathfrak{p}_1 \colon \mathfrak{M}(H\varpi^r, q)(w) \to \mathfrak{M}(H\varpi^r)(w)$, a formal model of $\pi_{1,\mathrm{rig}}$.

PROPOSITION 5.1.4. *Let R be a normal and ϖ-adically complete V-algebra. There is a natural bijection between $\mathfrak{M}(H\varpi^r, q)(w)(R)$ and the set of isomorphism classes of sextuples $(\mathcal{A}, i, \theta, \bar{\alpha}, Y, \mathcal{D})$, where:*

- *$(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$ is an object of the moduli problem, with $\mathcal{A}$ defined over R, of $\mathcal{M}(H\varpi^r)(w)$;*
- *$\mathcal{D}$ is a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule of $\mathcal{A}[\varpi^r]_1^{2,1}$ of rank r that intersects trivially the canonical subgroup of $\mathcal{A}[\varpi^r]_1^{2,1}$.*

PROOF. This Lemma 3.11 of [**AIS11**]. Let x be in $\mathfrak{M}(H\varpi^r, q)(w)(R)$. We have that $\mathfrak{p}_1(x) \in \mathfrak{M}(H\varpi^r)(w)(R)$, hence we obtain the quintuple $(\mathcal{A}, i, \theta, \bar{\alpha}, Y)$ of the proposition by the moduli description of $\mathfrak{M}(H\varpi^r)(w)$. It remains to construct $\mathcal{D}$. Let $A := \mathcal{A}_K$. Taking the base change of x to R_K, we obtain a point of

$\mathfrak{M}(H\varpi^r,q)(w)^{\mathrm{rig}}$, and in particular we have $D \subseteq A[\varpi^r]$, a finite and flat subgroup scheme that intersects trivially the canonical subgroup of $A[\varpi^r]$. We need to prove that D extends to $\mathcal{D}$. Let z be the R-point of $\mathcal{M}(H)$ corresponding to $(\mathcal{A}, i, \theta, \bar{\alpha})$ (here we are forgetting part of the level structure). We have that D gives t_K, an R_K point of $M(H,\varpi)$ that maps to the base change of z to R_K. Since the natural morphism $\mathcal{M}(H,\varpi) \to \mathcal{M}(H)$ is finite and flat (see [**Kas09**], Section 5), and R is normal, we have that t_K extends to a unique $t \in \mathcal{M}(H,\varpi)(R)$ that maps to z. In particular we obtain the required $\mathcal{D}$. Conversely, if we have a quintuple as in the statement of the proposition, we obtain an R_K-valued point of $\mathfrak{M}(H\varpi^r,q)(w)^{\mathrm{rig}}$, that extends to an R-valued point by normality of R. □

LEMMA 5.1.5. *Let R be a normal and ϖ-adically complete V-algebra. Furthermore, let $(\mathcal{A}, i, \theta, \bar{\alpha}, Y, \mathcal{D})$ be in $\mathfrak{M}(H\varpi^r,q)(qw)(R)$. Taking the quotient over $t_2(\mathcal{D})$, we obtain an object of $\mathfrak{M}(H\varpi^r)(w)(R)$.*

PROOF. Using the forgetful morphisms $\mathfrak{M}(H\varpi^r,q)(qw) \to \mathfrak{M}(H\varpi,q)(qw)$ and $\mathfrak{M}(H\varpi^r)(w) \to \mathfrak{M}(H\varpi)(w)$ we can work with $r = 1$. We can assume that R is a discrete valuation ring, whose valuation extends the one of V and that $\mathcal{A}$ is supersingular. Let $\mathcal{B}$ be $\mathcal{A}/t_2(\mathcal{D})$. Forgetting the extra structure, we need to prove that the R-point corresponding to $\mathcal{B}$ lies in $\mathfrak{M}(H)(qw)$. To prove this, let us consider the commutative diagram

$$\begin{tikzcd}
(\mathcal{A}[\varpi]_1^{2,1})^\vee(\overline{R}_K) \arrow[r] & \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}_1 \arrow[r, "\sim"] & \overline{R}_1 \\
(\mathcal{B}[\varpi]_1^{2,1})^\vee(\overline{R}_K) \arrow[r] \arrow[u, two heads] & \underline{\omega}_{\mathcal{B}/R} \otimes_R \overline{R}_1 \arrow[r, "\sim"] \arrow[u] & \overline{R}_1 \arrow[u]
\end{tikzcd}$$

We use the notation of the proof of Proposition 3.1.7, except that we write $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ to specify whether we are talking about $\mathcal{A}$ or $\mathcal{B}$, so we need to prove that $\mathrm{v}(E_{\mathcal{B}}) \leq \frac{\mathrm{v}(E_{\mathcal{A}})}{q}$. The right vertical map is the reduction of the multiplication by an element of valuation $\frac{\mathrm{v}(E_{\mathcal{A}})}{q}$ by [**Far07**], Remarque 2. Looking at the proof of Proposition 3.1.7, we see that the image of the compositions of the horizontal maps are generated by elements of valuation, respectively, $\frac{\mathrm{v}(E_{\mathcal{A}})}{q-1}$ and $\frac{\mathrm{v}(E_{\mathcal{B}})}{q-1}$, so $\frac{\mathrm{v}(E_{\mathcal{B}})}{q-1} + \frac{\mathrm{v}(E_{\mathcal{A}})}{q} = \frac{\mathrm{v}(E_{\mathcal{A}})}{q-1}$ as required. □

Using Propositions 2.3.2 and 5.1.4, together with Lemma 5.1.5, we define the morphism
$$\mathfrak{p}_2 \colon \mathfrak{M}(H\varpi^r,q)(qw) \to \mathfrak{M}(H\varpi^r)(w),$$
on points, taking the quotient over $\mathcal{D}$.

