

Threshold Resummation in $B \rightarrow X_c l \nu_l$ Decays

U. Aglietti ^a, L. Di Giustino ^b, G. Ferrera ^c, A. Renzaglia ^a, G. Ricciardi ^d, L. Trentadue ^b

^a*Dipartimento di Fisica, Università di Roma I “La Sapienza”*

and

INFN Sezione di Roma, Roma, Italy

^b*Dipartimento di Fisica, Università di Parma,*

and

INFN, Gruppo Collegato di Parma, Parma, Italy

^c *Departament ECM, Universitat de Barcelona, Barcelona, Spain*

and

Dipartimento di Fisica, Università di Firenze, Firenze, Italy

^d *Dipartimento di Scienze Fisiche, Università di Napoli “Federico II”*

and

INFN, Sezione di Napoli, Napoli, Italy

Abstract

We compute the QCD form factor resumming threshold logarithms in $B \rightarrow X_c + l + \nu_l$ decays to next-to-leading logarithmic approximation. We present an interpolation formula including soft as well as collinear effects softened by the non-vanishing charm mass which, in the N -space, reads:

$$\sigma_N = \exp \int_0^1 dy \left[(1-y)^{N-1} - 1 \right] \left\{ \frac{1}{y} \int_{\frac{Q^2 y^2}{1+\rho}}^{\frac{Q^2 y^2}{y+\rho}} A[\rho; \alpha_S(k^2)] \frac{dk^2}{k^2} + \frac{1}{y} D \left[\alpha_S \left(\frac{Q^2 y^2}{1+\rho} \right) \right] + \left(\frac{1}{y} - \frac{1}{y+\rho} \right) \Delta \left[\alpha_S \left(\frac{Q^2 y^2}{y+\rho} \right) \right] + \frac{1}{y+\rho} B \left[\alpha_S \left(\frac{Q^2 y^2}{y+\rho} \right) \right] \right\},$$

where $y = (m_{X_c}^2 - m_c^2) / (Q^2 - m_c^2)$, $\rho = m_c^2 / (Q^2 - m_c^2)$ and $Q = E_{X_c} + |\vec{p}_{X_c}|$. The function $A(\rho; \alpha_S)$ describes soft radiation collinearly enhanced off the charm quark and reduces to the standard double-logarithmic function $A(\alpha_S)$ in the massless limit $\rho \rightarrow 0$. The function $B(\alpha_S)$ describes hard collinear emission off the charm quark, being the standard jet function. $D(\alpha_S)$ and $\Delta(\alpha_S)$ describe soft radiation not collinearly enhanced off the beauty and the charm quark respectively.

1 Introduction and Summary of Results

Distributions in semileptonic B decays

$$B \rightarrow X_c + l + \nu_l, \quad (1)$$

such as for example the hadron-mass event fraction, often receive logarithmic contributions of infrared origin of the form

$$\alpha_S^n \log^k \left(\frac{m_{X_c}^2 - m_c^2}{m_b^2 - m_c^2} \right) \log^l \left(\frac{m_{X_c}^2}{m_b^2} \right), \quad (2)$$

where m_{X_c} is the final hadron mass and α_S is the QCD coupling evaluated at the hard scale. Even in weak coupling regime $\alpha_S \ll 1$ (perturbative phase), the terms above tend to spoil the convergence of the perturbative series in the threshold region, the latter being defined as the one having parametrically

$$m_{X_c} \ll m_b. \quad (3)$$

The large infrared logarithms are a “remnant” of an incomplete cancellation between real corrections to the decay distributions, integrated on the border of the phase space, and virtual ones. An all-order resummation is needed in order to have a reliable description of the spectra in the whole kinematical range. The infrared logarithms coming from soft-gluon emission are of the form

$$\log \left(\frac{m_{X_c}^2 - m_c^2}{m_b^2 - m_c^2} \right) \quad (4)$$

and formally diverge for $m_{X_c} \rightarrow m_c^+$. Since there is at most one soft logarithm for each gluon emission, i.e. for each power of α_S , it follows that $k \leq n$ in eq. (2). The infrared logarithms of collinear origin,

$$\log \left(\frac{m_{X_c}^2}{m_b^2} \right), \quad (5)$$

never become infinite but may become large in the threshold region (3). Since there are at most two infrared logarithms for each power of α_S , $k + l \leq 2n$ in eq. (2).

Our main result is the following expression for the resummed QCD form factor in N -space (also called moment or Mellin space):

$$\begin{aligned} \sigma_N(\rho, Q^2) = & \exp \int_0^1 dy \left[(1-y)^{N-1} - 1 \right] \left\{ \frac{1}{y} \int_{\frac{Q^2 y^2}{1+\rho}}^{\frac{Q^2 y^2}{y+\rho}} A[\rho; \alpha_S(k^2)] \frac{dk^2}{k^2} + \frac{1}{y} D \left[\alpha_S \left(\frac{Q^2 y^2}{1+\rho} \right) \right] + \right. \\ & \left. + \left(\frac{1}{y} - \frac{1}{y+\rho} \right) \Delta \left[\alpha_S \left(\frac{Q^2 y^2}{y+\rho} \right) \right] + \frac{1}{y+\rho} B \left[\alpha_S \left(\frac{Q^2 y^2}{y+\rho} \right) \right] \right\}. \quad (6) \end{aligned}$$

We have defined the hadron variable with unitary range

$$y \equiv \frac{m_{X_c}^2 - m_c^2}{Q^2 - m_c^2}, \quad (7)$$

which equals zero in the Born kinematics, and the mass-correction parameter

$$\rho \equiv \frac{m_c^2}{Q^2 - m_c^2}, \quad (8)$$

where Q , the hard scale in the heavy flavor decay, is given by

$$Q \equiv E_{X_c} + |\vec{p}_{X_c}|, \quad (9)$$

with E_{X_c} and \vec{p}_{X_c} being the total energy and 3-momentum of the final hadron state X_c . The function $A(\rho; \alpha_S)$ has an expansion in power of α_S ,

$$A(\rho; \alpha_S) = \sum_{n=1}^{\infty} A^{(n)}(\rho) \alpha_S^n, \quad (10)$$

and describes soft radiation collinearly enhanced off the charm quark; it reduces to the standard double-logarithmic function $A(\alpha_S)$ in the massless limit:

$$A(\rho; \alpha_S) \rightarrow A(\alpha_S) \quad \text{for } \rho \rightarrow 0^+. \quad (11)$$

The first-order coefficient is:

$$A^{(1)}(\rho) \equiv \frac{C_F}{\pi} (1 + 2\rho), \quad (12)$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$ for $N_c = 3$. The second-order coefficient $A^{(2)}(\rho)$ (as well as the third-order one $A^{(3)}(\rho)$) is only known in the massless limit (see later). The function $B(\alpha_S)$ is the standard massless jet-function, describing hard collinear emission off the charm quark, while $D(\alpha_S)$ and $\Delta(\alpha_S)$ describe soft radiation not collinearly enhanced off the beauty and charm quarks respectively. All these functions have a perturbative expansion in α_S ¹:

$$B(\alpha_S) = \sum_{n=1}^{\infty} B^{(n)} \alpha_S^n; \quad D(\alpha_S) = \sum_{n=1}^{\infty} D^{(n)} \alpha_S^n; \quad \Delta(\alpha_S) = \sum_{n=1}^{\infty} \Delta^{(n)} \alpha_S^n. \quad (13)$$

The $\mathcal{O}(\alpha_S)$ coefficients read:

$$B^{(1)} = -\frac{3}{4} \frac{C_F}{\pi}; \quad D^{(1)} = -\frac{C_F}{\pi}; \quad \Delta^{(1)} = -\frac{C_F}{\pi}. \quad (14)$$

The form factor (6) aims at describing the three different dynamical regions in the semileptonic decay (1), which are identified by the value of the (ordinary) charm velocity $u_c = p_c/E_c$ — without any generality loss, we can work in the beauty rest frame. It holds:

$$\rho \simeq \frac{1 - u_c}{2u_c}. \quad (15)$$

These regions are:

¹For a compilation of the massless coefficients, see for example [1], while for the heavy flavor case [2].

