THRESHOLD RESUMMED SPECTRA IN $B \to X_u l\nu$ DECAYS IN NLO (III)

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Abstract

We resum to next-to-leading order (NLO) the distribution in the light-cone momentum $p_+ = E_X - |\vec{p}_X|$ and the spectrum in the electron energy E_e in the semileptonic decays $B \to X_u l\nu$, where E_X and $\vec{p}_X$ are the total energy and three-momentum of the final hadron state X_u respectively. By expanding our formulae, we obtain the coefficients of all the infrared logarithms $\alpha_S^n L^k$ at $O(\alpha_S^2)$ and at $O(\alpha_S^3)$, with the exception of the $\alpha_S^3 L$ coefficient. By comparing our $O(\alpha_S^2 n_f)$ result for the p_+ distribution with a recent Feynman diagram computation, we obtain an explicit and non trivial verification of our resummation scheme at the two-loop level (n_f is the number of massless quark flavors). We also discuss the validity of the so-called Brodsky-Lepage-Mackenzie (BLM) scheme in the evaluation of the semileptonic spectra by comparing it with our results, finding that it is a reasonable approximation. Finally, we show that the long-distance phenomena, such as Fermi motion, are expected to have a smaller effect in the electron spectrum than in the hadron mass distribution.

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1 Introduction

The structure of final hadronic states in semileptonic B decays

$$B \rightarrow X_u + l + \nu, \tag{1}$$

where X_u is any hadronic final state coming from the fragmentation of the *up* quark and $l + \nu$ is a lepton-neutrino pair, is usually studied by measuring different spectra, such as the charged-lepton energy distribution, the dilepton invariant mass spectrum, the hadron mass distribution, etc. The computation of these spectra is often rather involved because of the occurrence of different long-distance effects, both perturbative and non-perturbative, such as large logarithms, Fermi motion, hadronization, etc. These effects are often substantial in the end-point regions relevant for the experimental analysis.

Let us summarize in physical terms the main properties of semi-inclusive B decays [1, 2, 3]. We begin by considering the radiative decays

$$B \rightarrow X_s + \gamma \tag{2}$$

which have a simpler long-distance structure than the semileptonic decays (1). That is because, in the radiative case, the tree-level process is the two-body decay

$$b \rightarrow s + \gamma \tag{3}$$

having the large final hadron energy

$$2\,E_X \;=\; m_b \left(1 - \frac{m_s^2}{m_b^2} \right) \;\cong\; m_b, \tag{4}$$

where in the last member we have neglected the small (compared with the beauty mass m_b) strange mass m_s. The final hadronic energy E_X fixes the hard scale Q in the decay:

$$Q \;=\; 2\,E_X, \tag{5}$$

so that $Q \cong m_b$. We are interested in the threshold region

$$m_X \;\ll\; E_X, \tag{6}$$

which can be considered a kind of "perturbation" of the tree-level process due to soft-gluon effects, where m_X is the final hadron mass. The hadron mass m_X, which vanishes in lowest order, remains indeed small in higher orders because of the condition (6), while the large tree-level hadron energy (4) is only mildly increased by soft emissions. The perturbative expansion is therefore controlled by the coupling

$$\alpha\,(Q) \;=\; \alpha\,(m_b) \;\cong\; 0.22 \;\ll\; 1, \tag{7}$$

where $\alpha \,=\, \alpha_S$ is the QCD coupling, and it is therefore legitimate.

In semileptonic decays, the tree-level process is instead the *three-body* decay

$$b \rightarrow u + l + \nu \tag{8}$$

and the hadron energy E_X (i.e. the energy of the *up* quark) can become substantially smaller than half of the beauty mass $m_b/2$. In other words, kinematical configurations are possible with

$$E_X \;\approx\; \frac{m_b}{2} \tag{9}$$

as well as with [4]

$$E_X \;\ll\; m_b. \tag{11}$$

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[4]It is clear that the condition

$$E_X \;\gg\; \Lambda \tag{10}$$

must always be verified in order to deal with a hard process and to be justified in the use of perturbation theory. Since the beauty mass m_b is larger than the hadronic scale Λ by an order of magnitude only, it is in practise not easy to satisfy all the conditions above. In any case, the theory is constructed by taking the limit $m_b \rightarrow \infty$.

Consider for example the kinematical configuration, in the b rest frame, with the electron and the neutrino parallel to each other, for which condition (9) holds true, or the configuration with the leptons back to back, each one with an energy $\approx m_b/2$, for which condition (11) is instead verified. This fact is the basic additional complication in going from the radiative decays to the semileptonic ones: the hard scale is no more fixed by the heavy flavor mass but it depends on the kinematics according to eq. (5).

Before going on, let us give the definition of a short-distance quantity in perturbation theory. Let us consider a process characterized by a hard scale

$$Q \gg \Lambda, \tag{12}$$

where Λ is the QCD scale, and by the infrared scales

$$\frac{m_X^4}{Q^2}, \quad m_X^2 \ll Q^2 \tag{13}$$

— the soft scale and the collinear scale respectively. If infrared effects are absent in the quantity under consideration, logarithmic terms of the form

$$\alpha^n \, \log^k \, \frac{Q^2}{m_X^2} \qquad (n = 1, 2, \cdots \infty, \ k = 1, 2, \cdots 2n) \tag{14}$$

must not appear in its perturbative expansion. The presence of terms of the form (14) would indeed signal significant contributions from small momentum scales. That is because these terms originate from the integration of the infrared-enhanced pieces of the QCD matrix elements from the hard scale down to one of the infrared scales:

$$\int_{m_X^4/Q^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}, \qquad \int_{m_X^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \qquad \Rightarrow \qquad \log \frac{Q^2}{m_X^2}. \tag{15}$$

If perturbation theory shows a significant contribution from small momentum scales to some cross section or decay width, we believe there is no reason to think that the same should not occur in a non-perturbative computation.

On the contrary, a quantity such as a spectrum or a ratio of spectra is long-distance dominated if it contains infrared logarithmic terms in the perturbative expansion of the form (14). These terms must be resummed to all orders of perturbation theory in the threshold region (6). For hadronic masses as small as

$$m_X \approx \sqrt{\Lambda \, Q}, \tag{16}$$

the non-perturbative effects related to the soft interations of the b quark in the B meson (the so-called Fermi motion) also come into play and have to be included by means of a structure function [5]. Our criterion to establish whether a quantity is short-distance or it is not is rather "narrow"; we believe that it is a fundamental one and we use it sistematically, i.e. we derive all its consequences. The consequences, as we are going to show, are in some cases not trivial.

Factorization and resummation of threshold logarithms in the radiative case is similar to that of shape variables in e^+e^- annihilations to hadrons. One can write:

$$\frac{1}{\Gamma_r} \int_0^{t_s} \frac{d\Gamma_r}{dt_s'} \, dt_s' = C_r \left[\alpha(m_b)\right] \Sigma \left[t_s; \, \alpha(m_b)\right] + D_r \left[t_s; \, \alpha(m_b)\right], \tag{17}$$

where Γ_r is the inclusive radiative width, we have defined

$$t_s \equiv \frac{m_{X_s}^2}{m_b^2} \qquad (0 \leq t_s \leq 1) \tag{18}$$

and:

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[5]The first and the second integral on the l.h.s. of eq. (15) are related to the structure function and to its coefficient function respectively.

- $C_r(\alpha)$ is a short-distance, process-dependent coefficient function, independent on the hadron variable t_s and having an expansion in powers of α:

$$C_r(\alpha) = 1 + \sum_{n=1}^{\infty} C_r^{(n)} \alpha^n. \tag{19}$$

The explicit expression of the first-order correction $C_r^{(1)}$ has been given in [1, 4];

- $\Sigma(u; \alpha)$ is the universal QCD form factor for heavy flavor decays and has a double expansion of the form:

$$
\begin{aligned}
\Sigma[u; \alpha] &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{nk} \, \alpha^n \, \log^k \frac{1}{u} = \\
&= 1 - \frac{1}{2} \frac{\alpha \, C_F}{\pi} \log^2 \frac{1}{u} + \frac{7}{4} \frac{\alpha \, C_F}{\pi} \log \frac{1}{u} + \frac{1}{8} \left(\frac{\alpha \, C_F}{\pi} \right)^2 \log^4 \frac{1}{u} + \cdots , \tag{20}
\end{aligned}
$$

where $C_F = (N_C^2 - 1)/(2N_C)$ is the Casimir of the fundamental representation of the group $SU(3)$ and $N_C = 3$ is the number of colors. In higher orders, as it is well known, Σ contains at most two logarithms for each power of α, coming from the overlap of the soft and the collinear region in each emission;

- $D_r(t_s; \alpha)$ is a short-distance-dominated, process-dependent remainder function, not containing infrared logarithms and vanishing for $t_s \to 0$ as well as for $\alpha \to 0$:

$$D_r(t_s; \alpha) = \sum_{n=1}^{\infty} D_r^{(n)}(t_s) \alpha^n. \tag{21}$$

The first-order correction $D_r^{(1)}(t_s)$ has been explicitly given in [1, 4].

Factorization and resummation of threshold logarithms in the semileptonic case is conveniently made starting with distributions not integrated over E_X, i.e. not integrated over the hard scale Q. The most general distribution in process (1) is a triple distribution, which has a resummed expression of the form [1, 2, 3]:

$$\frac{1}{\Gamma} \int_0^u \frac{d^3\Gamma}{dx\,dw\,du'} \, du' = C\left[x, w; \alpha(w \, m_b)\right] \Sigma\left[u; \alpha(w \, m_b)\right] + D\left[x, u, w; \alpha(w \, m_b)\right], \tag{22}$$

where we have defined the following kinematical variables:

$$x = \frac{2E_l}{m_b} \ (0 \le x \le 1); \quad w = \frac{Q}{m_b} \ (0 \le w \le 2); \quad u = \frac{1 - \sqrt{1 - (2m_X/Q)^2}}{1 + \sqrt{1 - (2m_X/Q)^2}} \simeq \left(\frac{m_X}{Q}\right)^2 \ (0 \le u \le 1). \tag{23}$$

In the last member we have kept the leading term in the threshold region $m_X \ll Q$ only. Q, the hard scale, is given by eq. (5). Γ is the total semileptonic width,

$$\Gamma = \Gamma_0 \left[1 + \frac{\alpha \, C_F}{\pi} \left(\frac{25}{8} - \frac{\pi^2}{2} \right) + O(\alpha^2) \right], \tag{24}$$

where $\Gamma_0 = G_F^2 m_b^5 |V_{ub}|^2 / (192\pi^3)$ is the tree-level width and m_b is the pole mass. Eq. (22) is a "kinematical" generalization of eq. (17) together with the quantities involved, which all have an expansion in powers of α:

- $C[x, w; \alpha]$, a short-distance, process-dependent coefficient function, dependent on the hadron and lepton energies but independent on the hadron variable u. The explicit expressions of the first two orders have been given in [1, 2];

- $\Sigma[u; \alpha]$, the universal QCD form factor for heavy flavor decays, which now is evaluated for a coupling with a general argument $Q = w \, m_b$;

- $D[x, u, w; \alpha]$, a short-distance dominated, process-dependent remainder function, depending on all the kinematical variables, not containing infrared logarithms and vanishing for $u \to 0$ as well as for $\alpha \to 0$. The first-order correction has been explicitly given in [2].

