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Identification of a source term and a coefficient in a parabolic degenerate problem

MAT/05 - Analisi Matematica

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Contents

Notations

1. X is some Banach space endowed with the norm $\|\cdot\|_X$.

2. X^* - the space of all linear and bounded functionals on X.

3. $A : D(A) \subset X \to X$ is a linear closed operator.

4. $D(A)$ is the domain of the operator A, endowed with the graph-norm of operator A i.e. $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$.

5. $\rho(A)$ is the resolvent set of the operator A.

9. $L(X;Y)$, X and Y being Banach spaces, denotes the Banach space of all linear bounded operators from X to Y.

6. $L^p(\Omega;X)$ is the space of all p-integrable X -valued functions over $\Omega \subset \mathbb{R}^n, p \in [1,+\infty]$.

7. $W^{1,p}(0,T;Y)$, Y being a Banach space, is the space of all functions $f \in L^p(0,T;Y)$ whose distributional derivatives f' belong to $L^p(0,T;Y)$.

8. $W^{1,p}(0,T;Y)$ is endowed with a norm $\|f\|_{W^{1,p}(0,T;Y)} = \|f\|_{L^p(0,T;Y)} + \|f'\|_{L^p(0,T;Y)}$.

9. $\mathcal{D}_A(\gamma,\infty)$ is the interpolation space between $D(A)$ and X.

10. μ is a positive finite Borel measure on $[0,T]$, whose support is *not* concentrated at $t = 0$.

11. $\arg\lambda \in (-\pi,\pi]$ is the principal argument of a complex number λ.

Abstract

The globally in time existence and uniqueness of solutions to inverse problems is one of the most difficult questions to be answered. Even though the direct problems are well-posed in the sense of Hadamard (i.e. existence, uniqueness and stability results hold true), the inverse ones generally are not. The situation gets more complicated if the equation contains more than one unknown coefficient, and even more if the unknown functions depend on different variables.

We consider the following identification abstract problem in a general Banach space X: find a function $u : [0,T] \to X$, a coefficient $a_1 : [0,T] \to \mathbb{R}$ and a vector $z \in X$ such that the initial-value problem

$$\frac{1}{a_0(t)}\, u'(t) - Au(t) - a_1(t)u(t) = f(t)z + g(t), \qquad u(0) = u_0 \tag{0.1}$$

is fulfilled, where $a_0(t) > 0$ and $a_0(t) = 0$ only in some negligible set, while $A : D(A) \subset X \to X$ is a closed linear operator, f is scalar functions and g is a X-valued source term. The occurrence of two unknowns require to introduce two additional conditions. We choose the first as nonlocal one in the integral form $\int_0^T \varphi(t)u(t)d\mu(t) = h$, where μ is a Borel measure on the interval $[0,T]$. The latter is of the following form: $\Phi[u(t)] = k(t),\ t \in [0,T]$, where Φ is a prescribed linear continuous functional. Here the functions $h,\ k, \varphi$ are scalar. So, we investigate the problem (0.1) along with these additional conditions. We study explicitly the case of the *Dirac measure* concentrated at $t = T_1, 0 < T_1 \leq T$ and the one of an *absolutely continuous measure* μ.

This thesis is devoted to investigation of inverse problems for degenerate parabolic equations aiming at the determination of one time-dependent coefficient a_1 and a spatial source term z. So, the goal of this work is to find sufficient conditions on our data and operator A under which the problem turns out to be well-posed.

By means of Semigroup Theory and the Banach fixed-point theorem, we can find out sufficient conditions on the data (f, g, u_0, h, k) ensuring *global-in-time* existence and uniqueness for the solution $(a_1, u, z) \in L^1(0,T;\mathbb{R}) \times \big[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\big] \times X$. Moreover, a continuous dependence of Lipschitz type of the solution on the data is provided. We stress that we are obliged to introduce an unusual distance involving the data accounting for the degeneracy of function a_0. Finally, using a suitable metric for the data, we apply such results to a concrete parabolic problem.

Introduction

We consider the following *identification problem*: let Ω be an open bounded domain in $\mathbb{R}^n$, with boundary of C^2-class and let us consider the problem of *recovering the unknown functions* $\mathbf{a_1} : [0,T] \to \mathbb{R}$, $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}$ *and* $\mathbf{z} : \Omega \to \mathbb{R}$ satisfying

$$\beta(t)D_t u(t,x) - \mathcal{A}(x,D_x)u(t,x) - \mathbf{a_1}(\mathbf{t})u(t,x) = f(t)\mathbf{z}(\mathbf{x}) + g(t,x), \ (t,x) \in [0,T] \times \Omega, \tag{0.2}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \quad Bu(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{0.3}$$

$$\int_0^T \varphi(t)u(t,x)d\mu(t) = h(x), \quad x \in \Omega, \quad \int_\Omega \psi(x)u(t,x)dx = k(t), \quad t \in [0,T]. \tag{0.4}$$

where $\beta(t) > 0$ for a.e. $t \in (0,T)$ and $\mathcal{A}$ is a second-order uniformly elliptic linear differential operator. The linear operator B in (0.3) is defined by either Dirichlet or Robin boundary condition. Let us note that similar problem was investigated by A. Lorenzi and Y. Anikonov in their work [1] in the case when $\beta, \varphi = 1$ and $a_1, g = 0$.

The problem (0.2), (0.3), (0.4) faces two different aspects involving both direct and inverse problems: *1) parabolic degenerate problems, 2) identification of two functions depending on different variables, involving time and space.* The main difficulty arising in our problem is that the unknowns belong to different function spaces which are – as will be shown in the thesis – *competitive* in some sense.

We emphasize that determining a coefficient depending only on t is a problem simpler than finding out a spatial dependent element. In the literature many authors widely studied the problem of identifying the time dependent coefficients in the regular case, and in several cases they deal with one dimensional problems. For instance, we cite the works of G.Snitko, Jones [27, 10].

Inverse problems for degenerate parabolic equations of type

$$u_t = a(t)u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t), 0 < x < h(t), 0 < t < T$$

when the principal coefficient $a = a(t)$ is unknown and $a(t) > 0, t \in (0,T]$, $a(0) = 0$, were studied in the papers [26], [8], [9].

Results concerning uniqueness and stability when determining the coefficient $a(x)$ in the parabolic equation

$$\partial_t u + Au + a(x)u = 0,$$

endowed with zero initial condition and nonzero boundary condition $Bu = \phi$ in $\partial Q \times (0,T)$ were obtained in [4] by the so-called final measurement by using an optimization control framework.

The problem of identifying the unknown right-hand side $q(x)$ in the parabolic equation

$$u_t - \triangle u + a(x,t)u = h(x,t)q(x) + f(x,t)$$

when u satisfies the following conditions

$$u(x,0) = 0, x \in \Omega, \quad u(x,t)|_{\Gamma\times(0,T)} = 0, \quad (\Phi u)(x) = \varphi(x), x \in \Omega.$$

was developed in [12]. Regarding the notations, Φ is a linear operator from $L^\infty(0,T;L^2(\Omega))$ to $L^2(\Omega)$ and the functions a, f, h, φ are given. Using the method of continuation with respect to a parameter, some theorem of existence and uniqueness for solutions in Sobolev space are proved. The continuous dependence is not established.

Some researchers have made attempts to solve the inverse problems consisting of simultaneously identifying unknowns depending on space and time variables. We can refer reader, e. g. to the works [5], [6], where M. Ivanchov investigated an inverse problem in a bounded domain $Q_T = \{(x,t) : x \in D \in \mathbb{R}^n, 0 < t < T\}$ for the multidimensional heat equation

$$u_t = \Delta u + g_0(x,t) + f_1(x)g_1(t) + f_2(t)g_2(x), \quad (x,t) \in D \subset R^n \times (0,T),$$

with *two unknown terms* $f_1(x)$ *and* $f_2(t)$ *in the source* under the conditions

$$u(x,0) = \varphi(x), \quad x \in \overline{D}, \qquad u(x,t)\big|_{S_T} = \mu(x,t), \quad (x,t) \in S_T \equiv S \times [0,T],\ S = \partial D,$$

$$u(x_0,t) = \varkappa(t), \quad x_0 \in D,\ t \in [0,T], \qquad \int_0^T u(x,t)dt = \psi(x),\ x \in \overline{D}, \quad f_1(x_0) = f_0,$$

where f_0 is a given constant. The solution (f_1, f_2, u) of the last problem is found from the Hölder class $H^\gamma(\overline{D}) \times H^{\gamma/2}[0,T] \times H^{2+\gamma,1+\gamma/2}(\overline{Q}_T), 0 < \gamma < 1$ and, for sufficiently small $T > 0$, that solution exists and is unique. On the contrary, *in this thesis we shall search for a global in time solution from Sobolev type spaces* for the problem (0.2)–(0.4).

The identification of only unknown source terms, under various additional conditions, have been analyzed by many authors (e.g. cf.[11, 19, 21, 22, 23, 24, 2]). The results established in those articles can be classified by the generality of the equation, dependence of the unknown parameter of the non-homogeneous term either on spatial variables or on time, and on the form of the overdetermination conditions. In this regard, we also refer the reader to the works, i.e., by A.I. Prilepko and A. B. Kostin established in [22, 23] dealing with the existence and uniqueness of the generalized solution $(f(x), u(x,t))$ of the inverse problem for the equation

$$u_t - L(x, D_x)u = f(x)h(x,t) + g(x,t), \quad (x,t) \in Q_T \equiv \Omega \times (0,T),$$

where either final or integral overdetermination are available.

A similar result in the classes of Hölder functions was obtained by A.I. Prilepko and V. V. Solovyev in [25]. As for the problems of determination of a time-dependent multiplier in the source term, the corresponding inverse problems for the equation in general form are considered only in the one-dimensional case by O. I. Prilepko and V.V. Solovyev [24] with the first boundary condition and the overdetermination condition of the form

$$u(x_0, t) = \chi(t), \quad 0 < x_0 < l.$$

In the case of several dimensions the corresponding inverse problem was investigated by A. I. Prylepko, A. L. Ivankov, and V. V. Soloviev [21] for the equation

$$u_t - \mathcal{L}u = f(t)h(x,t), \quad (x,t) \in D_T = G \times (0,T),$$

where G is a bounded domain in $\mathbb{R}^n, \mathcal{L}u = \operatorname{div}(k\nabla u) + a(x)u$, with the third boundary condition and the integral overdetermination condition.

A big amount of inverse problems related to determining unknown coefficients is devoted to the case when the coefficients are regular. However, far too little attention has been paid to the problems concerning the singular coefficients in parabolic equations. In particular, M. Ivanchov, A. Lorenzi and N. Saldina in [7] analyze such a case in the following problem

$$\begin{aligned}
&D_t v(t,x) = a(t)\mathcal{A}v(t,x) + f(t,x), \ (t,x) \in [0,T] \times \Omega, \\
&v(t,x) = v_0(x), \ x \in \Omega, \qquad Bv(t,x) = Bv_0(x), \ (t,x) \in [0,T] \times \partial\Omega, \\
&a(t)\Phi[v(t,\cdot)] = g(t), \ t \in [0,T],
\end{aligned}$$

where the unknown coefficient $a_0(t) \sim const \cdot t^\beta, \beta \in (-1,1)$ is singular at $t = 0$ if $\beta \neq 0$. They assume classical in time (and not Sobolev type as in our thesis case) regularity on data and they prove local-in-time existence and global uniqueness results for the solution of the previous problem in the space

$$\left[C([0,\tau];X) \cap C^1((0,\tau];X) \cap C((0,\tau];D(A))\right] \times C([0,\tau];\mathbb{R}_+).$$

Among problems of identifying the unknown coefficients in a degenerate parabolic equation we can quote also Zui-Cha Denga, Liu Yangb [3], where numerical results are achieved, the proposed there method is stable and the unknown function is recovered very well.

Before stating our problem we quote the problem studied in [14]. The author deals with the following *linear* identification problem in a Banach space X : *find a function* $u : [0, T] \to X$ *and an element* $z \in X$ *such that*

$$u'(t) - a_0(t)Au(t) - a_1(t)u(t) = a_0(t)f(t)z + g(t), \quad t \in [0, T], \tag{0.5}$$

$$u(0) = u_0, \quad \int_0^T u(t)d\mu(t) = h, \tag{0.6}$$

where μ is a finite positive Borel measure on $[0, T], T \in \mathbb{R}_+$, whose support is *not* concentrated at $t = 0$, while $f : [0, T] \to \mathbb{R}$ and $g : [0, T] \to X$, and $u_0, h \in X$ are given functions. In the case, when a_0 is a continuous, strictly positive function in $[0, T]$, the problem of recovering the element $z \in X$ in (1.1), is a well-studied problem in the Theory of Inverse Problems (e.g. ([15, 16, 20]). Instead, when the coefficient $a_0 \in L^1(0, T; [0, +\infty))$ and $z \in X$ is looked for, only partial results are available [14]. The authors in [1, 14] are particularly interested in the case when $a_0 \in L^1(0, T; [0, +\infty)), a_1 \in L^1(0, T; \mathbb{R})$. They make the following *fundamental assumptions involving operator A:*

(**A1**) Let $A : D(A) \subset X \to X$ be a linear closed operator with $\overline{D(A)} = X$;

(**A2**) A is a sectorial operator, i.e. the resolvent set $\rho(A)$ contains the closed sector $\overline{\Sigma}_\theta$, where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}, \quad \theta \in (\pi/2, \pi),$$

whose resolvent operator satisfies the estimate

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M(1 + |\lambda|)^{-1}, \quad \lambda \in \overline{\Sigma}_\theta.$$

Then it is known that A generates an analytical semigroup $\{e^{tA}\}_{t>0}$ of contractions decaying exponentially at $+\infty$, i.e. the following estimates hold:

(**A3**) $\left\|e^{tA}\right\|_{\mathcal{L}(X)} \leq e^{-\rho_0 t}$, for all $t \in [0, +\infty)$;

(**A4**) $\|Ae^{tA}\|_{\mathcal{L}(X)} \leq const(t^{-1} + 1)e^{-\rho_0 t}$, $t \in (0, +\infty)$,

for *some positive* ρ_0, which may be even very small. Here $\mathcal{L}(X)$ denotes the Banach space of all linear bounded operators from X to X. Under suitable conditions on the data a_0, a_1, f, g, u_0 and h, the author proved in [14] the existence and uniqueness of the solution (u, z) to the identification problem (1.1), (0.6) in the space

$$[W^{1,1}(0, T; X) \cap L^1(0, T; D(A); a_0 dt)] \times X.$$

Moreover, in that paper explicit operator integral representations of u and z in the terms of a_0, a_1, f, g, u_0 and h were determined.

This thesis will focus on extending the two works [1, 14], i.e. the problem (1.1), (0.6) in the case when the differential equation (1.1) and the integral condition in (0.6) are replaced with

$$u'(t) - a_0(t)Au(t) - a_0(t)a_1(t)u(t) = a_0(t)(f(t)z + g(t)), \quad t \in [0,T], \tag{0.7}$$

$$u(0) = u_0, \quad \int_0^T \varphi(t)u(t)\,d\mu(t) = h \tag{0.8}$$

correspondingly and *both* a_1 *and* z *are unknown.* Consequently, we are forced to add to the condition (1.3) also the following

$$\Phi[u(t)] = k(t), \quad t\in[0,T], \tag{0.9}$$

where Φ is prescribed linear continuous functional. In the our problem (0.7), (1.3), (1.4) the functions $a_0 : [0,T] \to [0,+\infty]$, $g : [0,T] \to X$, $f, \varphi, k : [0,T] \to \mathbb{R}$, $u_0, h \in X$ are given, while

$$z \in X, \quad a_1 : [0,T] \to \mathbb{R}, \quad u : [0,T] \to X$$

are to be determined. We stress that φ may be a non-positive function, so, that it cannot be absorbed into the positive measure μ occurring in the condition (1.3). The objectives of this research are to consider two different cases for the measure μ, called **Case 1** and **Case 2**, which correspond, respectively, to the case of the *Dirac* measure concentrated at the time $t = T_1, 0 < T_1 < T$ and to the one of an *absolutely continuous positive Borel measure,* i.e. $d\mu(t) = \psi(t)dt$, $\psi(t) \geq 0, t \in [0,T]$, with $\psi \in L^{p_1}_{|\varphi|}(0,T;\mathbb{R})$, where $L^{p_1}_{|\varphi|}(0,T;\mathbb{R})$ stands for the space of all measurable functions v on $[0,T]$ such that $\varphi v \in L^{p_1}(0,T;\mathbb{R})$. We have assumed that the operator A satisfies the properties (**A1**), (**A3**), (**A4**) and

(**A2$'$**) the resolvent set $\rho(A)$ contains the sector $-\mu_1 + \Sigma_\theta$, where $\mu_1\in\mathbb{R}_+$ and

$$\Sigma_\theta = \{\lambda\in\mathbb{C} : |\arg\lambda| < \theta\}, \quad \theta\in(\pi/2,\pi),$$

but, differently from [14], the number ρ_0 in the condition (A3) must be *a (large) positive constant* in order to guarantee the existence and uniqueness of the solution (a_1, u, z) to the problem (0.7), (1.3), (1.4) in the space

$$L^1_{a_0}(0,T;\mathbb{R}) \times \left[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\right] \times X.$$

The notation $L^1_{a_0}(0,T;\mathbb{R})$ stands for the space of all measurable functions v on $[0,T]$ such that $a_0 v \in L^1(0,T;\mathbb{R})$.

This paper has been divided into three chapters. The **Chapter 1** is split into four sections. The *first section* deals just with stating our abstract problem. The *second section* introduces the admissible spaces (of Sobolev type) for the solution and our data. More precisely, we need the following *main* assumptions on our data:

$a_0(t) > 0$ for a.e. $t\in(0,T)$, $a_0 \in L^{p_1}(0,T;[0,+\infty))$, $u_0, h \in D(A)$, $f, k \in W^{1,p_1}(0,T;\mathbb{R})$,

$k(t) \neq 0,\ t \in [0,T]$: where p_1 such that $1/p_1+1/p_1'=1, g \in L^{p_1}(0,T;X)$,

$a_0^{1/p_1} g \in L^{p_1}(0,T;\mathcal{D}_A(\gamma,\infty))$, for some $\gamma \in (0,1), \Phi \in X^*, \Phi[u_0] = k(0)$

and $(A1), (A2'), (A3), (A4)$ on operator A in order to establish the main abstract results. Here $\mathcal{D}_A(\gamma,\infty)$ is the interpolation space of order γ and $p=\infty$ between $D(A)$ and X [18]. We recall that in the problem (0.2), (0.3), (0.4) two difficulties occur: searching for the two unknown functions depending on different variables (time and space) and the singularity of the leading coefficient a_0. Thus, for the data space we are obliged to introduce the unusual distance

$$\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2) = \|f_1 - f_2\|_{W^{1,p_1}(0,T;\mathbb{R})} + \|g_1-g_2\|_{L^\infty(0,T;X)} + \|a_0^{1/p_1}(g_2-g_1)\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}$$
$$+ \|u_{0,2} - u_{0,1}\|_{D(A)} + \|h_2 - h_1\|_{D(A)} + \Big\|\frac{k_2'}{k_2} - \frac{k_1'}{k_1}\Big\|_{L^{p_1}(0,T;\mathbb{R})} + |k_2(0) - k_1(0)|$$
$$+ \Big\|\frac{a_0}{k_1} - \frac{a_0}{k_2}\Big\|_{L^{p_1'}(0,T;\mathbb{R})} + \Big\|\frac{1}{k_2} - \frac{1}{k_1}\Big\|_{L^\infty(0,T;\mathbb{R})},$$

accounting for the degeneracy of function a_0. To show the singularity due to a_0, during all thesis we exhibit the exemplar case $a_0(t) = t^{\alpha-1}$, with $\alpha \in (0,+\infty)\backslash\{1\}$. In this particular case, to ensure the well-posedness of our problem the power α must satisfy some unusual bounds, due to our assumptions on a_0 in the Banach space treatment: $1/a_0 \in L^{p_2/p_3}(0,T;\mathbb{R})$, $a_0 \in L^{p_2/p_3'}(0,T;[0,+\infty))$, $1/p_3 + 1/p_3' = 1$, $a_0 \in L^{1/(q-1)}(0,T;[0,+\infty))$, for some $q \in (1,+\infty)$, $a_0 \in L^{p_1'}(0,T;[0,+\infty))$.

The conclusion of the second section is existence, uniqueness and stability result for the identification problem (0.7), (1.3), (1.4) in a general Banach space.

In the *third section* we apply our abstract results to the concrete parabolic problem (0.2), (0.3), (0.4). More precisely, to solve (0.2)–(0.4) we apply our abstract results choosing $\beta(t) = 1/a_0(t) = t^{\alpha-1}$, the reference space $X_s = L^s(\Omega)$, $s \in (1,+\infty)$, and defining

$$\mathcal{D}(A_s) = \Big\{w \in W^{2,s}(\Omega) : Bw = 0 \text{ on } \partial\Omega\Big\}, \qquad A_s w = \mathcal{A}(\cdot, D_x)w, \quad w \in \mathcal{D}(A_s).$$

Assume that $(f,g,u_0,h,k) \in W^{1,p_1}(0,T;\mathbb{R}) \times L^{p_1}(0,T;L^s(\Omega)) \times \mathcal{D}(A_s)^2 \times W^{1,p_1}(0,T;\mathbb{R})$ and the conditions on our data from the abstract case are fulfilled. Then we can conclude that *problem (0.2)–(0.4) admits a unique solution*

$$(a_1,u,z) \in L^1_{a_0}(0,T;\mathbb{R}) \cap \big[W^{1,1}(0,T;L^s(\Omega)) \cap L^\infty(0,T;D(A_s))\big] \times L^s(\Omega),\ s \in (1,+\infty),$$

continuously depending on the data with respect to the norms pointed out. We stress again that we succeed to obtain the existence and uniqueness of the solution *globally in time* thanks to the choice of an enough large parameter ρ_0 involved in the conditions $(A3), (A4)$ and $(A21)(cf.pag.21)$. Besides, in the case when ρ_0 is not large we might obtain at most local in time existence and uniqueness results.

The main questions have been raised in this thesis is how to modify the assumptions on the coefficient a_0a_1 with respect to the ones on a_1 needed to solve the direct problem in [14]? We remind that Lorenzi's result [14] concerns the problem (1.1), (0.6) with the *only* unknown z and there the scalar (known) coefficients a_0 and a_1 must satisfy the following condition

$$a_1 \in L^1((0,T);\mathbb{R}), \ \int_s^t a_1(\sigma)\,d\sigma \le \rho_1 \int_s^t a_0(\sigma)\,d\sigma \quad \forall t \in (s,T], \ s \in [0,T), \ \rho_1 < \rho_0.$$

In our problem (0.7), (1.3), (1.4) with the additional unknown a_1 this conditions cannot be satisfied in the form above. By additional assumptions and choice the number ρ_0 *enough large* we solve this question using the method illustrated in the second section of **Chapter 2**, where also the proof of the existence and uniqueness theorem is given. This is based on an application of the Banach fixed-point theorem to an operator equation, equivalent to the inverse problem (0.7), (1.3), (1.4). For this purpose we proceed according to the following steps:

1) in the *first section* of Chapter 2, assuming that the pair (a_1, z) is known and using well-known results from the (analytic) Semigroup Theory [18], we deduce a suitable representation for u;

2) writing a chain of equivalent problems, we arrive at a solving operator system $(\overline{w}, \overline{z}) = \mathcal{N}(\overline{w}, \overline{z})$ involving the auxiliary unknowns $\overline{w} := \Phi[Au(t)]$ and $\overline{z} := \Phi[z]$;

3) we build the admissible set $\mathcal{K}$ for unknowns $\overline{z}$ and $\overline{w}$ and in the Sections 2-6 we prove the necessary conditions requested for the Banach theorem, i.e. that the operator $\mathcal{N}$ maps $\mathcal{K}$ into itself and is a contracting mapping;

3) then we solve the previous operator system by the classical Banach fixed-point theorem.

The last **Chapter 3** is devoted to the proof of the continuous dependence of the solution $(a_1, u, z) \in L^1_{a_0}(0,T;\mathbb{R}) \times \left[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\right] \times X$ to problem (0.7)–(1.4) on the data $\mathbf{d} = (f, g, u_0, h, k)$. To do that we have to strengthen the smoothness of the function g. For this purpose the condition $g \in L^{p_1}(0,T;X)$ is changed to $g \in L^\infty(0,T;X)$. Then, in the *first* section of the Chapter 3 we apply the existence results and in the *third* one we estimate the increments of the involved operators for different $\mathbf{d} \in W^{1,p_1}(0,T;\mathbb{R}) \times L^\infty(0,T;X) \times D(A)^2 \times W^{1,p_1}(0,T;\mathbb{R})$. As a result, in the *second* section we prove a continuous dependence estimate of Lipschitz-type and deduce the well-posedness of the the problem (0.7)–(1.4) in Hadamard's sense.

Chapter 1

Statement of the problem and main results

In this chapter we formulate our abstract problem, we give some hypothesis on data and operator of the equation needed for resolubility. After that, we apply the obtained results to some parabolic boundary-value problem.

1.1 Statement of the abstract problem

Let X be a Banach space endowed with the norm $\|\cdot\|_X$. and look for a triplet

$$(a_1, u, z) \in L^1(0,T;\mathbb{R}) \times \big[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\big] \times X$$

satisfying the following problem

$$u'(t) - a_0(t)Au(t) - a_0(t)a_1(t)u(t) = a_0(t)f(t)z + a_0(t)g(t), \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

$$\int_0^T \varphi(t)u(t)d\mu(t) = h, \tag{1.3}$$

$$\Phi[u(t)] = k(t),\ t\in[0,T]. \tag{1.4}$$

Here the functions

$$a_0 : [0,T] \to [0,+\infty], \quad g : [0,T] \to X, \quad f, \varphi, k : [0,T] \to \mathbb{R}, \quad u_0, h \in X$$

are given data and $\Phi \in X^*$ is a prescribed linear functional. We remark that the space $L^1(0,T;\mathbb{R}) \times \big[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\big] \times X$ is endowed with the norm

$$\|a_1\|_{L^1(0,T;\mathbb{R})} + \|u\|_{W^{1,1}(0,T;X)} + \|Au\|_{L^\infty(0,T;X)} + \|z\|_X.$$

1.2 Assumptions on the linear closed operator A and data

Let us state all sufficient assumptions needed to solve the problem (1.1)–(1.4).

Assumptions on the operator A

(A 1) Let $A : D(A) \subset X \to X$ be a linear closed operator with $\overline{D(A)} = X$, $D(A)$ being endowed with the graph-norm of A :

$$\|x\|_{D(A)} = \|x\| + \|Ax\|$$

and A generates an analytic semigroup

$$\left\{e^{tA}\right\}_{t\in[0,\infty)} \subset \mathcal{L}(X),$$

(A 2) the resolvent set $\rho(A)$ contains the sector $-\mu_1 + \Sigma_\theta$, where $\mu_1 \in \mathbb{R}_+$ and

$$\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}, \quad \theta \in (\pi/2, \pi];$$

(A 3) the semigroup $\left\{e^{tA}\right\}_{t\in[0,\infty)}$ fulfills the estimate:

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq e^{-\rho_0 t},$$

for all $t \in [0, +\infty)$ and *some large* $\rho_0 \in (0, +\infty)$; *

(A 4) $\|Ae^{tA}\|_{\mathcal{L}(X)} \leq C_1(t^{-1} + 1)e^{-\rho_0 t}, \qquad t \in \mathbb{R}_+.$

Remark 1.1 *Assume now that the operator A satisfies the following condition, weaker than* (A2) :

(A2′) *the resolvent set* $\rho(A) \supset \mu_2 + \Sigma_\theta$, *where* $\mu_2 \in \mathbb{R}_+$ *and*

$$\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\} \cup \{0\}, \quad \theta \in (\pi/2, \pi].$$

Then such a condition implies that $A - 2\mu_2 I$ *is invertible and* $(A - 2\mu_2 I)^{-1} \in L(X; D(A))$. *Define* $\widetilde{A} = A - 2\mu_2 I$ *and make the change of the unknown function defined by*

$$u(t) = v(t) \exp\left(-2\mu_2 \int_0^t a_0(\sigma) d\sigma\right).$$

*ρ_0 will be exactly chosen later on in the condition (A21)

So, the problem (1.1)-(1.4) can be rewritten in the following form:

$$v'(t) - a_0(t)\widetilde{A}v(t) - a_0(t)a_1(t)v(t) = \exp\left(-2\mu_2 \int_0^t a_0(\sigma)d\sigma\right)$$
$$\times\,(a_0(t)f(t)z + a_0(t)g(t)),$$
$$v(0) = u_0,$$
$$\int_0^T \varphi(t)\exp\left(\mu_1 \int_0^t a_0(\sigma)d\sigma\right) v(t)d\mu(t) = h,$$
$$\Phi[v(t)] = k(t)\exp\left(-2\mu_2 \int_0^t a_0(\sigma)d\sigma\right), \quad t\in[0,T].$$

Here the operator $\widetilde{A}$ *is continuously invertible and satisfies* (*A*2).

Assumptions on data

Let $(p_1,p_2,p_3)\in[1,+\infty)^3$ be a triplet such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \ \frac{1}{p_2} + \frac{1}{p_3} < 1. \tag{1.5}$$

Assume that the following conditions hold:

(A 5) $a_0(t) > 0$ for a.e. $t\in(0,T)$;

(A 6) $\dfrac{1}{a_0} \in L^{p_2/p_3}(0,T;\mathbb{R})$;

(A 7) $a_0 \in L^{p_2/p_3'}(0,T;[0,+\infty))$, $\quad \dfrac{1}{p_3} + \dfrac{1}{p_3'} = 1$;

(A 8) $a_0 \in L^{1/(q-1)}(0,T;[0,+\infty))$, for some $q\in(1,+\infty)$; †

(A 9) $a_0 \in L^{p_1'}(0,T;[0,+\infty))$;

(A 10) $u_0 \in D(A)$;

(A 11) $h \in D(A)$, $\varphi \in C(0,T;\mathbb{R})$;

(A 12) $f \in W^{1,p_1}(0,T;\mathbb{R})$;

---

†the number q is arbitrarily fixed. If we set $q = 1 + \dfrac{1}{p_1'}$ the condition (A8) follows from (A9).

(A 13) $k \in W^{1,p_1}(0,T;\mathbb{R}), \quad k(t) \neq 0$ for $t \in [0,T]$;

(A 14) $g \in L^{p_1}(0,T;X)$;

(A 15) $a_0^{1/p_1} g \in L^{p_1}(0,T;\mathcal{D}_A(\gamma,\infty))$, for some $\gamma \in \left(1-\dfrac{1}{p_2},1\right)$;

(A 16) $fk' \in L^{p_1}(0,T;\mathbb{R})$;

(A 17) $\Phi \in X^*$;

(A 18) $\Phi[u_0] = k(0)$.

Here $\mathcal{D}_A(\gamma,\infty)$ is the interpolation space between $D(A)$ and X. Precisely $\mathcal{D}_A(\gamma,p)$ is the class of intermediate spaces between X and $D(A)$ $(0<\gamma\leq 1, 1\leq p\leq +\infty)$ defined by

$$\mathcal{D}_A(\gamma,p) = \{x \in X : t \to \|t^{1-\gamma} A e^{tA} x\| \in L^p(0,1)\}$$

and normed by

$$\|x\|_{\mathcal{D}_A(\gamma,p)} = \|x\| + \|t^{1-\gamma} A e^{tA} x\|_{L^p(0,1)}.$$

With such a norm $\mathcal{D}_A(\gamma,p)$ turns out to be a Banach space. According to Lunardi [18, p.44] and Lions-J.Peetre [13]: the space $\mathcal{D}_A(\gamma,\infty)$ may be easily defined in terms of the semigroup e^{tA}, as the set of all $x \in X$ such that $t^{1-\gamma}\|Ae^{tA}x\|$ is bounded near $t=0$.

In this work we consider two cases for the measure μ in the condition (1.3): **Case 1** and **Case 2** which will correspond respectively to the cases of the ***Dirac measure*** concentrated at $t=T_1, 0<T_1\leq T$ and to the one of an ***absolutely continuous Borel measure,*** i.e. $d\mu(t)=\psi(t)dt$, with

$$\psi \in L^{p_1}_{|\varphi|}(0,T;\mathbb{R}). \tag{1.6}$$

where

$$L^p_\nu(0,T;\mathbb{R}) = \{v : v \text{ is a measurable function on } [0,T] \text{ such that } \nu v \in L^p(0,T;\mathbb{R})\},$$

where $p=1$ or $p=p_1$ and we endow it with the norms

$$\|v\|_{L^p_\nu(0,T;\mathbb{R})} = \|\nu v\|_{L^p(0,T;\mathbb{R})}.$$

Remark 1.2 *The notation $(Pi)_k$ corresponds to the k-equation of the problem Pi.*

Remark 1.3 In the case when $a_0(t) = t^{\alpha-1}$, $\alpha \in (0, +\infty)\backslash\{1\}$ the conditions $(A6)$, $(A7)$, $(A8)$, $(A9)$ simplify to the following inequalities:

$$\begin{cases} \dfrac{p_2(\alpha-1)}{p_3} + 1 > 0, \\ \dfrac{(1-\alpha)p_2}{p_3'} < 1, \\ \dfrac{\alpha-1}{q-1} + 1 > 0, \\ 1 + (\alpha-1)p_1' > 0, \end{cases} \tag{1.7}$$

where p_1, p_2, p_3, p_1' satisfy (1.5) and remind that all $p_i > 1$.

First we consider the case when α is a variable of the system (1.7) and the other parameters are fixed. In such a case we must solve that system for α. It is easy to see that first two inequalities yield a limitation for the power α :

$$\max\left(0, 1 - \frac{p_3'}{p_2}\right) < \alpha < 1 + \frac{p_2}{p_3}, \quad \alpha \neq 1, \tag{1.8}$$

while the last ones give

$$\alpha > 1 - \frac{1}{p_1'} = \frac{1}{p_1},$$

$$\alpha > 2 - q, \text{ if } q \in (1,2), \qquad \text{for all } \alpha > 0 \text{ if } q \geq 2.$$

As a result, the common solution varies dependently on q :

$$\begin{cases} \max\left(1 - \dfrac{p_3'}{p_2}, \dfrac{1}{p_1}, 2-q\right) < \alpha < 1 + \dfrac{p_2}{p_3}, \quad q \in (1,2) \\ \max\left(1 - \dfrac{p_3'}{p_2}, \dfrac{1}{p_1}\right) < \alpha < 1 + \dfrac{p_2}{p_3}, \quad q \in (2,+\infty). \end{cases}$$

As a conclusion, we conclude that in order to satisfy the conditions $(A6)$, $(A7)$, $(A8)$, $(A9)$ the number α must satisfy (1.8).

Second, we consider the case when α is fixed in (1.7) and the other parameters are variable. Note that the system (1.7) can be rewritten as

$$\begin{cases} \dfrac{1}{p_2} > (\alpha - 1)\dfrac{1}{p_3}, \\ (1-\alpha)\left(1 - \dfrac{1}{p_3}\right) < \dfrac{1}{p_2}, \\ \dfrac{1}{p_1} < \alpha. \end{cases}$$

Let us solve this system for the variables p_2 and p_3. For this aim we remind (1.5) and use the last condition from (1.11) to obtain

$$1 > \frac{1}{p_2} + \frac{1}{p_3} = 1 - \frac{1}{p_1} > 1 - \alpha.$$

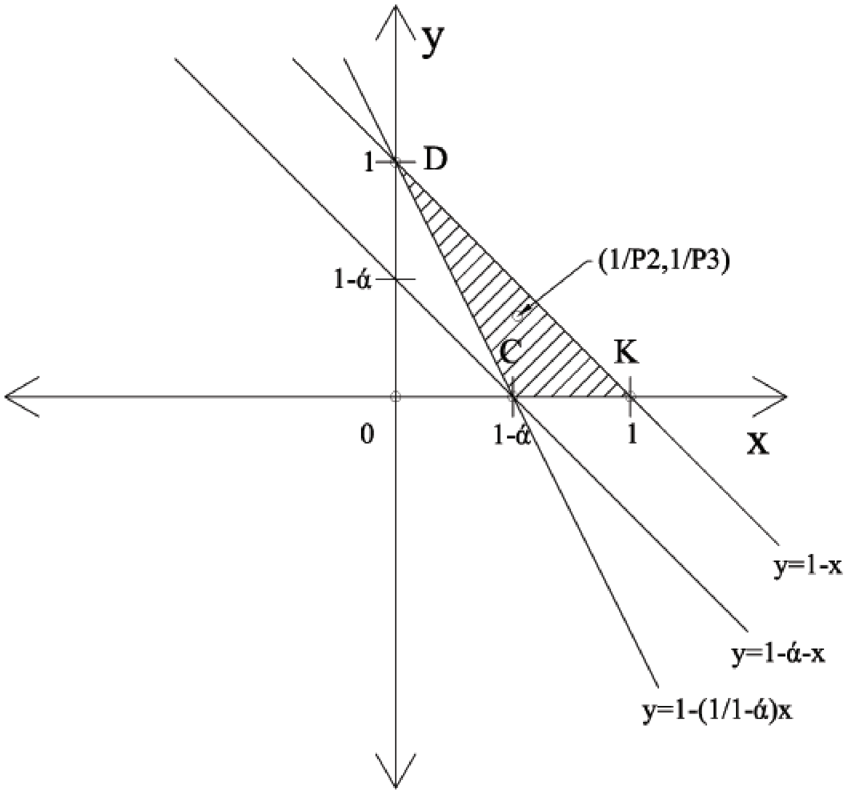
So, the system to be solved is

$$\begin{cases} \dfrac{1}{p_2} > (\alpha - 1)\dfrac{1}{p_3}, \\ (1-\alpha)\left(1 - \dfrac{1}{p_3}\right) < \dfrac{1}{p_2}, \\ \dfrac{1}{p_2} + \dfrac{1}{p_3} < 1, \\ \dfrac{1}{p_2} + \dfrac{1}{p_3} > 1 - \alpha, \end{cases} \tag{1.9}$$

We note that in the **case** $0 < \alpha < 1$ the system (1.9) is equivalent to the following:

$$\begin{cases} (1-\alpha)\left(1 - \dfrac{1}{p_3}\right) < \dfrac{1}{p_2}, \\ \dfrac{1}{p_2} + \dfrac{1}{p_3} < 1, \\ \dfrac{1}{p_2} + \dfrac{1}{p_3} > 1 - \alpha, \end{cases} \tag{1.10}$$

Consequently, the pair $(1/p_2, 1/p_3)$ must belong to the open triangle with vertices C, D, K on the figure below.

