Università degli Studi di Milano
Dipartimento di Matematica
"Federigo Enriques"

Universidad Complutense de Madrid
Facultad de Ciencias Matemáticas
Departamento de Álgebra

Scuola di Dottorato in Scienze Matematiche
Dottorato di Ricerca in Matematica – Ciclo XXIV
Programa de Doctorado en Investigación Matemática

Jumping spaces in Steiner bundles

Ph.D. Thesis

Advisors
Prof. Enrique Arrondo Esteban
Universidad Complutense de Madrid
Prof. Antonio Lanteri
Università degli Studi di Milano

Candidate
Simone Marchesi
Mat. n. R08228
MAT/03

Coordinatore del Dottorato
Prof. Marco Peloso
Università degli Studi di Milano

Academic Year 2010/2011

Ad Alba

Acknowledgements

I would like to thank Enrique Arrondo, first of all for taking me as his student and allowing me to have a wonderful "spanish adventure". I thank him for all his suggestions, all the time spent on me, all the discussions we had (about math and not) and all the possibilities he has given me.
I also wish to thank Antonio Lanteri and Cristina Turrini, for their huge availability and their valuable advices during these three years; each talk with them has always been an incentive to do my best.
I would like to thank all the people I have met during my Ph.D. as for widening my academic formation as for the great relationships made . In particular, I want to thank Laura Costa, Rosa Maria Miró-Roig, Giorgio Ottaviani and Jean Vallès, for giving me the important chance to present my work.
I would also like to thank the referees of this thesis for many helpful comments and suggestions.
To my family, Giuseppe, Franca, Federica and Pina, thank you, for supporting me in each one of my decisions; in spite of the distance, I have always felt them very close.
To Jose and Antonia, thank you, I will be always grateful to them for welcoming me into their home and giving me all the help I needed.
To all my friends, the old ones I left in Italy and make me feel home each time I return, and the new ones made in Spain who were fantastic mates even if "I have stolen their jobs and their women", thank you.
To Alba, thank you for your support, for giving me balance, for believing in me, for your love.

Per tre cose vale la pena di vivere:
la matematica, la musica e l'amore.

Renato Caccioppoli

Contents

Introduction

The problem of the classification of vector bundles on algebraic varieties has always been of huge interest in algebraic geometry. Due to the ampleness of the subject, it has always been studied focusing on families of bundles defined by specific characteristics. In this work we will concentrate on the studying of Steiner and Schwarzenberger bundles on the Grassmannian.

In 1961 (see [Sch61]), Schwarzenberger introduced a family of bundles F of rank n related to the secant space of rational normal curves and defined by a resolution of the type

$$0 \longrightarrow \mathcal{O}^s_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}^t_{\mathbb{P}^n} \longrightarrow F \longrightarrow 0.$$

Since then, many people have studied such family of bundles, most of the times trying to find geometrical configurations in the projective space in order to define the bundle, and also trying to prove Torelli type theorems, i.e. recovering the configuration from a given bundle. For instance, in 1993 (see [DK93]), Dolgachev and Kapranov, who were the first to denominate such bundles Steiner, investigate logarithmic bundles on the projective space defined by 1-differential forms on the union of an arrangement of hyperplanes with normal crossing. In their paper, they define the families of Steiner and Schwarzenberger bundles as subfamilies of the logarithmic one, and they prove results that concern the relations among the three considered sets of bundles. In particular they prove that a logarithmic bundle can be described as either Steiner or Schwarzenberger under specific hypothesis for the arrangement of hyperplanes.

In 2000 (see [Val00b]), Vallès proves a more general result that characterizes when a Steiner bundle F can be described also as a Schwarzenberger. He focuses on a particular family of hyperplanes $\{H_i\}$, which satisfy the condition $h^0(F^\vee_{H_i}) \neq 0$ and which are called *unstable hyperplanes*, and proves that such hyperplanes, seen as points in the dual projective space, always belong to a rational normal curve and this allows to state that the bundle F is Schwarzenberger in the sense of [DK93].

In 2001(see [AO01]), Ancona and Ottaviani reinforce the importance of the set of the unstable hyperplanes for a Steiner bundle F, proving that if we have a sufficient number of independent ones, the bundle F is also logarithmic.
The property of stability for rank n Steiner bundles over $\mathbb{P}^n$ was proved by Bohnhorst and Splinder, see [BS92], and Brambilla, see [Bra08], proved the stability of exceptional Steiner vector bundles. Moreover, in her PhD thesis (see [Bra04]), she characterizes simple and exceptional general Steiner vector bundles over the projective space.
In [Val00a], Vallès proposes a first generalization of logarithmic and Schwarzenberger bundles of higher rank. However, the first full generalization of Schwarzenberger bundles to arbitrary rank in projective spaces appears in [Arr10a]. In his paper, Arrondo basically generalizes two notions: the one of Schwarzenberger bundle, which he associates to a triple (X, L, M), where X is a projective variety and L, M two globally generated vector bundles over X, and the one of unstable hyperplane for a Steiner bundle F, which he denominates *jumping hyperplane*. Studying the locus of the jumping pairs, Arrondo manages to classify the Steiner bundles, whose locus has maximal dimension, and describe them as Schwarzenberger.
The study of Steiner bundles over varieties different from the projective space was given by Miró-Roig and Soares. First, in [Soa07] Soares defines Steiner vector bundles over the smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$, with $n \geq 3$. Moreover, she characterizes exceptional and simple Steiner bundles on the smooth hyperquadric and she proves that exceptionality in this case implies stability. In [MRS09] and [Soa08], Miró-Roig and Soares give the definition of Steiner bundle on any algebraic variety and they prove homological characterizations for such bundles. The proposed definition depends on the choice of a strongly exceptional pair of vector bundles over a projective variety.
In this thesis we obtain the results of Arrondo when choosing the definition of Steiner bundle on Grassmannians which has the most geometric meaning. We can state the problems we will solve in the following list.
Problem 1 *Finding the most natural and geometric definition of Schwarzenberger bundle for Grassmannians.*
Problem 2 *Generalizing the definition of jumping pair and give a description of their locus for Steiner bundles on Grassmannians.*
Problem 3 *Describing Steiner bundles on $\mathbb{G}(k, n)$ with jumping locus of maximal dimension as Schwarzenberger bundles and giving a classification for such case.*
In chapter 1 we will give the necessary preliminaries, recalling the definition and properties of the Grassmannians. We will also give an introduction on Schubert calculus.
In chapter 2 we will state the general definition of a Steiner bundle for Grassmannians,

that fits the one given by Miró-Roig and Soares and it is the natural generalization of the one stated by Arrondo.

Definition 1. *Let S, T be two vector spaces over $\mathbb{K}$, respectively s and t-dimensional. We will call an (s,t)-Steiner bundle, over $\mathbb{G}(k,n)$, the vector bundle defined by the resolution*

$$0 \longrightarrow S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0,$$

where $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}(k,n)}$ is the trivial bundle and $\mathcal{U} \to \mathbb{G}(k,n)$ is the universal bundle of rank $k+1$.
This is equivalent to fix a linear application

$$T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee) = \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$$

such that, for every $u_1, \ldots, u_{k+1} \in H^0(\mathcal{U}^\vee)^*$ linearly independent and for every $v_1, \ldots, v_{k+1} \in S^*$, there exists an $f \in \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$ such that $f \in \operatorname{Im}\varphi$ and $f(u_j) = v_j$ for each $j = 1, \ldots, k+1$.
If φ is injective F is said to be reduced or else we will call F_0 the reduced summand of F associated to the linear injective map

$$\varphi(T^*) = T_0^* \hookrightarrow S^* \otimes H^0(\mathcal{U}^\vee).$$

After showing a geometrical interpretation of the definition, we will then give a lower bound for the possible ranks of the bundles just defined, indeed we will prove the following result.

Theorem 2. *Let F be a Steiner bundle over $\mathbb{G}(k,n)$; then it will have rank*

$$\operatorname{rk} F \geq \min((k+1)(n-k), (n-k)\dim S).$$

In order to solve Problem 1, in chapter 3 we will give the definition of Schwarzenberger bundle, generalizing the one given in [Arr10a].

Definition 3. *Let us consider two globally generated vector bundles L, M over a projective variety X, with $h^0(M) = n+1$ and with the identification $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. The Schwarzenberger bundle on $\mathbb{G}(k,n)$ associated to the triple (X, L, M) will be the bundle defined by the resolution*

$$0 \longrightarrow H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0.$$

Observe that the injectivity of the first morphism of the resolution holds under specific conditions on a particular restriction of the multiplication map

$$H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M),$$

as we will see better in Definition 3.1.1.
As in the case of the projective space, the most significant examples of Schwarzenberger bundle will be when L has rank one and M has rank $k+1$.
We will then give the definition of jumping pair for a Steiner bundle and also give an algebraic structure to the set of all of them. We will bound the dimension of its locus through the description of its tangent space, which will give us information of the jumping locus as a projective variety.
Considering the bundle given by the triple (X, L, M), we notice that from a point $x \in X$ the image of $H^0(L \otimes M)^*$ through the dual of the multiplication map (which is the map φ in this case) restricted to the fiber of x has the particular form $H^0(L_x)^* \otimes H^0(M_x)^*$, i.e. it is the tensor product of two vector subspaces. This observation will lead us to define a similar object for Steiner bundles and the locus of such objects will give us information that will allow us to construct a Schwarzenberger bundle triple, starting from a Steiner bundle.

Definition 4. *Let F be a Steiner bundle over $\mathbb{G}(k, n)$. A pair (a, Γ), with $\dim a = 1$ and $\dim \Gamma = k+1$, such that $a \otimes \Gamma \subset S^* \otimes H^0(\mathcal{U}^\vee)$, is called a jumping pair if, considering the map $T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee)$, the tensor product $a \otimes \Gamma$ belongs to $\operatorname{Im} \varphi$.*

In order to solve Problem 3, our goal is to describe and study the locus of the jumping pairs associated to a Steiner bundle F, we will denote such locus by $\tilde{J}(F)$ (by abuse of notation we will use $\tilde{J}(F)$ both for the vector space locus and for the projective locus). This will allow us to use the following result to classify Steiner bundles.

Theorem 5. *Let $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ be the universal bundles of ranks respectively $1, k+1$ and $k+1$ over $G(1, S^*), G(k+1, H^0(\mathcal{U}^\vee))$ and $G(k+1, T_0^*)$.*
Notice we have two natural projections

$$\tilde{J}(F) \xrightarrow{\pi_1} G(1, S^*)$$

$$\tilde{J}(F) \xrightarrow{\pi_2} G(k+1, H^0(\mathcal{U}^\vee))$$

and that $\tilde{J}(F) \subset G(k+1, T_0^*)$. Assume that the natural maps

$$\begin{array}{llcl} \alpha: & H^0(G(1,S^*),\mathcal{A}) & \longrightarrow & H^0(\tilde{J}(F),\pi_1^*\mathcal{A}) \\ \beta: & H^0(G(k+1,H^0(\mathcal{U}^\vee)),\mathcal{B}) & \longrightarrow & H^0(\tilde{J}(F),\pi_2^*\mathcal{B}) \\ \gamma: & H^0(G(k+1,T_0^*),\mathcal{Q}) & \longrightarrow & H^0(\tilde{J}(F),\mathcal{Q}_{|\tilde{J}(F)}) \end{array}$$

are all isomorphisms. Then the Steiner bundle F_0, reduced summand of F, is a Schwarzenberger bundle given by the triple

$$(\tilde{J}(F),\pi_1^*\mathcal{A},\pi_2^*\mathcal{B}).$$

We will manage to give a geometrical description to this locus, seeing it as a projective variety. In fact, if we consider the generalized Segre map

$$\begin{array}{lccccc} \nu: & \mathbb{P}(S) & \times & \mathbb{G}(k,\mathbb{P}(H^0(\mathcal{U}^\vee))^*) & \longrightarrow & \mathbb{G}(k,\mathbb{P}(S\otimes H^0(\mathcal{U}^\vee)^*)) \\ & \mathbb{P}(l) & , & \mathbb{P}(\Lambda) & \mapsto & \mathbb{P}(l\otimes\Lambda) \end{array}$$

then it will be possible to define

$$\tilde{J}(F) = \operatorname{Im}\nu \cap \mathbb{G}(k,\mathbb{P}(T_0))$$

where, as usual, $T_0^* = \varphi(T^*)$, vector space associated to the reduced summand of F. Our goal is to study the dimension of such variety. Observe that a lower bound is given computing the expected dimension of the intersection, getting

$$\dim \tilde{J}(F) \geq (k+1)(t-k-sn-s+n)+s-1.$$

To get an upper bound we will study the tangent space of $\tilde{J}(F)$ at a point Λ representing a jumping pair.
After giving a description of the tangent space of the generalized Segre variety at the point Λ through linear algebra, we will prove a technical result of linear algebra that will give us the requested bound.

Theorem 6. *Let F be a Steiner bundle over $\mathbb{G}(k,n)$ and $\tilde{J}(F)$ its jumping pair locus; then, considering $\Lambda \in \tilde{J}(F)$, we obtain*

$$\dim \tilde{J}(F) \leq \dim T_\Lambda \tilde{J}(F) \leq (k+1)(t-(k+1)(s+n-k-1)-k).$$

In chapter 4, we will classify the Steiner bundles whose jumping locus has maximal

dimension.
We observe that given an $(s,t)-$Steiner bundle F on $\mathbb{G}(k,n)$, which we suppose to be reduced, with jumping locus of maximal dimension, if we fix a jumping pair $s_0 \otimes \Gamma = \varphi(\Lambda)$ then we can induce a $(s-1, t-k-1)$- Steiner bundle F' on $\mathbb{G}(k,n)$, that may not be reduced, with jumping locus also of maximal dimension. Such induction is consequence of the following commutative diagram

$$\begin{tikzcd}
T^* \arrow[r, hook, "\varphi"] \arrow[d, two heads] & S^* \otimes H^0(\mathcal{U}^\vee) \arrow[d, two heads] \\
\frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{\langle s_0 \rangle} \otimes H^0(\mathcal{U}^\vee)
\end{tikzcd}$$

Given a Steiner bundle on the Grassmannian, we will induce Steiner bundles for as many steps as we need to get to the most basic case $s = k+2$, which is classified by the following result.

Theorem 7. *Let F be a reduced Steiner bundle over $\mathbb{G}(k,n)$, with $\dim S = k+2$, then F can be seen as the Schwarzenberger bundle given by the triple $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$, where E is the vector bundle defined as the kernel of the surjective morphism*

$$H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)}.$$

Then, looking at the constructed diagram, we will sort the possible cases and throughout Theorem 5 we will manage to find the triple that describes the bundle as Schwarzenberger. We will get the following classification.

Theorem 8. *Let F be a reduced Steiner bundle on $\mathbb{G}(k,n)$ for which $\dim \tilde{J}(F)$ is maximal; then we are in one of the following cases:*

(i) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$. In this case $k=0$ and $t = n+s$.*

(ii) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^1, E(-1), \mathcal{O}_{\mathbb{P}^1}(1))$, where $E = \oplus_{i=1}^{t-s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 1$. In this case $k=0$ and $n=1$.*

(iii) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$ defined in Theorem 7. In this case $s = k+2$.*

(iv) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$. In this case $k = 0, n = 2, s = 3$ and $t = 6$.*

Chapter 1

Preliminaries

In this chapter we will show the notions we will need throughout this work. We will introduce the concept of Grassmannian and we will prove that it is a projective variety. We will define its canonical bundles, which will be necessary for the concept of Steiner bundle. Moreover, we will investigate its subvarieties and their intersection, using the theory of the Schubert calculus.

1.1 The Grassmannian

Let V be a $(n+1)$-dimensional vector space over an algebraically closed field $\mathbb{K}$ of characteristic zero and let the projective space $\mathbb{P}^n = \mathbb{P}(V)$ be constructed as the set of the equivalence classes of hyperplanes in V, or equivalently of lines of V^*. We will denote by $P(V^*)$ the projectivization given by the equivalence classes of lines in V^*.
We will now give the definition of Grassmannian, accordingly with the notation we have fixed.

Definition 1.1.1. *We will denote by $\mathbb{G}(k,n) = \mathbb{G}(k,\mathbb{P}^n)$ the variety, which we will call Grassmannian, consisting of the k-dimensional linear subspaces of $\mathbb{P}^n$.*

Notice that each element of the Grassmannian is in correspondence with a $(k+1)$-dimensional vector subspace of V^*, or equivalently with an $(n-k)$-dimensional subspace of V, and we will denote the Grassmannian in the vectorial case by $G(k+1, V^*)$. Moreover it is possible to construct a canonical isomorphism $\mathbb{G}(k,\mathbb{P}^n) \simeq \mathbb{G}(n-k-1,(\mathbb{P}^n)^*)$, associating of course each linear subspace to its orthogonal.

Proposition 1.1.2. *There exists a canonical embedding*

$$\rho : \mathbb{G}(k,n) \longrightarrow \mathbb{P}^{\binom{n+1}{k}-1},$$

called the Plücker embedding, whose coordinates of the arrival space are called Plücker coordinates. Moreover, the image of $\mathbb{G}(k,n)$ through ρ is a projective variety in the considered ambient projective space.

Proof. We can give a further description of the Grassmannian considering $(k+1)$-dimensional linear subspaces $W \subset V^*$, spanned by the vectors $v_0, \dots, v_k$ and the associated multivector

$$\omega = v_0 \wedge \ldots \wedge v_k \in \bigwedge^k (V^*).$$

Observe that the multivector is determined up to scalar multiplication by elements of the field $\mathbb{K}$, such scalar multiplication depending on the basis we have chosen to describe the vector space. This fact suggests us that the object we have considered must be in correspondence with a projective object. We get the following morphism

$$\rho : G(k+1, V^*) \longrightarrow P\left(\bigwedge^k V^*\right).$$

Notice that the given morphism is an inclusion, which is called *Plücker embedding*, because for each element $[\omega]$ in the image of ρ we can recover its unique preimage, given by all $v \in V^*$ such that $v \wedge \omega = 0 \in \bigwedge^{k+1} V$. The coordinates of the projective space $P\left(\bigwedge^k V^*\right)$ are called *Plücker coordinates*.

An explicit way to see such objects is considering the $(k+1) \times (n+1)$ matrix associated to W, constructed considering the coordinates in V^* of $k+1$ independent points that span the subspace, i.e. where the i-th row is given by the coordinates of the vector v_i; the k-minors of the matrix are our Plücker coordinates, that we will denote by $p_{i_0,\dots,i_k}$ with the subindex depending by the $k+1$ colums we have chosen to get the minor.

Once managed to see our Grassmannian as a subset of the projective space $P\left(\bigwedge^k V^*\right)$, we would like now to give a description of it as a projective subvariety. Observe that the elements of the Grassmannian in the wedge product are the totally decomposable elements, i.e. every element can be represented as a combination $\omega = v_0 \wedge \ldots \wedge v_k$. We can notice that an element $v \in V^*$ is one of the elements in the wedge combination, which describes ω, if and only if $v \wedge \omega = 0$, so the elements totally decomposable are the ones for which we can find $k+1$ independent vectors with such property. That means that an element belongs to the Grassmannian if and only if the rank of the map

$$\begin{array}{rccc} \tau(\omega): & V^* & \longrightarrow & \bigwedge^{k+2} V^* \\ & v & \mapsto & \omega \wedge v \end{array}$$

is $n-k$, which gives us that we can find exactly $k+1$ independent vectors in the kernel. If we consider the matrix associated to the map

$$\bigwedge^{k+1} V^* \longrightarrow \mathrm{Hom}(V^*, \bigwedge^{k+2} V^*)$$

that sends an element ω to the morphism constructed before, we have that the Grassmannian is defined by the vanishing of particular minors of the matrix associated to the map; this gives us the description as a subvariety. $\square$

We would like to give to such variety an appropriate "canonical" covering by open sets. In order to do so let us consider the Plücker matrix we described before, that is the matrix whose rows are a set of $k+1$ independent points that span the element $\Gamma \in \mathbb{G}(k,n)$. As we have already noticed, this description is not unique because it depends by the basis we choose for the vector space, so after multiplying by a matrix belonging to $GL(k+1, \mathbb{K})$, which represents the base change, we can suppose that the Plücker matrix has the following form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & y_{0,k+1} & \cdots & y_{0,n} \\ 0 & 1 & \cdots & 0 & y_{1,k+1} & \cdots & y_{1,n} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & y_{k,k+1} & \cdots & y_{k,n} \end{pmatrix}$$

We notice straightforwardly that we are considering the sets of all matrices such that the minor given by the first $k+1$ rows and columns is not zero, and remembering that we also defined the Plücker coordinates as the $k+1$ minors of the matrix, it means we are considering the points which belong to the open subset described as $\{p_{1,\ldots,k+1} \neq 0\}$. The other open subsets will be of course of type $\{p_{i_0,\ldots,i_k} \neq 0\}_{1 \leq i_0 < \ldots < i_k \leq n+1}$.
Observe that $\dim \mathbb{G}(k,n) = (k+1)(n-k)$, which is the number of free coordinates we have in on open subset of the variety.
Let us give an example that will help us to understand the definitions we have given.

Example 1.1.3. Let us consider the variety $\mathbb{G}(1,3) \subset \mathbb{P}^5$. Each line in $\mathbb{P}^3$ can be represented by two independent points spanning it and so the matrix associated will be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

where a_i and b_j are the coordinates of the two points. In this case the six Plücker coordinates will be given by

$$p_{0,1} = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \quad p_{0,2} = \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \quad p_{0,3} = \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix}$$

$$p_{1,2} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad p_{1,3} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \quad p_{2,3} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

The vanishing of the determinant of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

allows us to write the equation that defines the Grassmannian, seen as embedded in $\mathbb{P}^5$, which is

$$p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0.$$

1.1.1 Projective and Grassmann bundles

Definition 1.1.4. *Given a vector bundle $F \xrightarrow{\pi} X$, of rank r we will call the projective bundle of F the variety $P(F) \xrightarrow{q} X$ defined as follows: consider the covering $X = \bigcup U_i$, such that for every open subset U_i we have a commutative diagram*

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\simeq"] \arrow[dr] & & U_i \times \mathbb{K}^r \arrow[dl] \\ & U_i & \end{tikzcd}$$

given by the definition of vector bundle.
For the projective bundle we take $q^{-1}(U_i) \simeq U_i \times P(\mathbb{K}^r)$ and we manage to glue $U_i \times P(\mathbb{K}^r)$ with $U_j \times P(\mathbb{K}^r)$, when $U_i \cap U_j \neq 0$, using the transition matrices of the bundle F.

Example 1.1.5. We would like to study the particular commutative diagram

$$\begin{tikzcd} q^*(F) \arrow[r] \arrow[d] & F \arrow[d, "\pi"'] \\ P(F) \arrow[r, "q"] & X \end{tikzcd}$$

Consider the incidence variety $\mathcal{U} = \{(p, v) | v \in \vec{p}\} \subset \mathbb{P}^n \times \mathbb{K}^{n+1}$ and let us write the previous diagram locally in this particular case, observing that we can add a specific line bundle, induced by the incidence variety, with a natural inclusion

$$\begin{tikzcd} U_i \times \mathbb{P}^{r-1} \times \mathbb{K} \arrow[r, "\subset"] \arrow[dr] & U_i \times \mathbb{P}^{r-1} \times \mathbb{K}^r \arrow[r] \arrow[d] & U_i \times \mathbb{K}^r \arrow[d] \\ & U_i \times \mathbb{P}^n \arrow[r] & U_i \end{tikzcd}$$

Observe that the pieces $U_i \times \mathbb{P}^{r-1} \times \mathbb{K}$ can be glued together in order to form a line subbundle of q^*F. We will denote such vector bundle as $\mathcal{O}_F(-1) \hookrightarrow q^*F$. Notice that if L is a line bundle on X we have $P(F \otimes L) = P(F)$ and we obtain

$$\mathcal{O}_{F \otimes L}(-1) = \mathcal{O}_F(-1) \otimes q^*L \subset q^*(F \otimes L).$$

In the same way we are going to give the following definition.

Definition 1.1.6. *Given a vector bundle $F \xrightarrow{\pi} X$, of rank r we will call the Grassmann bundle of F the space $\mathbb{G}(k, F) \xrightarrow{q} X$ defined as follows: consider the covering $X = \bigcup U_i$, we take $q^{-1}(U_i) \simeq U_i \times \mathbb{G}(k, P(\mathbb{K}^r))$ and we manage to glue $U_i \times \mathbb{G}(k, P(\mathbb{K}^r))$ with $U_j \times \mathbb{G}(k, P(\mathbb{K}^r))$, when $U_i \cap U_j \neq 0$ using the transition matrices of the bundle F.*

Remark 1.1.7. Notice that the Grassmann bundle $\mathbb{G}(k, F)$ we just defined is different from the Grassmannian $\mathbb{G}(k, P(F))$ constructed taking the k-linear subspaces of the projective variety $P(F)$. Indeed, in $\mathbb{G}(k, F)$ we consider the k-linear subspaces of the projectivization of the fiber of the bundle for each point of the base.

