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THRESHOLD RESUMMED SPECTRA IN $B \to X_u l\nu$ DECAYS IN NLO (II)

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Abstract

We resum to next-to-leading order the distribution in the ratio of the invariant hadron mass m_X to the total hadron energy E_X and the distribution in m_X in the semileptonic decays $B \to X_u l\nu$. By expanding our formulas, we obtain the coefficients of all the infrared logarithms at $O(\alpha_S^2)$ and of the leading ones at $O(\alpha_S^3)$. We explicitly show that the relation between these semileptonic spectra and the photon spectrum in the radiative decay $B \to X_s\gamma$ is not a purely short-distance one. There are long-distance effects in the semileptonic spectra which are not completely factorized by the structure function as measured in the radiative decay and have to be modelled in some way.

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1 Introduction

Semileptonic B decays

$$B \rightarrow X_u + l + \nu, \tag{1}$$

where X_u is any hadron final state coming from the fragmentation of the *up* quark, are interesting processes for the study of strong interactions as well as of weak interactions. The computation of the spectra in (1) is often non trivial because of the presence of double infrared logarithms in the perturbative expansion, which formally diverge in the endpoints and therefore must be resummed to all orders in the QCD coupling α_S [1]. In all generality, large logarithms come from the so-called threshold region, defined as the one having

$$m_X \ll E_X \leq m_b, \tag{2}$$

where m_X and E_X are the invariant mass and total energy of the final hadron state X_u and m_b is the beauty mass. We find useful to summarize here the main results of [2, 3]. The infrared logarithms in process (1) can be organized in a series of the form

$$\Sigma[u;\, \alpha(Q)] \,=\, 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{nk}\, \alpha^n(Q)\, \log^k\left(\frac{1}{u}\right), \tag{3}$$

and can be factorized into the universal QCD form factor Σ. The Σ_{nk}'s are numerical coefficients whose explicit expressions can be obtained from [3] and $\alpha = \alpha_S$ is the QCD coupling. Q is the hard scale of the process and is determined by the final hadron energy:

$$Q \,=\, 2E_X. \tag{4}$$

We have defined the hadron variable

$$u \,=\, \frac{1-\sqrt{1-(2m_X/Q)^2}}{1+\sqrt{1-(2m_X/Q)^2}} \simeq \left(\frac{m_X}{Q}\right)^2 \qquad (0 \leq u \leq 1), \tag{5}$$

involving the ratio of the invariant hadron mass to the hard scale, where in the last member we have taken the leading term in the threshold region (2) only. As it is well known, the QCD form factor in eq. (3) contains at most two logarithms for each power of α, coming from the overlap of the soft and the collinear region in each emission. Note that the hard scale Q enters the argument of the infrared logarithms $1/u \simeq Q^2/m_X^2$ as well as the argument of the running coupling $\alpha = \alpha(Q)$. One can obtain a factorized form for the triple differential distribution — which is the most general distribution in process (1) — from which all other spectra can be obtained by integration:

$$\frac{1}{\Gamma}\int_0^u \frac{d^3\Gamma}{dx dw du'}\, du' \,=\, C\left[x, w; \alpha(w\, m_b)\right]\, \Sigma\left[u; \alpha(w\, m_b)\right] \,+\, D\left[x, u, w; \alpha(w\, m_b)\right], \tag{6}$$

where

$$w \,=\, \frac{2E_X}{m_b} \qquad (0 \leq w \leq 2); \qquad\qquad x \,=\, \frac{2E_l}{m_b} \qquad (0 \leq x \leq 1). \tag{7}$$

Γ is the total semileptonic width, $C\left[x, w; \alpha\right]$ is a short-distance coefficient function independent on u and $D\left[x, u, w; \alpha\right]$ is a remainder function not containing infrared logarithms (i.e. short-distance dominated) and vanishing for $u \to 0$ as well as for $\alpha \to 0$. The explicit expressions of these functions have been given in [3].

The properties of semileptonic decay spectra are best understood comparing them with the simpler radiative decay

$$B \rightarrow X_s + \gamma. \tag{8}$$

In such decay we have indeed:

$$Q \,=\, m_b\left(1 - \frac{q^2}{m_b^2} + \frac{m_X^2}{m_b^2}\right) \,=\, m_b\left(1 + \frac{m_X^2}{m_b^2}\right) \,\simeq\, m_b, \tag{9}$$

where q^μ is the 4-momentum of the real photon — in general of the probe. In the radiative decays (8) the hard scale is therefore independent on the kinematics and is fixed by the beauty mass. In the semileptonic decay (1), q^μ is the dilepton momentum and we have the more general situation $0 \le q^2 \le m_b^2$; the hard scale is given in this case by

$$Q \simeq m_b \left(1 - \frac{q^2}{m_b^2}\right) \tag{10}$$

and depends on the dilepton invariant mass squared q^2: it cannot be identified with the heavy flavor mass m_b. Kinematic configurations with

$$m_X \ll Q \approx m_b \tag{11}$$

as well as with

$$m_X \ll Q \ll m_b \tag{12}$$

are possible. In fact, in the radiative decays, for example, the average hadron energy is

$$\langle E_X \rangle_{rd} = \frac{1}{2} m_b \left[1 + O(\alpha)\right], \tag{13}$$

while in the semileptonic ones we have the smaller value [3]

$$\langle E_X \rangle_{sl} = \frac{7}{20} m_b \left[1 + O(\alpha)\right]. \tag{14}$$

In general, semileptonic spectra may or may not involve integration over the hadron energy E_X and, according to [2, 3], have a different infrared structure in the two cases. In [3] we have studied in detail the simpler case of the distributions not integrated over the hadron energy, i.e. over the hard scale Q, which have the same infrared structure of the hadron invariant mass distribution in the radiative decay (8):

$$\frac{1}{\Gamma_R} \int_0^{t_s} \frac{d\Gamma_R}{dt'_s} dt'_s = C_R(\alpha) \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{nk}\, \alpha(m_b)^n \log^k \frac{1}{t_s}\right) + D_R(t_s;\alpha). \tag{15}$$

We have defined $t_s = m_{X_s}^2/m_b^2 = 1 - x_\gamma$, with $x_\gamma = 2E_\gamma/m_b$, Γ_R is the total radiative width, $C_R(\alpha)$ is a short-distance coefficient function and $D_R(t_s;\alpha)$ is a short-distance remainder function.

In this paper, we attack the distributions integrated over the hadron energy, which have a more complicated infrared structure than the one in (8).

In sec. 2 we resum to next-to-leading order (NLO) the distribution in the hadron variable u defined before. The infrared logarithms appearing in the perturbative expansion of this spectrum,

$$\frac{1}{\Gamma} \int_0^{u} \frac{d\Gamma}{du'} du' = C_U(\alpha) \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{Unk}\, \alpha\,(m_b)^n \log^k \frac{1}{u}\right) + D_U(u;\alpha), \tag{16}$$

coincide in first order with those in the decay (8): $\Sigma_{U12} = \Sigma_{12}$ and $\Sigma_{U11} = \Sigma_{11}$, while they differ in higher orders. $C_U(\alpha)$ is a short-distance coefficient function and $D_U(u;\alpha)$ is a short-distance remainder function whose explicit expressions will be given in sec. (2).