1. *very slow charm quark*:

$$u_c \gtrsim 0 \quad \text{or, equivalently,} \quad \rho \gg 1. \quad (16)$$

For $u_c = 0$ there is no *soft-gluon* emission to any order in perturbation theory [3]². That is a coherence effect, namely destructive interference between emission off the initial and the final state. The physical picture is the following. The b decays into the c at rest and with the same color state of the b . Soft gluons “view” the b as well as the c as static color charges, and therefore do not see any acceleration or color-spin flip, i.e. any change occurring in the decay. As a consequence, there is no radiation. Furthermore, for small u_c , radiation is proportional to u_c^2 , i.e. first order terms $\mathcal{O}(u_c)$ vanish [4]. Radiative corrections to our form factor vanish in the no-recoil point:

$$\sigma_N(\rho; Q^2) \rightarrow 1 \quad \text{for} \quad \rho \rightarrow +\infty, \quad (17)$$

implying that this region is correctly described by the form factor;

2. *non-relativistic charm quark*. Schematically:

$$u_c \approx \frac{1}{3} \quad \text{or} \quad \rho \approx 1. \quad (18)$$

The final state X_c contains the final charm quark together with soft gluons emitted at any angle with respect to it, because there is a significant soft emission. Since the QCD matrix elements do not have any collinear enhancement, X_c does not have a jet structure. In the threshold region $y \ll 1$, so that $y \ll \rho$ and the form factor (6) simplifies into:

$$\begin{aligned} \sigma_{S,N}(\rho, Q^2) = \exp \int_0^1 \frac{dy}{y} [(1-y)^{N-1} - 1] \left\{ \int_{\frac{Q^2 y^2}{1+\rho}}^{\frac{Q^2 y^2}{\rho}} A[\rho; \alpha_S(k^2)] \frac{dk^2}{k^2} + D \left[\alpha_S \left(\frac{Q^2 y^2}{1+\rho} \right) \right] \right. \\ \left. + \Delta \left[\alpha_S \left(\frac{Q^2 y^2}{\rho} \right) \right] \right\}; \quad (19) \end{aligned}$$

3. *fast charm quark*:

$$u_c \lesssim 1 \quad \text{or} \quad \rho \ll 1. \quad (20)$$

Soft gluons are mostly radiated at small angle with respect to the charm quark and there is a jet structure of the final state. For $\rho \rightarrow 0^+$ the form factor (6) reduces to the standard massless expression [2, 5–7]:

$$\sigma_N(0, Q^2) = \exp \int_0^1 \frac{dy}{y} [(1-y)^{N-1} - 1] \left\{ \int_{Q^2 y^2}^{Q^2 y} A[\alpha_S(k^2)] \frac{dk^2}{k^2} + D[\alpha_S(Q^2 y^2)] + \right.$$

²By “soft” we mean a gluon with energy $E_g < m_c, m_b$.

$$+ B [\alpha_S (Q^2 y)] \Big\}, \quad (21)$$

where:

$$y \rightarrow u = \frac{E_X - |\vec{p}_X|}{E_X + |\vec{p}_X|} \simeq \frac{m_X^2}{4E_X^2}; \quad Q \rightarrow E_X + |\vec{p}_X| \simeq 2E_X, \quad (22)$$

where on the last members we have expanded for $m_X \ll E_X$. The inclusion of first-order corrections in ρ on the r.h.s. of eq. (21) basically amounts to take into account the dead-cone effect in gluon radiation off the charm [8]. The form factor factorizes into the product of the form factor for a massless charm times the universal mass-jet correction:

$$\sigma_N(\rho, Q^2) \simeq \sigma_N(0, Q^2) \delta_N(\rho, Q^2) \quad \text{for } \rho \ll 1, \quad (23)$$

where ³

$$\delta_N(\rho, Q^2) = \exp \int_0^1 dy \frac{(1-y)^{\rho(N-1)} - 1}{y} \left\{ - \int_{\rho Q^2 y^2}^{\rho Q^2 y} \frac{dk_{\perp}^2}{k_{\perp}^2} A [\alpha(k_{\perp}^2)] - B [\alpha(\rho Q^2 y)] + \right. \\ \left. + D [\alpha(\rho Q^2 y^2)] \right\}. \quad (24)$$

Let us note that in [8] the *universal* correction $\delta_N(\rho, Q^2)$ was found for $\rho \ll 1$, while in this work we consider a *specific* process for *any* ρ .

Since the ratio $m_c/m_b \approx 1/3 \div 1/4$ in the real world, i.e. it is not so small, region 3. is tiny. With typical values of the on-shell masses, $m_b = 4.7$ GeV and $m_c = m_b - m_B + m_D \simeq 1.29$ GeV [9], one has a maximal charm velocity $u_{c \max} \simeq 0.86$ corresponding to a Lorentz factor $\gamma_{c \max} \simeq 2$, or $0.081 \lesssim \rho \leq \infty$. With hadron kinematics, i.e. with $m_b = m_B$ and $m_c = m_D$, one obtains instead: $u_{c \max} = 0.78$, $\gamma_{c \max} \simeq 1.6$ and $\rho_{\min} \simeq 0.14$.

Let us comment further upon the properties of the form factor.

Since the y and k^2 integrals on the r.h.s. of eq. (6) extend down to zero, one hits the infrared singularity in the QCD couplings $\alpha_S(k^2)$ and $\alpha_S(\dots y^2)$ — the well-known Landau pole: some prescription is needed in order to render the integrands well behaved. As it is usually the case with resummation formulae, eq. (6), as it stands, has an “algebraic” sense: upon expansion in powers of α_S , it allows to predict the logarithmic corrections which would be explicitly found in higher-order Feynman diagram computations.

³The variable ρ used in this work and the variable $r \equiv m_c^2/Q^2 = \rho/(1+\rho)$ used in [8] coincide to first order.

In general, we expect less radiation to be emitted in the decay (1), because of the rather large charm mass, compared to the charmless channel

$$B \rightarrow X_u + l + \nu_l. \quad (25)$$

As a consequence, the typical Sudakov effects, namely suppression of non-radiative channels and broadening of sharp structures, are expected to be less pronounced for our process. In principle, for the decay (1), one has single-logarithmic corrections, which are not strong enough to shift the peak of tree-level distributions. For $y \rightarrow 0^+$ they produce indeed a form factor of the form

$$\sigma(y) \approx \frac{d}{dy} e^{-\alpha_S \log \frac{1}{y}} = \frac{\alpha_S}{y^{1-\alpha_S}}, \quad (26)$$

which still has an (infinite) peak in $y = 0$, like the lowest-order form factor $\sigma^{(0)}(y) = \delta(y)$. That is to be contrasted with a double-logarithmic form factor $\approx d/dy \exp(-\alpha_S \log^2 y)$.

It is remarkable that the limits of the k^2 -integral on the r.h.s. of eq. (6), $Q^2 y^2/(1+\rho)$ and $Q^2 y^2/(y+\rho)$, are equal to the arguments of the QCD couplings entering the “sub-leading” terms (i.e. those without the explicit k^2 integration); that seems a general feature of resummation formulae. A consequence of this fact is that only 3 out of the 4 resummation functions A , B , D and Δ entering σ_N are actually independent (see sec. 3.3). Similar considerations can be repeated for the soft form factor in eq. (19), in which the scales $Q^2 y^2/(1+\rho)$ and $Q^2 y^2/\rho$ appear as limits of the k^2 integral as well as arguments of α_S in the D and Δ terms. In the latter case, only 2 out of the 3 functions A , D and Δ are independent.