We write $\mathfrak{A}(H\varpi^r,q)(w)$ for the base change, via $\mathfrak{p}_1$, to $\mathfrak{M}(H\varpi^r,q)(w)$, of $\mathfrak{A}(H\varpi^r)(w)$. By definition, we have that $\mathfrak{A}(H\varpi^r,q)(w)$ is equipped with $\mathfrak{D}$, a subgroup of order q of its ϖ^r-torsion, that has trivial intersection with the canonical subgroup. The isogeny
$$\pi_{\mathfrak{D}} \colon \mathfrak{A}(H\varpi^r,q)(qw) \to \mathfrak{A}(H\varpi^r,q)(qw)/\mathfrak{D}$$
is defined over $\mathfrak{M}(H\varpi^r,q)(qw)$. Since $\mathfrak{A}(H\varpi^r,q)(qw)/\mathfrak{D}$ is the base change, via $f_{w,qw} \circ \mathfrak{p}_2$, to $\mathfrak{M}(H\varpi^r,q)(qw)$, of $\mathfrak{A}(H\varpi^r)(qw)$, we obtain, using Remark 4.2.20 and Lemma 4.2.17, a morphism
$$\tilde{\pi}_{\mathfrak{D}}^{\chi} \colon \mathfrak{p}_2^* \tilde{\Omega}_w^{\chi} \to \mathfrak{p}_1^* \tilde{\Omega}_{qw}^{\chi}.$$
To define the U operator, we can now follow [**AIS11**], Section 3.1.1. First of all consider
$$\tilde{\mathrm{U}} \colon S^D(K,w,K(H\varpi^r),\chi) \to S^D(K,w,K(H\varpi^r),\chi),$$

defined as the composition

$$\mathrm{H}^0(\mathfrak{M}(H\varpi^r)(qw), \tilde{\Omega}^\chi_{qw} \otimes_V K) \xrightarrow{\tilde{\rho}^{\mathrm{rig}}_{qw,w}} \mathrm{H}^0(\mathfrak{M}(H\varpi^r)(w), \tilde{\Omega}^\chi_w \otimes_V K) \xrightarrow{\mathfrak{p}_2^*}$$
$$\to \mathrm{H}^0(\mathfrak{M}(H\varpi^r, q)(qw), \mathfrak{p}_2^*\tilde{\Omega}^\chi_w \otimes_V K) \xrightarrow{\tilde{\pi}^\chi_{\mathfrak{D}}}$$
$$\to \mathrm{H}^0(\mathfrak{M}(H\varpi^r, q)(w), \mathfrak{p}_1^*\tilde{\Omega}^\chi_{qw} \otimes_V K) \xrightarrow{(\pi_{1,\mathrm{rig}})_*} \mathrm{H}^0(\mathfrak{M}(H\varpi^{er+1})(qw), \tilde{\Omega}^\chi_{qw} \otimes_V K),$$

where $(\pi_{1,\mathrm{rig}})_*$ is the map induced by the trace, that is well defined since $\pi_{1,\mathrm{rig}}$ is finite and flat. All the maps used to define $\tilde{\mathrm{U}}$ are G_r-equivariant, so the same holds for $\tilde{\mathrm{U}}$. Taking G_r-invariants we obtain, from $\tilde{\mathrm{U}}$, a map $S^D(K, w, K(H), \chi) \to S^D(K, w, K(H), \chi)$.

Definition 5.1.6. Let χ be an r-accessible character. The map

$$\begin{aligned}\mathrm{U}\colon S^D(K, qw, K(H), \chi) &\to S^D(K, qw, K(H), \chi)\\ f &\mapsto f_{|\,\mathrm{U}}\end{aligned}$$

is defined as $1/q$ times the map induced by $\tilde{\mathrm{U}}$.

Proposition 5.1.7. *The operator* U *is completely continuous.*

Proof. We claim that $\tilde{\mathrm{U}}$ is completely continuous. Since $\tilde{\mathrm{U}}$ factors through $\tilde{\rho}^{\mathrm{rig}}_{qw,w}$, it is enough to prove that $\tilde{\rho}^{\mathrm{rig}}_{qw,w}$ is completely continuous, and this can be done in exactly the same way as **[Kas09]**, Proposition 2.20. The proposition follows. □

Remark 5.1.8. Let us suppose that $r = 1$, so χ is an accessible character. Using Proposition 4.2.29 and Corollary 4.2.30, we can give a more concrete description of the U operator, using test objects. Let f be an element of $S^D(K, w, K(H), \chi)$. Take any test object $T = (\mathcal{A}/S, i, \theta, \bar{\alpha}, Y, \eta)$ as in Proposition 4.2.29. Let S' be a normal and ϖ-adically complete S-algebra such that

- $S_K \to S'_K$ is finite and étale;
- all finite and flat subgroup schemes of $\mathcal{A}_{\overline{S},K}[\varpi]^{2,1}_1$ are defined over S'_K.

Repeating what we have done in the proof of Proposition 5.1.4, we see that any finite and flat subgroup scheme of $\mathcal{A}_{S',K}[\varpi]^{2,1}_1$ extends to a subgroup scheme of $\mathcal{A}_{S'}[\varpi]^{2,1}_1$. Let $\mathcal{D}$ be any such subgroup, and suppose that $\mathcal{D}$ intersects trivially the canonical subgroup of $\mathcal{A}_{S'}[\varpi]^{2,1}_1$. We have that T gives a test object $((\mathcal{A}_{S'}/t_2(\mathcal{D}))/S', i', \theta', \bar{\alpha}', Y', \eta')$. Indeed the only non trivial thing to define is η'. Let $i_1, i_2\colon \mathrm{Spf}(S) \to \mathfrak{M}(H\varpi)(w)$ be the morphisms corresponding to $\mathcal{A}$ and $\mathcal{A}/t_2(\mathcal{D})$, respectively. In Remark 4.2.20 we showed that there is an isomorphism between the global sections of $i_1^*\mathcal{F}'$ and $i_2^*\mathcal{F}'$. We define η' as the image of η under this isomorphism. We have

$$\widetilde{f_{|\,\mathrm{U}}}(T) = \frac{1}{q}\sum_{\mathcal{D}} \tilde{f}(((\mathcal{A}_{S'}/t_2(\mathcal{D}))/S', i', \theta', \bar{\alpha}', Y', \eta')),$$

where the sum is taken over all $\mathcal{D}$'s as above.

For various w's, the norms defined on $S^D(K, w, K(H), \chi)$ are compatible, so $S^D_\dagger(K, K(H), \chi)$ is naturally a Fréchet space, and we obtain a continuous operator

$$\mathrm{U}\colon S^D_\dagger(K, K(H), \chi) \to S^D_\dagger(K, K(H), \chi).$$

Using the maps $\tilde{\pi}_{\mathfrak{D},r}$ defined in Remark 4.4.5, we can work with families: for any integer $r \geq 0$, we obtain an operator

$$\tilde{\mathrm{U}}_r\colon \tilde{\Omega}_{r,w}(\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}) \to \tilde{\Omega}_{r,w}(\mathcal{W}_r \times \mathfrak{M}(H\varpi^r)(w)^{\mathrm{rig}}),$$

such that the pullback $(\chi, \mathrm{id})^*(\tilde{\mathrm{U}}_r)$, for $\chi \in \mathcal{W}_r(K)$, is the $\tilde{\mathrm{U}}$ operator defined above. Everything we did above can be repeated for families, in particular we have the U_r operator and the following proposition, where $A_r := \mathcal{O}_{\mathcal{W}_r}(\mathcal{W}_r)$.