The resummed formula on the r.h.s. of eq. (22) has been derived with general arguments in an effective field theory which hold to any order in α; one basically considers the infinite-mass limit for the beauty quark

$$m_b \to \infty, \tag{25}$$

while keeping the hadronic energy E_X and the hadronic mass m_X fixed. The main result is that the hard scale Q enters:

1. the argument of the infrared logarithms factorized in the QCD form factor $\Sigma[u; \alpha]$,

$$L \equiv \log \frac{1}{u} \cong \log \frac{Q^2}{m_X^2}; \tag{26}$$

2. the argument of the running coupling $\alpha = \alpha(Q)$, from which the form factor $\Sigma[u; \alpha(Q)]$ depends (as well as the coefficient function and the remainder function).

An explicit check of property 1. has been obtained verifying the consistency between the resummed expression on the r.h.s. of eq. (22) expanded up to $O(\alpha)$ and the triple distribution computed to the same order in [5]. Since the dependence of the coupling on the scale is a second-order effect, point 2. cannot be verified with the above computation. The possibility for instance that the hard scale $Q = w \, m_b$ in (22) is fixed instead by the beauty mass m_b,

$$Q = m_b \quad (???), \tag{27}$$

cannot be explicitly ruled out by comparing with the $O(\alpha)$ triple distribution, because

$$\alpha(w \, m_b) = \alpha(m_b) + O\left(\alpha^2\right). \tag{28}$$

A second-order computation of the triple distribution is not available at present and therefore a direct check of 2. is not possible. An indirect check can however be obtained as explained in the following. Any semileptonic spectrum can be obtained from the triple distribution in (22) by phase-space integration. In general, different results are obtained for a spectrum if the coupling is evaluated at the scale (5) or instead for example at the scale (27). It is therefore sufficient to compute a spectrum in a single variable with our resummation scheme and to compare with an explicit second-order calculation. As far as we known, the only available second-order computation is that of the $O(\alpha^2 \, n_f)$ corrections to the distribution in the light-cone momentum p_+ [10]. To compute the fermionic corrections, one simply inserts in the gluon lines of the first-order diagrams a fermion bubble, avoiding therefore the computation of the complicated two-loop topologies. It is however not straightforward to extract information about resummation from a fermionic computation, as the latter is not consistent "at face value" with the exponentiation property of the form factors [6].

In sec. 2 we resum the spectrum in p_+ in next-to-leading order (NLO), which involves an integration over the hard scale Q. We also give the coefficients of the large logarithms $\alpha^n \, \log^k m_b/p_+$ up to third order included. The only coefficient that we are not able to compute is that of the single logarithm in third order, $\alpha^3 \, \log m_b/p_+$, for which a three-loop computation is required. These logarithmic terms can be used in phenomenological analyses which do not implement the full resummed formulas or can be compared with fixed-order computations as soon

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[6]Exponentiation in QED originates from the independence of mutiple soft-photon emissions. In QCD, multiple gluon emissions are instad not independent because of non-zero gluon color charge, but exponentiation still holds because of the cancellation of the non-abelian correlation effects related to the inclusive character of the form factors.

as the latter become available. By comparing the fermionic contributions to the coefficients of the infrared logarithms with the result of the Feynman diagram computation, we obtain an explicit and non-trivial check of our resummation scheme.

In sec. 3 we resum to NLO the charged lepton energy spectrum. Also in this case there is an integration over the hard scale Q and many considerations made for the p_+ distributions can be repeated for this case as well. The electron spectrum is one of the first quantities studied in semileptonic decays both theoretically and experimentally [14, 15, 16], because it allows for a determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$. To avoid the large backgroud from the decay

$$B \rightarrow X_c + l + \nu, \tag{29}$$

it is usually necessary to restrict the analysis to electron energies above the threshold of the previous process — recently somehow below — i.e. to lepton energies in the interval

$$\frac{m_b}{2} \left(1 - \frac{m_c^2}{m_b^2} \right) \leq E_e \leq \frac{m_b}{2}. \tag{30}$$

The actual hadron kinematics is obtained with the replacements $m_c \rightarrow m_D$ and $m_b \rightarrow m_B$. That gives a rather tight window in the end-point region,

$$\Delta E_e = \frac{m_c^2}{2 m_b} \rightarrow \frac{m_D^2}{2 m_B} = 330 \text{ MeV}, \tag{31}$$

which is dominated by large logarithms and Fermi-motion effects. We will show however that these long-distance phenomena are expected to have a smaller effect on this distribution than for example on the hadron mass spectrum.

In secs. 2 and 3 we also discuss the validity of the so-called Brodsky-Lepage-Mackenzie (BLM) scheme [11] for the evaluation of spectra in the threshold region — see also [12] for a discussion on the radiative case (2). In general, this scheme aims at an estimate of the full second-order corrections $O(\alpha^2)$ by means of the evaluation of the fermionic corrections, i.e. of the $O(\alpha^2 n_f)$ terms only. The idea is that the fermionic corrections carry with them a factor β_0, so after the fermionic computation, one makes the replacement:

$$n_f \rightarrow -\frac{3}{2} \left(4\pi \beta_0 \right) = n_f - \frac{33}{2}, \tag{32}$$

where $4\pi \beta_0 = (11/3 \, C_A - 4/3 \, T_R \, n_f)$ is the first coefficient of the β-function. $C_A = N_C = 3$ is the Casimir of the adjoint representation of the color group $SU(3)_C$ and $T_R = 1/2$ is the trace normalization of the fundamental generators. The BLM approximation is often a rather good one in estimating total cross section and inclusive decay widths. The total semileptonic decay width for example is known at present to full order α^2:

$$\frac{\Gamma}{\Gamma_0} = K(\alpha) = 1 + K^{(1)} \alpha + \left(n_f \hat{K}^{(2)} + \delta K^{(2)} \right) \alpha^2. \tag{33}$$

The explicit value of $K^{(1)}$ has been given in eq. (24) and the values of $\hat{K}^{(2)}$ and $\delta K^{(2)}$ can be found in [13]. With the BLM ansatz, $K^{(2)}$ is over-estimated by $\approx 15\%$ with respect to the exact result (with $n_f = 4$), i.e. this approximation works pretty well in this case.

In sec. 4 we study the BLM ansatz for the radiative and semileptonic distributions in the hadron invariant mass and for the semileptonic distribution in the variable u defined above. The conclusion is that The BLM ansatz works in general rather well, typically within an accuracy of 25%.

Finally, in sec. 5 we draw our conclusions and we discuss the forthcoming application of our formalism to phenomenology.

2 $\hat{p}_+$ distribution

Recently, the spectrum has been computed in the normalized light-cone momentum $\hat{p}_+$ [10] defined as

$$\hat{p}_+ = \frac{E_X - |\vec{p}_X|}{m_b} = \frac{u\,w}{1+u} \simeq u\,w \qquad (0 \le \hat{p}_+ \le 1), \tag{34}$$

where in the last member we have kept the leading term in the threshold region $u \ll 1$ only. Even though this variable is not special in the present framework, let us discuss its resummation for the reasons discussed in the introduction. For notational simplicity let us make everywhere in this section the replacement $\hat{p}_+ \to p$. The derivation is similar to that of the resummed hadron mass distribution made in [3]. The distribution in p is obtained integrating the distribution in the hadronic variables u and w:

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\hat{p}_+} = \int_0^2 dw \int_{\max(0,w-1)}^1 du\, \frac{1}{\Gamma}\frac{d^2\Gamma}{du\,dw}\, \delta\left(p - \frac{u\,w}{1+u}\right). \tag{35}$$

For techinical reasons, it is actually more convenient to compute the partially-integrated distribution given by:

$$R_P(p) \equiv \frac{1}{\Gamma}\int_0^p \frac{d\Gamma}{dp'}\,dp' = \int du\,dw\, \frac{1}{\Gamma}\frac{d^2\Gamma}{du\,dw}\, \theta\left(p - \frac{u\,w}{1+u}\right). \tag{36}$$

Let us replace the resummed expression for the distribution in the hadronic variables u and w [2] on the r.h.s. of eq. (36):

$$
\begin{aligned}
R_P(p) &= \int_0^2 dw \int_{\max(0,w-1)}^1 du\, C_H(w;\alpha)\,\sigma\left[u,\alpha(w\,m_b)\right]\theta\left(p - \frac{u\,w}{1+u}\right) + \\
&+ \int_0^2 dw \int_{\max(0,w-1)}^1 du\, d_H(u,w;\alpha)\,\theta\left(p - \frac{u\,w}{1+u}\right),
\end{aligned}
\tag{37}
$$

where:

- $C_H(w;\alpha)$ is a short-distance coefficient function having the following expansion in powers of α:

$$C_H(w;\alpha) = C_H^{(0)}(w) + \alpha\,C_H^{(1)}(w) + \alpha^2\,C_H^{(2)}(w) + O(\alpha^3), \tag{38}$$

  with

$$C_H^{(0)}(w) = 2w^2(3-2w); \tag{39}$$

$$C_H^{(1)}(w) = \frac{C_F}{\pi}\,2w^2(3-2w)\left[\mathrm{Li}_2(w) + \log w\log(1-w) - \frac{35}{8} - \frac{9-4w}{6-4w}\log w\right]. \tag{40}$$

  $\mathrm{Li}_2(w) = \sum_{n=1}^\infty w^n/n^2$ is the standard dilogarithm;

- $\sigma\left[u;\alpha\right]$ is the differential QCD form factor:

$$\sigma\left[u,\alpha\right] = \frac{d}{du}\Sigma[u;\alpha] = \delta(u) + O(\alpha); \tag{41}$$

- $d_H(u,w;\alpha)$ is a short-distance remainder function whose expansion starts at $O(\alpha)$ and which has an integrable singularity for $u \to 0$. The first-order correction for example has a logarithmic singularity:[7]

$$d_H^{(1)}(u,w) \approx \log u \qquad \text{for } u \to 0. \tag{42}$$

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[7]The explicit expression can be found in [2].