In sec. 3 we compute to NLO the distribution in the variable

$$t = \frac{m_X^2}{m_b^2} \qquad (0 \le t \le 1), \tag{17}$$

i.e. the distribution in the invariant hadron mass squared. The infrared logarithms appearing in this distribution,

$$\frac{1}{\Gamma} \int_0^{t} \frac{d\Gamma}{dt'} dt' = C_T(\alpha) \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{Tnk}\, \alpha\,(m_b)^n \log^k \frac{1}{t}\right) + D_T(t;\alpha), \tag{18}$$

differ also at the $O(\alpha)$ single logarithm level from the corresponding ones in (8): $\Sigma_{T11} \neq \Sigma_{11}$. We also define a non-minimal factorization-resummation scheme which seems to have better convergence properties of the perturbative series than the minimal one.

We define for both spectra effective form factors which resum the large logarithmic corrections. These form factors involve a convolution of a process-dependent coefficient function with the universal QCD form factor Σ entering the radiative decay (8) and the triple differential distribution in the decay (1). The definition of the effective form factors we give is, in a sense, non perturbative, because we do not look at the explicit perturbative expansion of Σ.

Finally, in sec. 4, we draw our conclusions and we consider generalizations of our results.

2 Distribution in the hadron mass/energy ratio

In this section we compute the resummed distribution in the variable u (defined in the introduction) to next-to-leading order (NLO). That is accomplished by integrating the resummed double distribution in u and w obtained in sec. (4) of [3] over w:

$$\frac{1}{\Gamma}\frac{d\Gamma}{du} = \int_0^{1+u} dw\, \frac{1}{\Gamma}\frac{d^2\Gamma}{dwdu}. \tag{19}$$

By replacing the resummed expression on the r.h.s. of eq. (19), we obtain:

$$\frac{1}{\Gamma}\frac{d\Gamma}{du} = \int_0^{1+u} dw\, C_H(w;\,\alpha)\,\sigma\left[u;\,\alpha(w\,m_b)\right] + \int_0^{1+u} dw\, d_H(u,w;\,\alpha). \tag{20}$$

$C_H(w;\,\alpha)$ is a short-distance coefficient function having an expansion in powers of α:

$$C_H(w;\,\alpha) = C_H^{(0)}(w) + \alpha\, C_H^{(1)}(w) + \alpha^2\, C_H^{(2)}(w) + O(\alpha^3), \tag{21}$$

with

$$C_H^{(0)}(w) = 2w^2(3-2w); \tag{22}$$

$$C_H^{(1)}(w) = \frac{C_F}{\pi}\, 2w^2(3-2w)\left[\mathrm{Li}_2(w) + \log w \log(1-w) - \frac{35}{8} - \frac{9-4w}{6-4w}\log w\right], \tag{23}$$

where $C_F = (N_c^2-1)/(2N_c)$, $N_c = 3$ is the number of colors and $\alpha = \alpha(m_b)$. $\sigma\,(u;\,\alpha)$ is the differential QCD form factor:

$$\sigma(u;\,\alpha) = \frac{d}{du}\Sigma(u;\,\alpha), \tag{24}$$

with $\Sigma(u;\,\alpha)$ being the cumulative form factor considered in the introduction,

$$\Sigma(u;\,\alpha) = 1 + \alpha\,\Sigma^{(1)}(u) + \alpha^2\,\Sigma^{(2)}(u) + O(\alpha^3) \tag{25}$$

and

$$\Sigma^{(1)}(u) = -\frac{C_F}{\pi}\left[\frac{1}{2}\log^2 u + \frac{7}{4}\log u\right]. \tag{26}$$

Finally, $d_H(u,w;\,\alpha)$ is a short-distance remainder function whose explicit expression is not needed here. We are interested in the threshold region (2), which can also be defined as the one having

$$u \ll 1. \tag{27}$$

Since large logarithms originate only from the first integral on the r.h.s. of eq. (20), in order to isolate them, let us neglect at first the contribution from the remainder function $d_H(u,w;\,\alpha)$. The small terms for $u \to 0$

will be included later on, by expanding the resummed expression and comparing with fixed-order spectrum, as discussed at the end of sec. 2 of [3]. The integration of the first term on the r.h.s. of eq. (20) over the hadron energy is not trivial because the coefficient function $C_H(w;\,\alpha)$ depends on w as well as the QCD form factor $\sigma[u;\,\alpha(w\,m_b)]$, which depends on w via the scale of the coupling $\alpha = \alpha(w\,m_b)$. Unlike the distributions considered in [3], we cannot factor out in this case the universal form factor σ.

Since it is technically simpler to deal with partially-integrated form factors rather than with differential ones, let us define the event fraction

$$R_U(u) \;=\; \int_0^u \frac{1}{\Gamma}\frac{d\Gamma}{du'}du', \tag{28}$$

which has the end-point values:

$$R_U(0) \;=\; 0; \qquad\qquad R_U(1) \;=\; 1. \tag{29}$$

The spectrum in u is trivially obtained by differentiation:

$$\frac{1}{\Gamma}\frac{d\Gamma}{du} \;=\; \frac{d}{du}\,R_U(u). \tag{30}$$

Integrating both sides of eq. (20), we obtain:

$$\begin{aligned} R_U\left[u;\alpha(m_b)\right] \;&=\; \int_0^u du' \int_0^{1+u'} dw\, C_H(w;\alpha)\,\sigma[u';\alpha(w\,m_b)] \;+\; O(u,\alpha) \\ &=\; \int_0^u du' \int_0^{1} dw\, C_H(w;\alpha)\,\sigma[u';\alpha(w\,m_b)] + \int_0^u du' \int_1^{1+u'} dw\, C_H(w;\alpha)\,\sigma[u';\alpha(w\,m_b)] + \\ &\quad + O(u;\alpha), \end{aligned} \tag{31}$$

where by $O(u;\alpha)$ we denote terms which vanish for $u \to 0$ as well as for $\alpha \to 0$. The second integral in the last member of the r.h.s. extends to a kinematic region in w which is $O(u)$ and therefore can be dropped in the threshold region; we can therefore assume tree-level kinematics:

$$0 \;\leq\; w \;\leq\; 1. \tag{32}$$

By exchanging the order of the integrations in the remaining integral, we obtain:

$$R_U\left[u;\alpha(m_b)\right] \;=\; \int_0^1 dw\, C_H(w;\alpha)\,\Sigma[u;\alpha(w\,m_b)] \;+\; O(u;\alpha). \tag{33}$$

Substituting the explicit expressions for the coefficient function given in eq. (21) and of the QCD form factor given in eq. (25), we obtain:

$$\int_0^1 dw\, C_H(w;\alpha)\,\Sigma(u;\,\alpha) \;=\; 1 \;-\; \frac{\alpha\,C_F}{\pi}\left[\frac{1}{2}\log^2 u \;+\; \frac{7}{4}\log u \;+\; \frac{335}{144}\right] \;+\; O(\alpha^2). \tag{34}$$

The next step is to factorize the event fraction $R_U(u;\,\alpha)$ into:

- a QCD form factor $\Sigma_U(u;\,\alpha)$ containing the long-distance contributions, i.e. the $\log 1/u$ terms diverging for $u \to 0$;
- a coefficient function $C_U(\alpha)$ containing the constant terms for $u \to 0$;
- a remainder function $D_U(u;\,\alpha)$, collecting the left-over small contributions $O(u;\alpha)$, vanishing for $u \to 0$ and for $\alpha \to 0$.