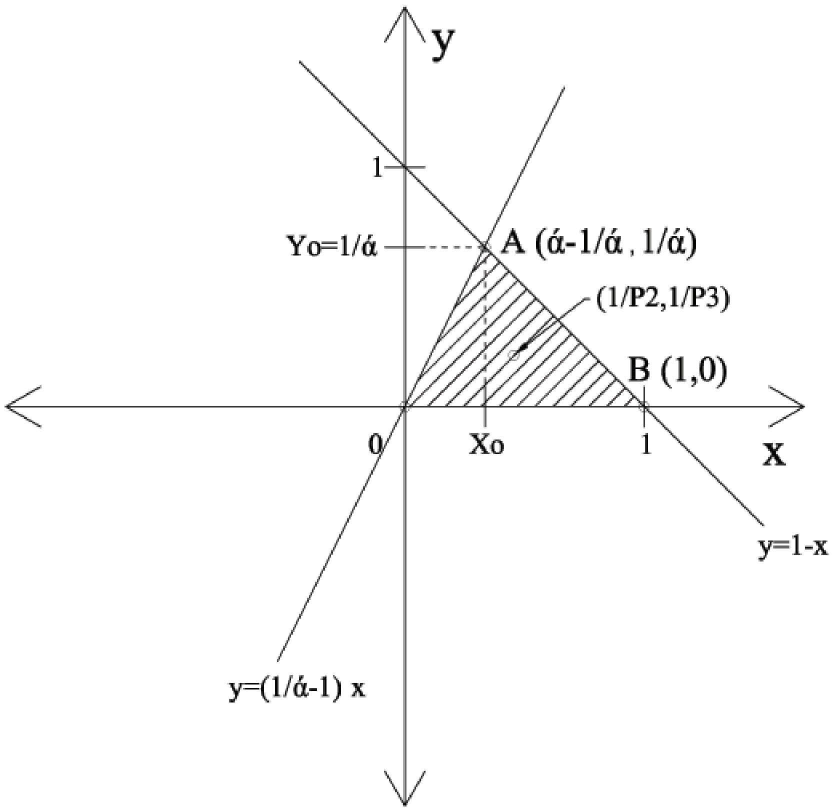


The Case ά<1

Concerning the **case** $\alpha > 1$ we observe that the system (1.9) is equivalent to the following:

$$\begin{cases} \dfrac{1}{p_3} < \dfrac{1}{\alpha - 1}\dfrac{1}{p_2}, \\ \dfrac{1}{p_2} + \dfrac{1}{p_3} < 1. \end{cases}$$

In this case the pair $(1/p_2, 1/p_3)$ must belong to the open triangle with vertices O, A, B, look the figure above.



The Case when ά>1

Now we introduce the space for our data and the appropriate distance for them.

The space of our data and the appropriate metrics

Let us associate with $r = (r_1, \dots, r_9) \in (\mathbb{R}_+)^9$ the space of data

$$\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}) = \Big\{ \mathbf{d} = (f, g, u_0, h, k) \in W^{1,p_1}(0, \widetilde{T}; \mathbb{R}) \times L^{p_1}(0, \widetilde{T}; X) \times D(A)^2 \times W^{1,p_1}(0, \widetilde{T}; \mathbb{R}),$$
$$k(t) > 0, \text{ for all } t \in [0, T], \ \left| \int_0^T \varphi(t) f(t) d\mu(t) \right| \geq r_1, \ \|f\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \leq r_2,$$
$$\|g\|_{L^{p_1}(0,\widetilde{T};X)} \leq r_3, \ \|a_0^{1/p_1} g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))} \leq r_4, \ \|Au_0\|_X \leq r_5, \|Ah\|_X \leq r_6,$$
$$\left\| \frac{k'}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq r_7, \ \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} \leq r_8, \left\| \frac{1}{k} \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq r_9 \Big\}. \qquad (1.11)$$

We endow $\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ with the distance

$$\begin{aligned}\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2) =&\|f_1-f_2\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+\|g_1-g_2\|_{L^{p_1}(0,\widetilde{T};X)}+\|a_0^{1/p_1}(g_2-g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}\\ &+\|u_{0,2}-u_{0,1}\|_{D(A)}+\|h_2-h_1\|_{D(A)}+\left\|\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\\ &+|k_2(0)-k_1(0)|+\left\|\frac{a_0}{k_1}-\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^{\infty}(0,\widetilde{T};\mathbb{R})}, \qquad (1.12)\end{aligned}$$

where

$$\widetilde{T} := \begin{cases} T_1, & \textbf{Case 1}, \\ T, & \textbf{Case 2}. \end{cases} \qquad (1.13)$$

1.3 Main abstract results

Existence and uniqueness

To state our theorem we also make the following additional assumptions:

(A 19) $\lambda(\mathbf{d}) := \int_0^T \varphi(t)f(t)d\mu(t) \neq 0,$

(A 20) $\left\|\frac{\Phi[a_0 g]}{k}\right\|_{L^1(s,t;\mathbb{R})} \leq C_2(\mathbf{d})\int_s^t a_0(\sigma)d\sigma, \qquad \left\|\frac{a_0 f}{k}\right\|_{L^1(s,t;\mathbb{R})} \leq C_3(\mathbf{d})\int_s^t a_0(\sigma)d\sigma,$

for any $0 \leq s < t \leq T$, and some positive continuous functions $C_2(\mathbf{d}),\ C_3(\mathbf{d})$,

(A 21) fix a pair $(M_1, M_2)\in\mathbb{R}_+^2$ such that

$$\begin{cases} M_1 > \dfrac{\|Ah\|_X\|\Phi\|_{X^*}}{|\lambda(\mathbf{d})|}\ \|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}, \\ M_2 > \dfrac{\|Ah\|_X\|\Phi\|_{X^*}}{|\lambda(\mathbf{d})|}. \end{cases} \qquad (1.14)$$

(A 22) The number ρ_0 is chosen so as to satisfy $(A3)$ and

$$\rho_0 > C_2(\mathbf{d}) + C_3(\mathbf{d})M_2 + 1 =: K(\mathbf{d}) + 1. \qquad (1.15)$$

(A 23) We also assume that

$$J(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) \leq \frac{|\lambda(\mathbf{d})|}{2}, \qquad (1.16)$$

where

$$J(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) := \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p'_1}(0,\widetilde{T};\mathbb{R})}\right)$$

$$\times\Bigg\{\tau_1(\rho_0,\mathbf{d},\widetilde{T})|f(0)| + \rho_0^{-1/p_3}\frac{\tau_2(\widetilde{T})}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}$$

$$\times\left[\left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|a_0^{1/p'_3}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\right.$$

$$\left.\times\left(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\right)\right]\Bigg\}, \tag{1.17}$$

$$\lambda(\mathbf{d}) = \begin{cases} \varphi(T_1)f(T_1), & \textbf{Case 1}, \\ \displaystyle\int_0^T \varphi(t)f(t)\psi(t)dt, & \textbf{Case 2}, \end{cases} \tag{1.18}$$

$$\tau_1(\rho_0,\mathbf{d},\widetilde{T}) := \begin{cases} |\varphi(T_1)|\exp\left(-(\rho_0-K(\mathbf{d}))\displaystyle\int_0^{T_1}a_0(\sigma)d\sigma\right), & \textbf{Case 1}, \\ \left\|\dfrac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,T;\mathbb{R})}\dfrac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}\rho_0^{-1/p_3}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}, & \textbf{Case 2}, \end{cases} \tag{1.19}$$

$$\tau_2(\widetilde{T}) := \begin{cases} |\varphi(T_1)|, & \textbf{Case 1}, \\ T^{1/p'_1}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}, & \textbf{Case 2}. \end{cases} \tag{1.20}$$

Theorem 1.1 *Suppose that the conditions* $(A1)-(A23)$ *hold. Then for any* $\boldsymbol{d} \in \mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$, *the problem (1.1)–(1.4) admits a unique solution*

$$(a_1,u,z)\in L^1_{a_0}(0,\widetilde{T};\mathbb{R})\times\left[W^{1,1}(0,\widetilde{T};X)\cap L^\infty(0,\widetilde{T};D(A))\right]\times X,$$

satisfying the estimates:

$$\|a_0a_1\|_{L^1(0,T;\mathbb{R})} \le C_4(\boldsymbol{d},M_1,M_2), \tag{1.21}$$

$$\|z\|_X \le C_5(\rho_0,\boldsymbol{d},M_1,M_2,\widetilde{T}), \tag{1.22}$$

$$\|Au\|_{L^\infty(0,\widetilde{T};X)} \le C_6(\rho_0,\boldsymbol{d},M_1,M_2,\widetilde{T}), \tag{1.23}$$

$$\|u\|_{L^\infty(0,\widetilde{T};X)} \leq \|A^{-1}\|_{\mathcal{L}(X)} C_6(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}), \tag{1.24}$$

$$\|u\|_{W^{1,1}(0,\widetilde{T};X)} \leq C_7(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}). \tag{1.25}$$

Here the functions $C_4,\ C_5,\ C_6,\ C_7$ are continuous in their variables and, moreover, $C_5 \to \dfrac{2\|Ah\|_X}{|\lambda(\boldsymbol{d})|}$, $C_6 \to \exp\Big(\Big\|\dfrac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\dfrac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)\|Au_0\|_X$ *when* $\rho_0 \to +\infty$.

Continuous dependence

To state our theorem first we recall the definition of the space $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$ and that fact that $|\lambda(\mathbf{d})| > r_1$. Than we rewrite the assumptions $(A20)$ for any $\mathbf{d}_1, \mathbf{d}_2 \in \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$:

$$\begin{aligned} &\Big\|\frac{\Phi[a_0 g_i]}{k_i}\Big\|_{L^1(s,t;\mathbb{R})} \leq C_2(\mathbf{d}_i) \int_s^t a_0(\sigma) d\sigma,\ i = 1, 2, \\ &\Big\|\frac{a_0 f_i}{k_i}\Big\|_{L^1(s,t;\mathbb{R})} \leq C_3(\mathbf{d}_i) \int_s^t a_0(\sigma) d\sigma,\ i = 1, 2. \end{aligned} \tag{1.26}$$

We rewrite the condition (A21) as follows: $(M_1, M_2) \in \mathbb{R}_+^2$ such that

$$\begin{cases} M_1 > \dfrac{r_6\|\Phi\|_{X^*}}{r_1}\ r_2, \\ M_2 > \dfrac{r_6\|\Phi\|_{X^*}}{r_1}, \end{cases} \tag{1.27}$$

a LARGE number ρ_0, satisfying $(A3), (A4)$ and

$$\rho_0 > r_9\big(\|\Phi\|_{X^*} r_3 + K_1(T) r_2 M_2\big) + 1 =: K_1(\mathbf{r}) + 1 \tag{1.28}$$

for some positive function K_1. We also assume that (cf. (1.16))

$$K_5(\rho_0, r) \leq \frac{r_1}{2}, \tag{1.29}$$

where the function K_5[‡] is positive, continuous, tends to 0 when $\rho_0 \to +\infty$ and satisfies the estimate

$$J(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) \leq K_5(\rho_0, r).$$

We can now state our continuous dependence result.

[‡]the function K_5 will be defined during the proof of continuous dependence (cf. page 103)

Theorem 1.2 *Under the assumptions (A1)-(A19),* (1.26), (1.27), (1.28), (1.29) *and*

(A 24) $g \in L^\infty(0,\widetilde{T};X)$,

the solution $(a_1,u,z)\in L^1_{a_0}(0,\widetilde{T};\mathbb{R})\times\big[W^{1,1}(0,\widetilde{T};X)\cap L^\infty(0,\widetilde{T};D(A))\big]\times X$ *to problem* (1.1)-(1.4) *is unique. Moreover, such a solution continuously depends on the data* $\boldsymbol{d}=(f,g,u_0,h,k)\in\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$, *i.e. for any* $\boldsymbol{d}_1,\boldsymbol{d}_2\in\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$, *with* $\boldsymbol{r}=(r_1,\dots,r_9)\in(\mathbb{R}_+)^9$, *the following estimates hold:*

$$
(IC)\begin{cases}
\|a_0(\cdot)a_1(\cdot,\boldsymbol{d}_1)-a_0(\cdot)a_1(\cdot,\boldsymbol{d}_2)\|_{L^1(0,\widetilde{T};\mathbb{R})}\\
\qquad\le\Big[L_1(\boldsymbol{r})+\rho_0^{-1/p_3}L_2(\boldsymbol{r})+L_3(\boldsymbol{r})\exp\Big(-\rho_0\displaystyle\int_0^{T_1}a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\boldsymbol{d}_1,\boldsymbol{d}_2),\\
\|u(\cdot,\boldsymbol{d}_1)-u(\cdot,\boldsymbol{d}_2)\|_{W^{1,1}(0,\widetilde{T};X)}+\|u(\cdot,\boldsymbol{d}_1)-u(\cdot,\boldsymbol{d}_2)\|_{L^\infty(0,\widetilde{T};D(A))}\\
\qquad\le\Big[L_4(\boldsymbol{r})+\rho_0^{-1/p_3}L_5(\boldsymbol{r})+L_6(\boldsymbol{r})\exp\Big(-\rho_0\displaystyle\int_0^{T_1}a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\boldsymbol{d}_1,\boldsymbol{d}_2),\\
\|z(\boldsymbol{d}_1)-z(\boldsymbol{d}_2)\|_X\\
\qquad\le\Big[L_7(\boldsymbol{r})+\rho_0^{-1/p_3}L_8(\boldsymbol{r})+L_9(\boldsymbol{r})\exp\Big(-\rho_0\displaystyle\int_0^{T_1}a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\boldsymbol{d}_1,\boldsymbol{d}_2).
\end{cases}
$$

Here the functions $L_i, i=1,\cdots,9$, *denote suitable continuous functions and* $\widetilde{T}$ *is defined by* (1.13).

1.4 An application of the abstract results

This section is devoted to an application of the abstract results stated in the previous sections to a concrete initial-boundary value problem arising in heat-conduction theory.

Let $\Omega\subseteq\mathbb{R}^n$ be an open bounded domain with a C^2-boundary $\partial\Omega$, and let us consider the following linear parabolic identification problem consisting in recovering the unknown function $u:[0,T]\times\Omega\to\mathbb{R}$, the coefficients $z:\Omega\to\mathbb{R}$ and $a_1:[0,T]\to\mathbb{R}$:

$$\beta(t)D_tu(t,x)-\mathcal{A}(x,D_x)u(t,x)-a_1(t)u(t,x)=f(t)z(x)+g(t,x),\ (t,x)\in[0,T]\times\Omega, \tag{1.30}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \qquad\qquad Bu(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{1.31}$$

$$\int_0^T \varphi(t) u(t,x) d\mu(t) = h(x), \quad x \in \Omega, \tag{1.32}$$

$$\Phi[u(t,x)] := \int_\Omega \chi(x) u(t,x) dx = k(t), \quad t \in [0,T]. \tag{1.33}$$

Here μ *is a positive finite Borel measure on* $[0,T]$, whose support is *not* concentrated at $t = 0$ and the functions

$$\beta : [0,T] \to [0,+\infty], \quad g : [0,T] \times \Omega \to \mathbb{R}, \quad f, \varphi, k : [0,T] \to \mathbb{R}, \quad \chi, h, u_0 : \Omega \to \mathbb{R}$$

are known. Moreover, $\mathcal{A}$ in the equation (1.30) denotes the following second-order operator

$$\mathcal{A}(x, D_x) = \sum_{j,k=1}^{n} D_{x_j}\big(a_{j,k}(x) D_{x_k}\big) - \rho_0, \tag{1.34}$$

for *a (large) positive constant*, ρ_0. [§] We assume that $a_{i,j} \in C^1(\overline{\Omega})$, $i, j = 1, .., d$, fulfill the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geq C_8 |\xi|^2 \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n,$$

where C_8 is a positive constant. In addition, the operator B in the condition (1.31) is defined by either Dirichlet or the Robin boundary operator, i.e. by one of the following relations:

$$\text{(D)} \quad Bw(x) = w(x), \qquad x \in \Omega, \tag{1.35}$$

$$\text{(R)} \quad Bw(x) = D_{\nu_{\mathcal{A}}} w(x) + b_0(x) w(x), \qquad x \in \Omega. \tag{1.36}$$

Here $b_0 \in C(\partial\Omega)$, $b_0(x) \geq 0$ for all $x \in \partial\Omega$, and $\nu_{\mathcal{A}}$ is the conormal vector associated with $\mathcal{A}(x, D_x)$, i.e.

$$(\nu_{\mathcal{A}}(x))_j = \sum_{j,k=1}^{n} \nu_i(x) a_{j,k}(x),$$

where $\nu(x)$ denotes the outward normal vector at $x \in \partial\Omega$. We now define

$$\beta(t) = \frac{1}{a_0(t)}, \ t \in (0,T)$$

[§] Moreover, ρ_0 satisfies the conditions (A3), (A4), (1.15), (1.16).

and assume that a_0 satisfies $(A5)$. Consequently, the problem (1.30)–(1.33) coincides with our initial problem (1.1)–(1.4). Therefore, in order to apply our abstract results, taking into account (1.34), we rewrite the equation (1.1) in the following form

$$D_t u(t,x) - a_0(t)\sum_{j,k=1}^{n} D_{x_j}\big(a_{j,k}(x)D_{x_k}\big)u(t,x) - a_0(t)(a_1(t)-\rho_0)u(t,x)$$
$$= a_0(t)(f(t)z(x)+g(t,x)).$$

Then, we define the reference function space to be $X_s = L^s(\Omega)$, $s\in(1,+\infty)$, and the operator A_s as the realization of $\mathcal{A}(\cdot,\partial_x)$, i.e.

$$A_s w = \mathcal{A}(\cdot, D_x)w, \quad w\in\mathcal{D}(A_s)$$

with domain

$$\mathcal{D}(A_s) = \Big\{w\in W^{2,s}(\Omega) : Bw = 0 \text{ on } \partial\Omega\Big\},$$

where B is defined in (1.35), (1.36) According to [29, p.321], the interpolation space $\mathcal{D}_A(\gamma,\infty)$ between X_s and $\mathcal{D}(A_s)$ can be identified with

$$\mathcal{D}_A(\gamma,\infty) = B^{2\gamma}_{s,\infty,B}(\Omega), \quad 2\gamma - 1/s \neq ordB,$$

where $B^{2\gamma}_{s,\infty,B}$ stands for Besov spaces [28, p.87]. Now we have to verify the conditions $(A1)-(A4)$ on the operator A_s.

Lemma 1.1 *The linear operator A_s fulfills assumption $(A1)-(A4)$.*

Proof . It is well-known [1, Section 3] that $T_s = A_s + \rho_0 I$ generates in $L^s(\Omega)$ the semigroup $\{e^{T_s t}\}_{t\geq 0}$, which is analytic in the sector $\Sigma_{(\pi/2)-\omega_p}$ and satisfies the estimate

$$\|e^{tT_s}\|_{\mathcal{L}(X)} \leq 1, \qquad \|T_s e^{tT_s}\|_{\mathcal{L}(X)} \leq c_1 t^{-1}, \qquad t\in\mathbb{R}_+.$$

and c_1 is some positive constant. This implies that A_s enjoys a similar property and $e^{tA_s} = e^{-\rho_0 t}e^{tT_s}$ satisfies

$$\|e^{tA_s}\|_{\mathcal{L}(X)} \leq e^{-\rho_0 t}, \qquad t\in\mathbb{R}_+.$$

Observe now that

$$\|A_s e^{tA_s}\|_{\mathcal{L}(X)} = e^{-\rho_0 t}\|(T_s-\rho_0)e^{tT_s}\|_{\mathcal{L}(X)} \leq (c_1 t^{-1}+\rho_0)e^{-\rho_0 t}, \qquad t\in(0,1];$$
$$\|A_s e^{tA_s}\|_{\mathcal{L}(X)} \leq \|A_s e^{A_s}\|_{\mathcal{L}(X)}\|e^{(t-1)A_s}\|_{\mathcal{L}(X)} \leq (c_1+\rho_0)e^{-\rho_0 t}, \qquad t\in(1,+\infty).$$

Whence we get

$$\|A_s e^{tA_s}\|_{\mathcal{L}(X)} \leq c_2(t^{-1}+1)e^{-\rho_0 t}, \qquad t\in\mathbb{R}_+.$$

with some positive constant c_2. This verifies the condition $(A4)$. ■

We now define the space of data

$$\mathbf{D}_s^{p_1} = W^{1,p_1}(0,T;\mathbb{R})\times L^{p_1}(0,T;L^s(\Omega))\times \mathcal{D}(A_s)^2\times W^{1,p_1}(0,T;\mathbb{R})$$

for $p_1 \in (1,+\infty)$. As a consequence of our abstract Theorems 1.1, 1.2, we obtain

Theorem 1.3 *Let $\chi \in L^{s'}(\Omega)$, the operator A_s satisfies $(A1)-(A4)$, the data $\mathbf{d} = (f,g,u_0,h,k) \in \boldsymbol{D}_s^{p_1}$ fulfill assumptions $(A5)-(A23)$ with a (**large**) positive constant, ρ_0, depending on $r_1,\cdots,r_9,M_2$, only. Then, the problem (1.30)–(1.33) admits a unique solution*

$$(a_1,u,z)\in L^1_{a_0}(0,\widetilde{T};(0,+\infty))\cap\left[W^{1,1}(0,\widetilde{T};L^s(\Omega))\cap L^\infty(0,\widetilde{T};D(A_s))\right]\times L^s(\Omega),$$

where $s,p_1\in(1,+\infty)$, s' being conjugated to s.

Theorem 1.4 *Let the data $\mathbf{d} = (f,g,u_0,h,k) \in \boldsymbol{D}_s^{p_1}$ and the operator A_s fulfill assumptions $(A1)-(A18),(A24)$, (1.26), (1.27), (1.28), (1.29) with $\chi \in L^{s'}(\Omega)$ and a (**large**) positive constant, ρ_0, depending on $r_1,\cdots,r_9,M_2$, only. Then, the problem (1.30)–(1.33) admits a unique solution*

$$(a_1,u,z)\in L^1_{a_0}(0,\widetilde{T};(0,+\infty))\cap\left[W^{1,1}(0,\widetilde{T};L^s(\Omega))\cap L^\infty(0,\widetilde{T};D(A_s))\right]\times L^s(\Omega),$$

where $s\in(1,+\infty)$, that continuously depends on the data $\mathbf{d}$ and satisfies the estimates (IC).

Chapter 2

Existence and uniqueness of the solution in the Banach space X

2.1 Reducing the abstract problem to an equivalent fixed-point system of operator equations $(\overline{w}, \overline{z}) = \mathcal{N}(\overline{w}, \overline{z})$

In this section we are going to reduce our initial abstract problem (1.1)–(1.4) to a system of operator equations in order to apply the Banach fixed-point theorem to the last one.

Problem 1

We denote the problem (1.1)-(1.4) as (P1), i.e. *look for a triplet* $(a_1, u, z) \in L^1_{a_0}(0,T;\mathbb{R})$ $\times[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))] \times X$ *such that*:

$$(P1) \quad \begin{cases} u'(t) - a_0(t)Au(t) - a_0(t)a_1(t)u(t) = a_0(t)f(t)z + a_0(t)g(t), \ t \in [0,T], \\ u(0) = u_0, \\ \displaystyle\int_0^T \varphi(t)u(t)d\mu(t) = h, \\ \Phi[u(t)] = k(t), \quad t \in [0,T]. \end{cases}$$

Problem 2

Let us change in the problem $(P1)_1, (P1)_2$ the unknown function u according to the following relation

$$u(t) = \exp\left(\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\right)v(t). \tag{2.1}$$

Therefore our problem can be rewritten as

$$v'(t) - a_0(t)Av(t) = a_0(t)f(t)\exp\left(-\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\right) z$$
$$+ a_0(t)\exp\left(-\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\right)g(t), \tag{2.2}$$
$$v(0) = u_0. \tag{2.3}$$

Using well-known results [18], under the following assumptions on our data

$$g \in L^p(0,T;\mathcal{D}_A(\gamma,\infty)),\ z \in \mathcal{D}_A(\gamma,\infty),\ f \in L^p(0,T),\ u_0 \in D(A),$$

the Cauchy problem (2.2), (2.3) admits a unique solution ([14, 1])

$$v \in L^1(0,T;\mathbb{R}) \times \left[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\right]$$

given by

$$v(t) = \exp\left(\int_0^t a_0(\sigma)d\sigma A\right)u_0 + \int_0^t a_0(s)f(s)\exp\left(-\int_0^s a_0(\sigma)a_1(\sigma)d\sigma\right)$$
$$\times \exp\left(\int_s^t a_0(\sigma)d\sigma A\right)zds + \int_0^t a_0(s)\exp\left(-\int_0^s a_0(\sigma)a_1(\sigma)d\sigma\right)$$
$$\times \exp\left(\int_s^t a_0(\sigma)d\sigma A\right)g(s)ds.$$

Therefore, taking into account (2.1), we have shown the equivalence of the problem (P1) to the following (P2): *look for a triplet*

$$(a_1, u, z) \in L^1_{a_0}(0,T;\mathbb{R}) \times \left[W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))\right] \times X$$

such that

$$
(P2)\quad \begin{cases} u(t) = \exp\Big(\displaystyle\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)u_0 \\ \qquad + \displaystyle\int_0^t a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)zds \\ \qquad + \displaystyle\int_0^t a_0(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds, \\ \displaystyle\int_0^T \varphi(t)u(t)d\mu(t) = h, \\ \Phi[u(t)] = k(t),\ t\in[0,T]. \end{cases}
$$

Problem 3

In order to deduce the equation for the unknown coefficient a_1 we apply the linear continuous functional Φ to the equation (1.1). According to (1.4) we find the equation

$$
k'(t) - a_0(t)\Phi[Au(t)] - a_0(t)f(t)\Phi[z] - a_0(t)\Phi[g(t)] = a_0(t)a_1(t)k(t).
$$

By assumption $(A13)$, we have $k(t) \neq 0$ for all $t\in[0,T]$, so that we get

$$
\begin{aligned} a_0(t)a_1(t) &= \frac{1}{k(t)}\Big(k'(t) - a_0(t)\Phi[Au(t)] - a_0(t)f(t)\Phi[z] - a_0(t)\Phi[g(t)]\Big) \\ &:= \overline{k}(t) - \frac{a_0(t)}{k(t)}\Phi[Au(t)] - \frac{a_0(t)f(t)}{k(t)}\Phi[z], \end{aligned} \tag{2.4}
$$

where

$$
\overline{k}(t) := \frac{k'(t)}{k(t)} - \frac{a_0(t)\Phi[g(t)]}{k(t)}. \tag{2.5}
$$

As a result, we have transformed the problem (P2) into the following: *look for a triplet* $(a_1, u, z)\in L^1_{a_0}(0,T;\mathbb{R}) \times [W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))] \times X$ *such that*

$$
(P3)\quad \begin{cases} u(t) = \exp\Big(\displaystyle\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)u_0 \\ \qquad + \displaystyle\int_0^t a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)zds \\ \qquad + \displaystyle\int_0^t a_0(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds,\ t\in[0,T], \\ a_0(t)a_1(t) = \overline{k}(t) - \dfrac{a_0(t)}{k(t)}\Phi[Au(t)] - \dfrac{a_0(t)f(t)}{k(t)}\Phi[z], \quad t\in[0,T], \\ \displaystyle\int_0^T \varphi(t)u(t)d\mu(t) = h. \end{cases}
$$

Let us now show the equivalence of the problems (P2) and (P3). We need to prove that the relation $(P2)_3$ holds true. Taking (2.4) into account and using the equation (1.1), we get

$$\begin{aligned} k'(t) - k(t)a_0(t)a_1(t) &= \Phi[a_0(t)Au(t) + a_0(t)f(t)z + a_0(t)g(t)] \\ &= \Phi[u'(t) - a_0(t)a_1(t)u(t)]. \end{aligned} \tag{2.6}$$

for a.e $t \in (0,T)$. From (2.6) and the consistency condition (A18) we arrive at the Cauchy problem:

$$\begin{cases} D_t\left(\Phi[u(t)] - k(t)\right) - a_0(t)a_1(t)\left(\Phi[u(t)] - k(t)\right) = 0, \\ \Phi[u(0)] - k(0) = 0, \end{cases}$$

where $t \to \Phi[u(t)] - k(t)$ belongs to $W^{1,1}(0,T;\mathbb{R})$. Due to the homogeneity of the previous problem, we easily deduce that

$$\Phi[u(t)] - k(t) = 0, \quad t \in [0,T].$$

Therefore the condition $(P2)_3$ is fulfilled.

Problem 4

If we substitute $(P3)_2$ into $(P3)_1$, we come to the new problem: *look for a pair* $(u,z) \in [W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))] \times X$ *such that*

$$(P4)\quad \begin{cases} u(t) = \exp\left(\int_0^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\right) \\ \qquad \times \exp\left(\int_0^t a_0(\sigma)d\sigma A\right)u_0 \\ \qquad + \int_0^t a_0(s)f(s)\exp\left(\int_s^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\right) \\ \qquad \times \exp\left(\int_s^t a_0(\sigma)d\sigma A\right)zds \\ \qquad + \int_0^t a_0(s)\exp\left(\int_s^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\right) \\ \qquad \times \exp\left(\int_s^t a_0(\sigma)d\sigma A\right)g(s)ds, \quad t \in [0,T], \\ \int_0^T \varphi(t)u(t)d\mu(t) = h. \end{cases}$$

The equivalence of problems (P3) and (P4) is obvious. Indeed, once we have solved problem (P4), we define a_0a_1 by equation (2.4).

Problem 5

The next problem can be obtained by replacing the right-hand side of $(P4)_1$ into the left-hand side in $(P4)_2$. In such a way the problem (P4) is equivalent to the following: *look for a pair* $(u, z) \in [W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))] \times X$ *such that*

$$
(P5)\quad \begin{cases}
u(t) = \exp\Big(\displaystyle\int_0^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad \times \exp\Big(\displaystyle\int_0^t a_0(\sigma)d\sigma A\Big)u_0 \\
\quad + \displaystyle\int_0^t a_0(s)f(s)\exp\Big(\int_s^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad \times \exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)z ds \\
\quad + \displaystyle\int_0^t a_0(s)\exp\Big(\int_s^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad \times \exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds, \\
h = \displaystyle\int_0^T \varphi(t)\exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad \times \exp\Big(\displaystyle\int_0^t a_0(\sigma)d\sigma A\Big)u_0 d\mu(t) + \int_0^T \varphi(t)d\mu(t)\int_0^t a_0(s)f(s) \\
\quad \times \exp\Big(\displaystyle\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad \exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)z\, ds + \int_0^T \varphi(t)d\mu(t) \\
\quad \displaystyle\int_0^t a_0(s)\exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[Au(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad \times \exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds.
\end{cases}
$$

Problem 6

Applying the operator A to the equation $(P6)_1$ and $(P6)_2$, using its closedness and linearity, setting

$$w := Au, \tag{2.7}$$

we get the new problem: *look for a pair* $(w,z)\in L^\infty(0,T;X)\times X$ *such that*

$$
(P6)\quad\begin{cases}
w(t) = \exp\Big(\int_0^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 \\
\quad + A\Big(\int_0^t a_0(s)f(s)\exp\Big(\int_s^t(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds\Big)z \\
\quad + A\Big(\int_0^t a_0(s)\exp\Big(\int_s^t(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds\Big), \quad t\in[0,T], \\
Ah = \int_0^T \varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 d\mu(t) + \int_0^T\varphi(t) \\
\quad\times A\Big(\int_0^t a_0(s)f(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)zds\Big)d\mu(t) + \int_0^T\varphi(t) \\
\quad\times A\Big(\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds\Big)d\mu(t).
\end{cases}
$$

The equivalence of problems (P6) and (P5) is obvious if we take into account the assumption $(A2)$ implying the invertibility of the operator A.

Problem 7

Let us rewrite $(P6)_2$ in the more convenient form:

$$
Ah - \int_0^T \varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big)
$$
$$
\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 d\mu(t)
$$

$$- \int_0^T \varphi(t) A \Big(\int_0^t a_0(s) \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \Phi[w(\sigma)] - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \Phi[z] \Big) d\sigma \Big)$$

$$\times \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) g(s) ds \Big) d\mu(t)$$

$$= \int_0^T \varphi(t) A \Big(\int_0^t a_0(s) f(s) \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \Phi[w(\sigma)] - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \Phi[z] \Big) d\sigma \Big)$$

$$\times \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) \, z ds \Big) d\mu(t). \tag{2.8}$$

Lemma 2.1 *For every $f \in W^{1,p}(0,T;X)$ the following formula holds:*

$$A\Big(\int_0^t a_0(s) f(s) \exp\Big(\int_s^t a_0(\sigma) a_1(\sigma) d\sigma \Big) \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) v_1 ds \Big)$$

$$= -f(t) v_1 + f(0) \exp\Big(\int_0^t a_0(\sigma) a_1(\sigma) d\sigma \Big) \exp\Big(\int_0^t a_0(\sigma) d\sigma A \Big) v_1$$

$$+ \int_0^t \Big[f'(s) - f(s) a_0(s) a_1(s) \Big] \exp\Big(\int_s^t a_0(\sigma) a_1(\sigma) d\sigma \Big)$$

$$\times \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) v_1 ds, \quad \forall v_1 \in X, \quad \forall t \in [0,T]. \tag{2.9}$$

The proof of this lemma is postponed until the end of this section.

In order to compute

$$A\Big(\int_0^t a_0(s) f(s) \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \Phi[w(\sigma)] - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \Phi[z] \Big) d\sigma \Big)$$

$$\times \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) \, z \, ds \Big)$$

we use (2.9) and take into account the equation (2.4). As a result, we get

$$A\Big(\int_0^t a_0(s) f(s) \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \Phi[w(\sigma)] - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \Phi[z] \Big) d\sigma \Big)$$

$$\times \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) z ds \Big) = -f(t) z + f(0)$$

$$\times \exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \Phi[w(\sigma)] - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \Phi[z] \Big) \, d\sigma \Big) \exp\Big(\int_0^t a_0(\sigma) d\sigma A \Big) z$$

$$+\int_0^t \Big[f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(\sigma)}\,\Phi[w(s)] - \frac{a_0(s)f(s)}{k(s)}\Phi[z]\Big)\Big]$$
$$\times \exp\bigg(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\bigg)\exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)zds. \tag{2.10}$$

Now we introduce the linear operator R defined by

$$R(\Phi[w],\Phi[z],\mathbf{d},t)$$
$$:= f(0)\exp\bigg(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\bigg)$$
$$\times \exp\bigg(\int_0^t a_0(\sigma)d\sigma A\bigg) + \int_0^t \Big[f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(s)}\,\Phi[w(s)] - \frac{a_0(s)f(s)}{k(s)}\Phi[z]\Big)\Big]$$
$$\times \exp\bigg(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\bigg)\exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)ds. \tag{2.11}$$

Consequently, from (2.10) and (2.11) we obtain the equality

$$\int_0^T \varphi(t)A\bigg(\int_0^t a_0(s)f(s)\exp\bigg(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\bigg)$$
$$\times \exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)ds\, z\bigg)d\mu(t) = \int_0^T \varphi(t)\,[-f(t)z + R(\Phi[w],\Phi[z],\mathbf{d},t)z]\,d\mu(t)$$
$$= \bigg(-\int_0^T \varphi(t)f(t)d\mu(t) + \int_0^T \varphi(t)R(\Phi[w],\Phi[z],\mathbf{d},t)\;d\mu(t)\bigg)z$$
$$=: Q(\Phi[w],\Phi[z],\mathbf{d},T)z. \tag{2.12}$$

Inserting such a result into the right-hand side of the equation (2.8), we get

$$Ah - \int_0^T \varphi(t)\exp\bigg(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\bigg)$$
$$\times \exp\bigg(\int_0^t a_0(\sigma)d\sigma A\bigg)Au_0 d\mu(t)$$
$$- \int_0^T \varphi(t)\bigg(\int_0^t a_0(s)\exp\bigg(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma$$
$$\times A\exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)g(s)ds\bigg)d\mu(t) = Q(\Phi[w],\Phi[z],\mathbf{d},T)z. \tag{2.13}$$

Plugging the right-hand side of the formula (2.10) into (2.8) and $(P6)_1$, we derive the new problem: *look for a pair* $(w,z)\in L^\infty(0,T;X)\times X$ *such that*

$$
(P7)\begin{cases}
w(t) = \exp\Big(\int_0^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad \times \exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 \\
\quad +\Big[-f(t)I + f(0)\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad \times \exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big) \\
\quad + \int_0^t (f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(s)}\,\Phi[w(s)] - \frac{a_0(s)f(s)}{k(s)}\Phi[z]\Big)) \\
\quad \times \exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\quad \times \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds\Big]z+ \\
\quad + \int_0^t a_0(s)\exp\Big(\int_s^t (\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\quad \times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds, \\
Ah - \int_0^T \varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\times \exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 d\mu(t) \\
- \int_0^T \varphi(t)\Big(\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma \\
\times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds\Big)d\mu(t) \\
= Q(\Phi[w],\Phi[z],\mathbf{d},T)z.
\end{cases}
$$

Proof of the Lemma 2.1 The proof of this lemma is analogous to [1, Proof of Theorem 2.2]. Observe that, for all $t\in[0,T], \varepsilon_0\in(0,1)$ and $v_1\in X$ we get

$$
\int_0^{\varepsilon_0 t} a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds
$$
$$
\to \int_0^t a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds \text{ in } X \text{ as } \varepsilon_0\to 1- .
$$

Computing the integral

$$
\begin{aligned}
&A\Big(\int_0^{\varepsilon_0 t} a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds\Big)\\
&=\int_0^{\varepsilon_0 t} a_0(s)f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds\\
&=-\int_0^{\varepsilon_0 t} f(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)D_s\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds\\
&=-f(\varepsilon_0 t)\exp\Big(\int_{\varepsilon_0 t}^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_{\varepsilon_0 t}^t a_0(\sigma)d\sigma A\Big)v_1\\
&+f(0)\exp\Big(\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)v_1\\
&+\int_0^{\varepsilon_0 t}\Big[f'(s)-f(s)a_0(s)a_1(s)\Big]\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)v_1 ds
\end{aligned}
$$

and let ε_0 tend to $\to 1-$, we get the desired formula (2.9). ■

Problem 8

Let us suppose that the operator Q is invertible for suitable pairs $(w,z)\in L^\infty(0,T;X)\times X$. This will be proved afterwards in the Subsection 2.3. This assumption let us find out the representation for z from (2.13). Therefore, the problem (P7) is reduced to the following: *look for a pair* $(w,z)\in L^\infty(0,T;X)\times X$ *such that*

$$
(P8)\quad
\begin{cases}
w(t) = \exp\Big(\int_0^t(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\Phi[w(\sigma)]-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big)\\
\qquad\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0\\
\qquad+\Big[-f(t)+f(0)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\Phi[w(\sigma)]-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big)\\
\qquad\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)\\
\qquad+\int_0^t\Big(f'(s)-f(s)\Big(\overline{k}(s)-\frac{a_0(s)}{k(s)}\Phi[w(s)]-\frac{a_0(s)f(s)}{k(s)}\Phi[z]\Big)\Big)
\end{cases}
$$

$$
(P8)\begin{cases}
\qquad \times \exp\Big(\displaystyle\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\qquad \times \exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)ds\Big]z \\
\qquad + \displaystyle\int_0^t a_0(s)\exp\Big(\int_s^t(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z])d\sigma\Big) \\
\qquad \times A\exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds =: W(\Phi[z],\Phi[w],\mathbf{d},t) - f(t)z, \quad t\in[0,T], \\
z = Q(\Phi[w],\Phi[z],\mathbf{d},T)^{-1} \\
\qquad \times\Big\{Ah - \displaystyle\int_0^T\varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\qquad \times \exp\Big(\displaystyle\int_0^t a_0(\sigma)d\sigma A\Big)Au_0d\mu(t) \\
\qquad - \displaystyle\int_0^T\varphi(t)\Big(\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\Phi[w(\sigma)] - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\Phi[z]\Big)d\sigma\Big) \\
\qquad \times A\exp\Big(\displaystyle\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds\Big)d\mu(t)\Big\} =: Z(\Phi[z],\Phi[w],\mathbf{d},T).
\end{cases}
$$

The equivalence of the problems $(P7)$ and $(P8)$ is easy to check: it is enough to apply the operator Q to the both sides of $(P8)_2$.