We are interested in the region

$$p \ll 1. \tag{43}$$

The integral of the remainder function on the r.h.s. of eq. (37) vanishes in the limit $p \to 0$, because:

$$\lim_{p \to 0} \int dw\, du\, d_H(u, w;\, \alpha)\, \theta\left(p - \frac{u\, w}{1 + u}\right) = \int dw\, du\, d_H(u, w;\, \alpha) \lim_{p \to 0} \theta\left(p - \frac{u\, w}{1 + u}\right) = 0. \tag{44}$$

The limit $p \to 0$ can be taken inside the integral because the integrand has, as already said, at most an integrable singularity for $u \to 0$. The first integral on the r.h.s. of eq. (37) can be decomposed as:

$$\int_0^1 dw \int_0^1 du C_H(w;\, \alpha)\sigma\left[u, \alpha(w\, m_b)\right] \theta\left(p - \frac{uw}{1 + u}\right) + \int_0^1 du \int_1^{1+u} dw C_H(w;\, \alpha)\sigma\left[u, \alpha(w\, m_b)\right] \theta\left(p - \frac{uw}{1 + u}\right). \tag{45}$$

The second integral vanishes in the limit $p \to 0$ because it extends to an infinitesimal domain in w. The lowest-order term, obtained replacing for $\sigma[u;\, \alpha]$ the last member of eq. (41), identically vanishes.

To summarize, as far as logarithmically enhanced terms and constants terms in the limit $p \to 0$ are concerned, we can make the following approximations:[8]

1. neglect the remainder function $d_H(u, w;\, \alpha)$, whose contribution will be includere later on together with contributions of similar size;

2. approximate the integration domain with a unit square:

$$0 \le u,\, w \le 1; \tag{46}$$

3. simplify the kinematical constraint according to the last member in eq. (34), since large logarithms of p can only come from the region $u \ll 1$.

We then obtain:

$$R_P(p;\, \alpha) = \int_0^p dw\, C_H(w;\, \alpha) + \int_p^1 dw\, C_H(w;\, \alpha) \Sigma\left[p/w;\, \alpha(w\, m_b)\right] + O(p;\, \alpha), \tag{47}$$

where $O(p;\, \alpha)$ denote non-logarithmic, small terms to be included later on by matching with the fixed-order distribution. To the same approximation, i.e. up to terms which $O(p;\, \alpha)$, we can further simplify the distribution by neglecting the first integral and integrating the second integrand down to $w = 0$:

$$R_P(p;\, \alpha) = \int_0^1 dw\, C_H(w;\, \alpha) \Sigma\left[p/w;\, \alpha(w\, m_b)\right] + O(p;\, \alpha), \tag{48}$$

where $\alpha = \alpha(m_b)$. The neglected term is indeed:

$$\int_0^p dw\, C_H(w;\, \alpha) \left\{1 - \Sigma\left[p/w;\, \alpha(w\, m_b)\right]\right\}. \tag{49}$$

Expanding the form factor in powers of $\alpha(m_b)$, the above expression reduces to a linear combination of integrals of the form

$$\int_0^p dw\, C_H(w;\, \alpha)\, \alpha^n \log^k(p) \log^l(w) \to 0 \qquad \text{for } p \to 0 \qquad (n \ge 1;\, k,\, l \ge 0). \tag{50}$$

We have therefore shown that the neglected term is $O(p;\, \alpha)$.

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[8]It is remarkable that the variable p keeps unitary range after the simultaneous approximations 2. and 3.

By insering the first-order expressions for the coefficient function $C_H(w; \alpha)$ and the universal QCD form factor $\Sigma(u; \alpha)$, we obtain for the event fraction $R_P(p; \alpha)$, apart from small terms $O(p; \alpha)$:

$$\int_0^1 dw\, C_H(w; \alpha)\, \Sigma\left[p/w; \alpha(w\, m_b)\right] = 1 - \frac{C_F\, \alpha}{\pi} \left(\frac{1}{2} \log^2 p + \frac{13}{6} \log p + \frac{463}{144}\right) + O(\alpha^2), \qquad (51)$$

where $C_F = \left(N_C^2 - 1\right)/(2N_C)$ is the Casimir of the fundamental representation of the color group $SU(3)_c$ and $N_C = 3$ is the number of colors. Let us note that we have a different coefficient for the single logarithm with respect to those ones in the hadron mass distribution in the semileptonic decay (1) or the radiative decay (2).

2.1 Minimal scheme

We introduce a resummed form of the event fraction as:

$$R_P(p; \alpha) = C_P(\alpha)\, \Sigma_P(p; \alpha) + D_P(p; \alpha). \qquad (52)$$

All the functions above have an expansion in powers of α:

$$C_P(\alpha) = 1 + \alpha\, C_P^{(1)} + \alpha^2\, C_P^{(2)} + O(\alpha^3); \qquad (53)$$
$$\Sigma_P(u; \alpha) = 1 + \alpha\, \Sigma_P^{(1)}(p) + \alpha^2\, \Sigma_P^{(2)}(p) + O(\alpha^3); \qquad (54)$$
$$D_P(p; \alpha) = \alpha\, D_P^{(1)}(p) + \alpha^2\, D_P^{(2)}(p) + O(\alpha^3). \qquad (55)$$

Let us begin considering a minimal scheme, where only logarithms in p are included in the effective form factor Σ_P. Since we will consider a different scheme later on, let us denote the quantities in the minimal scheme with a bar. The first-order corrections to the form factor and the coefficient function in the minimal scheme read:

$$\bar{\Sigma}_P^{(1)}(p) = -\frac{C_F}{\pi} \left(\frac{1}{2} \log^2 p + \frac{13}{6} \log p\right);$$
$$\bar{C}_P^{(1)} = -\frac{C_F}{\pi} \frac{463}{144} = -1.36461. \qquad (56)$$

Note that the correction to the coefficient function is rather large: for $\alpha(m_b) = 0.22$ it amounts to $\approx -30\%$.

We now expand the resummed result in powers of α and compare with the fixed order result, which is known to full order α [10]:

$$R_P(p; \alpha) = 1 + \alpha\, R_P^{(1)}(p) + \alpha^2\, R_P^{(2)}(p) + O(\alpha^3), \qquad (57)$$

with

$$R_P^{(1)}(p) = -\frac{C_F}{\pi} \left[\frac{1}{2} \log^2 p + \frac{13}{6} \log p + \frac{1}{2} \log^2(p)\, p^3\, (2-p) + \frac{1}{12} \log(p)\, p\, (4 - 23\, p - 14\, p^2 + 7\, p^3) + \right.$$
$$\left. + \frac{1}{144}(1-p)^2\, (463 + 246\, p - 240\, p^2 + 64\, p^3 - 7\, p^4 + 2\, p^5)\right]. \qquad (58)$$

We obtain for the leading-order remainder function:[9]

$$\bar{D}_P^{(1)}(p) = -\frac{C_F}{\pi} \left[\frac{1}{2} \log^2(p)\, p^3\, (2-p) + \frac{1}{12} \log(p)\, p\, (4 - 23\, p - 14\, p^2 + 7\, p^3) + \right.$$

[9]To our knowledge, this is the only first-order remainder function possessing a double logarithmic term (multiplied by positive powers of the variable p).

$$+ \frac{1}{144}(1-p)^2\, p\, (246 - 240\, p + 64\, p^2 - 7\, p^3 + 2\, p^4) - \frac{463}{144} p\, (2-p) \bigg]. \tag{59}$$

Since

$$\bar{\Sigma}_P(1;\, \alpha) = 1, \tag{60}$$

taking $p = 1$ in eq. (52), we obtain the following relation between the coefficient function and the remainder function in the upper endpoint:

$$\bar{C}_P(\alpha) = 1 - \bar{D}_P(1;\, \alpha). \tag{61}$$

It is a trivial matter to verify that the above relation holds true for our first-order expressions.

A definition of the minimal scheme which does not make reference to the explicit perturbative expansion of C_H and Σ can be given as follows. Neglecting $O(p;\, \alpha)$ terms, we have that

$$\int_0^1 dw\, C_H(w; \alpha)\, \Sigma\, [p/w;\, \alpha(w\, m_b)] = \bar{C}_P(\alpha)\, \bar{\Sigma}_P\, (p;\, \alpha), \tag{62}$$

so that the coefficient function in the minimal scheme can be defined taking $p = 1$ in the above equation and using eq. (60):

$$\bar{C}_P(\alpha) \equiv \int_0^1 dw\, C_H(w; \alpha)\, \Sigma\, [1/w;\, \alpha(w\, m_b)]. \tag{63}$$

The effective form factor then reads:

$$\bar{\Sigma}_P\, (p;\, \alpha) \equiv \frac{\int_0^1 dw\, C_H(w; \alpha)\, \Sigma\, [p/w;\, \alpha(w\, m_b)]}{\int_0^1 dw\, C_H(w; \alpha)\, \Sigma\, [1/w;\, \alpha(w\, m_b)]}. \tag{64}$$

Let us comment upon the above result:

- The universal form factor $\Sigma(u;\, \alpha)$ is evaluated for an unphysical arguement $u > 1$ in eq. (64), but this does not cause any problem because the perturbative form factor is an analytic function of u which can be continued in principle to any (complex) value of u; [10]

- The effective form factor $\bar{\Sigma}_P(p; \alpha)$ defined in eq. (64) has a similar form to the effective form factor for the t distribution $\bar{\Sigma}_T(t; \alpha)$ considered in [3], where $t = m_X^2/m_b^2$: the only differences are that in the numerator of the definition of $\bar{\Sigma}_T$, t/w^2 is replaced by p/w and in the denominator $1/w^2$ is replaced by $1/w$. The same considerations made for the t-distribution therefore can be repeated for the p-distribution. There are long-distance effects which cannot be extracted from the radiative decay (2), related to the integration over the hadron energy w in the parametric region

$$p \ll w \ll 1. \tag{66}$$

The effects of this region are however suppressed by the coefficient function:

$$C_H(w) \approx w^2 = \frac{Q^2}{m_b^2} \ll 1. \tag{67}$$

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[10] the form factor $\Sigma(u;\, \alpha)$ has been given to $NNLO$ in eq.(125) of [2], where it is written as a function of

$$\tau = \beta_0\, \alpha(Q)\, \log \frac{1}{u}. \tag{65}$$

For $u > 1$ we have that $\tau < 0$, but that does not produce any singularity in $\Sigma(u;\, \alpha)$.