1.1.2 The universal bundles

Let us consider a vector bundle, over the Grassmannian $\mathbb{G}(k, n)$, defined in the following way: $\mathcal{U} \subset \mathbb{G}(k, n) \otimes (\mathbb{K}^{n+1})^*$ where $\mathcal{U}$ is given by the incidence variety $\mathcal{U} = \{(\Lambda, v) | v \in \vec{\Lambda}\}$, with $\vec{\Lambda}$ representing the $(k+1)$-dimensional vector subspace associated to Λ. Denoting by π the map defining the bundle $\mathcal{U} \xrightarrow{\pi} \mathbb{G}(k,n)$, we have that the fiber over a point is $\pi^{-1}(\Lambda) = \vec{\Lambda}$. If we consider the covering of the Grassmannian given by the Plücker coordinates, then we will get that $\pi^{-1}(\{p_{i_o,\dots,i_k} \neq 0\}) \simeq \{p_{i_o,\dots,i_k} \neq 0\} \times \mathbb{K}^{k+1}$. We have a rank $k+1$ bundle over $\mathbb{G}(k,n)$. We can obtain an exact sequence

$$0 \longrightarrow F \longrightarrow \mathbb{G}(k,n) \times \mathbb{K}^{n+1} \longrightarrow \mathcal{U}^{\vee} \longrightarrow 0$$

where $\operatorname{rk} F = n-k$. In the case $k=0$ it can be proved that $F = \Omega_{\mathbb{P}^n} \otimes \mathcal{U}^{\vee}$, from which we get $F \otimes \mathcal{U} = \Omega_{\mathbb{P}^n}$. For every k we thus obtain $\Omega_{\mathbb{G}(k,n)} = F \otimes \mathcal{U}$. We will call $\mathcal{U}$ the *universal bundle* and $\mathcal{Q} = F^{\vee}$ the *universal quotient bundle.*
Let us observe that the dual bundle $\mathcal{U}^{\vee} \xrightarrow{\pi^*} \mathbb{G}(k,n)$ has fibers of type $(\pi^*)^{-1}(\Lambda) = \vec{\Lambda}^*$, consisting of the linear forms that vanish on Λ.
In an analogous way, considering $\mathbb{G}(k,n) \times \mathcal{S}^d(\mathbb{K}^{n+1})^* \longrightarrow \mathcal{S}^d\mathcal{U}^{\vee}$, we are considering the forms of degree d which vanish on Λ.
Let us prove that the canonical bundles $\mathcal{U}$ and $\mathcal{Q}$ give us strongly exceptional pairs for the Grassmannian, which we will need to define a Steiner bundle. We will denote by $\mathcal{O}_{\mathbb{G}}$ the line trivial bundle on $\mathbb{G}(k,n)$. Let us start recalling the definition.

Definition 1.1.8. *Let X be a smooth algebraic variety. A coherent sheaf E on X is simple if*

$$\operatorname{Hom}(E,E) \simeq \mathbb{K}.$$

If E is simple and, furthermore, is satisfies

$$\operatorname{Ext}^i(E,E) = 0, \textit{ for all } i \geq 1$$

then E is exceptional.
An ordered collection $(E_1,\dots,E_m)$ of coherent sheaves on a smooth algebraic variety X is an exceptional collection if all sheaves E_i are exceptional and

$$\operatorname{Ext}^p(E_i,E_j) = 0 \textit{ for all } i > j \textit{ and } p \geq 0.$$

If, in addition,

$$\operatorname{Ext}^p(E_j,E_i) = 0 \textit{ for all } j \leq i \textit{ and } p \neq 0,$$

the collection $(E_1,\dots,E_m)$ is a strongly exceptional collection.

Proposition 1.1.9. *The two pairs $(\mathcal{U},\mathcal{O}_{\mathbb{G}})$ and $(\mathcal{O}_{\mathbb{G}},\mathcal{Q})$ of vector bundles over $\mathbb{G}(k,n)$ are strongly exceptional.*

Proof. In order to prove the proposition we have to verify that

$$\begin{array}{ll} h^p(\mathcal{U}) = h^p(\mathcal{Q}^\vee) = 0 & \text{for all } p \geq 0, \\ h^p(\mathcal{U}^\vee) = h^p(\mathcal{Q}) = h^p(\mathcal{U} \otimes \mathcal{U}^\vee) = h^p(\mathcal{Q} \otimes \mathcal{Q}^\vee) = 0 & \text{for all } p > 0, \end{array}$$

where $h^p(E)$ denotes the dimension of $H^p(F)$ for the vector bundle F.
Let us compute the cohomology of the bundles $\mathcal{U}$ and $\mathcal{Q}$, or at least their dimension. Consider $I = \{(P, \Lambda) \in \mathbb{P}^n \times \mathbb{G}(k, n) | P \in \Lambda\}$ and its two natural projections $I \xrightarrow{p} \mathbb{P}^n$ and $I \xrightarrow{q} \mathbb{G}(k, n)$. Remembering the definition given for the universal bundle $\mathcal{U} \subset \mathbb{G}(k, n) \times \mathbb{K}^{n+1}$, we recover that $I = P(\mathcal{U})$, the projective bundle, and $\mathcal{O}_{\mathcal{U}}(-1) = p^*\mathcal{O}_{\mathbb{P}^n}(-1)$. Notice the fiber of p over $x \in \mathbb{P}^n$ is $p^{-1}(x) = \{\Lambda \in \mathbb{G}(k, n) | \Lambda \ni x\}$ which is isomorphic to $\mathbb{G}(k-1, n-1)$.
In order to calculate the cohomology we will use the following lemmas; for a complete reference, see for example [Har77].

Lemma 1.1.10. *Let $f : X \longrightarrow Y$ be a morphism between projective varieties and let F be a vector bundle over X. If $R^i f_*(F) = 0$, for each $i > 0$, where R denotes the derived right functor, then*

$$H^i(X, F) \simeq H^i(Y, f_*F) \text{ for each } i \geq 0.$$

Lemma 1.1.11. *Let Y a projective variety and F a rank $n+1$ vector bundle on it, with $n \geq 1$. Consider $X = P(F)$, with the bundle $\mathcal{O}_F(1)$ and the projection $\pi : X \longrightarrow Y$; then $\pi_*(\mathcal{O}_F(l)) \simeq \mathcal{S}^l(F^\vee)$, for $l \geq 0$, $\pi_*(\mathcal{O}_F(l)) = 0$, for $l < 0$; $R^i\pi_*(\mathcal{O}_F(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$ and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.*

In our case, noticing that $p_*(\mathcal{O}_{\mathcal{U}}(1)) = p_*(p^*(\mathcal{O}_{\mathbb{P}^n}(1)))$, the previous lemmas allow us to state that

$$H^i(\mathbb{G}(k, n), \mathcal{U}^\vee) \simeq H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).$$

From the duality of Grassmannians, $\mathbb{G}(k, n) \simeq \mathbb{G}(n-k-1, n)$, we manage to state the lemma.

Lemma 1.1.12. *On the Grassmannian $\mathbb{G}(k, n)$, the following holds*

$$h^i(\mathcal{U}^\vee) = h^i(\mathcal{Q}) = \begin{cases} n+1 & \text{for } i = 0, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and } h^i(\mathcal{U}) = h^i(\mathcal{Q}^\vee) = 0 \text{ for each } i.$$

Moreover, it is also possible to prove the following results.

Lemma 1.1.13. *On the Grassmannian $\mathbb{G}(k,n)$, the following holds*

$$h^i(\mathcal{U}\otimes\mathcal{Q}) = \begin{cases} \binom{n+1}{2} & \textit{for } i=0, k=n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

$$h^i(\mathcal{U}^\vee\otimes\mathcal{Q}^\vee) = \begin{cases} \binom{n+1}{2} & \textit{for } i=0, k=0, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 1.1.14. *On the Grassmannian $\mathbb{G}(k,n)$, the following holds*

$$h^i(\mathcal{U}\otimes\mathcal{U}^\vee) = h^i(\mathcal{Q}\otimes\mathcal{Q}^\vee) = \begin{cases} 1 & \textit{for } i=0, \\ 0 & \textit{otherwise.} \end{cases}$$

For a complete proof, see for example [Ott87]. ☐

1.1.3 Schubert subvarieties

We will introduce a way to describe particular subvarieties of the Grassmannian and then we will focus on some special examples.

Definition 1.1.15. *Let us fix a non empty flag of $\mathbb{P}^n$ made of $k+1$ linear subspaces $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k$. We call the Schubert variety associated to the flag the set*

$$\Omega(\Lambda_0,\dots,\Lambda_k) = \{\Lambda \in \mathbb{G}(k,n) \mid \dim(\Lambda\cap\Lambda_i) \geq i \, for \, i = 0,\dots,k\}.$$

Let us now consider some particular examples.

Example 1.1.16. Suppose that we have an inclusion $W \subset V$, this leads to an inclusion

$$G(k+1, W) \hookrightarrow G(k+1, V);$$

in the same way if we have a quotient $V \longrightarrow V/U$, where $\dim U = l$, then we get an inclusion of type

$$G(k+1-l, V/U) \hookrightarrow G(k+1, V).$$

If we consider $W \subset V$ and hyperplane, the subgrassmannian can be seen as the vanishing locus of a global section of the dual of the universal bundle $\mathcal{U}^\vee$.

It is possible to prove that such subvarieties in the projective space $P(\bigwedge^k V^*)$ are constructed considering the intersection of the original Grassmannian, embedded in the

projective space, with properly chosen linear subpaces.
We can also take a look to another particular subspace of the Grassmannian: if we fix a linear subspace of $\mathbb{P}^n$, we can consider all the elements of $\mathbb{G}(k, n)$ that have non empty intersection with the fixed one; we could also ask for those ones whose intersection has a specific dimension.

Example 1.1.17. Consider the set of k-planes meeting a given m-dimensional linear subspace $\Lambda' \subset \mathbb{P}^n$. We can describe this set, which we denote $\Sigma(\Lambda')$, by

$$\Sigma(\Lambda') = \{[\omega] : \omega \wedge v_1 \wedge \cdots \wedge v_{m+1} = 0 \ \forall \, v_1, \ldots, v_{m+1} \in \Lambda'\}.$$

Such sets are very important when we investigate the structure of the Grassmannians, as we will see in the next section.
If we consider a point $p \in \mathbb{P}^n$, then $\Sigma(p)$ can be seen as the vanishing locus of a global section of the universal quotient bundle $\mathcal{Q}$.

Notice that also in the Grassmannian context we have projection maps as we have for projective spaces. Indeed, consider W an l-dimensional subspace of V. If $k + 1 \leq l$, then we are able to construct a map

$$\pi : U \longrightarrow G(k + 1, V/W),$$

which is only defined in the open subset $U \subset G(k + 1, V)$, of all $(k + 1)$-linear spaces meeting W only at the (0), by taking their image; if $k + 1 \geq l$, we get a map

$$\tau : U' \longrightarrow G(k + 1 - l, W)$$

defined, in the open subset U' of planes transverse to W, by taking the intersection.
Such projections can also be seen as restriction of projections in the ambient space $P(\bigwedge^{k+1} V)$ given by the Plücker embedding. For example, the map π is nothing but the restriction of the projection

$$P(\bigwedge^{k+1} V) \longrightarrow P(\bigwedge^{k+1} (V/W)),$$

induced by the projection $V \longrightarrow V/W$ between the two vector spaces.

1.1.4 Vector bundles and projective morphisms

Theorem 1.1.18. *Let F be a vector bundle of rank r over X and let $V \subset H^0(X, F)$ that generates F, then there exists a map*

$$\varphi_V : X \longrightarrow \mathbb{G}(r-1, \dim V - 1)$$

such that $\varphi_V^* \mathcal{U}^\vee = F$ and $H^0(\mathbb{G}(r-1, \dim V-1), \mathcal{U}^\vee) = V$.

Proof. The fact that F is globally generated by V means that the morphism

$$\begin{array}{rccc} ev_V : & X \times V & \longrightarrow & F \\ & (x, s) & \mapsto & s(x) \end{array}$$

is surjective. We consider the dual morphism $ev_V^* : F^\vee \longrightarrow X \times V^*$ which is injective, that means is injective in each fiber. Therefore, for every $x \in X$ we get $F_x^* \hookrightarrow V^*$, vector subspace of dimension r. We define

$$\varphi_V(x) = P(F_x^*) \subset P(V^*) = \mathbb{P}^{\dim V - 1}.$$

Let us see such association explicitly choosing coordinates: let $s_1, \dots, s_n$ be a basis for V and let $\lambda_1, \dots, \lambda_n$ the coordinates with respect to such basis. For every $x \in X$ consider an open subset $U \subset X$ with $U \ni x$ such that the bundle is trivial in the open subset, i.e.

$$\begin{tikzcd} F_{|U} \simeq U \times \mathbb{K}^r \\ U \arrow[u, bend left, "s_{i|U}"] \arrow[u, bend right, "(f_1,\dots,f_r)"'] \end{tikzcd}$$

Because of the isomorphism, giving a section is equivalent to giving r regular functions $f_{i_1}, \dots, f_{i_r}$. We will have then

$$\begin{array}{cccc} ev_{V|U} : & U \times V & \longrightarrow & U \times \mathbb{K}^r \\ & \left(x', \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \right) & \mapsto & \left(x', \begin{pmatrix} f_{11}(x') & \cdots & f_{1n}(x') \\ \vdots & & \vdots \\ f_{r1}(x') & \cdots & f_{rn}(x') \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \right) \end{array}$$

We pass to the dual map and we notice that in the dual basis the map ev_V^* restricted to

U is given by the transposed matrix

$$\begin{pmatrix} f_{11}(x') & \cdots & f_{r1}(x') \\ \vdots & & \vdots \\ f_{1n}(x') & \cdots & f_{rn}(x') \end{pmatrix}.$$

This means that $\varphi_V(x')$ is the vector subspace generated by the points whose coordinates are the columns of the matrix and the Plücker coordinates of the subspace are the minors of order r of the matrix, which are also regular functions on U.
Let us recall that the pullback of a bundle is by definition

$$\varphi_V^*(\mathcal{U}^\vee) = \{(x,v) \mid v \in \mathcal{U}^\vee_{\varphi(x)}\};$$

in order to prove that $\varphi_V^*\mathcal{U}^\vee = F$ we apply the definition and prove the equality for the dual bundles. Remember that we have defined the universal bundle as the incidence variety

$$\mathcal{U} = \{(\Lambda, v) \mid v \in \vec{\Lambda}\} \subset \mathbb{G}(r-1, n-1) \times V^*$$

and consider the fiber of the pulback

$$\varphi_V^*(\mathcal{U})_x = \{(x,v) \mid v \in \vec{\varphi}_V(x) = F_x^*\} \subset X \times V^*;$$

we observe that we have the same fibers, so we can identify the bundles. Moreover, we know that $H^0(\mathbb{G}(r-1, \dim V - 1), \mathcal{U}^\vee)$ is the space of linear forms of $P(V^*)$, i.e. V. $\square$

Proposition 1.1.19. *Given a $\Lambda' \subset \mathbb{P}^n$ of dimension k' the set*

$$\Omega = \{\Lambda \in \mathbb{G}(k,n) \mid \dim(\Lambda \cap \Lambda') \geq l\}$$

has codimension $(n-k-k'+l)(l+1)$.

Observe that this is a particular case of a special Schubert cycle, which we will study in the next section.

Proof. Let us consider the incidence variety

$$I = \{(\Lambda, \Lambda'') \in \mathbb{G}(k,n) \times \mathbb{G}(l,k') \mid \Lambda'' \subset \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{G}(l,k'),$$

where of course $\mathbb{G}(l,k')$ represents the l-subspaces $\Lambda'' \subset \Lambda'$. We have two projections $I \xrightarrow{p_1} \mathbb{G}(k,n)$ and $I \xrightarrow{p_2} \mathbb{G}(l,k')$. Observe that the morphism p_2 is surjective and that

$\operatorname{Im} p_1 = \Omega$ because such image is exactly the set of Λ's such that there exists $\Lambda'' \subset \Lambda \cap \Lambda'$. We expect the dimension of Ω to be equal to the dimension of I, because p_1 is a finite morphism. We know, being p_2 surjective, that the dimension of I is equal to the dimension of the base plus the dimension of the fiber, i.e. $\dim I = \dim \mathbb{G}(l, k') + \dim$ fiber, where $\dim \mathbb{G}(l, k') = (l+1)(k'-l)$. The wanted fiber is isomorphic to $\mathbb{G}(k-1-l, n-1-l)$; indeed, having $\dim \Lambda'' = l$, we manage to find a linear subspace $\mathbb{P}^{n-1-l} \subset \mathbb{P}^n$ disjoint from Λ'' and, in order to raise the dimension till the wanted one $k = \dim \Lambda$, we need to consider a subspace $S \subset \mathbb{P}^{n-1-l}$, such that $\dim S = k-1-l$, and consider the linear span $< \Lambda'', S >$, which has the requested dimension.
This tells us that $\dim I = (l+1)(k'-l) + (k-l)(n-k)$ and moreover $\Omega = p_1(I)$ is irreducible because I irreducible, being all the fibers of p_2 irreducible. Observing that

$$(k+1)(n-k) - (n-k-k'+l)(l+1) = (l+1)(k'-l) + (k-l)(n-k),$$

the expected codimension is equal to the computed one, and we have proved the proposition. □

With the last two results we are able to prove the following theorem.

Theorem 1.1.20. *Let F be a vector bundle of rank r globally generated by $V \subset H^0(X, F)$ and given $s_1, \dots, s_m$ global sections (which correspond to a bundle morphism $X \times \mathbb{K}^m \stackrel{\varphi}{\to} F$), the set*

$$X_k = \{x \in X \mid \operatorname{rk}(s_1(x), \dots, s_m(x)) \leq k\} = \{x \in X \mid \operatorname{rk} \varphi_x \leq k\}$$

has codimension $(m-k)(r-k)$.

Proof. Considering the map

$$\varphi_V : X \longrightarrow \mathbb{G}(r-1, \dim V - 1),$$

we can see $s_1, \dots, s_m$ as sections of $\mathcal{U}^\vee$ and we can rewrite

$$X_k = \varphi^{-1}\left(\{\Lambda \in \mathbb{G}(r-1, \dim V - 1) \mid \dim(\Lambda \cap \Lambda') \geq r-1-k\}\right),$$

where $\Lambda' = V(s_1, \dots, s_m)$. Let us consider the incidence variety

$$I = \{(x, \Lambda') \mid \dim(\varphi_V(x) \cap \Lambda') \geq r-1-k\} \subset X \times \mathbb{G}(k', \dim V - 1)$$

with its natural projections $I \stackrel{p_1}{\to} X$ and $I \stackrel{p_2}{\to} \mathbb{G}(k', \dim V - 1)$; we will rewrite the theorem in terms of this diagram, because the general sets X_k are exactly the fibers of p_2.

We need to compute the dimension of I and we would like it to be

$$\dim I = \dim \mathbb{G}(k', \dim V - 1) + \dim X - (m-k)(r-k),$$

where $\dim X - (m-k)(r-k)$ is the wanted dimension for the fiber. We thus have to study the other projection computing $\dim I = \dim X + \dim p_1^{-1}(x)$, but we already know that $\dim p_1^{-1}(x) = \dim \mathbb{G}(k', \dim V - 1) - (r-k)(\dim V - 1 + r - 1 - k)$, see Proposition 1.1.19, which concludes the proof. $\square$

Let us now consider a vector bundle M of rank $k+1$ over a projective variety X and the associate Grassmannian $\mathbb{G}(k, \mathbb{P}(H^0(M)))$. Moreover, we will consider the hyperplane sections of the Grassmannian and show that a special family of them has preimage divisors of X. Recall that there exist particular hyperplanes H_A of the Grassmannian which can be described as the set of all the elements $\Gamma \in \mathbb{G}(k, \mathbb{P}(H^0(M)))$ that have non empty intersection with a fixed subspace $A \subset \mathbb{P}(H^0(M))$ of codimension $k+1$. The subspace A can be seen as the zero locus of $k+1$ independent sections of the bundle $\mathcal{U}^\vee$, hence the hyperplane H_A can be seen as the locus of points for which the bundle morphism $\mathcal{O}_{\mathbb{G}}^{k+1} \longrightarrow \mathcal{U}^\vee$ has not maximal rank. We can apply the pullback to this construction on the variety X, obtaining

$$\begin{tikzcd} \mathcal{O}_X^{k+1} \arrow[rr] \arrow[dr] & & M \arrow[dl] \\ & X & \end{tikzcd}$$

The locus of points in X for which the map has not maximal rank is a divisor $D \subset X$, whose image is exactly the considered hyperplane. Notice that such divisor is the locus of points whose restriction of the bundle morphism

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr] \arrow[dr] & & \bigwedge^{k+1} M \arrow[dl] \\ & X & \end{tikzcd}$$

vanishes.

1.1.5 Chern classes

In this section we recall the notation and the basic properties of the Chern classes; for a full reference, see for example [Har77].

Consider a projective variety X of dimension n and its Chow ring $A(X) = \bigoplus_{i=0}^{n} A^i$, where A^i is spanned by all the irreducible subvarieties of X of codimension i. The main properties of the Chern classes are recalled in the following proposition.

Proposition 1.1.21.

- *If L is a line bundle, $c_1(L)$ is the class of any divisor associated to it.*

- *Any vector bundle of rank r on X has Chern classes $c_i(F) \in A^i(F)$, for $i = 1, \dots, \min\{r, n\}$. We will call $c_y(F) = 1 + c_1(F)y + \dots c_r(F)y^r$ the* Chern polynomial *of F, using the convention $c_0(F) = 1$ and $c_i(F) = 0$ for $i > r$ or $i > n$.*

- *For any exact sequence $0 \to F' \to F \to F'' \to 0$, we have $c_y(F) = c_y(F')c_y(F'')$.*

- *If F is a rank r vector bundle over X with a section which vanishes in a subvariety $Z \subset X$ of codimension r, then $c_r(F) = [Z]$, i.e. the maximal Chern class is equal to the class of the subvariety in the Chow ring.*

1.1.6 The tangent bundle

Let us conclude this section with an interpretation of the tangent space at a point Λ of the Grassmannian. We have already noticed that the tangent bundle on $\mathbb{G}(k, n)$ can be described as $T_{\mathbb{G}} \simeq \mathcal{U}^\vee \otimes \mathcal{Q}$. Observe that a further way to construct an open cover for $\mathbb{G}(k, n)$ is the following: take any $(n - k)$-plane $\Gamma \subset \mathbb{K}^{n+1}$ and obtain the open set U_Γ, defined as the subset of all linear spaces $\Lambda \subset \mathbb{K}^{n+1}$ complementary to Γ. Such open subsets are isomorphic to the affine space $\mathbb{A}^{(k+1)(n-k)}$ in the following way: fixing any subspace $\Lambda \in U_\Gamma$, any other subspace $\Lambda' \in U_\Gamma$ is the graph of a homomorphism $\varphi : \Lambda \longrightarrow \Gamma$, so that

$$U_\Gamma = \mathrm{Hom}(\Lambda, \Gamma)$$

and such isomorphism induces the one between the tangent spaces

$$T_\Lambda(\mathbb{G}(k, n)) = \mathrm{Hom}(\Lambda, \Gamma).$$

Notice that this is equivalent to define the tangent space as

$$T_\Lambda(\mathbb{G}(k, n)) = \mathrm{Hom}(\Lambda, \mathbb{K}^{n+1}/\Lambda),$$

which is exactly the fiber of $T_{\mathbb{G}}$ over a point $\Lambda \in \mathbb{G}(k, n)$.

1.2 Schubert calculus

We recall the main results of the Schubert calculus, which is the instrument that allows us to know the behavior of the intersections in the Grassmannian. An introduction to such argument can be found in [GH78]. We will define the Schubert cycles, which we will see they are a special subset of the elements of the Grassmannians, and we will discover how such cycles generate the Chow ring of the Grassmannian. Let us start with the following definition.

Definition 1.2.1. *Let $\Omega(\Lambda_0, \ldots, \Lambda_k)$ be a Schubert variety. We will call a Schubert cycle the equivalence class of a Schubert variety under projectivities. We will denote a Schubert cycle by $\Omega(l_0, \ldots, l_k)$, where l_i is the dimension of the linear space Λ_i of the flag which defines the Schubert variety.*
It is possible to prove that $\dim \Omega(l_0, \ldots, l_k) = \sum_{i=0}^{k} (l_i - i)$.

Let us give a look to a particular family of Schubert cycles.

Definition 1.2.2. *We will define a special Schubert cycle as*

$$\sigma_a = \Omega(n-k-a, n-k+1, \ldots, n).$$

Notice that it is the cycle that represents the k-dimensional linear subspaces which have non empty intersection with a fixed linear space of dimension $n-k-a$, for $0 \leq a \leq n-k$.

Example 1.2.3. Let us consider the example of the $\mathbb{G}(1, 3)$, that we have studied in the previous section, and explicit the Schubert division of the variety through its cycle; the cycles taken will give us a cellular decomposition of the variety. Consider the first open subset covering the variety, given by $p_{0,1} \neq 0$, and we obtain a four dimensional cell of points that can be represented by a Plücker matrix of type

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix},$$

where a, b, c and d represent the free entries of the matrix; if we fix such values, then we will have a point of the open subset. We now want to look for the points that do not belong to the cell we just considered. Denoting by $(x_0 : x_1 : x_2 : x_3)$ the coordinates of $\mathbb{P}^3$, we can state that such points are the ones whose correspondent line in the projective space

space has non empty intersection with the subvariety $W := \{x_0 = x_1 = 0\}$. We want to cover with cells the points we have just underlined. Let us consider a three dimensional cell representing the lines in $\mathbb{P}^3$ that intersect W in a point different from $P = (0:0:0:1)$ and that are not entirely contained in the plane $Y := \{x_0 = 0\}$. Such lines can be represented by the Plücker matrix

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix},$$

where, as before, a, b, c represents the free entries of the matrix, giving us the dimension of the cell. We need to consider now the lines that either pass through P or which are contained in Y. Let us notice that the lines, satisfying the first request but not the second one, can be represented by the matrix

$$\begin{pmatrix} 1 & a & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and these are the lines passing through P but not contained in Y. On the other hand the lines contained in Y, but not passing through P, can be represented by

$$\begin{pmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{pmatrix}.$$

Observe that this last two cells are 2-dimensional. We now need to get the lines satisfying both conditions; such lines can be represented by the matrix

$$\begin{pmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can conclude considering the line W, in correspondence with the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have thus completed the study of the cellular decomposition of the Grassmannian $\mathbb{G}(1,3)$, given by Schubert cycles: we have a 4-cell which is $\Omega(2,3)$, one 3-cell which is $\sigma_1 = \Omega(1,3)$, two 2-cells which are $\sigma_2 = \Omega(0,3)$ and $\Omega(1,2)$, one 1-cell which is $\Omega(0,2)$ and finally the 0-cell given by $\Omega(0,1) = W$.