Let us write therefore:

$$R_U\,(u;\,\alpha)\,=\,C_U(\alpha)\,\Sigma_U(u;\,\alpha)\,+\,D_U(u,\,\alpha). \tag{35}$$

The coefficient function, the form factor and the remainder function can all be expanded in powers of α:

$$\begin{aligned}
C_U(\alpha) &= 1\,+\,\alpha\,C_U^{(1)}\,+\,\alpha^2\,C_U^{(2)}\,+\,O(\alpha^3); && (36)\\
\Sigma_U(u;\,\alpha) &= 1\,+\,\alpha\,\Sigma_U^{(1)}(u)\,+\alpha^2\,\Sigma_U^{(2)}(u)\,+\,O(\alpha^3); && (37)\\
D_U(u;\,\alpha) &= \alpha\,D_U^{(1)}(u)\,+\alpha^2\,D_U^{(2)}(u)\,+\,O(\alpha^3). && (38)
\end{aligned}$$

The knowledge of soft-gluon dynamics allows the resummation of the dominant terms to all orders in α in Σ_U; this is instead not possible for the coefficient function and the remainder function, which are not long-distance dominated and for them one has to use truncated expansions.

The above conditions do not completely specify the form of the form factor, of the coefficient function and of the remainder function, so we have to select a factorization scheme. Let us choose a minimal scheme, in which the form factor contains *only* logarithmic terms:

$$\Sigma_U(u;\,\alpha)\,=\,1\,+\,\sum_{n=1}^{\infty}\sum_{k=1}^{2n}\Sigma_{Unk}\,\alpha^n\,L^k. \tag{39}$$

where

$$L\,\equiv\,\log\frac{1}{u}. \tag{40}$$

From the above definition, it follows that Σ_U has the same normalization as Σ:

$$\Sigma_U(1;\,\alpha)\,=\,1, \tag{41}$$

which holds to any order in α. We obtain in first order:

$$\begin{aligned}
\Sigma_U^{(1)}(u) &= -\frac{C_F}{\pi}\left[\frac{1}{2}\log^2 u\,+\,\frac{7}{4}\log u\right]; && (42)\\
C_U^{(1)} &= -\frac{335}{144}\frac{C_F}{\pi}. && (43)
\end{aligned}$$

Note that

$$\Sigma_U^{(1)}(u) = \Sigma^{(1)}(u). \tag{44}$$

As we are going to explicitly show later in this section, this property does not hold in higher orders of α. For $\alpha(m_b) = 0.22$, we have a first order correction to the coefficient function of $-\,21.7\%$.

The remainder function is obtained by imposing consistency between the resummed expression and the fixed-order one (matching). By expanding to first order in α the resummed expression (35) and imposing the equality with the full $O(\alpha)$ result, one obtains [4]:

$$D_U^{(1)}(u)\,=\,\frac{C_F}{\pi}\Big[\frac{u(15624-2688u-1352u^2+141u^3)}{5040}-\frac{21-84u-29u^2+6u^3}{210}u\log u\Big]. \tag{45}$$

As required, the remainder function goes to zero in the elastic point $u\,=\,0$ (as $u\log u$). Taking $u\,=\,1$ in eq. (35), we obtain the following relation between the coefficient function and the remainder function in the upper endpoint:

$$C_U(\alpha)\,=\,1\,-\,D_U(1;\alpha), \tag{46}$$

which holds to any order in α and is verified in first order (see eqs. (43) and (45)).

The minimal scheme has been defined above by looking at the explicit form of the event fraction $R_U(u;\,\alpha)$ as a power series in α: one reorganizes the series picking up the logarithmic terms and putting them into the effective form factor Σ_U. It is also possible to give a different definition of the minimal scheme which does not make use of the explicit expansion of the event fraction. Since

$$\int_0^1 dw\, C_H(w;\alpha)\, \Sigma\left[u;\,\alpha(w\, m_b)\right] \,=\, C_U(\alpha)\, \Sigma_U(u;\,\alpha) \tag{47}$$

and

$$\Sigma_U(1;\,\alpha) \,=\, 1, \tag{48}$$

we can define the coefficient function to all orders as:

$$C_U(\alpha) \,=\, \int_0^1 dw\, C_H(w;\,\alpha). \tag{49}$$

The effective form factor is therefore written as:

$$\Sigma_U\,(u;\,\alpha) \,=\, \frac{\int_0^1 dw\, C_H(w;\,\alpha)\, \Sigma[u;\,\alpha(w\, m_b)]}{\int_0^1 dw\, C_H(w;\,\alpha)}. \tag{50}$$

The definition (50) of the minimal scheme has the following phenomenological advantage. One may wish to use for $\Sigma(u;\alpha)$, instead of the perturbative expression, for example the result of a fit to some experimental data or a non-perturbative model[4]. In these cases, $\Sigma(u;w)$ does not depend on the coupling and therefore cannot be expanded in α, but the effective form factor Σ_U in the minimal scheme can still be computed by means of eq. (50).

The representation of the effective form factor Σ_U given in eq. (50) allows us to make a few general comments:

- $\Sigma_U(u;\,\alpha)$ factorizes all the threshold logarithms in the spectrum but is, unlike $\Sigma(u)$, process dependent. That is because it involves the convolution of the universal form factor $\Sigma(u;\,\alpha)$ with the process-dependent coefficient function over all the hadron energies. $C_H(w;\,\alpha)$ has the role of a probability distribution: it gives the probability for the hadronic subprocess with hard scale $Q \,=\, w\, m_b$ to occur. In the u distribution hadronic subprocesses with all the possible hard scales Q from zero up to m_b do contribute, while in the radiative decay (8) Q is kinematically fixed to the upper value m_b. The relation between these two spectra therefore is not a purely short-distance one: to relate these two distributions one has to model in some way the variation of the form factor $\Sigma[u;\alpha(Q)]$ with Q ranging from m_b down to zero. In agreement with physical intuition, the problem in the computation of the u distribution is the estimate of the contributions from small hard scales, $Q \ll m_b$, where perturbation theory is expected to fail. However, since $C_H \,\propto\, Q^2$, small hard scales give a small contribution to the total. We may say that the u spectrum is "protected" from long-distance effects related to large logarithms with a large coupling, i.e. related to the region in eq. (12);
- since $\alpha(w\, m_b) \,=\, \alpha(m_b) \,+\, O(\alpha^2)$, the effective form factor and the universal form factor coincide in first order, as already found by explicit computation (see eq. (44));
- were not for the dependence of the form factor $\Sigma \,=\, \Sigma[u;\,\alpha(w\, m_b)]$ on w through the running coupling, $\alpha \,=\, \alpha(w\, m_b)$, the effective form factor Σ_U would coincide with the universal one Σ to all orders.

Let us now discuss the higher orders in the perturbative expansion of $\Sigma_U(u;\,\alpha)$. One has to insert in eq. (50):

- the QCD form factor Σ, whose next-to-next-to-leading order corrections (NNLO) are now well established due to the recent revaluation of the resummation constant D_2 in [5];

[4]For the coefficient function, one can still use the perturbative result. That is because the coefficient function, unlike the form factor, is short-distance dominated, and therefore its perturbative evaluation is more reliable.