2.2   Higher orders

Let us now discuss the higher orders. In order to remove the exponential (trivial) iterations, let us define the exponent of the effective form factor by means of the relation:

$$\bar{\Sigma}_P = e^{\bar{G}_P}, \tag{68}$$

with

$$\bar{G}_P\left(p; \alpha\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{n+1} \bar{G}_{Pnk}\, \alpha^n\, L_p^k \tag{69}$$

where we have defined:

$$L_p \equiv \log\frac{1}{p} \geq 0. \tag{70}$$

We find:

$$\bar{G}_{P12} = G_{12}; \tag{71}$$

$$\bar{G}_{P11} = G_{11} + \frac{5}{12}\, A_1; \tag{72}$$

$$\bar{G}_{P23} = G_{23}; \tag{73}$$

$$\bar{G}_{P22} = G_{22} + \frac{7}{96}\, A_1^2 + \frac{5}{24}\, A_1\, \beta_0; \tag{74}$$

$$\bar{G}_{P21} = G_{21} + \frac{5}{12}\, A_2 - \frac{5}{12}\, B_1\, \beta_0 + A_1^2\left(-\frac{47}{432} + \frac{5}{12}\, z(2)\right) +$$
$$+ \frac{23}{144}\, A_1\, \beta_0 + \frac{7}{48}\, A_1\,(B_1 + D_1) + A_1\frac{C_F}{\pi}\left(\frac{547}{216} - 2\, z(3)\right); \tag{75}$$

$$\bar{G}_{P34} = G_{34}; \tag{76}$$

$$\bar{G}_{P33} = G_{33} + \frac{83}{5184}\, A_1^3 + \frac{7}{96}\, A_1^2\, \beta_0 + \frac{5}{36}\, A_1\, \beta_0^2; \tag{77}$$

$$\bar{G}_{P32} = G_{32} + \frac{5}{12}\, A_2\, \beta_0 - \frac{5}{12}\, B_1\, \beta_0^2 + A_1^3\left(-\frac{1117}{20736} + \frac{7}{48}\, z(2) - \frac{5}{12}\, z(3)\right) + \frac{7}{48}\, A_1\, A_2 +$$
$$+ A_1\left(\frac{23}{144}\, \beta_0^2 + \frac{5}{24}\, \beta_1\right) - \frac{7}{96}\, A_1\, \beta_0\,(B_1 - D_1) + A_1\, \beta_0\frac{C_F}{\pi}\left(\frac{547}{432} - z(3)\right) +$$
$$+ \frac{83}{1728}A_1{}^2\,(B_1 + D_1) + A_1{}^2\, \beta_0\left(\frac{47}{864} + \frac{25}{24}\, z(2)\right) + A_1{}^2\frac{C_F}{\pi}\left(\frac{22747}{6912} - 3z(4)\right). \tag{78}$$

The G_{ij}'s are the coefficients of the threshold logarithms in the exponent of the form factor G for the radiative decay (2). They have been explicitly given, together with all the constants A_i, B_i, D_i and β_i entering the above equations, in [2][11]. $z(s) = \sum_{n=1}^{\infty} 1/n^s$ is the Riemann Zeta-function with $z(2) = \pi^2/6 = 1.64493\cdots$, $z(3) = 1.20206\cdots$ and $z(4) = \pi^4/90 = 1.08232\cdots$.

Replacing explicit values for the coefficients $\bar{G}_{Pij} \neq G_{ij}$, we obtain:

$$\bar{G}_{P11} = \frac{C_F}{\pi}\frac{13}{6}; \tag{79}$$

$$\bar{G}_{P22} = \frac{C_F}{2\pi^2}\left[C_A\left(\frac{25}{24} + \frac{z(2)}{2}\right) + C_F\left(\frac{7}{48} - z(2)\right) - \frac{n_f}{4}\right]; \tag{80}$$

$$\bar{G}_{P21} = \frac{C_F}{2\pi^2}\left[C_A\left(\frac{1253}{144} - \frac{13}{4}z(2) - \frac{z(3)}{2}\right) + C_F\left(\frac{1303}{288} + \frac{17}{6}z(2) - 3z(3)\right) - n_f\left(\frac{113}{72} - \frac{z(2)}{3}\right)\right]; \tag{81}$$

---

[11]The next-to-next-to-leading-order (NNLO) coefficients A_3, B_2 and D_2, which are a necessary input for the determination of the G_{ij}'s, have been computed in the past few years in refs. [17], [18] and [19] respectively.

$$\bar{G}_{P33} = \frac{C_F}{\pi^3}\left[\frac{n_f^2}{648} + C_F C_A\left(\frac{77}{1152} - \frac{11}{8}z(2)\right) + C_F n_f\left(\frac{29}{576} + \frac{z(2)}{4}\right) + \right.$$

$$\left. + C_A n_f\left(\frac{185}{1296} - \frac{z(2)}{12}\right) + C_A^2\left(-\frac{1589}{2592} + \frac{11}{24}z(2)\right) + C_F^2\left(\frac{83}{5184} + \frac{z(3)}{3}\right)\right]; \tag{82}$$

$$\bar{G}_{P32} = \frac{C_F}{\pi^3}\left[\left(\frac{359}{2592} - \frac{z(2)}{36}\right)n_f^2 + C_F C_A\left(\frac{64331}{41472} + \frac{313}{96}z(2) - \frac{11}{3}z(3) + \frac{5}{4}z(4)\right) + \right.$$

$$+ C_F n_f\left(-\frac{3829}{20736} - \frac{2}{3}z(2) + \frac{5}{12}z(3)\right) + C_F^2\left(\frac{65381}{20736} + \frac{7}{48}z(2) - \frac{13}{6}z(3) - \frac{11}{4}z(4)\right) +$$

$$\left. + C_A n_f\left(-\frac{949}{648} + \frac{53}{144}z(2) - \frac{z(3)}{24}\right) + C_A^2\left(\frac{34907}{10368} - \frac{293}{288}z(2) + \frac{77}{48}z(3) - \frac{11}{16}z(4)\right)\right]. \tag{83}$$

Let us make a few remarks:

- As in the case of the distributions in the variables u and t studied in [3], the leading logarithms $\alpha^n L^{n+1}$ have the same coefficients as in the hadron mass spectrum in the radiative decay (2): differences only occur in NLO and beyond;

- The differences between the coefficients G_{Pij} and the coefficients G_{ij} of the radiative spectrum are much smaller than for the u or the t distribution resummed in [3]. That is an indication that the long-distance effects in the p distribution and in the radiative decay (2) are not very different [10].

2.3   Check of Resummation Formula

In this section we check our results for the resummed p spectrum against an explicit computation of the fermionic corrections, i.e. of the $O(\alpha^2 n_f)$ contributions [10]. The computation with Feynman diagrams of the $O(\alpha^2 n_f)$ terms involves:

1. neglecting double emissions off the primary charges, which would give contributions proportional to α^2 without any n_f factor. The expansion up to $O(\alpha^2)$ of the r.h.s. of eq. (68) can be made by expanding it in powers of $\bar{G}_P$,

$$\bar{\Sigma}_P = 1 + \bar{G}_P + \frac{1}{2}\bar{G}_P^2 + \cdots \tag{84}$$

and keeping terms up to second order in α in $\bar{G}_P$ and up to first order in $\bar{G}_P^2$. There is not any multiple emission left in the fermonic computation, because one neglects the terms $\bar{G}_{P12}^2$, $\bar{G}_{P11}^2 = O(n_f^0)$ in eq. (84). That means that relation (68) is truncated to its (trivial) first order:

$$\hat{\Sigma}_P \simeq 1 + \hat{G}_P \qquad\qquad \left[O(\alpha^2 n_f) \text{ computation}\right]; \tag{85}$$

2. taking into account the secondary branching of a gluon into a (real or virtual) quark-antiquark pair,

$$g^* \to q + \bar{q}, \tag{86}$$

while neglecting the (non-abelian) gluon splitting,

$$g^* \to g + g, \tag{87}$$

Only a *single* secondary branching of abelian kind (86) is therefore allowed.

According to eq. (85), the best interpretation of a fermionic computation is that of relating this result directly to G_P and not to Σ_P.

Because of 2., a non-trivial strucure of G emerges:

$$\hat{G}_P = G_{P12}\,\alpha\,L_p^2 + G_{P11}\,\alpha\,L_p + \hat{G}_{P23}\,\alpha^2\,L_p^3 + \hat{G}_{P22}\,\alpha^2\,L_p^2 + \hat{G}_{P21}\,\alpha^2\,L_p. \tag{88}$$

An $O(\alpha^2 n_f)$ computation therefore does not provide any check of the exponentiation property, but it gives some information about the structure of G. The exponentiation property of form factors has already been checked with many second-order and third-order computations and it is not our concern. We are only interested in the structure of G which is, as we have already said, sensitive to the specific integration over the hard scale.

Our second-order coefficients $\bar{G}_{P2i}$ have a contribution proportional to n_f which is in agreement with the explicit computation:

$$\bar{G}_{P23} = \frac{1}{9\,\pi^2}\,n_f + O\left(n_f^0\right); \tag{89}$$

$$\bar{G}_{P22} = -\frac{1}{6\,\pi^2}\,n_f + O\left(n_f^0\right); \tag{90}$$

$$\bar{G}_{P21} = \left(\frac{1}{27} - \frac{113}{108\,\pi^2}\right)\,n_f + O\left(n_f^0\right), \tag{91}$$

where the numerical values of $C_F = 4/3$ and $C_A = 3$ have been replaced. If we assume instead that the hard scale Q in the coupling $\alpha = \alpha(Q)$ entering the resummed formula (22) is equal to the heavy flavor mass m_b and not to (two-times) the hadron energy $2E_X$, we obtain different fermionic contributions to $\bar{G}_{P22}$ and $\bar{G}_{P21}$ which are in disagreemnt with the fixed-order computation:

$$\bar{G}_{P22}^{(Q=m_b)} = -\frac{7}{27\,\pi^2}\,n_f + O\left(n_f^0\right); \tag{92}$$

$$\bar{G}_{P21}^{(Q=m_b)} = \left(\frac{1}{27} - \frac{47}{81\,\pi^2}\right)\,n_f + O\left(n_f^0\right). \tag{93}$$

This provides us with an explicit check of the resummation formula for the triple-differential distribution at the two-loop level. Let us remark that this check is not related to the study of the BLM scheme: we just compared the coefficients of two independent $O(\alpha^2 n_f)$ computations and we never made the BLM ansatz. Our aim indeed was simply that of studying the effects of eq. (5) in a partial contribution, not that of estimating full second-order corrections.