By looking at the y -integration on the r.h.s. of eq. (6), one finds that the threshold region cannot be described within a “pure” perturbative framework when the QCD coupling comes close to the infrared (Landau) singularity, namely when

$$\frac{Q^2 y^2}{1+\rho} \approx \Lambda_{QCD}^2, \quad (27)$$

where Λ_{QCD} is the QCD scale, i.e. when the final hadron mass becomes as small as ⁴

$$m_{X_c}^2 \Big|_{NP} \approx m_c^2 + \Lambda_{QCD} \sqrt{Q^2 - m_c^2}. \quad (28)$$

These effects are related to soft interactions only, because hard collinear terms ($\propto B(\alpha_S)$) are controlled by a smaller coupling, evaluated at the larger scale $Q^2 y^2/(y+\rho)$. These soft interactions are non-perturbative in region (28), and can be factorized into the shape function for a massive final quark [10]. Note that for $m_c \rightarrow 0$, the slice (28) reduces to the well-known

⁴For $Q = m_b$ and $\Lambda_{QCD} = 300$ MeV one obtains for the current values of the b and c masses $m_{X_c} \Big|_{NP} \simeq 1.7$ GeV.

one $m_X^2 \approx \Lambda_{QCD} Q$. In region (28), also soft interactions between the b quark and the light degrees of freedom in the B -meson, namely the valence quark, are important. That is the well-known “Fermi-motion”, characterized by momentum exchanges k_μ in the B -meson of the order of the hadronic scale,

$$|k_\mu| \approx \Lambda_{QCD}. \quad (29)$$

Physical intuition would suggest that Fermi motion is related to the initial B -meson state only, i.e. that it is independent on the mass of the final quark. In quantum field theory that is not true and it is not easy to find a general relation between eq. (21) with the B term dropped, describing soft interactions for $m_c \rightarrow 0$, and eq. (19), describing soft interactions for $m_c \neq 0$.

It is technically simpler to compute first the form factor for the radiative case

$$B \rightarrow X_s + \gamma \quad (30)$$

for $m_s \neq 0$ and then to extend the result to the semileptonic case (1), rather than to deal directly with the latter process. That is the strategy we will follow.

The plan of this note is the following.

In sec. 2 we consider soft effects only, which are in some sense “leading”, as they become formally infinite at the border of the phase space, for the radiative decay (30).

In sec. 3 we include hard collinear effects, which never become infinite but are logarithmically enhanced for small strange mass, as discussed at the beginning of this section.

In sec. 4 we derive the resummed form factor for the process of main physical interest, namely the semileptonic decay (1), by means of a generalization of the radiative case.

Finally, in sec. 5 we draw our conclusions and we give an outlook to possible developments of the results, both of theoretical and phenomenological nature.

2 Soft-Gluon Resummation

In this section we consider the resummation of threshold logarithms coming from multiple soft-gluon emission to all orders of perturbation theory. The QCD form factor factorizing such terms has a perturbative expansion of the form:

$$\sigma_S(y; \rho, Q^2) = \delta(y) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_{nk}(\rho) \alpha_S^n(Q^2) \left(\frac{\log^k y}{y} \right)_+$$

$$= \delta(y) + \frac{C_F \alpha_S}{\pi} \left[(1 + 2\rho) \log \left(\frac{1 + \rho}{\rho} \right) - 2 \right] \left(\frac{1}{y} \right)_+ + \mathcal{O}(\alpha_S^2). \quad (31)$$

As discussed in the introduction, we first consider the simpler radiative decay (30), for which y is naturally defined as

$$y \equiv \frac{m_{X_s}^2 - m_s^2}{m_b^2 - m_s^2} \quad (32)$$

and the mass-correction parameter ρ as

$$\rho \equiv \frac{m_s^2}{m_b^2 - m_s^2}. \quad (33)$$

The $c_{nk}(\rho)$'s are functions of ρ and the plus distributions are defined as usual as:

$$P(y)_+ \equiv \lim_{\epsilon \rightarrow 0^+} \left[\theta(y - \epsilon) P(y) - \delta(y - \epsilon) \int_\epsilon^1 dy' P(y') \right]. \quad (34)$$

Let us note that we do not have here a double-logarithmic problem as in the massless case ($\rho = 0$), but a *single-logarithmic* one: there is at most one infrared logarithm for each power of α_S . The additional singularities for $\rho \rightarrow 0^+$ are “hidden” in the coefficients $c_{nk}(\rho)$'s, which indeed diverge logarithmically in that limit — see the last member in eq. (31). Note that the square bracket on the last member of eq. (31) vanishes for $\rho \rightarrow \infty$ (see region 1. in the introduction). By leading logarithmic soft (LL_S) approximation, we mean the resummation of all the terms in eq. (31) with $k = n - 1$, by next-to-leading logarithmic (NLL_S) approximation also those with $k = n - 2$ and so on.

2.1 Single-Gluon Emission

In this section we consider the emission of a real gluon, i.e. the process:

$$b \rightarrow s + g + \gamma. \quad (35)$$

We will recover the soft contributions factorized in σ_S above.

2.1.1 Kinematics

The energy of the strange quark in the tree-level process

$$b \rightarrow s + \gamma \quad (36)$$

is fixed to

$$E_s^{(0)} = \frac{m_b}{2} \frac{1+2\rho}{1+\rho} \quad (37)$$

and its 3-momentum to

$$p_s^{(0)} \equiv |\vec{p}_s^{(0)}| = E_\gamma^{(0)} = \frac{m_b}{2(1+\rho)}. \quad (38)$$

The strange quark and the photon are emitted back to back in the decay of the b quark at rest. Note that $E_s^{(0)} + p_s^{(0)} = m_b$. Let us now consider kinematics of single-gluon emission. The maximum value of the gluon energy is

$$E_g^{\max} = \frac{m_b}{2(1+\rho)} \quad (39)$$

and it corresponds to the configuration with the strange quark and the gluon back to back with an accompanying soft photon. Let us define the unitary gluon energy ω as:

$$\omega \equiv \frac{2E_g}{m_b}(1+\rho) \quad (40)$$

and the unitary angular variable t as

$$t \equiv \frac{1 - \cos \theta}{2} \simeq \left(\frac{\theta}{2}\right)^2 \quad \text{for } \theta \ll 1, \quad (41)$$

where θ is the emission angle of the gluon with respect to the final (on-shell) strange quark. Let us now restrict our attention to kinematics *in the soft limit*. E_s and p_s are only mildly modified by the emission of a soft gluon, so that we can neglect recoil effects and set

$$E_s \simeq E_s^{(0)}; \quad p_s \simeq p_s^{(0)}. \quad (42)$$

Furthermore, the direction of motion of the strange quark is modified by gluon emission in negligible way. In terms of the variables defined above, we can write

$$y = \omega \frac{t + \rho}{1 + \rho}. \quad (43)$$

Note the linear relation between y and ω , which is a consequence of the soft limit, and that

$$y \rightarrow 0 \quad \iff \quad \omega \rightarrow 0. \quad (44)$$

That implies that the threshold region $y \ll 1$ selects kinematical configurations which are dynamically enhanced by the soft divergencies. At fixed y , one has for the angular variable t the whole range $0 < t < 1$. Finally, note that for any hadron final state:

$$E_{X_s} + |\vec{p}_{X_s}| = m_b. \quad (45)$$

That is a consequence of the massless “probe”, namely the photon ($q^2 = 0$ with $q = p_\gamma$).

2.1.2 Eikonal Current

In the soft limit $\omega \ll 1$, the emission of a real gluon off the b and the s quarks is described by the eikonal current:

$$J^\mu(k) = ig \left(\mathbf{T}_b \frac{p_b^\mu}{p_b \cdot k} - \mathbf{T}_s \frac{p_s^\mu}{p_s \cdot k} \right), \quad (46)$$

where \mathbf{T}_b and \mathbf{T}_s are the color generators of the b and s quarks respectively [8]. By squaring the eikonal current, integrating the phase space over the azimuth and summing/averaging over helicities and polarizations, we have the following contribution to the differential form factor from the emission of a real gluon:

$$\sigma_S^{(R)}(y) = \frac{C_F \alpha_S}{\pi} \int_0^1 dt \int_0^1 \frac{d\omega}{\omega} \left\{ \frac{1+2\rho}{t+\rho} - 1 - \frac{\rho(1+\rho)}{(t+\rho)^2} \right\} \delta \left(y - \omega \frac{t+\rho}{1+\rho} \right). \quad (47)$$

Let us briefly comment upon the result above. The soft singularity for $\omega \rightarrow 0$ of the QCD matrix element squared is screened by the kinematical constraint, which is infrared safe ($y \neq 0 \Rightarrow \omega \neq 0$) [11]. Also the collinear singularity is screened by the kinematical constraint, which is collinear safe ($y \neq 0 \Rightarrow t \neq 0$ for $\rho = 0$), as well as by the strange mass, because the expression (47) diverges for $t \rightarrow -\rho$ only, which is outside the physical domain ($t \geq 0$) for $\rho > 0$.