We will now state some results concerning the intersection between Schubert cycles and their role in the Chow ring of $\mathbb{G}(k,n)$, that we will denote by $A(G) = \oplus \mathcal{A}^i(G)$, where $A^i(G)$ represents the cycles of codimension i (see [Arr96]).

Theorem 1.2.4. *For any $i = 0, \dots, (k+1)(n-k)$, the group $A^i(\mathbb{G}(k,n))$ is freely generated by all Schubert cycles $\Omega(l_0, \dots, l_k)$ such that the dimension is the expected one, i.e. $\sum_{j=0}^{k}(l_j - j) = (k+1)(n-k) - i$.*

Theorem 1.2.5 (Pieri's formula)**.** *The intersection product between a Schubert cycle and a special Schubert cycle is given by the following formula:*

$$\Omega(l_0, \dots, l_k).\sigma_a = \sum \Omega(m_0, \dots, m_k),$$

where the sum is taken over all the m_i verifying the conditions $l_{i-1} \leq m_i \leq l_i$ and $\sum m_i = \sum l_i - a$.

Knowing the intersection structure of the cycles we can state the following result.

Theorem 1.2.6. *The Chow ring of a Grassmannian variety is generated by its special Schubert cycles.*

Let us consider our usual example, using the cellular decomposition of the Grassmannian $\mathbb{G}(1,3)$ we have already discribed. In this case we have three special Schubert cycles: $\sigma_0 = \Omega(2,3)$, $\sigma_1 = \Omega(1,3)$ and $\sigma_2 = \Omega(0,3)$. Observe that we obtain $\sigma_1^2 = \Omega(0,3) + \Omega(1,2) = \sigma_2 + \Omega(1,2)$, $\sigma_2^2 = \Omega(0,1)$, $\sigma_1.\sigma_2 = \Omega(0,2)$, and, remembering the given decomposition, it is natural to think and possible to prove that

$$\begin{aligned}
&A^0(G) = \mathbb{Z}\sigma_0 \\
&A^1(G) = \mathbb{Z}\sigma_1 \\
&A^2(G) = \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_1^2 \\
&A^3(G) = \mathbb{Z}(\sigma_1.\sigma_2) \\
&A^4(G) = \mathbb{Z}\sigma_2^2
\end{aligned}$$

Notice that Pieri's formula allowed us to have the result of one intersection with a special Schubert cycle. We wonder what we can say about multiple intersections, concentrating on the particular case when we consider only special Schubert cycles.

Theorem 1.2.7. *The multi-intersection of special Schubert cycles $\sigma_{a_1}.\sigma_{a_2}.\dots.\sigma_{a_s}$, with $s \geq 1$ has always non negative coefficients. Moreover if the codimension given by the intersection does not exceed the dimension of the Grassmannian, there always exists at least one positive coefficient.*

Proof. Recall that a special Schubert cycle σ_i can be defined as

$$\sigma_i = \Omega(n-k-i, n-k+1, \dots, n-1, n),$$

where

$$\Omega(\alpha_1, \dots, \alpha_k) = \left\{ \Lambda \,|\, \dim(\Lambda \cap \Lambda'_j) \geq j, \text{ with} \dim \Lambda'_j = \alpha_j \right\},$$

taking Λ and the Λ'_j's linear subspaces of $\mathbb{P}^n$, with $\dim \Lambda = k$.
Let us see what happens if we intersect two special Schubert cycles σ_a and σ_j.
If $a+j \leq n-k$, then we know that we have at least one element in the intersection, which happens because in this case $\sigma_{a+j} \in \sigma_a.\sigma_j$.
If $a+j > n-k$, we can use the following algorithm. Let us suppose that $n-k < a+j \leq 2(n-k)$ and $a+j = n-k+\beta$, where of course $\beta \leq n-k$. In this case we have that $\Omega(0, n-k+1-\beta, n-k+2, \dots, n) \in \sigma_a.\sigma_j$ and we get a non empty intersection.
If $a+j > 2(n-k)$, we look for more general bounds

$$p(n-k) < a+j \leq (p+1)(n-k)$$

with $a+j = p(n-k)+\beta$ and in this case we have that

$$\Omega(0, 1, \dots, p, n-k+p+1-\beta, \dots, n) \in \sigma_a.\sigma_j,$$

so we always get a non empty intersection as long as we do not go further than the dimension of the Grassmannian.
For the next step we use the same technique. We are going to take the result of our previous general intersection $\Omega(0, 1, \dots, p, n-k+p+1-\beta, \dots, n)$ and intersect it again with another special Schubert cycle σ_q. We only need to check the bounds of $\beta+q$ and iterate the process we have explained before.
We have a finite number of steps (in the longest case we have as many steps as the dimension of the Grassmannian), until we arrive to the cycle $\Omega(0, 1, 2, \dots, k)$.
We have thus proved that all the coefficients of subscript less or equal than the dimension of the Grassmannian must be positive, as the result of several intersections, and that they are zero if we exceed the dimension of the Grassmannian with the codimension of the cycle given by the intersection. □

Corollary 1.2.8. *Let $\mathcal{Q}$ be the universal quotient bundle on $\mathbb{G}(k,n)$. Every coefficient of* *the polynomial $c_y(\mathcal{Q})^s$ is non negative and it is of the form*

$$\alpha_q = \sum \sigma_{p_1}^{\beta_1} \dots \sigma_{p_s}^{\beta_s},$$

with $\sum_{i=1}^{s} p_i \beta_i = q$ and $\sum_{i=1}^{s} \beta_i = s$ for $q \leq (k+1) \dim S$. Moreover, if $q \leq \dim \mathbb{G}(k,n)$, the α_q are all not zero and hence positive.

Proof. Notice that we can write

$$c_y(\mathcal{Q})^s = \left(1 + \sigma_1 y + \sigma_2 y^2 + \dots + \sigma_{n-k} y^{n-k}\right)^s,$$

The coefficients of the polynomial are given by multi-intersection of special Schubert cycles, hence we can conclude by Theorem 1.2.7. $\square$

Chapter 2

Steiner bundles on Grassmannians

In this chapter we will first give an introduction of the problem we are studying, listing the known results, specifically the ones which will be generalized in this work. We will continue setting the definitions and the notations that we will use for Steiner bundles.

2.1 Historical introduction

In [Sch61] Schwarzenberger introduced a family of bundles F of rank n related to the secant space of rational normal curves and defined by a resolution of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^s(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^t \longrightarrow F \longrightarrow 0,$$

which will be called the family of (s,t)-Steiner bundles on the projective space.
In [DK93], Dolgachev and Kapranov consider a particular set of hyperplanes in the projective space in order to construct a family of vector bundles. In fact, they consider $\mathcal{H} = (H_1, \ldots, H_m)$ an arrangement of hyperplanes in $\mathbb{P}^n$, in general position, such that the divisor $\bigcup H_i$ has normal crossing. They call the sheaf $\Omega^1_{\mathbb{P}^n}(\log \mathcal{H})$ of the differential 1-forms, with logarithmic poles along $\mathcal{H}$, a *logarithmic bundle*, which they denote by $F(\mathcal{H})$. In the second section of this work, after introducing the concept of Steiner bundle, the authors give a criterion which allows us to relate Steiner and logarithmic bundles ([DK93]-3.5).

Theorem 2.1.1. *Let $\mathcal{H}$ be an arrangement of m hyperplanes in general position in $\mathbb{P}^n$. If $m \geq n+2$, then the logarithmic bundle $F(\mathcal{H})$ is a Steiner bundle over $\mathbb{P}^n$.*

The authors also prove a result which gives a lower bound for the possible ranks of a Steiner vector bundle over the projective space ([DK93]-3.9).

Proposition 2.1.2. *The rank of a non-trivial Steiner bundle on $\mathbb{P}^n$ is greater or equal than n.*

Let us recall the classical definition of a Schwarzenberger bundle (see [Sch61], [DK93] and [Val00b]), which is associated to a rational normal curve $C_n \subset (\mathbb{P}^n)^*$. Consider a two dimensional vector space U on $\mathbb{K}$. Denote by $\mathcal{S}_i = \mathrm{Sym}^i U$ the symmetric powers of U, by $(X_0, \dots, X_n)$ the coordinates of $\mathbb{P}^n = \mathbb{P}(\mathcal{S}_n^*)$ and by $C_n \subset (\mathbb{P}^n)^* = \mathbb{P}(\mathcal{S}_n^*)^*$ the image of $\mathbb{P}(\mathcal{S}_1)$ given by the Veronese embedding.

Definition 2.1.3. *For each integer m, with $m \geq n$, the Steiner bundle $F_m(C_n)$ defined by the following resolution*

$$0 \longrightarrow \mathcal{S}_{m-n} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{M} \mathcal{S}_m \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F_m(C_n) \longrightarrow 0$$

where the transpose of M is given by

$$M^t = \begin{pmatrix} X_0 & X_1 & \cdots & X_n & 0 & \cdots & \cdots & 0 \\ 0 & X_0 & X_1 & \cdots & X_n & 0 & & \vdots \\ \vdots & 0 & X_0 & X_1 & \cdots & X_n & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & X_0 & X_1 & \cdots & X_n \end{pmatrix}$$

is called a Schwarzenberger bundle associated to the rational normal curve C_n.

Dolgachev and Kapranov prove then a list of results that, in the same spirit of Theorem 2.1.1, link different families of vector bundles. Some of the following results will be generalized in this thesis; indeed, we will be concerned with investigating the relations between the two families of Steiner and Schwarzenberger bundles, but in the much more general case of bundles over Grassmannians.

Theorem 2.1.4. *Let $\mathcal{H} = (H_1, \dots, H_m)$ be an arrangement of m hyperplanes of $\mathbb{P}^n$ in general position. Suppose that all H_i, considered as points of $(\mathbb{P}^n)^*$, lie in a rational normal curve of the dual projective space, then the logarithmic bundle $F(\mathcal{H})$ is also a Schwarzenberger bundle.*

Theorem 2.1.5. *Any Steiner bundle F on $\mathbb{P}^n$, of rank n and with $s = 2$, is a Schwarzenberger bundle.*

Moreover, Dolgachev and Kapranov prove a Torelli type theorem ([DK93]-7.2) where, under specific hypothesis, it is possible to recover the rational normal curve from the given logarithmic bundle. In [Val00b], Vallès generalizes this last result, proving the following theorem ([Val00b]-3.1).

Theorem 2.1.6. *Let F be a Steiner bundle over $\mathbb{P}^n$, with $\operatorname{rk} F = n$, and $H_1, \dots, H_{n+s+2}$ distinct hyperplanes, such that, for each i, $h^0(F^\vee_{H_i}) \neq 0$. Then it exists a rational normal curve $C_n \subset (\mathbb{P}^n)^*$ such that $H_i \in C_n$, for $i = 1, \dots, n+s+2$, and F is the Schwarzenberger bundle associated to the rational normal curve.*

In [AO01] , Ancona and Ottaviani reinforce the importance of the particular family of hyperplanes $W(F) = \{H \in (\mathbb{P}^n)^* | h^0(F^\vee_H) \neq 0\}$, which Vallés already considered and which is called the family of *unstable hyperplanes*, proving the following result ([AO01]-5.11).

Theorem 2.1.7. *Let F be a rank n Steiner bundle over $\mathbb{P}^n$. If $W(F)$ contains at least $n+s+1$ hyperplanes, then for every subset $\mathcal{H} \subset W(F)$ consisting of $n+s+1$ hyperplanes $F \simeq \Omega(\log \mathcal{H})$, i.e. F is also a logarithmic bundle.*

Notice that all the results stated are given under the hypothesis that the rank of the Steiner bundle, over $\mathbb{P}^n$, is equal to n.
In [Arr10a], Arrondo gives the generalization of the definition of Schwarzenberger bundle, which we will recall in Section 3.1. Moreover, he introduces new objects, called *jumping pairs* and *jumping hyperplanes*, which are meant to substitute the idea of unstable hyperplane considered in the previous works. Through the study of the locus of such pairs, Arrondo gives the classification of the Steiner bundles (with arbitrary rank) which have jumping locus of maximal dimension. The classification is listed in the following result ([Arr10a]-3.7).

Theorem 2.1.8. *Let F be a Steiner bundle over $\mathbb{P}^n$ with $s \geq 2$ and such that the locus of the jumping pairs has maximal dimension; then F is a Schwarzenberger bundle, defined by the choice of one of the following triples:*

- $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$,
- $(\mathbb{P}^1, E(-1), \mathcal{O}_{\mathbb{P}^1}(1))$, *where* $E = \oplus_{i=1}^{t-s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ *with all* $a_i \geq 1$ *and* $\sum_{i=1}^{t-s} a_i = s$,
- $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), E'(-1))$, *where* $E' = \oplus_{i=1}^{t-n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$ *with all* $a_i \geq 1$ *and* $\sum_{i=1}^{t-n-1} a_i = n+1$,

- $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$.

Arrondo's article represents the starting point of our research; in fact, we will try to generalize the constructions he presented, passing from projective spaces to Grassmannians. We will always be careful to recover his results from the general statements we will present.
The most general definition of a Steiner bundle has been given in [Soa08] and [MRS09] by Rosa Maria Miró-Roig and Helena Soares. They consider the following setting (recall Definition 1.1.8).

Definition 2.1.9. *A vector bundle E on a smooth irreducible algebraic variety X is called a Steiner vector bundle if it is defined by an exact sequence of the form*

$$0 \longrightarrow F_0^s \longrightarrow F_1^t \longrightarrow E \longrightarrow 0,$$

where $s, t \geq 1$ and (F_0, F_1) is an ordered pair of vector bundles satisfying the two following conditions:

(i) *(F_0, F_1) is strongly exceptional;*

(ii) *$F_0^\vee \otimes F_1$ is generated by its global sections.*

Let us remark the importance of the choice of the strongly exceptional pair (F_0, F_1), which is not unique for a fixed algebraic variety X.

2.2 Definitions and first properties

In this section, we will set the notation that will be used throughout the rest of this chapter and we will continue by giving the notions that we will take as definitions and the properties that we need.
Let $\mathbb{K}$ be an algebraically closed field with $\operatorname{char} \mathbb{K} = 0$ and let V be a vector space over $\mathbb{K}$. Let us construct the projective space $\mathbb{P}^n = \mathbb{P}(V)$ as the equivalence classes of the hyperplanes of V, or equivalently the equivalence classes of the lines of the dual vector space V^*. So our projective Grassmannian will be given as

$$\mathbb{G}(k, \mathbb{P}(V)) = \mathbb{G}(k, n) = G(k+1, V^*),$$

the set of the $(k+1)$-dimensional linear subspaces of V^*.
Throughout this work, a morphism between vector bundles, when not specified, will always

refer to a bundle map.
Recalling Definition 2.1.9 we notice that in order to have a Steiner bundle we need to find a strongly exceptional pair on the Grassmannian. We have already proved in Proposition 1.1.9 that the ordered pair $(\mathcal{U}, \mathcal{O}_{\mathbb{G}})$, where $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}(k,n)}$ is the trivial bundle and $\mathcal{U} \to \mathbb{G}(k,n)$ is the universal bundle of rank $k+1$, and the ordered pair $(\mathcal{O}_{\mathbb{G}}, \mathcal{Q})$, where $\mathcal{Q} \to \mathbb{G}(k,n)$ is the universal quotient bundle of rank $n-k$, are strongly exceptional pairs. Moreover, $\mathcal{U}^\vee$ and $\mathcal{Q}$ are globally generated by their global sections, so the two pairs define a Steiner bundle. We will take as our definition the bundle defined by the first pair, since it is the natural generalization of the classical Steiner bundles on the projective space; however sometimes we will also use the second pair. We are ready to present one of our main definitions.

Definition 2.2.1. *Let S, T be two vector spaces over $\mathbb{K}$, respectively s and t-dimensional. We will call an (s,t)-Steiner bundle, over $\mathbb{G}(k,n)$, the vector bundle defined by the resolution*

$$0 \longrightarrow S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0.$$

It is trivial to determine the rank of a Steiner bundle, which is

$$\operatorname{rk} F = t - s(k+1).$$

We will discuss later in Section 2.3 the problem of the possible ranks of a Steiner bundle.

Geometrical interpretation of Steiner bundles

An injective morphism

$$\mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}}$$

is equivalent to fix a $(n+1)$-codimensional space in $\mathbb{P}(T)$. Indeed we have a surjective morphism

$$T^* \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{U}^\vee$$

and the induced map on the global sections is

$$T^* \xrightarrow{f} \mathbb{K}^{n+1}$$

which we suppose to be also surjective.
So $\ker f$ has codimension $n+1$, which gives us the requested subspace $\Lambda \subset \mathbb{P}(T)$ of codimension $n+1$ in the projective space.

The hyperplanes of $\mathbb{P}(T)$ that contain Λ are in a one-to-one correspondence with the points of a linear subspace of maximal dimension, disjoint from Λ and that generates with Λ all $\mathbb{P}(T)$. Due to the codimension we can build a one-to-one correspondence between such hyperplanes and the ambient space $\mathbb{P}^n$.
In the same way each injective map

$$S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}}$$

defines s subspaces $\Lambda_1, \ldots, \Lambda_s \subset \mathbb{P}(T)$ of codimension $n+1$. As before we can build several correspondences in order to have

$$\mathbb{P}(T)^*_{\Lambda_i} := \{\text{hyperplanes of } \mathbb{P}(T) \text{ that contain } \Lambda_i\} \simeq \mathbb{P}^n,$$

for each i from 1 to s. In order to describe the projectivization of a fiber of F we need to fix a point $\Gamma \in \mathbb{G}(k,n)$. Let us observe that every fixed point Γ gives us a subspace, one for each i, of codimension $k+1$ in $\mathbb{P}(T)$, that contains Λ_i and described by the intersection of the $k+1$ hyperplanes of $\mathbb{P}(T)^*_{\Lambda_i}$ given by the $k+1$ independent points that span the vector subspace in correspondence with Γ.
As a consequence of what discussed previously, we have the following equivalences

$$A_i = \{\text{subspaces of codimension } k+1 \text{ in } \mathbb{P}(T) \text{ that contains } \Lambda_i\} \simeq \mathbb{G}(k,n),$$

for every $i = 1, \ldots, s$.
For every $\Gamma \in \mathbb{G}(k,n)$ we have s linear subspaces, one for each A_i. The projectivization of the fiber F_Γ is given by the intersection of such s spaces. We notice that even if we are considering linear subspace whose intersection may be empty, we can assure that we have a non empty space for each fiber due to the fact that we are starting with a bundle.

Example 2.2.2. Let us consider the case $k = 0, n = 1$ and $t = s+1$. This gives us the following resolution

$$0 \longrightarrow \mathcal{O}^s_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^{s+1}_{\mathbb{P}^1} \longrightarrow F \longrightarrow 0.$$

In this case we have that the projectivization of a fiber is given by the intersection of s hyperplanes in $\mathbb{P}^s$ and for every point such fiber is non empty. Considering all the points and the respective fibers we get the rational normal curve $\mathcal{C}_s \subset \mathbb{P}^s$, indeed, $F = \mathcal{O}_{\mathbb{P}^1}(s)$.

Some properties of Steiner bundles

A very useful tool in the study of Steiner bundles is represented by the following lemma, which gives us an equivalent definition that only concerns linear algebra. In this work, when we will consider Steiner bundles, we will refer to any of the two proposed definitions.

Lemma 2.2.3. *Given S and T two vector spaces over $\mathbb{K}$, the followings are equivalent:*

(i) *a Steiner bundle F on $\mathbb{G}(k,n)$ given by the resolution*

$$0 \longrightarrow S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0$$

(ii) *a linear application*

$$T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee) = \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$$

such that, for every $u_1, \dots, u_{k+1} \in H^0(\mathcal{U}^\vee)^*$ linearly independent and for every $v_1, \dots, v_{k+1} \in S^*$ (no further hypothesis), there exists an $f \in \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$ such that $f \in \operatorname{Im}\varphi$ and $f(u_j) = v_j$ for each $j = 1, \dots, k+1$.

Proof. Let us consider the map

$$S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}}$$

and its dual one

$$\psi : T^* \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow S^* \otimes \mathcal{U}^\vee = \mathcal{H}om(\mathcal{U}, S^* \otimes \mathcal{O}_{\mathbb{G}})$$

that gives us the induced map on the global sections

$$\varphi : T^* \longrightarrow S^* \otimes H^0(\mathcal{U}^\vee) = \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*).$$

We need to characterize φ in order to have the map ψ surjective, that is equivalent to ask for ψ to be surjective in each fiber.
A point $\Gamma \in \mathbb{G}(k,n)$ is in correspondence with $k+1$ independent vectors $u_1, \dots, u_{k+1}$ in $H^0(\mathcal{U}^\vee)^*$, so that the bundle morphism in the fiber associated to Γ corresponds to the restriction of φ of the type

$$\tilde{\varphi} : T^* \longrightarrow \operatorname{Hom}(< u_1, \dots, u_k + 1 >, S^*).$$

The requested characterization will be exactly the one stated in point (ii), because $\tilde{\varphi}$ is surjective for every fiber if and only if for every $k+1$ independent vectors of $H^0(\mathcal{U}^\vee)^*$, related with the fixed point in the Grassmannian, and $k+1$ vectors of S^*, that can be chosen as image of the previous set, there exists $f \in \mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$ with $f \in \mathrm{Im}\,\varphi$ such that $f(u_j) = v_j$ for every j from 1 to $k+1$. □

2.2.1 Steiner bundles and linear algebra

In this paragraph we will prove some general results of linear algebra, that, because of what we proved in Lemma 2.2.3, we will be able to apply to our specific setting of Steiner bundles. Let us introduce the notation we need.
Let U and V be two vector spaces, of dimension respectively r and s.
Let $W \subset \mathrm{Hom}(U, V)$ be the vector subspace characterized by the following property, which we will denote P_k: for every $k+1$ independent vectors $\tilde{u}_1, \ldots, \tilde{u}_{k+1} \in U$ and for every $\tilde{v}_1, \ldots, \tilde{v}_{k+1} \in V$ there exists an element $f \in W$ such that $f(\tilde{u}_j) = \tilde{v}_j$, for every $j = 1, \ldots, k+1$. This is equivalent to ask that for every vector subspace $U' \subset U$ with $\dim U' = k+1$, we have a diagram

$$\begin{tikzcd} W \arrow[d, hook] \arrow[dr, "\alpha"] & \\ \mathrm{Hom}(U,V) \arrow[r, two heads] & \mathrm{Hom}(U',V) \end{tikzcd}$$

where the induced map α will always be surjective.
We can immediately prove the following lemma.

Lemma 2.2.4. *Let $W \subset \mathrm{Hom}(U, V)$ be a vector subspace that satisfies the property P_k. If $\dim V = k+1$, then the map $W \longrightarrow Hom(U, V)$ is also surjective.*

Proof. Consider an element in $U^* \otimes V$, that we can also write as a linear combination $\sum \lambda_{i,j} u_i^* \otimes v_j$, given by the choice of a basis for each of the vector spaces U^* and V. Collect all the elements of the combination using the basis of V, in order to write the combination as

$$\tilde{u_1}^* \otimes v_1 + \ldots + \tilde{u}_{k+1}^* \otimes v_{k+1}.$$

Notice that having at most $k+1$ independent $\tilde{u_i}^*$ and completing a basis of U^* with independent vectors that must vanish because the rank of the chosen element is determined, we can see the previous sum as an element of W. □

We can apply the previous lemma to prove the following result

Lemma 2.2.5. *Let $W \subset \mathrm{Hom}(U, V)$ be a vector subspace that satisfies the property P_k. Then, considering the induced map*

$$W \longrightarrow \mathrm{Hom}(V^*, U^*),$$

we have that for every $k+1$ independent vectors $\tilde{v}_1^*, \dots, \tilde{v}_{k+1}^* \in V^*$ and for every $\tilde{u}_1^*, \dots, \tilde{u}_{k+1}^* \in U^*$ there exists an element $\bar{f} \in W$ such that $\bar{f}(\tilde{v}_j^*) = \tilde{u}_j^*$, for every $j = 1, \dots, k+1$. We say that W also satisfy the reciprocal of the property P_k.

Proof. Let us consider an arbitrary $(k+1)$-dimensional quotient Q of the vector space V, so we are able to construct the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[rr, bend left, "f"] \arrow[dr, "\varphi'"'] \arrow[drr, bend right, "g"'] & \mathrm{Hom}(U,V) \arrow[r, two heads] & \mathrm{Hom}(U,Q) \arrow[d, "\simeq"] \\
& \mathrm{Hom}(V^*,U^*) \arrow[r, two heads] & \mathrm{Hom}(Q^*,U^*)
\end{tikzcd}$$

We know the map f to be surjective by Lemma 2.2.4, hence we have that for every $Q^* \subset V^*$ with $\dim Q = k+1$ the map g is also surjective. This means that the map φ' makes W satisfy also the reciprocal of the property P_k, because every morphism induced by φ' given by the restriction of V^* to a $(k+1)$-dimensional subspace is surjective. $\square$

Let us now apply what we have proved to our particular case.

Remark 2.2.6. By the description of the property P_k, we have that P_k implies P_i, for each $i \leq k$. Moreover, Lemma 2.2.3 and Lemma 2.2.5 tell us that having a Steiner bundle F on $\mathbb{G}(k, n)$ is equivalent to have a Steiner bundle $\tilde{F}$ on $\mathbb{G}(k, \mathbb{P}(S))$. Therefore considering a Steiner bundle actually means considering a family of $2(k+1)$ Steiner bundles.

We will now prove a lemma which will allow us to define the concept of *reduced Steiner bundle* on $\mathbb{G}(k, n)$. Such result represents a generalization of Lemma 1.3 in [Arr10a].

Lemma 2.2.7. *With the usual notation, the followings are equivalent:*

(i) *a linear subspace $K \subset T^*$ contained in the kernel of φ,*

(ii) *an epimorphism* $F \longrightarrow K^* \otimes \mathcal{O}_{\mathbb{G}}$,

(iii) *a splitting* $F = F' \oplus (K^* \otimes \mathcal{O}_{\mathbb{G}})$.

Proof.