2.4 The BLM scheme for the form factor

In this section we compare our exact results for the resummed spectrum in p with those ones coming from the BLM ansatz. Let us make a comparison of the coefficients G_{P2i} in QCD in the minimal scheme defined above with the BLM ones $\hat{G}_{P2i}$ starting from the leading terms :

1. $\alpha^2\,L_p^3$; the QCD coefficient

$$\bar{G}_{P23} = -\frac{11}{6\,\pi^2} + \frac{n_f}{9\,\pi^2} = -0.185756 + 0.0112579\,n_f \tag{94}$$

is in complete agreement with the BLM estimate:

$$\hat{G}_{P23} = \bar{G}_{P23}. \tag{95}$$

The approximation turns out to be exact because the QCD coefficient is proportional, as it is well known [20], to the first coefficient of the β-function β_0;

2. $\alpha^2 L_p^2$; the QCD coefficient is

$$\bar{G}_{P22} = \frac{1}{54} + \frac{239}{108\,\pi^2} - \frac{n_f}{6\,\pi^2} = 0.242739 - 0.0168869\,n_f \tag{96}$$

to be compared to the BLM estimate:

$$\hat{G}_{P22} = -\frac{1}{6\pi}\left(n_f - \frac{33}{2}\right) = 0.278633 - 0.0168869\,n_f. \tag{97}$$

The BLM ansatz over-estimates the non-abelian contribution by $\approx$ 15%, i.e. it works rather well;

3. $\alpha^2 L_p$; in QCD we have the value:

$$\bar{G}_{P21} = -\frac{215}{324} + \frac{13883}{648\,\pi^2} - \frac{11\,z(3)}{3\,\pi^2} + \left(\frac{1}{27} - \frac{113}{108\,\pi^2}\right)n_f = 1.06059 - 0.0689749\,n_f \tag{98}$$

to be compared to the BLM estimate

$$\hat{G}_{P21} = \left(\frac{1}{27} - \frac{113}{108\,\pi^2}\right)\left(n_f - \frac{33}{2}\right) = 1.13809 - 0.0689749\,n_f. \tag{99}$$

The BLM ansatz works with an accuracy better than 10%. Let us note also that the BLM cannot reproduce the terms proportional to the trascendental constant $z(3)$ defined in the previous section.

We conclude that the BLM scheme works rather well for the p spectrum.

2.5   Non minimal scheme

In this section we consider a non-minimal factorization scheme which has various advantages. The event fraction has a resummed form equal to the one in the minimal scheme,

$$R_P(p;\,\alpha) = C_P(\alpha)\,\Sigma_P(p;\,\alpha) + D_P(p;\,\alpha), \tag{100}$$

with different expressions for the coefficient function, the form factor and the remainder function. As in the minimal scheme, all these function have a perturbative expansion in α:

$$C_P(\alpha) = 1 + \alpha\,C_P^{(1)} + \alpha^2\,C_P^{(2)} + O\left(\alpha^3\right); \tag{101}$$

$$\Sigma_P(p;\,\alpha) = 1 + \alpha\,\Sigma_P^{(1)}(p) + \alpha^2\,\Sigma_P^{(2)}(p) + O\left(\alpha^3\right); \tag{102}$$

$$D_P(p;\,\alpha) = \alpha\,D_P^{(1)}(p) + \alpha^2\,D_P^{(2)}(p) + O\left(\alpha^3\right). \tag{103}$$

The coefficient function is defined, to all orders, as:

$$C_P(\alpha) = \int_0^1 dw\,C_H(w;\,\alpha), \tag{104}$$

and the expression for the effective form factor is:

$$\Sigma_P(p;\,\alpha) = \frac{\int_0^1 dw\,C_H(w;\,\alpha)\,\tilde{\Sigma}\left[p/w;\,\alpha(w\,m_b)\right]}{\int_0^1 dw\,C_H(w;\,\alpha)}. \tag{105}$$

$\tilde{\Sigma}[u; \alpha]$ is the extended form factor defined in [3], equal to the standard one $\Sigma[u; \alpha]$ for argument less than one, $u < 1$, and equal to one for a larger argument, $u \geq 1$. Explicitly (cfr. eq. (47)) [12]:

$$\Sigma_P (p; \alpha) = \frac{\int_0^p dw\, C_H (w; \alpha) + \int_p^1 dw\, C_H (w; \alpha)\, \Sigma[p/w; \alpha(w\, m_b)]}{\int_0^1 dw\, C_H (w; \alpha)}. \tag{106}$$

This effective form factor is normalized, to all orders, as in the minimal scheme:

$$\Sigma_P (1; \alpha) = 1. \tag{107}$$

By inserting the expansion for $C_H(w; \alpha)$ and for $\tilde{\Sigma}(u; \alpha)$ and integrating over the hadronic energy w, we obtain for the first-order terms:

$$C_P^{(1)} = -\frac{C_F}{\pi} \frac{335}{144} = -0.98735 \tag{108}$$

and

$$\Sigma_P^{(1)}(p) = -\frac{C_F}{\pi} \left(\frac{1}{2} \log^2 p + \frac{13}{6} \log p + \frac{8}{9} - \frac{25}{18} p^3 + \frac{1}{2} p^4 \right). \tag{109}$$

The coefficient function has a smaller value in this scheme than in the minimal one and the effective form factor contains not only a constant term for $p \to 0$ but also infinitesimal terms in the same limit. Matching with the fixed-order distribution as we have done in the minimal case, we obtain for the remainder function:

$$\begin{aligned} D_P^{(1)}(p) =\ & \frac{C_F}{\pi} \bigg[-\frac{1}{2}(2 - p)\, p^3 \log^2 p - \frac{1}{12}(4 - 23\, p - 14\, p^2 + 7\, p^3)\, p \log p + \\ & + \frac{p}{144} \left(680 + 269\, p - 990\, p^2 + 447\, p^3 - 80\, p^4 + 11\, p^5 - 2\, p^6 \right) \bigg]. \end{aligned} \tag{110}$$

Let us note that, with our scheme choices,

$$C_P(\alpha) = C_T(\alpha) = C_U(\alpha) \tag{111}$$

to all orders in α, where $C_T(\alpha)$ and $C_U(\alpha)$ are the coefficient functions for the distributions in the variables u and t computed in [3].

The exponent of the effective form factor in this non-minimal scheme G_P contains, in the limit $p \to 0$, also constants terms ($\bar{G}_{Pij} = G_{Pij}$ for $j \geq 1$):

$$G_{P10} = -\frac{23}{144} A_1 + \frac{5}{12} (B_1 + D_1); \tag{112}$$

$$\begin{aligned} G_{P20} =\ & -\frac{23}{144} A_2 - \frac{101}{576} A_1 \beta_0 + \frac{23}{48} \beta_0 \left(B_1 + \frac{2}{3} D_1 \right) - \frac{47}{432} A_1 (B_1 + D_1) + \frac{7}{96} (B_1 + D_1)^2 + \\ & + \frac{5}{12} (B_2 + D_2) + \frac{5}{12} z(2) A_1 (B_1 + D_1) + B_1 \frac{C_F}{\pi} \left(\frac{547}{216} - 2\, z(3) \right) + D_1 \frac{C_F}{\pi} \left(\frac{547}{216} - 2\, z(3) \right) + \\ & + A_1^2 \left(\frac{2057}{41472} - \frac{23}{144} z(2) + \frac{5}{12} z(3) \right) + A_1 \frac{C_F}{\pi} \left(-\frac{90121}{20736} + \frac{5}{6} z(3) + 3\, z(4) \right). \end{aligned} \tag{113}$$

Explicitly:

$$G_{P10} = -\frac{8}{9} \frac{C_F}{\pi}; \tag{114}$$

---

[12] Replacing the extended form factor $\tilde{\Sigma}$ with the usual one Σ in eq. (105) amounts to dropping the infinitesimal terms for $p \to 0$ in Σ_P.

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$$G_{P20} = \frac{C_F}{\pi^2}\left[n_f\left(\frac{2243}{5184} - \frac{5}{72}z(2)\right) + C_F\left(-\frac{14435}{1728} - \frac{83}{144}z(2) + \frac{19}{4}z(3) + 3\,z(4) - \frac{5}{8}z(3)\right) + \right.$$
$$\left. + C_A\left(-\frac{24775}{10368} + \frac{193}{288}z(2) + \frac{5}{48}z(3)\right)\right]. \tag{115}$$

As in the case of the t distribution considered in [3], the coefficients functions are related in the two schemes by the following relations:

$$C_P^{(1)} = \bar{C}_P^{(1)} - G_{P10}; \tag{116}$$
$$C_P^{(2)} = \bar{C}_P^{(2)} - \bar{C}_P^{(1)}G_{P10} + \frac{1}{2}G_{P10}^2 - G_{P20}. \tag{117}$$

The first of the above equations is easily verified by inserting our first-order expressions.

3 Electron spectrum

In our framework, the electron spectrum is obtained by integrating the distribution in the hadron and electron energies w and $\bar{x} \equiv 1 - x$, resummed to NLO in [2], over the hadron energy:

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\bar{x}} = \int_{\bar{x}}^{1+\bar{x}} dw\,\frac{1}{\Gamma}\frac{d^2\Gamma}{dw\,d\bar{x}}. \tag{118}$$

In the previous case, in order to avoid generalized functions entering the differential form factor $\sigma[u;\alpha]$, we have considered the partially integrated distribution or event fraction. That in not necessary in this case because the differential electron spectrum, as we are going to show, already contains the partially integrated form factor $\Sigma[u;\alpha]$. Inserting the resummed expression for the double distribution, we obtain:[13]

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\bar{x}} = \int_{\bar{x}}^{1} dw\,C_L(\bar{x},w;\alpha)\Sigma\left[\frac{\bar{x}}{w};\alpha(w\,m_b)\right] + \int_{1}^{1+\bar{x}} dw\,C_{XW1}(\bar{x};\alpha)\left\{1 - C_{XW2}(\bar{x};\alpha)\Sigma\left[w-1;\alpha(m_b)\right]\right\}$$
$$+ \int_{\bar{x}}^{1} dw\,d_<(\bar{x},w;\alpha) + \int_{1}^{1+\bar{x}} dw\,d_>(\bar{x},w;\alpha). \tag{119}$$