PROPOSITION 5.1.9. *For any integer $r \geq 1$ and any rational $w \leq 1/(q^{r-2}(q+1))$, we have that*

$$\mathrm{H}^0(\Omega_{r,w}, \mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}})$$

is a Banach A_r-module that satisfies the property (Pr). Furthermore the operator

$$\mathrm{U}_r \colon \mathrm{H}^0(\Omega_{r,w}, \mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}) \to \mathrm{H}^0(\Omega_{r,w}, \mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}})$$

is completely continuous.

REMARK 5.1.10. Kassaei has proved a result of classicality for modular forms of level $K(H)$ and integral weight k. Let f be in $S^D(K, w, K(H), k)$ and suppose that $f_{|\,\mathrm{U}} = af$, for some $a \in K$. If a satisfies $\mathrm{v}(a) < k - ef$, then f is classical, i.e. it can be extended to a global section of $\underline{\omega}^{\otimes k}_{\mathfrak{M}(H,\varpi)^{\mathrm{rig}}}$. See **[Kas09]**, Theorem 5.1.

Let $\chi \colon \mathcal{O}^*_{\mathcal{P}} \to K^*$ be a continuous character and let $\nu \in \mathbb{R}$. Let $\mathcal{V} \subseteq \mathcal{W}$ be an affinoid that contains the point of $\mathcal{W}$ given by χ. We write F for the characteristic power series of U_r restricted to $\mathrm{H}^0(\mathcal{V} \times \mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega_{r,w})$. Using the notations of **[Bel09]**, page 31, we have that F is ν-adapted if $\mathcal{V}$ is sufficiently small. In particular $\mathrm{H}^0(\mathcal{V} \times \mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega_{r,w})^{\leq \nu}$ makes sense. Let $\mathcal{V} = \mathrm{Spm}(R)$. Note that χ gives a morphism $R \to K$. We have an isomorphism

$$\mathrm{H}^0(\mathcal{V} \times \mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega_{r,w})^{\leq \nu} \otimes_R K \cong S^D(K, w, K(H), \chi)^{\leq \nu}.$$

In particular we have the following proposition, that, together with Remark 5.1.10, gives the analogue of Coleman's Theorem of the Introduction.

PROPOSITION 5.1.11. *Let ν be in $\mathbb{R}$ and let f be in $S^D(K, w, K(H), \chi)^{\leq v}$. Then there is an affinoid $\mathcal{V} \subseteq \mathcal{W}$ such that f can be deformed to a family of modular forms over $\mathcal{V}$. Furthermore, the* U*-operator acts with slope $\leq \nu$ on this family.*

5.2. Other Hecke operators

We now sketch the definition of other Hecke operators, see the beginning of Section 1.3 for the notations. Let $l \neq p$ be a rational prime. We write $\mathcal{L}_1, \ldots, \mathcal{L}_k$ for the primes of F lying over l. As in the case of the prime p, let $\mathcal{L}$ be $\mathcal{L}_1$. We assume that l splits in $\mathbb{Q}(\sqrt{\lambda})$, and that B is split at $\mathcal{L}$. We denote the completion of F at $\mathcal{L}_i$ with $F_{\mathcal{L}_i}$. We have

$$G(\mathbb{Q}_l) \cong \mathbb{Q}_l^* \times \mathrm{GL}_2(F_{\mathcal{L}}) \times \mathrm{GL}_2(F_{\mathcal{L}_2}) \times \cdots \times \mathrm{GL}_2(F_{\mathcal{L}_k}).$$

Recall that we are only considering compact open subgroups of $G(\mathbb{A}^f)$ of the form $K = \mathbb{Z}_p^* \times K_{\mathcal{P}} \times H$. In this section, we make the further assumption that H is of the form

$$H = \mathbb{Z}_l^* \times \mathrm{GL}_2(\mathcal{O}_{F_{\mathcal{L}}}) \times H',$$

where H' is compact open. Let ϖ_l be a uniformizer of $\mathcal{O}_{F_{\mathcal{L}}}$. If A is an abelian scheme that is part of an object of any of the moduli problems we have studied, our assumptions on $\mathcal{L}$ imply that we have a decomposition of $A[\varpi_l]$ similar to that of $A[\varpi]$, so $A[\varpi_l]^{2,1}_1$ is defined and it has an action of κ_l, the residue field $\mathcal{O}_{F_{\mathcal{L}}}/\varpi_l$.

Let $\chi \colon \mathcal{O}^*_{\mathcal{P}} \to K^*$ be an r-accessible character, so we have the sheaf $\tilde{\Omega}^{\chi}_w$ on $\mathfrak{M}(H\varpi^r)(w)$. As in the case of the U operator, we are going to change the level structure, but this time at $\mathcal{L}$. Let $H_{\mathcal{L}}$ be the set of invertible 2×2 matrices with left lower corner congruent to 0 modulo ϖ_l. The Shimura curve corresponding to the case $K_{\mathcal{P}} = K(H\varpi^r)$ and $H = \mathbb{Z}_l^* \times H_{\mathcal{L}} \times H'$ will be denoted with X. We have that X parametrizes objects of the moduli problem of $M(H\varpi^r)$ plus a finite and flat subgroup of $A[\varpi_l]_1^{2,1}$ of order $|\kappa_l|$, stable under the action of $\mathcal{O}_{F_{\mathcal{L}}}$ (see the description of the level structure in Section 1.3). If D is such a subgroup, we can define $t_2(D)$ as in the case of subgroups of $A[\varpi]_1^{2,1}$, and also the quotient of A by $t_2(D)$ can be defined as in **[Kas04]**, Section 4.4 (since $l \neq p$, we have that A and its quotient will have the same degree of overconvergence). We can repeat everything we have done for the U operator. In particular we obtain, with the obvious notations, two morphisms

$$\mathfrak{p}_1, \mathfrak{p}_2 \colon \mathfrak{X}(w) \to \mathfrak{M}(H\varpi^r)(w).$$

Furthermore, we have a morphism $\tilde{\pi}_{\mathfrak{D}} \colon \mathfrak{p}_2^* \tilde{\Omega}_w^{\chi} \to \mathfrak{p}_1^* \tilde{\Omega}_w^{\chi}$.

Definition 5.2.1. We define the operator

$$\tilde{\mathrm{T}}_{\mathcal{L}} \colon S^D(K, w, K(H\varpi^r), \chi) \to S^D(K, w, K(H\varpi^r), \chi)$$

exactly as in the case of $\tilde{\mathrm{U}}$. Taking G_r-invariants and dividing by $|\kappa_l|+1$, we obtain the operator

$$\mathrm{T}_{\mathcal{L}} \colon S^D(K, w, K(H), \chi) \to S^D(K, w, K(H), \chi).$$

Remark 5.2.2. Note that $\tilde{\mathrm{T}}_{\mathcal{L}}$ is a continuous operator, but, since it does not change the degree of overconvergence, it is not completely continuous.