Problem 9

Our next task consists in deriving a system for the pair of scalar functions

$$(\overline{w},\overline{z}) := (\Phi[w],\Phi[z]). \tag{2.14}$$

For this purpose, we rewrite $(P8)_1$ and $(P8)_2$ in the form:

$$w(t) = W(\overline{w},\overline{z},\mathbf{d},t) - f(t)Z(\overline{w},\overline{z},\mathbf{d},T), \tag{2.15}$$

$$z = Z(\overline{w},\overline{z},\mathbf{d},T), \tag{2.16}$$

where

$$W(\overline{w},\overline{z},\mathbf{d},t)=\sum_{j=1}^{4}W_j(\overline{w},\overline{z},\mathbf{d},t)\quad\text{and}\quad Z(\overline{w},\overline{z},\mathbf{d},T)=\sum_{j=1}^{3}\mathcal{Z}_j(\overline{w},\overline{z},\mathbf{d},T),\qquad(2.17)$$

each $\mathcal{Z}_j$ and W_j being defined by

$$\mathcal{Z}_1(\overline{w},\overline{z},\mathbf{d})=Q(\overline{w},\overline{z},\mathbf{d},T)^{-1}Ah,\qquad(2.18)$$

$$\begin{aligned}\mathcal{Z}_2(\overline{w},\overline{z},\mathbf{d})=&-Q(\overline{w},\overline{z},\mathbf{d},T)^{-1}\\&\times\int_0^T\varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\\&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0d\mu(t),\end{aligned}\qquad(2.19)$$

$$\begin{aligned}\mathcal{Z}_3(\overline{w},\overline{z},\mathbf{d})=&-Q(\overline{w},\overline{z},\mathbf{d},T)^{-1}\int_0^T\varphi(t)d\mu(t)\\&\times\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\\&\times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds,\end{aligned}\qquad(2.20)$$

$$\begin{aligned}W_1(\overline{w},\overline{z},\mathbf{d},t)=&\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\\&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0,\end{aligned}\qquad(2.21)$$

$$\begin{aligned}W_2(\overline{w},\overline{z},\mathbf{d},t)=&f(0)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\\&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)\,Z(\overline{w},\overline{z},\mathbf{d},T),\end{aligned}\qquad(2.22)$$

$$\begin{aligned}W_3(\overline{w},\overline{z},\mathbf{d},t)=&\int_0^t\Big[f'(s)-f(s)\Big(\overline{k}(s)-\frac{a_0(s)}{k(s)}\,\overline{w}(s)-\frac{a_0(s)f(s)}{k(s)}\overline{z}\Big)\Big]\\&\times\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\\&\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\,Z(\overline{w},\overline{z},\mathbf{d},T)ds,\end{aligned}\qquad(2.23)$$

$$W_4(\overline{w},\overline{z},\mathbf{d},t)=\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)$$

$$\times A \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) g(s)ds. \tag{2.24}$$

Applying the functional Φ to both sides of $(P8)_1$ and $(P8)_2$ and taking (2.14) into account, we get the new problem: *look for a pair* $(\overline{w},\overline{z})\in L^\infty(0,T;\mathbb{R})\times\mathbb{R}$ *such that*

$$(P9)\quad\begin{cases} \overline{w}(t) = \Phi\left[W(\overline{w},\overline{z},\mathbf{d},t)\right] - f(t)\Phi\left[Z(\overline{w},\overline{z},\mathbf{d})\right] =: \mathcal{N}_1(\overline{w},\overline{z},\mathbf{d},t),\ t\in[0,T], \\ \\ \overline{z} = \Phi\left[Z(\overline{w},\overline{z},\mathbf{d})\right] =: \mathcal{N}_2(\overline{w},\overline{z},\mathbf{d}). \end{cases}$$

Let us show the equivalence of the problems $(P8)$ and $(P9)$. Once we have solved the problem $(P9)$ and have found out the scalar pair $(\overline{w},\overline{z})\in L^\infty(0,T;\mathbb{R})\times\mathbb{R}$, we can define the vector pair $(w,z)\in L^\infty(0,T;X)\times X$ by the formulae

$$\begin{aligned} w(t) =&\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0 \\ &\Big[-f(t)I+f(0)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big) \\ &\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)+\int_0^t\Big(f'(s)-f(s)\Big(\overline{k}(s)-\frac{a_0(s)}{k(s)}\,\overline{w}(s)-\frac{a_0(s)f(s)}{k(s)}\overline{z}\Big)\Big) \\ &\times\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big) \\ &\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds\Big]Z(\overline{w},\overline{z},\mathbf{d},T) \\ &+\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big) \\ &\times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds,\quad \text{for a.e}\quad t\in(0,T), \end{aligned} \tag{2.25}$$

$$\begin{aligned} z =&Q(\overline{w},\overline{z},\mathbf{d},T)^{-1}\Big\{Ah-\int_0^T\varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big) \\ &\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0d\mu(t) \\ &-\int_0^T\varphi(t)\Big(\int_0^t a_0(s)\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma \end{aligned}$$

$$\times A \exp\left(\int_s^t a_0(\sigma)d\sigma A\right)g(s)ds\Bigg)d\mu(t)\Bigg\}. \tag{2.26}$$

Now we show that the pair $(w, z) \in L^\infty(0, T; X) \times X$ defined by (2.25) and (2.26) satisfies problem (P9). So, it suffices to prove that $\Phi[w] = \overline{w}$ and $\Phi[z] = \overline{z}$. For this purpose, let us apply the linear functional Φ to both sides of (2.25), (2.26) and let us compare obtained relations with $(P9)_1$ and $(P9)_2$ to get

$$\begin{aligned}\Phi[w(t)] =& \Phi\left[\sum_{j=1}^{4} W_j(\overline{w}, \overline{z}, \mathbf{d}, t) - f(t)Z(\overline{w}, \overline{z}, \mathbf{d}, T)\right] = \overline{w},\\ \Phi[z] =& \Phi\left[\sum_{j=1}^{3} \mathcal{Z}_j(\overline{w}, \overline{z}, \mathbf{d}, t)\right] = \overline{z}.\end{aligned}$$

Thus, we have proved that the pair (w, z) defined by (2.25), (2.26) satisfies (2.14). Consequently, replacing in (2.25), (2.26) the pair $(\overline{w}, \overline{z}) \in L^\infty(0, T; \mathbb{R}) \times \mathbb{R}$ by $(\Phi[w(t)], \Phi[z])$, we deduce that the pair $(w, z) \in L^\infty(0, T; X) \times X$ solves problem (P9).

Finally, collecting all the results in this section, we conclude that problems (P1) and (P9) are equivalent.

We have reduced the problem (1.1)-(1.4) to the equivalent system (P9) for $(\overline{w}, \overline{z})$ in the case when

$$(\overline{w}, \overline{z}) \in L^\infty(0, T; \mathbb{R}) \times \mathbb{R}.$$

In order to prove the existence of the solution to that system we shall first limit ourselves to looking for:

$$(\overline{w}, \overline{z}) \in L^{p_1}(0, T; \mathbb{R}) \times \mathbb{R}.$$

After the proof of the existence we shall show that actually $\overline{w} = \Phi[Au]$ belongs to $L^\infty(0, T; \mathbb{R})$.

Let us introduce the complete metric space

$$\mathcal{K}(M_1, M_2, T) := \{(\overline{w}, \overline{z}) \in L^{p_1}(0, T; \mathbb{R}) \times \mathbb{R} : \; \|\overline{w}\|_{L^{p_1}(0,T;\mathbb{R})} \le M_1, \quad |\overline{z}| \le M_2\}, \tag{2.27}$$

whose metric is induced by the norm

$$\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|.$$

In order to show the existence and uniqueness of the solution to the system (P9), we will use the Banach fixed-point theorem. To satisfy all the conditions of the theorem we need to establish that the vector-mapping $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the operators in the right-hand side in (P9), maps the space $\mathcal{K}(M_1, M_2, T)$ into itself and turns out to be a contracting mapping for *large enough* ρ_0.

2.2 Fundamental lemmas

In this section we give some preliminary results.

In order to obtain the existence and uniqueness of a solution to problem $(P9)$ we must estimate the right part of the system $(P9)$. For this purpose we have to know the estimate of term $\exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)$, which enters into the representation of the solution. The following lemma gives us this estimate.

Lemma 2.2 *Under the assumption of Theorem 1.1, for all* $(\overline{w}, \overline{z}, \boldsymbol{d})$ *in* $\mathcal{K}(M_1, M_2, T)$ $\times\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$ *the following estimate holds:*

$$\exp\Big(\int_s^t \Big|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big|d\sigma\Big)\,\Big\|\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\Big\|_X$$
$$\leq \exp\Big(-(\rho_0 - K(\boldsymbol{d}))\int_s^t a_0(\sigma)d\sigma\Big)\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,t;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,t;\mathbb{R})}\Big).$$

Here the function $\overline{k}$ *defined by* (2.5).

Proof . In order to prove this lemma, let us consider and estimate the integral $\int_s^t |a_0(\sigma)a_1(\sigma)|\,d\sigma$ for any $s, t \in (0,T), s < t$. From (2.5), taking into account the assumptions $(A9),(A11)-(A14)$, we get

$$\int_s^t |a_0(\sigma)a_1(\sigma)|\,d\sigma \le \int_s^t \Big|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big| d\sigma$$
$$\le \int_s^t |\overline{k}(\sigma)|\;d\sigma + \int_s^t \Big|\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)\Big|d\sigma + \int_s^t \left|\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\right| d\sigma$$
$$\le \left\|\frac{k'(t)}{k(t)} - \frac{\Phi[a_0(t)g(t)]}{k(t)}\right\|_{L^1(s,t)} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(s,t)} + M_2 \left\|\frac{a_0 f}{k}\right\|_{L^1(s,t)}$$
$$\le \left\|\frac{k'}{k}\right\|_{L^1(s,t;\mathbb{R})} + \left\|\frac{\Phi[a_0 g]}{k}\right\|_{L^1(s,t;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(s,t;\mathbb{R})} + M_2 \left\|\frac{a_0 f}{k}\right\|_{L^1(s,t;\mathbb{R})}$$
$$=: I(s,t,M_1,M_2), \tag{2.28}$$

where $p_1' \in (1,+\infty)$ is the index conjugate to p_1, i.e. $\dfrac{1}{p_1}+\dfrac{1}{p_1'}=1$. Let us remind now that in the conditions $(A20)$ we have assumed:

$$\left\|\frac{\Phi[a_0 g]}{k}\right\|_{L^1(s,t;\mathbb{R})} \le C_2(\mathbf{d})\int_s^t a_0(\sigma)d\sigma, \qquad \left\|\frac{a_0 f}{k}\right\|_{L^1(s,t;\mathbb{R})} \le C_3(\mathbf{d})\int_s^t a_0(\sigma)d\sigma,$$

for any $0 \le s < t \le T$, and some positive continuous and non-decreasing functions $C_2(\mathbf{d}),\ C_3(\mathbf{d})$. Summing up these inequalities, we get

$$\left\|\frac{\Phi[a_0 g]}{k}\right\|_{L^1(s,t;\mathbb{R})} + M_2 \left\|\frac{a_0 f}{k}\right\|_{L^1(s,t;\mathbb{R})} \le (C_2(\mathbf{d}) + C_3(\mathbf{d})M_2)\int_s^t a_0(\sigma)d\sigma. \tag{2.29}$$

Thus, under these hypotheses, we get

$$I(s,t,M_1,M_2) \le \left\|\frac{k'}{k}\right\|_{L^1(s,t;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(s,t;\mathbb{R})} + (C_2(\mathbf{d}) + C_3(\mathbf{d})M_2)\int_s^t a_0(\sigma)d\sigma. \tag{2.30}$$

Then we assume that

$$(C_2(\mathbf{d}) + C_3(\mathbf{d})M_2)\int_s^t a_0(\sigma)d\sigma \le \varepsilon(\rho_0,d)\rho_0 \int_s^t a_0(\sigma)d\sigma, \quad \varepsilon \in (0,1), \tag{2.31}$$

where

$$\rho_0 > K(\mathbf{d}) + 1$$

and

$$\varepsilon(\rho_0, \mathbf{d}) := \frac{C_2(\mathbf{d}) + C_3(\mathbf{d})M_2}{\rho_0} =: \frac{K(\mathbf{d})}{\rho_0} \in (0,1).$$

This explains the choice of (1.15). It is easy to see from (2.28), (2.30), (2.31) that

$$\int_s^t |a_0(\sigma)a_1(\sigma)|\, d\sigma \leq \varepsilon(\rho_0, \mathbf{d})\rho_0 \int_s^t a_0(\sigma)\, d\sigma + \left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}. \quad (2.32)$$

The previous inequality implies

$$\exp\left(\int_s^t |a_0(\sigma)a_1(\sigma)|\, d\sigma\right) = \exp\left(\int_s^t \left|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\right| d\sigma\right)$$

$$\leq \exp\left(K(\mathbf{d}) \int_s^t a_0(\sigma)d\sigma\right) \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right), \quad (2.33)$$

for all $s < t,\ s,\ t \in [0,T]$.

Let us remark that the assumption $(A3)$, in the case of the problem (1.1)–(1.3), can be written as

$$\left\|\exp\left(\int_s^t a_0(\sigma)d\sigma A\right)\right\|_{\mathcal{L}(X)} \leq \exp\left(-\rho_0 \int_s^t a_0(\sigma)d\sigma\right), \quad 0 \leq s < t \leq T. \quad (2.34)$$

Taking (2.33) and (2.34) into account, for any $0 \leq s < t \leq T$ we get the assertion of Lemma 2.2. ■

Remark 2.1

Now we are going to explain the origin of the unusual inequality (2.32). Let us consider the problem (1.1)–(1.3) when a_1 *is known*. In such a case we can use Lorenzi's result [14] in the case when the pair (a_1, ρ_0) there is replaced with $(a_0a_1, \varepsilon\rho_0)$. Hence, the scalar coefficient $a_1(t)$ should satisfy the following relations

$$a_1 \in L^1_{a_0}(0,T;\mathbb{R}), \int_s^t |a_0(\sigma)a_1(\sigma)|\, d\sigma \leq \varepsilon\rho_0 \int_s^t a_0(\sigma)\, d\sigma, \quad \forall t \in (s,T],\ s \in [0,T). \quad (2.35)$$

Unfortunately, in our problem the coefficient a_1 *is unknown* and the requirement (2.35) has to be changed. In which way it must be changed we shall understand from the next reasoning. Let us assume that the inequality (2.35) holds and prove the contradiction. In fact, the limitation

$$I(s,t,M_1,M_2) \le \varepsilon\rho_0 \int_s^t a_0(\sigma)\,d\sigma, \quad \forall t \in (s,T],\ s \in [0,T),$$

where I is defined by (2.3), or

$$\rho_0 \ge \left(\varepsilon \int_s^t a_0(\sigma)\,d\sigma\right)^{-1} I(s,t,M_1,M_2) =: I_1(s,t,M_1,M_2), \quad \forall t \in (s,T],\ s \in [0,T), \tag{2.36}$$

similar to (2.35), *fails* and it is easy to check why.

Let us suppose that the inequality (2.36) holds. Therefore, taking the supremum in (2.36) with respect to s,t, we obtain

$$\rho_0 \ge \sup_{0\le s<t\le T} I_1(s,t,M_1,M_2) =: \widetilde{I}_1(M_1,M_2). \tag{2.37}$$

However, in the general case $\widetilde{I}_1(M_1,M_2)$ may be $+\infty$ for arbitrary choice of our data. Let us analyze $\widetilde{I}_1(M_1,M_2)$ in the *case of smooth functions* $g \in C^1([0,T],X), k, f \in C^1([0,T],\mathbb{R})$ and $a_0(t) = t^{\alpha-1}$. We now aim to evaluate

$$\begin{aligned} I_1(s,t,M_1,M_2) &= \frac{1}{\varepsilon\|a_0\|_{L^1(s,t;\mathbb{R})}} \Bigg[\left\|\frac{k'}{k}\right\|_{L^\infty(s,t;\mathbb{R})} (t-s) + \|\Phi\|_{X^*} \|a_0\|_{L^1(s,t;\mathbb{R})} \left\|\frac{g}{k}\right\|_{L^\infty(0,T;X)} \\ &\quad + M_1 \left\|\frac{1}{k}\right\|_{L^\infty(0,T;\mathbb{R})} \|a_0\|_{L^{p_1'}(s,t;\mathbb{R})} + M_2 \|a_0\|_{L^1(s,t;\mathbb{R})} \left\|\frac{f}{k}\right\|_{L^\infty(s,t;\mathbb{R})} \Bigg] \\ &= \frac{\alpha(t-s)}{\varepsilon(t^\alpha - s^\alpha)} \left\|\frac{k'}{k}\right\|_{L^\infty(0,T;\mathbb{R})} + \frac{1}{\varepsilon}\|\Phi\|_{X^*} \left\|\frac{g}{k}\right\|_{L^\infty(0,T;X)} + \frac{\alpha M_1}{\varepsilon(t^\alpha - s^\alpha)} \left\|\frac{1}{k}\right\|_{L^\infty(0,T;\mathbb{R})} \\ &\quad \times \left(\frac{1}{p_1'(\alpha-1)+1}\left(t^{p_1'(\alpha-1)+1} - s^{p_1'(\alpha-1)+1}\right)\right)^{1/(p_1')} + \frac{M_2}{\varepsilon} \left\|\frac{f}{k}\right\|_{L^\infty(s,t;\mathbb{R})}. \end{aligned} \tag{2.38}$$

Let us distinguish the following two cases: $\alpha \in (0,1)$ and $\alpha \in (1,+\infty)$.

Case $\alpha \in (0,1)$

Let us set $s = mt$, where $m \in (0,1)$, and analyze the right-hand side in (2.38). We notice that the fraction

$$\frac{t-s}{t^\alpha - s^\alpha} = \frac{t^{1-\alpha}(1-m)}{1-m^\alpha} \leq C_9(\alpha)T^{1-\alpha}$$

is bounded in $0 < s < t < T$, while the quotient

$$\frac{\left[t^{p_1'(\alpha-1)+1} - s^{p_1'(\alpha-1)+1}\right]\Big)^{1/(p_1')}}{t^\alpha - s^\alpha} = t^{-1/p_1}\frac{[1 - m^{p_1'(\alpha-1)+1}]^{1/p_1'}}{1-m^\alpha}$$

is unbounded in the same set. This implies that $\widetilde{I}_1(M_1, M_2) = +\infty$ for any $\alpha \in (0,1)$.

Case $\alpha \in (1,+\infty)$

In this case it is unbounded also the function

$$(s,t) \to \frac{t-s}{t^\alpha - s^\alpha}.$$

This implies that $\widetilde{I}_1(M_1, M_2) = +\infty$ for any $\alpha > 0$.

It is easy to see from (2.28), (2.29), (2.31) that we are obliged to replace the hypothesis (2.35) with (2.32). ∎

The following lemma gives two important estimates on some integrals and will be used in the proof of existence results.

Lemma 2.3 *For any pair of functions* $m_1, m_2 \in L^{p_1}(0,t;\mathbb{R})$, *and* a_0 *satisfying* $(A6), (A7), (A9)$, *the following estimates hold:*

$$\int_0^t |m_1(s)| \exp\Big(-(\rho_0 - K(\boldsymbol{d}))\int_s^t a_0(\sigma)d\sigma\Big)ds$$
$$\leq \rho_0^{-1/p_3}\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,t;\mathbb{R})}\frac{\|m_1\|_{L^{p_1}(0,t;\mathbb{R})}}{[p_3(1-K(\boldsymbol{d})/\rho_0)]^{1/p_3}}, \qquad (2.39)$$

$$\int_0^t a_0(s)|m_2(s)|\exp\Big(-(\rho_0 - K(\boldsymbol{d}))\int_s^t a_0(\sigma)d\sigma\Big)ds$$
$$\le \rho_0^{-1/p_3}\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,t;\mathbb{R})}\frac{\|m_2\|_{L^{p_1}(0,t;\mathbb{R})}}{[p_3(1-K(\boldsymbol{d})/\rho_0)]^{1/p_3}}, \tag{2.40}$$
$$0 \le s < t \le T, \quad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1.$$

Here ρ_0, K are defined in (1.15).

Proof. Taking the assumptions $(A6), (A9)$ into account, we estimate via Hölder's inequality the integral

$$\int_0^t |m_1(s)|\frac{1}{a_0^{1/p_3}(s)}a_0^{1/p_3}(s)\exp\Big(-(\rho_0-K(\mathbf{d}))\int_s^t a_0(\sigma)d\sigma\Big)ds$$
$$\le \|m_1\|_{L^{p_1}(0,t;\mathbb{R})}\left(\int_0^t \frac{1}{a_0^{p_2/p_3}(s)}\,ds\right)^{1/p_2}$$
$$\times\left(\int_0^t a_0(s)\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_s^t a_0(\sigma)d\sigma\Big)ds\right)^{1/p_3} = \|m_1\|_{L^{p_1}(0,t;\mathbb{R})}$$
$$\times\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,t;\mathbb{R})}\left(\int_0^t a_0(s)\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_s^t a_0(\sigma)d\sigma\Big)ds\right)^{1/p_3}. \tag{2.41}$$

Observe now that

$$\int_0^t a_0(s)\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_s^t a_0(\sigma)d\sigma\Big)ds$$
$$=\frac{1}{(1-K(\mathbf{d})/\rho_0)p_3}\left[1-\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_0^t a_0(\sigma)d\sigma\Big)\right]$$
$$\le \frac{\rho_0^{-1/p_3}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}. \tag{2.42}$$

Similarly to (2.42), we get

$$\int_0^T a_0(t)\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_0^t a_0(\sigma)d\sigma\Big)dt \le \frac{\rho_0^{-1/p_3}}{(1-K(\mathbf{d})/\rho_0))p_3}. \tag{2.43}$$

As an immediate consequence, from (2.41) and (2.42) we get the estimate (2.39).

Assessing as in (2.41), taking into account $(A7)$ and using Hölder's inequality, we obtain the following estimate

$$\int_0^t a_0^{1/p_3'}(s)a_0^{1/p_3}(s)|m_2(s)|\exp\Big(-(\rho_0-K(\mathbf{d}))\int_s^t a_0(\sigma)d\sigma\Big)ds \le \|m_2\|_{L^{p_1}(0,t;\mathbb{R})}$$
$$\times\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,t;\mathbb{R})}\Big(\int_0^t a_0(s)\exp\Big(-(\rho_0-K(\mathbf{d}))p_3\int_s^t a_0(\sigma)d\sigma\Big)ds\Big)^{1/p_3}.$$

Thanks to (2.42), we easily verify the estimate (2.40). ■

2.3 Existence of the inverse to the operator Q and related estimates

Let us recall that in order to reduce the problem (1.1)-(1.4) the system $(P9)$ we assumed the operator Q defined by (2.12) to be invertible. In this section we shall study the invertibility of the operator Q.

First we prove the following lemma related to the continuous invertibility of the operator Q.

Lemma 2.4 *Let $\widetilde{R}$ be the operator defined by*

$$\widetilde{R}(\overline{w},\overline{z},\boldsymbol{d},T) := f(0)\int_0^T\varphi(t)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)$$
$$\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)d\mu(t)$$
$$+\int_0^T\varphi(t)d\mu(t)\int_0^t\Big(f'(s)-f(s)\Big(\overline{k}(s)-\frac{a_0(s)}{k(s)}\,\overline{w}(s)-\frac{a_0(s)f(s)}{k(s)}\overline{z}\Big)\Big)$$
$$\times\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds. \quad (2.44)$$

If for any $(\overline{w},\overline{z},\boldsymbol{d})\in\mathcal{K}(M_1,M_2,T)\times\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ the assumption

$$\|\widetilde{R}(\overline{w},\overline{z},\boldsymbol{d},T)\|_{\mathcal{L}(X)}\le\frac{|\lambda(\boldsymbol{d})|}{2}, \quad (2.45)$$

holds, where $\lambda(\boldsymbol{d})$ *defined by (1.18), then the operator* Q *is defined by (2.12) is continuously invertible. Moreover, the following estimate holds:*

$$\|Q(\overline{w}, \overline{z}, \boldsymbol{d}, T)^{-1}\|_{\mathcal{L}(X)} \leq \frac{2}{|\lambda(\boldsymbol{d})|}. \tag{2.46}$$

Proof . We remind that the operator Q is defined by (2.12), i.e.

$$Q(\overline{w}, \overline{z}, \mathbf{d}, T) = -\lambda(\mathbf{d})I + \widetilde{R}(\overline{w}, \overline{z}, \mathbf{d}, T) = -\lambda(\mathbf{d})\left(I - \frac{1}{\lambda(\mathbf{d})}\widetilde{R}(\overline{w}, \overline{z}, \mathbf{d}, T)\right). \tag{2.47}$$

To show that Q is continuously invertible it suffices to apply Neuman's theorem [17] under the condition (2.45). The formulae (2.47) implies

$$Q(\overline{w}, \overline{z}, \mathbf{d}, T)^{-1} = -\frac{1}{\lambda(\mathbf{d})}\left(1 - \frac{1}{\lambda(\mathbf{d})}\widetilde{R}(\overline{w}, \overline{z}, \mathbf{d}, T)\right)^{-1}.$$

Whence we deduce

$$\|Q(\overline{w}, \overline{z}, \mathbf{d}, T)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{|\lambda(\mathbf{d})|\left(1 - \frac{\|\widetilde{R}(\overline{w}, \overline{z}, \mathbf{d}, T)\|_{\mathcal{L}(X)}}{|\lambda(\mathbf{d})|}\right)} = \frac{1}{|\lambda(\mathbf{d})| - \|\widetilde{R}(\overline{w}, \overline{z}, \mathbf{d}, T)\|_{\mathcal{L}(X)}}.$$

Then, taking into account the assumption (2.45), we easily deduce the estimate (2.46).

■

Lemma 2.5 *Let the assumptions* $(A1) - (A21)$ *hold. Then for all*

$$(\overline{w}, \overline{z}, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$$

the operator Q *admits an inverse* $Q^{-1} \in \mathcal{L}(X)$ *satisfying the estimate (2.46), i.e.*

$$\|Q(\overline{w}, \overline{z}, \boldsymbol{d}, T)^{-1}\|_{\mathcal{L}(X)} \leq \begin{cases} \dfrac{2}{|\varphi(T_1)f(T_1)|}, & \text{Case 1,} \\ \dfrac{2}{\left|\int_0^T \varphi(t)f(t)\psi(t)dt\right|}, & \text{Case 2.} \end{cases} \tag{2.48}$$

Proof . In order to prove the existence and the estimate for the operator Q^{-1}, we have to estimate $\widetilde{R}$ (cf. (2.44)) and show the inequality (2.45). Let us prove the estimate (2.48) only in the Case 2, which seems to be technically more complicated. The Case 1 is analogous, it is enough only to apply the estimate (2.34) as well as (2.5), Lemma 2.2, (2.33), (2.39) and (2.40) to get the desired estimate in Case 1.

For the sake of convenience, let us rewrite (2.44) as

$$\widetilde{R}(\overline{w},\overline{z},\mathbf{d},T) = \widetilde{R}_1(\overline{w},\overline{z},\mathbf{d},T) + \widetilde{R}_2(\overline{w},\overline{z},\mathbf{d},T), \tag{2.49}$$

where

$$\widetilde{R}_1(\overline{w},\overline{z},\mathbf{d},T) = f(0)\int_0^T \varphi(t)\exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big) \\ \times \exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)d\mu(t)$$

and

$$\widetilde{R}_2(\overline{w},\overline{z},\mathbf{d},T) = \int_0^T \varphi(t)d\mu(t)\int_0^t \Big(f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(s)}\,\overline{w}(s) - \frac{a_0(s)f(s)}{k(s)}\overline{z}\Big)\Big) \\ \times \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds.$$

We remind that in the Case 2 we have $d\mu(t) = \psi(t)dt$ with $\psi(t) > 0$ satisfying (1.6). We first start to estimate $\widetilde{R}_1$. Applying the Lemma 2.2, we get

$$\|\widetilde{R}_1(\overline{w},\overline{z},\mathbf{d},T)\|_{\mathcal{L}(X)} \leq |f(0)|\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big) \\ \times \int_0^T |\varphi(t)|\psi(t)\exp\Big(-(\rho_0 - K(\mathbf{d}))\int_0^t a_0(\sigma)d\sigma\Big)dt. \tag{2.50}$$

Using Hölder's inequality and (2.43), let us first evaluate the integral

$$\int_0^T |\varphi(t)|\psi(t)\exp\Big(-(\rho_0 - K(\mathbf{d}))\int_0^t a_0(\sigma)d\sigma\Big)dt \\ = \int_0^T |\varphi(t)|\psi(t)a_0(t)^{1/p_3}\exp\Big(-(\rho_0 - K(\mathbf{d}))\int_0^t a_0(\sigma)d\sigma\Big)\frac{1}{a_0(t)^{1/p_3}}dt \\ \leq \Big(\int_0^T [|\varphi(t)|\psi(t)]^{p_1}dt\Big)^{1/p_1}\Big(\int_0^T \frac{dt}{a_0^{p_2/p_3}(t)}\Big)^{1/p_2}$$

$$\times \left(\int_0^T a_0(t) \exp\Big(-(\rho_0 - K(\mathbf{d}))p_3 \int_0^t a_0(\sigma)d\sigma \Big) dt \right)^{1/p_3}$$

$$\leq \rho_0^{-1/p_3} \left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,T;\mathbb{R})} \frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}. \tag{2.51}$$

As a result, from (2.50) we get

$$\|\widetilde{R}_1(\overline{w},\overline{z},\mathbf{d},T)\|_{\mathcal{L}(X)} \leq |f(0)|\rho_0^{-1/p_3} \left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,T;\mathbb{R})} \frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}$$
$$\times \exp\left(\left\| \frac{k'}{k} \right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,T;\mathbb{R})} \right). \tag{2.52}$$

Next, we estimate the second summand of $\widetilde{R}$. By means of notation (2.5) and Lemma 2.2 we get

$$\|\widetilde{R}_2(\overline{w},\overline{z},\mathbf{d},T)\|_{\mathcal{L}(X)} \leq \exp\left(\left\| \frac{k'}{k} \right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,T;\mathbb{R})} \right)$$
$$\times \int_0^T |\varphi(t)|\psi(t)dt \int_0^t \Big| f'(s) - \frac{f(s)k'(s)}{k(s)} + \frac{f(s)\Phi[a_0(s)g(s)]}{k(s)} + \frac{a_0(s)f(s)}{k(s)}\,\overline{w}(s)$$
$$+ \frac{a_0(s)f^2(s)}{k(s)}\overline{z} \Big| \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma)d\sigma \Big) ds. \tag{2.53}$$

First we use Lemma 2.3 to assess the integral

$$\int_0^t \Big| f'(s) - \frac{f(s)k'(s)}{k(s)} + \frac{a_0(s)f(s)\Phi[g(s)]}{k(s)} + \frac{a_0(s)f(s)}{k(s)}\,\overline{w}(s) + \frac{a_0(s)f^2(s)}{k(s)}\overline{z}\Big) \Big|$$
$$\times \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma)d\sigma \Big) ds \leq \rho_0^{-1/p_3} \frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}$$
$$\times \Bigg[\Big(\|f'\|_{L^{p_1}(0,T;\mathbb{R})} + \left\| \frac{fk'}{k} \right\|_{L^{p_1}(0,T;\mathbb{R})} \Big) \left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,T;\mathbb{R})} + \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,T;\mathbb{R})}$$
$$\times \Big(\|\Phi\|_{X^*} \left\| \frac{fg}{k} \right\|_{L^{p_1}(0,T;X)} + \left\| \frac{f}{k} \right\|_{L^\infty(0,T;\mathbb{R})} M_1 + \left\| \frac{f^2}{k} \right\|_{L^{p_1}(0,T;\mathbb{R})} M_2 \Big) \Bigg]. \tag{2.54}$$

Second, taking into account (2.39), (2.40), we estimate the following integral:

$$\int_0^T |\varphi(t)|\psi(t)dt \int_0^t |\kappa_1(s)| \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_0^t a_0(\sigma)d\sigma \Big) ds$$

$$\leq \Big(\int_0^T [|\varphi(t)|\psi(t)]^{p_1} dt \Big)^{p_1}$$

$$\times \Big(\int_0^T \Big(\int_0^t |\kappa_1(s)| \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma) d\sigma \Big) ds \Big)^{p_1'} dt \Big)^{1/p_1'}$$

$$\leq \Big(\int_0^T [|\varphi(t)|\psi(t)]^{p_1} dt \Big)^{p_1}$$

$$\times \Big(\int_0^T \Big(\|\kappa_1\|_{L^{p_1}(0,T;\mathbb{R})} \Big\| \frac{1}{a_0^{1/p_3}} \Big\|_{L^{p_2}(0,T;\mathbb{R})} \frac{1}{[\rho_0 p_3 (1 - K(\mathbf{d})/\rho_0)]^{1/p_3}} \Big)^{p_1'} dt \Big)^{1/p_1'}$$

$$= \rho_0^{-1/p_3} \Big\| \frac{1}{a_0^{1/p_3}} \Big\|_{L^{p_2}(0,T;\mathbb{R})} \frac{T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})} \|\kappa_1\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}}. \tag{2.55}$$

Analogously, we get

$$\int_0^T |\varphi(t)|\psi(t) dt \int_0^t a_0(s)|\kappa_2(s)| \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma) d\sigma \Big) ds$$

$$\leq \frac{T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})} \|\kappa_2\|_{L^{p_1}(0,T;\mathbb{R})} \Big\| a_0^{1/p_3'} \Big\|_{L^{p_2}(0,T;\mathbb{R})}}{[\rho_0 p_3 (1 - K(\mathbf{d})/\rho_0)]^{1/p_3}}. \tag{2.56}$$

Consequently, from (2.55) and (2.56) we deduce

$$\int_0^T |\varphi(t)|\psi(t)| dt \int_0^t (|\kappa_1(s)| + a_0(s)|\kappa_2(s)|) \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma) d\sigma \Big) ds$$

$$\leq \frac{T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{(\rho_0 p_3 (1 - K(\mathbf{d})/\rho_0)))^{1/p_3}} \Bigg(\Big\| \frac{1}{a_0^{1/p_3}} \Big\|_{L^{p_2}(0,T;\mathbb{R})} \|\kappa_1\|_{L^{p_1}(0,T;\mathbb{R})}$$

$$+ \|\kappa_2\|_{L^{p_1}(0,T;\mathbb{R})} \Big\| a_0^{1/p_3'} \Big\|_{L^{p_2}(0,T;\mathbb{R})} \Bigg). \tag{2.57}$$

Furthermore, taking advantage of (2.54) and (2.57), we get

$$\|\widetilde{R}_2(\overline{w}, \overline{z}, \mathbf{d}, T)\|_{\mathcal{L}(X)} \leq \rho_0^{-1/p_3} \frac{T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}}$$

$$\times \Bigg(\Big\| \frac{1}{a_0^{1/p_3}} \Big\|_{L^{p_2}(0,T;\mathbb{R})} \Big(\|f'\|_{L^{p_1}(0,T;\mathbb{R})} + \Big\| \frac{fk'}{k} \Big\|_{L^{p_1}(0,T;\mathbb{R})} \Big) + \Big\| a_0^{1/p_3'} \Big\|_{L^{p_2}(0,T;\mathbb{R})}$$

$$\times \Big(\|\Phi\|_{X^*} \Big\| \frac{fg}{k} \Big\|_{L^{p_1}(0,T;\mathbb{R})} + \Big\| \frac{f}{k} \Big\|_{L^{\infty}(0,T;\mathbb{R})} M_1 + \Big\| \frac{f^2}{k} \Big\|_{L^{p_1}(0,T;\mathbb{R})} M_2 \Big) \Bigg). \tag{2.58}$$

Taking into account (2.54), (2.51), (2.58), from (2.50) we finally get

$$\|\widetilde{R}(\overline{w},\overline{z},\mathbf{d},T)\|_{\mathcal{L}(X)}$$
$$\leq \rho_0^{-1/p_3}\frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})}+M_1\left\|\frac{a_0}{k}\right\|_{L^{p'_1}(0,T;\mathbb{R})}\right)$$
$$\times\left[|f(0)|\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,T;\mathbb{R})}+T^{1/p'_1}\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,T;\mathbb{R})}\Big(\|f'\|_{L^{p_1}(0,T;\mathbb{R})}\right.$$
$$+\left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,T;\mathbb{R})}\Big)+T^{1/p'_1}\Big(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,T;X)}+\left\|\frac{f}{k}\right\|_{L^\infty(0,T;\mathbb{R})}M_1$$
$$\left.+\left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,T;\mathbb{R})}M_2\Big)\left\|a_0^{1/p'_3}\right\|_{L^{p_2}(0,T;\mathbb{R})}\right]=J(\rho_0,\mathbf{d},M_1,M_2,T)$$
$$\leq\frac{1}{2}\Big|\int_0^T\varphi(t)f(t)\psi(t)dt\Big|,$$

according to the definition (1.17) and the assumption ($A21$). ■

2.4 $\mathcal{N}$ maps the metric space $\mathcal{K}$ into itself

In order the operator defined by the right-hand side of the equations in ($P9$) to map the set $\mathcal{K}(M_1,M_2,T)$, defined in (2.27), into itself, the following conditions

$$\|\mathcal{N}_1(\overline{w},\overline{z},\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\leq\|\Phi\|_{X^*}\sum_{j=1}^{4}\|W_j(\overline{w},\overline{z},\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$+\|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\|\Phi\|_{X^*}\sum_{j=1}^{3}\left\|Z_j(\overline{w},\overline{z},\mathbf{d},\widetilde{T})\right\|_X\leq M_1, \tag{2.59}$$

$$|\mathcal{N}_2(\overline{w},\overline{z},\mathbf{d})|\leq\|\Phi\|_{X^*}\sum_{j=1}^{3}\left\|Z_j(\overline{w},\overline{z},\mathbf{d},\widetilde{T})\right\|_X\leq M_2, \tag{2.60}$$

have to be satisfied. To prove them, we must estimate each $\mathcal{Z}_j, j=1,2,3$ and $W_j, j=1,2,3,4$. We assess firstly $\mathcal{N}_2$. To this aim we need the following lemma.