2.2 Inclusive Gluon-Decay Effects

In the previous section we considered the $\mathcal{O}(\alpha_S)$ correction to the tree-level decay (36), namely the emission of a single gluon. Higher order corrections in α_S basically involve *multiple* emissions of the following two kinds:

1. *primary* emissions, i.e. direct emissions of gluons directly attached to the colored $b \rightarrow s$ line;
2. *secondary* emissions, produced by the splitting of a gluon into a gg , $q\bar{q}$ pair, or by gluon emission off secondary quarks or antiquarks, etc.

As we are going to explicitly show in sec. (2.4), the main effect of primary emissions is to exponentiate the single-gluon distribution. Let us consider in this section secondary emissions, namely the “decay” of a primary (virtual) gluon with momentum p_g into secondary partons.

The amplitude for the emission of one gluon off the strange quark is enhanced when the scalar part of the s -quark propagator,

$$\Delta_s(p_s^*) = \frac{1}{p_s^{*2} - m_s^2}, \quad (48)$$

becomes large, i.e. when the virtual s quark is close to its mass-shell. This denominator reads in the case of the emission of one virtual gluon with momentum p_g and invariant mass squared $m_g^2 = p_g^2$:

$$\Delta_s(p_s + p_g) = \frac{1}{2p_s \cdot p_g + m_g^2} \simeq \frac{1 + \rho}{E_g m_b (1 + 2\rho) \left[1 - \sqrt{1 - \frac{m_g^2}{E_g^2} \frac{1-2t}{1+2\rho}} \right]}, \quad (49)$$

where we have taken the strange quark after gluon emission to be on-shell, i.e. $p_s^2 = m_s^2$, since we are considering a *single primary* emission. On the last member we have made the soft approximations for $E_g \ll E_s$ discussed above and we have dropped the m_g^2 term, which comes out to be negligible *a posteriori* (see later). Since infrared singularities originate from quasi-real partons, i.e. from partons with a small virtuality, we can assume that $m_g^2 \ll E_g^2$, to obtain [12]:

$$\Delta_s(p_s + p_g) \simeq \frac{(1 + \rho)^2}{m_b^2 \omega (t + \rho) \left[1 + \frac{(1+\rho)^2 m_g^2}{(t+\rho)\omega^2 m_b^2} \right]}, \quad (50)$$

where we have made the small-angle approximation $1 - 2t \simeq 1$ in the term multiplying m_g^2 , since the latter is already a correction. Now it comes the main point: the $\mathcal{O}(\alpha_S)$ correction involves a real gluon, having $m_g^2 = 0$, and is logarithmically enhanced in the infrared regions discussed in the previous section. In order not to spoil the lowest-order enhancement, the denominator has to remain small, i.e. the correction term in the square bracket has to be much smaller than one:

$$m_g^2 < m_b^2 \omega^2 \frac{t + \rho}{(1 + \rho)^2}, \quad (51)$$

where we have replaced the strong inequality with an ordinary inequality since we work within logarithmic accuracy. The condition above specifies the upper limit on the gluon mass which does not spoil the leading logarithmic structure of the cascade. It is immediate at this point to justify neglecting the m_g^2 term on the last member of eq. (49):

$$\frac{m_g^2}{2p_s \cdot p_g} \lesssim \omega \ll 1. \quad (52)$$

The amplitude for the emission of a soft gluon off the initial b quark (at rest) contains the scalar part of the b propagator

$$\Delta_b(p_b - p_g) = \frac{1}{(p_b - p_g)^2 - m_b^2} = \frac{1}{-2 m_b E_g + m_g^2}, \quad (53)$$

which has a different enhancement with respect to the s one; in particular, it involves only the gluon energy and not the angle with respect to the s quark. However, for $m_g^2 \neq 0$, the kinematical constraint of fixed y is no more linear in ω (cfr. eq. (43)) and reads:

$$y \simeq \frac{\omega}{1+\rho} \left[t + \rho + \frac{(1+\rho)^2 m_g^2}{m_b^2 \omega^2} \right]. \quad (54)$$

In particular, for $y \rightarrow 0^+$ one does not select anymore the limit $\omega \rightarrow 0^+$ unless the gluon mass satisfies the constraint (51). As a consequence, for kinematical reasons, also amplitudes for soft emission off the b are enhanced only if condition (51) holds.

The conclusion is that, as long as the restriction (51) is satisfied, one can sum over all possible gluon decay channels. The y -distribution is an example of “*inclusive gluon decay*” quantity. By including *real* as well as *virtual* gluon-splitting processes, the sum of cuts of the gluon propagator can be written as its discontinuity, so that the tree-level QCD coupling is replaced by the following momentum-dependent effective coupling [13]:

$$\alpha_S \rightarrow \tilde{\alpha}_S(k^2) = \frac{i}{2\pi} \int_0^{k^2} dm_g^2 \text{Disc} \left[\frac{1}{m_g^2 \beta_0 \log \left(\frac{-m_g^2}{\Lambda_{QCD}^2} \right)} \right] \simeq \alpha_S(k^2), \quad (55)$$

where $\beta_0 = (11/3C_A - 2/3n_f)/(4\pi)$ with $C_A = N_c = 3$ and we have defined the transverse momentum squared as the largest possible m_g^2 :

$$k^2 \equiv \max m_g^2 = m_b^2 \omega^2 \frac{t + \rho}{(1 + \rho)^2}. \quad (56)$$

On the last member of eq. (55), absorptive effects (the “ $-i\pi$ ” terms in gluon polarization function) have been neglected [14]⁵. Eq. (55) implies that, in order to include leading gluon-decay effects, one has to evaluate the running coupling at the transverse momentum squared. The latter has the range:

$$\frac{m_b^2 y^2}{1 + \rho} < k^2 < \frac{m_b^2 y^2}{\rho}. \quad (57)$$

Let us note that this range shrinks to zero for $\rho \rightarrow \infty$, killing any collinear enhancement. Finally, let us observe that in the massless limit for the strange quark, $\rho = 0$, one recovers the definition of transverse momentum given in [6].

2.3 Effective Single-Gluon Distribution

In this section we include gluon branching effects, i.e. secondary emission, in the single-gluon distribution by means of the replacement (55). The real contribution to the form factor therefore

⁵The over-all sign on the r.h.s. of eq. (11) of [14] should be minus.

reads:

$$\sigma_S^{(R)}(y) = \frac{C_F}{\pi} \int_{\rho}^{1+\rho} d\tau \int_0^1 \frac{d\omega}{\omega} \left[\frac{1+2\rho}{\tau} - 1 - \frac{\rho(1+\rho)}{\tau^2} \right] \alpha_S \left[\frac{m_b^2 \omega^2 \tau}{(1+\rho)^2} \right] \delta \left(y - \frac{\omega \tau}{1+\rho} \right), \quad (58)$$

where we have changed variable from t to

$$\tau \equiv t + \rho. \quad (59)$$

Let us consider the integration of the above terms in turn:

1. after integrating over τ and changing variable from ω to k^2 , the first term in the square bracket on the r.h.s. of eq. (58) reads:

$$\frac{1}{y} \frac{C_F}{\pi} (1+2\rho) \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{\rho}} \frac{dk^2}{k^2} \alpha_S(k^2). \quad (60)$$

Note that for $\rho \gg 1$ the coupling becomes large and we leave the perturbative phase of QCD even outside the threshold region, i.e. for large $y \sim \mathcal{O}(1)$;

2. the second term on the square bracket (related to soft-gluon emission off the b), after integrating over ω the δ -function, reads:

$$-\frac{1}{y} \frac{C_F}{\pi} \int_{\rho}^{1+\rho} d\tau \alpha_S \left(\frac{m_b^2 y^2}{\tau} \right) \cong -\frac{1}{y} \frac{C_F}{\pi} \alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right); \quad (61)$$

3. the last term in the square bracket (related to soft gluon emission off the s quark) is treated in a similar manner and reads:

$$-\frac{C_F}{\pi} \frac{\rho(1+\rho)}{y} \int_{\rho}^{1+\rho} \frac{d\tau}{\tau^2} \alpha_S \left(\frac{m_b^2 y^2}{\tau} \right) \cong -\frac{1}{y} \frac{C_F}{\pi} \alpha_S \left(\frac{m_b^2 y^2}{\rho} \right). \quad (62)$$

Let us note that in 1. we have rewritten the logarithmic integration over τ in terms of k^2 , while in 2. and in 3. we have explicitly computed the non-logarithmic integrals over τ .