Also the operators $\tilde{\mathrm{T}}_{\mathcal{L}}$ can be put in families. Furthermore, if χ is accessible, we have a description of $\tilde{\mathrm{T}}_{\mathcal{L}}$ in terms of testing objects similar to that of Remark 5.1.8, taking quotient over subgroups of $\mathcal{A}[\varpi_l]_1^{2,1}$.

5.3. The eigencurve

Let $r \geq 1$ be an integer, and assume that $0 < w$ is a rational number sufficiently small. Let $\mathcal{Z}_r$ be the spectral variety associated to the U-operator acting on $\mathrm{H}^0(\mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}}, \Omega_{r,w})$. We have proved that all assumptions needed to use the machine developed by Buzzard in **[Buz07]** are satisfied, so we have the following

Theorem 5.3.1. *We have a rigid space $\mathcal{C}_r \subseteq \mathcal{W}_r \times \mathbb{A}_K^{1,\mathrm{rig}}$ equipped with a finite morphism $\mathcal{C}_r \to \mathcal{Z}_r$. If L is a finite extension of K, then the points of $\mathcal{C}_r(L)$ correspond to systems of eigenvalues of modular forms with growth condition w and coefficients in L. If $x \in \mathcal{C}(L)$, let $\mathcal{M}(w)_x$ be the set of modular forms corresponding to x. Then all the elements of $\mathcal{M}(w)_x$ have weight $\pi_1(x) \in \mathcal{W}(L)$ and the* U-*operator acts on $\mathcal{M}(w)_x$ with eigenvalue $\pi_2(x)^{-1}$. For various r and w, these construction are compatible. Letting $r \to \infty$ we have $w \to 0$ and we obtain the global eigencurve $\mathcal{C}_r \subseteq \mathcal{W}_r \times \mathbb{A}_K^{1,\mathrm{rig}}$.*

APPENDIX

Raynaud theory

We will work locally throughout the appendix. We fix a rational number $0 \leq w < \frac{q}{q+1}$. Let $\operatorname{Spf}(R) \to \mathfrak{M}(H)(w)$ be an affine. We denote with $\operatorname{Spf}(S)$ the inverse image of $\operatorname{Spf}(R)$ to $\mathfrak{M}(H\varpi)(w)$. Let $\mathcal{A}$ be the abelian scheme associated to $\operatorname{Spf}(R)$. We know that we have a canonical subgroup $\mathcal{C}_1^{2,1} \subseteq \mathcal{A}[\varpi]_1^{2,1}$. We are going to obtain some specific results about the group scheme structure of $\mathcal{C}_1^{2,1}$. In [**Col05**], Coleman studied the canonical subgroup of an elliptic curve in the spirit of the classification given in [**OT70**]. We do not need such an explicit study of the canonical subgroup, but it is interesting in itself. Since our group is of order q, we need to follow the paper [**Ray74**]. Recall that V is a finite extension of $\mathcal{O}_{\mathcal{P}}$, with field of fraction K.

In this appendix we find it convenient to denote the Teichmüller character $[\cdot]$ with $\chi_1(\cdot)$ (see below). Let W be an R-algebra (we will be interested in the cases $W = R$, $W = S$, and $W = S_K$) and let T be a W-algebra such that $\operatorname{Spec}(T)$ is a κ-vector space scheme of order q over W, in particular we can choose T such that $\operatorname{Spec}(T) = \mathcal{C}_1^{2,1}$. We write $c\colon T \to T \otimes_{R'} T$ for its comultiplication. Let T' be the W-linear dual of T, so $\operatorname{Spec}(T)^{\mathrm{D}} = \operatorname{Spec}(T')$. We have that $\operatorname{Spec}(T')$ is a κ-vector space scheme too. Indeed, the action of κ is given, on points, as follows. Let U be a W-algebra and let $z \in \kappa$. If $u \in \operatorname{Spec}(T)^{\mathrm{D}}(U) = \operatorname{Hom}(\operatorname{Spec}(T)_U, \mathbb{G}_{\mathrm{m},U})$, then zu is the homomorphism $\operatorname{Spec}(T)_U \to \mathbb{G}_{\mathrm{m},U}$ given by

$$(zu)(x) = u(zx),$$

for all U-algebra X and all $x \in \operatorname{Spec}(T)(X)$. We denote with I and I' the augmentation ideal of T and T'. Since κ acts on $\operatorname{Spec}(T)$, if $z \in \kappa$, we have a morphism, as W-algebras, $[z]\colon T \to T$: these maps satisfy the obvious compatibility properties. We write $[z]'$ for the corresponding morphisms for T'.

Let M be the set of multiplicative characters $\chi\colon \kappa^* \to \mathcal{O}_{\mathcal{P}}^*$, extended to the whole κ by $\chi(0) = 0$ (we will often see χ as taking values in some $\mathcal{O}_{\mathcal{P}}$-algebra, using the natural morphism from $\mathcal{O}_{\mathcal{P}}$). Following Raynaud in [**Ray74**], we say that $\chi \in M$ is a *fundamental character* if the map

$$\kappa \to \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}/\varpi\mathcal{O}_{\mathcal{P}} = \kappa$$

is a field homomorphism. If χ satisfies this condition, all fundamental characters are of the form $z \mapsto \chi(z)^{p^i}$, with $z \in \kappa$. It follows that we can denote all fundamental characters as χ_{p^i}, where $i \in \mathbb{Z}/f\,\mathbb{Z}$ (recall that $q = p^f$). In [**Ray74**], χ_{p^i} is denoted with χ_i, the reason why we write the subscript in this way will become clear later on. Furthermore we can assume that $\chi_{p^{i+1}} = \chi_{p^i}^p$ and that χ_1 is the Teichmüller character. Any $\chi \in M$ can be decomposed as

$$\chi = \prod_{i \in \mathbb{Z}/f\,\mathbb{Z}} \chi_{p^i}^{n_i},$$

with $0 \leq n_i \leq p-1$, and, if $\chi \neq 1$, this decomposition is unique.

Given a character $\chi \in M$, we define the W-linear operator on T

$$i_\chi = \frac{1}{q-1} \sum_{z \in \kappa^*} \chi^{-1}(z)[z].$$

We also have its analogue for T', denoted in the same way. The operator i_χ preserves both I and I', we write $I_\chi := i_\chi(I)$ and similarly for I'.

LEMMA A.1. *We have a decomposition, as R-modules,*

$$I = \bigoplus_{\chi \in M} I_\chi.$$

Furthermore, I_χ is the set of all $a \in I$ such that $[z]a = \chi(z)a$ for all $z \in \kappa^*$, and similarly for I'. Each I_χ and I'_χ is a projective R-module of rank 1.