Lemma 2.6 *Under the assumption ($A4$) the following estimate holds:*

$$\|A\exp(tA)z\|_X\leq C_{10}(\gamma,T)t^{-1+\gamma}\exp(-\rho_0 t)\|z\|_{\mathcal{D}_A(\gamma,\infty)},\ z\in\mathcal{D}_A(\gamma,\infty),\ t\in\mathbb{R}_+. \tag{2.61}$$

Here $C_{10}(\gamma,T)$ *is some positive constant,* $0<\theta<1$, $\gamma>0$.

Proof . From $(A4)$ for $z\in X$ we have:

$$\|Ae^{tA}z\|_X\le C_1(t^{-1}+1)e^{-\rho_0 t}\|z\|_X.$$

For $z\in D(A)$ use $(A3)$:

$$\|Ae^{tA}z\|_X=\|e^{tA}Az\|_X\le e^{-\rho_0 t}\|Az\|_X\le e^{-\rho_0 t}\|z\|_{D(A)}.$$

Then the operator

$$\widetilde{B}(t):=Ae^{tA}$$

firstly is continuous in $\mathcal{L}(X)$ with norm not exceeding $C_1(t^{-1}+1)e^{-\rho_0 t}$ and, secondly, continuous from $D(A)$ in X with norm not exceeding $e^{-\rho_0 t}$. Using interpolation results [18, Chapter 1, page 12], we get the inclusion

$$\widetilde{B}(t)\in\mathcal{L}(\mathcal{D}_A(\gamma,\infty);X),\ t\in\mathbb{R}_+$$

and the estimate

$$\|\widetilde{B}(t)\|_{L(\mathcal{D}_A(\gamma,\infty);X)}\le C_{11}\|\widetilde{B}(t)\|^{1-\gamma}_{\mathcal{L}(X)}\|\widetilde{B}(t)\|^{\gamma}_{L(D(A);X)}\le C_{11}(C_1(t^{-1}+1)e^{-\rho_0 t})^{1-\gamma}e^{-\gamma\rho_0 t}$$
$$=C_{11}C_1^{1-\gamma}(t^{-1}+1)^{1-\gamma}e^{-\rho_0 t}\le C_{11}C_1^{1-\gamma}\frac{(1+T)^{1-\gamma}}{t^{1-\gamma}}\ e^{-\rho_0 t}=:C_{10}(\gamma,T)t^{\gamma-1}e^{-\rho_0 t}.\ \blacksquare$$

Lemma 2.7 *Under the conditions (A5), (A9), (A12), (A13), (A15), for any* $(\overline{w},\overline{z},\boldsymbol{d})\in\mathcal{K}(M_1,M_2,T)\times\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ *the following estimate holds:*

$$I(t):=\int_0^t a_0(s)\exp\Big(\int_s^t\Big|\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big|\,d\sigma\Big)$$
$$\times\Big\|A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)\Big\|_X\,ds$$
$$\le\rho_0^{-1/p_3}C_{10}(\gamma,T)\frac{\|a_0\|^{(1-(1-\gamma)p_2)/p_2}_{L^1(0,T;\mathbb{R})}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}$$

$$\times \exp\Big(\Big\| \frac{k'}{k} \Big\|_{L^1(0,T;\mathbb{R})} + M_1 \Big\| \frac{a_0}{k} \Big\|_{L^{p_1'}(0,T;\mathbb{R})} \Big), \quad 0 < t \le T. \tag{2.62}$$

Here $\overline{k}$ is defined by (2.5) and C_{10} by (2.61).

Proof . The estimate (2.61) implies

$$\Big\| A \exp\Big(\int_s^t a_0(\sigma)d\sigma A \Big) g(s) \Big\|_X \le C_{10}(\gamma, T) \|g(s)\|_{\mathcal{D}_A(\gamma,\infty)}$$
$$\times \left(\int_s^t a_0(\sigma)d\sigma \right)^{-1+\gamma} \exp\left(-\rho_0 \int_s^t a_0(\sigma)d\sigma \right), \ 0 \le s < t \le T. \tag{2.63}$$

By means of identity $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} = 1$, from (2.33) and (2.63) we deduce

$$I(t) \le C_{10}(\gamma, T) \exp\Big(\Big\| \frac{k'}{k} \Big\|_{L^1(0,T;\mathbb{R})} + M_1 \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,T;\mathbb{R})} \Big)$$
$$\times \int_0^t a_0(s) \exp\Big(K(\mathbf{d}) \int_s^t a_0(\sigma)d\sigma \Big) \left(\int_s^t a_0(\sigma)d\sigma \right)^{-1+\gamma}$$
$$\times \exp\Big(-\rho_0 \int_s^t a_0(\sigma)d\sigma \Big) \|g(s)\|_{\mathcal{D}_A(\gamma,\infty)} ds$$
$$= C_{10}(\gamma, T) \exp\Big(\Big\| \frac{k'}{k} \Big\|_{L^1(0,T;\mathbb{R})} + M_1 \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,T;\mathbb{R})} \Big)$$
$$\times \int_0^t a_0^{1/p_3}(s) \exp\Big(-(\rho_0 - K(\mathbf{d})) \int_s^t a_0(\sigma)d\sigma \Big)$$
$$\times a_0^{1/p_2}(s) \left(\int_s^t a_0(\sigma)d\sigma \right)^{-1+\gamma} a_0^{1/p_1}(s) \|g(s)\|_{\mathcal{D}_A(\gamma,\infty)} ds. \tag{2.64}$$

Then applying Hölder's inequality, taking into account the inequality (2.42) and the equality

$$\int_0^t a_0(s) \left(\int_s^t a_0(\sigma)d\sigma \right)^{(\gamma-1)p_2} ds = \frac{1}{1 - (1-\gamma)p_2} \left(\int_0^t a_0(\sigma)d\sigma \right)^{1-(1-\gamma)p_2},$$

from (2.64) we conclude

$$I(t) \le \Big[\int_0^t a_0(s) \exp\Big(-(1 - K(\mathbf{d})/\rho_0) p_3 \rho_0 \int_s^t a_0(\sigma)d\sigma \Big) ds \Big]^{1/p_3}$$

$$\times\left[\int_0^t a_0(s)\left(\int_s^t a_0(\sigma)d\sigma\right)^{(\gamma-1)p_2}ds\right]^{1/p_2}\|a_0^{1/p_1}g\|_{L^{p_1}(0,t;D_A(\gamma,\infty))}$$
$$\le C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})}+M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)$$
$$\times\frac{\rho_0^{-1/p_3}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\|a_0^{1/p_1}g\|_{L^{p_1}(0,t;D_A(\gamma,\infty))}. \quad\blacksquare \qquad (2.65)$$

Remark 2.2 *The estimate (2.65) implies the following*

$$\int_0^T|\varphi(t)||\psi(t)|dt\int_0^t a_0(s)\exp\left(\int_s^t\left|\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\right|d\sigma\right)$$
$$\times\left|A\exp\left(\int_s^t a_0(\sigma)d\sigma A\right)g(s)\right|ds\le\left(\int_0^T|\varphi(t)|\psi(t)^{p_1}dt\right)^{1/p_1}$$
$$\times\left[\int_0^T\left(C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})}+M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)\right.\right.$$
$$\left.\left.\times\frac{\rho_0^{-1/p_3}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}\right)^{p_1'}dt\right]^{1/p_1'}$$
$$\le\rho_0^{-1/p_3}T^{1/p_1'}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$
$$\times\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})}+M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right). \qquad (2.66)$$

2.4.1 Estimate of $Z(\overline{w},\overline{z},\mathbf{d})$

In this section we are going to asses Z. The following lemma ensures (1.22).

Lemma 2.1 *Assume that the conditions of Theorem 1.1 hold. Then the following estimate holds true for any* $(\overline{w},\overline{z},\boldsymbol{d})\in\mathcal{K}(M_1,M_2,T)\times\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$:

$$\|Z(\overline{w},\overline{z},\boldsymbol{d},\widetilde{T})\|_X\le\frac{2}{|\lambda(\boldsymbol{d})|}\left\{\|Ah\|_X+\left[\tau_1(\rho_0,\boldsymbol{d},\widetilde{T})\|Au_0\|_X\right.\right.$$
$$\left.+\rho_0^{-1/p_3}\tau_2(\widetilde{T})C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\cdot\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}\right]$$

$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)\Bigg\} = C_5(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}) \qquad (2.67)$$

according to the notation (1.22). Here $Z, \lambda, \tau_1, \tau_2$ are defined by formulas (2.16), (1.18), (1.19), (1.20).

Proof . We shall prove this lemma only in the Case 2, which seems to be a bit more complicated. Let us take into account (2.16) for the case of the continuous measure and first establish an estimate for $\mathcal{Z}_1$ defined by (2.18). Using (2.48), we get

$$\|\mathcal{Z}_1(\overline{w},\overline{z}, \mathbf{d})\|_X \leq \|Q(\overline{w}, \overline{z}, \mathbf{d}, T)^{-1}\|_{\mathcal{L}(X)}\|Ah\|_X \leq \frac{2\|Ah\|_X}{\left|\int_0^T \varphi(t)f(t)\psi(t)dt\right|}. \qquad (2.68)$$

By means of Lemma 2.2, estimate (2.51) and the representation (2.19), we get

$$\|\mathcal{Z}_2(\overline{w},\overline{z}, \mathbf{d})\|_X \leq 2\rho_0^{-1/p_3} \frac{\|Au_0\|_X\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{\left|\int_0^T \varphi(t)f(t)\psi(t)dt\right|}\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,T)} \frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}$$

$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right). \qquad (2.69)$$

In order to get the estimate for $\mathcal{Z}_3$ we use (2.66). Hence, from (2.20), taking into account (2.48), we conclude

$$\|\mathcal{Z}_3(\overline{w},\overline{z}, \mathbf{d})\|_X \leq \rho_0^{-1/p_3} \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T_1;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}$$

$$\times \frac{2T^{1/p_1'}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}C_{10}(\gamma,T)}{\left|\int_0^T \varphi(t)f(t)\psi(t)dt\right|} \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right). \qquad (2.70)$$

Summing up the inequalities (2.68), (2.69) and (2.70), we finally obtain (2.67).

Concerning the Case 1, we investigate it analogously to the Case 2. In order to obtain an estimate for $\mathcal{Z}_2$ we use (2.48), Lemma 2.2. Then, from (2.19) we get the desired estimate. From (2.20), taking into account (2.48), (2.62), we obtain the estimate of $\mathcal{Z}_3$.

The assertion of Lemma 2.1 is then achieved by adding together the estimates for $Z_j, j = 1, 2, 3$. ■

2.4.2 Estimate of $W(\overline{w}, \overline{z}, \mathbf{d})$

In the previous section we have gotten an estimate of Z. In order to deal with $\mathcal{N}_2$ defined in $(P9)_2$, we need also to assess

$$W(\overline{z}, \overline{w}, \mathbf{d}, t) = \sum_{j=1}^{4} W_j(\overline{w}, \overline{z}, \mathbf{d}, t), \tag{2.71}$$

where the operators W_j are defined by (2.21)–(2.24).

Lemma 2.1 *In view of the conditions of the Theorem 1.1 for any*

$$(\overline{w}, \overline{z}, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$$

the following estimate holds:

$$\|W(\overline{w}, \overline{z}, \boldsymbol{d})\|_{L^{p_1}(0,\widetilde{T};X)} \leq C_{12}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}), \tag{2.72}$$

where $C_{12} \to 0,$ *as* $\rho_0 \to +\infty$.

Proof . We note that the difference of Case 1 and Case 2 here is only in the estimate (2.67) of Z, involved in the representation of W.

Firstly we estimate $W_1(\overline{w}, \overline{z}, \mathbf{d})$ defined in (2.21). Exploiting (2.33), we get:

$$\begin{aligned}
\|W_1(\overline{w}, \overline{z}, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} &= \Bigg(\int_0^{\widetilde{T}} \Bigg| \exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z} \Big) d\sigma \Bigg) \\
&\quad \times \exp\Bigg(\int_0^t a_0(\sigma) d\sigma A \Bigg) A u_0 \Bigg|^{p_1} dt \Bigg)^{1/p_1} \\
&\quad \leq \|Au_0\|_X \exp\Bigg(\Big\| \frac{k'}{k} \Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\| \frac{a_0}{k} \Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} \Bigg) \\
&\quad \times \Bigg(\int_0^{\widetilde{T}} \exp\Bigg(-(\rho_0 - K(\mathbf{d})) p_1 \int_0^t a_0(\sigma) d\sigma \Bigg) dt \Bigg)^{1/p_1}.
\end{aligned} \tag{2.73}$$

In order to assess the last integral we introduce a new independent parameter $q > 1$ and use the Hölder inequality:

$$\begin{aligned}
&\int_0^{\widetilde{T}} \exp\Big(-(\rho_0 - K(\mathbf{d}))p_1 \int_0^t a_0(\sigma)d\sigma\Big)dt \\
&\quad = \int_0^{\widetilde{T}} \frac{1}{a_0^{1/q}(t)} \exp\Big(-(\rho_0 - K(\mathbf{d}))p_1 \int_0^t a_0(\sigma)d\sigma\Big) a_0^{1/q}(t)dt \\
&\quad \le \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/q'} \left(\int_0^{\widetilde{T}} \exp\Big(-(\rho_0 - K(\mathbf{d}))qp_1 \int_0^t a_0(\sigma)d\sigma\Big) a_0(t)dt\right)^{1/q} \\
&\quad = \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/q'} \frac{1}{[\rho_0 qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/q}} \\
&\quad \times \Big(1 - \exp\Big(-(\rho_0 - K(\mathbf{d}))qp_1 \int_0^{\widetilde{T}} a_0(\sigma)d\sigma\Big)\Big)^{1/q} \\
&\quad \le \rho_0^{-1/q} \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/q'} \frac{1}{[qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/q}}. \qquad (2.74)
\end{aligned}$$

Using the inequality (2.74) in (2.73), we then deduce

$$\begin{aligned}
\|W_1(\overline{w}, \overline{z}, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} &\le \rho_0^{-1/(q'p_1)} \|\Phi\|_{X^*} \|Au_0\|_X \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/(q'p_1)} \\
\times \frac{1}{[qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/(qp_1)}} &\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right). \qquad (2.75)
\end{aligned}$$

In order to assess the second summand W_2 in W, we use estimates (2.67) and (2.74). Proceeding analogously to the estimate of W_1, we get:

$$\begin{aligned}
&\|W_2(\overline{w},\overline{z}, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \\
&\quad = |f(0)| \|Z(\overline{w}, \overline{z}, \mathbf{d}, \widetilde{T}\|_X \left(\int_0^{\widetilde{T}} \left| \exp\left(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\right)\right.\right. \\
&\quad \times \left.\left. \exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)\right|^{p_1} dt\right)^{1/p_1} \le \rho_0^{1/(qp_1)} |f(0)| \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/(q'p_1)} \\
&\quad \times \frac{C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})}{[qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/(qp_1)}} \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right). \qquad (2.76)
\end{aligned}$$

An immediate consequence of the estimate of $\widetilde{R}_2$ is the following:

$$\begin{aligned}
&\|W_3(\overline{w},\overline{z},\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\\
&\quad=\bigg(\int_0^{\widetilde{T}}\bigg|\int_0^t\Big(f'(s)-f(s)\Big(\overline{k}(s)-\frac{a_0(s)}{k(s)}\,\overline{w}(s)-\frac{a_0(s)f(s)}{k(s)}\overline{z}\Big)\Big)\\
&\quad\times\exp\bigg(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\bigg)\\
&\quad\times\exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)ds\,Z(\overline{w},\overline{z},\mathbf{d},\widetilde{T})\bigg|^{p_1}dt\bigg)^{1/p_1}\\
&\quad\le\rho_0^{-1/p_3}\widetilde{T}^{1/p_1}\frac{C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\;\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\bigg)\\
&\quad\times\bigg[\bigg(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\bigg)\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\bigg(\|\Phi\|_{X^*}\Big\|\frac{fg}{k}\Big\|_{L^{p_1}(0,\widetilde{T};X)}\\
&\quad+\Big\|\frac{f}{k}\Big\|_{L^{\infty}(0,\widetilde{T};\mathbb{R})}M_1+\Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\bigg)\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\bigg].
\end{aligned}\tag{2.77}$$

Applying the inequality (2.66), from (2.24) we obtain

$$\begin{aligned}
&\|W_4(\overline{w},\overline{z},\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\\
&\quad\le\bigg[\int_0^{\widetilde{T}}\bigg|\int_0^t a_0(s)\exp\bigg(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}\Big)d\sigma\bigg)\\
&\quad\times A\exp\bigg(\int_s^t a_0(\sigma)d\sigma A\bigg)g(s)ds\bigg|^{p_1}dt\bigg]^{1/p_1}\le\rho_0^{-1/p_3}\widetilde{T}^{1/p_1}\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\\
&\quad\times\frac{C_{10}(\gamma,T)\|a_0^{1/p_1}g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\bigg).
\end{aligned}\tag{2.78}$$

Finally, summing up the estimates (2.75), (2.76), (2.77), (2.78), we deduce

$$\begin{aligned}
\|W(\overline{w},\overline{z},\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\le\bigg\{&\rho_0^{-1/(qp_1)}\bigg(\int_0^{\widetilde{T}}\frac{dt}{a_0^{q'/q}(t)}\bigg)^{1/(q'p_1)}\frac{\|Au_0\|_X}{[qp_1(1-K(\mathbf{d})/\rho_0)]^{1/(qp_1)}}\\
&+\rho_0^{-1/(qp_1)}C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\bigg(\int_0^{\widetilde{T}}\frac{dt}{a_0^{q'/q}(t)}\bigg)^{1/(q'p_1)}\frac{|f(0)|}{[qp_1(1-K(\mathbf{d})/\rho_0)]^{1/(qp_1)}}
\end{aligned}$$

$$+ \rho_0^{-1/p_3} \frac{\widetilde{T}^{1/p_1}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\, C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})$$

$$\times \Bigg[\left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\| \frac{fk'}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \right) + \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$

$$\times \left(\|\Phi\|_{X^*} \left\| \frac{fg}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\| \frac{f}{k} \right\|_{L^{\infty}(0,\widetilde{T};\mathbb{R})} M_1 + \left\| \frac{f^2}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2 \right) \Bigg]$$

$$+ \rho_0^{-1/p_3} \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{\widetilde{T}^{1/p_1} C_{10}(\gamma, T)}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\, \|a_0^{1/p_1} g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))} \Bigg\}$$

$$\times \exp\left(\left\| \frac{k'}{k} \right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} \right) =: C_{12}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}). \blacksquare \qquad (2.79)$$

Conclusions

Thus we have obtained all estimates needed to prove that $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$ maps the metric space $\mathcal{K}(M_1, M_2, T)$ into itself. Taking into account (2.67) and (2.72) we can conclude that operators $\mathcal{N}_1$ and $\mathcal{N}_2$ can be estimated (cf.(2.59), (2.60)) in the following way:

$$\|\mathcal{N}_1(\overline{w}, \overline{z}, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \le \|\Phi\|_{X^*} C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})$$
$$+ \|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \|\Phi\|_{X^*} C_{12}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}), \qquad (2.80)$$

$$|\mathcal{N}_2(\overline{w}, \overline{z}, \mathbf{d})| \le \|\Phi\|_{X^*} C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}). \qquad (2.81)$$

We now remind that $C_5 \to \dfrac{2\|Ah\|_X \|\Phi\|_{X^*}}{|\lambda(\mathbf{d})|}$ and $C_{12} \to 0$ as ρ_0 tend to $+\infty$. Then, evaluating the right-hand side in (2.80), (2.81) as ρ_0 tend to $+\infty$, we obtain that the pair (M_1, M_2) must satisfy the inequalities:

$$\begin{cases} \dfrac{2\|Ah\|_X \|\Phi\|_{X^*}}{|\lambda(\mathbf{d})|} \le M_2, \\ \dfrac{2\|Ah\|_X \|\Phi\|_{X^*}}{|\lambda(\mathbf{d})|} \|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \le M_1. \end{cases}$$

Summing up: in order that the vector-mapping $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$ in (P9) maps the set $\mathcal{K}(M_1, M_2, T)$ (cf. (2.27)) into itself, we must choose the numbers $M_1 = M_1(\mathbf{d}, \Phi)$ and

$M_2 = M_2(\mathbf{d}, \Phi)$ so as to satisfy the stricter system of inequalities (1.14). Taking this choice into account, we can find so a large ρ_0 to satisfy the system (2.60), (2.59).

2.5 Contractivity of mapping $\mathcal{N}$

To apply the Banach theorem on contracting mappings in the complete metric space $\mathcal{K}(M_1, M_2, T)$ we need to prove that the mapping $\mathcal{N}$ into itself is contracting, that is the inequalities

$$\begin{aligned}
&\|\mathcal{N}_1(\overline{w}_2, \overline{z}_2,\mathbf{d}) - \mathcal{N}_1(\overline{w}_1, \overline{z}_1, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \\
&\qquad \leq \|\Phi\|_{X^*} \sum_{j=1}^{4} \|W_j(\overline{w}_2, \overline{z}_2, \mathbf{d}) - W_j(\overline{w}_1, \overline{z}_1, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \\
&\qquad + \|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \|\Phi\|_{X^*} \|Z(\overline{w}_2, \overline{z}_2, \mathbf{d}, \widetilde{T}) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, \widetilde{T})\|_X \\
&\qquad \leq \mathbf{q}_1(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \qquad (2.82)
\end{aligned}$$

$$\begin{aligned}
&|\mathcal{N}_2(\overline{w}_2, \overline{z}_2,\mathbf{d}) - \mathcal{N}_2(\overline{w}_1, \overline{z}_1, \mathbf{d})| \leq \|\Phi\|_{X^*} \|Z(\overline{w}_2, \overline{z}_2, \mathbf{d}, \widetilde{T}) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, \widetilde{T})\|_X \\
&\qquad \leq \mathbf{q}_2(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|) \qquad (2.83)
\end{aligned}$$

hold for all $(\overline{w}_i, \overline{z}_i) \in \mathcal{K}(M_1, M_2, T), i = 1, 2,$ for some positive constants $\mathbf{q}_1 < 1, \mathbf{q}_2 < 1$. In order to get the estimate of the increments of Z in $(\overline{w}, \overline{z})$ we need to find the appropriate estimate of the increments for Q^{-1}. To this aim we need an additional lemma.

Lemma 2.2 *For all $(\overline{w}_i, \overline{z}_i, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2,$ and $\overline{k}$ defined by (2.5) the following estimate holds:*

$$\begin{aligned}
&\Bigg\| \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1 \Big) d\sigma \Big) \\
&- \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2 \Big) d\sigma \Big) \Bigg\|_X \leq \max \Big(\Big\| \frac{a_0}{k} \Big\|_{L^{p_1'}(s,t;\mathbb{R})}, \Big\| \frac{a_0 f}{k} \Big\|_{L^1(s,t;\mathbb{R})} \Big)
\end{aligned}$$

$$\times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(s,t;\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(s,t;\mathbb{R})}\Big) \exp\Big(K(\boldsymbol{d}) \int_s^t a_0(\sigma) d\sigma\Big)$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(s,t;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad \forall s \in [0,t],\ \forall t \in [0,T]. \tag{2.84}$$

Proof . In order to deal with the difference

$$I_2(s,t) := \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big) d\sigma\Big)$$
$$- \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2\Big) d\sigma\Big)$$

we need the following relation:

$$e^x - e^y = -\int_0^1 \frac{\partial}{\partial \theta_1} [\exp\Big((1-\theta_1)x + \theta_1 y\Big)] d\theta_1 = -(y-x) \int_0^1 \exp[(1-\theta_1)x + \theta_1 y] d\theta_1. \tag{2.85}$$

Taking this equality into account, we achieve

$$I_2(s,t) = -\int_s^t \Big(\frac{a_0(\sigma)}{k(\sigma)}\,(\overline{w}_2(\sigma) - \overline{w}_1(\sigma)) + \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} (\overline{z}_2 - \overline{z}_1)\Big) d\sigma$$
$$\times \int_0^1 \exp\Big((1-\theta_1) \int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2\Big) d\sigma\Big)$$
$$\times \exp\Bigg(\theta_1 \int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big) d\sigma\Bigg) d\theta_1,\ s \in [0,t).$$

Using Lemma 2.2, we get

$$\big\|I_2(s,t)\big\|_X \le \int_s^t \Big|\frac{a_0(\sigma)}{k(\sigma)}\,(\overline{w}_2(\sigma) - \overline{w}_1(\sigma)) + \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} (\overline{z}_2 - \overline{z}_1)\Big| d\sigma$$
$$\times \int_0^1 \exp\Big((1-\theta_1) \int_s^t \Big|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2\Big| d\sigma\Big)$$
$$\times \exp\Bigg(\theta_1 \int_s^t \Big|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big| d\sigma\Bigg) d\theta_1$$
$$\le \int_s^t \Big|\frac{a_0(z)}{k(z)}\,(\overline{w}_2(z) - \overline{w}_1(z)) + \frac{a_0(z) f(z)}{k(z)} (\overline{z}_2 - \overline{z}_1)\Big| dz$$

$$\times \int_0^1 \exp\Big(K(\mathbf{d}) \int_s^t a_0(\sigma) d\sigma\Big) \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(s,t;\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(s,t;\mathbb{R})}\Big) d\theta_1.$$

Applying Hölder's inequality, we easily obtain

$$\begin{aligned}
\big\|I_2(s,t)\big\|_X &\le \Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(s,t;\mathbb{R})} \|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(s,t;\mathbb{R})} + \Big\|\frac{a_0 f}{k}\Big\|_{L^1(s,t;\mathbb{R})} |\overline{z}_2 - \overline{z}_1|\Big) \\
&\quad \times \exp\Big(K(\mathbf{d}) \int_s^t a_0(\sigma) d\sigma\Big) \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(0,T;\mathbb{R})}\Big) \\
&\le \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(s,t;\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(s,t;\mathbb{R})}\Big) \exp\Big(K(\mathbf{d}) \int_s^t a_0(\sigma) d\sigma\Big) \\
&\quad \times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(s,t;\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(s,t;\mathbb{R})}\Big) \\
&\quad \times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(s,t;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad \forall s \in [0,t). \qquad \blacksquare
\end{aligned}$$

2.5.1 Estimate of $Q^{-1}(\overline{w}_2, \overline{z}_2, \mathbf{d}) - Q^{-1}(\overline{w}_1, \overline{z}_1, \mathbf{d})$

Let $(\overline{w}_1, \overline{z}_1), (\overline{w}_2, \overline{z}_2) \in \mathcal{K}(M_1, M_2, T)$. Taking into account (2.5), the definitions (2.44) and (2.47), after some simple computations we obtain

$$\begin{aligned}
Q(\overline{w}_2,&\overline{z}_2, \mathbf{d}, T) - Q(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) = \widetilde{R}(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) - \widetilde{R}(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) \\
&= f(0) \int_0^T \varphi(t) \Big[\exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big) d\sigma\Big) \\
&- \exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2\Big) d\sigma\Big)\Big] \exp\Big(\int_0^t a_0(\sigma) d\sigma A\Big) d\mu(t) \\
&+ \int_0^T \varphi(t) d\mu(t) \\
&\times \int_0^t \Big(f'(s) - \frac{f(s) k'(s)}{k(s)} + a_0(s) \Big(\frac{f(s)\Phi[g(s)]}{k(s)} + \frac{f(s)}{k(s)} \overline{w}_1(s) + \frac{f^2(s)}{k(s)} \overline{z}_1\Big)\Big) \\
&\times \Big[\exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big) d\sigma\Big) \\
&- \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2\Big) d\sigma\Big)\Big] \exp\Big(\int_s^t a_0(\sigma) d\sigma A\Big) ds
\end{aligned}$$

$$+\int_0^T \varphi(t)d\mu(t)\int_0^t a_0(s)\left[\frac{f(s)}{k(s)}(\overline{w}_1(s)-\overline{w}_2(s))+\frac{f^2(s)}{k(s)}(\overline{z}_1-\overline{z}_2)\right]$$
$$\times\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds$$
$$=:\sum_{i=1}^{3}R_i(\overline{z},\overline{w},\mathbf{d},T). \tag{2.86}$$

Lemma 2.1 *Under the conditions of Theorem 1.1 the following estimate holds for any* $(\overline{w}_i,\overline{z}_i,\boldsymbol{d})\in\mathcal{K}(M_1,M_2,T)\times\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}}),\ i=1,2:$

$$\|Q^{-1}(\overline{w}_2,\overline{z}_2,\boldsymbol{d},\widetilde{T})-Q^{-1}(\overline{w}_1,\overline{z}_1,\boldsymbol{d},\widetilde{T})\|_{\mathcal{L}(X)}$$
$$\leq C_{13}(\rho_0,\boldsymbol{d},M_1,M_2,\widetilde{T})(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|), \tag{2.87}$$

where

$$C_{13}(\rho_0,\boldsymbol{d},M_1,M_2,\widetilde{T}):=\frac{4}{\lambda^2(\boldsymbol{d})}\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times\Bigg\{\tau_1(\rho_0,\boldsymbol{d},\widetilde{T})|f(0)|\max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\left\|\frac{a_0f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)$$
$$+\rho_0^{-1/p_3}\tau_2(\widetilde{T})\ \frac{\max\Big(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\left\|\frac{a_0f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)}{[p_3(1-K(\boldsymbol{d})/\rho_0)]^{1/p_3}}$$
$$\times\Bigg[\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big)$$
$$+\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)}+\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1$$
$$+\left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\Big)\Bigg]+\rho_0^{-1/p_3}\tau_2(\widetilde{T})\ \frac{\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}}{[p_3(1-K(\boldsymbol{d})/\rho_0)]^{1/p_3}}$$
$$\times\max\Big(\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})},\left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big)\Bigg\}. \tag{2.88}$$

Remark 2.1 *We note that the positive function* C_{13} *tends to 0 as* $\rho_0\to+\infty$.

Proof . The proof of this lemma follows immediately from the relation

$$Q^{-1}(\overline{w}_2,\overline{z}_2,\mathbf{d},T)-Q^{-1}(\overline{w}_1,\overline{z}_1,\mathbf{d},T)$$

$$=Q^{-1}(\overline{w}_1,\overline{z}_1,\mathbf{d},T)\cdot\sum_{i=1}^{3}R_i(\overline{z},\overline{w},\mathbf{d},T)\cdot Q^{-1}(\overline{w}_2,\overline{z}_2,\mathbf{d},T) \tag{2.89}$$

as soon as we have an estimate of each $R_i, i=1,2,3$. Let us prove the inequality (2.87) in the Case 2.

First, using (2.34), (2.84), (2.51), we get

$$\begin{aligned}
&\|R_1(\overline{z},\overline{w},\mathbf{d},T)\|_{\mathcal{L}(X)}\\
&\quad\le |f(0)|\int_0^T|\varphi(t)|\psi(t)\Big\|\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\\
&\quad-\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\Big\|_X\\
&\quad\times\Big\|\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)\Big\|_{\mathcal{L}(X)}dt\le\rho_0^{-1/p_3}|f(0)|\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,T;\mathbb{R})}\\
&\quad\times\frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\;\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})},\Big\|\frac{a_0f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)\\
&\quad\times\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})}+|\overline{z}_2-\overline{z}_1|).
\end{aligned} \tag{2.90}$$

Applying the estimates (2.34), (2.84), (2.58), we deduce

$$\begin{aligned}
&\|R_2(\overline{z},\overline{w},\mathbf{d},T)\|_{\mathcal{L}(X)}\le\int_0^T|\varphi(t)|\psi(t)dt\\
&\quad\times\int_0^t\Big|f'(s)-\frac{f(s)k'(s)}{k(s)}+a_0(s)\Big(\frac{f(s)\Phi[g(s)]}{k(s)}+\frac{f(s)}{k(s)}\,\overline{w}_1(s)+\frac{f^2(s)}{k(s)}\overline{z}_1\Big)\Big|\\
&\quad\times\Big\|\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\\
&\quad-\exp\Big(\int_s^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\Big\|_X\\
&\quad\times\Big\|\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\Big\|_{\mathcal{L}(X)}ds\le\rho_0^{-1/p_3}\;\frac{T^{1/p_1'}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\\
&\quad\times\Big[\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,T;\mathbb{R})}\Big(\|f'\|_{L^{p_1}(0,T;\mathbb{R})}+\Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,T;\mathbb{R})}\Big)+\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,T;\mathbb{R})}
\end{aligned}$$

$$\times \Big(\|\Phi\|_{X^*} \left\|\frac{fg}{k}\right\|_{L^{p_1}(0,T;X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,T;\mathbb{R})} M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,T;\mathbb{R})} M_2\Big)\Big]$$
$$\times \max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,T;\mathbb{R})}\right)$$
$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.91}$$

Taking into account (2.34), (2.56), we obtain

$$\|R_3(\overline{z},\overline{w},\mathbf{d},T)\|_{\mathcal{L}(X)}$$
$$\leq \int_0^T |\varphi(t)|\psi(t)dt \int_0^t a_0(s) \left|\frac{f(s)}{k(s)} (\overline{w}_1(s) - \overline{w}_2(s)) + \frac{f^2(s)}{k(s)} (\overline{z}_1 - \overline{z}_2)\right|$$
$$\times \exp\left(-(\rho_0 - K(\mathbf{d})) \int_0^t a_0(\sigma)d\sigma\right) ds \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)$$
$$\leq \rho_0^{-1/p_3} \frac{T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})} \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,T;\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}}$$
$$\times \left(\left\|\frac{f}{k}\right\|_{L^\infty(0,T_1;\mathbb{R})} \|\overline{w}_1 - \overline{w}_2\|_{L^{p_1}(0,T_1;\mathbb{R})} + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,T_1;\mathbb{R})} |\overline{z}_1 - \overline{z}_2|\right)$$
$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right). \tag{2.92}$$

Applying (2.90), (2.91), (2.92) to (2.86), in view of (2.89) and taking into account (2.48), we get (2.88).