(ii) $\Leftrightarrow$ (iii)
Let us observe that F is generated by its global sections (that is because we have a surjective map from a vector space tensor the canonical bundle to it). Let us consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S \otimes \mathcal{U} \arrow[r] & T \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] \arrow[dr] & F \arrow[r] \arrow[d, "f"] & 0 \\
 & & & K^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

The fact that F is generated by its global sections tells us that a basis will be sent to a basis, so, let us take a basis of $H^0(F)$ and suppose that f is surjective. What we need to do is to take a basis of $K^* \otimes \mathcal{O}_{\mathbb{G}}$ and take its preimage in F. We can complete the basis and with the elements added we can generate a vector bundle F' that gives us the splitting.
If we already know we have a splitting, of course the map f is surjective.

(i) $\Rightarrow$ (ii)

If (i) is true, then, because of the fact that $K \subset \ker \varphi$, we have a morphism of the form $\bar{\varphi} : T^*/K \longrightarrow S^* \otimes H^0(\mathcal{U}^\vee)$. We know well by now that such morphism is associated to a Steiner bundle F'. This gives us the following commutative diagram, that we can construct starting from the two rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S \otimes \mathcal{U} \arrow[r] \arrow[d, equal] & (T^*/K)^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] \arrow[d, hook, "\alpha"] & F' \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & S \otimes \mathcal{U} \arrow[r] & T \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\gamma"] & 0 \\
 & & K^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[r, equal] \arrow[d] & K^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The morphism α is injective by hypothesis and it gives us that also β is injective. Because of the commutativity of the diagram, the map γ must be surjective. We thus have (ii).

(ii) $\Rightarrow$ (i)

By hypothesis we have an epimorphism $F \longrightarrow K^* \otimes \mathcal{O}_{\mathbb{G}}$ and taking the resolution of F we can construct a further surjective morphism g in the following way:

$$\begin{tikzcd}
\cdots \arrow[r] & T \otimes \mathcal{O}_{\mathbb{G}} \arrow[r, two heads] \arrow[dr, "g"] & F \arrow[r] \arrow[d] & 0 \\
 & & K^* \otimes \mathcal{O}_{\mathbb{G}} &
\end{tikzcd}$$

Having such surjective morphism g tells us that $K \subset T^*$ must be a vector subspace. Starting from the dual of the map g we are able to construct another commutative diagram, this time starting from the two columns. We get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \otimes \mathcal{O}_{\mathbb{G}} \arrow[r, equal] \arrow[d] & K \otimes \mathcal{O}_{\mathbb{G}} \arrow[d] & & \\
0 \arrow[r] & F^{\vee} \arrow[r] \arrow[d] & T^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[r, "\varphi"] \arrow[d] & S^* \otimes \mathcal{U}^{\vee} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (F')^{\vee} \arrow[r] \arrow[d] & (T^*/K) \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] \arrow[d] & S^* \otimes \mathcal{U}^{\vee} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Let us observe that the bundle morphism $\varphi : T^* \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow S^* \otimes \mathcal{U}^{\vee}$ has a factorization through $(T^*/K) \otimes \mathcal{O}_{\mathbb{G}}$. This means that K is contained in the kernel of φ and we get (i). $\square$

Remark 2.2.8. The bundle F', by Lemma 2.2.3, is the Steiner bundle associated to the map $T^*/K \longrightarrow S^* \otimes H^0(\mathcal{U}^{\vee})$. If we consider $T_0^* = \operatorname{Im}\varphi$, we obtain an inclusion $T_0^* \hookrightarrow S^* \otimes H^0(\mathcal{U}^{\vee})$ associated to a Steiner bundle F_0. If we take a look at the long exact sequence of the dual of the resolution of F_0, we get $H^0(F_0^{\vee}) = 0$ and $F = F_0 \oplus (T/T_0 \otimes \mathcal{O}_{\mathbb{G}})$. In particular $H^0(F_0^{\vee}) = 0$ if and only if φ is injective.

All the Steiner bundles satisfying the property stated in the previous remark form a particular subset.

Definition 2.2.9. *Using the usual notation, a Steiner bundle over $\mathbb{G}(k,n)$ is said to be reduced if $H^0(F^{\vee}) = 0$. In general, we will denote by F_0 the reduced summand of a Steiner bundle.*

2.3 The classification of the case $1 \leq \dim S \leq k+1$ and the rank limit

In this section we will prove a classification theorem for (s,t)-Steiner bundles with $s \leq k+1$ and then we will show which are the possible ranks for a Steiner bundle over $\mathbb{G}(k,n)$. The solution of this last problem for projective spaces was given by Dolgachev and Kapranov

in [DK93], see Proposition 2.1.2, where they prove that the rank of a non trivial Steiner bundle over $\mathbb{P}^n$ is at least n.
We have the following result.

Theorem 2.3.1. *Let F be an (s,t)-Steiner bundle on $\mathbb{G}(k,n)$ as defined in 2.2.1, with $s \leq k+1$. Then, if F is reduced, it will be of the type $F = S \otimes \mathcal{Q}$; if it is not it will be of the type $F = (S \otimes \mathcal{Q}) \oplus \mathcal{O}_{\mathbb{G}}^p$ for some $p > 0$.*

Proof. From Remark 2.2.8, it is enough to prove the result for reduced Steiner bundles; and, in this case, we need to show that the injective map $T^* \to S^* \otimes H^0(\mathcal{U}^\vee)$ is also surjective. Consider the following commutative diagram, constructed starting from the dual of the sequence that defines the bundle,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F^\vee \arrow[r] \arrow[d] & T^* \otimes \mathcal{O}_{\mathbb{G}(k,n)} \arrow[r] \arrow[d] & S^* \otimes \mathcal{U}^\vee \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & S^* \otimes \mathcal{Q}^\vee \arrow[r] \arrow[d] & S^* \otimes H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{G}(k,n)} \arrow[r] \arrow[d] & S^* \otimes \mathcal{U}^\vee \arrow[r] & 0 \\
 & \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{G}(k,n)} \arrow[r, "\simeq"] \arrow[d] & \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{G}(k,n)} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Observe that, if $T^* \neq S^* \otimes H^0(\mathcal{U}^\vee)$, the given diagram allows us to have an injective morphism of type

$$\mathcal{O}_{\mathbb{G}} \longrightarrow (\mathcal{Q})^s \simeq S \otimes \mathcal{Q}.$$

If $s \leq k+1$ this is impossible; indeed, as we observed in Example 1.1.17, a global section of $\mathcal{Q}$ vanishes in all $\Lambda \in \mathbb{G}(k,n)$ passing through a fixed point of $\mathbb{P}^n$. Considering s sections of $\mathcal{Q}$ means considering all Λ's passing through s independent points of $\mathbb{P}^n$. If $s \leq k+1$ then we have at least one element with such property. Hence $T^* = S^* \otimes H^0(U^\vee)$, which completes the proof. $\square$

Remark 2.3.2. Observe that, from the previous diagram, the Steiner bundle F can be also defined considering the strongly exceptional pair $(\mathcal{O}_{\mathbb{G}}, \mathcal{Q})$. This proves that all the theory presented in this work does not depend on the choice of the strongly exceptional pair, as we pointed out at the beginning of Section 2.2.

Theorem 2.3.3. *Let F be a Steiner bundle over $\mathbb{G}(k,n)$; then it will have rank*

$$\operatorname{rk} F \geq \min((k+1)(n-k),(n-k)\dim S).$$

Proof. Let us consider the morphism

$$S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \dashrightarrow F, \tag{2.1}$$

and let us write $r = \operatorname{rk} F$. We already know that $r = \dim T - (k+1)\dim S$. We want to discover for which points of the Grassmannian such morphism is not injective and in order to do so we will use Porteous' formula (see for example [ACGH85] for a reference). The expected codimension of such set, given by the formula, is equal to $r+1$, so if we have $\dim \mathbb{G}(k,n) < r+1$ then there will be no points that drop the rank of the morphism and we can ensure its injectivity. In the case the set is not empty, let us compute the fundamental class obtained applying Porteous: if the class is empty, we will still ensure surjectivity and thus have a Steiner bundle, otherwise, we will not.
The fundamental class can be computed as

$$\Delta_{r+1,1}\left(c_y\left(T \otimes \mathcal{O}_{\mathbb{G}} - S \otimes \mathcal{U}\right)\right) = \Delta_{r+1,1}\left(c_y\left(\frac{1}{S \otimes \mathcal{U}}\right)\right) = \Delta_{r+1,1}\left(c_y\left(S \otimes \mathcal{Q}\right)\right),$$

where c_y denotes the Chern polynomial of the bundle and, following the notation of [ACGH85], the symbol $\Delta_{r+1,1}$ just denotes the coefficient of the term of degree $r+1$ in the polynomial.
We thus need to describe the coefficients of

$$c_y(\mathcal{Q})^{\dim S^*} = \left(1 + \sigma_1 y + \sigma_2 y^2 + \ldots + \sigma_{n-k} y^{n-k}\right)^{\dim S^*},$$

where each Chern class $c_i(\mathcal{Q})$ is equal to σ_i, the special Schubert cycle of codimension i for the Grassmannian $\mathbb{G}(k,n)$. Let us observe that, having a power of the polynomial, we need to know how the Schubert cycles intersect among each other and the solution is given by Corollary 1.2.8. Recall that the corollary tells us that if we do not exceed the dimension of the Grassmannian in the Chow ring, the intersection of special Schubert cycles is always non empty and we prove the theorem. ∎

Chapter 3

Jumping locus of a Steiner bundle

In this chapter we will recall the definition of a Schwarzenberger bundle for the case of the projective space, the one proposed in [Arr10a], noticing that its natural generalization is straightforward for the Grassmannian case, but paying attention to the fact that with the new definition it will be very hard to find examples.
We will notice immediately, from the definition itself, that a Schwarzenberger bundle is a particular case of a Steiner bundle and we will wonder if we can say something about the other way around.
Question A *When is a Steiner bundle F also a Schwarzenberger bundle?*
In order to answer to such question we will introduce the concept of *jumping pair* for a Steiner bundle, which will represent the link between the two considered families.

3.1 Schwarzenberger bundles on Grassmannians

Let X be a projective variety and L, M be globally generated vector bundles on X. Let us take M such that $h^0(M) = n+1$, so that we can identify $\mathbb{P}^n \simeq \mathbb{P}(H^0(M)^*)$. Consider the natural composition

$$H^0(L) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow H^0(L) \otimes H^0(M) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{P}^n}$$

constructed in the following way. For each $\sigma \in H^0(M)$ we associate the point $[\sigma] \in \mathbb{P}^n$, given by the above identification, and the fiber map over such point is

$$H^0(L) \otimes <\sigma> \longrightarrow H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M).$$

Identifying $H^0(L)\otimes <\sigma>$ with $H^0(L)$ (we have just fixed one global section), we have that the composition is injective because it is simply

$$H^0(L) \xrightarrow{\cdot\sigma} H^0(L\otimes M)$$

given by the moltiplication with the global section.
We manage to construct a Steiner bundle of the form

$$0 \longrightarrow H^0(L)\otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow H^0(L\otimes M)\otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F \longrightarrow 0.$$

The generalization seems very natural, because fixing a point in the Grassmannian should correspond to fixing $k+1$ independent global sections and apply the multiplication with their linear span. The difficulty of finding examples is given by the fact that taking $k+1$ sections breaks very easily the injectivity of the map, given by the multiplication through elements in its span, which is necessary to have a Steiner bundle.
To generalize a Schwarzenberger bundle in the case of Grassmannians we must construct the following setting.
Let us consider two globally generated vector bundles L, M over a projective variety X, with $h^0(M) = n+1$ and with the identification $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. The Schwarzenberger bundle on $\mathbb{G}(k,n)$ associated to the triple (X, L, M) will be the bundle defined as the cokernel of the composition

$$H^0(L)\otimes\mathcal{U} \longrightarrow H^0(L)\otimes H^0(M)\otimes\mathcal{O}_{\mathbb{G}} \longrightarrow H^0(L\otimes M)\otimes\mathcal{O}_{\mathbb{G}}$$

for which we need to require the injectivity in each fiber, in order to have a resolution.
This is equivalent to fix $k+1$ independent global sections $\{\sigma_1,\dots,\sigma_{k+1}\}$ in $H^0(M)$ in correspondence to the point $\Gamma = [<\sigma_1,\dots,\sigma_{k+1}>] \in \mathbb{G}(k,n)$ and require the injectivity of the following composition, given by the multiplication with the global section subspace we fixed

$$H^0(L)\otimes <\sigma_1,\dots,\sigma_{k+1}> \longrightarrow H^0(L)\otimes H^0(M)\otimes\mathcal{O}_{\mathbb{G}} \longrightarrow H^0(L\otimes M)\otimes\mathcal{O}_{\mathbb{G}}.$$

Definition 3.1.1. *Let X be a projective variety and let L and M be globally generated vector bundle over X, with $h^0(M) = n+1$.*
The bundle $F = F(X, L, M)$ defined by the resolution

$$0 \longrightarrow H^0(L)\otimes\mathcal{U} \longrightarrow H^0(L\otimes M)\otimes\mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0$$

is called a Schwarzenberger bundle over $\mathbb{G}(k,n)$.

The map φ of Lemma 2.2.3 will be the dual of the multiplication map

$$H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M).$$

In particular F is reduced if and only if the multiplication map is surjective.
We give an example to show that a big family of Schwarzenberger bundle on the projective space, one of the most important in Arrondo's classification (see [Arr10a]), is not valid anymore in the Grassmannian case.

Example 3.1.2. Consider $X = \mathbb{P}^n$, with $n \geq 2$, $L = \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^n}(a_i)$, with $a_i \geq 0$ and $M = \mathcal{O}_{\mathbb{P}^n}(1)$. The triple (X, L, M) gives a Schwarzenberger bundle on $\mathbb{P}^n$ but we will not be able to get one on the Grassmannians, because if we restrict $H^0(M)$ to a subspace of dimension $k+1$, the multiplication fails to be injective.

Nevertheless we manage to find an example which ensures that the definition is correct.

Lemma 3.1.3. *Consider a triple* (X, L, M) *such that* L *and* M *are globally generated vector bundles of rank respectively 1 and* $k+1$ *over a projective variety* X*, with* $\dim X \geq k+1$, $h^0(M) = k+2$ *and* $c_{k+1}(M) \neq 0$. *Then, the triple defines a Schwarzenberger bundle on* $\mathbb{G}(k, k+1)$.

Proof. Notice that for each $\Delta \subset H^0(M)$ of dimension $k+1$, the map $\Delta \otimes \mathcal{O}_X \longrightarrow M$ is injective seen as a morphism of sheaves. Suppose that it is not and take $s_1 \dots, s_{k+1}$ generators of the subspace, then we will have that for each $x \in X$ the vectors given by $s_1(x), \dots, s_{k+1}(x)$ are linearly dependent. Consider a further section $s_{k+2} \in H^0(M)$ which completes a basis for the space of the global sections, we know that its zero locus is non empty by the assumptions on the dimension of X and on $c_{k+1}(M)$. Taking a point of this locus, we obtain that the evaluation map $H^0(M) \otimes \mathcal{O}_X \longrightarrow M$ cannot be surjective, which leads to contradiction because we have supposed that M is globally generated.
So, tensorizing the morphism with L, we obtain an injective morphism of sheaves $L \otimes \Delta \otimes \mathcal{O}_X \longrightarrow L \otimes M$ and, taking global sections, we get an injective linear map

$$H^0(L) \otimes \Delta \longrightarrow H^0(L \otimes M),$$

which ensures we have a Schwarzenberger bundle. □

Remark 3.1.4. Notice that the hypothesis $\dim X \geq k+1$ and $c_{k+1}(M) \neq 0$ are necessary. Indeed, if we consider $X = \mathbb{P}^1$ and $M = \mathcal{O}_{\mathbb{P}^1}(1) \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}$, and taking Δ generated by the two independent global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ and $k-1$ independent sections given by the other summands, then the morphism of sheaves $\Delta \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}$ is not injective.

A particular case of the considered family is given by taking the triple $(X, L, M) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1), T_{\mathbb{P}^k}(-1))$. This example will be of extreme importance in the classification, as we will see in Section 4.1.

Geometrical interpretation of Schwarzenberger bundles

Consider the Schwarzenberger bundle $F = F(X, L, M)$. Moreover, let us consider a point $\Gamma \in \mathbb{G}(k, n)$ which is associated to a $(k+1)$-dimensional subspace Δ of independent global sections of M, and hence to a subspace $A \subset \mathbb{P}(H^0(M))$ of codimension $k+1$. Let us suppose that A is defined by the equations $y_1 = \ldots = y_{k+1} = 0$, with an appropriate choice of the coordinates $(y_1 : \ldots : y_{n+1})$ for the projective space $\mathbb{P}(H^0(M))$. Moreover, to each $f \in H^0(L)$ we can obtain an hyperplane $H_f \subset \mathbb{P}(H^0(L))$ and we can suppose it is defined by the equation $x_1 = 0$, after choosing proper coordinates $(x_1 : \ldots : x_s)$ for $\mathbb{P}(H^0(L))$. These two subspaces define a further linear subspace of codimension $k+1$ in $\mathbb{P}(H^0(L) \otimes H^0(M))$ and hence a subspace $\tilde{A}_f$ of codimension $k+1$ in $\mathbb{P}(H^0(L \otimes M))$. Indeed, $\tilde{A}_f$ is defined by the vanishing of the equations $\{x_1 y_j\}_{j=1}^{k+1}$. Recall that by hypothesis we have that $f \cdot \Delta \longrightarrow H^0(L \otimes M)$ is injective, which ensures that all products $x_1 y_j$ are independent seen as linear forms of $\mathbb{P}(H^0(L \otimes M))$.
We are able to define $\tilde{A}_f$ considering the Segre map

$$\nu : \mathbb{P}(H^0(L)) \times \mathbb{P}(H^0(M)) \longrightarrow \mathbb{P}(H^0(L) \otimes H^0(M)),$$

indeed we have that

$$\tilde{A}_f = \left\langle \nu\left(H_f \times \mathbb{P}(H^0(M))\right) \cup \nu\left(\mathbb{P}(H^0(L)) \times A\right) \right\rangle \cap \mathbb{P}(H^0(L \otimes M)),$$

where we look at $\mathbb{P}(H^0(L \otimes M))$ as a linear subspace of $\mathbb{P}(H^0(L) \otimes H^0(M))$.
Recalling the geometric interpretation of a Steiner bundle, we get that the projectivization of the fiber of F over the point Γ is given as the intersection of all $\tilde{A}_f$ for each $f \in H^0(L)$. We thus obtain that

$$\mathbb{P}(F_\Gamma) = \left\langle \nu\left(\mathbb{P}(H^0(L)) \times A\right) \right\rangle \cap \mathbb{P}(H^0(L \otimes M)).$$

Example 3.1.5. Consider the Schwarzenberger bundle F over $\mathbb{G}(k, k+1)$ given by the triple $(X, L, M) = (\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), T_{\mathbb{P}^{k+1}}(-1))$.
Let us first observe that $H^0(T_{\mathbb{P}^{k+1}}(-1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^{k+1}}(1))^*$ and therefore the hyperplane section of the Segre variety with

$$\mathbb{P}(H^0(T_{\mathbb{P}^{k+1}})) \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{k+1}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^{k+1}}(1))^*)$$

is given considering all the pairs

$$\left\{(x, H) \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{k+1}}(1))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{k+1}}(1))^*) \mid x \in H\right\}.$$

Therefore the projectivization of the fiber F_Γ, for a point $\Gamma \in \mathbb{G}(k, k+1)$, is given by

$$\nu(H_x \times x) \subset \mathbb{P}(H^0(T_{\mathbb{P}^{k+1}})),$$

where x is the subspace of codimension $k+1$ in $\mathbb{P}(H^0(M))$ (in this case we have a point) and H_x represents the set of all hyperplanes containing x. The dimension of such projectivization is k, which confirms our interpretation.

3.2 Jumping pairs of a Steiner bundle

Let us notice that if we have a Schwarzenberger bundle given by the triple (X, L, M), with $\operatorname{rk} L = 1$ and $\operatorname{rk} M = k+1$, then at every point $x \in X$ we can associate a $k+1$ dimensional subspace considering the short exact sequence

$$0 \longrightarrow H^0(M \otimes \mathcal{I}_x) \longrightarrow H^0(M) \longrightarrow H^0(M_x) \longrightarrow 0$$

and taking $H^0(M_x)^*$. Remember now that in this case the map φ was given by the dual of the multiplication map and we can restrict the arrival space with the fixed subspace

$$H^0(L \otimes M)^* \longrightarrow H^0(L)^* \otimes H^0(M)^* \supset H^0(L)^* \otimes H^0(M_x)^*.$$

We observe that under such hypothesis we are always able to find a line a in $H^0(L)^* = S^*$ such that $a \otimes H^0(M_x)^*$ belongs to the image of φ. The natural choice will be $a = H^0(L_x)^*$, having $H^0(L_x)^* \otimes H^0(M_x)^* = H^0(L_x \otimes M_x)^*$. These considerations lead us to the following definition.

Definition 3.2.1. *Let F be a Steiner bundle over $\mathbb{G}(k, n)$. A pair (a, Γ), with* $\dim a = 1$

and $\dim \Gamma = k+1$*, such that* $a \otimes \Gamma \subset S^* \otimes H^0(\mathcal{U}^\vee)$*, is called a jumping pair if, considering the map* $T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee)$*, the tensor product* $a \otimes \Gamma$ *belongs to* $\operatorname{Im} \varphi$*.*

Let us fix some notation.

Definition 3.2.2. *We will denote by* $\tilde{J}(F)$ *the set of jumping pairs of* F*. We have two natural projections*

$$\begin{tikzcd} & \tilde{J}(F) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G(1, S^*) & & G(k+1, H^0(\mathcal{U}^\vee)) \end{tikzcd}$$

and we define $\Sigma(F) := \pi_1(\tilde{J}(F))$ *and* $J(F) := \pi_2(\tilde{J}(F))$ *the set of jumping spaces.*

Let us now show some useful properties.

Lemma 3.2.3. *Let* F *be a Steiner bundle over* $\mathbb{G}(k,n)$ *and let* F_0 *be its reduced summand, then* $\tilde{J}(F) = \tilde{J}(F_0)$*.*

Proof. Let us consider the following commutative diagram, where $T_0^* = \operatorname{Im} \varphi$.

$$\begin{tikzcd} & & 0 \arrow[d] & \\ & 0 \arrow[r] & T_0^* \arrow[r, "\varphi_0"] \arrow[d] & S^* \otimes H^0(\mathcal{U}^\vee) \arrow[d, equal] \\ 0 \arrow[r] & T^*/T_0^* \arrow[r] \arrow[dr] & T^* \arrow[r, "\varphi"] \arrow[d] & S^* \otimes H^0(\mathcal{U}^\vee) \\ & & T^*/T_0^* \arrow[d] & \\ & & 0 & \end{tikzcd}$$

Recall that we have a splitting of the bundle as $F = F_0 \oplus ((T/T_0) \otimes \mathcal{O}_{\mathbb{G}})$ that gives $F^* = F_0^* \oplus ((T^*/T_0^*) \otimes \mathcal{O}_{\mathbb{G}})$ and hence

$$H^0(F^*) = \underbrace{H^0(F_0^*)}_{=0} \oplus (T^*/T_0^*).$$

Obviously if (a, Γ) is a jumping pair for F_0, then it is also a jumping pair for F. Using the definition we prove the viceversa. Indeed, we know that a jumping pair is defined as $0 \neq (a \otimes \Gamma) \subset S^* \otimes H^0(\mathcal{U}^\vee)$ such that it exists $\Lambda \subset T^*$ with $\varphi(\Lambda) = a \otimes \Gamma$. If $\Lambda \cap (T^*/T_0^*) \neq \emptyset$ then $\dim a \otimes \Gamma < k+1$ because of the exact sequence defining φ. We thus have that $\Lambda \subset T_0^*$ and $a \otimes \Gamma$ is a jumping pair also for F_0. $\square$

The previous lemma tells us that from now on we can concentrate only on reduced Steiner bundles.
Having noticed how important the jumping pairs are in order to check if a Steiner bundle is also Schwarzenberger, we would like to give an appropriate description of the locus of the jumping pairs, seeing it as a variety.