PROOF. The various i_χ's are orthogonal idempotents in the group algebra $W[\kappa^*]$, whose sum is 1, so we get the decomposition. Since we have $[z]i_\chi = \chi(z)i_\chi$, if $a \in I$ satisfies $a = i_\chi(b)$ for some $b \in I$, we must have $[z]a = \chi(z)a$ for all $z \in \kappa^*$. Conversely, if a satisfies this condition, we have $i_\chi(a) = a$. The last part of the lemma is **[Ray74]**, Proposition 1.2.2. □

By Remark 2.3.6, we know that $\mathcal{C}_1^{2,1}$ is, as a scheme, $\operatorname{Spec}(R[x]/(x^q + \frac{\varpi}{E}x))$ and that the base change to S_K of $\mathcal{C}_1^{2,1}$ is a constant group scheme, with associated abstract group κ. Since κ acts on the S_K-points of $\mathcal{C}_1^{2,1}$, that correspond to 0 and the $q-1$-th roots of $-\frac{\varpi}{E}$, we see that there is $\overline{\chi} \in M$ such that the action of $z \in \kappa$ on $S_K[x]/(x^q + \frac{\varpi}{E}x)$ is given by $x \mapsto \overline{\chi}(z)x$ (clearly $\overline{\chi}(z)$, if not 0, is a $q-1$-th root of unity).

LEMMA A.2. *We have that $\overline{\chi} = \chi_1$.*

PROOF. Let $\tilde{z} \in \mathcal{O}_{\mathcal{P}}$ be a lifting of $z \in \kappa$, by condition (1a) of the moduli problem, we have $\tilde{z}\,\mathrm{d}\,x = \mathrm{d}(\overline{\chi}(z)x) = \overline{\chi}(z)\,\mathrm{d}\,x$, so $\tilde{z} \equiv \overline{\chi}(z) \bmod \varpi$ (we can restrict to the ordinary locus, so we may assume that E is a unit). Since $\overline{\chi}$ is multiplicative, the lemma follows. □

The Hopf algebra of $(\mathcal{C}_1^{2,1})_{S_K}$ is isomorphic to the algebra of S_K-valued functions on κ. Let ε_z, for $z \in \kappa$, denote the characteristic function of $\{z\}$, we then have a natural isomorphism (of S_K-modules)

$$S_K[x]/(x^q + \frac{\varpi}{E}x) \to \bigoplus_{z \in \kappa} S_K\varepsilon_z$$
$$x \mapsto \sum_{z \in \kappa} \chi_1(z)\alpha\varepsilon_z$$

where α is a chosen $q-1$-th root of $-\frac{\varpi}{E}$. The ε_z's, with $z \neq 0$, form a basis of I. If χ is in M, we write ε_χ for $\sum_{z \in \kappa} \chi(z)\varepsilon_z$: it is the generator of I_χ defined in **[Ray74]**, page 249. Via the above isomorphism, we have $x = \alpha\varepsilon_{\chi_1}$.

LEMMA A.3. *We have that $\Omega^1_{\mathcal{C}_1^{2,1}/R}$ is generated, as $R[x]/(x^q + \frac{\varpi}{E}x)$-module, by $\mathrm{d}(x)$, with the unique relation $\frac{\varpi}{E}\mathrm{d}(x) = 0$. In particular, $\underline{\omega}_{\mathcal{C}_1^{2,1}/R}$, the module of invariant differentials of $\mathcal{C}_1^{2,1}$, is isomorphic, as R-module, to $R/\frac{\varpi}{E}R\,\mathrm{d}(x)$.*

PROOF. Let $B := R[x]/(x^q + \frac{\varpi}{E}x)$. We have $\Omega^1_{B/R} = B/\frac{\varpi}{E}B\,\mathrm{d}(x)$. Indeed, let $b \in B$ such that $\frac{\varpi}{E}b = q$, we have

$$\mathrm{d}(x^q + \frac{\varpi}{E}x) = \frac{\varpi}{E}(1 + bx^{q-1})\,\mathrm{d}(x).$$

But $(1 + bx^{q-1})(1 - \frac{bx^{q-1}}{1-q}) = 1$, the lemma follows. □

We can now give an explicit formula for the comultiplication. By Lemma A.2, we have $x^{p^i} = \alpha^{p^i}\varepsilon_{\chi_{p^i}}$. Let J be the set of all f-tuples of integers $\underline{n} = (n_0, \ldots, n_{f-1})$ such that $0 \leq n_i \leq p-1$ for all i and $\underline{n} \neq (0, \ldots, 0)$. Using the above decomposition of $\chi \in M$, we have a bijection between M and J, we write $\chi_{\underline{n}}$ for the character associated to $\underline{n} \in J$. Given $\underline{n} \in J$, we write $-\underline{n}$ for the unique f-tuple in J such that $\chi_{\underline{n}}\chi_{-\underline{n}} = \chi_1$. By [**Ray74**], we see that the comultiplication has the form

$$c(\varepsilon_{\chi_1}) = \varepsilon_{\chi_1} \otimes 1 + 1 \otimes \varepsilon_{\chi_1} + \sum_{\underline{n}\in J} w_{\chi_{\underline{n}},\chi_{-\underline{n}}} \varepsilon_{\chi_{\underline{n}}} \otimes \varepsilon_{\chi_{-\underline{n}}},$$

where $w_{\chi,\chi'} \in \mathcal{O}_{\mathcal{P}}$ are the universal constants defined in [**Ray74**]. We have

$$\varepsilon_{\chi_{\underline{n}}} = \prod_{i=0}^{f-1} \varepsilon_{\chi_{p^i}}^{n_i} = \prod_{i=0}^{f-1} \varepsilon_{\chi_1}^{n_i p^i} = \varepsilon_{\chi_1}^{|\underline{n}|},$$

where $\underline{n} = (n_0, \dots, n_{f-1})$ and $|\underline{n}| = \sum_{i=0}^{f-1} n_i p^i$. Note that $|\underline{n}| + |-\underline{n}| = q \equiv 1 \bmod (q-1)$ and that $|\cdot|$ gives a bijection between J and $\{1, \dots, q-1\}$. By [**Ray74**], page 257, we have

$$w_{\chi_{\underline{n}},\chi_{-\underline{n}}} = \frac{w^{h_{\underline{n}}}}{w_{\underline{n}} w_{-\underline{n}}},$$

where w and $w_{\underline{n}}$ are universal constants in $\mathcal{O}_{\mathcal{P}}$ and $h_{\underline{n}}$ is the smallest integer, with $0 < h_{\underline{n}} \leq f$, such that $p^{f-h_{\underline{n}}}$ divides $|\underline{n}|$. It follows that we can write

$$c(\varepsilon_{\chi_1}) = \varepsilon_{\chi_1} \otimes 1 + 1 \otimes \varepsilon_{\chi_1} + \sum_{i=1}^{q-1} \frac{w^{h_i}}{w_i w_{q-i}} \varepsilon_{\chi_i} \otimes \varepsilon_{\chi_{q-i}},$$