We prove (2.88) in the Case 1 analogously to the one in the Case 2. First, using (2.34), (2.84), we get the estimate of R_1; second, taking into account (2.34), (2.84) and (2.54) we deduce the estimate for R_2; third, using (2.33), (2.40), we obtain the estimate of R_3. ■

2.5.2 Estimate of $Z(\overline{w}_2, \overline{z}_2, \mathbf{d}) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d})$

Let us take into account the identities (cf. (2.16))

$$Z(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) = \sum_{j=1}^{3} (\mathcal{Z}_j(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - \mathcal{Z}_j(\overline{w}_1, \overline{z}_1, \mathbf{d}, T))$$

and

$$\begin{aligned} &Q(\overline{w}_2, \overline{z}_2, \mathbf{d})^{-1} a - Q(\overline{w}_1, \overline{z}_1, \mathbf{d})^{-1} b = Q(\overline{w}_2, \overline{z}_2, \mathbf{d})^{-1}(a-b) \\ &+ (Q(\overline{w}_2, \overline{z}_2, \mathbf{d})^{-1} - Q(\overline{w}_1, \overline{z}_1, \mathbf{d})^{-1}) b, \quad a, b \in X. \end{aligned}$$

Then, according to definition (2.18)–(2.20), the next equalities hold:

$$\mathcal{Z}_1(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - \mathcal{Z}_1(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) = (Q(\overline{w}_2, \overline{z}_2, \mathbf{d}, T)^{-1} - Q(\overline{w}_1, \overline{z}_1, \mathbf{d}, T)^{-1}) Ah,$$

$$\begin{aligned} &\mathcal{Z}_2(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - \mathcal{Z}_2(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) = \\ &\quad = Q(\overline{w}_2, \overline{z}_2, \mathbf{d}, T)^{-1} \\ &\quad \times \int_0^T \varphi(t) \Bigg(\exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1 \Big) d\sigma \Bigg) \\ &\quad - \exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \overline{w}_2(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_2 \Big) d\sigma \Bigg) \Bigg) \\ &\quad \times \exp\Bigg(\int_0^t a_0(\sigma) d\sigma A \Bigg) A u_0 d\mu(t) + (Q(\overline{w}_1, \overline{z}_1, \mathbf{d}, T)^{-1} - Q(\overline{w}_2, \overline{z}_2, \mathbf{d}, T)^{-1}) \\ &\quad \times \int_0^T \varphi(t) \exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1 \Big) d\sigma \Bigg) \\ &\quad \times \exp\Bigg(\int_0^t a_0(\sigma) d\sigma A \Bigg) A u_0 d\mu(t) \\ &\quad =: \mathcal{Z}_{2,1}(\overline{w}_2, \overline{w}_1, \overline{z}_2, \overline{z}_1, \mathbf{d}, T) + \mathcal{Z}_{2,2}(\overline{w}_2, \overline{w}_1, \overline{z}_2, \overline{z}_1, \mathbf{d}, T), \end{aligned} \tag{2.93}$$

$$\begin{aligned} &\mathcal{Z}_3(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - \mathcal{Z}_3(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) = Q(\overline{w}_2, \overline{z}_2, \mathbf{d}, T)^{-1} \\ &\quad \times \int_0^T \varphi(t) \Bigg[\int_0^t a_0(s) \Bigg(\exp\Bigg(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)} \, \overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1 \Big) d\sigma \Bigg) \end{aligned}$$

$$
\begin{aligned}
&- \exp\biggl(\int_s^t \Bigl(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Bigr) d\sigma\biggr)\biggr)\\
&\times A\exp\biggl(\int_s^t a_0(\sigma)d\sigma A\biggr) g(s)ds\biggr] d\mu(t)\\
&+ (Q(\overline{w}_1, \overline{z}_1, \mathbf{d}, T)^{-1} - Q(\overline{w}_2, \overline{z}_2, \mathbf{d}, T)^{-1})\\
&\times \int_0^T \varphi(t)\biggl(\int_0^t a_0(s)\exp\biggl(\int_s^t \Bigl(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Bigr) d\sigma\biggr)\\
&\times A\exp\biggl(\int_s^t a_0(\sigma)d\sigma A\biggr) g(s)ds\biggr) d\mu(t)\\
&=: \mathcal{Z}_{3,1}(\overline{w}_2, \overline{w}_1, \overline{z}_2, \overline{z}_1, \mathbf{d}, T) + \mathcal{Z}_{3,2}(\overline{w}_2, \overline{w}_1, \overline{z}_2, \overline{z}_1, \mathbf{d}, T). \qquad (2.94)
\end{aligned}
$$

Lemma 2.1 *In view of the conditions of Theorem 1.1, the following estimate holds for any* $(\overline{w}_i, \overline{z}_i, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2:$

$$
\begin{aligned}
&\|Z(\overline{w}_2, \overline{z}_2, \boldsymbol{d}, \widetilde{T}) - Z(\overline{w}_1, \overline{z}_1, \boldsymbol{d}, \widetilde{T})\|_X\\
&\qquad \le C_{14}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T})(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \qquad (2.95)
\end{aligned}
$$

Here

$$
\begin{aligned}
C_{14}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}) &= \|Ah\|_X C_{13}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}) + \biggl\{\|Au_0\|_X \tau_1(\rho_0, \boldsymbol{d}, \widetilde{T})\\
&+ \rho_0^{-1/p_3}\tau_2(\widetilde{T})\ \frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\ C_{10}(\gamma, T)\ \frac{\|a_0^{1/p_1} g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}\biggr\}\\
&\times \biggl\{\frac{2}{|\lambda(\boldsymbol{d})|}\max\biggl(\Bigl\|\frac{a_0}{k}\Bigr\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Bigl\|\frac{a_0 f}{k}\Bigr\|_{L^1(0,\widetilde{T};\mathbb{R})}\biggr) + C_{13}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T})\biggr\}\\
&\times \exp\biggl(\Bigl\|\frac{k'}{k}\Bigr\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\Bigl\|\frac{a_0}{k}\Bigr\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\biggr) \qquad (2.96)
\end{aligned}
$$

and $\widetilde{T}, C_{10}, C_{13}, \tau_1, \tau_2$ *defined in (1.13), (2.63), (2.88), (1.19), (1.20).*

Remark 2.1 Taking into account Remark 2.1 and the definition (1.19) of τ_1, we conclude that

$$C_{14} \to 0 \quad \text{as} \quad \rho_0 \to +\infty.$$

Proof . We prove the estimate (2.95) for the Case 2, i.e. when $d\mu(t) = \psi(t)dt$. Using (2.87) we achieve

$$\|\mathcal{Z}_1(\overline{w}_2,\overline{z}_2,\mathbf{d},T) - \mathcal{Z}_1(\overline{w}_1,\overline{z}_1,\mathbf{d},T)\|_X \leq \|Q(\overline{w}_2,\overline{z}_2,\mathbf{d},T)^{-1} - Q(\overline{w}_1,\overline{z}_1,\mathbf{d},T)^{-1}\|_{\mathcal{L}(X)}$$
$$\times \|Ah\|_X \leq \ C_{13}(\rho_0,\mathbf{d},M_1,M_2,T)\|Ah\|_X(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.97}$$

Now, by means of the estimates (2.84), (2.48), from (2.93) we have

$$\|\mathcal{Z}_{2,1}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X \leq \|Q(\overline{w}_2,\overline{z}_2,\mathbf{d},T)^{-1}\|_{\mathcal{L}(X)}$$
$$\times \int_0^T |\varphi(t)|\psi(t)\Big\|\exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)$$
$$- \exp\Big(\int_0^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\Big\|_{\mathcal{L}(X)}$$
$$\times \Big\|\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0\Big\|_X dt$$
$$\leq \frac{2\|Au_0\|_X \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)}{\big|\int_0^T \varphi(t)f(t)\psi(t)dt\big|}$$
$$\times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|)$$
$$\times \int_0^T |\varphi(t)|\psi(t)\exp\Big(-(\rho_0 - K(\mathbf{d}))\int_0^t a_0(\sigma)d\sigma\Big)dt.$$

Then, from (2.51) we get

$$\|\mathcal{Z}_{2,1}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X \leq \rho_0^{1/p_3}\frac{2\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)}{\big|\int_0^T \varphi(t)f(t)\psi(t)dt\big|}\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,T;\mathbb{R})}$$
$$\times \frac{\|Au_0\|_X\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}}\ \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.98}$$

Exploiting the estimates (2.88) and (2.51), we have

$$\begin{aligned}
&\|\mathcal{Z}_{2,2}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X \le \|Q(\overline{w}_1,\overline{z}_1,\mathbf{d},T)^{-1} - Q(\overline{w}_2,\overline{z}_2,\mathbf{d},T)^{-1}\|_{\mathcal{L}(X)} \\
&\qquad \times \int_0^T |\varphi(t)|\psi(t) \exp\Big(\int_0^t \Big|\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big| d\sigma\Big) \\
&\qquad \times \Big\|\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0\Big\|_X dt \le \rho_0^{-1/p_3}\|Au_0\|_X C_{13}(\rho_0,\mathbf{d},M_1,M_2,T) \\
&\qquad \times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,T;\mathbb{R})} \\
&\qquad \times \frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\,(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \qquad (2.99)
\end{aligned}$$

Summing up (2.98), (2.99), we get

$$\begin{aligned}
&\|\mathcal{Z}_2(\overline{w}_2,\overline{z}_2,\mathbf{d},T) - \mathcal{Z}_2(\overline{w}_1,\overline{z}_1,\mathbf{d},T)\|_X \\
&\qquad \le \rho_0^{1/p_3}\left\{\frac{2\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)}{\big|\int_0^T \varphi(t)f(t)\psi(t)dt\big|} + C_{13}(\rho_0,\mathbf{d},M_1,M_2,T)\right\} \\
&\qquad \times \|Au_0\|_X \Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,T;\mathbb{R})} \frac{\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}} \\
&\qquad \times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).
\end{aligned} \tag{2.100}$$

From (2.94), using (2.48), (2.84), we deduce the next estimate:

$$\begin{aligned}
&\|\mathcal{Z}_{3,1}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X \le \|Q(\overline{w}_2,\overline{z}_2,\mathbf{d},T)^{-1}\|_{\mathcal{L}(X)} \int_0^T |\varphi(t)|\psi(t)dt \\
&\qquad \times \int_0^t a_0(s)\Big\|\exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big) \\
&\qquad - \exp\Big(\int_s^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\Big\|_X \\
&\qquad \times \Big\|A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)\Big\|_X ds
\end{aligned}$$

$$\leq \frac{2\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)}{\big|\int_0^T \varphi(t)f(t)\psi(t)dt\big|}\,(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})}+|\overline{z}_2-\overline{z}_1|)$$
$$\times\int_0^T|\varphi(t)|\psi(t)dt\int_0^t a_0(s)\exp\Big(K(\mathbf{d})\int_s^t a_0(\sigma)d\sigma\Big)$$
$$\times\Big\|A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)\Big)\Big\|_X\,ds\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big).$$

Owing to (2.66), we get

$$\|\mathcal{Z}_{3,1}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X\leq\frac{2\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,T;\mathbb{R})}\Big)}{\big|\int_0^T \varphi(t)f(t)\psi(t)dt\big|}$$
$$\times\rho_0^{-1/p_3}T^{1/p_1'}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$
$$\times\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)$$
$$\times(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})}+|\overline{z}_2-\overline{z}_1|).\tag{2.101}$$

In order to obtain an estimate of $\mathcal{Z}_{3,2}$, we use (2.87), (2.66). Thus, from (2.94) we have

$$\|\mathcal{Z}_{3,2}(\overline{w}_2,\overline{w}_1,\overline{z}_2,\overline{z}_1,\mathbf{d},T)\|_X\leq\|Q(\overline{w}_1,\overline{z}_1,\mathbf{d},T)^{-1}-Q(\overline{w}_2,\overline{z}_2,\mathbf{d},T)^{-1}\|_{\mathcal{L}(X)}$$
$$\times\int_0^T|\varphi(t)|\psi(t)dt\int_0^t a_0(s)\exp\Big(\int_s^t\Big|\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big|d\sigma$$
$$\times\|A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)\|_X ds$$
$$\leq\rho_0^{-1/p_3}C_{13}(\rho_0,\mathbf{d},M_1,M_2,T)T^{1/p_1'}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$
$$\times\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\;\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)$$
$$\times(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})}+|\overline{z}_2-\overline{z}_1|).\tag{2.102}$$

Now from (2.101), (2.102) we deduce the estimate

$$\|\mathcal{Z}_3(\overline{w}_2,\overline{z}_2,\mathbf{d},T)-\mathcal{Z}_3(\overline{w}_1,\overline{z}_1,\mathbf{d},T)\|_X$$

$$\leq \left\{ \frac{2\max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,T;\mathbb{R})}\right)}{\left|\int_0^T \varphi(t)f(t)\psi(t)dt\right|} + C_{13}(\rho_0, \mathbf{d}, M_1, M_2, T) \right\}$$

$$\times \rho_0^{-1/p_3} T^{1/p_1'} \|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})} C_{10}(\gamma, T) \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$

$$\times \frac{\|a_0^{1/p_1} g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}} \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \qquad (2.103)$$

Finally, summing up (2.97), (2.100), (2.103), we conclude (2.95).

In the analogous way we reason for the Case 1. Using (2.87), we easily estimate the increments of $\mathcal{Z}_1$ in $(\overline{w}, \overline{z})$. In order to estimate $\mathcal{Z}_{2,1}$ we take into account (2.48), (2.84). Using (2.87), Lemma 2.2, we get the estimate of $\mathcal{Z}_{2,2}$. We exploit the relations (2.48), (2.84), (2.62) to estimate $\mathcal{Z}_{3,1}$. For the estimate of $\mathcal{Z}_{3,2}$ we use (2.87) and (2.62).■

2.5.3 Estimate of $W(\overline{w}_2, \overline{z}_2, \mathbf{d}) - W(\overline{w}_1, \overline{z}_1, \mathbf{d})$

Lemma 2.1 *Under the conditions of the Theorem 1.1 the following estimates hold for any* $(\overline{w}_i, \overline{z}_i, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2:$

$$\|W(\overline{w}_2, \overline{z}_2, \boldsymbol{d}) - W(\overline{w}_1, \overline{z}_1, \boldsymbol{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$\leq C_{15}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T})(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \qquad (2.104)$$

where

$$C_{15} \to 0 \quad as \quad \rho_0 \to +\infty. \qquad (2.105)$$

Moreover,

$$\|W(\overline{w}_2, \overline{z}_2, \boldsymbol{d}) - W(\overline{w}_1, \overline{z}_1, \boldsymbol{d})\|_{L^{\infty}(0,\widetilde{T};X)}$$
$$\leq C_{16}(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T})(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \qquad (2.106)$$

and $C_{16} \nrightarrow 0$ *as* $\rho_0 \to +\infty$.

Remark 2.1 *We shall need the estimate (2.106) exclusively only in the proof of continuous dependencies.*

Proof . Let us estimate the increments $W_j(\overline{w}, \overline{z}, \mathbf{d}, t)$ in $(\overline{w}, \overline{z})$. We remind that the difference between the estimate of these increments in the two cases of measure μ is only in W_2, W_3. In fact, the representation of last vectors involves Z, which contains the measure μ. Now we consider the first difference. Using (2.34), (2.84), (2.74), similarly to (2.75), we obtain

$$\begin{aligned}
&\|W_1(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_1(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \\
&\quad = \Bigg(\int_0^{\widetilde{T}} \Bigg| \Bigg[\exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big) d\sigma\Bigg) \\
&\quad - \exp\Bigg(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big) d\sigma\Bigg)\Bigg] \\
&\quad \times \exp\Bigg(\int_0^t a_0(\sigma) d\sigma A\Bigg) A u_0 \Bigg|^{p_1} dt\Bigg)^{1/p_1} \\
&\quad \le \rho_0^{-1/(qp_1)} \|Au_0\|_X \frac{\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)}{[qp_1(1-K(\mathbf{d})/\rho_0)]^{1/(qp_1)}} \Bigg(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\Bigg)^{1/(p_1 q')} \\
&\quad \times \exp\Bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Bigg) (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).
\end{aligned} \tag{2.107}$$

Likewise we estimate the same increments in $L^\infty(0,\widetilde{T};X)$. We get

$$\begin{aligned}
&\|W_1(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_1(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^\infty(0,\widetilde{T};X)} \\
&\quad \le \|Au_0\|_X \exp\Bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Bigg) \\
&\quad \times \max\Bigg(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Bigg) (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).
\end{aligned} \tag{2.108}$$

Let us write down the increments of W_2 in a convenient form:

$$\begin{aligned}
W_2(\overline{w}_2,&\overline{z}_2,\mathbf{d},t)-W_2(\overline{w}_1,\overline{z}_1,\mathbf{d},t)\\
&=f(0)\Big[\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\\
&-\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\Big]\\
&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Z(\overline{w}_2,\overline{z}_2,\mathbf{d},T)\\
&+f(0)\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\\
&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)(Z(\overline{w}_2,\overline{z}_2,\mathbf{d},T)-Z(\overline{w}_1,\overline{z}_1,\mathbf{d},T))\\
&=:W_{2,1}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d},t)+W_{2,2}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d},t).
\end{aligned}\tag{2.109}$$

Analogously to (2.107), applying (2.84), (2.67), we obtain

$$\begin{aligned}
\|W_{2,1}(\overline{w}_1,&\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\\
&\le|f(0)|\Big(\int_0^{\widetilde{T}}\Big|\Big[\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big)\\
&-\exp\Big(\int_0^t\Big(\overline{k}(\sigma)-\frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma)-\frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\Big]\\
&\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Z(\overline{w}_2,\overline{z}_2,\mathbf{d},T)\Big|^{p_1}ds\Big)^{1/p_1}\\
&\le\rho_0^{1/(qp_1)}|f(0)|C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)\\
&\times\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|)\\
&\times\Big(\int_0^{\widetilde{T}}\frac{dt}{a_0^{q'/q}(t)}\Big)^{1/(q'p_1)}\frac{1}{[qp_1(1-K(\mathbf{d})/\rho_0)]^{1/(qp_1)}}.
\end{aligned}$$

Likewise we achieve

$$\|W_{2,1}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}$$

$$\leq |f(0)| C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)$$

$$\times \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).$$

Using (2.95), similarly to (2.76) we can deduce

$$\|W_{2,2}(\overline{w}_1,\overline{w}_2, \overline{z}_1, \overline{z}_2, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$

$$\leq |f(0)| \Bigg(\int_0^{\widetilde{T}} \Bigg|\exp\Big(\int_0^t \Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma) f(\sigma)}{k(\sigma)} \overline{z}_1\Big) d\sigma\Big)$$

$$\times \exp\Big(\int_0^t a_0(\sigma) d\sigma A\Big)(Z(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, T))\Bigg|^{p_1} ds\Bigg)^{1/p_1}$$

$$\leq \rho_0^{-1/(qp_1)} |f(0)| \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)$$

$$\times C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|)$$

$$\times \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/(q'p_1)} \frac{1}{[qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/(qp_1)}}. \tag{2.110}$$

Likewise we get

$$\|W_{2,2}(\overline{w}_1,\overline{w}_2, \overline{z}_1, \overline{z}_2, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$

$$\leq |f(0)| \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)$$

$$\times C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.111}$$

As a result, taking into account Minkovskiy's inequality and the estimates (2.110), (2.111), from (2.109) we get the final estimate

$$\|W_2(\overline{w}_2,\overline{z}_2, \mathbf{d}) - W_2(\overline{w}_1, \overline{z}_1, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$

$$\leq \rho_0^{-1/(qp_1)} |f(0)| \left(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\right)^{1/(q'p_1)} \frac{1}{[qp_1(1 - K(\mathbf{d})/\rho_0)]^{1/(qp_1)}}$$

$$\times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)\Big[C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})$$

$$\times \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big) + C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})\Big]$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.112}$$

Likewise we get

$$\begin{aligned}
&\|W_2(\overline{w}_2,\overline{z}_2, \mathbf{d}) - W_2(\overline{w}_1, \overline{z}_1, \mathbf{d})\|_{L^\infty(0,\widetilde{T};X)} \\
&\quad \le \Big(C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big) \\
&\quad + C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})\Big)|f(0)| \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big) \\
&\quad \times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).
\end{aligned} \tag{2.113}$$

Let us now estimate the increments for W_3. We remind that

$$\begin{aligned}
&W_3(\overline{w}_2,\overline{z}_2, \mathbf{d}, t) - W_3(\overline{w}_1, \overline{z}_1, \mathbf{d}, t) \\
&\quad = \int_0^t \Big(f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(s)}\,\overline{w}_2(s) - \frac{a_0(s)f(s)}{k(s)}\overline{z}_2\Big)\Big) \\
&\quad \times \exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big) \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds \\
&\quad \times (Z(\overline{w}_2, \overline{z}_2, \mathbf{d}, T) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, T)) \\
&\quad + \int_0^t \Big(f'(s) - f(s)\Big(\overline{k}(s) - \frac{a_0(s)}{k(s)}\,\overline{w}_2(s) - \frac{a_0(s)f(s)}{k(s)}\overline{z}_2\Big)\Big) \\
&\quad \times \Big(\exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\Big)d\sigma\Big) \\
&\quad - \exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)\Big) \\
&\quad \times \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds\ Z(\overline{w}_1, \overline{z}_1, \mathbf{d}, T) \\
&\quad + \int_0^t a_0(s)\Big(\frac{f(s)}{k(s)}\,(\overline{w}_2(s) - \overline{w}_1(\sigma)) + \frac{f^2(s)}{k(s)}(\overline{z}_2 - \overline{z}_1)\Big) \\
&\quad \times \exp\Big(\int_s^t\Big(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\Big)d\sigma\Big)
\end{aligned}$$

$$\times \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds\ Z(\overline{w}_1,\overline{z}_1,\mathbf{d},T) =: \sum_{j=1}^{3} W_{3,j}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d},t).$$

Using (2.54) and (2.95), similarly to (2.77), the next estimate is satisfied:

$$\begin{aligned}
&\|W_{3,1}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \\
&\quad\le \rho_0^{-1/p_3}\widetilde{T}C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}} \\
&\quad\times \exp\Bigg(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Bigg)(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|) \\
&\quad\times \Bigg[\Bigg(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Bigg)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \\
&\quad\times \Bigg(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)}+\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1+\left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\Bigg)\Bigg].
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\|W_{3,1}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d})\|_{L^{\infty}(0,\widetilde{T};X)} \\
&\quad\le \rho_0^{-1/p_3}\frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\ C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) \\
&\quad\times \Bigg[\Bigg(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Bigg)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \\
&\quad\times \Bigg(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)}+\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1+\left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\Bigg)\Bigg] \\
&\quad\times \exp\Bigg(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Bigg)(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|).
\end{aligned}$$

In order to estimate $W_{3,2}$ we use (2.91). So, we achieve

$$\begin{aligned}
&\|W_{3,2}(\overline{w}_1,\overline{w}_2,\overline{z}_1,\overline{z}_2,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \\
&\quad\le \rho_0^{-1/p_3}\ \frac{\widetilde{T}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\ C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) \\
&\quad\times \Bigg[\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Bigg(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Bigg)+\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}
\end{aligned}$$

$$\times \Big(\|\Phi\|_{X^*} \left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\Big)\Big]$$
$$\times \max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).$$

In the similar way we obtain

$$\|W_{3,2}(\overline{w}_1, \overline{w}_2, \overline{z}_1, \overline{z}_2, \mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}$$
$$\leq \rho_0^{-1/p_3} \frac{C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}} \max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times \left[\left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\right.$$
$$\times \Big(\|\Phi\|_{X^*} \left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\Big)\Big]$$
$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|).$$

To assess $W_{3,3}$ we proceed like in (2.92) and we get:

$$\|W_{3,3}(\overline{w}_1, \overline{w}_2, \overline{z}_1, \overline{z}_2, \mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$\leq \rho_0^{-1/p_3} \frac{T\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}} C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})$$
$$\times \max\left(\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}, \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right) \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times \Big(\|\overline{w}_1 - \overline{w}_2\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_1 - \overline{z}_2|\Big).$$

Likewise we obtain the analogous estimate in L^∞ :

$$\|W_{3,3}(\overline{w}_1, \overline{w}_2, \overline{z}_1, \overline{z}_2, \mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}$$
$$\leq \rho_0^{-1/p_3} \frac{\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}}{[p_3(1 - K(\mathbf{d})/\rho_0)]^{1/p_3}} C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})$$

$$\times \max\left(\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}, \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right) \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times \left(\|\overline{w}_1 - \overline{w}_2\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_1 - \overline{z}_2|\right).$$

Consequently the following estimate holds::

$$\|W_3(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_3(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)} \le \widetilde{T} C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})$$
$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right) \frac{\rho_0^{-1/p_3}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}$$
$$\times \left[\left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right) \left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\right.$$
$$+ \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \left(\|\Phi\|_{X^*} \left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1\right.$$
$$\left.\left. + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\right)\right] (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|)$$
$$+ \rho_0^{-1/p_3} \widetilde{T} \frac{C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}} \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)$$
$$\times \left\{ \max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)\right.$$
$$\times \left[\left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right) \left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\right.$$
$$\left.\times \left(\|\Phi\|_{X^*} \left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\right)\right]$$
$$\left. + \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,T;\mathbb{R})} \max\left(\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}, \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\right\}$$
$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.114}$$

Analogously we come to the estimate:

$$\|W_3(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_3(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}$$
$$\le \rho_0^{-1/p_3} \frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}} \left\{ \left[C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) \right.\right.$$

$$
\begin{aligned}
&+ C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})\ \max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)\Bigg]\\
&\times\Bigg[\left(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{fk'}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\\
&\times\left(\|\Phi\|_{X^*}\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\right)\Bigg]\\
&+ \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\ C_5(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T})\\
&\times\max\left(\left\|\frac{f}{k}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}, \left\|\frac{f^2}{k}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\Bigg\}\\
&\times\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)\left(\|\overline{w}_1 - \overline{w}_2\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_1 - \overline{z}_2|\right).
\end{aligned}
\tag{2.115}
$$

Now, by means of the estimate (2.101), let us assess

$$
\begin{aligned}
&\|W_4(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_4(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}\\
&\quad= \left(\int_0^{\widetilde{T}}\left\|\int_0^t a_0(s)\Bigg[\exp\left(\int_s^t\left(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_2(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_2\right)d\sigma\right)\right.\right.\\
&\quad- \exp\left(\int_s^t\left(\overline{k}(\sigma) - \frac{a_0(\sigma)}{k(\sigma)}\,\overline{w}_1(\sigma) - \frac{a_0(\sigma)f(\sigma)}{k(\sigma)}\overline{z}_1\right)d\sigma\right)\Bigg]\\
&\quad\times\left.\left. A\exp\left(\int_s^t a_0(\sigma)d\sigma A\right)g(s)ds\right\|_X^{p_1} dt\right)^{1/p_1}\\
&\quad\le \rho_0^{-1/p_3} T C_{10}(\gamma, T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\\
&\quad\times\exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)\max\left(\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \left\|\frac{a_0 f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)\\
&\quad\times\left(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,T;\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|\right).
\end{aligned}
\tag{2.116}
$$

The following estimate is obtained similarly to the previous one:

$$
\|W_4(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W_4(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}
$$

$$\le \rho_0^{-1/p_3} C_{10}(\gamma,T) \frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{\|a_0^{1/p_1} g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}$$
$$\times \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big) \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)$$
$$\times (\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2-\overline{z}_1|). \tag{2.117}$$

From (2.107), (2.112), (2.114), (2.116) we deduce

$$\|W(\overline{w}_2,\overline{z}_2,\mathbf{d}) - W(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$\le \Big\{\rho_0^{-1/(qp_1)} \frac{1}{[qp_1(1-K(\mathbf{d})/\rho_0)]^{1/(qp_1)}} \Big(\int_0^{\widetilde{T}} \frac{dt}{a_0^{q'/q}(t)}\Big)^{1/(p_1 q')}$$
$$\times \Bigg(\|Au_0\|_X \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)$$
$$+ \Bigg(C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) \max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)$$
$$+ C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\Bigg)\Bigg)$$
$$+ \rho_0^{-1/p_3} \frac{\widetilde{T}}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}} \Bigg[C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})$$
$$\times \Bigg[\Big(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big) \Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$
$$\times \Big(\|\Phi\|_{X^*} \Big\|\frac{fg}{k}\Big\|_{L^{p_1}(0,\widetilde{T};X)} + \Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\Big)\Bigg]$$
$$+ C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}) \Big\{\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)$$
$$\times \Bigg[\Big(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + \Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big) \Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$
$$\times \Big(\|\Phi\|_{X^*} \Big\|\frac{fg}{k}\Big\|_{L^{p_1}(0,\widetilde{T};X)} + \Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})} M_1 + \Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} M_2\Big)\Bigg]$$
$$+ \Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \max\Big(\Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})}, \Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big)\Big\}$$

$$+\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{[1-(1-\gamma)p_2]^{1/p_2}}$$

$$\times\,\|a_0^{1/p_1}g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}\Big]\Big\}\exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Big)$$

$$\times\,(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|)$$

$$=:C_{15}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})(\|\overline{w}_2-\overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+|\overline{z}_2-\overline{z}_1|).$$

Analogously, from (2.108), (2.113), (2.115), we get:

$$\|W(\overline{w}_2,\overline{z}_2,\mathbf{d})-W(\overline{w}_1,\overline{z}_1,\mathbf{d})\|_{L^\infty(0,\widetilde{T};X)}$$

$$\leq\Bigg\{\|Au_0\|_X\max\left(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)$$

$$+\,|f(0)|\Big[C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)$$

$$+\,C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\Big]+\rho_0^{-1/p_3}\frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\Bigg\{\Big[C_{14}(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})$$

$$+\,C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\;\max\left(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\right)\Big]$$

$$\times\Bigg[\Big(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big)\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$

$$\times\,\left(\|\Phi\|_{X^*}\Big\|\frac{fg}{k}\Big\|_{L^{p_1}(0,\widetilde{T};X)}+\Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1+\Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\right)\Bigg]$$

$$+\,\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\;C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})$$

$$\times\max\left(\Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})},\Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right)\Bigg\}$$

$$+\,\rho_0^{-1/p_3}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}$$

$$\times\max\Big(\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},\Big\|\frac{a_0 f}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}\Big)\Bigg\}$$

$$\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)(\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|). \tag{2.118}$$

Here C_2, C_5, C_{14} are defined by (2.61), (2.67), (2.96) respectively. It is easy to verify (2.105). ∎

To apply the Banach theorem on contracting mappings in the complete metric space $\mathcal{K}(M_1, M_2, T)$ we need to prove that the mapping $\mathcal{N}$ is contracting i.e. the inequalities (2.82) and (2.83) hold for some positive constants $\mathbf{q}_1 < 1, \mathbf{q}_2 < 1$. In fact, taking (2.95) and (2.104) into account, from (2.82) and (2.83) we obtain the system of the inequalities:

$$\mathbf{q}_1 := \|\Phi\|_{X^*} C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) + \|\Phi\|_{X^*}\|f\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} C_{15}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) < 1,$$

$$\mathbf{q}_2 := \|\Phi\|_{X^*} C_{14}(\rho_0, \mathbf{d}, M_1, M_2, \widetilde{T}) < 1,$$

where C_{14}, C_{15} are defined in (2.96), (2.104). Since $\mathbf{q}_1, \mathbf{q}_2 \to 0$ as $\rho_0 \to +\infty$ (cf. Remark 2.1 and (2.105)) let us choose so a large ρ_0 such that

$$\mathbf{q}_1 \leq 1/2, \quad \mathbf{q}_2 \leq 1/2.$$

This shows that the operator $\mathcal{N}$ is a contracting mapping from $\mathcal{K}(M_1, M_2, T)$ into itself. Consequently, our problem (P9) admits unique solution in $\mathcal{K}(M_1, M_2, T)$.

2.6 Existence and uniqueness of the solution to Problem 1

We shall now prove Theorem 1.1 by means of a fixed-point argument in the space $\mathcal{K}(M_1, M_2, T)$. So, all previous sections we were preparing to prove the existence and uniqueness of the solution to problem (1.1)-(1.4).

Assume that the conditions $(A1)-(A21)$ hold. We now need to show that there exist and unique solution

$$(a_1, u, z) \in L^1_{a_0}(0,T;\mathbb{R}) \times [W^{1,1}(0,T;X) \cap L^\infty(0,T;D(A))] \times X.$$

The existence and uniqueness of the solution to the Problem 1 follows immediately from the results of the Sections 2.3, 2.4 applying the Banach fixed-point Theorem.

The regularity of Au is studied once again and the corresponding estimates are derived.

Let us prove the estimate (1.21). From the relations (2.4) we have

$$a_0(t)a_1(t) = \frac{k'(t)}{k(t)} - \frac{a_0(t)\Phi[g(t)]}{k(t)} - \frac{a_0(t)}{k(t)}\overline{w}(t) - \frac{a_0(t)f(t)}{k(t)}\overline{z}. \tag{2.119}$$

Taking into account that $(\overline{w}, \overline{z}) \in \mathcal{K}(M_1, M_2, T)$ and the assumption $(A9)$, $(A11)-(A14)$, we easily deduce (1.21).

The estimate (1.22) follows immediately from (2.67).

Lemma 2.2 *For $u \in L^\infty(0,\widetilde{T};D(A))$ the following inequality holds:*

$$\|Au\|_{L^\infty(0,\widetilde{T};X)} \le C_6(\rho_0, \boldsymbol{d}, M_1, M_2, \widetilde{T}),$$

where $C_6, \widetilde{T}$ are defined by (1.23) and (1.13). Moreover, $\boldsymbol{d} \in \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$, M_1, M_2, ρ_0 are satisfying (1.14), (1.15), (A3), (A4) and (1.16).

Proof . We now prove (1.23). Reminding from (2.7), (2.15), (2.17) that

$$Au(t) = w(t) = \sum_{j=1}^{4} W_j(\overline{w}, \overline{z}, \mathbf{d}, t) - f(t)Z(\overline{w}, \overline{z}, \mathbf{d}, T),$$

using Lemma 2.2, (2.67), (2.39), (2.40), (2.54), we obtain

$$\|Au\|_{L^\infty(0,\widetilde{T};X)} \le \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1\left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)\|Au_0\|_X$$

$$
\begin{aligned}
&+\Bigg[\|f\|_{L^\infty(0,\widetilde{T},\mathbb{R})}+|f(0)|\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\bigg)\\
&+\rho_0^{-1/p_3}\frac{1}{[p_3(1-K(\mathbf{d})/\rho_0)]^{1/p_3}}\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\bigg)\\
&\times\bigg\{\Big(\|f'\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+\Big\|\frac{fk'}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\Big)\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\\
&+\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(\|\Phi\|_{X^*}\Big\|\frac{fg}{k}\Big\|_{L^{p_1}(0,\widetilde{T};X)}+\Big\|\frac{f}{k}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})}M_1\\
&+\Big\|\frac{f^2}{k}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}M_2\Big)\bigg\}\Bigg]C_5(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T})\\
&+\rho_0^{-1/p_3}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{1}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\\
&\times\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\bigg)\\
&\times\|a_0^{1/p_1}g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}=C_6(\rho_0,\mathbf{d},M_1,M_2,\widetilde{T}).\quad\blacksquare
\end{aligned}
\tag{2.120}
$$

Remark 2.2 *Estimate (2.120) implies that* $\overline{w}=\Phi[Au]\in L^\infty(0,T;\mathbb{R})$. *In such a way our existence result is now in accordance with the equivalence results proved in the Section 2.*

Remark 2.3 *It is easy to see that (1.24) holds: it is enough to use the assumption on the invertibility of the operator* A, *the identity*

$$u=A^{-1}(Au)$$

and apply (2.120).

Let us remind the the representation of the solution to problem (1.1), (1.2) is the following

$$u(t)=\exp\Big(\int_0^t a_0(s)a_1(s)ds\Big)\exp\Big(\int_0^t a_0(s)dsA\Big)u_0$$

$$+\int_0^t a_0(s)f(s)\exp\Big(\int_s^t a_0(s)a_1(s)ds\Big)\exp\Big(\int_s^t a_0(s)dsA\Big)zds$$
$$+\int_0^t a_0(s)\exp\Big(\int_s^t a_0(s)a_1(s)ds\Big)\exp\Big(\int_s^t a_0(s)dsA\Big)g(s)ds$$
$$=: u_{01}(t)+u_{02}(t)+u_{03}(t),\ \ t\in(0,T). \tag{2.121}$$

Similarly to [14], we shall prove that the solution (2.121) belongs to $W^{1,1}(0,T;X)$ and the estimate (1.25). For this aim we compute the derivative of $u(t)$ and estimate it in $L^1(0,T)$. The following three lemmas give this result.

Lemma 2.3 *The function u_{01} is differentiable a.e. in $(0,T)$, i.e. its derivative belongs to $L^1(0,T;X)$, and is given by the following formula:*

$$u_{01}'(t)=a_0(t)\exp\Big(\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)Au_0$$
$$+\,a_1(t)a_0(t)\exp\Big(\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)u_0.$$

Moreover, the following estimate holds:

$$\|u_{01}'\|_{L^1(0,T;X)}\le\exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})}+M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\bigg)$$
$$\times\left(\|Au_0\|_X\|a_0\|_{L^1(0,T;\mathbb{R})}+\|u_0\|_X C_4(\boldsymbol{d},M_1,M_2)\right),$$

where C_4 is defined by (1.21).

Lemma 2.4 *Let the following assumptions hold:*

$$I_{11}(a_0):=\int_0^T a_0(t)dt\int_0^t\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1}|a_0(t)-a_0(s)|ds<+\infty,$$
$$I_{12}(a_0):=\int_0^T a_0(t)dt\int_0^t\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1}a_0(s)(t-s)^{1/p_1'}ds<+\infty,$$
$$I_{13}(a_0):=\int_0^T |a_1(t)|a_0(t)dt\int_0^t|a_0(t)-a_0(s)|ds<+\infty,$$
$$I_{14}(a_0):=\int_0^T a_0(t)dt\int_0^t|a_0(t)-a_0(s)|\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1}ds<+\infty,$$

$$I_{15}(a_0) := \int_0^T a_0(t)dt \int_0^t \big|a_1(t)a_0(t) - a_1(s)a_0(s)\big|ds < +\infty.$$

Then the function u_{02} *is differentiable a.e. in* $(0,T)$, *i.e. its derivative belongs to* $L^1(0,T;X)$, *and is given by the following formula:*

$$\begin{aligned} u_{02}'(t) =& \int_0^t a_0(t)\Big(A\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) \\ &+ a_1(t)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\Big)[a_0(s)f(s)z - a_0(t)f(t)z]\,ds \\ &+ \int_0^t \Big[(a_0(t)-a_0(s))A\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)ds\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) \\ &+ (a_1(t)a_0(t) - a_1(s)a_0(s))\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)ds\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\Big]a_0(t)f(t)z\,ds \\ &+ \exp\Big(\int_0^t a_0(\sigma)a_1(\sigma)d\sigma\Big)\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)a_0(t)f(t)z. \end{aligned}$$

Moreover the following estimate holds:

$$\begin{aligned} \|u_{02}'\|_{L^1(0,T;X)} \quad &\le \exp\Big(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\Big)K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})} \\ &\times\|z\|_X\Big[C_1\big[I_{14}(a_0) + I_{15}(a_0)\big] + \big\{I_{11}(a_0) + I_{12}(a_0) + I_{13}(a_0) \\ &+ T^{1/p_1'}\|a_0\|_{L^1(0,T;\mathbb{R})}C_4(\boldsymbol{d}, M_1, M_2)\big\} + \|a_0\|_{L^1(0,T;\mathbb{R})}\Big]. \end{aligned}$$

Here C_1, C_4 *are defined in (A4),* (1.21).