By summing all three terms, we obtain:

$$\sigma_S^{(R)}(y; \rho, m_b^2) = \frac{1}{y} \left\{ \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{\rho}} \frac{dk^2}{k^2} A^{(1)}(\rho) \alpha_S(k^2) + D^{(1)} \alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right) + \Delta^{(1)} \alpha_S \left(\frac{m_b^2 y^2}{\rho} \right) \right\}. \quad (63)$$

In the frozen coupling limit $\alpha_S(\dots) \rightarrow \alpha_S$ or, equivalently, in first order, the above expression reduces to the real contribution in the last member of eq. (31):

$$\sigma_S^{(R)}(y, \rho, \alpha_S) = \frac{C_F \alpha_S}{\pi} \frac{1}{y} \left[(1+2\rho) \log \left(\frac{1+\rho}{\rho} \right) - 2 \right] + \mathcal{O}(\alpha_S^2). \quad (64)$$

2.4 N -space and Exponentiation

The exponentiation of threshold form factors in gauge theories is a consequence both of dynamical and kinematical factorization properties [11, 15, 16]. In QED, multiple soft-photon matrix elements factorize into the product of single-photon matrix elements because of independent emission (Poisson distribution). In QCD that is no longer true: soft gluons are not emitted independently because of gluon splitting, but gluon correlations largely cancel in the form factor, leading to an “effective” matrix element factorization. On the kinematical side, to have exponentiation, the constraint for multiple soft emission must factorize into the product of single-gluon (or single-photon) kinematical constraints. The latter property does not hold in the physical, energy-momentum, space and, to have a consistent resummation, one has to transform the form factor to N -space:

$$\sigma_{S,N} \equiv \int_0^1 (1-y)^{N-1} \sigma_S(y) dy. \quad (65)$$

The QCD form factor in the original (physical) space is recovered by means of a standard inverse transform:

$$\sigma_S(y) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN (1-y)^{-N} \sigma_{S,N}, \quad (66)$$

where C is a real constant chosen so that all the singularities of $\sigma_{S,N}$ lie to the left of the integration contour.

Up to now we computed only real emission contributions to the form factor. Virtual corrections do not need to be explicitly evaluated: they can be included by imposing the vanishing of the first moment ($N = 1$) of the form factor, implying the replacement

$$(1-y)^{N-1} \rightarrow (1-y)^{N-1} - 1. \quad (67)$$

By inserting the expression for $\sigma_S^{(R)}(y)$ obtained in the previous section and exponentiating the result in order to take into account multiple primary emissions (real and virtual ones), we obtain:

$$\begin{aligned} \sigma_{S,N}(\rho, m_b^2) = & \exp \int_0^1 \frac{dy}{y} \left[(1-y)^{N-1} - 1 \right] \left\{ \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{\rho}} A^{(1)}(\rho) \alpha_S(k^2) \frac{dk^2}{k^2} + \right. \\ & \left. + D^{(1)} \alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right) + \Delta^{(1)} \alpha_S \left(\frac{m_b^2 y^2}{\rho} \right) \right\}. \quad (68) \end{aligned}$$

Eq. (68) consistently resums soft logarithms in leading logarithmic (LL_S) approximation ⁶, i.e.

⁶The form of the resummation coefficients given in eq. (68) is not unique. An alternative form involves a

all the soft terms having the form in N -space

$$\alpha_S^n \log^n N. \quad (70)$$

For a complete next-to-leading logarithmic (NLL_S) approximation, i.e. the inclusion of all the terms

$$\alpha_S^n \log^{n-1} N, \quad (71)$$

one has to include second-order corrections to the above first-order terms, i.e.

$$A^{(1)}(\rho) \alpha_S \rightarrow A^{(1)}(\rho) \alpha_S + A^{(2)}(\rho) \alpha_S^2; \quad D^{(1)} \alpha_S \rightarrow D^{(1)} \alpha_S + D^{(2)} \alpha_S^2; \quad \Delta^{(1)} \alpha_S \rightarrow \Delta^{(1)} \alpha_S + \Delta^{(2)} \alpha_S^2. \quad (72)$$

These higher-order corrections describe residual multi-parton interactions which are not taken into account neither by the exponentiation of the one-gluon distribution nor by the insertion of the effective gluon-jet coupling $\alpha_S(k^2)$ ⁷. The second-order corrections are, as far as we know, not available at present in QCD⁸. However, an improvement of eq. (68) can be obtained by using the two-loop approximation for the QCD coupling α_S and by adding the second-order correction $A^{(2)}(\rho)$ in the limit $\rho \rightarrow 0$, which is known since a long time [17–19]:

$$A^{(2)}(\rho) \approx A^{(2)}(0) = \frac{C_F}{\pi^2} \left[C_A \left(\frac{67}{36} - \frac{z(2)}{2} \right) - \frac{5}{18} n_f \right], \quad (73)$$

where $z(2) = \pi^2/6$ and n_f is the number of active quark flavors. Let us note that the term proportional to $A^{(2)}$ gives a contribution on the r.h.s. of eq. (68)

$$\approx A^{(2)} \alpha_S^2 \log \left(\frac{1+\rho}{\rho} \right), \quad (74)$$

which vanishes in the limit $\rho \rightarrow \infty$, and therefore does not spoil the vanishing of the exponent of the form factor in that limit (see eq. (17)). Soft logarithms can formally be resummed to any accuracy by including higher order corrections to the above functions, namely:

$$\sigma_{S,N}(\rho, m_b^2) = \exp \int_0^1 \frac{dy}{y} \left[(1-y)^{N-1} - 1 \right] \left\{ \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{\rho}} A[\rho; \alpha_S(k^2)] \frac{dk^2}{k^2} + \right.$$

coefficient $\tilde{A}^{(1)}$ equal to the massless one and coefficients $\tilde{D}^{(1)}$ and $\tilde{\Delta}^{(1)}$ depending on ρ :

$$\tilde{A}^{(1)} = \frac{C_F}{\pi}; \quad \tilde{D}^{(1)}(\rho) = -\frac{C_F}{\pi} \left(1 - \rho \log \frac{1+\rho}{\rho} \right); \quad \tilde{\Delta}^{(1)}(\rho) = -\frac{C_F}{\pi} \left(1 - \rho \log \frac{1+\rho}{\rho} \right). \quad (69)$$

Note that $\tilde{D}^{(1)}(\rho)$ and $\tilde{\Delta}^{(1)}(\rho)$ vanish for $\rho \rightarrow \infty$.

⁷That is similar to the (more familiar) inclusion of higher-order corrections to a one-loop anomalous dimension: $\gamma^{(0)} \alpha_S \rightarrow \gamma(\alpha_S) = \gamma^{(0)} \alpha_S + \gamma^{(1)} \alpha_S^2 + \dots$.

⁸Virtual corrections to heavy flavor decay have been computed in two-loop order in soft approximation within an effective theory framework for $n_f = 0$ (purely gluonic case) in [4]. The relation to QCD distributions is straightforward to $\mathcal{O}(\alpha_S)$, but beyond leading order we do not fully understand the regularization-renormalization scheme effects.

$$+ \left. D \left[\alpha_S \left(\frac{m_b^2 y^2}{1 + \rho} \right) \right] + \Delta \left[\alpha_S \left(\frac{m_b^2 y^2}{\rho} \right) \right] \right\}. \quad (75)$$

The functions $D(\alpha_S)$ and $\Delta(\alpha_S)$ coincide up to first order and, on the basis of physical arguments, it was conjectured in [8] that this property extends to higher orders, i.e. that ⁹

$$\Delta(\alpha_S) = D(\alpha_S). \quad (76)$$

We do not deal with this problem here because $A^{(2)}(\rho)$ is not available at present and therefore our analysis is limited to including $D^{(1)}$ and $\Delta^{(1)}$ only.

The soft resummation formula in eq. (75) is the main result of this section. Let us comment upon it. The term proportional to $A(\rho; \alpha_S)$ contains a logarithmic integration over the gluon transverse momentum squared and becomes large for ρ small: it contains a kind of “ ρ -evolution”, while in the massless case one has an evolution in y (cfr. eq. (86)). The contribution in $D(\alpha_S)$ comes from soft emission off the beauty quark, while that one in $\Delta(\alpha_S)$ from the strange quark.