Lemma 3.2.4. *Let F be a Steiner bundle on $\mathbb{G}(k, n)$ and $T_0^* \subset S^* \otimes H^0(\mathcal{U}^\vee)$ be the image of φ (or equivalently the vector space associated with the reduced summand F_0 of F). Consider the Segre generalized embedding*

$$\begin{array}{cccccc} \nu: & G(1, S^*) & \times & G(k+1, H^0(\mathcal{U}^\vee)) & \longrightarrow & G(k+1, S^* \otimes H^0(\mathcal{U}^\vee)) \\ & a & , & \Gamma & \mapsto & a \otimes \Gamma \end{array}$$

Then

(i) *we have that*
$$\tilde{J}(F) = \operatorname{Im} \nu \cap G(k+1, T_0^*),$$

(ii) *Let $\mathcal{A}, \mathcal{B}, \mathcal{H}$ be the universal bundles of ranks respectively $1, k+1$ and $k+1$ over $G(1, S^*), G(k+1, H^0(\mathcal{U}^\vee))$ and $G(k+1, T_0^*)$.*
Consider the projections

$$\begin{array}{ccccc} & & \tilde{J}(F) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ G(1, S^*) & & & & G(k+1, H^0(\mathcal{U}^\vee)) \end{array}$$

and that $\tilde{J}(F) \subset G(k+1, T_0^*)$. Assume that the natural maps

$$\begin{array}{lll} \alpha: & H^0(G(1, S^*), \mathcal{A}) & \longrightarrow \; H^0(\tilde{J}(F), \pi_1^* \mathcal{A}) \\ \beta: & H^0(G(k+1, H^0(\mathcal{U}^\vee)), \mathcal{B}) & \longrightarrow \; H^0(\tilde{J}(F), \pi_2^* \mathcal{B}) \\ \gamma: & H^0(G(k+1, T_0^*), \mathcal{H}) & \longrightarrow \; H^0(\tilde{J}(F), \mathcal{H}_{|\tilde{J}(F)}) \end{array}$$

are all isomorphisms. Then the Steiner bundle F_0, reduced summand of F, is a Schwarzenberger bundle given by the triple

$$(\tilde{J}(F), \pi_1^*\mathcal{A}, \pi_2^*\mathcal{B}).$$

Proof. Notice that part (i) is just the geometrical interpretation of the definition of jumping pair we have given.
To prove part (ii), consider the commutative diagram

$$\begin{tikzcd} S \otimes H^0(\mathcal{U}^\vee)^* \arrow[r] \arrow[d] & T_0 \arrow[d] \\ H^0(\tilde{J}(F), \pi_1^*\mathcal{A}) \otimes H^0(\tilde{J}(F), \pi_2^*\mathcal{B}) \arrow[r] & H^0(\tilde{J}(F), \pi_1^*\mathcal{A} \otimes \pi_2^*\mathcal{B}) \end{tikzcd}$$

The top map is the dual of the inclusion $T_0^* \hookrightarrow S^* \otimes H^0(\mathcal{U}^\vee)$ that defines the reduced summand of a Steiner bundle; such map can be identified with the following

$$H^0(G(1, S^*), \mathcal{A}) \otimes H^0(G(k+1, H^0(\mathcal{U}^\vee)), \mathcal{B}) \longrightarrow H^0(G(k+1, T_0^*), \mathcal{H})$$

defined by the composition of the multiplication map and the restriction of the global sections, due to the fact that $T_0^* \subset S^* \otimes H^0(\mathcal{U}^\vee)$,

$$H^0(G(k+1, S^* \otimes H^0(\mathcal{U}^\vee)), \tilde{\mathcal{H}}) \longrightarrow H^0(G(k+1, T_0^*), \mathcal{H}),$$

where of course $\tilde{\mathcal{H}}$ denotes the universal bundle over $G(k+1, S^* \otimes H^0(\mathcal{U}^\vee))$. The vertical maps, due to the last identification, are $\alpha \otimes \beta$ and γ, which are isomorphisms by hypothesis. The bottom map is the multiplication map whose dual defines a Schwarzenberger bundle. Because of the isomorphisms in the diagram, we can state that F_0 is Schwarzenberger defined by the triple $(\tilde{J}(F), \pi_1^*\mathcal{A}, \pi_2^*\mathcal{B})$ and this concludes the proof. □

3.3 The tangent space $T_\Lambda \tilde{J}(F)$

Our next goal is to compute the dimension of $\tilde{J}(F)$. In order to do so, we will consider the projective Segre generalized map and we will look at $\tilde{J}(F)$ as a projective variety; with an abuse of notation we will denote the vector space and the projective variety in the same way. The technique we will use to answer this question will be to consider a jumping pair $\Lambda \in \tilde{J}(F)$ and study the tangent space $T_\Lambda \tilde{J}(F)$ at the point Λ. This will give us an

upper bound for the dimension; moreover, if we manage to prove that the dimension of the variety and the one of the tangent space coincide, we will have that $\tilde{J}(F)$ is smooth.
Let us consider the projectivization of the generalized Segre map

$$\begin{array}{rcccc} \nu : & \mathbb{P}(S) & \times & \mathbb{G}(k, \mathbb{P}(H^0(\mathcal{U}^\vee))^*) & \longrightarrow & \mathbb{G}(k, \mathbb{P}(S \otimes H^0(\mathcal{U}^\vee)^*)) := \bar{\mathbb{G}} \\ & \mathbb{P}(a) & , & \mathbb{P}(\Gamma) & \mapsto & \mathbb{P}(a \otimes \Gamma) \end{array}$$

and we get $\tilde{J}(F) = \operatorname{Im} \nu \cap \mathbb{G}(k, \mathbb{P}(T_0))$.
Let us notice that $\operatorname{codim}_{\bar{\mathbb{G}}}(\operatorname{Im} \nu) = (k+1)(sn+s-n-1)-s+1$ and of course $\dim \mathbb{G}(k, \mathbb{P}(T_0)) = (k+1)(t_0-k-1)$ where $t_0 = \dim T_0$. This allows us to give a lower bound for the dimension of $\tilde{J}(F)$, obtained in the case we get a complete intersection, which is

$$\dim \tilde{J}(F) \geq (k+1)(t-k-sn-s+n)+s-1. \tag{3.1}$$

Remark 3.3.1. The previous inequality, which is an equality for the general Steiner bundle, tells us that if we consider s, n and k such that its right term is negative, then we will have that the general Steiner bundle with the chosen parameters cannot be Schwarzenberger, because it has no jumping pairs.

To limit the dimension from above we want to study the tangent space $T_\Lambda \tilde{J}(F)$ at a jumping pair Λ.
First of all, we need to find a proper description for the tangent space we are looking for; the result obtained is the following, to whom we will devote the rest of the section and it will be a direct consequence of Theorem 3.3.4.

Theorem 3.3.2. *Let F be a Steiner bundle over $\mathbb{G}(k,n)$ and let $\Lambda \in \tilde{J}(F)$ be one of its jumping pairs; then*

$$T_\Lambda \tilde{J}(F) = \left\{ \psi \in \operatorname{Hom}\left(\Lambda, \frac{T_0^*}{\Lambda}\right) \mid \begin{array}{c} (\psi(\varphi_i))(\ker \varphi_i) \subset < v_1 > \\ (\psi(\varphi_i))(u_i) \equiv (\psi(\varphi_j))(u_j) \mod v_1 \end{array} \right\}, \tag{3.2}$$

where $v_1 \in S^*$ and the sets $\{\varphi_i\}_{i=1}^{k+1}$ and $\{u_j\}_{j=1}^{n+1}$ are basis respectively of Λ and $H^0(\mathcal{U}^\vee)^*$, properly chosen as we will see in the proof of the next theorem.

Let us recall, see Lemma 3.2.4, that we can also define the jumping locus as $\tilde{J}(F) = \operatorname{Im} \nu \cap \mathbb{G}(k, \mathbb{P}(T_0))$ where $T_0^* = \operatorname{Im} \varphi$ is the vector subspace associated to the reduced summand of the bundle. Denoting $\operatorname{Seg} := \operatorname{Im} \nu$, what we actually need to find now is the description of the tangent space of Seg in a point Λ representing a jumping pair.
Let us start considering the following example.

Example 3.3.3. Take $s = 2, k = 1, n = 3$, i.e. we are dealing with the map

$$\mathbb{P}^1 \times \mathbb{G}(1,3) \longrightarrow \mathbb{G}(1,7).$$

We will work with the equations that define the Segre variety locally and we will induce the equations defining the tangent space at a fixed jumping point Λ.
We can suppose, without loss of generality, that Λ, seen as a subset of maps belonging to $\mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$, is represented by the $(s \times (n+1)) = (3,4)$-matrices of type

$$\left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ with } \lambda_1, \lambda_2 \in \mathbb{K} \right\},$$

simply by the definition of Λ.
Seen as a point of the $\mathbb{G}(1,7)$ we can represent it as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

point belonging to the open set defined by $p_{0,1} \neq 0$. All the points in this open neighbourhood will be of the form

$$\begin{bmatrix} 1 & 0 & x_{002} & x_{003} & x_{010} & x_{011} & x_{012} & x_{013} \\ 0 & 1 & x_{102} & x_{103} & x_{110} & x_{111} & x_{112} & x_{113} \end{bmatrix}.$$

Notice that all the variables x_{ijp} can already be considered as Plücker coordinates, taking proper order 2 minors.
Now we consider the two independent matrices related to the general point of the open neighbourhood. We notice that the image of both matrices has rank 1 and moreover both matrices map into the same vector. Therefore, we can translate this condition as

$$\mathrm{rk} \left[\begin{array}{cccc|cccc} 1 & 0 & x_{002} & x_{003} & 0 & 1 & x_{102} & x_{103} \\ x_{010} & x_{011} & x_{012} & x_{013} & x_{110} & x_{111} & x_{112} & x_{113} \end{array} \right] < 2.$$

The conditions that we get (for which we underline their linear part) are

$$\underline{x_{011}} = 0, \quad \underline{x_{110}} = 0, \quad \underline{x_{010} - x_{111}} = 0,$$

$$\begin{aligned}
&x_{002}x_{010} - \underline{x_{012}} = 0, && x_{003}x_{010} - \underline{x_{013}} = 0,\\
&x_{102}x_{111} - \underline{x_{112}} = 0, && x_{103}x_{111} - \underline{x_{113}} = 0,\\
&x_{002}x_{013} - x_{012}x_{003} = 0, && x_{102}x_{113} - x_{112}x_{103} = 0,\\
&x_{002}x_{112} - x_{012}x_{102} = 0, && x_{002}x_{113} - x_{012}x_{103} = 0,\\
&x_{003}x_{112} - x_{013}x_{102} = 0, && x_{003}x_{113} - x_{013}x_{103} = 0.
\end{aligned}$$

To define the tangent space we need to take the linear components of the equations we previously obtained. In this case we get 7 independent conditions in $\mathbb{A}^{12}$, which is the tangent space at a point of $\mathbb{G}(1,7)$, getting dimension 5 and that ensures us we are proceeding correctly.

The technique used in the example above can be generalized and this allows us to state the following theorem.

Theorem 3.3.4. *Let* Seg *be the image of the generalized projective Segre embedding and let* $\tilde{J}(F) \subset \mathrm{Seg}$ *be the set of the jumping pairs of a Steiner vector bundle* F *over* $\mathbb{G}(k,n)$*. Fixing a point* $\Lambda = s_0 \otimes \Gamma \in \tilde{J}(F)$*, we have that*

$$\begin{aligned}
T_\Lambda \,\mathrm{Seg} := \ &\Big\{\psi \in \mathrm{Hom}\Big(\Lambda, \tfrac{\mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)}{\Lambda}\Big) \mid \forall\, \varphi \in \Lambda, (\psi(\varphi))(\ker\varphi) \subset < s_0 > \\
&\textit{and}\, \exists\, A \supset < s_0 > \ \textit{with}\, A \subset S^*, \dim A = 2 \textit{ such that } \mathrm{Im}\,\psi(\varphi) \subset A\}
\end{aligned}$$

where $T_\Lambda \,\mathrm{Seg}$ *denotes the tangent of the Segre image at the point* Λ.

Proof. In order to prove the lemma, we will divide it in two different steps. In the first one we will consider the local definition of the Segre generalized variety and from those equations we will induce the ones defining its tangent space in a jumping point Λ, that belongs to the open subset we are considering. In the second part we will prove that the equations obtained generate the ideal associated to the tangent space. In order to do so we will prove that the dimension of the ideal generated by the equations found is equal to the dimension of the ideal of the tangent space.

Step 1 In this part we will assume that the point Λ we have fixed is the origin, looking at it belonging to its affine open set of the cover, and we will look for the linear forms of the equations that define locally the Segre variety in order to induce the equations that define its tangent space.

The point Λ can be represented by matrices

$$\left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{bmatrix} \text{ with } \lambda_1, \dots, \lambda_{k+1} \in \mathbb{K} \right\}$$

so it can be seen in the Grassmannian as the point

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

and the points in the corresponding neighbourhood will be of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & x_{0,0,k+1} & \cdots & x_{0,s-1,n} \\ 0 & 1 & \ddots & \vdots & x_{1,0,k+1} & \cdots & x_{1,s-1,n} \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & x_{k,0,k+1} & \cdots & x_{k,s-1,n} \end{bmatrix}$$

belonging, as before, to the open subset $p_{01\dots k} \neq 0$ and all $x_{i,j,p}$ can be seen as Plücker coordinates defined by proper minors. As we did in the example we consider the $k+1$ independent matrices related to the general point of the neighbourhood. The image of every morphism is of rank 1 and moreover the image for all of them will be the same vector by hypothesis. The condition is translated as

$$\operatorname{rk} \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & x_{0,0,k+1} & \cdots & x_{0,0,n} & \\ x_{0,1,0} & x_{0,1,1} & \cdots & x_{0,1,k} & x_{0,1,k+1} & \cdots & x_{0,1,n} & \\ \vdots & & & & & & \vdots & \cdots \\ x_{0,s-1,0} & x_{0,s-1,1} & \cdots & x_{0,s-1,k} & x_{0,s-1,k+1} & \cdots & x_{0,s-1,n} & \end{array} \right.$$

$$\left. \begin{array}{c|ccccccc} & 0 & \cdots & 0 & 1 & x_{k,0,k+1} & \cdots & x_{k,0,n} \\ & x_{k,1,0} & & & & & & x_{k,1,n} \\ \cdots & \vdots & & & & & & \vdots \\ & x_{k,s-1,0} & \cdots & & & & \cdots & x_{k,s-1,n} \end{array} \right) < 2.$$

We can notice that we get, from the minors of order 2, three types of linear conditions:

- the ones obtained fixing one column that starts with 1 and one column that starts with 0, of the type
$$x_{i,j,p} = 0 \text{ with } j = 1, \dots, s-1 \text{ for all } p \leq k \text{ with } p \neq i;$$
we have $k(s-1)(k+1)$ independent conditions in this case
- the ones obtained fixing one column that starts with a 1 and for each block take the columns from $k+1$ to n and the row starting with 1 fixed and the others with index from 1 to $s-1$, which are of type
$$x_{i,j,p} = x_{i,0,p} x_{i,j,0}$$
and we have $(k+1)(n-k)(s-1)$ independent conditions in this case
- the ones obtained fixing two columns starting with a 1, which are of the type
$$x_{i,j,i} = x_{p,j,p} \text{ with } p \neq i$$
and we have $k(s-1)$ independent conditions in this case.

All the conditions taken are independent among each other because the linear part is given every time by different coordinates (they come from different blocks). Summing all the conditions we have $(s-1)(nk+n+k)$ of them.
We can observe that the number of conditions we found is exactly the one needed to define the tangent space. Indeed, we got independent conditions in the tangent space of the variety $\mathbb{G}(k, \mathbb{P}(S \otimes H^0(\mathcal{U}^\vee)^*))$ and the difference

$$(k+1)(s(n+1)-1-k) - (s-1)(nk+n+k) = s-1+(k+1)(n-k),$$

which is exactly the dimension of the generalized Segre variety. We are ready to give a good candidate for the tangent space of the Segre at one point, which will be a set satisfying all the conditions we have listed. We are going now to prove that the equations obtained are sufficient to define the required tangent space.

The notation used is the following: a jumping pair Λ is equal to a tensor product $s_0 \otimes \Gamma \subset S^* \otimes H^0(\mathcal{U}^\vee) = \mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$. We can suppose that, once we fixed Λ, for every choice of a basis $v_1, \dots, v_s$ of S^* we can take $v_1 = s_0$. Such vector represents the image of all the rank 1 elements in Λ, seen as morphism from $H^0(\mathcal{U}^\vee)^*$ to S^*.

Step 2 To prove that our candidate, given by all the linear forms extracted, is actually the tangent space we need to compute its dimension. In order to do so we will compute instead the dimension of

$$\begin{aligned}\tilde{T} := \ & \{\psi \in \operatorname{Hom}\left(\Lambda, \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)\right) \mid \forall\, \varphi \in \Lambda, (\psi(\varphi))(\ker \varphi) \subset < v_1 > \\ & \text{and}\ \exists\, A \supset < v_1 > \ \text{with}\ \dim A = 2\ \text{such that}\ \operatorname{Im} \psi(\varphi) \subset A\}\end{aligned}$$

Observe that the vector space A is the same for each $\varphi \in \Lambda$, which guarantees that the set $\tilde{T}$ is itself a vector space seen as a vector subspace of $\operatorname{Hom}\left(\Lambda, \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)\right)$.
The technique we will use to calculate the dimension it is to consider proper subsets of $\tilde{T}$, whose dimension is known, and then study the cokernel given by the exact sequence induced by the inclusion.
Let us consider the proper subspace of $\tilde{T}$ given by

$$K = \left\{\psi \in \operatorname{Hom}(\Lambda, \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, < v_1 >))\right\}$$

with of course $\dim K = (k+1)(n+1)$. We obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook] & \tilde{T} \arrow[r] \arrow[dr] & P \arrow[r] \arrow[d, hook] & 0 \\
& & & \operatorname{Hom}\left(\Lambda, \operatorname{Hom}\left(H^0(\mathcal{U}^\vee)^*, \frac{S^*}{<v_1>}\right)\right) &
\end{tikzcd}$$

where $P \subset \operatorname{Hom}\left(\Lambda, \operatorname{Hom}\left(H^0(\mathcal{U}^\vee)^*, \frac{S^*}{<v_1>}\right)\right)$ represents the subset of morphisms whose images are all rank 1 elements of the set of morphisms
$\operatorname{Hom}\left(H^0(\mathcal{U}^\vee)^*, \frac{S^*}{<v_1>}\right)$.
Observe that P is also a vector space, being the quotient of two vector spaces. Our next goal is to compute the dimension of P.
Consider a family of morphisms $g_i : S^* \longrightarrow S^*$ such that $g_i(v_1) = v_i$, for $i = 2, \dots, s$ and define $s-1$ morphisms $\psi_i(\varphi) := g_i \circ \varphi$. We get then the commutative diagrams, one for

each i,

$$\begin{tikzcd}
H^0(\mathcal{U}^\vee)^* \arrow[r, "\varphi"] \arrow[d, "\simeq"'] & <v_1>\subset S^* \arrow[d, "g_i"] \\
H^0(\mathcal{U}^\vee)^* \arrow[r, "\psi_i(\varphi)"] \arrow[dr] & <v_i>\subset S^* \arrow[d, "\pi"] \\
& <\bar{v}_i>\subset \frac{S^*}{<v_1>}
\end{tikzcd}$$

where π simply represents the quotient of S^* by the subspace $< v_1 >$. Observe that all the images of the elements $g_i(v_1)$ through the projection π are independent by definition. In this way we have constructed the maps $\psi_2, \dots, \psi_s$ and we want to prove they form a basis for P.

Let us fix a basis $u_1, \dots, u_{n+1}$ for $H^0(\mathcal{U}^\vee)^*$ and let us build a basis $\varphi_1, \dots, \varphi_{k+1}$ of Λ with the property that $\varphi_i(u_i) = v_1$ and $\varphi_i(u_j) = 0$, for every i from 1 to $k+1$ and every j from 1 to $n+1$, with $j \neq i$.

Let us take the generic element $\psi \in P$ and check how it behaves when applied to the elements of the chosen basis of Λ. Consider two such elements φ_i and φ_j, with $i \neq j$. By hypothesis we know that $(\psi(\varphi_i))(H^0(\mathcal{U}^\vee)^*) \subset< \tilde{v}_i >$ with $\tilde{v}_i \in \frac{V}{<v_1>}$ and also $(\psi(\varphi_j))(H^0(\mathcal{U}^\vee)^*) \subset< \tilde{v}_j >$ with $\tilde{v}_j \in \frac{V}{<v_1>}$. Furthermore we know that, considering the element given by the sum $\varphi_i + \varphi_j$, we have $(\psi(\varphi_i + \varphi_j))(H^0(\mathcal{U}^\vee)^*) \subset< \tilde{v} >$ with $\tilde{v} \in \frac{V}{<v_1>}$, because the morphism $\psi(\varphi_i + \varphi_j)$ must have rank one and therefore exists a $\tilde{v}$ that gives us the last inclusion. We repeat this process for every pair of independent elements in the basis of Λ, so that we can state that we must get the same vector $\tilde{v} \in \frac{V}{<v_1>}$ for every $\varphi \in \Lambda$, i.e. $((\psi(\varphi))(H^0(\mathcal{U}^\vee)^*) \subset< \tilde{v} >$.

We have that $\tilde{v} = \lambda_2\tilde{v}_2 + \lambda_3\tilde{v}_3 + \dots \lambda_s\tilde{v}_s$ where $\lambda_i \in \mathbb{K}$ and $\tilde{v}_i$, for $i = 2, \dots, s$, are the vectors found defining the maps ψ_i and moreover the previous equivalence is true for every $u \in H^0(\mathcal{U}^\vee)^*$ and for every $\phi \in \Lambda$, which guarantees us that $\psi = \lambda_2\psi_2 + \dots + \lambda_s\psi_s$ for every $\psi \in P$.

We have thus proved that we have a basis for P, whose morphisms ψ_p have the following behavior: we know that $(\psi_p(\varphi_i))(\ker\varphi_i) \subset< v_1 >$ for every $i = 1, \dots, k+1$ and to every morphism we can associate a vector v_p with $\pi(v_p) = \tilde{v}_p \in \frac{S^*}{<v_1>}$ which is not zero, such that $(\psi_p(\varphi_i))(u_i) \subset< v_p >$. Considering the sum of two morphisms and its kernel

$$\ker(\varphi_i + \varphi_j) =< u_i - u_j, \{u_{\bar{k}}\}_{\bar{k}\neq i,j} >$$

we also get $(\psi_p(\varphi_i+\varphi_j))(\ker(\varphi_i+\varphi_j)) \subset < v_1 >$; so we can extend such properties to every element of Λ.
To conclude we have proved that $\dim P = s-1$, hence

$$\dim \tilde{T} = (k+1)(n+1)+s-1.$$

The remaining part to prove is compute the dimension of the tangent, but we can easily get that from the following commutative diagram

$$\begin{tikzcd}
0 & 0 \\
T_\Lambda \operatorname{Seg} \arrow[u] \arrow[r, hook] & \operatorname{Hom}\left(\Lambda, \frac{\operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)}{\Lambda}\right) \arrow[u] \\
\tilde{T} \arrow[u] \arrow[r, hook] & \operatorname{Hom}(\Lambda, \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)) \arrow[u] \\
\operatorname{End}(\Lambda) \arrow[u] \arrow[r, "="] & \operatorname{End}(\Lambda) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where, rewriting the conditions in an equivalent way, we can describe

$$T_\Lambda \operatorname{Seg} = \left\{ \psi \in \operatorname{Hom}\left(\Lambda, \frac{\operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)}{\Lambda}\right) \mid \begin{array}{c} (\psi(\varphi_i))(\ker\varphi_i) \subset < v_1 > \\ (\psi(\varphi_i))(u_i) \equiv (\psi(\varphi_j))(u_j) \mod v_1 \end{array} \right\}$$

and of course

$$\tilde{T} = \left\{ \psi \in \operatorname{Hom}\left(\Lambda, \operatorname{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)\right) \mid \begin{array}{c} (\psi(\varphi_i))(\ker\varphi_i) \subset < v_1 > \\ (\psi(\varphi_i))(u_i) \equiv (\psi(\varphi_j))(u_j) \mod v_1 \end{array} \right\}$$

We get that $\dim T_\Lambda \operatorname{Seg} = \dim \tilde{T} - \dim \operatorname{End}(\Lambda) = (k+1)(n-k)+s-1 = \dim(\mathbb{P}(S) \times \mathbb{G}(k,n))$; so it is actually the tangent space we were looking for and this completes the proof. $\square$

3.4 The technical lemmas and the bound

As we mentioned before our goal is to find now an upper bound for the dimension of the tangent space of $\tilde{J}(F)$ in a point representing a jumping pair. In order to do so we will use the description of the tangent we have given, seeing its elements as morphisms. The bound we are looking for will be the consequence of a more general result of linear algebra that we will be able to apply to our particular case.
Let us recall the notation fixed in Subsection 2.2.1, which we will use also in this section. Moreover, we will try to recover the same one we fixed in the proof of the description of the tangent space, in order to underline the correspondences between the general result and our setting.
Notation. Let U be a vector space of dimension $r > k$ and V be a vector space of dimension s. Let $v_1, \dots, v_s$ be a basis for V. Consider the vector space

$$\Lambda := \left\{ \varphi \in \mathrm{Hom}(U,V) | \, \mathrm{Im}\, \varphi \subset < v_1 > \text{ and } \dim \left(\bigcap_{\varphi \in \Lambda} \ker \varphi \right) = n - k \right\}.$$

We are able to construct a basis $u_1, \dots, u_r$ of U and a basis $\varphi_1, \dots, \varphi_{k+1}$ of Λ in order to have

$$\ker \varphi_i =< u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_r >=: U_i',$$

i.e. its kernel is the span of all vectors except u_i.
Recall that we denoted by $W \subset \mathrm{Hom}(U,V)$ the vector subspace characterized by the following property, that we called the property P_k: for every $k+1$ independent vectors $\tilde{u}_1, \dots, \tilde{u}_{k+1} \in U$ and for every $\tilde{v}_1, \dots, \tilde{v}_{k+1} \in V$ there exists an element $f \in W$ such that $f(\tilde{u}_j) = \tilde{v}_j$, for every $j = 1, \dots, k+1$.
The main result we need to prove will be accomplished by two steps, each one stated as a technical lemma. Let us start with the first one.

Lemma 3.4.1. *Let U, V, Λ and W be defined as before with the basis previously constructed. Let us consider*

$$\tilde{\Gamma} := \{\psi \in \mathrm{Hom}(\Lambda, W) \mid (\psi(\varphi_i))(\ker \varphi_i) \subset < v_1 >, \; i = 1, \dots, k+1\},$$

then $\dim \tilde{\Gamma} \leq (k+1)(t - (k+1)(s + r - k - 3))$ *where we denote by* $t = \dim W$.