where $\chi_j := \chi_{\underline{n}}$, with $|\underline{n}| = j$, $w_j = w_{\chi_j}$ and similarly for h_j (this notation is not used in [**Ray74**]). Note that $\chi_{q-1}(z) = 1$ for all $z \in \kappa^*$. Now we use $x^{p^i} = \alpha^{p^i} \varepsilon_{\chi_{p^i}}$: by the second point of [**Ray74**], Proposition 1.3.1, there is a unit $u \in \mathcal{O}_{\mathcal{P}}^*$ such that $w = pu = \varpi^e u$. Since $h_i \geq 1$, we can write

$$c(x) = x \otimes 1 + 1 \otimes x - \varpi^{e-1} u E \sum_{i=1}^{q-1} \frac{w^{h_i - 1}}{w_i w_{q-i}} x^i \otimes x^{q-i}. \tag{A.1}$$

PROPOSITION A.4. *The comultiplication in* $\mathcal{C}_1^{2,1}$ *is given by formula* (A.1).

PROOF. Since S_K is finite and étale over R_K, the above formula gives the comultiplication of $(\mathcal{C}_1^{2,1})_K$. By flatness of R over $\mathcal{O}_{\mathcal{P}}$, the same holds for $\mathcal{C}_1^{2,1}$. $\square$

REMARK A.5. Do not confuse our w_i's with Raynaud's ones, that are all equal and are denoted with w here. Since χ_{p^i} is a fundamental character for each i, we have $w_{p^i} = 1$ for $i = 0, \dots, f-1$. In general we have that $F_{\mathcal{P}} \neq \mathbb{Q}_p$, but it is interesting to see what happens to the formulas in the case $\varpi = p$ and $f = 1$ (hence $e = 1$ and $q = p$), that is the situation studied in [**OT70**]. Let us write w_i' for the universal constants introduced there. We have

$$w_i = (-1)^{i+1} \frac{w_i'}{(p-1)^{i-1}} \text{ and } w = \frac{w_p'}{(p-1)^{p-1}}.$$

Furthermore $h_1 = 1$ for all i in this case. In particular we find that

$$c(x) = x \otimes 1 + 1 \otimes x - \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{w_{p-1}' E}{w_i' w_{p-i}'} x^i \otimes x^{p-i},$$

so our description of the comultiplication is exactly the same as the one given in [**Col05**] for the canonical subgroup of an elliptic curve.

Following [**AIS11**], we now find an explicit S-point of $(\mathcal{C}_1^{2,1})^{\mathrm{D}}$. Note that, by Proposition A.8 below, this point does not allow to develop our theory. As above, it is convenient to start working over S_K. Since $(\mathcal{C}_1^{2,1})_{S_K}$ is constant, the Hopf algebra of $(\mathcal{C}_1^{2,1})_{S_K}^{\mathrm{D}}$ is isomorphic to $S_K[\kappa]$, the group algebra of κ with coefficients in S_K. The canonical base of $S_K[\kappa]$ will be denoted $\{\mathbf{z}\}_{z\in\kappa}$, so we have $\mathbf{z}_1\mathbf{z}_2 = \mathbf{z}$, with $z = z_1 + z_2$. Using ζ_p, our fixed primitive p-th root of unity, we can identify $\mathbb{F}_p$ with $\mu_p(V)$. In particular, the trace map $\mathrm{Tr}_{\kappa/\mathbb{F}_p}$ can be seen as a morphism $\Psi\colon \kappa \to \mu_p(V) \subseteq V^*$. In this way, we obtain a morphism of group schemes $(\mathcal{C}_1^{2,1})_{S_K} \to \mu_p$. This morphism corresponds to the S_K-point of $(\mathcal{C}_1^{2,1})^{\mathrm{D}}_{S_K}$ given by $\mathbf{z} \mapsto \Psi(z)$ and comes from

$$\eta\colon S_K[y]/(y^p - 1) \to \bigoplus_{z\in\kappa} S_K\varepsilon_z \cong S_K[x]/(x^q + \frac{\varpi}{E}x)$$
$$y \mapsto \sum_{z\in\kappa} \Psi(z)\varepsilon_z$$

We are going to explicitly write $\eta(y) = \sum_{z\in\kappa} \Psi(z)\varepsilon_z$ in terms of x. Let e_{χ_i}, with $0 < i < q-1$, be $\sum_{z\in\kappa} \chi_i^{-1}(z)\mathbf{z}$ and let $e_{\chi_{q-1}}$ be $\sum_{z\in\kappa} \mathbf{z} - q\mathbf{0}$. We have that $\left\{\frac{e_i}{q-1}\right\}_i$ is a basis of I', dual to $\{\varepsilon_{\chi_i}\}$. It follows that we have

$$\sum_{z\in\kappa} \Psi(z)\varepsilon_z = \varepsilon_0 + \frac{1}{q-1}\sum_{i=1}^{q-1} e_{\chi_i}\left(\sum_{z\in\kappa^*} \Psi(z)\varepsilon_z\right)\varepsilon_{\chi_i} =$$
$$= \varepsilon_0 + \varepsilon_{\chi_{q-1}} + \frac{1}{q-1}\sum_{i=1}^{q-1} g(\chi_i)\varepsilon_{\chi_i} = 1 + \frac{1}{q-1}\sum_{i=1}^{q-1} g(\chi_i)\alpha^{-i}x^i,$$

where α is the root of $-\frac{\varpi}{E}$ given in the proof of Lemma 2.3.4, and $g(\chi_i)$ is the Gauss sum associated to the multiplicative character χ_i^{-1} and to the additive character Ψ, i.e.

$$g(\chi_i) = \begin{cases} -q & \text{if } i = q-1, \\ \sum_{z\in\kappa} \chi_i^{-1}(z)\Psi(z) & \text{otherwise.} \end{cases}$$

If $0 \leq i \leq q-1$ is an integer, written in base p as $i = \sum_{k=0}^{f-1} i_k p^k$, we define $s(i)$ to be $i_0 + \ldots + i_{f-1}$. If $i \neq 0$, by [**Ray74**], page 251, we have

$$w_i = w_{\chi_i} = w_{\underbrace{\chi_1, \ldots, \chi_1}_{i_0 \text{ times}}, \ldots, \underbrace{\chi_{p^{f-1}}, \ldots, \chi_{p^{f-1}}}_{i_{f-1} \text{ times}}} = \frac{g(\chi_1)^{i_0}\cdots g(\chi_{p^{f-1}})^{i_{f-1}}}{(q-1)^{s(i)-1} g(\chi_i)}.$$

Since $g(\chi_{p^k}) = g(\chi_1)$ for every k, we obtain

$$g(\chi_i) = \frac{1}{(q-1)^{s(i)-1}}\frac{g(\chi_1)^{s(i)}}{w_i}. \tag{A.2}$$

For $k = 0, \ldots, f-1$, let

$$\beta_k := g(\chi_1)\alpha^{-p^k},$$

so we have

$$\eta(y) = 1 + \sum_{i=1}^{q-1} \frac{1}{(q-1)^{s(i)}}\frac{x^i}{w_i}\prod_{k=0}^{f-1}\beta_k^{i_k},$$

where $i = \sum_{k=0}^{f-1} i_k p^k$.