- the coefficient function $C_H(w;\, \alpha)$, which is known to $O(\alpha)$, i.e. to NLO (cfr. eq. (21)),

and perform the integration over w. To obtain the truncated expansion of Σ_U, one simply replaces the truncated expansions of Σ and of C_H, expands the product and integrates term by term.

The exponential structure of the threshold logarithms in $\Sigma(u;\, \alpha)$ is partially spoiled in $\Sigma_U(u;\, \alpha)$ because of the integration over the hard scale $Q = w\, m_b$, but it is not completely ruined. In order to simplify the representation of the large logarithms, it is therefore convenient to introduce the exponent of the form factor also in the effective case, i.e. to define G_U as:

$$\Sigma_U \;=\; e^{G_U}. \tag{51}$$

Expanding the exponent in powers of α up to third order included, we have an expansion of the same form of G, defined in [3], that is:

$$G_U\,(u;\,\alpha) \;=\; \sum_{n=1}^{\infty}\sum_{k=1}^{n+1} G_{Unk}\,\alpha^n\,L^k \;=\; G_{U12}\alpha L^2 + G_{U11}\alpha L + G_{U23}\alpha^2 L^3 + G_{U22}\alpha^2 L^2 + G_{U21}\alpha^2 L + \cdots\,, \tag{52}$$

where

$$L \,\equiv\, \log\frac{1}{u} \,\geq\, 0 \tag{53}$$

and $\alpha \,\equiv\, \alpha(m_b)$. In order to explicitly see the effects of the integration over the hard scale, let us give the coefficients G_{Uij} up to the third order in terms of the corresponding coefficients G_{ij} of the radiative decay (8) and of the resummation constants $A_i,\; B_i,\; D_i$ and of the β-function coefficients β_i [3]:

$$\begin{aligned}
G_{U12} &= G_{12}; && (54)\\
G_{U11} &= G_{11}; && (55)\\
G_{U23} &= G_{23}; && (56)\\
G_{U22} &= G_{22} - \frac{5}{12}A_1\beta_0; && (57)\\
G_{U21} &= G_{21} - \frac{5}{6}\beta_0\Big(B_1 + D_1\Big); && (58)\\
G_{U34} &= G_{34}; && (59)\\
G_{U33} &= G_{33} - \frac{5}{6}A_1\beta_0^2; && (60)\\
G_{U32} &= G_{32} - \frac{5}{6}\,A_2\,\beta_0 - \frac{1}{36}\,A_1\left(23\,\beta_0{}^2 + 15\,\beta_1\right) - \frac{5}{6}\beta_0{}^2\left(B_1 + 2\,D_1\right) - \frac{5}{6}\,A_1{}^2\,\beta_0\,z(2) + \\
&- \left(\frac{547}{216} - 2\,z(3)\right)\frac{C_F}{\pi}\,A_1\,\beta_0\,, && (61)
\end{aligned}$$

where $z(a) = \sum_{n=1}^{\infty} 1/n^a$ is the Riemann Zeta function and the explicit expressions of the G_{ij} have been given in sec. (2) of [3]. For a reference use in phenomenological analysis, let us also give the explicit expressions for the coefficients, which can be checked against the second and third order computations of the u spectrum as soon as the latter will become available. We report only the coefficients differing from the starting ones, $G_{Uij} \neq G_{ij}$:

$$G_{U22} = \frac{C_F}{\pi^2}\left[-\frac{n_f}{48} - \frac{C_F\, z(2)}{2} + C_A\left(-\frac{5}{96} + \frac{z(2)}{4}\right)\right]; \tag{62}$$

$$G_{U21} = \frac{C_F}{\pi^2}\left[n_f\left(-\frac{5}{6} + \frac{z(2)}{6}\right) + C_A\left(\frac{215}{48} - \frac{17\, z(2)}{12} - \frac{z(3)}{4}\right) + C_F\left(\frac{3}{32} + z(2) + \frac{z(3)}{2}\right)\right]; \tag{63}$$

$$\begin{aligned} G_{U33} &= \frac{C_F}{\pi^3}\left[-\frac{11}{432}n_f^2 + C_A\, n_f\left(\frac{95}{216} - \frac{z(2)}{12}\right) + C_A^2\left(-\frac{2471}{1728} + \frac{11\, z(2)}{24}\right) + \right. \\ &+ \left. C_F\, n_f\left(\frac{1}{16} + \frac{z(2)}{4}\right) - \frac{11}{8}C_F\, C_A\, z(2) + \frac{1}{3}C_F^2\, z(3)\right]; \end{aligned} \tag{64}$$

$$\begin{aligned} G_{U32} &= \frac{C_F}{\pi^3}\left[n_f^2\left(\frac{49}{432} - \frac{z(2)}{36}\right) + C_F\, C_A\left(-\frac{23177}{10368} + \frac{155\, z(2)}{96} - \frac{11\, z(3)}{12} + \frac{5\, z(4)}{4}\right) + \right. \\ &+ C_F\, n_f\left(\frac{2971}{5184} - \frac{17\, z(2)}{48} - \frac{z(3)}{12}\right) + C_F^2\left(-\frac{7\, z(3)}{4} + \frac{z(4)}{4}\right) + C_A\, n_f\left(-\frac{1709}{1728} + \frac{19 z(2)}{72} - \frac{z(3)}{24}\right) + \\ &+ \left. C_A^2\left(\frac{1541}{864} - \frac{4\, z(2)}{9} + \frac{77\, z(3)}{48} - \frac{11\, z(4)}{16}\right)\right], \end{aligned} \tag{65}$$

where $C_A = N_c = 3$ and n_f is the number of active flavors. Let us make a few comments about the results obtained:

- it is remarkable that we could explicitly compute G_{U32} with the knowledge of the first two orders of the coefficient function $C_H(w;\,\alpha)$ only (cfr. eqs. (22) and (23)). We can compute also the coefficients $G_{Un,n-1}$ for $n > 3$, i.e. the exponent G_U to NNLO. As explained in detail in [6, 3], NNLO means indeed that for each power of α we can compute the three principal logarithms:[5]

$$G_{Un,n+1}\,\alpha^n\, L^{n+1}, \qquad G_{Un,n}\,\alpha^n\, L^n, \qquad G_{Un,n-1}\,\alpha^n L^{n-1}. \tag{67}$$

  By general counting arguments, one would expect that the NNLO corrections to Σ_U also require the knowledge of the NNLO contribution to the coefficient function, i.e. of the $O(\alpha^2)$ term $C_H^{(2)}(w)$. That is actually not the case because the NNLO contributions proportional to $C_H^{(2)}(w)$ cancel between the numerator and the denominator in the definition of the effective form factor in eq. (50). However, a complete NNLO resummation of the u-spectrum also requires the knowledge of the second-order correction to the coefficient function $C_U^{(2)}$, and for that $C_H^{(2)}(w)$ is needed (see eq.(49));

- Since the coefficients of the threshold logarithms in the u distribution and in the photon spectrum in the radiative decay (8) differ from two loops on in NLO, the cancellation of long-distance effects in the ratio considered in [4]

$$R(u) = \frac{d\Gamma_R/dt_s(t_s = u)}{d\Gamma/du(u)} \tag{68}$$

  is not exact but occurs only in leading order. As previously discussed, the u distribution has additional long distance effects with respect to the radiative decay related to small hadron energies.