Remark 2.4 Let us check the conditions appearing in the Lemma 2.4.

We remind that $a_0(t) = t^\alpha - 1$ and consider

$$\begin{aligned} I_{11}(a_0) &= \int_0^T t^{\alpha-1}dt \int_0^t \Big(\int_s^t \sigma^{\alpha-1}d\sigma\Big)^{-1}|t^{\alpha-1} - s^{\alpha-1}|ds \\ &= \alpha\int_0^T t^{\alpha-1}dt \int_0^t \frac{(t^{\alpha-1} - s^{\alpha-1})}{t^\alpha - s^\alpha}\,ds. \end{aligned}$$

Let us consider separately the integral

$$\int_0^t \frac{t^{\alpha-1} - s^{\alpha-1}}{t^\alpha - s^\alpha}\,ds = |s = \sigma t,\ ds = td\sigma| = \int_0^1 \frac{1-\sigma^{\alpha-1}}{1-\sigma^\alpha}\,d\sigma < +\infty. \qquad (2.122)$$

As a result, taking into account that $a_0 \in L^1(0,T;\mathbb{R})$, we deduce

$$I_{11}(a_0) < +\infty.$$

The next integral we assess likewise:

$$\begin{aligned}
I_{12}(a_0) &= \alpha \int_0^T t^{\alpha-1} dt \int_0^t \frac{s^{\alpha-1}(t-s)^{1/p_1'}}{t^\alpha - s^\alpha} ds = |s = \sigma t,\ ds = t d\sigma| \\
&= \alpha \int_0^T t^{\alpha-1} dt \int_0^1 t \frac{(\sigma t)^{\alpha-1}(t-\sigma t)^{1/p_1'}}{t^\alpha - (\sigma t)^\alpha}\, d\sigma = \alpha \int_0^T t^{\alpha-1+1/p_1'} dt \\
&\times \int_0^1 \frac{\sigma^{\alpha-1}(1-\sigma)^{1/p_1'}}{1-\sigma^\alpha}\, d\sigma < +\infty
\end{aligned}$$

since the function $\dfrac{(1-\sigma)^{1/p_1'}}{1-\sigma^\alpha}$ is integrable in a neighborhood of $\sigma = 1$.

Making again the same change of variable like in (2.122), we obtain

$$\begin{aligned}
I_{13}(a_0) &= \int_0^T |a_1(t)| t^{\alpha-1} dt \int_0^t (t^{\alpha-1} - s^{\alpha-1}) ds = |s = t\sigma| \\
&= \int_0^T |a_1(t)| t^{2(\alpha-1)+1} dt \int_0^1 (1-\sigma^{\alpha-1}) d\sigma \leq |t^{2\alpha} \leq T^\alpha t^\alpha| \\
&\leq T^\alpha \int_0^T |a_1(t)| t^{\alpha-1} dt \int_0^1 (1-\sigma^{\alpha-1}) d\sigma < +\infty. \qquad (2.123)
\end{aligned}$$

Using (2.122), we easily get

$$I_{14}(a_0) = \alpha \int_0^T t^{\alpha-1} dt \int_0^t \frac{t^{\alpha-1} - s^{\alpha-1}}{t^\alpha - s^\alpha} ds < +\infty,$$

Splitting $I_{15}(a_0)$ into two integrals and reminding (2.123), we have the following

$$\begin{aligned}
I_{15}(a_0) &= \int_0^T a_0(t) dt \int_0^t \big|a_1(t)a_0(t) - a_1(s)a_0(s)\big| ds \\
&\leq I_{13}(a_0) + \int_0^T a_0(t) dt \int_0^t a_0(s)|a_1(t) - a_1(s)| ds \\
&\leq I_{13}(a_0) + \int_0^T t^{\alpha-1}|a_1(t)| dt \int_0^T s^{\alpha-1} ds + \int_0^T t^{\alpha-1} dt \int_0^T s^{\alpha-1}|a_1(s)| ds < +\infty.
\end{aligned}$$

Lemma 2.5 *The function u_{03} is differentiable a.e. in $(0,T)$, i.e. its derivative belongs to $L^1(0,T;X)$ and is given by the following formula:*

$$u'_{03}(t) = a_0(t)\int_0^t a_0(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big)A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds$$
$$+ a_1(t)a_0(t)\int_0^t a_0(s)\exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)ds\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g(s)ds.$$

Moreover the following estimate holds:

$$\|u'_{03}\|_{L^1(0,T;X)} \le \Bigg(\rho_0^{-1/p_3}T^{1/p_1}C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\boldsymbol{d})/\rho_0)p_3]^{1/p_3}}$$
$$+ C_4(\boldsymbol{d},M_1,M_2)\|g\|_{L^{p_1}(0,T;X)}\Bigg)\|a_0\|_{L^{p'_1}(0,T;\mathbb{R})}$$
$$\times\exp\Bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p'_1}(0,T;\mathbb{R})}\Bigg).$$

Here C_4, C_{10} is defined by (1.21), (2.61).

Now we prove these lemmas. We start from Lemma 2.3.

Proof (of Lemma 2.3). Let

$$\omega(T) = \{(t,s)\in\mathbb{R}^2 : 0 < s < t < T\},\quad \omega_1(T) = \{(t,s)\in\mathbb{R}^2 : 0 \le s < t \le T\}.$$

Introduce the operator-valued function $B:\omega(T)\to\mathcal{L}(X)$ defined by

$$B(t,s) = \exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)ds\Big)\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big).$$

It is immediate to check that $B\in C(\overline{\omega(T)};\mathcal{L}(X))\cap C(\omega_1(T);\mathcal{L}(X;\mathcal{D}(A))$. Moreover, $D_tB\in C(\omega(T);\mathcal{L}(X))$ and can be continuously extended to $\omega_1(T)$.
By simple computations, we verify that B solves the differential equations

$$D_tB(t,s) = a_0(t)(AB(t,s)+a_1(t)B(t,s)),\ \text{for a.e. } t\in(s,T),\ s\in[0,T), \tag{2.124}$$

$$D_sB(t,s) = -a_0(s)(AB(t,s)+a_1(s)B(t,s)),\ \text{for a.e. } s\in(0,t),\ t\in(0,T], \tag{2.125}$$

and the initial condition

$$B(s+,s) = B(t,t-) = I, \qquad s,t \in (0,T).$$

Moreover, according to (2.33), (2.34), it is easy to check that for $(t,s) \in \omega(T)$ the operator B satisfies the estimates

$$\|B(t,s)\|_{\mathcal{L}(X)} \le \exp\Big(-(\rho_0 - K(\mathbf{d}))\int_s^t a_0(\sigma)\,d\sigma\Big) C_{17}(\mathbf{d}) \le C_{17}(\mathbf{d}), \qquad (2.126)$$

where we have set

$$C_{17}(\mathbf{d}) := \exp\bigg(\Big\|\frac{k'}{k}\Big\|_{L^1(0,T;\mathbb{R})} + M_1\Big\|\frac{a_0}{k}\Big\|_{L^{p_1'}(0,T;\mathbb{R})}\bigg).$$

Obviously, the relations (2.124) and (2.125) imply

$$D_tB(t,s) + D_sB(t,s) = (a_0(t) - a_0(s))AB(t,s)$$
$$+ (a_1(t)a_0(t) - a_1(s)a_0(s))B(t,s), \quad (t,s) \in \omega(T). \qquad (2.127)$$

Taking into account (2.124), we get

$$u_{01}'(t) = D_t\big(B(t,0)u_0\big) = a_0(t)B(t,0)Au_0 + a_1(t)a_0(t)B(t,0)u_0.$$

We now use (2.126), (1.21) to estimate the right-hand side above in $L^1(0,T)$. As a result, we obtain the following bound:

$$\begin{aligned}\|u_{01}'\|_{L^1(0,T;X)} \le{}& \int_0^T a_0(t)\|B(t,s)\|_{\mathcal{L}(X)}\|Au_0\|_X dt \\ &+ \int_0^T |a_1(t)|a_0(t)\|B(t,s)\|_{\mathcal{L}(X)}\|u_0\|_X dt \\ &\le C_{17}(\mathbf{d})\Big(\|Au_0\|_X\int_0^T a_0(t)dt + \|u_0\|_X\int_0^T |a_1(t)|a_0(t)dt\Big) \\ &\le C_{17}(\mathbf{d})\big(\|Au_0\|_X\|a_0\|_{L^1(0,T;\mathbb{R})} + \|u_0\|_X C_4(\mathbf{d},M_1,M_2)\big) < +\infty\end{aligned}$$

since $a_1 \in L^1_{a_0}(0,T;\mathbb{R})$ by definition of the solution and the conditions $(A5), (A9)$ hold.

■

Proof (of Lemma 2.4).

Introduce the family of functions $u_{02\varepsilon} : [0,T] \to X$, $\varepsilon \in (0,1)$, defined by

$$u_{02\varepsilon}(t) := \int_0^{\varepsilon t} \exp\Big(\int_s^t a_0(\sigma)a_1(\sigma)d\sigma\Big) \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) a_0(s)f(s)z\,ds.$$

Such functions are a.e. differentiable in $(0,T)$ and their derivatives are given by

$$\begin{aligned} u'_{02\varepsilon}(t) &= \int_0^{\varepsilon t} D_t B(t,s)a_0(s)f(s)z\,ds + \varepsilon B(t,\varepsilon t)a_0(\varepsilon t)f(\varepsilon t)z \\ &= \int_0^{\varepsilon t} D_t B(t,s)[a_0(s)f(s)z - a_0(t)f(t)z]\,ds \\ &+ \int_0^{\varepsilon t} \Big[D_t B(t,s) + D_s B(t,s)\Big] a_0(t)f(t)z\,ds \\ &- B(t,\varepsilon t)\big[a_0(t)f(t)z - \varepsilon\, a_0(\varepsilon t)f(\varepsilon t)z\big] \\ &+ B(t,0)a_0(t)f(t)z, \end{aligned} \tag{2.128}$$

since

$$\int_0^{\varepsilon t} D_s B(t,s)a_0(t)f(t)z\,ds = \big[B(t,\varepsilon t) - B(t,0)\big]a_0(t)f(t)z, \quad \text{for a.e. } t \in (0,T).$$

Let us remind the known imbedding inequality:

$$\|m\|_{L^\infty(0,T;Y)} \le K_2(T)\|m\|_{W^{1,p_1}(0,T;Y)}, \tag{2.129}$$

$$|f(t) - f(s)| \le \|f\|_{W^{1,p_1}(0,T;\mathbb{R})}(t-s)^{1/p'_1}. \tag{2.130}$$

In fact, the last inequality follows from:

$$|f(t) - f(s)| = \Big|\int_s^t f'(r)dr\Big| \le \int_s^t |f'(r)|dr \le \Big(\int_s^t dr\Big)^{1/p'_1} \Big(\int_s^t |f'(r)|^{p_1}dr\Big)^{1/p_1}.$$

Consider the right-hand side of (2.128). Note now that for a.e. $t \in (0,T)$ and $(s,t) \in \omega(T)$ we have

$$\|a_0(t)f(t)z\|_X \le K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|z\|_X a_0(t), \tag{2.131}$$

$$\|a_0(t)f(t)z - a_0(s)f(s)z\| \le \|(a_0(t)-a_0(s))f(t)z\|$$

$$+ \|a_0(s)(f(t)-f(s))z\| \le K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\Big(|a_0(t)-a_0(s)|$$

$$+ a_0(s)(t-s)^{1/p_1'}\Big). \tag{2.132}$$

For the next estimates we shall use the assumption (A4) and (2.63). For any $h_1 : (0,T) \to X$ and $(t,s) \in \omega(T)$ from (2.124), (2.125), the following inequalities hold:

$$\|AB(t,s)\|_{\mathcal{L}(X)} \le C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} C_{17}(\mathbf{d}),$$

$$\|D_tB(t,s)h_1(s)\|_{\mathcal{L}(X)} \le a_0(t)C_{17}(\mathbf{d})\Big[C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(t)|\Big]\|h_1(s)\|_X, \tag{2.133}$$

$$\|D_sB(t,s)h_1(s)\|_{\mathcal{L}(X)} \le a_0(s)C_{17}(\mathbf{d})\Big(C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(s)|\Big)\|h_1(s)\|_X.$$

As a consequence, from the last two estimates, we get:

$$\|(D_tB(t,s)+D_sB(t,s))h_1(s)\|_{\mathcal{L}(X)} \le \Big(a_0(t)\big[C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(t)|\big]$$

$$+ a_0(s)\big[C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(s)|\big]\Big)C_{17}(\mathbf{d})\|h_1(s)\|_X. \tag{2.134}$$

We now make use of (2.132) and (2.133) and consider the following estimates holding for a.e. $(t,s) \in \omega(T)$:

$$\|D_tB(s,t)[a_0(t)f(t)z - a_0(s)f(s)z]\|$$

$$\le a_0(t)C_{17}(\mathbf{d})\Big[C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(t)|\Big]$$

$$\times \|(a_0(t)-a_0(s))f(t)z + a_0(s)(f(t)-f(s))z\|_X$$

$$\le a_0(t)C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\Big[C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + |a_1(t)|\Big]$$

$$\times \Big(|a_0(t)-a_0(s)| + a_0(s)(t-s)^{1/p_1'}\Big) = C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}$$

$$\times \Big\{\Big[C_1a_0(t)\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + a_0(t)|a_1(t)|\Big]|a_0(t)-a_0(s)|$$

$$+ a_0(s)(t-s)^{1/p_1'}\Big[C_1 a_0(t)\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + a_0(t)|a_1(t)|\Big]\Big\}$$

$$= C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}$$

$$\times\Big\{C_1 a_0(t)\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1}|a_0(t)-a_0(s)| + a_0(t)|a_1(t)||a_0(t)-a_0(s)|$$

$$+ C_1 a_0(t)a_0(s)(t-s)^{1/p_1'}\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + a_0(t)|a_1(t)|a_0(s)(t-s)^{1/p_1'}\Big\}.$$

It implies

$$\int_0^T dt\int_0^t \|D_tB(s,t)[a_0(t)f(t)z - a_0(s)f(s)z]\|ds \le C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}$$

$$\times\Big(I_{11}(a_0) + I_{12}(a_0) + I_{13}(a_0) + \int_0^T |a_1(t)|a_0(t)dt\int_0^t a_0(s)(t-s)^{1/p_1'}ds\Big)$$

$$\le C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\Big(I_{11}(a_0) + I_{12}(a_0) + I_{13}(a_0)$$

$$+T^{1/p_1'}\|a_0\|_{L^1(0,T;\mathbb{R})}\|a_1\|_{L^1_{a_0}(0,T;\mathbb{R})}\Big). \tag{2.135}$$

We have used here the identity (2.127).

By means of (2.134) and (2.131), we estimate now

$$\big\|[D_tB(t,s) + D_sB(t,s)]a_0(t)f(t)z\big\|_{\mathcal{L}(X)}$$

$$\le \Big(|a_0(t)-a_0(s)|C_1\Big(\int_s^t a_0(\sigma)d\sigma\Big)^{-1} + \big|a_1(t)a_0(t) - a_1(s)a_0(s)\big|\Big)$$

$$\times C_{17}(\mathbf{d})K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|z\|_X a_0(t).$$

Integrating the last inequality with respect to t over $(0,T)$, we get

$$\int_0^T dt\int_0^t \big\|[D_tB(s,t) + D_sB(t,s)]a_0(t)f(t)z\big\|\,ds \le C_1C_{17}(\mathbf{d})K_1(T)$$

$$\times\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|z\|_X\big[I_{14}(a_0) + I_{15}(a_0)\big]. \tag{2.136}$$

In order to assess the next integral we take into account the fundamental relation

$$\int_0^T |a_0(t) - a_0(\varepsilon t)|\,dt \to 0, \quad \text{as } \varepsilon \to 1-,$$

and (2.126). As a result we achieve:

$$\int_0^T \|B(t,\varepsilon t)\big[a_0(t)f(t)z - \varepsilon(a_0(\varepsilon t)f(\varepsilon t)z\big]\|dt$$

$$\leq \|z\|_X \int_0^T \big|a_0(t)f(t)z - \varepsilon a_0(\varepsilon t)f(\varepsilon t)\big|dt$$

$$\leq C_{17}(\mathbf{d})\|z\|_X \int_0^T \big|(a_0(t) - \varepsilon a_0(\varepsilon t))f(t) + \varepsilon a_0(\varepsilon t)(f(t) - f(\varepsilon t))\big|dt$$

$$\leq C_{17}(\mathbf{d})\|z\|_X \Big(K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})} \int_0^T |a_0(t) - \varepsilon a_0(\varepsilon t)|dt$$

$$+ C_{17}(\mathbf{d})\varepsilon\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}(1-\varepsilon)^{1/p_1} \int_0^T t^{1/p_1} a_0(\varepsilon t)dt\Big) \to 0 \quad \text{as} \quad \varepsilon \to 1- .$$

Exploiting (2.126) and (2.131), we easily obtain the estimate

$$\int_0^T \|B(t,s)a_0(t)f(t)z\|dt \leq K_1(T)C_{17}(\mathbf{d})\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|a_0\|_{L^1(0,T;\mathbb{R})}. \tag{2.137}$$

Introduce now the function

$$v(t) = \int_0^t D_t B(t,s)[a_0(s)f(s)z - a_0(t)f(t)z]\,ds$$
$$+ \int_0^t \Big[D_t B(t,s) + D_s B(t,s)\Big]a_0(t)f(t)z\,ds + B(t,0)a_0(t)f(t)z.$$

Summing up the all established estimates, we can easily show that

$$u_{02\varepsilon} \to u_{02}, \quad u'_{02\varepsilon} \to v \quad \text{in } L^1(0,T;X) \quad \text{as } \varepsilon \to 1- .$$

Then u_{02} is differentiable in the distribution sense and

$$u'_{02} = v \quad \text{a.e. in } (0,T).$$

In particular, from (2.137), (2.136), (2.135) we deduce the estimate

$$\|u'_{02}\|_{L^1(0,T;X)} = \|v\|_{L^1(0,T;X)} \leq C_1C_{17}(\mathbf{d})K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|z\|_X\big[I_{14}(a_0) + I_{15}(a_0)\big]$$

$$+C_{17}(\mathbf{d})K_1(T)\|z\|_X\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\Big(I_{11}(a_0) + I_{12}(a_0) + I_{13}(a_0)$$

$$+T^{1/p_1'}\|a_0\|_{L^1(0,T;\mathbb{R})}\|a_1\|_{L^1_{a_0}(0,T;\mathbb{R})}\Big) + C_{17}(\mathbf{d})K_1(T)\|f\|_{W^{1,p_1}(0,T;\mathbb{R})}\|z\|_X\|a_0\|_{L^1(0,T;\mathbb{R})}.$$

Finally, exploiting (1.21), we finish the proof of Lemma 2.4. ■

Proof (of Lemma 2.5)

Remind the formula (2.124) and observe that for any $h_2 \in L^{p_1}(0,T;\mathcal{D}_A(\gamma,\infty))$ and $(t,s) \in \omega(T)$ the following estimates hold

$$\|D_t B(t,s)h_2(s)\|_{\mathcal{L}(X)} \le a_0(t)\Big(C_{10}(\gamma,T)\|h_2(s)\|_{\mathcal{D}_A(\gamma,\infty)}\left(\int_s^t a_0(\sigma)d\sigma\right)^{-1+\gamma}$$
$$+ |a_1(t)|\|h_2(s)\|_X\Big)C_{17}(\mathbf{d})$$

$$\|AB(t,s)h(s)\|_X \le C_{10}(\gamma,T)\|h(s)\|_{\mathcal{D}_A(\gamma,\infty)}C_{17}(\mathbf{d})\left(\int_s^t a_0(\sigma)d\sigma\right)^{-1+\gamma}.$$

Let us compute the derivative of $u_{0,3}$. We find

$$u_{03}'(t) = D_t\Big(\int_0^t a_0(s)B(t,s)g(s)ds\Big) = a_0(t)\int_0^t a_0(s)AB(t,s)g(s)ds$$
$$+ a_1(t)a_0(t)\int_0^t a_0(s)B(s,t)g(s)ds. \tag{2.138}$$

Then, using (2.62), (2.66), (2.126), (1.21), we estimate (2.138) in $L^1(0,T;X)$:

$$\|u_{03}'\|_{L^1(0,T;X)} \le \int_0^T a_0(t)dt\int_0^t a_0(s)\|AB(t,s)g(s)\|_X ds$$
$$+ \int_0^T |a_1(t)|a_0(t)dt\int_0^t a_0(s)\|B(t,s)g(s)\|_X ds$$
$$\le \rho_0^{-1/p_3}T^{1/p_1}\|a_0\|_{L^{p_1'}(0,T;\mathbb{R})}C_{10}(\gamma,T)C_{17}(\mathbf{d})\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$
$$\times\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}} + C_{17}(\mathbf{d})\int_0^T|a_1(t)|a_0(t)dt\int_0^T a_0(s)\|g(s)\|_X ds$$
$$\le \rho_0^{-1/p_3}T^{1/p_1}\|a_0\|_{L^{p_1'}(0,T;\mathbb{R})}C_{10}(\gamma,T)C_{17}(\mathbf{d})\frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$
$$\times\frac{\|a_0^{1/p_1}g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}} + C_{17}(\mathbf{d})C_4(\mathbf{d},M_1,M_2)\|a_0\|_{L^{p_1'}(0,T;\mathbb{R})}$$

$$\times \|g\|_{L^{p_1}(0,T;X)} < +\infty$$

since $(A5), (A9), (A14), (A15)$ hold. Here we have used the inequality:

$$\int_0^T a_0(t)dt \int_0^t a_0(s) \exp\left(\int_s^t a_0(\sigma)|a_1(\sigma)|d\sigma\right) \left|A \exp\left(\int_s^t a_0(\sigma)d\sigma A\right) g(s)\right| ds$$

$$\leq \left(\int_0^T a_0(t)^{p_1'} dt\right)^{1/p_1'} \left[\int_0^T \left(C_{10}(\gamma, T) \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{\rho_0^{-1/p_3}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}\right.\right.$$

$$\left.\left.\times \exp\left(\left\|\frac{k'}{k}\right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\|\frac{a_0}{k}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right) \|a_0^{1/p_1} g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}\right)^{p_1} dt\right]^{1/p_1}$$

$$\leq \rho_0^{-1/p_3} T^{1/p_1} \|a_0\|_{L^{p_1'}(0,T;\mathbb{R})} C_{10}(\gamma, T) C_{17}(\mathbf{d}) \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$

$$\times \frac{\|a_0^{1/p_1} g\|_{L^{p_1}(0,T;D_A(\gamma,\infty))}}{[(1-K(\mathbf{d})/\rho_0)p_3]^{1/p_3}}. \qquad \blacksquare$$

The Lemma 2.4 has just been proved and we are now in the position to complete the proof of the Theorem 1.1 ■

Chapter 3

Continuous dependence of the solution on the data

We now aim at displaying a continuous dependence estimate of Lipschits type for the solution to problem (1.1)-(1.4) with respect to the structural data $\mathbf{d}$. To this purpose, basing on the definition of the data set $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$, we prove some useful estimates which will be used to assess Q^{-1} and Z, W and their increments in all variables. We also verify the estimate IC stated in the Theorem 1.2

3.1 Preliminary results

In order to prove the continuous dependence of the solution $(a_1, u, z) \in L^1_{a_0}(0, T; \mathbb{R}) \times [W^{1,1}(0, T; X) \cap L^\infty(0, T; D(A))] \times X$ to problem (1.1)–(1.4) on our data, we need to strengthen the smoothness of the function g. To this purpose the condition $(A14)$ is changed to $(A24)$, i.e.

$$g \in L^\infty(0, T; X).$$

Now we rewrite the definitions (1.11) and (1.12) with an arbitrarily fixed vector

$\mathbf{r} = (r_1, .., r_9) \in (\mathbb{R}_+)^9$:

$$\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}) = \Big\{ \mathbf{d} = (f, g, u_0, h, k) \in W^{1,p_1}(0, \widetilde{T}; \mathbb{R}) \times L^\infty(0, \widetilde{T}; X) \times D(A)^2 \times W^{1,p_1}(0, \widetilde{T}; \mathbb{R}) :$$

$$k(t) > 0,\ t \in [0, T],\ \left| \int_0^T \varphi(t) f(t) d\mu(t) \right| \geq r_1,\ \|f\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \leq r_2,$$

$$\|g\|_{L^\infty(0,\widetilde{T};X)} \leq r_3,\ \|a_0^{1/p_1} g\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))} \leq r_4,\ \|Au_0\|_X \leq r_5, \|Ah\|_X \leq r_6,$$

$$\left\| \frac{k'}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq r_7,\ \left\| \frac{a_0}{k} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} \leq r_8, \left\| \frac{1}{k} \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq r_9 \Big\},$$

$$\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2) = \|f_1 - f_2\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} + \|g_1 - g_2\|_{L^\infty(0,\widetilde{T};X)} + \|a_0^{1/p_1}(g_2 - g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}$$
$$+ \|u_{0,2} - u_{0,1}\|_{D(A)} + \|h_2 - h_1\|_{D(A)} + \left\| \frac{k_2'}{k_2} - \frac{k_1'}{k_1} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}$$
$$+ |k_2(0) - k_1(0)| + \left\| \frac{a_0}{k_1} - \frac{a_0}{k_2} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} + \left\| \frac{1}{k_2} - \frac{1}{k_1} \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}.$$

Remark 3.1 *We remind that the notation $\widetilde{T}$, defined by* (1.13)*, means that above we consider two data spaces and two distances corresponding to the Case 1 and 2.*

Remark 3.2 *The presence of the term $|k_2(0) - k_1(0)|$ in the definition of* $\mathbf{dist}(\boldsymbol{d}_1, \boldsymbol{d}_2)$ *is required by the definition of a distance.*

For a further use, we remind the known imbedding inequalities

$$\|m\|_{L^1(0,T;Y)} \leq T^{1/p_1'} \|m\|_{L^{p_1}(0,T;Y)}, \qquad \|m\|_{L^{p_1}(0,T;Y)} \leq T^{1/p_1} \|m\|_{L^\infty(0,T;Y)}$$

and (2.129).

Lemma 3.1 *Under the conditions $(A9) - (A16), (A22)$ the following inequalities hold:*

$$\left\| \frac{f}{k} \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq K_1(\widetilde{T}) r_2 r_9, \qquad \left\| f^2 \left(\frac{1}{k} \right) \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq K_1^2(\widetilde{T}) r_2^2 r_9,$$

$$\left\| \frac{f^2}{k} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq \widetilde{T}^{1/p_1} K_1^2(\widetilde{T}) r_2^2 r_9, \qquad \left\| f \left(\frac{k'}{k} \right) \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq K_1(\widetilde{T}) r_2 r_7,$$

$$\left\|\frac{fg}{k}\right\|_{L^{p_1}(0,\widetilde{T};X)} \leq \widetilde{T}^{1/p_1}K_1(\widetilde{T})r_2r_3r_9, \qquad \left\|\frac{a_0f}{k}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} \leq r_2r_8,$$

$$\left\|\frac{f_2}{k_2}-\frac{f_1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq K_1(\widetilde{T})\left[r_9\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+r_2\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\right],$$

$$\left\|\frac{f_2^2}{k_2}-\frac{f_1^2}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \leq K_1^2(\widetilde{T})\left[2r_2r_9\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+r_2^2\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\right],$$

$$\left\|\frac{f_2^2}{k_2}-\frac{f_1^2}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq \widetilde{T}^{1/p_1}K_1^2(\widetilde{T})\left(2r_2r_9\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+r_2^2\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\right),$$

$$\left\|\frac{a_0f_2}{k_2}-\frac{a_0f_1}{k_1}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} \leq r_8\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+r_2\left\|\frac{a_0}{k_2}-\frac{a_0}{k_1}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},$$

$$\left\|\frac{a_0g_1}{k_1}-\frac{a_0g_2}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} \leq r_8\widetilde{T}^{1/p_1}\|g_1-g_2\|_{L^\infty(0,\widetilde{T};X)}+\widetilde{T}^{1/p_1}r_3\left\|\frac{a_0}{k_1}-\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})},$$

$$\left\|\frac{f_1k_1'}{k_1}-\frac{f_2k_2'}{k_2}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \leq K_1(\widetilde{T})\left(r_7\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+r_2\left\|\frac{k_1'}{k_1}-\frac{k_2'}{k_2}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}\right),$$

$$\begin{aligned}\left\|\frac{f_2g_2}{k_2}-\frac{f_1g_1}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};X)} &\leq K_1(\widetilde{T})\Big[\widetilde{T}^{1/p_1}r_3r_9\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\\
&+r_2r_9\widetilde{T}^{1/p_1}\|g_2-g_1\|_{L^\infty(0,\widetilde{T};X)}+\widetilde{T}^{1/p_1}r_3r_2\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\Big],\end{aligned}$$

$$\begin{aligned}&\exp\left(\left\|\frac{k_2'}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}+\left\|\frac{k_1'}{k_1}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k_1}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)\\
&\qquad\leq \exp(2\widetilde{T}^{1/p_1'}r_7+2M_1r_8),\end{aligned}$$

$$\begin{aligned}\|\overline{k}_2-\overline{k}_1\|_{L^1(0,\widetilde{T};X)} &\leq \widetilde{T}^{1/p_1'}\left\|\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+r_8\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}\|g_1-g_2\|_{L^\infty(0,\widetilde{T};X)}\\
&+\widetilde{T}^{1/p_1}r_3\|\Phi\|_{X^*}\left\|\frac{a_0}{k_1}-\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}.\end{aligned}$$

Here $\overline{k}$ is defined by (2.5) and K_1 in (2.129).

Proof . We limit ourself only to proving two of the estimates announced in this

Lemma. The other ones will be treated analogously. By means of (2.129) and the definition of the set $\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$, we have

$$\left\|\frac{f_2g_2}{k_2}-\frac{f_1g_1}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};X)} \le \left\|\frac{g_2}{k_2}\,(f_2-f_1)\right\|_{L^{p_1}(0,\widetilde{T};X)} + \left\|f_1\,\left(\frac{g_2}{k_2}-\frac{g_1}{k_1}\right)\right\|_{L^{p_1}(0,\widetilde{T};X)}$$

$$\le \|g_2\|_{L^{p_1}(0,\widetilde{T};X)}\left\|\frac{1}{k_2}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|f_2-f_1\|_{L^\infty(0,\widetilde{T};\mathbb{R})}+\|f_1\|_{L^\infty(0,\widetilde{T};\mathbb{R})}$$

$$\times\left(\left\|\frac{1}{k_2}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|g_2-g_1\|_{L^{p_1}(0,\widetilde{T};X)}+\|g_1\|_{L^{p_1}(0,\widetilde{T};X)}\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\right)$$

$$\le \widetilde{T}^{1/p_1}r_3r_9K_1(\widetilde{T})\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}$$

$$+K_1(\widetilde{T})r_2\Big(r_9\widetilde{T}^{1/p_1}\|g_2-g_1\|_{L^\infty(0,\widetilde{T};X)}+\widetilde{T}^{1/p_1}r_3\left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\Big).$$

Taking into account the notation (2.5) and the estimates already proved in this Lemma, we get

$$\|\overline{k}_2(t)-\overline{k}_1(t)\|_{L^1(0,\widetilde{T})}\le\left\|\frac{k_2'(t)}{k_2(t)}-\frac{k_1'(t)}{k_1(t)}\right\|_{L^1(0,\widetilde{T})}+\left\|\Phi\left[\frac{a_0(t)g_1(t)}{k_1(t)}-\frac{a_0(t)g_2(t)}{k_2(t)}\right]\right\|_{L^1(0,\widetilde{T})}$$

$$\le\left\|\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+\|\Phi\|_{X^*}\left\|\frac{a_0g_1}{k_1}-\frac{a_0g_2}{k_2}\right\|_{L^1(0,\widetilde{T};X)}\le\widetilde{T}^{1/p_1'}\left\|\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}$$

$$+r_8\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}\|g_1-g_2\|_{L^\infty(0,\widetilde{T};X)}+\widetilde{T}^{1/p_1}r_3\|\Phi\|_{X^*}\left\|\frac{a_0}{k_1}-\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}. \qquad \blacksquare$$

Remark 3.3 *In order to distinguish the constants related to the continuous dependence result, we denote them with "K_i".*

Let us remind that in order to deduce the fundamental estimate (2.33) we have introduced the assumptions leading to constants C_2 and C_3. In order to avoid the dependence on $\mathbf{d}_i\in\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}}), i=1,2,$ we estimate

$$C_2(\mathbf{d}_i)=\sup_{0<s<t<T}\frac{\|\Phi[\frac{a_0(\sigma)g_i(\sigma)}{k_i(\sigma)}]\|_{L^1(s,t)}}{\int_s^t a_0(\sigma)d\sigma}\le\sup_{0<s<t<T}\frac{\|\Phi\|_{X^*}\|\frac{a_0(\sigma)g_i(\sigma)}{k_i(\sigma)}\|_{L^1(s,t)}}{\|a_0(\sigma)\|_{L^1(s,t)}}$$

$$\le\sup_{0<s<t<T}\left(\|\Phi\|_{X^*}\left\|\frac{g_i(\sigma)}{k_i(\sigma)}\right\|_{L^\infty(s,t)}\right)\le\|\Phi\|_{X^*}r_3r_9,$$

and

$$C_3(\mathbf{d}_i) = \sup_{0<s<t<T} \frac{\| \frac{a_0(\sigma) f_i(\sigma)}{k_i(\sigma)} \|_{L^1(s,t)}}{\int_s^t a_(\sigma) d\sigma} \leq \sup_{0<s<t<T} \left\| \frac{f_i(\sigma)}{k_i(\sigma)} \right\|_{L^\infty(s,t)} \leq K_1(T) r_2 r_9.$$

Therefore

$$C_2(\mathbf{d}_i) + C_3(\mathbf{d}_i) \leq (\|\Phi\|_{X^*} r_3 + K_1(T) r_2) r_9.$$

Now we choose

$$\rho_0 > r_9 \big(\|\Phi\|_{X^*} r_3 + K_1(T) r_2 M_2 \big) + 1 =: K_3(\mathbf{r}) + 1. \tag{3.1}$$

Reminding (2.32), for $i = 1, 2$ we deduce

$$\int_s^t \Big| \overline{k}_i(\sigma) - \frac{a_0(\sigma)}{k_i(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f_i(\sigma)}{k_i(\sigma)} \overline{z} \Big| d\sigma \leq K_2(\mathbf{r}) \int_s^t a_0(\sigma)\, d\sigma$$
$$+ \left\| \frac{k_i'}{k_i} \right\|_{L^1(0,T;\mathbb{R})} + M_1 \left\| \frac{a_0}{k_i} \right\|_{L^{p_1'}(0,T;\mathbb{R})}.$$

Therefore the fundamental estimate (2.33) is changed to the following.

Lemma 3.2 *For all* $(\overline{w}, \overline{z}, \boldsymbol{d}_i)$ *in* $\mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2,$ *the following estimate holds:*

$$\exp\Big(\int_s^t \Big| \overline{k}_i(\sigma) - \frac{a_0(\sigma)}{k_i(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f_i(\sigma)}{k_i(\sigma)} \overline{z} \Big| d\sigma \Big) \Big\| \exp\Big(\int_s^t a_0(\sigma) d\sigma A \Big) \Big\|_X$$
$$\leq \exp\bigg(-(\rho_0 - K_2(\boldsymbol{r})) \int_s^t a_0(\sigma) d\sigma \bigg) \exp(T^{1/p_1'} r_7 + M_1 r_8), \tag{3.2}$$

for all $s, t \in [0, T], s < t,$ *where* $\overline{k}_i, i = 1, 2,$ *are defined by* (2.5).

The next lemma will give an estimate for $a_0 a_1$ which will be useful to estimate the increments of u.

Lemma 3.3 *For* $a_1 \in L^1_{a_0}(0, T; \mathbb{R})$ *the following inequality holds:*

$$\|a_0 a_1\|_{L^1(0,T;\mathbb{R})} \leq T^{1/p_1'} r_7 + \|\Phi\|_{X*} T^{1/p_1} r_3 r_8 + r_8 M_1 + r_2 r_8 M_2 =: K_4(\boldsymbol{r}) \tag{3.3}$$

with an arbitrarily fixed vector $\boldsymbol{r} = (r_1, .., r_9) \in (\mathbb{R}_+)^9$.