PROPOSITION A.6. *The morphism*

$$\eta\colon R[y]/(y^p - 1) \to R[x]/(x^q + \frac{\varpi}{E}x)$$

induces a canonical S-point of $(\mathcal{C}_1^{2,1})^{\mathrm{D}}$. Its base change to S_K, denoted with γ', is a generator, as κ-vector space, of $(\mathcal{C}_1^{2,1})^{\mathrm{D}}(S_K)$.

PROOF. First of all we have to show that each β_k is in S. By equation (A.2), we see that the valuation of $g(\chi_1)$ is $\frac{e}{p-1}$, so we can write, in S_K,

$$\beta_k = g(\chi_1)\alpha^{-p^k} = v\varpi^{\frac{e}{p-1}-\frac{p^k}{q-1}}(\varpi^{\frac{1}{q-1}}\alpha^{-1})^{p^k},$$

where v is a unit of V. Since $\frac{e}{p-1} - \frac{p^{f-1}}{q-1} \geq 0$ and $\varpi^{1/(q-1)}\alpha^{-1} \in S$, we see that $\beta_k \in S$. All the various relations that our map must satisfy in order to induce a morphism of group schemes can be checked in S_K, so the proposition follows by the above discussion. $\square$

REMARK A.7. Using the relations between our w_i's and the universal constants of Oort and Tate given in Remark A.5, we see that, in the case $f = 1$ and $\varpi = p$, our morphism is exactly the one defined in [**AIS11**], Proposition 5.2 (see also [**AG07b**] to relate our formula with the one of [**AIS11**]).

We did a very detailed study of $\mathcal{C}_1^{2,1}$ the canonical subgroup of $\mathcal{A}[\varpi]_1^{2,1}$. The corresponding results are used, in [**AIS11**], to relate $\mathcal{C}_1^{2,1}$ with the modular form E_1. In particular, they show that the image of γ' under the map $\mathrm{d}\log$ (see Section 2 of [**AIS11**] or 3.1) is congruent, modulo p^{1-v}, to E_1. This is not possible in our situation.

PROPOSITION A.8. *If e is big enough, then γ' is in the kernel of the map*

$$\mathrm{d}\log\colon (\mathcal{C}_1^{2,1})^{\mathrm{D}}(S_K) \to \underline{\omega}_{(\mathcal{C}_1^{2,1})_S/S} \otimes_S S/pS.$$

PROOF. First of all note that $(\mathcal{C}_1^{2,1})^{\mathrm{D}}$, being killed by ϖ, is also killed by p, so the map in the proposition makes sense. By definition, γ' comes from the morphism $S[y]/(y^p - 1) \to S[x]/(x^q + \frac{\varpi}{E}x)$ that sends y to $\eta(y)$. We have $\mathrm{d}\log(\gamma') = d\eta(y)/\eta(y)$, so to prove the proposition we need to show that $d\eta(y) = 0$. By Lemma A.3, we have that $\varpi\,\mathrm{d}(x) = 0$ in $\Omega^1_{\mathcal{C}_1^{2,1}/R}$. In particular it is enough to prove that ϖ divides β_k for each $k = 0, \ldots, f-1$ (see above for the definition of β_k and η). In the proof of Proposition A.6 we have shown that $\varpi^{\frac{e}{p-1}-\frac{p^k}{q-1}}$ divides β_k. If e is big enough, this implies that ϖ divides β_k as required. $\square$

REMARK A.9. The above proposition shows that, in general, the analogue of Proposition 5.2 of [**AIS11**] is not true in our situation. This is one of the main reasons to work with ϖ-divisible groups.

Recall that in section 2.1 we have fixed a coordinate x on the formal group $\widehat{\mathcal{A}}_1^{2,1}$ and we have denoted with ω a differential dual to x.

PROPOSITION A.10. *Let $h\colon \mathcal{C}_1^{2,1} \to \widehat{\mathcal{A}}[\varpi]_1^{2,1}$ be the natural map. In $\Omega^1_{\mathcal{C}_1^{2,1}/R}$, we have the equality*

$$h^*(\omega) = \frac{\mathrm{d}(x)}{1 - \varpi^{e-1}uE\frac{w_{f-1}}{w_{q-1}}x^{q-1}}.$$

Furthermore, if we write $\underline{\omega}_{\mathcal{C}_1^{2,1}/R} \cong R/\frac{\varpi}{E}R\,\mathrm{d}(x)$ as in Lemma A.3, we have

$$h^*(\omega) = \mathrm{d}(x).$$

PROOF. It is convenient to write the comultiplication $c(x)$ of $\mathcal{C}_1^{2,1}$ as $F(X,Y)$, where $X = x \otimes 1$ and $Y = 1 \otimes x$. Let $f(X)\,\mathrm{d}(X)$ be an invariant differential, we have

$$f(X)\,\mathrm{d}(X) + f(Y)\,\mathrm{d}(Y) = f(F(X,Y))(\frac{\partial}{\partial X}F(X,Y) + \frac{\partial}{\partial Y}F(X,Y)),$$

so comparing the coefficients of $\mathrm{d}(Y)$ in the two sides of the equation and setting $Y = 0$ we find, by Lemma A.3,

$$f(0) \equiv f(X)(1 - QX^{q-1}) \bmod \frac{\varpi}{E},$$

where

$$Q := \varpi^{e-1} u E \frac{w^{f-1}}{w_{q-1}}.$$

Since $w = \frac{g(\chi_1)^{p-1}}{(q-1)^{p-1}}$ ([**Ray74**], page 251), with some calculations we find that

$$(1 - (q-1)QX^{q-1})(1 - QX^{q-1}) = 1,$$

so any invariant differential on $\mathcal{C}_1^{2,1}$ has the form

$$\frac{r\,\mathrm{d}(x)}{1 - Qx^{q-1}},$$

for some $r \in R/\frac{\varpi}{E}R$ (note that a priori it is not clear whether any r is possible). The canonical subgroup was originally defined as $\mathrm{Spec}(R[x]/(x^q - t_{\mathrm{can}}x))$, since with this presentation ω is a differential dual to x, we have $\omega = f(x)\,\mathrm{d}(x)$, with $f \equiv 1 \bmod x$. The isomorphisms we used to write $\mathcal{C}_1^{2,1} \cong \mathrm{Spec}(R[x]/(x^q + \frac{\varpi}{E}))$ (see the proof of Proposition 2.3.5) preserve this property, so the first part of the proposition follows. The last statement is a consequence of the fact that the counit of $\mathcal{C}_1^{2,1} \cong \mathrm{Spec}(R[x]/(x^q + \frac{\varpi}{E}x))$ is the map $x \mapsto 0$. □

REMARK A.11. If $\varpi = p$ and $f = 1$, we have $\omega = \frac{\mathrm{d}(x)}{1+\frac{E}{p-1}}$ (see Remark A.5), but $p\,\mathrm{d}(x) = 0$, so $h^*(\omega) = \frac{\mathrm{d}(x)}{1-E}$. In particular we see that our results generalize part of [**Col97a**].