3 Resumming Final-Mass Effects

In this section we derive a general resummation formula by using the dipole factorization formulae for the massive case derived in [20, 21] (see also [22]). By taking the photon in the final state as the spectator parton, one obtains from eq. (5.16) of [21]:

$$\begin{aligned} \sigma^{(R)}(y) &= \frac{1}{y} \frac{C_F \alpha_S}{\pi} \int_{\frac{\rho}{y+\rho}}^1 \left[\frac{1+2\rho}{1-(1-y)z} - \frac{1+z}{2} - \frac{\rho}{y} \right] dz + \\ &+ (\text{soft emission contributions off the } b, \propto D^{(1)}), \end{aligned} \quad (77)$$

where z is the normalized projection of the strange 4-momentum along the (reference) photon 4-momentum:

$$z = z_s \equiv \frac{p_s \cdot p_\gamma}{(p_s + p_g) \cdot p_\gamma}; \quad z_g \equiv \frac{p_g \cdot p_\gamma}{(p_s + p_g) \cdot p_\gamma} = 1 - z_s. \quad (78)$$

The factor $1 + 2\rho$ is not present in eq. (5.16) of [21]: we have inserted it by hand in order to take into account the specific *soft* structure of heavy flavor decay, namely one heavy leg in the

⁹An explicit check of the above relation requires a massive two-loop computation, which is outside the scope of the present paper.

initial state and one lighter leg in the final state; this process was not explicitly considered in [21].¹⁰ Let us now sketch the integrations of the above terms in turn:

1. by keeping logarithmic terms only, the first term in the square bracket on the r.h.s. of eq. (77) reads:

$$\frac{C_F\alpha_S}{\pi} \frac{1+2\rho}{y} \int_y^{\frac{(1+\rho)y}{y+\rho}} \frac{d\omega}{\omega} = -\frac{C_F\alpha_S}{\pi} \frac{1+2\rho}{y} \log\left(\frac{y+\rho}{1+\rho}\right), \quad (79)$$

where the unitary gluon energy reads in terms of light-cone variables:

$$\omega \equiv 1 - (1-y)z \simeq 1 - z + y. \quad (80)$$

Eq. (80) is easily proved by using the following expression for y :

$$y = y_{sg,\gamma} = \frac{p_s \cdot p_g}{p_s \cdot p_\gamma + p_g \cdot p_\gamma + p_s \cdot p_g}; \quad (81)$$

2. The second term on the r.h.s., related to hard collinear emission off the s quark, reads:

$$-\frac{C_F\alpha_S}{\pi} \left[\frac{3}{4} \frac{1}{y+\rho} + \frac{\rho}{4} \frac{1}{(y+\rho)^2} \right]. \quad (82)$$

The second term in the above equation is always smaller than the first one and it does not produce any logarithmic contributions. Therefore, it can safely be neglected. For $y \gg \rho$, we re-obtain the well-known massless contribution, $-3C_F\alpha_S/(4\pi y)$, while in the opposite regime, $y \ll \rho$, the above contribution can be neglected because there is no more a small denominator. Let us stress that all that is in complete agreement with physical intuition;

3. The last term on the r.h.s., related to soft emission not collinearly enhanced off the strange quark, reads:

$$-\frac{C_F\alpha_S}{\pi} \left(\frac{1}{y} - \frac{1}{y+\rho} \right). \quad (83)$$

3.1 Inclusive Gluon-Decay Effects

The gluon transverse momentum squared is conveniently written as

$$k^2 \equiv \frac{m_b^2 y \omega}{1+\rho}, \quad (84)$$

¹⁰Let us stress that $1+2\rho \rightarrow 1$ in the quasi-collinear limit ($k_\perp \rightarrow 0$, $m \rightarrow 0$ with $k_\perp/m \rightarrow \text{const}$). That implies that this factor cannot be determined with quasi-collinear factorization but only with soft factorization because it is process dependent.

where ω is expressed in terms of light-cone variables in eq. (80) (see also [23]). As in the soft case, we include gluon-branching effects by replacing the tree-level coupling with the effective coupling (55) in the one-gluon distribution (77):

1. The double-logarithmic term on the l.h.s. of eq. (79) then reads:

$$\frac{1}{y} \frac{C_F}{\pi} (1 + 2\rho) \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{y+\rho}} \frac{dk^2}{k^2} \alpha_S(k^2). \quad (85)$$

In the soft limit ($y \rightarrow 0$ at fixed $\rho \in (0, \infty)$), the above expression reduces to the corresponding eikonal contribution derived in eq. (60) of the previous section: the only difference lies in the upper limit. In the massless limit ($\rho \rightarrow 0$ with $y \neq 0$), the above formula reduces instead to the standard (massless) contribution:

$$\frac{1}{y} \int_{m_b^2 y^2}^{m_b^2 y} \frac{dk^2}{k^2} A^{(1)} \alpha_S(k^2); \quad (86)$$

2. The determination of the argument of the QCD coupling entering the subleading terms goes as follows:

$$\begin{aligned} & -\frac{1}{y} \frac{C_F}{\pi} \int_{\frac{\rho}{y+\rho}}^1 \left(\frac{1+z}{2} + \frac{\rho}{y} \right) \alpha_S \left[\frac{m_b^2 y (1 - (1-y)z)}{1+\rho} \right] dz \\ &= -\frac{1}{y} \frac{C_F}{\pi} \alpha_S \left(\frac{m_b^2 y^2}{y+\rho} \right) \int_{\frac{\rho}{y+\rho}}^1 \left(\frac{1+z}{2} + \frac{\rho}{y} \right) dz + \mathcal{O} \left(\frac{\alpha_S^2}{y} \right), \end{aligned} \quad (87)$$

where on the last member we have replaced the following expansion of the QCD coupling:

$$\alpha_S \left[\frac{m_b^2 y (1 - (1-y)z)}{1+\rho} \right] = \alpha_S \left(\frac{m_b^2 y^2}{y+\rho} \right) - \beta_0 \alpha_S^2 \left(\frac{m_b^2 y^2}{y+\rho} \right) \log \left\{ \frac{(y+\rho)[1 - (1-y)z]}{(1+\rho)y} \right\} + \dots \quad (88)$$

The z -integral on the r.h.s. of eq. (87) is equal to the one in lowest order we have already calculated and the higher-order terms in the expansion (88) produce terms beyond our accuracy, namely $\mathcal{O}(\alpha_S^2/y)$. Let us remark that the rule (55) can be applied also to describe the branching of a hard gluon emitted collinearly to the s -quark, because the enhancement is given again by the denominator of the s -quark propagator (see sec. 2.2).

By explicitly adding the soft emission contribution off the b obtained in the previous section, the complete distribution reads:

$$\sigma^{(R)}(y; \rho, m_b^2) = \frac{1}{y} \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{y+\rho}} A^{(1)}(\rho) \alpha_S(k^2) \frac{dk^2}{k^2} + D^{(1)} \alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right) \frac{1}{y} + \quad (89)$$

$$+ \Delta^{(1)}\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) \left(\frac{1}{y} - \frac{1}{y + \rho} \right) + B^{(1)}\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) \frac{1}{y + \rho}.$$

3.2 N -Space and Exponentiation

By repeating the computation made in the soft case, one obtains:

$$\begin{aligned} \sigma_N(\rho; m_b^2) = \exp \int_0^1 dy [(1-y)^{N-1} - 1] & \left\{ \frac{1}{y} \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{y+\rho}} A^{(1)}(\rho) \alpha_S(k^2) \frac{dk^2}{k^2} + \frac{1}{y} D^{(1)}\alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right) + \right. \\ & \left. + \left(\frac{1}{y} - \frac{1}{y + \rho} \right) \Delta^{(1)}\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) + \frac{1}{y + \rho} B^{(1)}\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) \right\}. \end{aligned} \quad (90)$$

What about the inclusion of higher orders? The first point is that there is a different counting in the soft region $\rho \gtrsim \mathcal{O}(1)$, which is a single logarithmic region, and in the collinear ($\rho = 0$) or quasi-collinear regions ($\rho \ll 1$), which are double logarithmic ones. For instance, for large $\rho \approx \mathcal{O}(1)$, the term proportional to $A^{(1)}(\rho)$ is of the same order of the $D^{(1)}$ and $\Delta^{(1)}$ terms. On the contrary, for $\rho \ll 1$ the $A^{(1)}$ term is leading because it is multiplied by a double log, while $D^{(1)}$ and $\Delta^{(1)}$ are subleading because they are multiplied by a single log. That complication arises because our formula is a smooth interpolation between two different dynamical regions. An improvement can be realized as in the soft case by means of the replacement:

$$A^{(1)}(\rho) \alpha_S \rightarrow A^{(1)}(\rho) \alpha_S + A^{(2)}(0) \alpha_S^2, \quad (91)$$

with $A^{(2)}(0)$ given in eq. (73). The above substitution allows a *complete NLL* resummation in the *massless* case ($\rho/y \rightarrow 0$) and a *partial NLL* resummation in the *soft* case ($y/\rho \rightarrow 0$).