[5] If we look instead at the form factor itself, Σ_U, NNLO means that we can compute for any n all the coefficients Σ_{Unk} with $n-1 \le k \le 2n$:

$$\Sigma_{Un,2n}\,\alpha^n\, L^{2n}, \qquad \Sigma_{Un,2n-1}\,\alpha^n\, L^{2n-1}, \qquad \cdots, \qquad \Sigma_{Un,n}\,\alpha^n\, L^n, \qquad \Sigma_{Un,n-1}\,\alpha^n\, L^{n-1}. \tag{66}$$

For instance for $n = 3$ we can compute the coefficients of the logarithms from power six down to power two included.

The differential spectrum in u is obtained from the event fraction $R_U(u)$ by differentiation:

$$\frac{1}{\Gamma}\frac{d\Gamma}{du} = C_U(\alpha)\,\sigma_U\,(u;\,\alpha) + d_U(u;\,\alpha), \tag{69}$$

where we have defined

$$\sigma_U\,(u;\alpha) \equiv \frac{d}{du}\Sigma_U\,(u;\alpha)\,; \qquad\qquad d_U(u;\alpha) \equiv \frac{d}{du}\,D_U(u;\alpha). \tag{70}$$

The coefficient function is clearly the same in the partially-integrated spectrum and in the differential one, while the remainder function is obtained by differentiation and reads:

$$d_U^{(1)}(u) = \frac{C_F}{\pi}\left[\frac{36-8u-8u^2+u^3}{12} - \frac{7-56u-29u^2+8u^3}{70}\log u\right]. \tag{71}$$

3 Hadron mass spectrum

In this section we resum to NLO the distribution in the invariant hadron mass squared, i.e. the distribution in the variable

$$t = \frac{m_X^2}{m_b^2} = \frac{u\,w^2}{(1+u)^2} \simeq u\,w^2 \qquad\qquad (0 \le t \le 1), \tag{72}$$

where in the last member we have kept the leading term for $u \to 0$ only. This distribution is obtained by integrating the distribution in the hadron variables u and w with the previous kinematic constraint:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dt} = \int_D dudw\,\frac{1}{\Gamma}\frac{d^2\Gamma}{dudw}\delta\left[t - \frac{u\,w^2}{(1+u)^2}\right]; \tag{73}$$

the integration covers the whole phase space D of the hadron variables:

$$\int_D dudw = \int_0^2 dw \int_{max[0,w-1]}^1 du = \int_0^1 du \int_0^{1+u} dw. \tag{74}$$

It is convenient to evaluate the event fraction, defined like in the previous distribution as:

$$R_T(t) = \int_0^t dt'\,\frac{1}{\Gamma}\frac{d\Gamma}{dt'}, \tag{75}$$

with the end-point values $R_T(0) = 0$ and $R_T(1) = 1$. Integrating both sides of eq. (73), one obtains:

$$R_T(t) = \int_D dudw\,\frac{1}{\Gamma}\frac{d^2\Gamma}{dudw}\theta\left[t - \frac{u\,w^2}{(1+u)^2}\right]. \tag{76}$$

By inserting the resummed form for the double hadron distribution and neglecting at first the remainder function, we obtain:

$$R_T(t;\,\alpha) = \int_D dudw\,C_H(w;\,\alpha)\,\sigma\,[u;\,\alpha(w\,m_b)]\;\theta\left[t - \frac{u\,w^2}{(1+u)^2}\right] + O(t,\alpha), \tag{77}$$

where by $O(t,\alpha)$ we denote terms which are zero at the threshold and vanish for $\alpha = 0$. In order to isolate the large logarithms, let us simplify the domain D into the unit square

$$0 \le u,\, w \le 1 \tag{78}$$

and simplify the kinematic constraint as well as:

$$\theta\left[t - u\,w^2/(1+u)^2\right] \;\rightarrow\; \theta[t - u\,w^2]. \tag{79}$$

Let us observe that the variable t keeps unitary range even after such approximations. We then obtain:

$$
\begin{aligned}
R_T(t;\alpha) &= \int_0^1\int_0^1 du dw\, C_H(w;\alpha)\,\sigma\left[u;\alpha(w\,m_b)\right]\,\theta\left[t-uw^2\right] + O(t,\alpha) \\
&= \int_0^{\sqrt{t}} dw\, C_H(w;\alpha) + \int_{\sqrt{t}}^1 dw\, C_H(w;\alpha)\,\Sigma\left[t/w^2;\,\alpha(w\,m_b)\right] + O(t,\alpha).
\end{aligned}
\tag{80}
$$

Let us remark that the present case is more involved with respect to the one treated in the previous section, because the hadron mass squared t is a combination of both variables used for threshold resummation, namely u and w. Neglecting infinitesimal terms for $t \to 0$, the expression above can be further simplified by neglecting the first integral and integrating the second integrand over w down to zero,

$$
R_T(t;\,\alpha) = \int_0^1 dw\, C_H(w;\alpha)\,\Sigma\left[t/w^2;\,\alpha(w\,m_b)\right] + O(\alpha,t).
\tag{81}
$$

By inserting the first-order expressions for the coefficient function and the form factor given in the previous section, we obtain:

$$
\int_0^1 dw\, C_H(w;\alpha)\,\Sigma\left[t/w^2;\,\alpha(w\,m_b)\right] = 1 - \frac{\alpha\, C_F}{\pi}\left[\frac{1}{2}\log^2 t + \frac{31}{12}\log t + \frac{637}{144}\right] + O(\alpha^2).
\tag{82}
$$

As anticipated in the introduction, we have a different coefficient for the single logarithm with respect to the hadron mass distribution in the radiative decay (8) or the u distribution of the previous section.

As with the u distribution, we introduce a resummed form of the event fraction as:

$$
R_T(t;\,\alpha) = C_T(\alpha)\,\Sigma_T(t;\,\alpha) + D_T(t;\,\alpha).
\tag{83}
$$

All the functions above have an expansion in powers of α:

$$
C_T(\alpha) = 1 + \alpha\, C_T^{(1)} + \alpha^2\, C_T^{(2)} + O(\alpha^3); \tag{84}
$$

$$
\Sigma_T(u;\,\alpha) = 1 + \alpha\,\Sigma_T^{(1)}(t) + \alpha^2\,\Sigma_T^{(2)}(t) + O(\alpha^3); \tag{85}
$$

$$
D_T(t;\,\alpha) = \alpha\, D_T^{(1)}(t) + \alpha^2\, D_T^{(2)}(t) + O(\alpha^3). \tag{86}
$$

Let us consider a minimal factorization scheme, where only logarithms in t are factorized in the effective form factor Σ_T. Since we will consider a different scheme later on in this section, let us denote the quantities in the minimal scheme with a bar. The first-order corrections to the form factor and the coefficient function in the minimal scheme read:

$$
\begin{aligned}
\bar{\Sigma}_T^{(1)}(t) &= -\frac{C_F}{\pi}\left[\frac{1}{2}\log^2 t + \frac{31}{12}\log t\right]; \\
\bar{C}_T^{(1)} &= -\frac{C_F}{\pi}\,\frac{637}{144} = -\,1.87744.
\end{aligned}
\tag{87}
$$

Note that the correction to the coefficient function is very large: for $\alpha(m_b) = 0.22$ it amounts to $-\,41.3\%$.