Proof. Let us fix an element φ_i of the basis of Λ and check out how many independent morphisms, to choose as its image, we can find. We will associate such possible morphisms

as the image of φ_i and choosing the identically zero morphism as the image of the other morphisms of the basis, we can construct independent elements of $\mathrm{Hom}(\Lambda, W)$ which are not in $\tilde{\Gamma}$ and independent modulo $\tilde{\Gamma}$. Then we will fix, one at a time, each element of the basis of Λ and we will repeat such procedure. This technique allows us to give an upper bound for the dimension of $\tilde{\Gamma}$.
Remember that at the moment we have fixed one element φ_i and we are looking for all its possible images. Let us take $\tilde{u}_1, \ldots, \tilde{u}_{k+1}$ independent elements of $\ker \varphi_i$ and let us build

$$\tilde{\varphi}_{ij} \begin{cases} \tilde{u}_i & \mapsto & v_j \\ \tilde{u}_l & \mapsto & 0 \end{cases} \quad \text{for } i, l = 1, \ldots, k+1;\ j = 2, \ldots, s \text{ and } l \neq i.$$

By counting them, we observe we have constructed $(k+1)(s-1)$ independent morphisms modulo $\tilde{\Gamma}$. Let us denote the family of morphisms we have found as

$$\Psi_{ij} := \begin{cases} \varphi_i \mapsto \tilde{\varphi}_{ij} \\ \varphi_l \mapsto 0 \end{cases} \quad \text{for } i, l = 1, \ldots, k+1,\ j = 2, \ldots, s,\ l \neq i.$$

We would now like to construct new independent morphisms by using the rest of the elements in the kernel of φ_i other that the $k+1$ independent chosen vectors as we have done until now. This is why we now want to add a further vector $\tilde{u}_{k+2}$, belonging to $\ker \varphi_i$ and independent from the ones we have fixed before. By the condition that defines the set W we are allowed to fix the image of $k+1$ independent vectors at a time; notice that we are dealing with $k+2$ independent vectors and this means we are actually considering as ambient space the vectorial Grassmannian $G(k+1, k+2)$.
Suppose we have $(k+1)(s-1)+p$ independent morphisms modulo $\tilde{\Gamma}$ and notice that such morphisms must belong to $\mathrm{Hom}(U, \frac{V}{<v_1>})$. Working with $(k+1)$-dimensional subspaces means that for every $\Lambda \in G(k+1, k+2)$ we choose, we want to restrict the given morphisms to the vector space spanned by Λ. Such restriction of morphism can be represented by considering the fibers of the vector bundle over the Grassmannian

$$\mathrm{Hom}\left(\mathcal{U}, \frac{V}{< v_1 >} \otimes \mathcal{O}_{\mathbb{G}}\right) \simeq \frac{V}{< v_1 >} \otimes \mathcal{U}^{\vee},$$

where $\mathcal{U}$ and $\mathcal{O}_{\mathbb{G}}$ respectively denote, as usual, the universal and the trivial bundle on the Grassmannian. Asking for more independent morphisms in W, to take as image of φ_i, is equivalent to asking that the following bundle morphism does not have maximal rank

$$\begin{tikzcd} \mathcal{O}_{\mathbb{G}}^{(k+1)(s-1)+p} \arrow[rr] \arrow[dr] & & \frac{V}{<v_1>} \otimes \mathcal{U}^{\vee} \arrow[dl] \\ & G(k+1,k+2) & \end{tikzcd}$$

Using Porteous formula we get that the expected codimension of the points which give us non maximal rank, is $p+1$, so as long as $p+1 \leq k+1$ we can always find at least one independent morphism.
In the general step, suppose that we have already found $(k+1)(s-1)+(k+1)q$ independent morphisms modulo $\tilde{\Gamma}$, with $q \leq r-k-2$; this means that we have already fixed independent vectors $\tilde{u}_1, \dots, \tilde{u}_{k+1}, \tilde{u}_{k+2}, \dots, \tilde{u}_{q+k+1}$ in $\ker \varphi_i$; subsequently, we add a new independent vector $\tilde{u}_{q+k+2}$ and consider the generalization of the diagram we have seen before. The ambient space will now be $G(k+1, q+k+2)$ because, as we explained before, we can control $k+1$ independent vectors among the $q+k+2$ we are considering.

$$\begin{tikzcd} \mathcal{O}_{\mathbb{G}}^{(k+1)(s-1)+(k+1)q+p} \arrow[rr] \arrow[dr] & & \frac{V}{<v_1>} \otimes \mathcal{U}^{\vee} \arrow[dl] \\ & G(k+1,q+k+2) & \end{tikzcd}$$

By Porteous formula, if we consider the points for which the restriction of the morphism between vector bundles has no maximal rank, then their expected codimension will be $(k+1)q+p+1$ in an ambient space of dimension $\dim G(k+1, q+k+2) = (k+1)(q+1)$. Like in the first case, as long as $p+1 \leq k+1$, we can add a new independent morphism; moreover we can repeat this process until we find enough independent vectors that span $\ker \varphi_i$. Let us denote by $\hat{\varphi}_{i,q,p}$ the independent morphism we obtain when we have already fixed q elements in $\ker \varphi_i$ and we have already found $p-1$ new morphisms for this step. Notice that by construction, at each step there exists a vector $\bar{u}_{iqp}$, not belonging to the span $\langle \tilde{u}_1, \dots, \tilde{u}_{k+1}, \tilde{u}_{k+2}, \dots, \tilde{u}_{q-1} \rangle$, such that $\hat{\varphi}_{iqp}(\bar{u}_{iqp})$ is linearly independent, modulo v_1, with respect to the set

$$\{\hat{\varphi}_{ihk}(\bar{u}_{iqp}), \tilde{\varphi}_{ij}(\bar{u}_{iqp}) \mid i = 1, \dots, k+1; h \leq q; k = 1, \dots, k+1 \text{ and } k < p \text{ if } h = q\}.$$

Using the morphisms obtained we are able to construct a family

$$\Delta_{iqs} := \begin{cases} \varphi_i \mapsto \hat{\varphi}_{iqp} \\ \varphi_l \mapsto 0 \end{cases} \text{for every} \quad \begin{array}{l} i, l = 1, \dots, k+1 \ \text{ with } \ l \neq i, \\ q = k+2, \dots, r-1 \text{ and } p = 1, \dots, k+1. \end{array}$$

Observe that we have found $(k+1)(s-1)+(k+1)(r-k-2)$ independent images for each fixed φ_i element of the basis of Λ and then we have decided to send the other elements of the basis to the zero morphism. Repeating such procedure for each fixed element φ_i, with i from 1 to $k+1$, we have obtained

$$(k+1)^2(r+s-k-3)$$

independent morphisms modulo $\tilde{\Gamma}$. Indeed, consider their linear combination

$$\sum_{i=1}^{k+1}\sum_{j=2}^{s}\lambda_{ij}\Psi_{ij} + \sum_{i=1}^{k+1}\sum_{q=k+2}^{r-1}\sum_{p=1}^{k+1}\mu_{iqp}\Delta_{iqp} = f,$$

with $f \in \tilde{\Gamma}$, and apply it to a fixed element φ_i of the basis of Λ, in order to get

$$\sum_{j=2}^{s}\lambda_{ij}\tilde{\varphi}_{ij} + \sum_{q=k+2}^{r-1}\sum_{p=1}^{k+1}\mu_{iqp}\hat{\varphi}_{iqp} = f(\varphi_i).$$

Notice that, by construction, applying such combination to the vectors $\bar{u}_{iqp}$, starting from the top values of the subindex p and q, makes all coefficients of type μ_{iqp} vanish. That is because the image of the corresponding morphism was independent modulo v_1 to the images of the previous ones and $(f(\varphi))(\bar{u}_{iqp}) \in < v_1 >$. We are left with the combination

$$\sum_{j=2}^{s}\lambda_{ij}\tilde{\varphi}_{ij} = f(\varphi_i)$$

which, applied to element $\tilde{u}_i$, gives us

$$\sum_{j=2}^{s}\lambda_{ij}v_j = (f(\varphi_i))(\tilde{u}_i) \subset < v_1 >,$$

hence $\lambda_{ij} = 0$. Fixing each element φ_i, we manage to vanish all the coefficients in the linear combination, hence we have independency modulo $\tilde{\Gamma}$.

We can thus state that

$$\dim \tilde{\Gamma} \leq (k+1)(t-(k+1)(r+s-k-3)),$$

which concludes the proof. □

Observe that the bound we just proved only involves one of the two conditions listed in 3.2 that we found in the definition the tangent space given in Theorem 3.3.2. That is why we will now demonstrate a second result that will also involve the second condition.

Lemma 3.4.2. *Let U, V, Λ and W defined as before and let $\tilde{\Gamma}$ be the vector subspace defined in Lemma 3.4.1. Let us consider*

$$\Gamma = \left\{ \psi \in \tilde{\Gamma} | (\psi(\varphi_i))(u_i) \equiv (\psi(\varphi_j))(u_j) \mod v_1, \ \textit{with}\, i \neq j \textit{ from}\, 1 \textit{ to}\, k+1 \right\},$$

then $\dim \Gamma \leq \dim \tilde{\Gamma} - k(k+1)$.

Proof. We will prove this lemma with the same idea we used to prove the previous one, i.e. we will look for morphisms in $\tilde{\Gamma}$ that are independent modulo Γ. Consider a fixed element φ_i of the basis of Λ and the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker \alpha \arrow[r] & W \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}\left(\ker \varphi_i, \frac{V}{<v_1>}\right) \arrow[d] \\ 0 \arrow[r] & \ker \beta \arrow[r] \arrow[u, hook] & W \arrow[r, "\beta"] & \mathrm{Hom}\left(U, \frac{V}{<v_1>}\right) \end{tikzcd}$$

Notice that the number of the morphisms we are looking for is equal to the number of independent morphisms of $\frac{\ker \alpha}{\ker \beta}$, so we must compute

$$\dim(\ker \alpha) - \dim(\ker \beta) = \dim(\mathrm{Im}\, \beta) - \dim(\mathrm{Im}\, \alpha).$$

For our vector space V of dimension s, let us consider a quotient

$$Q = \frac{V}{< v_1, \bar{v}_2, \ldots, \bar{v}_{s-k-1} >}$$

in order to have $\dim Q = k+1$ and construct elements $\{\tilde{v}_{s-k}, \ldots, \tilde{v}_s\}$ which are linearly independent modulo $< v_1, \bar{v}_2, \ldots, \bar{v}_{s-k-1} >$; then take a look at the composition

$$\begin{tikzcd} W \arrow[r, "\beta"] \arrow[rr, bend right, "f"'] & \mathrm{Hom}(U, \frac{V}{<v_1>}) \arrow[r, "\pi"] & \mathrm{Hom}(U, Q). \end{tikzcd}$$

By Lemma 2.2.4 we know that f is surjective, so if we consider the $k+1$ independent morphisms

$$\{\delta_j\}_{j=1}^{k+1} \subset \mathrm{Hom}\left(U, \frac{V}{< v_1 >}\right) \text{ with } \delta_j\left(\frac{U}{\ker \varphi_i}\right) = \tilde{v}_{s-k-1+j},$$

we have that the set $\{\pi(\delta_j)\}_{j=1}^{k+1}$ is also independent considered in $\mathrm{Hom}(U, Q)$ and each morphism of the set must have at least one preimage in W, independent among them modulo $\mathrm{Im}\,\alpha$ by construction. Observe that we found at least $k+1$ independent morphisms for the fixed element φ_i. If we use the explained technique for each φ_i fixed, with $i = 2, \ldots, k+1$, while we always send the element φ_1 to the zero morphism, in order to guarantee the independency modulo Γ, we get $k(k+1)$ morphisms that belong to $\tilde{\Gamma}$ and that are independent modulo Γ. We can thus say that

$$\dim \Gamma \leq \dim \tilde{\Gamma} - k(k+1)$$

which concludes the proof. □

Having proved this technical results we are finally ready to give the requested upper bound for the dimension of the tangent space.

Theorem 3.4.3. *Let F be a Steiner bundle on $\mathbb{G}(k, n)$ defined by the injective map $S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}}$ and let Λ be a jumping pair for F, i.e. $\Lambda \in \tilde{J}(F)$, then the dimension of the tangent space of the jumping locus variety at the point Λ is bounded by*

$$\dim T_\Lambda \tilde{J}(F) \leq (k+1)(t - (k+1)(s+n-k-1) - k),$$

where as usual $s = \dim S$ and $t = \dim T$.

Proof. Notice that the geometrical description of the tangent space given in the expression 3.2 is a set whose conditions satisfy exactly the ones requested in the hypothesis of Lemma 3.4.2. Applying the lemma, with the values set by the bundle, gives us the result. □

Chapter 4

The classification of Steiner bundles with jumping locus of maximal dimension

In this chapter we will give a complete classification of Steiner bundles whose jumping locus has maximal dimension. In fact, we will manage to describe them as Schwarzenberger bundles associated to a triple (X, L, M), generalizing the result given in [Arr10a]. Such classification will give a positive answer to *Question A* in this particular case.

4.1 The case $s = k + 2$

In Theorem 2.3.1 we have seen that every (s, t)-Steiner bundle with $s \leq k+1$ is essentially trivial. In this section we classify the first non trivial case, when $s = k + 2$.

Remark 4.1.1. Recall that Lemma 2.2.3 and Lemma 2.2.5 implied that every Steiner bundle F on $\mathbb{G}(k, n)$ is equivalent to a Steiner bundle $\tilde{F}$ on $\mathbb{G}(k, \mathbb{P}(S))$.

Let us consider the case $s = k + 2$, which, thanks to the previous observation, we are able to classify completely.

Theorem 4.1.2. *Let F be a reduced Steiner bundle over $\mathbb{G}(k, n)$, with $\dim S = k+2$, then F can be seen as the Schwarzenberger bundle given by the triple $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^{\vee}(-1))$, where we identify $\mathbb{P}(S) = \mathbb{P}^{k+1}$ and E is the vector bundle defined as the kernel of the* *surjective morphism*

$$H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)}.$$

Observe that the difference between the two bounds given in the inequality (3.1) and Theorem 3.4.3 is equal to $(s-k-2)(nk-k^2+n-k-1)$; so, in this case, $\tilde{J}(F)$ is a smooth complete intersection of dimension

$$\dim \tilde{J}(F) = (k+1)(t-(k+1)(n+1)-k).$$

Proof. Consider the following commutative diagram, observing that $\mathcal{O}_{\mathbb{G}(k,\mathbb{P}(S^*)} \simeq \mathcal{O}_{\mathbb{P}(S)}$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E \arrow[r] \arrow[d] & T^* \otimes \mathcal{O}_{\mathbb{P}(S)} \arrow[r] \arrow[d] & T_{\mathbb{P}(S)}(-1) \otimes H^0(\mathcal{U}^\vee) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \arrow[r] \arrow[d] & S^* \otimes H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)} \arrow[r] \arrow[d] & T_{\mathbb{P}(S)}(-1) \otimes H^0(\mathcal{U}^\vee) \arrow[r] & 0 \\
 & \frac{S^*\otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)} \arrow[r, equal] \arrow[d] & \frac{S^*\otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

that gives us the requested triple and thus the requested description. The classification is complete due to Remark 4.1.1. ☐

Some remarks: the interpretation of the fiber of E, for a point $s_0 \in \mathbb{P}(S)$, is the vector space

$$E_{s_0} = \left\{ \left(< s_0 > \otimes H^0(\mathcal{U}^\vee) \right) \cap T^* \right\},$$

that is the set we need to study in order to find jumping pairs. In fact we have that

$$\tilde{J}(F) = G(k+1, E)$$

seen as a Grassmann bundle (see Subsection 1.1.1).

Let us observe also that $E^\vee(-1)$ is a Steiner bundle on $\mathbb{P}(S)$ given by the resolution

$$0 \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)} \longrightarrow E^\vee(-1) \longrightarrow 0,$$

so basically, Lemma 2.2.3 and Lemma 2.2.5 tell us that a Steiner bundle over $\mathbb{G}(k,n)$ is equivalent to a Steiner bundle on $\mathbb{P}(S)$, with the equivalence expressed above. In the particular case $k=0$ we have $E^\vee(-1) = \bigoplus_{i=1}^{t-s+n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with degrees $a_i \geq 1$.
We would like to show a geometrical interpretation of this classification and we will do so in the particular case $n = k+1$.
We know that it is possible to take $\tilde{J}(F) \subset \mathbb{P}^{k+1} \times \mathbb{G}(k,k+1) \overset{\nu}{\hookrightarrow} \mathbb{G}(k,(k+2)^2-1)$. Considering the Segre embedding $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1} \hookrightarrow \mathbb{P}^{(k+2)^2-1}$ we are able to see the image of the generalized Segre embedding ν as the $k+1$ family (given by the first projective space) of $(k+1)$-linear spaces (subvarieties of the second projective space). In order to get the variety $\tilde{J}(F)$ we have to cut such family with the hyperplane $\mathbb{P}(T_0)$ and we observe that, according to the definition of Steiner bundle, the only non trivial case gives us $\dim T_0 = (k+2)^2 - 1$. We obtain that every linear space of the family is always cut by $\mathbb{P}(T_0)$ to a unique linear space of dimension at least k. This gives us an isomorphism $\tilde{J}(F) \simeq \mathbb{P}(S) \simeq \mathbb{P}^{k+1}$. Observe that the construction taken can be described as

$$\begin{tikzcd}
 & \tilde{J}(F) \subset \mathbb{P}^{k+1} \times (\mathbb{P}^{k+1})^* \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mathcal{O}_{\mathbb{P}^{k+1}}(1) \boxtimes T_{(\mathbb{P}^{k+1})^*}(-1)"] & & \mathbb{G}(k,(k+2)^2-1) \\
\mathbb{P}^{k+1} \arrow[d, "\simeq"'] & & (\mathbb{P}^{k+1})^* \arrow[d, "\simeq"] & \\
\mathbb{P}(S) & & \mathbb{G}(k, \mathbb{P}(H^0(\mathcal{U}^\vee))) &
\end{tikzcd}$$

moreover, we can prove that also π_2 is an isomorphism.
Indeed, the morphism $\mathcal{O}_{\mathbb{P}^{k+1}}(1) \boxtimes T_{(\mathbb{P}^{k+1})^*}(-1)$ restricted to the fiber of a point $H \in (\mathbb{P}^{k+1})^*$ is given by $\mathcal{O}^{k+1}_{\mathbb{P}^{k+1}}(1)$, with $c_{k+1}(\mathcal{O}^{k+1}_{\mathbb{P}^{k+1}}(1)) = 1$, hence Proposition 1.1.21 ensures us that the fiber is made by just one point and we have an isomorphism.
Since the linear space cut has dimension exactly k, we have that $\tilde{J}(F)$ is isomorphic to $\mathbb{P}^{k+1}$ and F is a Schwarzenberger bundle given by the triple $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), T_{\mathbb{P}^{k+1}}(-1))$.

4.2 The induction tecnique

Now that we have studied and understood the particular case $s = k + 2$, we are ready to explain the induction process that will allow us to classify Steiner bundles with maximal jumping locus.
The idea is that, fixed a jumping pair $s_0 \otimes \Gamma \in \tilde{J}(F)$, where F is a reduced (s,t)-Steiner bundle over $\mathbb{G}(k,n)$, we can induce a further Steiner bundle F': a vector bundle, still over $\mathbb{G}(k,n)$, which may not be reduced, that is of $(s-1, t-k-1)$-type. Observing that this type of induction lowers the parameter s, we may think that the right technique of classification will be to apply induction till arriving at the case $s = k+2$, which we fully know, in order to get information about the original bundle through the induction steps we have processed.
Let us consider the morphism $\varphi : T^* \longrightarrow S^* \otimes H^0(\mathcal{U}^\vee)$ which defines the Steiner bundle F and let us fix a jumping pair $s_o \otimes \Gamma \in \tilde{J}(F)$, with $s_0 \otimes \Gamma = \varphi(\Lambda)$. We can induce the commutative diagram

$$\begin{tikzcd} T^* \arrow[r, hook, "\varphi"] \arrow[d, "\tilde{pr}"'] & S^* \otimes H^0(\mathcal{U}^\vee) \arrow[d, "pr\otimes id"] \\ \frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{<s_0>} \otimes H^0(\mathcal{U}^\vee) \end{tikzcd} \tag{4.1}$$

First of all φ' defines a Steiner bundle F' of type $(s-1, t-k-1)$, because by the commutativity of the above diagram, the map φ' satisfies the Steiner properties. Nevertheless, we must be careful because the bundle F' may not be reduced.
Remember that we had two canonical projections: the map $\pi_1 : \tilde{J}(F) \to \mathbb{P}(S)$, with $\pi_1(\tilde{J}(F)) = \Sigma(F)$, and the map $\pi_2 : \tilde{J}(F) \to \mathbb{G}(k, \mathbb{P}(H^0(\mathcal{U}^\vee)^*))$, with $\pi_2(\tilde{J}(F)) = J(F)$.

Proposition 4.2.1. *Let F be a Steiner bundle over $\mathbb{G}(k,n)$ and F' the one induced as above by fixing $s_0 \otimes \Gamma$ jumping pair, then*

(i) $J(F) \subset J(F') \cup \pi_2(\pi_1^{-1}(s_0))$;

(ii) *considering the set of all jumping pairs corresponding to the fixed s_0, we denote by $\mathcal{L}_{s_0} = \mathbb{P}\left((s_0 \otimes H^0(\mathcal{U}^\vee) \cap T^*)^*\right)$, we obtain a projection*

$$pr_{(s_0,\Gamma)} : \mathbb{G}(k, \mathbb{P}(T)) \longrightarrow \mathbb{G}(k, \mathbb{P}(T_0'))$$

where $(T_0')^*$ denotes the image of the map φ' and whose center of projection is

$\mathbb{G}(k, \mathcal{L}_{s_0})$. *Moreover we have a natural projection*

$$pr_{s_0} : \mathbb{P}(S) \longrightarrow \mathbb{P}\left(\left(\frac{S^*}{< s_0 >}\right)^*\right).$$

and for every jumping pair $(s, \bar{\Gamma})$ *with* $s \neq s_0$ *we have that*

$$pr_{(s_0,\Gamma)}(s, \bar{\Gamma}) = \left(pr_{s_0}(s), \bar{\Gamma}\right);$$

(iii) $pr_{(s_0,\Gamma)}(\tilde{J}) \subset \tilde{J}(F_0')$ *where* $\tilde{J}$ *is an irreducible component of* $\tilde{J}(F)$ *not in* $\pi_1^{-1}(s_0)$*;*

(iv) $pr_{s_0}(\Sigma(F)) \subset \Sigma(F_0')$ *if* $\Sigma(F) \neq \{s_0\}$ *or if the generic component is different from* $\{s_0\}$.

Proof. Consider an element $s \otimes \bar{\Gamma} \in \tilde{J}(F)$, if $s \neq s_0$ then $pr_{s_0}(s) \otimes \tilde{\Gamma} \in \tilde{J}(F')$, hence $\bar{\Gamma} \in J(F')$; if $s = s_0$ then obviously $\bar{\Gamma} \in \pi_2(\pi_1^{-1}(s_0))$, so (i) is proved. To prove (ii) notice that the kernel of the map

$$(pr \otimes id) \circ \varphi : T^* \longrightarrow \frac{S^*}{< s_0 >} \otimes H^0(\mathcal{U}^\vee),$$

described in diagram 4.1, is exactly $\pi_1^{-1}(s_0)$, so its projectivization will be $\mathcal{L}_0$, that represents the center of the given projection. Parts (iii) and (iv) come automatically from the commutativity of diagram 4.1. $\square$

The next proposition shows us how the property of having a jumping locus of maximal dimension is maintained during the induction and explains the relations between the sets $\Sigma(F)$ and $J(F)$ and their respective sets $\Sigma(F')$ and $J(F')$ in the induced bundle.

Proposition 4.2.2. *Let* F *be a reduced Steiner bundle over* $\mathbb{G}(k, n)$ *defined by the morphism* φ *and let* $\tilde{J}(F)$ *have maximal dimension. Let* F' *be the bundle induced by* F*, once fixed the jumping pair* $s_0 \otimes \Gamma \in \tilde{J}(F)$*, and let* F_0' *be its reduced summand. Let* $\tilde{J}_0$ *be an irreducible component of* $\tilde{J}(F)$ *of maximal dimension such that* $s_0 \otimes \Gamma \in \tilde{J}_0$*, then*

(i) *the image of both* $\tilde{J}_0$ *and* $\tilde{J}(F)$ *under* $pr_{(s_0,\Gamma)}$ *has dimension*

$$(k+1)(t - (k+1)(s+n-k-1) - k) - (k+1)(l_0 - k),$$

where $l_0 = \dim \mathcal{L}_0$.

(ii) $\dim \tilde{J}(F_0') = (k+1)(t - (k+1)(s+n-k-1) - k) - (k+1)(l_0 - k)$*;*

(iii) *If $\tilde{J}(F_0')$ is irreducible then*

- **a)** $\tilde{J}(F_0') = pr_{(s_0,\Gamma)}(\tilde{J}(F))$,
- **b)** $\tilde{J}(F)$ *is irreducible,*
- **c)** $J(F) = J(F')$,
- **d)** $\Sigma(F') = pr_{s_0}(\Sigma(F))$, *so it is a projection from an inner point.*

Proof. From Theorem 3.4.3 we know that

$$\dim pr_{(s_0,\Gamma)}(\tilde{J}_0) \le (k+1)(t_0' - (k+1)(s+n-k-2) - k);$$

if we substitute $t_0' = t - l_0 - 1$, which we know to be true by Proposition 4.2.1 (ii), we obtain that

$$\dim pr_{(s_0,\Gamma)}(\tilde{J}_0) \le (k+1)(t-(k+1)(s+n-k-1)-k) - (k+1)(l_0-k).$$

We need to check that the dimension always reaches the top value. Let us suppose that this never occurs, i.e.

$$\dim pr_{(s_0,\Gamma)}(\tilde{J}_0) \le (k+1)(t-(k+1)(s+n-k-1)-k) - (k+1)(l_0-k) - 1.$$

If this happens, then it means that $\tilde{J}(F)$ must be a cone, because the dimension of the image of the projection is smaller than the dimension of $\tilde{J}(F)$ minus the dimension of the center of projection. We know that it is smooth, so it must be a projective space and hence it is contained in one of the fibers of the general Segre variety. This leads to contradiction because we cannot be either in $\pi_1^{-1}(s_0)$ or in $\pi_2^{-1}(\Gamma)$; this proves part (i).
Part (ii) is proved considering the combination of the fact that $\tilde{J}_0 \subset \tilde{J}(F_0')$ and Theorem 3.4.3.
Part (iii-a) comes directly by computing dimensions. To prove part b) suppose there exists another component $\tilde{J}_1$ different from $\tilde{J}_0$ and fix an element $s_1 \otimes \Gamma_1 \in \tilde{J}_1 \backslash \tilde{J}_0$. There must exist an element $\bar{s} \otimes \Gamma_1 \in \tilde{J}_0$ that has the same projection as the element fixed, so there is a line L connecting the two points that meets in $\pi_1^{-1}(s_0)$; such line must be contained in $\tilde{J}(F)$. By hypothesis we know that $L \not\subset \tilde{J}_0$ so it must belong to another component, this would tell us that the point $\bar{s} \otimes \Gamma_1$ is singular, which is a contradiction. Part (iii-c) and (iii-d) come automatically by irreducibility and the commutativity of diagram 4.1. ☐

Two consequences of what we just proved are

- $\Sigma(F)$ is a variety with minimal degree, i.e. it can be a rational normal scroll, the Veronese surface, a projective space or a cone over the previous three varieties;

- $J(F) = J(\bar{F}'_0)$ where by $\bar{F}'_0$ we denote the reduced summand of the induced bundle at the step $s = k+2$.