Infrared logarithms of subleading orders can be formally resummed by including higher-order corrections to the functions in the exponent, similarly to the soft case:

$$\begin{aligned} \sigma_N(\rho; m_b^2) = \exp \int_0^1 dy [(1-y)^{N-1} - 1] & \left\{ \frac{1}{y} \int_{\frac{m_b^2 y^2}{1+\rho}}^{\frac{m_b^2 y^2}{y+\rho}} A[\rho; \alpha_S(k^2)] \frac{dk^2}{k^2} + \frac{1}{y} D \left[\alpha_S \left(\frac{m_b^2 y^2}{1+\rho} \right) \right] + \right. \\ & \left. + \left(\frac{1}{y} - \frac{1}{y + \rho} \right) \Delta \left[\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) \right] + \frac{1}{y + \rho} B \left[\alpha_S \left(\frac{m_b^2 y^2}{y + \rho} \right) \right] \right\}. \end{aligned} \quad (92)$$

3.3 A Symmetry of the Resummation Functions

Eq. (92) is invariant under the simultaneous substitutions:

$$A(\rho; \alpha_S) \rightarrow A(\rho, \alpha_S) + \beta(\alpha_S) \frac{d}{d\alpha_S} F(\alpha_S); \quad D(\alpha_S) \rightarrow D(\alpha_S) + F(\alpha_S);$$

$$\Delta(\alpha_S) \rightarrow \Delta(\alpha_S) - F(\alpha_S); \quad B(\alpha_S) \rightarrow B(\alpha_S) - F(\alpha_S), \quad (93)$$

where $F(\alpha_S)$ is an arbitrary function having a power-series expansion in the QCD coupling,

$$F(\alpha_S) = \sum_{n=1}^{\infty} F^{(n)} \alpha_S^n. \quad (94)$$

We have defined the QCD β -function coefficients with an over-all minus sign:

$$k^2 \frac{d\alpha_S}{dk^2} = \beta(\alpha_S) = -\beta_0 \alpha_S^2 - \beta_1 \alpha_S^3 - \dots. \quad (95)$$

The above relation implies that only 3 out of the 4 functions entering eq. (92) are independent; in particular, one can choose $F(\alpha_S)$ in such a way as to make one out of the functions $B(\alpha_S)$, $D(\alpha_S)$ and $\Delta(\alpha_S)$ vanishing, or to reduce the function $A(\rho, \alpha_S)$ to its lowest-order term $A^{(1)}(\rho) \alpha_S$. The above symmetry is a generalization of that one of the massless formula ($\rho = 0$) involving only the functions A , B and D found in [24]. In NLL, the above substitutions read:

$$A^{(2)}(\rho) \rightarrow A^{(2)}(\rho) - \beta_0 F^{(1)}; \quad D^{(1)} \rightarrow D^{(1)} + F^{(1)}; \quad \Delta^{(1)} \rightarrow \Delta^{(1)} - F^{(1)}; \quad B^{(1)} \rightarrow B^{(1)} - F^{(1)}. \quad (96)$$

Finally, the resummation coefficients are also modified by a change of the renormalization scheme of $\alpha_S = \alpha_S'(1 + C\alpha_S' + \dots)$; this problem has been investigated in [25].

4 Semileptonic Decay

This is the central section of the paper, in which we extend threshold resummation from the radiative decay to the semileptonic one. By means of general field-theoretic arguments, one can show that the hard scale Q in heavy flavor decays is proportional to the total final hadron energy [2, 7]:

$$Q = 2 E_{X_f} + \mathcal{O}\left(\frac{m_{X_f}^2}{E_{X_f}^2}\right), \quad (97)$$

where $f = s, c$. The main difference between the radiative decay ($q^2 = 0$, $q = p_\gamma$) and the semileptonic one ($q^2 \geq 0$, $q = p_l + p_\nu$) is that in the latter case the lepton-neutrino pair can take away from the QCD final state a large fraction of the initial energy provided by the b -quark mass. We may fix the power-suppressed terms in eq. (97) by imposing that the hard scale Q exactly coincides with the beauty mass m_b when $q^2 = 0$ (see eq. (45)), so that:

$$Q = E_{X_f} + |\vec{p}_{X_f}| \Rightarrow m_b \quad \text{when } q^2 = 0. \quad (98)$$

The hadronic variable y and the mass-correction parameter ρ must therefore be defined by replacing m_b with the general (q^2 -dependent) hard scale Q :

$$y = \frac{m_{X_s}^2 - m_s^2}{m_b^2 - m_s^2} \rightarrow \frac{m_{X_c}^2 - m_c^2}{Q^2 - m_c^2}; \quad \rho = \frac{m_s^2}{m_b^2 - m_s^2} \rightarrow \frac{m_c^2}{Q^2 - m_c^2}. \quad (99)$$

One then obtains the resummed QCD form factor in N -space given in eq. (6) of the introduction, the main result of this work. This equation systematically resums all threshold logarithms occurring in the semileptonic B decays for any value of the ratio $m_c/m_b \in [0, 1]$. It interpolates in a smooth way dynamical regions 1., 2. and 3. described in the introduction.

If the charm quark is not (too) fast, as it happens in most of the phase space of semileptonic $b \rightarrow c$ transitions, one can use the simplified resummation formula (19), which resums soft logarithms only. The latter correctly describes regions 1. and 2. but, unlike the full result (6), cannot account for the massless limit $\rho \rightarrow \infty$, i.e. for region 3. In the soft limit:

$$E_{X_c} \simeq E_c \simeq E_c^{(0)} = \frac{m_b^2 + m_c^2 - q^2}{2m_b}; \quad |\vec{p}_{X_c}| \simeq |\vec{p}_c| \simeq |\vec{p}_c^{(0)}| = \frac{\sqrt{\lambda(m_b^2, m_c^2, q^2)}}{2m_b}, \quad (100)$$

where $\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. It is straightforward to check that the above formulae reduce to the one for the radiative case in the limit $q^2 \rightarrow 0$.

5 Conclusions

We have presented in eq. (6) an expression for the QCD form factor resumming threshold logarithms in the semileptonic decay

$$B \rightarrow X_c + l + \nu_l. \quad (101)$$

The resummation is valid for any chosen value of the ratio $m_c/m_b \in [0, 1]$ and reduces to the standard next-to-leading logarithmic approximation for $m_c \rightarrow 0$. Coherence effects, occurring for a small relative velocity of the c and b quarks and leading to destructive soft-gluon interference, are correctly incorporated in the form factor, whose corrections vanish in the no-recoil point. Our form factor is basically a smooth interpolation of a *soft* resummation and a *soft + collinear* one and constitutes a consistent description of a multi-scale process like semileptonic $b \rightarrow c$ decay; such interpolation is not unique and has been constructed on the basis of simplicity's requirements. The resummation formula in eq. (6) has been derived from "first principles", namely from the universal properties of QCD radiation. Much attention has been paid to the correct inclusion of gluon-branching effects by means of a proper definition of gluon

transverse momentum. We have also presented in eq. (19) a simplified resummation formula which takes into account soft effects only and which can be applied as long as the charm quark is not too fast, as is the case in most of the phase space of semileptonic $b \rightarrow c$ transitions.