We are interested in the region

$$\bar{x} \ll 1, \tag{120}$$

which selects hadron final states with a small invariant mass and produces large logarithms in the perturbative expansion. The integral of the first remainder function $d_<(\bar{x},w;\alpha)$ vanishes in the limit $\bar{x} \to 0$:

$$\int_{\bar{x}}^{1} dw\,d_<(\bar{x},w;\alpha) \simeq \int_{0}^{1} dw\,d_<(\bar{x},w;\alpha) \to 0 \qquad \text{for } \bar{x} \to 0, \tag{121}$$

because:

$$d_<(\bar{x},w;\alpha) \to 0 \qquad \text{for } \bar{x} \to 0 \qquad\qquad (0 < w < 1). \tag{122}$$

The integral of the second remainder function $d_>(\bar{x},w;\alpha)$ also vanishes in the same limit:

$$\int_{1}^{1+\bar{x}} dw\,d_>(\bar{x},w;\alpha) \to 0 \qquad \text{for } \bar{x} \to 0, \tag{123}$$

---

[13]The explcit expressions of the coefficients functions and the remainder functions can be found in [2].

because the integral extends to an infinitesimal region and the integrand vanishes for $w \to 1^+$:

$$d_>(\bar{x}, w; \alpha) \to 0 \qquad \text{for } w \to 1^+. \tag{124}$$

Finally, the integral involving $\Sigma[w-1; \alpha]$ also vanishes because, after expanding the form factor in powers of α, it produces terms of the form:

$$\int_1^{1+\bar{x}} dw \, \alpha^n \log^k(w-1) \to 0 \qquad \text{for } \bar{x} \to 0, \tag{125}$$

because the logarithm function to any positive power $k \geq 0$, $\log^k(u)$, has an integrable singularity in $u = 0$. We may write therefore:

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\bar{x}} = \int_{\bar{x}}^1 dw \, C_L(\bar{x}, w; \alpha) \, \Sigma\left[\frac{\bar{x}}{w}; \alpha(w \, m_b)\right] + O(\bar{x}; \alpha), \tag{126}$$

where by $O(\bar{x}; \alpha)$ we denote terms which vanish for $\bar{x} \to 0$ as well as for $\alpha \to 0$ (the neglected terms also vanish for $\alpha \to 0$). Let us conclude that, as long as logarithmically enhanced terms for $\bar{x} \to 0$ and constants are concerned, the integration can be made in the leading region

$$0 \leq w \leq 1, \tag{127}$$

which coincides with the phase-space domain at the tree level.

3.1 Minimal scheme

For clarity's sake let us consider at first a minimal factorization scheme. Since we are in region (120), we can take the limit $\bar{x} \to 0$ in the r.h.s. of eq. (126) anytime we do not encounter singularities:

$$\frac{1}{2\Gamma}\frac{d\Gamma}{d\bar{x}} = \frac{1}{2}\int_0^1 dw \, C_L(0, w; \alpha) \, \Sigma\left[\frac{\bar{x}}{w}; \alpha(w \, m_b)\right] + O(\bar{x}), \tag{128}$$

where we have divided by a factor two in order to simplify the forthcoming formulas. Let us note that, since we have taken the limit $\bar{x} \to 0$ in the coefficient function $C_L(\bar{x}, w; \alpha)$ which also has an $O(\alpha^0)$ contribution and we have integated down to $w = 0$, we have modified the integral above by terms $O(\bar{x})$ not multiplied by α: we will take care of them in a moment. Replacing the first-order expressions for the coefficient function [3],

$$C_L(0, w; \alpha) = 12 \, w \, (1-w)\left\{1 + \frac{\alpha \, C_F}{\pi}\left[\text{Li}_2(w) + \log w \log(1-w) - \frac{3}{2}\log w - \frac{w \log w}{2(1-w)} - \frac{35}{8}\right] + O\left(\alpha^2\right)\right\}, \tag{129}$$

and the QCD form factor given in eq. (20), we obtain:

$$\frac{1}{2}\int_0^1 dw \, C_L(0, w; \alpha) \, \Sigma\left[\frac{\bar{x}}{w}; \alpha(w \, m_b)\right] = 1 - \frac{\alpha_S \, C_F}{\pi}\left(\frac{1}{2}\log^2 \bar{x} + \frac{31}{12}\log \bar{x} + \frac{15}{4}\right) + O(\alpha^2). \tag{130}$$

In order to avoid a remainder function $O(\alpha^0)$, let us introduce an over-all factor equal to the lowest-order spectrum, so that the resummed form for the electron spectrum reads:

$$\frac{1}{2\,\Gamma}\frac{d\Gamma}{d\bar{x}} = (1-\bar{x})^2 \, (1 + 2\,\bar{x})\left[\bar{C}_X(\alpha)\,\bar{\Sigma}_X(\bar{x}; \alpha) + \bar{d}_X(\bar{x}; \alpha)\right]. \tag{131}$$

We require that the remainder function vanishes for $\bar{x} \to 0$:

$$\lim_{\bar{x} \to 0} \bar{d}_X(\bar{x}; \alpha) = 0. \tag{132}$$

The coefficient function,

$$\bar{C}_X(\alpha) = 1 + \alpha \, \bar{C}_X^{(1)} + \alpha^2 \, \bar{C}_X^{(2)}, + O(\alpha^3) \tag{133}$$

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has the leading correction:

$$\bar{C}_X^{(1)} = -\frac{15 \, C_F}{4 \, \pi} = -\frac{5}{\pi} = -1.59155. \tag{134}$$

Let us note that this correction is very large as it amounts to $\approx -35\%$ for $\alpha(m_b) = 0.22$. The effective electron form factor,

$$\bar{\Sigma}_X(\bar{x}; \alpha) = 1 + \alpha \, \bar{\Sigma}_X^{(1)}(\bar{x}) + \alpha^2 \, \bar{\Sigma}_X^{(2)}(\bar{x}) + O(\alpha^3), \tag{135}$$

has a first-order expression:

$$\bar{\Sigma}_X^{(1)}(\bar{x}) = -\frac{C_F}{2 \, \pi} \log^2 \bar{x} - \frac{31 \, C_F}{12 \, \pi} \log \bar{x}. \tag{136}$$

Let us now compute the remainder function, which is necessary to describe also the region $\bar{x} \sim O(1)$ and to have a uniform approximation in the whole $\bar{x}$ domain. The remainder function has, as usual, an expansion of the form:

$$\bar{d}_X(\bar{x}; \alpha) = \alpha \, \bar{d}_X^{(1)}(\bar{x}) + \alpha^2 \, \bar{d}_X^{(2)}(\bar{x}) + O(\alpha^3). \tag{137}$$

We match with the fixed-order spectrum,

$$\frac{1}{2 \, \Gamma} \frac{d\Gamma}{d\bar{x}} = (1 - \bar{x})^2 \, (1 + 2 \, \bar{x}) \left[1 + \alpha_S \, H^{(1)}(\bar{x}) + \alpha_S^2 \, H^{(2)}(\bar{x}) + O(\alpha^3) \right], \tag{138}$$

whose first order correction is well known [21, 5]:

$$H^{(1)}(\bar{x}) = \frac{C_F}{\pi} \left\{ -\frac{1}{2} \log^2 \bar{x} - \left[\frac{2}{3} + \frac{41}{36 \, (1 - \bar{x})^2} - \frac{13}{54 \, (1 - \bar{x})} + \frac{55}{54 \, (1 + 2 \, \bar{x})} \right] \log \bar{x} + \right.$$
$$\left. - \frac{4}{3} - \frac{41}{36 \, (1 - \bar{x})} - \frac{23}{18 \, (1 + 2 \, \bar{x})} + \log \bar{x} \log(1 - \bar{x}) + \text{Li}_2(\bar{x}) \right\}. \tag{139}$$

The leading-order remainder function reads:

$$\bar{d}_X^{(1)}(\bar{x}) = \frac{C_F}{\pi} \left\{ \log \bar{x} \log(1 - \bar{x}) + \text{Li}_2(\bar{x}) + \frac{29}{12} - \frac{41}{36(1 - \bar{x})} - \frac{23}{18(1 + 2\bar{x})} + \right.$$
$$\left. + \left[\frac{23}{12} - \frac{41}{36 \, (1 - \bar{x})^2} + \frac{13}{54 \, (1 - \bar{x})} - \frac{55}{54 \, (1 + 2 \, \bar{x})} \right] \log \bar{x} \right\}. \tag{140}$$

It vanishes, as requested, for $\bar{x} \to 0$ and, despite appearances, it has not any $1/(1 - \bar{x})$ singularity.

The coefficient function can be defined to all orders as:

$$\bar{C}_X(\alpha) = \frac{1}{2} \int_0^1 dw \, C_L(0, w; \alpha) \, \Sigma \left[\frac{1}{w}; \alpha(w \, m_b) \right] \tag{141}$$

and the effective form factor as:

$$\bar{\Sigma}_X(\bar{x}; \alpha) = \frac{\int_0^1 dw \, C_L(0, w; \alpha) \, \Sigma \left[\bar{x}/w; \alpha(w \, m_b) \right]}{\int_0^1 dw \, C_L(0, w; \alpha) \, \Sigma \left[1/w; \alpha(w \, m_b) \right]}. \tag{142}$$

The latter is normalized to all orders as:

$$\bar{\Sigma}_X(1; \alpha) = 1. \tag{143}$$

The effective form factor (142) factorizes the long-distance effects related to the threshold region, a part of which cannot be extracted from the radiative decay (2). An integration over the hard scale $Q = w \, m_b$ is indeed