We now expand the resummed result in powers of α and compare with the fixed order result, which is known to full order α [7, 8]:

$$
R_T(t;\alpha) = 1 + \alpha\, R_T^{(1)}(t) + \alpha^2\, R_T^{(2)}(t) + O(\alpha^3),
\tag{88}
$$

with

$$
R_T^{(1)}(t) = -\frac{C_F}{\pi}\left[\frac{1}{2}\log^2 t + \frac{31}{12}\log t + \frac{637}{144} - \frac{97}{18}t + \frac{25}{18}t^3 - \frac{61}{144}t^4 + \left(\frac{5}{3}t - \frac{3}{2}t^2 + \frac{1}{6}t^4\right)\log t\right].
\tag{89}
$$

We obtain for the remainder function in first order:

$$\bar{D}_T^{(1)}(t) \, = \, \frac{97}{18}t - \frac{25}{18}t^3 + \frac{61}{144}t^4 - \left(\frac{5}{3}t - \frac{3}{2}t^2 + \frac{1}{6}t^4\right)\log t. \tag{90}$$

Since

$$\bar{\Sigma}_T(1;\,\alpha) \, = \, 1, \tag{91}$$

taking $t = 1$ in eq. (83), we obtain a relation between the coefficient function and the remainder function in the endpoint:

$$\bar{C}_T(\alpha) \, = \, 1 \, - \, \bar{D}_T(1;\,\alpha). \tag{92}$$

It is a trivial matter to verify that the above relation holds true for our first-order expressions.

A compact definition of the minimal scheme can be given as follows. Since

$$\int_0^1 dw\, C_H(w;\alpha)\, \Sigma\left[t/w^2;\, \alpha(w\, m_b)\right] \, = \, \bar{C}_T(\alpha)\, \bar{\Sigma}_T\,(t;\,\alpha)\,, \tag{93}$$

the coefficient function in the minimal scheme can be defined taking $t = 1$ in the above equation and using (91):

$$\bar{C}_T(\alpha) \, \equiv \, \int_0^1 dw\, C_H(w;\alpha)\, \Sigma\left[1/w^2;\, \alpha(w\, m_b)\right]. \tag{94}$$

The effective form factor then reads:

$$\bar{\Sigma}_T\,(t;\,\alpha) \, \equiv \, \frac{\int_0^1 dw\, C_H(w;\alpha)\, \Sigma\left[t/w^2;\, \alpha(w\, m_b)\right]}{\int_0^1 dw\, C_H(w;\alpha)\, \Sigma\left[1/w^2;\, \alpha(w\, m_b)\right]}. \tag{95}$$

Let us comment the above result. The effective form factor involves a convolution over the hadron energy w of the coefficient function and the universal form factor, which cannot be reduced to an ordinary product by the standard moment transform. That is because the variable w enters not only in the first argument of $\Sigma = \Sigma\left[t/w^2;\, \alpha(w\, m_b)\right]$ but also in the scale of the coupling $\alpha = \alpha(w\, m_b)$. Analogously to the u distribution, there are long-distance effects in the effective form factor related to small hadron energies $w \ll 1$, which are suppressed by the coefficient function. In the present case, however, there is an additional mechanism suppressing the small energy contributions: since Σ is evaluated in t/w^2, small w's correspond to a large argument $u = t/w^2$ of $\Sigma(u)$, where there are no large logarithms, $\log 1/u \sim O(1)$, and one is inclusive at the parton level. We may say that this spectrum is "double protected" from the non-perturbative long-distance effects related to small hadron energies.

The systematic expansion of the form factor is easily obtained by writing as usual:

$$\bar{\Sigma}_T \, = \, e^{\bar{G}_T}, \tag{96}$$

one obtains:

$$\bar{G}_T(t;\alpha) \, = \, \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} \bar{G}_{Tnk}\, \alpha^n\, L_t^k, \tag{97}$$

where

$$L_t \, \equiv \, \log\frac{1}{t} \tag{98}$$

and

$$\bar{G}_{T12} = G_{12}; \tag{99}$$
$$\bar{G}_{T11} = G_{11} + \frac{5}{6}A_1; \tag{100}$$
$$\bar{G}_{T23} = G_{23}; \tag{101}$$
$$\bar{G}_{T22} = G_{22} + \frac{7}{24}A_1^2 + \frac{5}{6}A_1\beta_0; \tag{102}$$
$$\begin{aligned}\bar{G}_{T21} &= G_{21} + \frac{5}{6}A_2 + A_1^2\left[\frac{5}{6}z(2) - \frac{47}{54}\right] - \frac{23}{36}A_1\beta_0 + \frac{7}{12}A_1(B_1+D_1) + \\ &+ \frac{5}{6}\beta_0 D_1 + A_1\frac{C_F}{\pi}\left[\frac{547}{108} - 4z(3)\right];\end{aligned} \tag{103}$$
$$\bar{G}_{T34} = G_{34}; \tag{104}$$
$$\bar{G}_{T33} = G_{33} + \frac{83}{648}A_1^3 + \frac{7}{12}A_1^2\beta_0 + \frac{10}{9}A_1\beta_0^2; \tag{105}$$
$$\begin{aligned}\bar{G}_{T32} &= G_{32} + \frac{5}{3}A_2\,\beta_0 + \frac{5}{3}\beta_0{}^2\,D_1 + \frac{83}{216}A_1{}^2\,(B_1\,+\,D_1) + A_1{}^2\,\beta_0\left(\frac{35\,z(2)}{12} - \frac{47}{27}\right) \\ &+ A_1{}^2\,\frac{C_F}{\pi}\left(\frac{22747}{1728} - 12\,z(4)\right) + \frac{7}{12}A_1\,A_2 - \frac{1}{18}A_1\left(23\,\beta_0{}^2 - 15\,\beta_1\right) + \frac{7}{12}A_1\,\beta_0\left(B_1 + 2D_1\right) + \\ &+ A_1\,\beta_0\,\frac{C_F}{\pi}\left(\frac{547}{108} - 4\,z(3)\right) + A_1{}^3\left(-\frac{1117}{1296} + \frac{7\,z(2)}{12} - \frac{5\,z(3)}{6}\right).\end{aligned} \tag{106}$$

For comparison with future higher-order computations, let us give the explicit values of the coefficients $\bar{G}_{Tij} \neq G_{ij}$:

$$\bar{G}_{T11} = \frac{31\,C_F}{12\,\pi}; \tag{107}$$
$$\bar{G}_{T22} = \frac{C_F}{\pi^2}\left[-\frac{11\,n_f}{48} + C_F\left(\frac{7}{24} - \frac{z(2)}{2}\right) + C_A\left(\frac{35}{32} + \frac{z(2)}{4}\right)\right]; \tag{108}$$
$$\bar{G}_{T21} = \frac{C_F}{\pi^2}\left[n_f\left(\frac{z(2)}{6} - \frac{83}{144}\right) + C_F\left(\frac{941}{288} + \frac{11}{6}z(2) - \frac{7}{2}z(3)\right) + C_A\left(\frac{107}{32} - \frac{11}{6}z(2) - \frac{z(3)}{4}\right)\right]; \tag{109}$$
$$\begin{aligned}\bar{G}_{T33} &= \frac{C_F}{\pi^3}\left[\frac{37}{1296}\,n_f^2 + C_F\,C_A\left(\frac{77}{144} - \frac{11\,z(2)}{8}\right) + C_F\,n_f\left(-\frac{5}{144} + \frac{z(2)}{4}\right) + \right. \\ &\left. + C_A\,n_f\left(-\frac{25}{162} - \frac{z(2)}{12}\right) + C_A^2\left(\frac{1057}{5184} + \frac{11\,z(2)}{24}\right) + C_F^2\left(\frac{83}{648} + \frac{z(3)}{3}\right)\right];\end{aligned} \tag{110}$$
$$\begin{aligned}\bar{G}_{T32} &= \frac{C_F}{\pi^3}\left[\left(\frac{263}{2592} - \frac{z(2)}{36}\right)n_f^2 + C_F\,C_A\left(\frac{28493}{10368} + \frac{457\,z(2)}{96} - \frac{77\,z(3)}{12} + \frac{5\,z(4)}{4}\right) + \right. \\ &+ C_F\,n_f\left(-\frac{2353}{5184} - \frac{47\,z(2)}{48} + \frac{11\,z(3)}{12}\right) + C_F^2\left(\frac{60287}{5184} + \frac{7\,z(2)}{12} - \frac{31\,z(3)}{12} - \frac{47\,z(4)}{4}\right) + \\ &\left. + C_A\,n_f\left(-\frac{6515}{5184} + \frac{17z(2)}{36} - \frac{z(3)}{24}\right) + C_A^2\left(\frac{31841}{10368} - \frac{229\,z(2)}{144} + \frac{77\,z(3)}{48} - \frac{11\,z(4)}{16}\right)\right].\end{aligned} \tag{111}$$

Let us make a few remarks:

- The coefficient of the single logarithm at $O(\alpha)$ is different from the previous case as well as from the radiative decay:

$$\bar{G}_{T11} \neq G_{U11} = G_{11}, \tag{112}$$

because $\bar{G}_{T11}$ takes a kinematic contribution from A_1, i.e. from the double logarithm at one loop in Σ.

The logarithmic structure of $\bar{G}_T$ radically differs from that of G because the integration variable w enters not only the argument of the running coupling $\alpha = \alpha(w\, m_b)$ but also the argument of the logarithm $L = \log(t/w^2)$;

- If the hard scale was set by the heavy flavor mass, $Q = m_b$ (a kind of frozen coupling case with respect to the real case) we would have the following values for the coefficients: $\bar{G}_{T22}^{fr} = G_{22} + 7/24\ A_1^2 + 5/4\ A_1\beta_0$ and $\bar{G}_{T33}^{fr} = G_{33} + 83/648\ A_1^3 + 7/8\ A_1^2\beta_0 + 35/18\ A_1\beta_0^2$.

As discussed in the previous section, in phenomenological studies one may want to replace the perturbative expression of $\Sigma(u;\, \alpha)$ with a fit to some experimental data or with a non-perturbative model. The minimal scheme cannot be used directly in these circumstances because the effective form factor $\bar{\Sigma}(t;\, \alpha)$ involves the integration of $\Sigma(u;\, \alpha)$ in the unphysical region $u > 1$. There are various ways to deal with this problem. One way could be for example replacing $\Sigma(u;\, \alpha)$ with the non-perturbative quantity $\Sigma_{np}(u; w)$ for $u \leq 1$, while still keeping the perturbative $\Sigma(u;\, \alpha)$ in the unphysical region $u > 1$. The perturbative form factor is indeed an analytic function of u, which can be continued to any value of u. To avoid the inclusion of the perturbative $\Sigma(u;\, \alpha)$ for $u > 1$, let us consider instead a non-minimal scheme with an effective form factor defined as:

$$\Sigma_T\,(t;\, \alpha) \,=\, \frac{\int_0^1 dw\, C_H(w;\, \alpha)\, \tilde{\Sigma}\left[t/w^2;\, \alpha(w\, m_b)\right]}{\int_0^1 dw\, C_H(w, \alpha)}, \tag{113}$$

where the standard form factor $\Sigma(u;\alpha)$ has been extended to arguments larger than one, $u > 1$, since:

$$\tilde{\Sigma}(u;\alpha) \equiv \begin{cases} \Sigma(u;\alpha) & \text{for } u \,\leq\, 1; \\ 1 & \text{for } u \,>\, 1. \end{cases} \tag{114}$$

Because of the definition, it holds:

$$\Sigma_T(1;\, \alpha) \,=\, 1. \tag{115}$$

Explicitly one has (cfr. the r.h.s. of eq. (80)):

$$\int_0^1 dw\, C_H(w;\, \alpha)\, \tilde{\Sigma}\left[t/w^2;\, \alpha(w\, m_b)\right] \,=\, \int_0^{\sqrt{t}} dw\, C_H(w;\alpha) \,+\, \int_{\sqrt{t}}^1 dw\, C_H(w;\alpha)\, \Sigma\left[t/w^2;\, \alpha(w\, m_b)\right]. \tag{116}$$

The coefficient function is given in the new scheme by:

$$C_T(\alpha) \,=\, \int_0^1 dw\, C_H(w, \alpha). \tag{117}$$

By inserting in eq. (113) the perturbative expansions for C_H and for Σ, we obtain:

$$\Sigma_T^{(1)}(t) \,=\, \frac{C_F}{\pi}\left[-\frac{1}{2}\log^2 t - \frac{31}{12}\log t - \frac{151 - 232\, t^{3/2} + 81\, t^2}{72}\right] \tag{118}$$

and

$$C_T^{(1)} \,=\, -\frac{C_F}{\pi}\,\frac{335}{144} \,=\, -\,0.98735. \tag{119}$$

The first of the above equations shows that Σ_T, unlike $\bar{\Sigma}_T$, is not defined in a minimal factorization scheme, i.e. it does not contain only logarithmic terms $\alpha^n\, L_t^k$, but also contributions of a different form. Note that the coefficient function in (119) has a much smaller value than in the minimal scheme, giving a hint of better convergence of the perturbative series in the modified scheme.

Matching with the first-order spectrum, one obtains for the remainder function in the modified scheme:

$$D_T^{(1)}(t) \;=\; \frac{C_F}{\pi}\left[\frac{97}{18}t-\frac{29}{9}t^{3/2}+\frac{9}{8}t^2-\frac{25}{18}t^3+\frac{61}{144}t^4-\left(\frac{5}{3}t-\frac{3}{2}t^2+\frac{1}{6}t^4\right)\log t\right]. \tag{120}$$

The coefficients of the logarithms in the exponent of the form factor G_T are the same in the minimal scheme and in the modified one:

$$G_{Tij} \;=\; \bar{G}_{Tij} \qquad \text{for } j\geq 1. \tag{121}$$

The non logarithmic coefficients in the modified scheme are given by ($\bar{G}_{T10}=0$ and $\bar{G}_{T20}=0$ by definition of minimal scheme):

$$\begin{aligned} G_{T10} &= -\frac{23}{36}A_1+\frac{5}{6}(B_1+D_1); \qquad (122)\\ G_{T20} &= -\frac{23}{36}A_2+\frac{7}{24}(B_1+D_1)^2+\frac{5}{6}(B_2+D_2)+\frac{23}{36}\beta_0 B_1+\\ &+ A_1\,(B_1+D_1)\left(-\frac{47}{54}+\frac{5\,z(2)}{6}\right)+A_1^2\left(\frac{2057}{2592}-\frac{23\,z(2)}{36}+\frac{5\,z(3)}{6}\right)+\\ &+\frac{C_F}{\pi}\,(B_1+D_1)\left(\frac{547}{108}-4\,z(3)\right)+\frac{C_F}{\pi}A_1\left(-\frac{90121}{5184}+\frac{10\,z(3)}{3}+12\,z(4)\right). \qquad (123) \end{aligned}$$