4.3 The classification of bundles with maximal dimensional locus

This final section contains the statement and the proof of our main result, i.e. the theorem that classifies the Steiner bundles whose jumping locus is maximal. Notice that such result includes the classification given by Arrondo in [Arr10a] (see Theorem 2.1.8), even if proved with a different technique. Observe also that all the bundles in the classification are Schwarzenberger, thus providing in this case a positive answer to *Question A*.

Theorem 4.3.1. *Let F be a reduced Steiner bundle on $\mathbb{G}(k,n)$ for which $\dim \tilde{J}(F)$ is maximal; then we are in one of the following cases:*

(i) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$. In this case $k=0$ and $t = n+s$.*

(ii) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^1, E(-1), \mathcal{O}_{\mathbb{P}^1}(1))$, where $E = \oplus_{i=1}^{t-s}\mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 1$. In this case $k=0$ and $n=1$.*

(iii) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$, where E is a Steiner bundle defined by the following exact sequence*

$$0 \longrightarrow E \longrightarrow H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)} \longrightarrow 0.$$

In this case $s = k+2$.

(iv) *F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$. In this case $k=0, n=2, s=3$ and $t=6$.*

Recall the induction construction we showed in the previous section, that gives us the following commutative diagram, essential for the classification. During this section we will consider all of our sets as projective varieties. Let F be a Steiner bundle over $\mathbb{G}(k,n)$, then we have

$$\begin{tikzcd}
 & \tilde{J}(F) \arrow[dl, "\pi_1"'] \arrow[d, dashed] \arrow[dr, "\pi_2"] & \\
\Sigma(F) \arrow[d, dashed, "pr_l"'] & \tilde{J}(F_0') \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & J(F) \arrow[d, equal, "pr_r"] \\
\Sigma(F_0') & & J(F_0')
\end{tikzcd} \tag{4.2}$$

where as usual F_0' denotes the reduced summand of the induced bundle obtained fixing a jumping pair $\Lambda = s_0 \otimes \Gamma$.
Notice that $\dim \Sigma(F) = \dim \tilde{J}(F_0')$; indeed by Proposition 4.2.2 we know that

$$\dim \tilde{J}(F_0') = \dim \tilde{J}(F) - (k+1)(l_0 - k),$$

where $l_0 = \dim \mathbb{P}(((< s_0 > \otimes H^0(\mathcal{U}^\vee)) \cap T^*)^*)$ for s_0 generic point for $\Sigma(F)$, moreover also

$$\dim \Sigma(F) = \dim \tilde{J}(F) - (k+1)(l_0 - k)$$

because we know π_1 to be surjective and $(k+1)(l_0 - k)$ is the dimension of the generic fiber.
We also have that $\dim \Sigma(F_0') \leq \dim \Sigma(F) \leq \dim \Sigma(F_0') + 1$ because we proved that pr_l is a projection from an inner point.
Combining these two relations, we get that the fiber of the projection $\pi_1' : \tilde{J}(F_0') \longrightarrow \Sigma(F_0')$ has dimension at most one, but we know that the dimension of the fiber of such projection for a Steiner bundle has dimension either zero or is greater equal than $k+1$. This means that we need to divide the two cases $k = 0$ and $k \geq 1$.
A further division is given focusing on $\tilde{J}(F)$ and $\Sigma(F)$ and observe that we have two possibilities: $\dim \Sigma(F) = \dim \tilde{J}(F)$ or $\dim \Sigma(F) < \dim \tilde{J}(F)$. Let us study the several cases we have pointed out. Each case will give us a Schwarzenberger bundle, because of Theorem 3.2.4.

The case of the projective space $k = 0$

Case $\dim \Sigma(F) < \dim \tilde{J}(F)$
Supposing this inequality means that for every s_0 in $\Sigma(F)$ we have $\dim(\pi_1^{-1}(s_0)) \geq 1$. Let us suppose that $\dim \Sigma(F) = \dim \Sigma(F_0')$, which implies that $\tilde{J}(F_0')$ is birational to $\Sigma(F_0')$. In this setting fix an element $\bar{s} \in \Sigma(F)$ such that $0 \neq pr_l(\bar{s}) \in \Sigma(F_0')$.

By hypothesis there exist at least two points $\bar{s}\otimes v_1$ and $\bar{s}\otimes v_2$ which represent independent jumping pairs, so $v_1, v_2 \in J(F)$. By Proposition 4.2.1 (ii) we get the commutativity of the projections, so by one side we know that $pr_l(\bar{s}) \otimes v_1$ and $pr_l(\bar{s}) \otimes v_2$ belong to $\tilde{J}(F_0')$ and they are independent; this leads to contradiction because from the other side we know that the generic point $pr_l(s_0) \in \Sigma(F_0')$ has only one preimage.
Hence we get that it must be $\dim \Sigma(F) = \dim \Sigma(F_0') + 1$, which means that we are still in the case $\dim \tilde{J}(F_0') > \dim \Sigma(F_0')$ and we can iterate the induction until the step $s = 1$ that gives us $\Sigma = \mathbb{P}^1$. We discovered that if the fiber of π_1 has positive dimension, then it is also true for the induced bundle; this allows us to state that if $\dim \tilde{J}(F) > \dim \Sigma(F)$ then $\Sigma(F) = \mathbb{P}(S)$. Indeed, we have seen that the left projections, from an inner point, drop at each step the dimension by one and leave unchanged the degree.
We would like to exclude the case when the dimension of the fiber is greater or equal than two; in order to do so, we have the following theorem.

Theorem 4.3.2. *If* $\dim(\pi_1^{-1}(s_0)) \geq 2$ *for every* $s_0 \in \Sigma_F$*, then* $\tilde{J}(F) \simeq \mathbb{P}(S) \otimes \mathbb{P}^n$.

Proof. Notice that, looking at the diagram (4.2), the induced bundle F_0' will give us generic fiber, of the morphism π_1', of dimension one, so the only possible case not to get a contradiction, because of the commutativity of the diagram, is the trivial one. □

Case $\dim \Sigma(F) = \dim \tilde{J}(F)$
In this case we have that $\tilde{J}(F)$ is birational to $\tilde{J}(F_0')$. We now need to distinguish the case where the birationality is conserved throughout the induction or else if we have one step where $\dim \Sigma(F) = \dim \Sigma(F_0') + 1$, when we can recover the case we studied before.
If the birationality is conserved, then we can arrive at the step $s = 1$, where we know that $\Sigma = \mathbb{P}^1$, so all the left projections are isomorphisms, because they are birational maps between rational normal curves, and we get that $\Sigma(F) \subset \mathbb{P}(S)$ is nothing more that the rational normal curve. This allows us to state that in this situation is enough to study the step $s = 1$ and we obtain the triple $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$.
For what we have just proved, we can, without loss of generality, consider the situation where we start with $\tilde{J}(F)$ birational to $\Sigma(F)$ and the birationality is broken in the first step of the induction; we already know that the fiber will be of positive dimension in every further step. Let us take a look at the following diagram that explains better in what setting we are (close to the varieties we will see their dimension, close to the arrows we

will denote the birationality or the positivity of the dimension of the fiber).

$$\begin{tikzcd}
 & & \tilde{J}(F)^{s-1} \arrow[dl, "bir"'] \arrow[d, dashed, "bir"] \\
\text{step } s & \mathbb{P}^{s-1} \arrow[d, dashed] & \tilde{J}(F_0') \arrow[dl, "+1"'] \arrow[dd, dashed] \\
\text{step } s-1 & \mathbb{P}^{s-2} \arrow[dd, dashed] & \\
 & & \tilde{J}(\bar{F}) \arrow[dl, "+1"'] \\
\text{step } 2 & \mathbb{P}^1 &
\end{tikzcd}$$

Observe that $\dim \tilde{J}(F_0)$ lowers by one at each step of the induction, so we have that $\dim \tilde{J}(\bar{F}) = 2$ and this means that we can relate it, as the projectivization, to a rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a + b = n + 1$, the degree of the variety.
If one between a or b is greater or equal than 2, then we could relate $\tilde{J}(F_0')$ to a vector bundle of type $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(c_i)$, with $c_i \geq 1$. This would have allowed us to find another projection of $\tilde{J}(F_0')$ whose image is birational to $\tilde{J}(F_0')$ itself, which, however, it is not possible by our hyphotesis. Hence we get $a = b = 1$ which means that the only possible case is given by $n = 1$, so $\tilde{J}(\bar{F})$ is associated to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and $\tilde{J}(F_0')$ must be in correspondence with the bundle $\bigoplus^s \mathcal{O}_{\mathbb{P}^1}(1)$. The last birational step (and every other eventual birational step) only increases by one the degree of one of the bundle summands. At the end we always get a relation with a $\bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_i \geq 1$ and we obtain the triple defined in (ii). Notice that in order to have the situation we just described we need to ask for a starting point $s \geq 4$.
Let us deal now with the case $s = 3$. Such case is the one represented by the following

diagram

$$\begin{tikzcd}
 & \tilde{J}(F) \arrow[dl, "bir"'] \arrow[d, dashed, "bir"] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^2 \arrow[d, dashed, "pr_l"'] & \tilde{J}(F'_0) \arrow[dl, "\pi'_1"'] \arrow[dr, "\pi'_2"] & J(F) \arrow[d, equal, "pr_r"] \\
\mathbb{P}^1 & & J(F'_0)
\end{tikzcd}$$

Recalling the classification of the case $s = 2$, we must have that $\tilde{J}(F'_0)$ is associated to the bundle $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, with $a + b = n + 1$. Being $\tilde{J}(F'_0)$ a projection from an inner point of $\tilde{J}(F)$, we get that $\tilde{J}(F)$ can either be a rational normal scroll or the Veronese surface, in the special case $a = 2$ and $b = 1$ or viceversa, or the Hirzebruch surface. We exclude this last case because its projection would give us the trivial case $\mathbb{P}^1 \times \mathbb{P}^1$, which we have already studied. We can also exclude a rational normal scroll, because being a minimal surface, we would have that the projection on $\mathbb{P}^2$ given in the diagram is an isomorphism, which of course leads to a contradiction. The only case left is the Veronese surface and we can conclude that F is the Schwarzenberger bundle given by the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$. Notice that, in this particular case, $\tilde{J}(F'_0)$ is a cubic surface in $\mathbb{P}^4$

The Grassmannian case $k \geq 1$

Notice that in this case it is impossible to have $\dim \Sigma(F) = \dim \Sigma(F'_0) + 1$, or else we would obtain a morphism $\pi'_1 : \tilde{J}(F'_0) \longrightarrow \Sigma(F'_0)$ with 1-dimensional fiber. We already observed that the fiber can have dimension 0 or else dimension greater or equal than $k + 1$ and we would get a contradiction. So if we have $\dim \tilde{J}(F) > \dim \Sigma(F)$ then the only possible case is the trivial one, i.e. when $\tilde{J}(F) = \mathbb{P}(S) \times \mathbb{G}(k, n)$.
On the other hand, if $\dim \tilde{J}(F) = \dim \Sigma(F)$, this birational relation also remains in the subsequent steps of the induction.
Let us focus now on the last step of the induction, i.e. considering the case $s = k + 3$ and $s - 1 = k + 2$ and let us suppose that both $\tilde{J}(F)$ is birational to $\Sigma(F)$ and $\tilde{J}(F'_0)$ is birational to $\Sigma(F'_0)$. In order to do the induction we can manage to take a jumping pair

$s_0 \otimes \Gamma \in \tilde{J}(F)$ that is the unique point in the fiber $\pi_1^{-1}(s_0)$. Recall the induction diagram

$$\begin{tikzcd} & \tilde{J}(F) \arrow[dl, "\pi_1"'] \arrow[d, dashed, "pr_c"] \arrow[dr, "\pi_2"] & \\ \Sigma(F) \arrow[d, dashed, "pr_l"'] & \tilde{J}(F_0') \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & J(F) \arrow[d, equal, "pr_r"] \\ \Sigma(F_0') & & J(F_0') \end{tikzcd} \tag{4.3}$$

In this case we have that $\tilde{J}(F_0') \simeq \Sigma(F_0') \simeq \mathbb{P}^{k+1}$, because of the fact that, by the given classification $\tilde{J}(F_0')$ is the related to $G(k+1, E)$, where $\operatorname{rk} E = k+1$. We are in perfect situation in order to state the following result.

Lemma 4.3.3. *Let F be a Steiner bundle on $\mathbb{G}(k,n)$ and $\tilde{J}(F)$ its jumping locus. Suppose that $\tilde{J}(F)$ is birational to $\Sigma(F)$, and, fixed a jumping pair $s_0 \otimes \Gamma_0$, consider the first step of the induction (as described in diagram 4.3). If the morphism π_1' is an isomorphism, then also π_1 will be an isomorphism.*

Proof. Notice that for the generic $s_0 \in \Sigma(F)$, we have a unique associated jumping pair $s_0 \otimes \Gamma_0$. Suppose that π_1 is not an isomorphism, so we can find an element $s_1 \in \Sigma(F)$, with $s_1 \neq s_0$, associated with two independent jumping pairs $s_1 \otimes \Gamma_1$ and $s_1 \otimes \Gamma_2$. A consequence of this fact is that $pr_c(s_1 \otimes \Gamma_1) = pr_l(s_1) \otimes \Gamma_1$ and $pr_c(s_1 \otimes \Gamma_2) = pr_l(s_1) \otimes \Gamma_2$ are independent jumping pairs belonging to $\tilde{J}(F_0')$, associated with the non zero point $pr_l(s_1) \in \Sigma(F_0')$. This leads to contradiction because we supposed π_1' to be an isomorphism. □

Due to the lemma we get that $\tilde{J}(F) \simeq \Sigma(F)$ and, by Proposition 4.2.2 and its consequences, we know that $\Sigma(F) \simeq Q \subset \mathbb{P}^{k+2}$, where Q is the $(k+1)$-dimensional quadric in the projective space. Moreover we know that $J(F) = J(F_0') \simeq \mathbb{P}^{k+1}$, which tells us that the morphism π_2' is generically finite and the generic fiber consists of one point. Notice that π_2' is an isomorphism. Indeed, a morphism between two projective spaces of the same dimension, in this case $k+1$, is given by $k+1$ homogeneous independent forms in the coordinates of $\mathbb{P}^{k+1}$. Moreover, being generically finite and generically of degree 1 implies that the forms must be linear, hence we have an isomorphism.
Recalling diagram 4.3, which commutes, we are in the situation where π_2 is a morphism, π_2' is an isomorphism, while pr_c is not a morphism, because it is a projection from an inner point. This leads to a contradiction and we obtain that the only possible case is the one where $s = k+2$, which we have already classified.

I would like to observe that the contradiction could have also been obtained proving that a birational morphism between the $(k+1)$-dimensional quadric $Q \subset \mathbb{P}^{k+2}$ and $\mathbb{P}^{k+1}$ does not exists, which implies that π_2 cannot be defined. Indeed, if Q is a surface, we would have an isomorphism because of the minimality of the surface, which is impossible. If $\dim Q \geq 3$, then its Picard group is $\operatorname{Pic} Q = \mathbb{Z}[L]$, with L an ample divisor of the quadric. Notice that each fiber of π_2 is connected by Zariski's Main Theorem (see [Har77], Corollary 11.4), so if there exists a fiber of positive dimension, we must have a curve $C \subset Q$ which is contracted by π_2. Taking an hyperplane section $h \in |\mathcal{O}_{\mathbb{P}^{k+1}}(1)|$, we consider its pullback $\pi_2^* h = H = aL$ for some $a \geq 0$. Observing that the intersection $C.aL \neq 0$, it is a finite number of points, but $(\pi_{2*}C).H = 0$, we get a contradiction; so π_2 must be an isomorphism, which is impossible.

We have thus completed the classification, having considered all the possible cases.

Resumen en castellano
Espacios de salto en fibrados de Steiner

El problema de clasificación de fibrados vectoriales sobre variedades algebraicas ha sido siempre de gran interés en la geometría algebraica. Dada su amplitud, este tema ha sido siempre estudiado centrándose en familias de fibrados definidas por características específicas. En este trabajo nos centraremos en el estudio de fibrados de Steiner y de Schwarzenberger sobre la Grassmanniana.
En 1961 (véase [Sch61]), Schwarzenberger introdujo una familia de fibrados F de rango n relacionada con los espacios secantes de las curvas racionales normales y definida por una resolución del tipo

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^s(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^t \longrightarrow F \longrightarrow 0.$$

Desde entonces, muchos matemáticos han estudiado esta familia de fibrados, la mayoría intentando construir una configuración geométrica en el espacio proyectivo para definir el fibrado, y también intentando demostrar teoremas de tipo Torelli, i.e. recuperando la configuración desde un fibrado dado. Por ejemplo, en 1993 (véase [DK93]), Dolgachev and Kapranov, que fueron los primeros en denominar dichos fibrados Steiner, investigan fibrados logarítmicos sobre el espacio proyectivo definidos por 1-formas diferenciales sobre la unión de una colección de hiperplanos con normal crossing. En su artículo, definen las familias de fibrados de Steiner y Schwarzenberger como subfamilias de la logarítmica, y demuestran además resultados que relacionan las tres familias consideradas. En particular, demuestran que un fibrado logarítmico puede ser descrito como Steiner o Schwarzenberger bajo hipótesis especificas para la colección de hiperplanos.
En 2000 (véase [Val00b]), Vallès demuestra un resultado más general que caracteriza cuándo un fibrado de Steiner F puede ser descrito como un fibrado de Schwarzenberger.

Él se centra en una familia particular de hiperplanos $\{H_i\}$, que satisfacen la condición $h^0(F^{\vee}_{H_i}) \neq 0$ y que llama *hiperplanos inestables*, y demuestra que dichos hiperplanos, vistos como puntos en el espacio proyectivo dual, pertenecen siempre a una curva racional normal y esto permite ver el fibrado F como Schwarzenberger en el sentido dado en [DK93]. En 2001 (véase [AO01]), Ancona y Ottaviani renuevan la importancia del conjunto de hiperplanos inestables para un fibrado de Steiner F, demostrando que si tenemos un número suficiente de ellos, independientes dentro del conjunto, el fibrado F es también logarítmico.

La propiedad de estabilidad para fibrados de Steiner de rango n sobre $\mathbb{P}^n$ ha sido demostrada por Bohnhorst y Spindler, véase [BS92], y Brambilla, véase [Bra08], demuestra la estabilidad para fibrados vectoriales de Steiner excepcionales. Además, en su tesis de doctorado (véase [Bra04]), caracteriza los fibrados de Steiner generales, simples y excepcionales sobre el espacio proyectivo.

En [Val00a], Vallès propone una primera generalización de fibrados logarítmicos y de Schwarzenberger de rango mayor que la dimensión del espacio proyectivo base. Sin embargo, la primera generalización completa de fibrado de Schwarzenberger para rango arbitrario aparece en [Arr10a]. En su artículo, Arrondo generaliza principalmente dos nociones: la de fibrado de Schwarzenberger, que él asocia a una terna (X, L, M), donde X es una variedad proyectiva y L, M son dos fibrados vectoriales globalmente generados sobre X, y la noción de hiperplano inestable para un fibrado de Steiner F, al que él llama *hiperplano de salto*. Estudiando el lugar de los pares de salto, Arrondo consigue clasificar los fibrados de Steiner cuyo lugar tiene dimensión máxima y describirlos como Schwarzenberger.

El estudio de fibrados de Steiner sobre variedades distintas del espacio proyectivo lo han llevado a cabo Miró-Roig y Soares. Primero, en [Soa07] Soares define fibrados vectoriales de Steiner sobre hipercuádricas lisas $Q_n \subset \mathbb{P}^{n+1}$, con $n \geq 3$. Además, en su artículo caracteriza fibrados de Steiner excepcionales y simples sobre la hipercuádrica lisa y demuestra que en este caso excepcionalidad implica estabilidad. En [MRS09] y [Soa08], Miró-Roig y Soares dan una definición de fibrado de Steiner sobre una variedad algebraica cualquiera y demuestran una caracterización cohomológica de ellos. La definición propuesta depende de la elección de un par fuertemente excepcional de fibrados vectoriales sobre una variedad proyectiva.

En esta tesis obtendremos los resultados de Arrondo, eligiendo la definición de fibrado de Steiner sobre las Grassmannianas que tenga el mayor significado geométrico. Podemos resumir los problemas que queremos resolver en la siguiente lista.

Problema 1 *Encontrar la definición mas natural y geométrica de fibrado de Schwarzenberger para Grassmannianas.*

Problema 2 *Generalizar la definición de par de salto y dar una descripción del lugar de dichos pares para un fibrado de Steiner sobre la Grassmanniana.*

Problema 3 *Describir fibrados vectoriales de Steiner sobre* $\mathbb{G}(k,n)$ *con lugar de salto de dimensión máxima como fibrados de Schwarzenberger y dar una clasificación de este caso.*

En el capítulo 1 daremos los preliminares necesarios, recordando la definición y las propiedades de las Grassmannianas. Daremos también una introducción sobre el cálculo de Schubert.
En el capítulo 2 daremos la definición general de fibrado de Steiner para Grassmannianas, acorde con la dada por Miró-Roig y Soares y que es la generalización natural de la dada por Arrondo.

Definición 1. *Sean* S, T *dos espacios vectoriales sobre* $\mathbb{K}$*, respectivamente de dimensión* s *y* t*.*
Llamaremos un (s,t)*-fibrado de Steiner, sobre* $\mathbb{G}(k,n)$*, el fibrado vectorial definido por la resolución*

$$0 \longrightarrow S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0,$$

donde $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}(k,n)}$ *es el fibrado trivial y* $\mathcal{U} \to \mathbb{G}(k,n)$ *es el fibrado universal de rango* $k+1$*.*
Esto es equivalente a fijar una aplicación lineal

$$T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee) = \mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$$

tal que, para cada $u_1, \ldots, u_{k+1} \in H^0(\mathcal{U}^\vee)$ *linealmente independientes y para cada* $v_1, \ldots, v_s \in S^*$*, existe una* $f \in \mathrm{Hom}(H^0(\mathcal{U}^\vee), S^*)$ *tal que* $f \in \mathrm{Im}\,\varphi$ *y* $f(u_j) = v_j$ *para cada* $j = 1, \ldots, k+1$*.*
Si φ *es inyectiva, se dice que* F *es reducido y en general llamaremos* F_0 *al sumando reducido de* F *asociado a la aplicación lineal inyectiva*

$$\varphi(T^*) = T_0^* \hookrightarrow S^* \otimes H^0(\mathcal{U}^\vee).$$

Después de mostrar una interpretación geométrica de la definición, daremos una cota inferior para los posibles rangos de los fibrados que acabamos de definir, de hecho, demostraremos el siguiente resultado.

Teorema 2. *Sea* F *un fibrado de Steiner sobre* $\mathbb{G}(k,n)$*; entonces tendrá rango*

$$\mathrm{rk}\, F \geq \min((k+1)(n-k), (n-k)\dim S).$$

Para resolver el Problema 1, en el capítulo 3 daremos la definición de fibrado de Schwarzenberger, que generaliza la de [Arr10a].

Definición 3. *Consideramos dos fibrados vectoriales globalmente generados L, M sobre una variedad proyectiva X, con $h^0(M) = n+1$ y con la identificación $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. El fibrado de Schwarzenberger sobre $\mathbb{G}(k,n)$ asociado a la terna (X, L, M) será el fibrado definido por la resolución*

$$0 \longrightarrow H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0.$$

Como en el caso proyectivo, obtendremos los ejemplos mas importantes de fibrados de Schwarzenberger cuando L tiene rango uno y M tiene rango $k+1$.
Daremos después la definición de par de salto para un fibrado de Steiner y asociaremos una estructura algebraica al conjunto de todos las pares. Acotaremos la dimensión de dicho lugar a través de la descripción de su espacio tangente en un par fijado, que nos dará informaciones sobre el lugar de salto, visto como una variedad algebraica.
Considerando el fibrado dado por la terna (X, L, M), observamos que para cada punto $x \in X$, la imagen de $H^0(L \otimes M)^*$ a través del dual de la aplicación de multiplicación (la aplicación φ en este caso) restringida a la fibra sobre x tiene la forma particular $H^0(L_x)^* \otimes H^0(M_x)^*$, i.e. es el producto tensorial de dos subespacios vectoriales. Esta observación nos sugerirá la definición de un objeto similar para fibrados de Steiner y el lugar de dichos objetos nos dará información que nos permitirá construir una terna de Schwarzenberger, dado un fibrado de Steiner.

Definición 4. *Sea F un fibrado de Steiner sobre $\mathbb{G}(k,n)$. Un par (a, Γ), con* $\dim a = 1$ *y* $\dim \Gamma = k+1$, *tal que $a \otimes \Gamma \subset S^* \otimes H^0(\mathcal{U}^\vee)$, se llama par de salto si, considerando la aplicación $T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee)$, el producto tensorial $a \otimes \Gamma$ partenece a* $\operatorname{Im} \varphi$.