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involved in $\bar{\Sigma}_X$ while, in the radiative decay, kinematics fixes $w = 1$ and there is no such integration. The relevant integration region is, parametrically:

$$\bar{x} \ll w \ll 1, \tag{144}$$

where the large logarithms $|\log(\bar{x}/w)| \gg 1$ are multiplied by a coupling $\alpha(w\,m_b) \gg \alpha(m_b)$. As in the case of the p_+ distribution, the effects of this region are however suppressed by the coefficient function:

$$C_L(w) \approx w = \frac{Q}{m_b} \ll 1. \tag{145}$$

3.2 Higher orders

Let us write as usual

$$\bar{\Sigma}_X = e^{\bar{G}_X}, \tag{146}$$

where the exponent of the form factor has the perturbative expansion:

$$\bar{G}_X\,(\bar{x};\,\alpha) = \sum_{n=1}^{\infty}\sum_{k=1}^{n+1} \bar{G}_{Xnk}\,\alpha^n\,L_{\bar{x}}^k, \tag{147}$$

with

$$L_{\bar{x}} \equiv \log\frac{1}{\bar{x}} \geq 0. \tag{148}$$

By inserting the truncated expansions for the coefficient function C_L and the form factor Σ in eq. (142), expanding the product, integrating term by term and taking the logarithm, we obtain:

$$\bar{G}_{X12} = G_{12}; \tag{149}$$

$$\bar{G}_{X11} = G_{11} + \frac{5}{6}A_1; \tag{150}$$

$$\bar{G}_{X23} = G_{23}; \tag{151}$$

$$\bar{G}_{X22} = G_{22} + \frac{13}{72}A_1^2 + \frac{5}{12}A_1\beta_0; \tag{152}$$

$$\bar{G}_{X21} = G_{21} + \frac{5}{6}A_2 + A_1^2\left[\frac{5}{6}z(2) - \frac{25}{54}\right] + \frac{19}{36}A_1\beta_0 + \frac{13}{36}A_1(B_1 + D_1) +$$
$$- \frac{5}{6}\beta_0\,B_1 + A_1\frac{C_F}{\pi}\left[\frac{583}{216} - 2z(3)\right]; \tag{153}$$

$$\bar{G}_{X34} = G_{34}; \tag{154}$$

$$\bar{G}_{X33} = G_{33} + \frac{35}{648}A_1^3 + \frac{13}{72}A_1^2\beta_0 + \frac{5}{18}A_1\beta_0^2; \tag{155}$$

$$\bar{G}_{X32} = G_{32} + \frac{5}{6}A_2\beta_0 - \frac{5}{6}\beta_0^2\,B_1 + \frac{13}{36}A_1\,A_2 + A_1\left(\frac{19}{36}\beta_0^2 + \frac{5}{12}\beta_1\right) +$$
$$- \frac{13}{72}A_1\beta_0\,(B_1 - D_1) + A_1\beta_0\frac{C_F}{\pi}\left(\frac{583}{432} - z(3)\right) + A_1^2\frac{C_F}{\pi}\left(\frac{737}{216} - 3\,z(4)\right) +$$
$$+ A_1^3\left(-\frac{5}{16} + \frac{13}{36}z(2) - \frac{5}{6}z(3)\right) + \frac{35}{216}A_1^2\,(B_1 + D_1) + A_1^2\beta_0\left(\frac{25}{108} + \frac{25}{12}z(2)\right). \tag{156}$$

Explicitly, one has for the coefficients $\bar{G}_{Xij} \neq G_{ij}$:

$$\bar{G}_{X11} = \frac{31\,C_F}{12\,\pi}; \tag{157}$$

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$$\bar{G}_{X22} = \frac{C_F}{\pi^2}\left[-\frac{23}{144}n_f + C_F\left(\frac{13}{72} - \frac{z(2)}{2}\right) + C_A\left(\frac{205}{288} + \frac{z(2)}{4}\right)\right]; \tag{158}$$

$$\bar{G}_{X21} = \frac{C_F}{\pi^2}\left[n_f\left(-\frac{73}{72} + \frac{z(2)}{6}\right) + C_F\left(\frac{163}{96} + \frac{11}{6}z(2) - \frac{3}{2}z(3)\right) + C_A\left(\frac{23}{4} - \frac{11}{6}z(2) - \frac{z(3)}{4}\right)\right]; \tag{159}$$

$$\bar{G}_{X33} = \frac{C_F}{\pi^3}\left[\frac{7}{1296}n_f^2 + C_F C_A\left(\frac{143}{864} - \frac{11}{8}z(2)\right) + C_F n_f\left(\frac{7}{216} + \frac{z(2)}{4}\right) + C_A n_f\left(\frac{65}{648} - \frac{z(2)}{12}\right) + \right.$$
$$\left. + \ C_A^2\left(-\frac{2573}{5184} + \frac{11}{24}z(2)\right) + C_F^2\left(\frac{35}{648} + \frac{z(3)}{3}\right)\right]; \tag{160}$$

$$\bar{G}_{X32} = \frac{C_F}{\pi^3}\left[\left(\frac{229}{1296} - \frac{z(2)}{36}\right)n_f^2 + C_F C_A\left(\frac{2807}{1296} + \frac{1183}{288}z(2) - \frac{11}{3}z(3) + \frac{5}{4}z(4)\right) + \right.$$
$$+ \ C_F n_f\left(-\frac{401}{1296} - \frac{121}{144}z(2) + \frac{5}{12}z(3)\right) + C_F^2\left(\frac{811}{288} + \frac{13}{36}z(2) - \frac{31}{12}z(3) - \frac{11}{4}z(4)\right) + $$
$$\left. + \ C_A n_f\left(-\frac{10115}{5184} + \frac{29}{72}z(2) - \frac{z(3)}{24}\right) + C_A^2\left(\frac{6217}{1296} - \frac{29}{24}z(2) + \frac{77}{48}z(3) - \frac{11}{16}z(4)\right)\right]. \tag{161}$$

The coefficient of the single logarithm at $O(\alpha)$, $\bar{G}_{X11}$, is the same as in the distribution in the hadron mass squared t, while the higher-order coefficients are different. Therefore there is not a simple relation between these two spectra beyond leading order from two loops on, as was instead guessed in [22].

3.3 Study of the BLM scheme

In this section we compare the exact second-order coefficients in the exponent of the electron form factor $\bar{G}_X$ with those ones obtained by the BLM ansatz:

$$\bar{G}_{X22} = \frac{2053}{648\,\pi^2} + \frac{1}{54} - \frac{23}{108\,\pi^2}\,n_f = 0.339525 - 0.0215777\,n_f \tag{162}$$

$$\hat{G}_{X22} = -\frac{23}{108\,\pi^2}\left(n_f - \frac{33}{2}\right) = 0.356031 - 0.0215777\,n_f \tag{163}$$

$$\bar{G}_{X21} = -\frac{11\,z(3)}{3\,\pi^2} + \frac{1405}{54\,\pi^2} - \frac{55}{81} - \left(\frac{73}{54\,\pi^2} - \frac{1}{27}\right)n_f = 1.51064 - 0.0999342\,n_f \tag{164}$$

$$\hat{G}_{X21} = -\left(\frac{73}{54\,\pi^2} - \frac{1}{27}\right)\left(n_f - \frac{33}{2}\right) = 1.64891 - 0.0999342\,n_f \tag{165}$$

As in the previous case, this approximation works within $O(10\%)$, i.e. rather well.

3.4 Non-minimal scheme

As discussed in detail in [3], for phenomenological applications it is convenient to define a non-minimal scheme which involves the integration of the universal form factor $\Sigma(u;\alpha)$ only in the physical region $0 < u \le 1$. Let us introduce also for this scheme an over-all factor equal to the spectrum in lowest order:

$$\frac{1}{2\,\Gamma}\frac{d\Gamma}{d\bar{x}} = (1 - \bar{x})^2\,(1 + 2\,\bar{x})\left[C_X(\alpha)\,\Sigma_X(\bar{x};\,\alpha) + d_X(\bar{x};\,\alpha)\right]. \tag{166}$$

We choose to define the coefficient function as:

$$C_X(\alpha) \equiv \frac{1}{2}\int_0^1 dw\,C_L(0, w;\,\alpha). \tag{167}$$

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One immediately obtains for the first-order correction:

$$C_X^{(1)} = -\frac{C_F}{\pi}\frac{127}{72} = -0.748618. \tag{168}$$

The correction to the coefficient function is — as in all the cases we have considered — negative. For $\alpha(m_b) = 0.22$ the correction is $\approx -16.5\%$, i.e. it is basically half of that in the minimal scheme.

The effective electron form factor can be defined as:

$$
\begin{aligned}
\Sigma_X(\bar{x}; \alpha) &= \frac{\int_0^1 dw\, C_L(0, w; \alpha)\, \tilde{\Sigma}\, [\bar{x}/w;\, \alpha(w\, m_b)]}{\int_0^1 dw\, C_L(0, w; \alpha)} \\
&= \frac{\int_0^{\bar{x}} dw\, C_L(0, w; \alpha) + \int_{\bar{x}}^1 dw\, C_L(0, w; \alpha)\, \Sigma\, [\bar{x}/w;\, \alpha(w\, m_b)]}{\int_0^1 dw\, C_L(0, w; \alpha)},
\end{aligned}
\tag{169}
$$

where $\alpha = \alpha(m_b)$. In order to simplify the definition as much as possible, we have integrated down to $w = 0$, replacing the form factor $\Sigma[u; \alpha]$ with its extension $\tilde{\Sigma}[u; \alpha]$. The normalization is the same as in the minimal scheme:

$$\Sigma_X(1; \alpha) = 1. \tag{170}$$

The perturbative expansion of the form factor is:

$$\Sigma_X(\bar{x}; \alpha) = 1 + \alpha\, \Sigma_X^{(1)}(\bar{x}) + \alpha^2\, \Sigma_X^{(2)}(\bar{x}) + O(\alpha^3). \tag{171}$$

By replacing the explicit expressions for the coefficient function C_L and the form factor Σ, expanding in α and performing the integration, we obtain:

$$\Sigma_X^{(1)}(\bar{x}) = \frac{C_F}{\pi}\left(-\frac{1}{2}L_{\bar{x}}^2 + \frac{31}{12}L_{\bar{x}} - \frac{143}{72} + \frac{27}{8}\bar{x}^2 - \frac{25}{18}\bar{x}^3\right). \tag{172}$$

In general, the form factor contains constant terms for $\bar{x} \to 0$ as well as vanishing terms in the same limit. The former modify the coefficient function while the latter modify the remainder function with respect to the values in the minimal scheme.

The remainder function reads in this scheme:

$$
\begin{aligned}
\tilde{d}_X^{(1)}(\bar{x}) &= \frac{C_F}{\pi}\left\{\frac{29}{12} - \frac{27}{8}\bar{x}^2 + \frac{25}{18}\bar{x}^3 - \frac{41}{36\,(1 - \bar{x})} - \frac{23}{18\,(1 + 2\,\bar{x})} + \log\bar{x}\,\log(1 - \bar{x}) + \mathrm{Li}_2(\bar{x}) + \right. \\
&\quad \left. + \left[\frac{23}{12} - \frac{41}{36\,(1 - \bar{x})^2} + \frac{13}{54\,(1 - \bar{x})} - \frac{55}{54\,(1 + 2\,\bar{x})}\right]\log\bar{x}\right\}.
\end{aligned}
\tag{173}
$$

This remainder function also vanishes for $\bar{x} \to 0$ and differs from the one in the minimal scheme only for the subtraction of the infinitesimal terms in (172).