Proof . It immediately follows from (2.119), the definition of $\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ and Lemma 3.1 ■

Lemma 3.4 *For any* $(\overline{w},\overline{z},\boldsymbol{d})\in\mathcal{K}(M_1,M_2,T)\times\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ *the functionals* J,Z,W *defined by formula (1.17), (2.16),* $(P8)_1$*, accordingly, satisfy the next estimates:*

$$J(\rho_0,\boldsymbol{d},M_1,M_2,\widetilde{T})\le K_5(\rho_0,r), \tag{3.4}$$

$$\|Z(\overline{w},\overline{z},\boldsymbol{d},\widetilde{T})\|_X\le K_6(\boldsymbol{r})+\tau_4K_7(\boldsymbol{r})\exp\left(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\right)+\rho_0^{-1/p_3}K_8(\boldsymbol{r}), \tag{3.5}$$

$$\|W(\overline{w},\overline{z},\boldsymbol{d})\|_{L^{p_1}(0,T;\mathbb{R})}\le K_9(\boldsymbol{r})\rho_0^{-\min(1/(qp_1),1/p_3)}. \tag{3.6}$$

Moreover, choose ρ_0 *such as to satisfy an inequality stricter than* $(A23)_7$*:*

$$K_5(\rho_0,r)\le\frac{r_1}{2}.$$

Here we have denoted

$$\tau_4=\begin{cases}1, & \text{Case 1},\\ 0, & \text{Case 2},\end{cases} \tag{3.7}$$

while the function K_4 *is defined by*

$$\begin{aligned}K_5(\rho_0,r)=r_2\exp(T^{1/p_1'}r_7+M_1r_8)\Bigg\{&\tau_1(\rho_0,\boldsymbol{r},\widetilde{T})+\tau_2(\widetilde{T})\rho_0^{-1/p_3}\frac{(K_2(\boldsymbol{r})+1)^{1/p_3}}{[p_3]^{1/p_3}}\\ &\times\left[(1+K_1(T)r_7)\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}K_1(T)r_9\right.\\ &\left.\times\left(\|\Phi\|_{X^*}T^{1/p_1}r_3+M_1+K_1(T)r_2M_2\right)\right]\Bigg\}.\end{aligned}$$

and K_5,K_6,K_7,K_8 *are positive continuous functions in* r.

Proof . Let us note from (1.19) that $|\tau_1(\rho_0,\mathbf{d},\widetilde{T})|\le\tau_1(\rho_0,\mathbf{r},\widetilde{T})$, where

$$\tau_1(\rho_0,\mathbf{r},\widetilde{T}):=\begin{cases}|\varphi(T_1)|\exp\left(-(\rho_0-K_2(\mathbf{r}))\int_0^{T_1}a_0(\sigma)d\sigma\right), & \textbf{Case 1},\\ \rho_0^{-1/p_3}\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,T;\mathbb{R})}\|\varphi\psi\|_{L^{p_1}(0,T;\mathbb{R})}\frac{(K_2(\mathbf{r})+1)^{1/p_3}}{p_3^{1/p_3}}, & \textbf{Case 2}.\end{cases}$$

Taking then into account Lemma 3.1, the estimates (3.4)–(3.6) easily follow from (1.17), (2.67) and (2.79). ■

Now we are going to prove the estimates for the increments of Q^{-1}, Z, W in $(\overline{w}_i, \overline{z}_i), i = 1, 2$.

Lemma 3.5 *For any* $(\overline{w}_i, \overline{z}_i, \boldsymbol{d}) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2,$ *the following estimates hold:*

$$\|Q^{-1}(\overline{w}_2,\overline{z}_2, \boldsymbol{d}, \widetilde{T}) - Q^{-1}(\overline{w}_1, \overline{z}_1, \boldsymbol{d}, \widetilde{T})\|_{\mathcal{L}(X)} \leq \Big[K_{10}(\boldsymbol{r})\rho_0^{-1/p_3}$$

$$+ \tau_4 K_{11}(\boldsymbol{r}) \exp\left(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\right)\Big](\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad (3.8)$$

$$\|Z(\overline{w}_2,\overline{z}_2, \boldsymbol{d}, \widetilde{T}) - Z(\overline{w}_1, \overline{z}_1, \boldsymbol{d}, \widetilde{T})\|_X \leq \Big[K_{12}(\boldsymbol{r})\rho_0^{-1/p_3}$$

$$+ \tau_4 K_{13}(\boldsymbol{r}) \exp\left(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\right)\Big](\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad (3.9)$$

$$\|W(\overline{w}_2,\overline{z}_2, \boldsymbol{d}) - W(\overline{w}_1, \overline{z}_1, \boldsymbol{d})\|_{L^{p_1}(0,\widetilde{T};X)} \leq \rho_0^{-\min(1/(qp_1),1/p_3)} K_{14}(\boldsymbol{r})$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad (3.10)$$

$$\|W(\overline{w}_2,\overline{z}_2, \boldsymbol{d}) - W(\overline{w}_1, \overline{z}_1, \boldsymbol{d})\|_{L^{\infty}(0,\widetilde{T};X)} \leq \Big(K_{15}(\boldsymbol{r})$$

$$+ \tau_4 K_{16}(\boldsymbol{r}) \exp\left(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\right) + \rho_0^{-1/p_3} K_{17}(\boldsymbol{r})\Big)$$

$$\times (\|\overline{w}_2 - \overline{w}_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + |\overline{z}_2 - \overline{z}_1|), \quad (3.11)$$

where the positive functions $K_9 - K_{16}$ *are continuous.*

Proof . Taking into account the definition of $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$, (2.95), (2.96) and Lemma 3.1 from (2.87), (2.88), (2.104), (2.118) we get the estimates (3.8)–(3.11). ■

Lemma 3.6 *For any* $u \in W^{1,1}(0, T; X) \cap L^{\infty}(0, T; D(A))$ *the following estimate holds:*

$$\|Au\|_{L^{\infty}(0,\widetilde{T};X)} \leq K_{18}(\boldsymbol{r}) + \rho_0^{-1/p_3} K_{19}(\boldsymbol{r}) + \tau_4 K_{20}(\boldsymbol{r}) \exp\left(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\right), \quad (3.12)$$

where K_{17}, K_{18}, K_{19} are continuous functions.

Proof . It follows from (1.23), taking into account the definition of $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$ and Lemma 3.1. ■

3.2 Proof of continuous dependence

The aim of this section is to prove Theorem 1.2.

Let us first recall that in the Subsection 2.1 the problem $(P1)$ or (1.1)-(1.4) was shown to be equivalent to the system $(P9)$. We denote the related solutions to problem (1.1)-(1.4) corresponding to different data $\mathbf{d}_j \in \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), j = 1, 2$, with "tildas" and the corresponding one to a system (P9) with "overline bars", e.g.

$$\widetilde{u}_j = \widetilde{u}(\cdot, \mathbf{d}_j), \quad \widetilde{z}_j = \widetilde{z}(\mathbf{d}_j), \quad \widetilde{a}_{1,j} = \widetilde{a}_1(\cdot, \mathbf{d}_j), \; j = 1, 2,$$

and

$$\overline{w}_j := \overline{w}(\mathbf{d}_j), \quad \overline{z}_j := \overline{z}(\mathbf{d}_j).$$

Taking into account the notations introduced in the previous sections, i.e.

$$w(t) = Au(t) = W(\Phi[z], \Phi[w], \mathbf{d}, t) - f(t)z, \tag{3.13}$$

$$z = Z(\Phi[z], \Phi[w], \mathbf{d}), \tag{3.14}$$

$$\overline{w} = \Phi[w], \quad \overline{z} = \Phi[z], \tag{3.15}$$

we write down the corresponding equations for the increments related to (3.13)–(3.15):

$$\begin{aligned} \widetilde{w}_1(t) - \widetilde{w}_2(t) = A\widetilde{u}_1(t) - A\widetilde{u}_2(t) = W(\overline{z}_1, \overline{w}_1, \mathbf{d}_1, t) - W(\overline{z}_2, \overline{w}_2, \mathbf{d}_2, t) \\ + f_2(t)(\widetilde{z}_2 - \widetilde{z}_1) + (f_2(t) - f_1(t))\widetilde{z}_1, \end{aligned} \tag{3.16}$$

$$\widetilde{z}_1 - \widetilde{z}_2 = Z(\overline{z}_1, \overline{w}_1, \mathbf{d}_1) - Z(\overline{z}_2, \overline{w}_2, \mathbf{d}_2), \tag{3.17}$$

$$\overline{w}_1 - \overline{w}_2 = \Phi[\widetilde{w}_1 - \widetilde{w}_2], \quad \overline{z}_1 - \overline{z}_2 = \Phi[\widetilde{z}_1 - \widetilde{z}_2]. \tag{3.18}$$

In order to prove the continuous dependence on data we need to consider

$$\|a_0(\cdot)a_1(\cdot, \mathbf{d}_1) - a_0(t)a_1(\cdot, \mathbf{d}_2)\|_{L^1(0,\widetilde{T};\mathbb{R})} = \|a_0(\cdot)\widetilde{a}_{1,1} - a_0(t)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T},;\mathbb{R})}, \tag{3.19}$$

$$\begin{aligned} &\|u(\cdot, \mathbf{d}_1) - u(\cdot, \mathbf{d}_2)\|_{W^{1,1}(0,\widetilde{T};X)} + \|u(\cdot, \mathbf{d}_1) - u(\cdot, \mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};D(A))} \\ &\qquad = \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)} + \|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))}, \end{aligned} \tag{3.20}$$

$$\|z(\mathbf{d}_1) - z(\mathbf{d}_2)\|_X = \|\widetilde{z}_1 - \widetilde{z}_2\|_X. \tag{3.21}$$

Proof of Theorem 1.2 In order to continue the estimates (3.19), (3.20), (3.21), it is clear that we need firstly to assess $\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}$ and $\|\widetilde{z}_2 - \widetilde{z}_1\|_X$.

We divide the proof of this Theorem 1.2 into seven steps.

Step 1 (Estimate of $\|\widetilde{z}_2 - \widetilde{z}_1\|_X$)

The first step consists in evaluating (3.21) in terms of $\widetilde{z}_1 - \widetilde{z}_2$ and $\widetilde{w}_1 - \widetilde{w}_2$. From formula (3.17) we get:

$$\begin{aligned} \|\widetilde{z}_1 - \widetilde{z}_2\|_X \le \|Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_1, \widetilde{T}) - Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_2, \widetilde{T})\|_X. \\ + \|Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_2, \widetilde{T}) - Z(\Phi[\widetilde{z}_2], \Phi[\widetilde{w}_2], \mathbf{d}_2, \widetilde{T})\|_X \end{aligned} \tag{3.22}$$

or

$$\begin{aligned} &\|\widetilde{z}_1 - \widetilde{z}_2\|_X \le \|Z(\overline{w}_1, \overline{z}_1, \mathbf{d}_1, \widetilde{T}) - Z(\overline{w}_1, \overline{z}_1, \mathbf{d}_2, \widetilde{T})\|_X \\ &+ \|Z(\overline{w}_1, \overline{z}_1, \mathbf{d}_2, \widetilde{T}) - Z(\overline{w}_2, \overline{z}_2, \mathbf{d}_2, \widetilde{T})\|_X. \end{aligned} \tag{3.23}$$

Now we state one lemma which will be proved in the next section.

Lemma 3.7 *For any* $(\overline{w}, \overline{z}) \in \mathcal{K}(M_1, M_2, T)$, $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \boldsymbol{D}$ *the following estimate holds:*

$$\|Z(\overline{w},\overline{z}, \boldsymbol{d}_2, \widetilde{T}) - Z(\overline{w}, \overline{z}, \boldsymbol{d}_1, \widetilde{T})\|_X \leq \Big[K_{21}(\boldsymbol{r}) + \rho_0^{-1/p_3} K_{22}(\boldsymbol{r})$$
$$+ \tau_4 K_{23}(\boldsymbol{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \mathbf{dist}(\boldsymbol{d}_1, \boldsymbol{d}_2). \tag{3.24}$$

Here the positive functions K_{20}, K_{21}, K_{22} *are continuous and* τ_4 *is defined by (3.7).*

As a result, using (3.24), (3.15) and (3.9), from (3.23) we get

$$\|\widetilde{z}_1 - \widetilde{z}_2\|_X \leq \|\Phi\|_{X^*} \Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3} + \tau_4 K_{12}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big]$$
$$\times (\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + |\widetilde{z}_2 - \widetilde{z}_1|) + \Big[K_{20}(\mathbf{r}) + \rho_0^{-1/p_3} K_{21}(\mathbf{r})$$
$$+ \tau_4 K_{22}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$
$$=: K_{24}(\rho_0, r)(\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + |\widetilde{z}_2 - \widetilde{z}_1|_X) + K_{25}(\rho_0, r) \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \tag{3.25}$$

Step 2 (Estimate of $\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}$**)**

The second step consists in estimating (3.16):

$$\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^{p_1}(0,\widetilde{T};X)} \leq \|W(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_1) - \|W(\Phi[\widetilde{z}_2], \Phi[\widetilde{w}_2], \mathbf{d}_1)\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$+ \|f_2\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \|Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_2, \widetilde{T}) - Z(\Phi[\widetilde{z}_2], \Phi[\widetilde{w}_2], \mathbf{d}_2, \widetilde{T})\|_X$$
$$+ \|W(\Phi[\widetilde{z}_2], \Phi[\widetilde{w}_2], \mathbf{d}_1) - W(\Phi[\widetilde{z}_2], \Phi[\widetilde{w}_2], \mathbf{d}_2)\|_{L^{p_1}(0,\widetilde{T};X)}$$
$$+ \|f_2\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \|Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_1, \widetilde{T}) - Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_2, \widetilde{T})\|_X$$
$$+ \|f_2 - f_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \|Z(\Phi[\widetilde{z}_1], \Phi[\widetilde{w}_1], \mathbf{d}_1, \widetilde{T})\|_X, \tag{3.26}$$

Now we state one necessary lemma in order to continue the estimate (3.31), and the first one will be proved in the next section.

Lemma 3.8 *For any* $(\overline{w}, \overline{z}) \in \mathcal{K}(M_1, M_2, T)$, $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$ *the following estimate holds*

$$\|W(\overline{w},\overline{z}, \boldsymbol{d}_2) - W(\overline{w}, \overline{z}, \boldsymbol{d}_1)\|_{L^\infty(0,\widetilde{T};X)} \leq \Big[K_{26}(\boldsymbol{r}) + K_{27}(\boldsymbol{r})\rho_0^{-1/p_3}$$

$$+ \tau_4 K_{28}(\boldsymbol{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \ \mathbf{dist}(\boldsymbol{d}_1, \boldsymbol{d}_2). \tag{3.27}$$

Here the functions K_{25}, K_{26} *and* K_{27} *are positive and continuous and* τ_4 *is defined by (3.7).*

Thus, applying the formulas (3.27), (2.129), (3.15), (3.6), (3.9), (3.10) and definition of $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$ in (3.31), we get

$$\begin{aligned}
\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^{p_1}(0,\widetilde{T};X)} \leq\ & \rho_0^{-\min(1/(qp_1),1/p_3)} K_{13}(\mathbf{r}) \|\Phi\|_{X^*} (\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + |\widetilde{z}_2 - \widetilde{z}_1|) \\
& + r_2 \Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3} + \tau_4 K_{12}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \\
& \times \|\Phi\|_{X^*} (\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + |\widetilde{z}_2 - \widetilde{z}_1|) \\
& + \widetilde{T}^{1/p_1} \Big[K_{25}(\mathbf{r}) + K_{26}(\mathbf{r})\rho_0^{-1/p_3} + \tau_4 K_{27}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \\
& \times \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2) + r_2 \Big[K_{20}(\mathbf{r}) + \rho_0^{-1/p_3} K_{21}(\mathbf{r}) \\
& + \tau_4 K_{22}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big)\Big] \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2) \\
& + \|f_2 - f_1\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} \Big(K_5(\mathbf{r}) + \tau_4 K_6(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big) \\
& + \rho_0^{-1/p_3} K_7(\mathbf{r})\Big) =: K_{29}(\rho_0, r)(\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + |\widetilde{z}_2 - \widetilde{z}_1|_X) \\
& + K_{30}(\rho_0, r) \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2).
\end{aligned} \tag{3.28}$$

Exploiting the relations (3.16), (3.17), we deduce that in order to find the estimates of $\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}$ and $\|\widetilde{z}_2 - \widetilde{z}_1\|_X$, we must solve the system of two inequalities consisting of (3.25) and (3.28).

Step 3 (Solving of the system for $\|\widetilde{z}_2 - \widetilde{z}_1\|_X$ and $\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}$)

We write the system (3.25) and (3.28) in a more compact form:

$$(S1)\begin{cases} \|\widetilde{z}_2-\widetilde{z}_1\|_X \le K_{23}(\rho_0,r)(\|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}+|\widetilde{z}_2-\widetilde{z}_1|_X)+K_{24}(\rho_0,r)\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2), \\ \|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} \le K_{28}(\rho_0,r)(\|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}+|\widetilde{z}_2-\widetilde{z}_1|_X) \\ \qquad\qquad +K_{29}(\rho_0,r)\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \end{cases}$$

For the sake of simplicity, for now, we omit the dependence of (ρ_0, r) in functions K_{ij} from the system above. Let us note that the system (S1) implies

$$(1-K_{23}-K_{28})(\|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}+|\widetilde{z}_2-\widetilde{z}_1|_X) \le (K_{24}+K_{29})\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \quad (3.29)$$

Owing to the behavior of functions K_{23}, K_{28} from (3.25), (3.28) for (*large enough*) ρ_0, we have

$$1-K_{23}-K_{28}>0 \iff K_{23}+K_{28}<1.$$

This is true because the functions K_{23} and K_{28} in both our Cases 1 and 2 of measure tend to 0 as $\rho_0 \to +\infty$. As a result, from (3.29) we get the estimate

$$\|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}+|\widetilde{z}_2-\widetilde{z}_1|_X \le [1-K_{23}-K_{28}]^{-1}(K_{24}+K_{29})\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \quad (3.30)$$

Therefore from $(S1)$ we determine the final estimates

$$\begin{cases} \|\widetilde{z}_2-\widetilde{z}_1\|_X \le \Big[K_{23}[1-K_{23}-K_{28}]^{-1}(K_{24}+K_{29})+K_{24}\Big]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2), \\ \|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} \le \Big[K_{28}[1-K_{23}-K_{28}]^{-1}(K_{24}+K_{29})+K_{29}\Big]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \end{cases}$$

Step 4 (Estimate of $\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)}$)

Likewise to the Step 3 we estimate:

$$\|\widetilde{w}_1-\widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)} \le \|W(\Phi[\widetilde{z}_1],\Phi[\widetilde{w}_1],\mathbf{d}_1)-\|W(\Phi[\widetilde{z}_2],\Phi[\widetilde{w}_2],\mathbf{d}_1)\|_{L^\infty(0,\widetilde{T};X)}$$

$$+\|f_2\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|Z(\Phi[\widetilde{z}_1],\Phi[\widetilde{w}_1],\mathbf{d}_2,\widetilde{T})-Z(\Phi[\widetilde{z}_2],\Phi[\widetilde{w}_2],\mathbf{d}_2,\widetilde{T})\|_X$$
$$+\|W(\Phi[\widetilde{z}_2],\Phi[\widetilde{w}_2],d_1)-W(\Phi[\widetilde{z}_2],\Phi[\widetilde{w}_2],d_2)\|_{L^\infty(0,\widetilde{T};X)}$$
$$+\|f_2\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|Z(\Phi[\widetilde{z}_1],\Phi[\widetilde{w}_1],\mathbf{d}_1,\widetilde{T})-Z(\Phi[\widetilde{z}_1],\Phi[\widetilde{w}_1],\mathbf{d}_2,\widetilde{T})\|_X$$
$$+\|f_2-f_1\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|Z(\Phi[\widetilde{z}_1],\Phi[\widetilde{w}_1],\mathbf{d}_1,\widetilde{T})\|_X$$
$$\leq\bigg\{K_{14}(\mathbf{r})+\tau_4K_{15}(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)+\rho_0^{-1/p_3}K_{16}(\mathbf{r})$$
$$+K_1(T)r_2\bigg[\rho_0^{-1/p_3}K_{11}(\mathbf{r})+\tau_4K_{12}(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)\bigg]\bigg\}$$
$$\times\|\Phi\|_{X^*}(\|\widetilde{w}_2-\widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)}+|\widetilde{z}_2-\widetilde{z}_1|)$$
$$+\bigg\{K_{25}(\mathbf{r})+K_{26}(\mathbf{r})\rho_0^{-1/p_3}+\tau_4K_{27}(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)$$
$$+K_1(T)r_2\bigg[K_{20}(\mathbf{r})+\rho_0^{-1/p_3}K_{21}(\mathbf{r})$$
$$+\tau_4K_{22}(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)\bigg]\bigg\}\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)$$
$$+K_1(T)\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}$$
$$\times\bigg[K_5(\mathbf{r})+\tau_4K_6(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)+\rho_0^{-1/p_3}K_7(\mathbf{r})\bigg].\tag{3.31}$$

Now, applying the formula (3.30) to (3.31), we obtain

$$\|\widetilde{w}_1-\widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)}\leq\bigg[K_{31}(r)+\tau_4K_{32}(\mathbf{r})\exp\bigg(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\bigg)$$
$$+\rho_0^{-1/p_3}K_{33}(\mathbf{r})\bigg]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2).\tag{3.32}$$

Step 5 (Estimate of $\|a_0(\cdot)\widetilde{a}_{1,1}-a_0(\cdot)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})}$)

This step consists in the evaluating of (3.19) in terms of $\widetilde{z}_1-\widetilde{z}_2$ and $\widetilde{w}_1-\widetilde{w}_2$. Taking into account (3.19) and Lemma 3.1, we get:

$\|a_0(\cdot)\widetilde{a}_{1,1}-a_0(\cdot)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})}$

$$\leq \left\|\frac{k_1'(t)}{k_1(t)} - \frac{k_2'(t)}{k_2(t)}\right\|_{L^1(0,\widetilde{T})} + \left\|\frac{a_0(t)\Phi[g_2(t)]}{k_2(t)} - \frac{a_0(t)\Phi[g_1(t)]}{k_1(t)}\right\|_{L^1(0,\widetilde{T})}$$

$$+ \left\|\Big(\frac{a_0(t)}{k_2(t)} - \frac{a_0(t)}{k_1(t)}\Big)\overline{w}_1(t)\right\|_{L^1(0,\widetilde{T})} + \left\|\Big(\frac{a_0(t)f_2(t)}{k_2(t)} - \frac{a_0(t)f_1(t)}{k_1(t)}\Big)\overline{z}_1\right\|_{L^1(0,\widetilde{T})}$$

$$+ \left\|\frac{a_0(t)}{k_2(t)}\left(\overline{w}_2(t) - \overline{w}_1(t)\right)\right\|_{L^1(0,\widetilde{T})} + \left\|\frac{a_0(t)f_2(t)}{k_2(t)}(\overline{z}_2 - \overline{z}_1)\right\|_{L^1(0,\widetilde{T})}$$

$$\leq \left\|\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + \|\Phi\|_{X^*} r_8 \widetilde{T}^{1/p_1}\|g_1 - g_2\|_{L^\infty(0,\widetilde{T};X)}$$

$$+ \Big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1} r_3 + M_1 + r_2 M_2\Big)\left\|\frac{a_0}{k_1} - \frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}$$

$$+ r_8 M_2\|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} + r_8\left\|\overline{w}_2 - \overline{w}_1\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} + r_2 r_8|\overline{z}_2 - \overline{z}_1|. \qquad (3.33)$$

Then, recalling the notations (3.18), from (3.33) we obtain

$$\|a_0(\cdot)\widetilde{a}_{1,1} - a_0(\cdot)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})}$$

$$\leq \max(1, \|\Phi\|_{X^*} r_8 T^{1/p_1}, \big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1} r_3 + M_1 + r_2 M_2\big), r_8 M_2)$$

$$\times \left(\left\|\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + \|g_1 - g_2\|_{L^\infty(0,\widetilde{T};X)} + \left\|\frac{a_0}{k_1} - \frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right.$$

$$\left. + \|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\right) + r_8 \max(1, r_2)\|\Phi\|_{X^*}$$

$$\times (\|\widetilde{w}_2 - \widetilde{w}_1\|_{L^{p_1}(0,\widetilde{T};X)} + \|\widetilde{z}_2 - \widetilde{z}_1\|_X). \qquad (3.34)$$

Now, substituting (3.30) into (3.34), we get the final estimate in the form:

$$\|a_0(\cdot)\widetilde{a}_{1,1} - a_0(\cdot)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})}$$

$$\leq \max(1, \|\Phi\|_{X^*} r_8 \widetilde{T}^{1/p_1}, \big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1} r_3 + M_1 + r_2 M_2\big), r_8 M_2)$$

$$\times \left(\left\|\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})} + \|g_1 - g_2\|_{L^\infty(0,\widetilde{T};X)} + \left\|\frac{a_0}{k_1} - \frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right.$$

$$\left. + \|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\right) + r_8 \max(1, r_2)\|\Phi\|_{X^*}$$

$$\times [1 - K_{23} - K_{28}]^{-1}(K_{24} + K_{29})\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$=: \Big[K_{34}(\mathbf{r}) + \rho_0^{-1/p_3} K_{35}(\mathbf{r}) + K_{36}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big] \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \tag{3.35}$$

Step 6 (Estimate of $\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))} + \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)}$)

This step consists in estimating of (3.20) in terms of $\widetilde{z}_1 - \widetilde{z}_2$ and $\widetilde{w}_1 - \widetilde{w}_2$. First we consider

$$\begin{aligned}\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))} &= \|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};X)} + \|A\widetilde{u}_1 - A\widetilde{u}_2\|_{L^\infty(0,\widetilde{T};X)} \\ &\le (\|A^{-1}\|_{\mathcal{L}(X)} + 1)\|A\widetilde{u}_1 - A\widetilde{u}_2\|_{L^\infty(0,\widetilde{T};X)}. \end{aligned} \tag{3.36}$$

Using the equation (1.1), we get

$$\begin{aligned} D_t\widetilde{u}_1 - D_t\widetilde{u}_2 = a_0(t)[A\widetilde{u}_1 - A\widetilde{u}_2 + (\widetilde{a}_{1,1} - \widetilde{a}_{1,2})\widetilde{u}_1 + \widetilde{a}_{1,2}(\widetilde{u}_1 - \widetilde{u}_2) + (f_1(t) - f_2(t))z_1 \\ + f_2(t)(z_1 - z_2) + g_1(t) - g_2(t)]. \end{aligned} \tag{3.37}$$

As a consequence, using (3.37), we derive

$$\begin{aligned} \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)} &= \|\widetilde{u}_1 - \widetilde{u}_2\|_{L^1(0,\widetilde{T};X)} + \|D_t\widetilde{u}_1 - D_t\widetilde{u}_2\|_{L^1(0,\widetilde{T};X)} \\ &\le \|\widetilde{u}_1 - \widetilde{u}_2\|_{L^1(0,\widetilde{T};X)} + \|a_0(A\widetilde{u}_1 - A\widetilde{u}_2)\|_{L^1(0,\widetilde{T};X)} + \|a_0(a_{1,1} - a_{1,2})\widetilde{u}_1\|_{L^1(0,\widetilde{T};X)} \\ &+ \|a_0 a_{1,2}(\widetilde{u}_1 - \widetilde{u}_2)\|_{L^1(0,\widetilde{T};X)} + \|a_0(f_1 - f_2)\|_{L^1(0,\widetilde{T};X)}\|\widetilde{z}_1\|_X \\ &+ \|a_0 f_2\|_{L^1(0,\widetilde{T};X)}\|\widetilde{z}_1 - \widetilde{z}_2\|_X + \|a_0(g_1 - g_2)\|_{L^1(0,\widetilde{T};X)}. \end{aligned} \tag{3.38}$$

Taking into account the definition of $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$, Lemma 3.1, (3.34) and (3.3), from (3.36) and (3.38) we have

$$\begin{aligned} &\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))} + \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)} \\ &\le (\|A^{-1}\|_{\mathcal{L}(X)} + 1)\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)} + \widetilde{T}\|A^{-1}\|_{\mathcal{L}(X)}\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)} \\ &+ \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)} + \|a_0(a_{1,1} - a_{1,2})\|_{L^1(0,\widetilde{T};\mathbb{R})} T\|A^{-1}\|_{\mathcal{L}(X)} \end{aligned}$$

$$\times \|A\widetilde{u}_1\|_{L^\infty(0,\widetilde{T};X)} + T\|a_0a_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})}\|A^{-1}\|_{\mathcal{L}(X)}\|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)}$$

$$+ \|a_0\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\|g_1 - g_2\|_{L^{p_1}(0,\widetilde{T};X)} + \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}\|f_1 - f_2\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|\widetilde{z}_1\|_X$$

$$+ \|a_0\|_{L^1(0,\widetilde{T};X)}\|f_2\|_{L^\infty(0,\widetilde{T};X)}\|\widetilde{z}_1 - \widetilde{z}_2\|_X$$

$$\le \Big(1 + \|A^{-1}\|_{\mathcal{L}(X)} + T\|A^{-1}\|_{\mathcal{L}(X)} + \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})} + TK_3(\mathbf{r})\|A^{-1}\|_{\mathcal{L}(X)}\Big)$$

$$\times \|\widetilde{w}_1 - \widetilde{w}_2\|_{L^\infty(0,\widetilde{T};X)} + \Big[K_{33}(\mathbf{r}) + \rho_0^{-1/p_3}K_{34}(\mathbf{r}) + K_{35}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]$$

$$\times \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)T\|A^{-1}\|_{\mathcal{L}(X)}\|A\widetilde{u}_1\|_{L^\infty(0,\widetilde{T};X)} + \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}\|f_1 - f_2\|_{L^\infty(0,\widetilde{T};\mathbb{R})}\|\widetilde{z}_1\|_X$$

$$+ \|a_0\|_{L^1(0,\widetilde{T};X)}\|f_2\|_{L^\infty(0,\widetilde{T};X)}\|\widetilde{z}_1 - \widetilde{z}_2\|_X + \|a_0\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\|g_1 - g_2\|_{L^{p_1}(0,\widetilde{T};X)}. \tag{3.39}$$

We now apply the results of Step 3, (3.32), (3.12), (3.5) to (3.39):

$$\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))} + \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)}$$

$$\le \Big(1 + \|A^{-1}\|_{\mathcal{L}(X)} + \widetilde{T}\|A^{-1}\|_{\mathcal{L}(X)} + \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})} + \widetilde{T}K_3(\mathbf{r})\|A^{-1}\|_{\mathcal{L}(X)}\Big)$$

$$\times \Big[K_{30}(r) + \tau_4K_{31}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_{31}(\mathbf{r})\Big]\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$+ \Big[K_{33}(\mathbf{r}) + \rho_0^{-1/p_3}K_{34}(\mathbf{r}) + K_{35}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\widetilde{T}\|A^{-1}\|_{\mathcal{L}(X)}$$

$$\times [K_{17}(\mathbf{r}) + \rho_0^{-1/p_3}K_{18}(\mathbf{r}) + \tau_4K_{19}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)]\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$\times \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2) + \|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}K_1(T)\|f_1 - f_2\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\Big(K_5(\mathbf{r})$$

$$+ \tau_4K_6(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_7(\mathbf{r})\Big) + \|a_0\|_{L^1(0,\widetilde{T};X)}K_1(T)r_2$$

$$\times \Big[K_{23}[1 - K_{23} - K_{28}]^{-1}(K_{24} + K_{29}) + K_{24}\Big]\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$+ \|a_0\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\|g_1 - g_2\|_{L^{p_1}(0,\widetilde{T};X)}. \tag{3.40}$$

Step 7(Final estimates)

As a result, from (3.35), (3.40), (3.31), we conclude that

$$
\begin{aligned}
&\|a_0(\cdot)\widetilde{a}_{1,1} - a_0(\cdot)\widetilde{a}_{1,2}\|_{L^1(0,\widetilde{T};\mathbb{R})} \\
&\qquad \le \Big[K_{33}(\mathbf{r}) + \rho_0^{-1/p_3}K_{34}(\mathbf{r}) + K_{35}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2), \\
&\|\widetilde{z}_1 - \widetilde{z}_2\|_X \le \Big[K_{37}(\mathbf{r}) + \rho_0^{-1/p_3}K_{38}(\mathbf{r}) + K_{39}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2), \\
&\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^\infty(0,\widetilde{T};D(A))} + \|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{1,1}(0,\widetilde{T};X)} \\
&\qquad \le \Big[K_{40}(\mathbf{r}) + \rho_0^{-1/p_3}K_{41}(\mathbf{r}) + K_{42}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2),
\end{aligned}
$$

where $K_{33} - K_{41}$ are positive continuous functions. Consequently, inequalities (EC) are satisfied. ■

3.3 Estimates of the increments of Q^{-1}, Z, W for different data

The aim of this section is to estimate the increments of Q^{-1}, Z, W for different data. For this purpose, we need the following lemma.

Lemma 3.9 *For any* $(\overline{w}, \overline{z}, \boldsymbol{d}_i) \in \mathcal{K}(M_1, M_2, T) \times \mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}}), i = 1, 2,$ *the following inequality holds:*

$$
\begin{aligned}
&\Big\|\exp\Big(\int_s^t\Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big) \\
&\qquad - \exp\Big(\int_s^t\Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\overline{w}(\sigma) - \frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)\Big\|_X \\
&\qquad \le K_{43}(\boldsymbol{r})\exp\Big(K_2(\boldsymbol{r})\int_s^t a_0(\sigma)d\sigma\Big)\mathbf{dist}(\boldsymbol{d}_1, \boldsymbol{d}_2),
\end{aligned}
\tag{3.41}
$$

where K_{42} *is a positive continuous function and* K_2 *is defined in* (3.1).