Para resolver el Problema 3, nuestro objetivo es describir y estudiar el lugar de los pares de salto asociado a un fibrado F de Steiner, denotaremos dicho lugar por $\tilde{J}(F)$ (con un abuso de notación utilizaremos $\tilde{J}(F)$ para el lugar visto como espacio vectorial y también visto como variedad proyectiva). Esto nos permitirá utilizar el siguiente resultado para clasificar fibrados de Steiner.

Teorema 5. *Sean $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ los fibrados universales de rango respectivamente* $1, k+1$ *y* $k+1$ *sobre $G(1, S^*), G(k+1, H^0(\mathcal{U}^\vee))$ y $G(k+1, T_0^*)$.*
Observamos que tenemos dos proyecciones naturales

$$\tilde{J}(F) \xrightarrow{\pi_1} G(1, S^*)$$

$$\tilde{J}(F) \xrightarrow{\pi_2} G(k+1, H^0(\mathcal{U}^\vee))$$

y que $\tilde{J}(F) \subset G(k+1, T_0^*)$. *Supongamos que las aplicaciones naturales*

$$\begin{array}{llll} \alpha: & H^0(G(1,S^*),\mathcal{A}) & \longrightarrow & H^0(\tilde{J}(F), \pi_1^*\mathcal{A}) \\ \beta: & H^0(G(k+1,H^0(\mathcal{U}^\vee)),\mathcal{B}) & \longrightarrow & H^0(\tilde{J}(F), \pi_2^*\mathcal{B}) \\ \gamma: & H^0(G(k+1,T_0^*),\mathcal{Q}) & \longrightarrow & H^0(\tilde{J}(F), \mathcal{Q}_{|\tilde{J}(F)}) \end{array}$$

sean todas isomorfismos. Entonces el fibrado de Steiner F_0*, sumando reducido de* F*, es un fibrado de Schwarzenberger dado por la terna*

$$(\tilde{J}(F), \pi_1^*\mathcal{A}, \pi_2^*\mathcal{B}).$$

Conseguiremos dar una descripción geométrica del lugar de salto, visto como una variedad proyectiva. De hecho, si consideramos la aplicación de Segre generalizada

$$\begin{array}{cccccc} \nu: & \mathbb{P}(S) & \times & \mathbb{G}(k, \mathbb{P}(H^0(\mathcal{U}^\vee))^*) & \longrightarrow & \mathbb{G}(k, \mathbb{P}(S \otimes H^0(\mathcal{U}^\vee)^*)) \\ & \mathbb{P}(l) & , & \mathbb{P}(\Lambda) & \mapsto & \mathbb{P}(l \otimes \Lambda) \end{array}$$

entonces es posible definir

$$\tilde{J}(F) = \operatorname{Im}\nu \cap \mathbb{G}(k, \mathbb{P}(T_0))$$

donde, como siempre, $T_0^* = \varphi(T^*)$ es el espacio vectorial asociado al sumando reducido de F. Nuestro objetivo es investigar la dimensión de esta variedad. Observamos que obtenemos una cota inferior calculando la dimensión esperada de la intersección, que es

$$\dim \tilde{J}(F) \geq (k+1)(t-k-sn-s+n)+s-1.$$

Para conseguir una cota superior, estudiaremos el espacio tangente de $\tilde{J}(F)$ en un punto Λ que representa un par de salto.
Después de dar una descripción del tangente de una variedad de Segre generalizada en un punto Λ a través del álgebra lineal, demostraremos un resultado técnico de álgebra lineal que nos dará la cota buscada.

Teorema 6. *Sea* F *un fibrado de Steiner sobre* $\mathbb{G}(k,n)$ *y sea* $\tilde{J}(F)$ *su lugar de los pares de salto; entonces, considerando* $\Lambda \in \tilde{J}(F)$*, obtenemos*

$$\dim \tilde{J}(F) \leq \dim T_\Lambda \tilde{J}(F) \leq (k+1)(t-(k+1)(s+n-k-1)-k).$$

En el capítulo 4, clasificaremos los fibrados de Steiner cuyo lugar de salto tiene dimen-

sión máxima.
Observamos que dado un (s,t)-fibrado de Steiner F sobre $\mathbb{G}(k,n)$, que supongamos sea reducido, con lugar de salto de dimensión máxima, si fijamos un par de salto $s_0 \otimes \Gamma = \varphi(\Lambda)$ entonces podemos inducir un $(s-1, t-k-1)$-fibrado de Steiner F' sobre $\mathbb{G}(k,n)$, que puede no ser reducido, pero tendrá también lugar de salto de dimensión máxima. Dicha inducción es consecuencia del siguiente diagrama conmutativo

$$\begin{tikzcd} T^* \arrow[r, hook, "\varphi"] \arrow[d] & S^* \otimes H^0(\mathcal{U}^\vee) \arrow[d] \\ \frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{<s_0>} \otimes H^0(\mathcal{U}^\vee) \end{tikzcd}$$

Dado un fibrado de Steiner sobre la Grassmanniana, induciremos fibrados de Steiner para tantos pasos como necesitemos para llegar al caso mas básico $s = k+1$, que está clasificado por el siguiente resultado.

Teorema 7. *Sea F un fibrado de Steiner reducido sobre $\mathbb{G}(k,n)$, con $\dim S = k+2$, entonces F puede ser descrito como un fibrado de Schwarzenberger dado por la terna $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$, donde E es el fibrado vectorial definido como el núcleo del morfismo sobreyectivo*

$$H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)}.$$

Después, estudiando el diagrama construido, clasificaremos todos los posibles casos y gracias al Teorema 5 conseguiremos encontrar una terna que describa el fibrado como Schwarzenberger. Obtendremos la siguiente clasificación.

Teorema 8. *Sea F un fibrado de Steiner reducido sobre $\mathbb{G}(k,n)$ con $\dim \tilde{J}(F)$ maximal; entonces estamos en uno de los siguientes casos:*

(i) *F es el fibrado de Schwarzenberger dado por la terna $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$. En este caso $k=0$ y $t = n+s$.*

(ii) *F es el fibrado de Schwarzenberger dado por la terna $(\mathbb{P}^1, E(-1), \mathcal{O}_{\mathbb{P}^1}(1))$, donde $E = \oplus_{i=1}^{t-s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ con $a_i \geq 1$. En este caso $k=0$ y $n=1$.*

(iii) *F es el fibrado de Schwarzenberger dado por la terna $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$ definido en el Teorema 7. En este caso $s = k+2$.*

(iv) *F es el fibrado de Schwarzenberger dado por la terna $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$. En este caso $k = 0, n = 2, s = 3$ y $t = 6$.*

Riassunto in italiano

Spazi di salto in fibrati di Steiner

Il problema di classificazione dei fibrati vettoriali su varietà algebriche è sempre stato di grande interesse nella geometria algebrica. Vista la grandezza dell'argomento, è stato sempre studiato concentrandosi su famiglie di fibrati definite da caratterizzazioni specifiche. In questo lavoro ci concentreremo sullo studio dei fibrati di Steiner e di Schwarzenberger sulle Grassmanniane.
Nel 1961 (vedi [Sch61]), Schwarzenberger introduce una famiglia di fibrati F di rango n che sono in relazione con lo spazio delle secanti di una curva razionale normale e definiti da una risoluzione del tipo

$$0 \longrightarrow \mathcal{O}^s_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}^t_{\mathbb{P}^n} \longrightarrow F \longrightarrow 0.$$

Da allora, molte persone hanno studiato queste due famiglie di fibrati, cercando di costruire, nella maggior parte dei casi, delle configurazioni geometriche nello spazio proiettivo che definiscano il fibrato, e dimostrando anche dei teoremi di tipo Torelli, i.e. recuperando la configurazione da un dato fibrato. Per esempio, nel 1993 (vedi [DK93]), Dolgachev e Kapranov, che furono i primi a denotare tali fibrati come Steiner, studiano fibrati logaritmici sullo spazio proiettivo definiti da 1-forme differenziali sull'unione di una collezione di iperpiani aventi normal crossing. Nel loro lavoro, definiscono le famiglie dei fibrati di Steiner e di Schwarzenberger come sottofamiglie di quella dei logaritmici, e inoltre dimostrano dei risultati che mettono in relazione tra di loro i tre insiemi di fibrati considerati. In particolare dimostrano che un fibrato logaritmico può essere descritto come Steiner o Schwarzenberger sotto particolari ipotesi sulla collezione degli iperpiani.
Nel 2000 (vedi [Val00b]), Vallès dimostra un risultato più generale che caratterizza quando un fibrato di Steiner F può essere descritto anche come uno Schwarzenberger. La sua ricerca si concentra su una particolare famiglia di iperpiani $\{H_i\}$, che soddisfano la con-

dizione $h^0(F^\vee_{H_i}) \neq 0$ e che chiama *iperpiani instabili*, provando che tali iperpiani, visti come punti nello spazio proiettivo duale, appartengono sempre ad una curva razionale normale e ciò permette di vedere il fibrato F come uno Schwarzenberger secondo la definizione considerata in [DK93].

Nel 2001 (vedi [AO01]), Ancona ed Ottaviani rafforzano l'importanza dell'insieme degli iperpiani instabili di un fibrato di Steiner F, dimostrando che se abbiamo un numero sufficiente di essi indipendenti, allora il fibrato F può essere anche descritto come logaritmico.

La proprietà di stabilità per fibrati di Steiner di rango n su $\mathbb{P}^n$ è stata dimostrata da Bohnhorst e Spindler, vedi [BS92], mentre Brambilla, vedi [Bra08], la dimostra per fibrati vettoriali di Steiner eccezionali. Inoltre, nella sua tesi di dottorato (vedi [Bra04]), fornisce una caratterizzazione per fibrati di Steiner generali, semplici ed eccezionali sullo spazio proiettivo.

In [Val00b], Vallès propone una prima generalizzazione dei concetti di fibrato logaritmico e di Schwarzenberger di rango maggiore della dimensione dello spazio proiettivo di base. Tuttavia, la prima generalizzazione completa di fibrati di Schwarzenberger di rango arbitrario su spazi proiettivi compare in [Arr10a]. Nel suo lavoro, Arrondo generalizza principalmente due nozioni: quella di fibrato di Schwarzenberger, che viene associato ad una terna (X, L, M), dove X è una varietà proiettiva ed L ed M sono due fibrati vettoriali globalmente generati su X, e quella di iperpiano instabile per un fibrato di Steiner F, che viene chiamato *iperpiano di salto*. Studiando il luogo delle coppie di salto, Arrondo riesce a classificare i fibrati di Steiner il cui luogo di salto ha dimensione massima e a descriverli come fibrati di Schwarzenberger.

Lo studio dei fibrati di Steiner su varietà distinte dallo spazio proiettivo è stato dato da Miró-Roig e Soares. Come primo caso, in [Soa07] Soares definisce i fibrati vettoriali di Steiner su una iperquadrica liscia $Q_n \in \mathbb{P}^{n+1}$, con $n \geq 3$. Inoltre, caratterizza i fibrati di Steiner eccezionali e semplici sull'iperquadrica liscia e dimostra che in questo caso l'eccezionalità implica la stabilità. In [MRS09] e [Soa08], Miró-Roig e Soares presentano una definizione di fibrato di Steiner per una qualsiasi varietà algebrica e dimostrano una caratterizzazione coomologica per tali fibrati. La definizione proposta dipende dalla scelta di una coppia fortemente eccezionale di fibrati vettoriali su una varietà proiettiva.

In questa tesi otterremo i risultati di Arrondo scegliendo però la definizione di fibrato di Steiner sulla Grassmanniana che abbia il maggior significato geometrico.

Possiamo elencare i problemi che vogliamo risolvere nella seguente lista.

Problema 1 *Trovare la definizione più naturale e più geometrica di fibrato di Schwarzenberger per Grassmanniane.*

Problema 2 *Generalizzare la definizione di coppia di salto e dare una descrizione del luogo di tali coppie per un fibrato di Steiner sulla Grassmanniana.*

Problema 3 *Descrivere i fibrati di Steiner su $\mathbb{G}(k, n)$ con luogo di salto di dimensione massima come fibrati di Schwarzenberger e fornire una classificazione di questo caso.*

Nel capitolo 1 daremo i preliminari necessari, richiamando la definizione e le proprietà della Grassmanniana. Daremo inoltre un'introduzione sul calcolo di Schubert.
Nel capitolo 2 stabiliremo la definizione generale di fibrato di Steiner per Grassmanniane, in accordo con quella data da Miró-Roig e Soares e che rappresenta la generalizzazione naturale di quella data da Arrondo.

Definizione 1. *Siano S, T due spazi vettoriali su $\mathbb{K}$, di dimensione rispettivamente s e t. Chiameremo un (s, t)-fibrato di Steiner, su $\mathbb{G}(k, n)$, il fibrato vettoriale definito dalla risoluzione*

$$0 \longrightarrow S \otimes \mathcal{U} \longrightarrow T \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0,$$

dove $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}(k,n)}$ denota il fibrato di linea banale e $\mathcal{U} \longrightarrow \mathbb{G}(k, n)$ denota il fibrato universale di rango $k + 1$.
Ciò è equivalente a fissare un'applicazione lineare

$$T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee) = \mathrm{Hom}(H^0(\mathcal{U}^\vee)^*, S^*)$$

tale che, per ogni $u_1, \dots, u_{k+1} \in H^0(\mathcal{U}^\vee)^*$ linearmente indipendenti e per ogni $v_1, \dots, v_{k+1} \in S^*$, esista una $f \in \mathrm{Hom}(H^0(\mathcal{U}^\vee), S^*)$ tale che $f \in \mathrm{Im}\,\varphi$ e $f(u_j) = v_j$ per ogni $j = 1, \dots, k + 1$.
Se φ è iniettiva chiameremo F ridotto o in caso contrario denoteremo con F_0 l'addendo ridotto di F, associato all'applicazione lineare iniettiva

$$\varphi(T^*) = T_0^* \hookrightarrow S^* \otimes H^0(\mathcal{U}^\vee).$$

Dopo aver fornito un'interpretazione geometrica della definizione, daremo un limite dal basso per il rango dei fibrati appena definiti, dimostreremo infatti il seguente risultato.

Teorema 2. *Sia F un fibrato di Steiner su $\mathbb{G}(k, n)$. Allora F sarà di rango*

$$\mathrm{rk}\, F \geq \min((k + 1)(n - k), (n - k) \dim S).$$

Per risolvere il Problema 1, nel capitolo 3 proporremo la definizione di fibrato di

Schwarzenberger, che generalizza quella data in [Arr10a].

Definizione 3. *Consideriamo due fibrati vettoriali L, M globalmente generati su una varietà proiettiva X, con $h^0(M) = n + 1$ e con l'identificazione $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. Il fibrato di Schwarzenberger su $\mathbb{G}(k, n)$ associato alla terna (X, L, M) sarà il fibrato definito dalla risoluzione*

$$0 \longrightarrow H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0.$$

Come nel caso proiettivo, gli esempi più significativi di fibrato di Schwarzenberger si otterranno considerando L di rango uno ed M di rango $k + 1$.
Daremo poi la definizione di coppia di salto per un fibrato di Steiner e daremo inoltre una struttura algebrica all'insieme di tali coppie. Dimostreremo un limite dall'alto della dimensione del luogo delle coppie di salto attraverso la descrizione dello spazio tangente in una di esse, che ci darà informazioni sul luogo di salto, visto come una varietà algebrica.
Considerando il fibrato definito dalla terna (X, L, M), osserviamo che per ogni punto $x \in X$, l'immagine di $H^0(L \otimes M)^*$ attraverso la duale della mappa di moltiplicazione (che un questo caso è l'applicazione φ), ristretta alla fibra su x, ha la forma particolare $H^0(L_x)^* \otimes H^0(M_x)^*$, i.e. è data dal prodotto tensoriale di due sottospazi vettoriali. Questa osservazione ci porta a definire un oggetto simile per fibrati di Steiner e il luogo di tali oggetti ci darà informazioni che ci permetteranno costruire la terna di un fibrato di Schwarzenberger, partendo dal fibrato di Steiner.

Definizione 4. *Sia F un fibrato di Steiner su $\mathbb{G}(k, n)$. Una coppia (a, Γ), con $\dim a = 1$ e $\dim \Gamma = k + 1$, tale che $a \otimes \Gamma \subset S^* \otimes H^0(\mathcal{U}^\vee)$, è detta coppia di salto se, considerando l'applicazione $T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{U}^\vee)$, il prodotto tensoriale $a \otimes \Gamma$ appartiene a $\operatorname{Im} \varphi$.*

Per risolvere il Problema 3, il nostro obiettivo è descrivere e studiare il luogo delle coppie di salto associato ad un fibrato di Steiner F, che denoteremo con $\tilde{J}(F)$ (con un abuso di notazione, useremo $\tilde{J}(F)$ per il luogo visto sia come spazio vettoriale sia come varietà proiettiva). Questo ci permetterà utilizzare il seguente risultato, utile per la classificazione dei fibrati di Steiner.

Teorema 5. *Siano $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ i fibrati universali di rango rispettivamente 1 , $k+1$ e $k+1$ su $G(1, S^*), G(k+1, H^0(\mathcal{U}^\vee))$ e $G(k+1, T_0^*)$.*
Osserviamo che abbiamo due proiezioni naturali

$$\tilde{J}(F) \xrightarrow{\pi_1} G(1, S^*)$$

$$\tilde{J}(F) \xrightarrow{\pi_2} G(k+1, H^0(\mathcal{U}^\vee))$$

e che $\tilde{J}(F) \subset G(k+1, T_0^*)$*. Supponiamo che le applicazioni naturali*

$$\begin{array}{llcl} \alpha: & H^0(G(1,S^*),\mathcal{A}) & \longrightarrow & H^0(\tilde{J}(F),\pi_1^*\mathcal{A}) \\ \beta: & H^0(G(k+1,H^0(\mathcal{U}^\vee)),\mathcal{B}) & \longrightarrow & H^0(\tilde{J}(F),\pi_2^*\mathcal{B}) \\ \gamma: & H^0(G(k+1,T_0^*),\mathcal{Q}) & \longrightarrow & H^0(\tilde{J}(F),\mathcal{Q}_{|\tilde{J}(F)}) \end{array}$$

siano tutti isomorfismi. Allora il fibrato di Steiner F_0*, addendo ridotto di* F*, è un fibrato di Schwarzenberger dato dalla terna*

$$(\tilde{J}(F), \pi_1^*\mathcal{A}, \pi_2^*\mathcal{B}).$$

Riusciremo inoltre a dare una descrizione geometrica del luogo di salto, visto come una varietà proiettiva. Infatti, se consideriamo l'applicazione di Segre generalizzata

$$\begin{array}{cccccc} \nu: & \mathbb{P}(S) & \times & \mathbb{G}(k,\mathbb{P}(H^0(\mathcal{U}^\vee))^*) & \longrightarrow & \mathbb{G}(k,\mathbb{P}(S\otimes H^0(\mathcal{U}^\vee)^*)) \\ & \mathbb{P}(l) & , & \mathbb{P}(\Lambda) & \mapsto & \mathbb{P}(l\otimes\Lambda) \end{array}$$

allora è possibile definire

$$\tilde{J}(F) = \mathrm{Im}\,\nu \cap \mathbb{G}(k,\mathbb{P}(T_0))$$

dove, como al solito, $T_0^* = \varphi(T^*)$ denota lo spazio vettoriale associato all'addendo ridotto di F. Il nostro obiettivo è quello di studiare la dimensione di tale varietà. Osserviamo che un limite dal basso è dato calcolando la dimensione attesa dell'intersezione, ottenendo

$$\dim \tilde{J}(F) \geq (k+1)(t-k-sn-s+n)+s-1.$$

Per avere un limite dall'alto, studieremo lo spazio tangente a $\tilde{J}(F)$ in un punto Λ rappresentante una coppia di salto.
Dopo aver fornito una descrizione dello spazio tangente della varietà di Segre generalizzata nel punto Λ attraverso l'algebra lineare, dimostreremo un risultato tecnico, sempre di algebra lineare, che ci darà il limite richiesto.

Teorema 6. *Sia* F *un fibrato di Steiner su* $\mathbb{G}(k,n)$ *e sia* $\tilde{J}(F)$ *il suo luogo di salto; allora, considerando* $\Lambda \in \tilde{J}(F)$*, abbiamo che*

$$\dim \tilde{J}(F) \leq \dim T_\Lambda \tilde{J}(F) \leq (k+1)(t-(k+1)(s+n-k-1)-k).$$

Nel capitolo 4, classificheremo i fibrati di Steiner il cui luogo di salto ha dimensione massima.

Osserviamo che, dato un (s,t)-fibrato di Steiner F su $\mathbb{G}(k,n)$ che supponiamo sia ridotto, con luogo di salto di dimensione massima, se fissiamo una coppia di salto $s_0 \otimes \Gamma = \varphi(\Lambda)$ allora possiamo indurre un $(s-1, t-k-1)$-fibrato di Steiner F' su $\mathbb{G}(k,n)$, che potrebbe non essere ridotto, ma il cui luogo di salto sarà di dimensione massima. Tale induzione è una conseguenza del diagramma commutativo seguente

$$\begin{tikzcd} T^* \arrow[r, hook, "\varphi"] \arrow[d, two heads] & S^* \otimes H^0(\mathcal{U}^\vee) \arrow[d, two heads] \\ \frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{<s_0>} \otimes H^0(\mathcal{U}^\vee) \end{tikzcd}$$

Dato un fibrato di Steiner sulla Grassmanniana, indurremo ulteriori fibrati di Steiner per tanti passi quanti necessari ad arrivare al caso base $s = k+2$, che è classificato dal seguente risultato.

Teorema 7. *Sia F un fibrato di Steiner ridotto su $\mathbb{G}(k,n)$, con $\dim S = k+2$, allora F può essere descritto come il fibrato di Schwarzenberger dato dalla terna $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$, dove E è il fibrato vettoriale definito come il nucleo del morfismo suriettivo*

$$H^0(\mathcal{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \longrightarrow \frac{S^* \otimes H^0(\mathcal{U}^\vee)}{T^*} \otimes \mathcal{O}_{\mathbb{P}(S)}.$$

A questo punto, osservando il diagramma costruito, possiamo classificare tutti i possibili casi attraverso il Teorema 5 e riusciamo inoltre a costruire una terna che descriva tali fibrati come Schwarzenberger. Otterremo la seguente classificazione

Teorema 8. *Sia F un fibrato di Steiner ridotto su $\mathbb{G}(k,n)$ con $\dim \tilde{J}(F)$ massima; allora siamo in uno dei seguenti casi:*

(i) *F è il fibrato di Schwarzenberger dato dalla terna $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-1), \mathcal{O}_{\mathbb{P}^1}(n))$. In questo caso $k = 0$ e $t = n + s$.*

(ii) *F è il fibrato di Schwarzenberger dato dalla terna $(\mathbb{P}^1, E(-1), \mathcal{O}_{\mathbb{P}^1}(1))$, dove $E = \oplus_{i=1}^{t-s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ con $a_i \geq 1$. In questo caso $k = 0$ e $n = 1$.*

(iii) *F è il fibrato di Schwarzenberger dato dalla terna $(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(1), E^\vee(-1))$ definito nel Teorema 7. In questo caso $s = k + 2$.*

(iv) *F è il fibrato di Schwarzenberger dato dalla terna $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$. In questo caso $k = 0, n = 2, s = 3$ e $t = 6$.*

Bibliography

[ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris. *Geometry of Algebraic Curves, Volume I.* Springer-Verlag, 1985.

[AO01] V. Ancona and G. Ottaviani. Unstable hyperplanes for Steiner bundles and multidimensional matrices. *Adv. Geom.*, 1:165–192, 2001.

[Arr96] E. Arrondo. Subvarieties of Grassmannians. *Lecture Note Series Dipartimento di Matematica Univ. Trento*, 10, 1996.

[Arr10a] E. Arrondo. Schwarzenberger bundles of arbitrary rank on the projective space. *J. of Lond. Math. Soc.*, 82:697–716, 2010.

[Arr10b] E. Arrondo. Vector bundles in Algebraic Geometry, unpublished notes. 2010.

[Bra04] C. Brambilla. *Simplicity of vector bundles on $\mathbb{P}^n$ and exceptional vector bundles.* PhD thesis, University of Florence, 2004.

[Bra08] C. Brambilla. Cokernel bundles and Fibonacci bundles. *Math. Nachr.*, 208(4):499–516, 2008.

[BS92] G. Bohnhorst and H. Spindler. The stability of certain vector bundles on $\mathbb{P}^n$. *Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math.*, 1507:39–50, 1992.

[DK93] I. Dolgachev and M. Kapranov. Arrangements of hyperplanes and vector bundles on $\mathbb{P}^n$. *Duke Math. J.*, 71(3):633–664, 1993.

[Ful98] W. Fulton. *Intersection theory.* Springer, 1998.

[GH78] P.A. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley, 1978.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[Har92] J. Harris. *Algebraic Geometry. A First Course.* Springer-Verlag, 1992.

[MRS09] R.M. Miró-Roig and H. Soares. Cohomological characterisation of Steiner bundles. *Forum math.*, 21:871–891, 2009.

[Ott87] G. Ottaviani. A class of n-bundle on $\mathbb{G}(k, n)$. *J. Reine Angew. Math.*, 379:182–208, 1987.

[Sch61] R.L.E Schwarzenberger. Vector bundles on the projective plane. *Proc. London Math. Soc.*, 11:633–640, 1961.

[Soa07] H. Soares. Steiner bundles on the hyperquadric $Q_n \subset \mathbb{P}^{n+1}, n \geq 3$. *Commun. Algebra*, 35(12):4116–4136, 2007.

[Soa08] H. Soares. *Steiner vector bundles on algebraic varieties.* PhD thesis, Barcelona, 2008.

[Val00a] J. Vallès. Fibrés de Schwarzenberger et fibrés logarithmiques généralisés. *Bull. Soc. Math. France*, 128:433–449, 2000.

[Val00b] J. Vallès. Nombre maximal d'hyperplans instables pour un fibré de Steiner. *Math. Z.*, 233:507–514, 2000.