The coefficients of the infrared logarithms G_{Xij} in the exponent of the form factor G_X are the same as those in the minimal scheme computed in the previous section:

$$G_{Xij} = \bar{G}_{Xij} \qquad \text{for } j \geq 1. \tag{174}$$

In the limit $\bar{x} \to 0$ (i.e. neglecting $O(\bar{x})$ terms), G_X also contains the constant terms:

$$G_{X10} = -\frac{19}{36}A_1 + \frac{5}{6}(B_1 + D_1); \tag{175}$$

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$$G_{X20} = -\frac{19}{36} A_2 - \frac{65}{72} A_1 \beta_0 + \frac{13}{72} (B_1 + D_1)^2 + \frac{19}{36} \beta_0 (3 B_1 + 2 D_1) + \frac{5}{6} (B_2 + D_2) +$$

$$+ A_1 (B_1 + D_1) \left(-\frac{25}{54} + \frac{5}{6} z(2) \right) + (B_1 + D_1) \frac{C_F}{\pi} \left(\frac{583}{216} - 2 z(3) \right) +$$

$$+ A_1{}^2 \left(\frac{905}{2592} - \frac{19}{36} z(2) + \frac{5}{6} z(3) \right) + A_1 \frac{C_F}{\pi} \left(-\frac{7337}{1296} + \frac{5}{3} z(3) + 3 z(4) \right). \tag{176}$$

Explicitly:

$$G_{X10} = -\frac{C_F}{\pi} \frac{143}{72}; \tag{177}$$

$$G_{X20} = \frac{C_F}{\pi^2} \left[n_f \left(\frac{1507}{1296} - \frac{5}{36} z(2) \right) + C_A \left(-\frac{2101}{324} + \frac{13}{9} z(2) + \frac{5}{24} z(3) \right) + \right.$$

$$\left. + C_F \left(-\frac{90725}{10368} - \frac{49}{36} z(2) + \frac{19}{4} z(3) + 3 z(4) \right) \right]. \tag{178}$$

The relations between the coefficients functions in the two schemes have the same form as those for the $\hat{p}_+$ spectrum given in the previous section:

$$C_X^{(1)} = \bar{C}_X^{(1)} - G_{X10}; \tag{179}$$

$$C_X^{(2)} = \bar{C}_X^{(2)} - \bar{C}_X^{(1)} G_{X10} + \frac{1}{2} G_{X10}^2 - G_{X20}. \tag{180}$$

The first of the above equations can be directly verified by inserting the first-order quantities.

Long-distance phenomena (large logarithms, Fermi motion, hadronization, etc.) are expected to have a larger effect in the distributions in the variables $q = t, u$ or p than in the electron energy spectrum [14]. That is because the former variables have a tree-level distribution consisting of a peak in zero:

$$\frac{1}{\Gamma} \frac{d\Gamma}{dq} = \delta(q) + O(\alpha) \qquad\qquad (q = u, t, p), \tag{181}$$

while the electron spectrum has a broader distribution with a maximum in $\bar{x} = 0$:

$$\frac{1}{2\,\Gamma} \frac{d\Gamma}{d\bar{x}} = (1 - \bar{x})^2 (1 + 2\,\bar{x}) + O(\alpha). \tag{182}$$

Long-distance phenomena always have a smearing effect, which is more prounanced in a starting distribution of the form (181) than of the form (182). Technically, that is reflected by the fact that long-distance effects in the electron spectrum are tempered by the presence of the partially-integrated form factor Σ instead of the differential one σ. We therefore expect the electron spectrum to be a less sensitive quantity to threshold effects.

4 The BLM scheme for other spectra

The accuracy of the BLM ansatz can also be studied by comparing its predictions with the exact results obtained for the following distributions treated in [2] and in [3]:

---

[14]The u spectrum have been studied in [3].

1. hadron mass distribution in the radiative decay (2) (or, equivalently, semileptonic distributions not integrated over the hadron energy). Our results for the QCD coefficients and the BLM estimates read respectively:

$$G_{22} = \frac{95}{72\,\pi^2} + \frac{1}{54} - \frac{13}{108\,\pi^2}\,n_f = 0.152206 - 0.0121961\,n_f; \tag{183}$$

$$\hat{G}_{22} = -\frac{13}{108\,\pi^2}\left(n_f - \frac{33}{2}\right) = 0.201235 - 0.0121961\,n_f; \tag{184}$$

$$G_{21} = -\frac{z(3)}{9\,\pi^2} + \frac{917}{72\,\pi^2} - \frac{35}{54} - \left(\frac{85}{108\,\pi^2} - \frac{1}{27}\right)n_f = 0.628757 - 0.0427065\,n_f; \tag{185}$$

$$\hat{G}_{21} = -\left(\frac{85}{108\,\pi^2} - \frac{1}{27}\right)\left(n_f - \frac{33}{2}\right) = 0.704657 - 0.0427065\,n_f. \tag{186}$$

The estimate of the non-abelian contribution is accurate with $O(25\%)$;

2. distribution in the variable u defined in the introduction (basically the hadron mass normalized to the hadron energy). We obtain:

$$G_{U22} = -\frac{5}{24\,\pi^2} + \frac{1}{54} - \frac{1}{36\,\pi^2}\,n_f = -0.00259006 - 0.00281448\,n_f; \tag{187}$$

$$\hat{G}_{U22} = -\frac{1}{36\,\pi^2}\left(n_f - \frac{33}{2}\right) = 0.0464389 - 0.00281448\,n_f; \tag{188}$$

$$G_{U21} = -\frac{z(3)}{9\,\pi^2} + \frac{217}{12\,\pi^2} - \frac{35}{54} - \left(\frac{10}{9\,\pi^2} - \frac{1}{27}\right)n_f = 1.17054 - 0.0755421\,n_f; \tag{189}$$

$$\hat{G}_{U21} = -\left(\frac{10}{9\,\pi^2} - \frac{1}{27}\right)\left(n_f - \frac{33}{2}\right) = 1.24644 - 0.0755421\,n_f; \tag{190}$$

The agreement is good for the G_{U21} coefficient, within 10%, but not good for the G_{U22} coefficient, which is however very small because of some accidental cancellation;

3. hadron mass spectrum in the semileptonic case. We obtain:

$$G_{T22} = \frac{1057}{216\,\pi^2} + \frac{1}{54} - \frac{11}{36\,\pi^2}\,n_f = 0.514336 - 0.0309593\,n_f \tag{191}$$

$$\hat{G}_{T22} = -\frac{11}{36\,\pi^2}\left(n_f - \frac{33}{2}\right) = 0.510828 - 0.0309593\,n_f; \tag{192}$$

$$G_{T21} = -\frac{65\,z(3)}{9\,\pi^2} + \frac{12431}{648\,\pi^2} - \frac{55}{81} - \left(\frac{83}{108\,\pi^2} - \frac{1}{27}\right)n_f = 0.385075 - 0.0408302\,n_f; \tag{193}$$

$$\hat{G}_{T21} = -\left(\frac{83}{108\,\pi^2} - \frac{1}{27}\right)\left(n_f - \frac{33}{2}\right) = 0.673698 - 0.0408302\,n_f. \tag{194}$$

The estimate of G_{T22} is very accurate, while for G_{T21} it is not.

The over-all picture is that the BLM ansatz offers an estimate of the resummation coefficients with an acceptable relative error most of the times. There are "exceptional cases", when the coefficients are very small because of some accidental cancellation, which cannot be accounted for by the general BLM ansatz.

5 Conclusions

In this paper we have resummed to next-to-leading order the distributions in the light-cone momentum $p_+ = E_X - |\vec{p}_X|$ and in the charged lepton energy E_e in the semileptonic decays (1), where E_X and $\vec{p}_X$ are the total

energy and three-momentum of the final hadron state X_u. These spectra have a different threshold structure with respect to the hadron energy distribution in (1) or the photon spectrum in (2), because they involve integration over E_X, i.e. over the hard scale $Q = 2E_X$ of the hadronic subprocess in (1). We have also presented a qualitative discussion of specific aspects of the long-distance effects in the electron energy spectrum: they should be suppressed with respect to the distribution in the hadron mass m_X or in p_+ because of a smearing effect of the Born three-body kinematics.

We have explicitly checked our resummation formula for the p_+ form factor by expanding it to $O(\alpha^2)$, picking up the n_f contributions and comparing them with the $O(\alpha^2 \, n_f)$ corrections computed with Feynman diagrams. This is a highly non-trivial check at the two-loop level of the resummation formula for the triple differential distribution and makes us more confident of our formalism.

We have also discussed the validity of the so-called Brodsky-Lepage-Mackanzie (BLM) scheme for the estimate of the second-order corrections to the form factors for various spectra. The conclusions are that this scheme offers in general a rather good approximation: it exactly reproduces the coefficients of the $\alpha^2 \, \log^3$ terms and gives good numerical estimates of the $\alpha^2 \, \log^2$ and $\alpha^2 \, \log$ coefficients. In cases in which the QCD coefficients are anomaously small because of some accidental cancellation, the BLM scheme is not accurate, as in these cases the n_f-dependent and the n_f-independent contributions presumably cancel to a different degree. It would be interesting to extend the BLM analysis to the third-order coefficients which have been computed by us for all the distributions treated. Let us remark that the exponentiation property does not follow from the BLM ansata and it has to be imposed externally.

By expanding the resummed formulas up to $O(\alpha^3)$, we have computed the coefficients $\bar{G}_{Pij}$ and $\bar{G}_{Xij}$ of the infrared logarithms $\log m_b/p_+$ and $\log 1/\bar{x}$ in the exponent of the effective form factors G_P and G_X respectively. These coefficients constitute a true prediciton of our resummation schme and can be checked with second and third order computations as soon as they become available. These coefficients can also be used in phenomenological studies which do not implement the full resummed formulas. For both distributions we have defined (standard) minimal factorization schemes and also non-minimal schemes which can be easily implemented in phenomenological analyses. As in our previous work [3], we find that the perturbative expansion seems to have better convergence properties in the modified schemes than in the minimal one.

Analytic expressions for double distributions or for single distributions with some kinematical cut can be obtained in similar way as we have made. This work completes however our study of the general factorization-resummation properties of semileptonic decay spectra [1, 2, 3]: for each distribution considered we have computed the coefficient function and the remainder function in NLO and the (effective) form factor in NNLO. The next step is a phenomenological analysis of the experimental data on semileptonic and radiative decays by using a model for the QCD form factor such as that formulated in [23], combined eventually with non-perturbative components not calculable in perturbatve QCD.

Note added

A few days after the first version of this paper was put on the archive, another work on resummation in semileptonic decays has also appeared [24]. In this paper, expressions of the second-order coefficients of the p_+ form factor are derived, which are in complete agreement with ours.

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