Resummed perturbation theory breaks down for final hadron masses as small as

$$m_X^2 \Big|_{NP} \approx m_c^2 + \Lambda_{QCD} \sqrt{Q^2 - m_c^2}, \quad (102)$$

because of large soft effects controlled by the QCD coupling evaluated close to the Landau pole. Such effects can formally be factorized by the shape function for a massive quark in the final state [10]. In practice, one can model them with an effective ghost-less QCD coupling constructed on the basis of some analyticity requirement, along the lines of [26, 27]. Our resummed calculation can be combined with the $\mathcal{O}(\alpha_S)$ triple differential distribution computed in [28, 29] by a proper matching procedure in order to obtain an uniform approximation in the whole phase space.

We shortly mention here some phenomenological outputs of our results. Resummation effects are expected to be largely sensitive to the charm mass and a phenomenological analysis could probably results in a more precise determination of m_c . Unlike the charmless channel, the $b \rightarrow c$ rate is known to be dominated by few hadronic states; given the high-quality data of the B -factories, comparison with solid theoretical expectations could also provide a non-trivial check of local parton-hadron duality and, in general, it could give more information about hadron dynamics at the scale of few GeV's. The theoretical knowledge of $b \rightarrow c$ semileptonic spectra could also be of utility for the inclusive extraction of the CKM matrix element $|V_{cb}|$. Furthermore in [26] a model was formulated which is in disagreement with the charmless electron spectra below 2.2 GeV measured at the B -factories; a natural explanation is an under-subtracted charm background, which could be better estimated by means of our results.

Our formula can also be used to describe beauty-mass effects in semileptonic top decays $t \rightarrow b + W$. A natural extension of our formalism would also allow a smooth description of heavy-flavor production in e^+e^- annihilation,

$$e^+e^- \rightarrow Q\bar{Q}, \quad (103)$$

as function of the center-of-mass energy $E_{c.m.}$, from threshold, $E_{c.m.} = 2m_Q$, up to higher energies, $E_{c.m.} \gg 2m_Q$. As far as soft effects are concerned this process may be seen as a crossed semileptonic decay. Coherence effects play a similar role as in the heavy flavor decay also in the production process (103): one produces a $Q\bar{Q}$ pair in a color singlet state, which at the threshold, classically, sitting one on the top of the other, do screen their color and consequently emit no soft radiation.

Acknowledgments

We would like to thank V. Del Duca for discussions.

G. F. acknowledges support by the European Community's Marie-Curie Research Training Network Programme under contract MRTN-CT-2006-035505 "*Tools and Precision Calculations for Physics Discoveries at Colliders*".

References

- [1] S. Moch and A. Vogt, "*Higher-order soft corrections to lepton pair and Higgs boson production,*" Phys. Lett. B **631** (2005) 48 [arXiv:hep-ph/0508265].
- [2] U. Aglietti, G. Ricciardi and G. Ferrera, "*Threshold resummed spectra in $B \rightarrow X_u l \nu$ decays in NLO. I,*" Phys. Rev. D **74** (2006) 034004 [arXiv:hep-ph/0507285].
- [3] N. Isgur and M. B. Wise, "*Weak Decays Of Heavy Mesons In The Static Quark Approximation,*" Phys. Lett. B **232** (1989) 113. The conclusions for exclusive decays in the above paper are easily extended to inclusive ones.
- [4] G. P. Korchemsky and A. V. Radyushkin, "*Renormalization of the Wilson Loops Beyond the Leading Order,*" Nucl. Phys. B **283** (1987) 342.
- [5] G. Sterman, "*Summation of Large Corrections to Short Distance Hadronic Cross-Sections,*" Nucl. Phys. B **281** (1987) 310.
- [6] S. Catani and L. Trentadue, "*Resummation of the QCD Perturbative Series for Hard Processes,*" Nucl. Phys. B **327**, 323 (1989).
- [7] U. Aglietti, "*Resummed $B \rightarrow X_u l \nu$ decay distributions to next-to-leading order,*" Nucl. Phys. B **610** (2001) 293 [arXiv:hep-ph/0104020].
- [8] U. Aglietti, L. Di Giustino, G. Ferrera and L. Trentadue, "*Resummed mass distribution for jets initiated by massive quarks,*" Phys. Lett. B **651** (2007) 275 [arXiv:hep-ph/0612073v3].
- [9] U. Aglietti, "*Prediction of beauty particle masses with the heavy quark effective theory,*" Phys. Lett. B **281** (1992) 341.
- [10] T. Mannel and M. Neubert, "*Resummation of nonperturbative corrections to the lepton spectrum in inclusive $B \rightarrow X$ lepton anti-neutrino decays,*" Phys. Rev. D **50** (1994) 2037 [arXiv:hep-ph/9402288].

- [11] Y. Dokshitzer et al., *Basics of Perturbative QCD*, Editions Frontieres, Paris (1991); R. Ellis, W. Stirling and B. Webber, *QCD and Collider Physics*, Cambridge University Press, Cambridge (1996).
- [12] For a detailed derivation, see for example the discussion on infrared divergencies in: V. B. Berestetsky, E. M. Lifshitz and L. P. Pitaevsky, “*Quantum Electrodynamics*,” Oxford, Uk: Pergamon (1982) 652 P. (Course Of Theoretical Physics, 4).
- [13] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, “*A Treatment Of Hard Processes Sensitive To The Infrared Structure Of QCD*,” Nucl. Phys. B **173** (1980) 429.
- [14] U. Aglietti and G. Ricciardi, “*A model for next-to-leading order threshold resummed form factors*,” Phys. Rev. D **70** (2004) 114008 [arXiv:hep-ph/0407225].
- [15] A. Bassetto, M. Ciafaloni and G. Marchesini, “*Jet Structure And Infrared Sensitive Quantities In Perturbative QCD*,” Phys. Rept. **100** (1983) 201.
- [16] See for example: S. Catani, “*Soft-gluon resummation: A short review*,” arXiv:hep-ph/9709503.
- [17] J. Kodaira and L. Trentadue, “*Summing Soft Emission In QCD*,” Phys. Lett. B **112** (1982) 66.
- [18] J. Kodaira and L. Trentadue, “*Soft Gluon Effects In Perturbative Quantum Chromodynamics*,” SLAC preprint, SLAC-PUB-2934;
- [19] S. Catani, E. D’Emilio and L. Trentadue, “*The gluon form-factor to higher orders: gluon gluon annihilation at small Q-transverse*,” Phys. Lett. B **211** (1988) 335.
- [20] S. Dittmaier, “*A general approach to photon radiation off fermions*,” Nucl. Phys. B **565** (2000) 69 [arXiv:hep-ph/9904440].
- [21] S. Catani, S. Dittmaier, M. H. Seymour and Z. Trocsanyi, “*The dipole formalism for next-to-leading order QCD calculations with massive partons*,” Nucl. Phys. B **627** (2002) 189 [arXiv:hep-ph/0201036].
- [22] S. Keller and E. Laenen, “*Next-to-leading order cross sections for tagged reactions*,” Phys. Rev. D **59** (1999) 114004 [arXiv:hep-ph/9812415].
- [23] S. Gieseke, P. Stephens and B. Webber, “*New formalism for QCD parton showers*,” JHEP **0312** (2003) 045 [arXiv:hep-ph/0310083]: see eq. (2.5).

- [24] S. Catani and L. Trentadue, “*Comment on QCD exponentiation at large x ,*” Nucl. Phys. B **353** (1991) 183.
- [25] S. Catani, B. R. Webber and G. Marchesini, “*QCD coherent branching and semiinclusive processes at large x ,*” Nucl. Phys. B **349** (1991) 635.
- [26] U. Aglietti, G. Ferrera and G. Ricciardi, “*Semi-inclusive B decays and a model for soft-gluon effects,*” Nucl. Phys. B **768** (2007) 85 [arXiv:hep-ph/0608047].
- [27] U. Aglietti, G. Corcella and G. Ferrera, “*Modelling non-perturbative corrections to bottom-quark fragmentation,*” Nucl. Phys. B **775** (2007) 162 [arXiv:hep-ph/0610035].
- [28] M. Trott, “*Improving extractions of $|V_{cb}|$ and m_b from the hadronic invariant mass moments of semileptonic inclusive B decay,*” Phys. Rev. D **70** (2004) 073003 [arXiv:hep-ph/0402120].
- [29] V. Aquila, P. Gambino, G. Ridolfi and N. Uraltsev, “*Perturbative corrections to semileptonic b decay distributions,*” Nucl. Phys. B **719** (2005) 77 [arXiv:hep-ph/0503083].