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Bibliography

[AG07a] F. Andreatta and C. Gasbarri, *The canonical subgroup for families of abelian varieties*, Compos. Math. **143** (2007), no. 3, 566–602.

[AG07b] ———, *Torsors under some group schemes of order p^n*, J. Algebra **318** (2007), no. 2, 1057–1067.

[AIS11] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens, *On overconvergent modular forms*, preprint. Available at `http://www.mathstat.concordia.ca/faculty/iovita`, 2011.

[Bel09] Joël Bellaïche, *Critical p-adic L-functions*, preprint. To appear at Inventiones Math. Available at `http://people.brandeis.edu/~jbellaic`, 2009.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer-Verlag, 1984.

[BL93] Siegfried Bosch and Werner Lütkebohmert, *Formal and rigid geometry. I. Rigid spaces*, Math. Ann. **295** (1993), no. 2, 291–317.

[Bri08] Olivier Brinon, *Représentations p-adiques cristallines et de de Rham dans le cas relatif*, Mém. Soc. Math. Fr. (N.S.) (2008), no. 112.

[Buz03] Kevin Buzzard, *Analytic continuation of overconvergent eigenforms*, J. Amer. Math. Soc. **16** (2003), no. 1, 29–55.

[Buz07] ———, *Eigenvarieties*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, 2007, pp. 59–120.

[Car86] Henri Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Math. **59** (1986), no. 2, 151–230.

[CM98] R. Coleman and B. Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, 1998, pp. 1–113.

[Col97a] Robert F. Coleman, *Classical and overconvergent modular forms of higher level*, J. Théor. Nombres Bordeaux **9** (1997), no. 2, 395–403.

[Col97b] ———, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), no. 3, 417–479.

[Col05] ———, *The canonical subgroup of E is* $\operatorname{Spec} R[x]/(x^p + \frac{p}{E_{p-1}(E,\omega)}x)$, Asian J. Math. **9** (2005), no. 2, 257–260.

[Del71] Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Springer, 1971, pp. 123–165. Lecture Notes in Math., Vol. 244.

[Del79] ———, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., 1979, pp. 247–289.

[Dwo73] B. Dwork, *The U_p operator of Atkin on modular functions of level 2 with growth conditions*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, 1973, pp. 57–67. Lecture Notes in Math., Vol. 350.

[Fal87] Gerd Faltings, *Hodge-Tate structures and modular forms*, Math. Ann. **278** (1987), no. 1-4, 133–149.

[Fal02] ———, *Group schemes with strict $\mathcal{O}$-action*, Mosc. Math. J. **2** (2002), no. 2, 249–279.

[Far07] Laurent Fargues, *Application de Hodge-Tate duale d'un groupe de Lubin-Tate, immeuble de Bruhat-Tits du groupe linéaire et filtrations de ramification*, Duke Math. J. **140** (2007), no. 3, 499–590.

[Far10] ———, *La filtration de Harder-Narasimhan des schémas en groupes finis et plats*, J. Reine Angew. Math. **645** (2010).

[FGL08] Laurent Fargues, Alain Genestier, and Vincent Lafforgue, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in Mathematics, vol. 262, Birkhäuser Verlag, 2008.

[Fon82] Jean-Marc Fontaine, *Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux*, Invent. Math. **65** (1981/82), no. 3, 379–409.

[Gou88] Fernando Q. Gouvêa, *Arithmetic of p-adic modular forms*, Lecture Notes in Mathematics, vol. 1304, Springer-Verlag, 1988.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977, Graduate Texts in Mathematics, No. 52.

[Har80] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176.

[Hid86] Haruzo Hida, *Galois representations into* $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. **85** (1986), no. 3, 545–613.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, 2001.

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, 1975, Graduate Texts in Mathematics, No. 21.

[Kas99] Payman L. Kassaei, *p-adic modular forms over Shimura curves over* $\mathbb{Q}$, Phd-Thesis. Available at `http://www.mth.kcl.ac.uk/~kassaei/`, 1999.

[Kas04] ———, *$\mathcal{P}$-adic modular forms over Shimura curves over totally real fields*, Compos. Math. **140** (2004), no. 2, 359–395.

[Kas09] ———, *Overconvergence and classicality: the case of curves*, J. Reine Angew. Math. **631** (2009), 109–139.

[Kat73] Nicholas M. Katz, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, 1973, pp. 69–190. Lecture Notes in Mathematics, Vol. 350.

[Mes72] William Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, 1972.

[Mil05] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., 2005, Available at `http://www.jmilne.org/math/`, pp. 265–378.

[Mil06] ———, *Complex multiplication*, Available at `http://www.jmilne.org/math/`, 2006.

[OT70] Frans Oort and John Tate, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21.

[Pil09] Vincent Pilloni, *Formes modulaires surconvergentes*, preprint. Available at `http://perso.ens-lyon.fr/vincent.pilloni/`, 2009.

[Ray74] Michel Raynaud, *Schémas en groupes de type* $(p, \dots, p)$, Bull. Soc. Math. France **102** (1974), 241–280.

[Rob00] Alain M. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, vol. 198, Springer-Verlag, New York, 2000.

[Ser62] Jean-Pierre Serre, *Endomorphismes complètement continus des espaces de Banach p-adiques*, Inst. Hautes Études Sci. Publ. Math. (1962), no. 12, 69–85.

[Ser73] ———, *Formes modulaires et fonctions zêta p-adiques*, 191–268. Lecture Notes in Math., Vol. 350.

[Shi94] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, 2009.

[Tat67] J. T. Tate, *$p-$divisible groups.*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, 1967, pp. 158–183.

[Tat71] John Tate, *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289.

[Urb10] Eric Urban, *Eigenvarieties for reductive groups*, preprint. Available at `http://www.math.columbia.edu/~urban/`, 2010.

[Vig80] Marie-France Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 800, Springer, 1980.

[Wat79] William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, 1979.