Explicitly:

$$\begin{aligned} G_{T10} &= -\frac{C_F}{\pi}\frac{151}{72}; \qquad (124)\\ G_{T20} &= \frac{C_F}{\pi^2}\Bigg\{C_A\left(-\frac{344}{81}+\frac{3}{2}z(2)+\frac{5}{24}z(3)\right)+n_f\left(\frac{971}{1296}-\frac{5}{36}z(2)\right)+\\ &\quad + C_F\left(-\frac{79889}{3456}-\frac{53}{36}z(2)+\frac{119}{12}z(3)+12\,z(4)\right)\Bigg\}. \qquad (125) \end{aligned}$$

The relations between the coefficient function in the minimal scheme and in the modified one are obtained imposing that

$$C_T\,e^{G_T} \;=\; \bar{C}_T\,e^{\bar{G}_T}\;+\;O(t;\alpha) \tag{126}$$

and read:

$$\begin{aligned} C_T^{(1)} &= \bar{C}_T^{(1)}-G_{T10};\\ C_T^{(2)} &= \bar{C}_T^{(2)}-\bar{C}_T^{(1)}\,G_{T10}+\frac{1}{2}G_{T10}^2-G_{T20}. \qquad (127) \end{aligned}$$

The first equation can be directly verified by inserting our first-order expressions. The second order correction to the coefficient function is unknown at present in either scheme and its determination requires a full two-loop calculation; the second of the above equations simply allows us to transform the coefficient function from one scheme to another one.

The Babar collaboration has recently presented the differential spectrum in t [9], which is obtained from the previous one by differentiation:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dt} \;=\; \frac{d}{dt}R_T(t). \tag{128}$$

Due to their relevance, let us present explicit formulas. The resummed spectrum reads:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dt} \;=\; C_T(\alpha)\,\sigma_T(t;\,\alpha)\;+\;d_T(t;\,\alpha) \tag{129}$$

where

- the coefficient function is the same as in the event fraction, since it is independent on t;
- the effective form factor is

$$\sigma_T(t;\,\alpha)\;=\;\frac{d}{dt}\Sigma_T(t;\,\alpha). \tag{130}$$

  More explicitly:

$$\begin{aligned}\sigma_T(t;\,\alpha) &= \frac{\int_{\sqrt{t}}^{1} dw/w^2\, C_H(w;\,\alpha)\,\sigma[t/w^2;\,\alpha(w\,m_b)]}{\int_0^1 dw\, C_H(w;\,\alpha)}\\ &= \frac{\int_t^1 du/(2\sqrt{t\,u})\, C_H\left(\sqrt{t/u};\,\alpha(w\,m_b)\right)\,\sigma\left[u;\,\alpha\left(m_b\sqrt{t/u}\right)\right]}{\int_0^1 dw\, C_H(w;\,\alpha)},\end{aligned} \tag{131}$$

  where a double representation as an integral over w or over u respectively has been given;
- the remainder function is:

$$d_T(t;\,\alpha)\;=\;\frac{d}{dt}D_T(t;\,\alpha)\;=\;\alpha\,d_T^{(1)}(t)\;+\;\alpha^2\,d_T^{(2)}(t)\;+\;O(\alpha^3). \tag{132}$$

  The explicit value of the first-order correction is:

$$d_T^{(1)}(t)\;=\;\frac{C_F}{\pi}\left[\frac{67}{18}-\frac{29}{6}\sqrt{t}+\frac{15}{4}t-\frac{25}{6}t^2+\frac{55}{36}t^3-\frac{5}{3}\log t+3t\log t-\frac{2}{3}t^3\log t\right]. \tag{133}$$

Let us make a comment about the non-perturbative effects entering the differential mass distribution. The expression for the effective form factor $\sigma_T(t;\,\alpha)$ at the second member of eq. (131) involves an integration over w from $\sqrt{t}$ up to one. Since the running coupling is evaluated in $Q = w\,m_b$, the smallest hard scale contributing to the distribution is

$$Q_{min}\;=\;\sqrt{t}\;m_b\;=\;m_X. \tag{134}$$

In order to avoid the infrared pole in the coupling — the well-known Landau pole — implying a breakdown of the perturbative scheme, one has to impose the condition

$$m_X\;\gg\;\Lambda_{QCD}, \tag{135}$$

which is also very reasonable from the physical viewpoint. Resummed perturbation theory therefore signals that the hadron mass distribution cannot be computed for hadron masses of the order of the hadron scale because of the appearance of the Landau pole[6].

4 Conclusions

In this work we have presented next-to-leading resummed expressions for the distribution in the final hadron mass/energy ratio and for the distribution in the invariant hadron mass in the semileptonic decays

$$B\;\rightarrow\;X_u\;+\;l\;+\;\nu. \tag{136}$$

By expanding our formulas, we have obtained the coefficients of all the infrared logarithms to $O(\alpha^2)$ and of the leading ones to $O(\alpha^3)$. These two spectra have different logarithmic structures from each other, which are both

[6]The universal QCD form factor $\sigma\,[u;\,\alpha(Q)]$ has an infrared singularity at $u = \exp\left[-1/(2\,\beta_0\,\alpha(Q))\right]$, related to the Landau pole for $m_X \approx \sqrt{\Lambda_{QCD} Q} \gg \Lambda_{QCD}$. We assume that the latter has been regulated in some way; for a recent discussion see for example [10].

different also from that one in the radiative decay (8). That occurs because these spectra involve integration over the total hadron energy E_X, which sets the hard scale Q of the hadronic subprocess in (136):

$$Q = 2E_X. \tag{137}$$

Long distance effects manifest in perturbation theory in the form of large infrared logarithms, which are usually factorized and resummed in QCD form factors. Universality of long-distance effects therefore shows up in perturbation theory as the occurrence of equal form factors in different distributions. That implies that the spectra we have considered in this work have different long-distance effects from each other as well as from the radiative decay (8). There is no simple connection between these semileptonic spectra and the hadron mass distribution in radiative decays. For both spectra, we have introduced effective, i.e. process dependent, form factors, which factorize the large logarithms to all orders in perturbation theory. These effective form factors can also be computed in a phenomenological way by inserting, in place of the perturbative QCD form factor $\Sigma\left[u;\, \alpha(w\, m_b)\right]$, the form factor $\Sigma_{np}(u;\, w)$ computed with a non-perturbative model or a fit to experimental data. In the case of the distribution in the hadron mass squared t, we have also considered a non-minimal factorization-resummation scheme, which seems to have better convergence properties of the perturbative series with respect to the minimal one.

There are other important semileptonic spectra which have similar properties to those of the distributions considered here, i.e. long distance effects which cannot be factorized into a process independent form factor, to be extracted for example from the radiative decay [11].

Finally, we have also shown that the cancellation of long-distance effects in the ratio constructed in [4] occurs only in leading order while it is violated at the level of $\alpha^2 \log^2\left(\frac{1}{u}\right)$ terms.

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