Proof . Using the relation (2.85) and proceeding similarly to (2.84), with

$$
\begin{aligned}
&y - x\\
&= \int_s^t \Big[(\overline{k}_1(\sigma) - \overline{k}_2(\sigma) + \overline{w}(\sigma) \left(\frac{a_0(\sigma)}{k_1(\sigma)} - \frac{a_0(\sigma)}{k_2(\sigma)} \right) + a_0(\sigma)\overline{z} \left(\frac{f_2(\sigma)}{k_2(\sigma)} - \frac{f_1(\sigma)}{k_1(\sigma)} \right) \Big]\, d\sigma,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_2(s,t) :=& \exp\Big(\int_s^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)} \overline{z} \Big) d\sigma \Big)\\
&- \exp\Big(\int_s^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \overline{z} \Big) d\sigma \Big)\\
&= \int_s^t \Big(\overline{k}_1(\sigma) - \overline{k}_2(\sigma) + \overline{w}(\sigma) \left(\frac{a_0(\sigma)}{k_1(\sigma)} - \frac{a_0(\sigma)}{k_2(\sigma)} \right)\\
&+ \overline{z} \left(\frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)} - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \right) \Big) d\sigma\\
&\times \int_0^1 \exp\Big((1-\tau) \int_s^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)} \overline{z} \Big) d\sigma \Big)\\
&\times \exp\Big(\tau \int_s^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \overline{z} \Big) d\sigma \Big) d\tau, \tau \in (0,1). \quad (3.42)
\end{aligned}
$$

Let us notice that from (2.32) we have

$$
\begin{aligned}
&\int_s^t \Big| \overline{k}_i(\sigma) - \frac{a_0(\sigma)}{k_i(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_i(\sigma)}{k_i(\sigma)} \overline{z} \Big| d\sigma \le K_2(\mathbf{r}) \int_s^t a_0(\sigma)\, d\sigma\\
&+ \left\| \frac{k_i'}{k_i} \right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\| \frac{a_0}{k_i} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}, \quad i = 1, 2. \qquad (3.43)
\end{aligned}
$$

As a result, making use of (3.43), we get

$$
\begin{aligned}
&\exp\Big((1-\tau) \int_s^t \Big| \overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)} \overline{z} \Big| d\sigma \Big)\\
&\le \exp\Big((1-\tau) K_2(\mathbf{r}) \int_s^t a_0(\sigma) d\sigma \Big) \exp\Big((1-\tau) \Big(\left\| \frac{k_2'}{k_2} \right\|_{L^1(0,\widetilde{T};\mathbb{R})} + M_1 \left\| \frac{a_0}{k_2} \right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})} \Big) \Big)
\end{aligned}
$$

and

$$
\exp\Big(\tau \int_s^t \Big| \overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\, \overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \overline{z} \Big| \Big)
$$

$$\leq \exp\left(\tau K_2(\mathbf{r})\int_s^t a_0(\sigma)d\sigma\right)\exp\left(\tau\left(\left\|\frac{k_1'}{k_1}\right\|_{L^1(0,T;\mathbb{R})}+M_1\left\|\frac{a_0}{k_1}\right\|_{L^{p_1'}(0,T;\mathbb{R})}\right)\right).$$

Taking into account the last two inequalities, from (3.42) we achieve

$$\begin{aligned}\|I_2(s,t)\|_X \leq \int_s^t &\Big|\overline{k}_1(\sigma)-\overline{k}_2(\sigma)\\ &+\overline{w}(\sigma)\left(\frac{a_0(\sigma)}{k_1(\sigma)}-\frac{a_0(\sigma)}{k_2(\sigma)}\right)+\overline{z}\left(\frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}-\frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\right)\Big|d\sigma\\ &\times\exp\left(K_2(\mathbf{r})\int_s^t a_0(\sigma)d\sigma\right)\exp\left(\left\|\frac{k_2'}{k_2}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right)\\ &\times\exp\left(\left\|\frac{k_1'}{k_1}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k_1}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\right).\end{aligned}$$

Than, proceeding with Hölder inequality in the second term, we conclude that

$$\begin{aligned}\|I_2(s,t)\|_X \leq \Bigg(&\|\overline{k}_1-\overline{k}_2\|_{L^1(0,\widetilde{T};\mathbb{R})}+M_1\left\|\frac{a_0}{k_1}-\frac{a_0}{k_2}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\\ &+M_2\left\|\frac{a_0f_2}{k_2}-\frac{a_0f_1}{k_1}\right\|_{L^1(0,\widetilde{T};\mathbb{R})}\Bigg)\\ &\times\exp(2T^{1/p_1'}r_7+2M_1r_8)\exp\left(K_2(\mathbf{r})\int_s^t a_0(\sigma)d\sigma\right).\end{aligned}$$

By means of Lemma 3.1, we finally deduce the following estimate

$$\begin{aligned}\|I_2(s,t)\|_X \leq \Bigg[&T^{1/p_1'}\left\|\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}+r_8T^{1/p_1}\|\Phi\|_{X^*}\|g_1-g_2\|_{L^\infty(0,\widetilde{T};X)}\\ &+M_2r_8\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}+\left(T^{1/p_1}r_3\|\Phi\|_{X^*}+M_1+r_2M_2\right)\\ &\times\left\|\frac{a_0}{k_2}-\frac{a_0}{k_1}\right\|_{L^{p_1'}(0,\widetilde{T};\mathbb{R})}\Bigg]\exp(2T^{1/p_1'}r_7+2M_1r_8)\exp\left(K_2(\mathbf{r})\int_s^t a_0(\sigma)d\sigma\right).\quad\blacksquare\end{aligned}$$

3.3.1 Estimates of $Q^{-1}(\overline{z},\overline{w},\mathbf{d}_2)-Q^{-1}(\overline{z},\overline{w},\mathbf{d}_1)$

First we consider the relation

$$\begin{aligned}Q^{-1}(\overline{z},\overline{w},\mathbf{d}_2,T)-Q^{-1}(\overline{z},\overline{w},\mathbf{d}_1,T)&=Q^{-1}(\overline{z},\overline{w},\mathbf{d}_1,T)\\ &\times[Q(\overline{z},\overline{w},\mathbf{d}_1,T)-Q(\overline{z},\overline{w},\mathbf{d}_2,T)]Q^{-1}(\overline{z},\overline{w},\mathbf{d}_2,T).\end{aligned}\tag{3.44}$$

The definition (2.47) implies that

$$Q(\overline{z},\overline{w},\mathbf{d}_1,T)-Q(\overline{z},\overline{w},\mathbf{d}_2,T)=\lambda(\mathbf{d}_2)-\lambda(\mathbf{d}_1)+\widetilde{R}(\overline{z},\overline{w},\mathbf{d}_1,T)-\widetilde{R}(\overline{z},\overline{w},\mathbf{d}_2,T), \quad (3.45)$$

where λ and $\widetilde{R}$ are defined by (1.18), (2.44). Taking (2.5) into account, we obtain

$$\widetilde{R}(\overline{z},\overline{w},\mathbf{d}_1,T)-\widetilde{R}(\overline{z},\overline{w},\mathbf{d}_2,T)=\sum_{i=1}^{4}\widetilde{R}_i(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,T), \quad (3.46)$$

where

$$\begin{aligned}\widetilde{R}_1(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,T) &:= (f_1(0)-f_2(0))\\ &\times\int_0^T\varphi(t)\exp\Big(\int_0^t\Big(\overline{k}_1(\sigma)-\frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)\\ &\times\exp\Big(\int_0^t a_0(\sigma)d\sigma A\Big)d\mu(t), \end{aligned} \quad (3.47)$$

$$\begin{aligned}\widetilde{R}_2(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,T) &:= -\int_0^T\varphi(t)d\mu(t)\int_0^t\Big[f_2'(s)-f_1'(s)+\frac{f_1(s)k_1'(s)}{k_1(s)}-\frac{f_2(s)k_2'(s)}{k_2(s)}\\ &+a_0(s)\Big(\frac{f_2(s)\Phi[g_2(s)]}{k_2(s)}-\frac{f_1(s)\Phi[g_1(s)]}{k_1(s)}+\Big(\frac{f_2(s)}{k_2(s)}-\frac{f_1(s)}{k_1(s)}\Big)\overline{w}_2(s)\\ &+\Big(\frac{f_2^2(s)}{k_2(s)}-\frac{f_1^2(s)}{k_1(s)}\Big)\overline{z}_2\Big)\Big]\exp\Big(\int_s^t\Big(\overline{k}_2(\sigma)-\frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big)\\ &\times\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds, \end{aligned} \quad (3.48)$$

$$\begin{aligned}\widetilde{R}_3(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,T) &:= -\int_0^T\varphi(t)d\mu(t)\int_0^t\Big(f_1'(s)-\frac{f_1(s)k_1'(s)}{k_1(s)}\\ &+a_0(s)\Big[\frac{f_1(s)\Phi[g_1(s)]}{k_1(s)}+\frac{f_1(s)}{k_1(s)}\,\overline{w}(s)+\frac{f_1^2(s)}{k_1(s)}\overline{z}\Big]\Big)\\ &\times\Big[\exp\Big(\int_s^t\Big(\overline{k}_2(\sigma)-\frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big)\\ &-\exp\Big(\int_s^t\Big(\overline{k}_1(\sigma)-\frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)\Big]\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)ds, \end{aligned} \quad (3.49)$$

$$\widetilde{R}_4(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,T) := f_2(0)\int_0^T\varphi(t)\Big[\exp\Big(\int_0^t\Big(\overline{k}_1(\sigma)-\frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)$$

$$- \exp\left(\int_0^t\left(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\right)d\sigma\right)\right] \exp\left(\int_0^t a_0(\sigma)d\sigma A\right)d\mu(t). \tag{3.50}$$

Lemma 3.1 *For any* $(\overline{w},\overline{z})\in\mathcal{K}(M_1,M_2,T)$ *and* $\boldsymbol{d}_1,\boldsymbol{d}_2\in\mathbf{D}(\mathbf{r},\widetilde{\mathbf{T}})$ *the following estimate holds:*

$$\|Q^{-1}(\overline{w},\overline{z},\boldsymbol{d}_2,\widetilde{T}) - Q^{-1}(\overline{w},\overline{z},\boldsymbol{d}_1,\widetilde{T})\|_{\mathcal{L}(X)}$$
$$\le \left[K_{44}(\boldsymbol{r}) + \tau_4 K_{45}(\boldsymbol{r})\exp\left(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\right) + \rho_0^{-1/p_3}K_{46}(\boldsymbol{r})\right]\mathbf{dist}(\boldsymbol{d}_1,\boldsymbol{d}_2). \tag{3.51}$$

Here K_{43}, K_{44}, K_{45} *are continuous positive functions.*

Proof . The equalities (3.44), (3.45), (3.46) and (2.48) imply

$$\|Q^{-1}(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T}) - Q^{-1}(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T})\|_{\mathcal{L}(X)} \le \|Q^{-1}(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T})\|_{\mathcal{L}(X)}$$
$$\times\|Q(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T}) - Q(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)}\|Q^{-1}(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)}$$
$$\le \frac{4}{r_1^2}\left(\int_0^T |f_2(t)-f_1(t)||\varphi(t)|d\mu(t) + \sum_{i=1}^4\|\widetilde{R}_i(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)}\right). \tag{3.52}$$

Therefore, we proceed to estimating each $\widetilde{R}_i$, $i = 1,2,3,4$.

First we estimate the operator $\widetilde{R}_1$. Using (2.129), (2.34), (3.2) in the Case 1 and (2.34), (2.51), (2.129), (3.2) in the Case 2, from (3.47) we get the common estimate for both cases:

$$\|\widetilde{R}_1(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)} \le K_1(T)\|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\tau_1(\rho_0,\mathbf{r},\widetilde{T})$$
$$\times\exp(T^{1/p_1'}r_7 + M_1 r_8). \tag{3.53}$$

Now we consider $\widetilde{R}_2$ defined in (3.48). Analogously to (2.54), we estimate separately the integral

$$I_1(t) := \int_0^t\left|f_2'(s) - f_1'(s) + \frac{f_1(s)k_1'(s)}{k_1(s)} - \frac{f_2(s)k_2'(s)}{k_2(s)} + a_0(s)\left(\frac{f_2(s)\Phi[g_2(s)]}{k_2(s)}\right.\right.$$

$$
\begin{aligned}
&- \frac{f_1(s)\Phi[g_1(s)]}{k_1(s)} + \Big(\frac{f_2(s)}{k_2(s)} - \frac{f_1(s)}{k_1(s)}\Big)\overline{w}_2(s) + \Big(\frac{f_2^2(s)}{k_2(s)} - \frac{f_1^2(s)}{k_1(s)}\Big)\overline{z}_2\Big)\Big| \\
&\times \exp\Big(\int_s^t \Big|\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big| d\sigma\Big) \\
&\times \Big\|\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)\Big\|_{\mathcal{L}(X)} ds \\
&\le \rho_0^{-1/p_3}\frac{1}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\Big(\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,t;\mathbb{R})}\Big[\|f_2' - f_1'\|_{L^{p_1}(0,t;\mathbb{R})} \\
&+ \Big\|\frac{f_1k_1'}{k_1} - \frac{f_2k_2'}{k_2}\Big\|_{L^{p_1}(0,t;\mathbb{R})}\Big] + \Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,t;\mathbb{R})}\Big[\|\Phi\|_X\Big\|\frac{f_2g_2}{k_2} - \frac{f_1g_1}{k_1}\Big\|_{L^{p_1}(0,t;X)} \\
&+ \Big\|\frac{f_2}{k_2} - \frac{f_1}{k_1}\Big\|_{L^{\infty}(0,t;\mathbb{R})} M_1 + \Big\|\frac{f_2^2}{k_2} - \frac{f_1^2}{k_1}\Big\|_{L^{p_1}(0,t;\mathbb{R})} M_2\Big]\Big) \exp(T^{1/p_1'}r_7 + M_1r_8).
\end{aligned}
$$

Applying Lemma 3.1, we get

$$
\begin{aligned}
I_1(t) &\le \rho_0^{-1/p_3}\frac{1}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\Big[\|f_2 - f_1\|_{W^{1,p_1}(0,t;\mathbb{R})} \\
&\times \Big\{\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,t;\mathbb{R})}(1 + K_1(T)r_7) + K_1(T)r_9\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,t;\mathbb{R})} \\
&\times \Big(\|\Phi\|_{X^*}T^{1/p_1}r_3 + M_1r_9 + 2r_2M_2T^{1/p_1}K_1(T)\Big)\Big\} \\
&+ K_1(T)r_2\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,t;\mathbb{R})}\Big\|\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\Big\|_{L^{p_1}(0,t;\mathbb{R})} \\
&+ \Big\|\frac{1}{k_2} - \frac{1}{k_1}\Big\|_{L^{\infty}(0,t;\mathbb{R})}\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,t;\mathbb{R})} K_1(T)r_2 \\
&\times \Big(\|\Phi\|_{X^*}T^{1/p_1}r_3 + M_1 + M_2T^{1/p_1}K_1(T)r_2\Big)\Big] \exp(T^{1/p_1'}r_7 + M_1r_8). \qquad (3.54)
\end{aligned}
$$

Therefore, the operator $\widetilde{R}_2$ can be assessed by using the formulas (2.57) and (3.2). We obtain

$$
\begin{aligned}
\|\widetilde{R}_2(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)} &\le \rho_0^{-1/p_3}\frac{\tau_2(\widetilde{T})}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\Big\{\|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \\
&\times \Big[\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\big(1 + K_1(\widetilde{T})r_7\big) + K_1(\widetilde{T})r_9\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}
\end{aligned}
$$

$$\times \big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3 + M_1 r_9 + 2r_2 M_2 \widetilde{T}^{1/p_1}K_1(\widetilde{T})\big)\Big]$$

$$+ \left\|\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} K_1(\widetilde{T})r_2 \left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{1}{k_2} - \frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}$$

$$\times \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} K_1(\widetilde{T})r_2\big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3 + M_1 + M_2\widetilde{T}^{1/p_1}K_1(\widetilde{T})r_2\big)\Bigg\}$$

$$\times \exp(T^{1/p_1'}r_7 + M_1 r_8). \tag{3.55}$$

In order to obtain the estimate of $\widetilde{R}_3$, firstly we treat the following integral

$$I_2(t) := \int_0^t \left| f_1'(s) - \frac{f_1(s)k_1'(s)}{k_1(s)} + a_0(s)\Big[\frac{f_1(s)\Phi[g_1(s)]}{k_1(s)} + \frac{f_1(s)}{k_1(s)}\,\overline{w}(s) + \frac{f_1^2(s)}{k_1(s)}\overline{z}\Big]\right|$$

$$\times \left\| \exp\left(\int_s^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\right)\right.$$

$$\left. - \exp\left(\int_s^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\right)\right\|_X$$

$$\times \left\|\exp\left(\int_s^T a_0(\sigma)d\sigma A\right)\right\|_{\mathcal{L}(X)} ds \le K_{42}(\mathbf{r})\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$\times \int_0^t \left| f_1'(s) - \frac{f_1(s)k_1'(s)}{k_1(s)} + a_0(s)\Big[\frac{f_1(s)\Phi[g_1(s)]}{k_1(s)} + \frac{f_1(s)}{k_1(s)}\,\overline{w}(s) + \frac{f_1^2(s)}{k_1(s)}\overline{z}\Big]\right|$$

$$\times \exp\left(-(\rho_0 - K_2(\mathbf{r}))\int_s^t a_0(\sigma)d\sigma\right)ds$$

$$\le \rho_0^{-1/p_3}\frac{r_2 K_{42}(\mathbf{r})}{[p_3(1 - K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$\times \left[\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,t;\mathbb{R})}\big(1 + K_1(T)r_7\big) + K_1(T)r_9\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,t;\mathbb{R})}\right.$$

$$\left.\times \Big(\|\Phi\|_{X^*}T^{1/p_1}r_3 + M_1 + T^{1/p_1}K_1(T)r_2 M_2.\Big)\right]. \tag{3.56}$$

Then, similarly to (2.54), using (2.34), (3.41), (2.54) in the Case 1 and (3.41), (2.34), (2.58), Lemma 3.1 in the Case 2, from (3.49) we get the common estimate for both cases:

$$\|\widetilde{R}_3(\overline{z},\overline{w},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_{\mathcal{L}(X)} \le \rho_0^{-1/p_3}\frac{r_2 K_{42}(\mathbf{r})\tau_2(\widetilde{T})}{[p_3(1 - K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$\times \left[\left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} (1 + K_1(\widetilde{T}) r_7) + K_1(\widetilde{T}) r_9 \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \right.$$

$$\left. \times \left(\|\Phi\|_{X^*} \widetilde{T}^{1/p_1} r_3 + M_1 + \widetilde{T}^{1/p_1} K_1(\widetilde{T}) r_2 M_2 \right) \right]. \tag{3.57}$$

Exploiting (3.41), (2.34) in the Case 1 and (3.41), (2.34), (2.51) in the Case 2, from (3.50) we deduce

$$\|\widetilde{R}_4(\overline{z}, \overline{w}, \mathbf{d}_1, \mathbf{d}_2, \widetilde{T})\|_{\mathcal{L}(X)} \leq K_1(\widetilde{T}) r_2 K_{42}(\mathbf{r}) \tau_1(\rho_0, \mathbf{r}, \widetilde{T}) \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \tag{3.58}$$

As a result, using (3.53), (3.55), (3.57), (3.58) in (3.52), we conclude:

$$\|Q^{-1}(\overline{w}, \overline{z}, \mathbf{d}_2, \widetilde{T}) - Q^{-1}(\overline{w}, \overline{z}, \mathbf{d}_1, \widetilde{T})\|_{\mathcal{L}(X)} \leq \frac{4}{r_1^2} \Big(K_1(T) \|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}$$

$$\times \int_0^T |\varphi(t)| d\mu(t) + K_1(\widetilde{T}) \|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \tau_1(\rho_0, \mathbf{r}, \widetilde{T}) \exp(T^{1/p_1'} r_7 + M_1 r_8)$$

$$+ \rho_0^{-1/p_3} \frac{\tau_2(\widetilde{T})}{[p_3(1 - K_2(r)/\rho_0)]^{1/p_3}} \Big\{ \|f_2 - f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}$$

$$\times \left[\left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \big(1 + K_1(\widetilde{T}) r_7\big) + K_1(\widetilde{T}) r_9 \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \right.$$

$$\left. \times \big(\|\Phi\|_{X^*} \widetilde{T}^{1/p_1} r_3 + M_1 r_9 + 2 r_2 M_2 \widetilde{T}^{1/p_1} K_1(\widetilde{T}) \big) \right]$$

$$+ \left\| \frac{k_1'}{k_1} - \frac{k_2'}{k_2} \right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})} K_1(T) r_2 \left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\| \frac{1}{k_2} - \frac{1}{k_1} \right\|_{L^\infty(0,\widetilde{T};\mathbb{R})}$$

$$\times \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} K_1(\widetilde{T}) r_2 \big(\|\Phi\|_{X^*} \widetilde{T}^{1/p_1} r_3 + M_1 + M_2 T^{1/p_1} K_1(\widetilde{T}) r_2 \big) \Big\}$$

$$\times \exp(T^{1/p_1'} r_7 + M_1 r_8) + \rho_0^{-1/p_3} \frac{r_2 K_{42}(\mathbf{r}) \tau_2(\widetilde{T})}{[p_3(1 - K_2(r)/\rho_0)]^{1/p_3}} \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2)$$

$$\times \left[\left\| \frac{1}{a_0^{1/p_3}} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} (1 + K_1(\widetilde{T}) r_7) + K_1(\widetilde{T}) r_9 \left\| a_0^{1/p_3'} \right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} \right.$$

$$\left. \times \left(\|\Phi\|_{X^*} \widetilde{T}^{1/p_1} r_3 + M_1 + \widetilde{T}^{1/p_1} K_1(T) r_2 M_2 \right) \right]$$

$$+ K_1(\widetilde{T}) r_2 K_{42}(\mathbf{r}) \tau_1(\rho_0, \mathbf{r}, \widetilde{T}) \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2) \Big).$$

We have thus proved an inequality of the form (3.51) as in the statement of the lemma

■

3.3.2 Estimates of $Z(\overline{z}, \overline{w}, \mathbf{d}_2) - Z(\overline{z}, \overline{w}, \mathbf{d}_1)$

The goal of this subsection is to prove the Lemma 3.7.

Let us remind from (2.17) that

$$Z(\overline{w}, \overline{z}, \mathbf{d}_2, T) - Z(\overline{w}, \overline{z}, \mathbf{d}_1, T) = \sum_{j=1}^{3} (\mathcal{Z}_j(\overline{w}, \overline{z}, \mathbf{d}_2, T) - \mathcal{Z}_j(\overline{w}, \overline{z}, \mathbf{d}_1, T))$$

Making use of the algebraic relation

$$ab - cd = a(b - d) + (a - c)d \tag{3.59}$$

and the definition of $\mathcal{Z}_1$, we get

$$\begin{aligned} \mathcal{Z}_1(\overline{w},\overline{z}, \mathbf{d}_2, T) - \mathcal{Z}_1(\overline{w}, \overline{z}, \mathbf{d}_1, T) &= Q^{-1}(\overline{w}, \overline{z}, \mathbf{d}_2, T)(Ah_2 - Ah_1) \\ &+ (Q^{-1}(\overline{w}, \overline{z}, \mathbf{d}_2, T) - Q^{-1}(\overline{w}, \overline{z}, \mathbf{d}_1, T))Ah_1. \end{aligned} \tag{3.60}$$

Similarly, using (3.59), we obtain

$$\begin{aligned} &\mathcal{Z}_2(\overline{w},\overline{z}, \mathbf{d}_2, T) - \mathcal{Z}_2(\overline{w}, \overline{z}, \mathbf{d}_1, T) = Q(\overline{w}, \overline{z}, \mathbf{d}_2, T)^{-1} \\ &\quad \times \int_0^T \varphi(t) \Big[\exp\Big(\int_0^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \overline{z} \Big) d\sigma \Big) \\ &\quad \times \exp\Big(\int_0^t a_0(\sigma) d\sigma A \Big) (Au_{0,1} - Au_{0,2}) \\ &\quad + \Big(\exp\Big(\int_0^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)} \overline{z} \Big) d\sigma \Big) \\ &\quad - \exp\Big(\int_0^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)} \overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)} \overline{z} \Big) d\sigma \Big) \Big) \\ &\quad \times \exp\Big(\int_0^t a_0(\sigma) d\sigma A \Big) Au_{0,2} \Big] d\mu(t) + (Q(\overline{w}, \overline{z}, \mathbf{d}_2, T)^{-1} - Q(\overline{w}, \overline{z}, \mathbf{d}_1, T)^{-1}) \end{aligned}$$

$$\times \int_0^T \varphi(t) \exp\Big(\int_0^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big) d\sigma\Big)$$

$$\times \exp\Big(\int_0^t a_0(\sigma) d\sigma A\Big) A u_{0,1} d\mu(t) =: \sum_{i=1}^{3} \widetilde{\mathcal{Z}}_{2,i}(\overline{w}, \overline{z}, \mathbf{d}_1, \mathbf{d}_2, T) \tag{3.61}$$

and

$$\mathcal{Z}_3(\overline{w},\overline{z}, \mathbf{d}_2, T) - \mathcal{Z}_3(\overline{w}, \overline{z}, \mathbf{d}_1, T) = Q(\overline{w}, \overline{z}, \mathbf{d}_2, T)^{-1}$$

$$\times \Big[\int_0^T \varphi(t) \Big(\int_0^t a_0(s) \exp\Big(\int_s^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big) d\sigma\Big)$$

$$\times A \exp\Big(\int_s^t a_0(\sigma) d\sigma A\Big)(g_2(s) - g_1(s)) ds$$

$$+ \int_0^t a_0(s) \Big[\exp\Big(\int_s^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big) d\sigma\Big)$$

$$- \exp\Big(\int_s^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big) d\sigma\Big)\Big]$$

$$\times A \exp\Big(\int_s^t a_0(\sigma) d\sigma A\Big) g_1(s) ds\Big) d\mu(t)\Big] + (Q(\overline{w}, \overline{z}, \mathbf{d}_2, T)^{-1} - Q(\overline{w}, \overline{z}, \mathbf{d}_1, T)^{-1})$$

$$\times \int_0^T \varphi(t) \Big(\int_0^t a_0(s) \exp\Big(\int_s^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big) d\sigma\Big)$$

$$\times A \exp\Big(\int_s^t a_0(\sigma) d\sigma A\Big) g_1(s) ds\Big) d\mu(t) =: \sum_{i=1}^{3} \widetilde{\mathcal{Z}}_{3,i}(\overline{w}, \overline{z}, \mathbf{d}_1, \mathbf{d}_2, T). \tag{3.62}$$

Proof of Lemma 3.7. From (3.60), using the estimates (2.48), (3.51), and the definition of $\mathbf{D}(\mathbf{r}, \widetilde{\mathbf{T}})$, we get

$$\|\mathcal{Z}_1(\overline{w},\overline{z}, \mathbf{d}_2, \widetilde{T}) - \mathcal{Z}_1(\overline{w}, \overline{z}, \mathbf{d}_1, \widetilde{T})\|_X \le \frac{2}{r_1}\, \|Ah_2 - Ah_1\|_X + r_6\Big[K_{43}(\mathbf{r})$$

$$+ \tau_4 K_{44}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big) + \rho_0^{-1/p_3} K_{45}(\mathbf{r})\Big] \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \tag{3.63}$$

We now estimate $\|\mathcal{Z}_2(\overline{w}, \overline{z}, \mathbf{d}_2, \widetilde{T}) - \mathcal{Z}_2(\overline{w}, \overline{z}, \mathbf{d}_1, \widetilde{T})\|_X$. From the representation (3.61), using (2.48), (2.51), (3.2), we get

$$\|\widetilde{\mathcal{Z}}_{2,1}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X \le \frac{2}{r_1}\ \tau_1(\rho_0,\mathbf{r},\widetilde{T})\exp(T^{1/p_1'}r_7+M_1r_8)\|Au_{0,1}-Au_{0,2}\|_X. \tag{3.64}$$

Applying the formulae (2.48), (3.41) to $\widetilde{\mathcal{Z}}_{2,2}$, we obtain

$$\|\widetilde{\mathcal{Z}}_{2,2}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X \le K_{42}(\mathbf{r})\tau_1(\rho_0,\mathbf{r},\widetilde{T})\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)r_5. \tag{3.65}$$

Using Lemma 2.2 and (3.51), we deduce

$$\begin{aligned}\|\widetilde{\mathcal{Z}}_{2,3}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X &\le r_5\Big[K_{43}(\mathbf{r})+\tau_4K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)\\ &\quad+\rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big]\tau_1(\rho_0,\mathbf{r},\widetilde{T})\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2).\end{aligned} \tag{3.66}$$

In conclusion, from (3.64), (3.65), (3.66) we get the following estimate:

$$\begin{aligned}&\|\mathcal{Z}_2(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T})-\mathcal{Z}_2(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T})\|_X\\ &\quad\le\frac{2}{r_1}\ \tau_1(\rho_0,\mathbf{r},\widetilde{T})\exp(T^{1/p_1'}r_7+M_1r_8)\|Au_{0,1}-Au_{0,2}\|_X+\Big(K_{42}(\mathbf{r})\tau_1(\rho_0,\mathbf{r},\widetilde{T})r_5\\ &\quad+r_5\Big[K_{43}(\mathbf{r})+\tau_4K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)+\rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big]\\ &\quad\times\tau_1(\rho_0,\mathbf{r},\widetilde{T})\Big)\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2).\end{aligned} \tag{3.67}$$

Let us assess the increments of $\mathcal{Z}_3$ in $\mathbf{d}$. Using the estimates (2.48), (2.62), from (3.62) we obtain

$$\begin{aligned}\|\widetilde{\mathcal{Z}}_{3,1}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X &\le \rho_0^{-1/p_3}\ \frac{2}{r_1}\ \tau_2(\widetilde{T})C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\\ &\quad\times\frac{\|a_0^{1/p_1}(g_2-g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}}\ \exp(T^{1/p_1'}r_7+M_1r_8).\end{aligned} \tag{3.68}$$

Taking the estimates (3.41) and (2.62) into account, we have

$$\|\widetilde{\mathcal{Z}}_{3,2}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X \le \rho_0^{-1/p_3}\ \frac{2}{r_1}\ \tau_2(\widetilde{T})C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$

$$\times \frac{r_4 K_{42}(\mathbf{r})}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}} \ \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \tag{3.69}$$

Proceeding similarly, using (3.51) and (2.62), we get

$$\begin{aligned} \|\widetilde{\mathcal{Z}}_{3,3}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,\widetilde{T})\|_X &\leq \rho_0^{-1/p_3}\tau_2(\widetilde{T}) \frac{\|a_0\|_{L^1(0,T;\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{r_4 C_{10}(\gamma,T)}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}} \\ &\times \Big[K_{43}(\mathbf{r}) + \tau_4 K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big] \\ &\times \exp(T^{1/p_1'}r_7 + M_1 r_8)\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \end{aligned} \tag{3.70}$$

Subsequently, from (3.68), (3.69), (3.70) we come to the estimate

$$\begin{aligned} &\|\mathcal{Z}_3(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T}) - \mathcal{Z}_3(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T})\|_X \\ &\leq \rho_0^{-1/p_3}\tau_2(\widetilde{T})C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \frac{1}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}} \\ &\times \Big\{\frac{2}{r_1}\ \|a_0^{1/p_1}(g_2-g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}\ \exp(T^{1/p_1'}r_7 + M_1 r_8) \\ &+ r_4\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)\Big(\frac{2}{r_1}\ \ K_{42}(\mathbf{r}) + \exp(T^{1/p_1'}r_7 + M_1 r_8) \\ &\times \Big[K_{43}(\mathbf{r}) + \tau_4 K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big]\Big)\Big\}. \end{aligned} \tag{3.71}$$

Finally, summing up (3.63), (3.67), (3.71), we get

$$\begin{aligned} &\|Z(\overline{w},\overline{z},\mathbf{d}_2,\widetilde{T}) - Z(\overline{w},\overline{z},\mathbf{d}_1,\widetilde{T})\|_X \leq \frac{2}{r_1}\ \|Ah_2 - Ah_1\|_X \\ &+ \frac{2}{r_1}\ \tau_1(\rho_0,\mathbf{r},\widetilde{T})\exp(T^{1/p_1'}r_7 + M_1 r_8)\|Au_{0,1} - Au_{0,2}\|_X + \Big\{r_6\Big[K_{43}(\mathbf{r}) \\ &+ \tau_4 K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big] + \Big(K_{42}(\mathbf{r})\tau_1(\rho_0,\mathbf{r},\widetilde{T})r_5 \\ &+ r_5\Big[K_{43}(\mathbf{r}) + \tau_4 K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big] \\ &\times \tau_1(\rho_0,\mathbf{r},\widetilde{T})\Big)\Big\}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2) + \rho_0^{-1/p_3}\tau_2(\widetilde{T})C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}} \end{aligned}$$

$$\times \frac{1}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}}\Bigg\{\frac{2}{r_1}\ \|a_0^{1/p_1}(g_2-g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}$$

$$\times \exp(T^{1/p_1'}r_7+M_1r_8)+r_4\mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)\bigg(\frac{2}{r_1}\ K_{42}(\mathbf{r})+\exp(T^{1/p_1'}r_7+M_1r_8)$$

$$\times\Big[K_{43}(\mathbf{r})+\tau_4K_{44}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)+\rho_0^{-1/p_3}K_{45}(\mathbf{r})\Big]\bigg)\Bigg\},$$

which has exactly the form (3.24). ■

At this point we have just proved the Lemma 3.7, which was used in the Section 3.2 to prove the dependence continuous of the solution.

3.3.3 Estimates of $W(\overline{z},\overline{w},\mathbf{d}_2)-W(\overline{z},\overline{w},\mathbf{d}_1)$

The goal of this subsection is to prove Lemma 3.8, i.e. to estimate

$$\|W(\cdot,\mathbf{d}_1,\overline{w}_1,\overline{z}_1)-W(\cdot,\mathbf{d}_2,\overline{w}_2,\overline{z}_2)\|_{L^\infty(0,T;X)}.$$

Proof of Lemma 3.8. Let us recall (2.71), where W_j are defined by (2.21)–(2.24). First we need an estimate for $\|W_1(\overline{w},\overline{z},\mathbf{d}_2)-W_1(\overline{w},\overline{z},\mathbf{d}_1)\|_{L^\infty(0,T;X)}$. For this purpose we notice that from formula (2.21) we easily derive the identity

$$\begin{aligned}
&W_1(\overline{w},\overline{z},\mathbf{d}_2,t)-W_1(\overline{w},\overline{z},\mathbf{d}_1,t)\\
&\quad=\exp\bigg(\int_0^t\Big(\overline{k}_2(\sigma)-\frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\bigg)\exp\bigg(\int_0^t a_0(\sigma)d\sigma A\bigg)\\
&\quad\times\{Au_{0,2}-Au_{0,1}\}+\bigg\{\exp\bigg(\int_0^t\Big(\overline{k}_2(\sigma)-\frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\bigg)\\
&\quad-\exp\bigg(\int_0^t\Big(\overline{k}_1(\sigma)-\frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma)-\frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\bigg)\bigg\}\\
&\quad\times\exp\bigg(\int_0^t a_0(\sigma)d\sigma A\bigg)Au_{0,1}.
\end{aligned}$$

Using (3.2) and (3.41), we get

$$\|W_1(\overline{w},\overline{z},\mathbf{d}_2)-W_1(\overline{w},\overline{z},\mathbf{d}_1)\|_{L^\infty(0,\widetilde{T};X)}\le\exp(T^{1/p_1'}r_7+M_1r_8)\|Au_{0,2}-Au_{0,1}\|_X$$

$$+ r_5 K_{42}(\mathbf{r})\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2).$$

In order to estimate the increments of W_2 in $\mathbf{d}$, we use (2.22). Thus we achieve

$$\begin{aligned}
&W_2(\overline{w},\overline{z}, \mathbf{d}_2, t) - W_2(\overline{w}, \overline{z}, \mathbf{d}_1, t) \\
&\quad = (f_2(0) - f_1(0)) \exp\bigg(\int_0^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big) d\sigma\bigg) - \\
&\quad \times \exp\bigg(\int_0^t a_0(\sigma) d\sigma A\bigg)\, Z(\overline{w}, \overline{z}, \mathbf{d}_2, T) \\
&\quad + f_1(0)\bigg(\exp\bigg(\int_0^t \Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big) d\sigma\bigg) \\
&\quad - \exp\bigg(\int_0^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big) d\sigma\bigg)\bigg) \\
&\quad \times \exp\bigg(\int_0^t a_0(\sigma) d\sigma A\bigg)\, Z(\overline{w}, \overline{z}, \mathbf{d}_2, T) \\
&\quad + f_1(0) \exp\bigg(\int_0^t \Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big) d\sigma\bigg) \\
&\quad \times \exp\bigg(\int_0^t a_0(\sigma) d\sigma A\bigg)\, (Z(\overline{w}, \overline{z}, \mathbf{d}_2, T) - Z(\overline{w}, \overline{z}, \mathbf{d}_1, t)) \\
&\quad =: \sum_{i=1}^{3} \widetilde{W}_{2,i}(\overline{w}, \overline{z}, \mathbf{d}_1, \mathbf{d}_2, t).
\end{aligned}$$

Analogously to the estimate (3.53) for $\widetilde{R}_1$, using (3.5), we get the estimate

$$\begin{aligned}
\|\widetilde{W}_{2,1}(\overline{w},\overline{z}, \mathbf{d}_1, \mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\le K_1(T)\|f_1 - f_2\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \exp(T^{1/p_1'} r_7 + M_1 r_8) \\
&\times \Big[K_5(\mathbf{r}) + \tau_4 K_6(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big) + \rho_0^{-1/p_3} K_7(\mathbf{r})\Big]. \qquad (3.72)
\end{aligned}$$

Making use of (3.58) and (3.5), we deduce

$$\begin{aligned}
\|\widetilde{W}_{2,2}(\overline{w},\overline{z}, \mathbf{d}_1, \mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\le r_2 K_{42}(\mathbf{r})\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2) \\
&\times \Big[K_5(\mathbf{r}) + \tau_4 K_6(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma) d\sigma\Big) + \rho_0^{-1/p_3} K_7(\mathbf{r})\Big]. \qquad (3.73)
\end{aligned}$$

Taking into account (3.2) and (3.9), we have

$$\|\widetilde{W}_{2,3}(\overline{w},\overline{z}, \mathbf{d}_1, \mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} \le r_2 \exp(T^{1/p_1'} r_7 + M_1 r_8)$$

$$\times \Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3} + \tau_4 K_{12}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2). \qquad (3.74)$$

Therefore, summing up the estimates (3.72)–(3.74), we conclude

$$\begin{aligned}
&\|W_2(\overline{w},\overline{z}, \mathbf{d}_2) - W_2(\overline{w}, \overline{z}, \mathbf{d}_1)\|_{L^\infty(0,\widetilde{T};X)} \\
&\quad \le K_1(T)\|f_1 - f_2\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})} \exp(T^{1/p_1'} r_7 + M_1 r_8) \\
&\quad \times \Big[K_5(\mathbf{r}) + \tau_4 K_6(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3} K_7(\mathbf{r})\Big] \\
&\quad + \Bigg(r_2 K_{42}(\mathbf{r})\Big[K_5(\mathbf{r}) + \tau_4 K_6(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big) + \rho_0^{-1/p_3} K_7(\mathbf{r})\Big] \\
&\quad + r_2 \exp(T^{1/p_1'} r_7 + M_1 r_8) \\
&\quad \times \Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3} + K_{12}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]\Bigg)\mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2).
\end{aligned}$$

From (2.23), the next equality holds true:

$$\begin{aligned}
&W_3(\overline{w},\overline{z}, \mathbf{d}_2, t) - W_3(\overline{w}, \overline{z}, \mathbf{d}_1, t) \\
&\quad = \int_0^t \Bigg[\Big(f_2'(s) - f_2(s)\Big(\overline{k}_2(s) - \frac{a_0(s)}{k_2(s)}\,\overline{w}(s) - \frac{a_0(s) f_2(s)}{k_2(s)}\overline{z}\Big)\Big) \\
&\quad - \Big(f_1'(s) - f_1(s)\Big(\overline{k}_1(s) - \frac{a_0(s)}{k_1(s)}\,\overline{w}(s) - \frac{a_0(s) f_1(s)}{k_1(s)}\overline{z}\Big)\Big)\Bigg] \\
&\quad \times \exp\Big(\int_s^t\Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big) \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) \\
&\quad \times Z(\overline{w}, \overline{z}, \mathbf{d}_2, T)ds \\
&\quad + \int_0^t \Big(f_1'(s) - f_1(s)\Big(\overline{k}_1(s) - \frac{a_0(s)}{k_1(s)}\,\overline{w}(s) - \frac{a_0(s) f_1(s)}{k_1(s)}\overline{z}\Big)\Big) \\
&\quad \times \Bigg[\exp\Big(\int_s^t\Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big) \\
&\quad - \exp\Big(\int_s^t\Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma) f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)\Bigg] \exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big) \\
&\quad \times Z(\overline{w}, \overline{z}, \mathbf{d}_2, T)ds
\end{aligned}$$

$$+\int_0^t \left(f_1'(s) - f_1(s)\Big(\overline{k}_1(s) - \frac{a_0(s)}{k_1(s)}\,\overline{w}(s) - \frac{a_0(s)f_1(s)}{k_1(s)}\overline{z}\Big)\right)$$

$$\times \exp\left(\int_s^t\Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\right)\exp\left(\int_s^t a_0(\sigma)d\sigma A\right)ds$$

$$\times \left(Z(\overline{w},\overline{z},\mathbf{d}_2,T) - Z(\overline{w},\overline{z},\mathbf{d}_1,T)\right) =: \sum_{i=1}^{3}\widetilde{W}_{3,i}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,t).$$

In order to estimate the increment of W_3 we assess each $\widetilde{W}_{3,i}$ in view of (3.5). The estimate (3.54) of I_1 implies

$$\begin{aligned}
\|\widetilde{W}_{3,1}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\leq \rho_0^{-1/p_3}\frac{1}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}} \\
&\times\Bigg[\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\Bigg(\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(1+K_1(T)r_7\Big) \\
&+ K_1(T)r_9\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3 + M_1r_9 + 2r_2M_2\widetilde{T}^{1/p_1}K_1(T)\Big)\Bigg) \\
&+\left\|\frac{k_1'}{k_1}-\frac{k_2'}{k_2}\right\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}K_1(T)r_2\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})} + \left\|\frac{1}{k_2}-\frac{1}{k_1}\right\|_{L^\infty(0,\widetilde{T};\mathbb{R})} \\
&\times\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}K_1(T)r_2\big(\|\Phi\|_{X^*}T^{1/p_1}r_3 + M_1 + M_2T^{1/p_1}K_1(T)r_2\big)\Bigg] \\
&\times\left[K_5(\mathbf{r}) + \tau_4K_6(\mathbf{r})\exp\left(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\right) + \rho_0^{-1/p_3}K_7(\mathbf{r})\right] \\
&\times\exp(T^{1/p_1'}r_7 + M_1r_8).
\end{aligned} \tag{3.75}$$

Proceeding analogously to (3.56), we obtain the estimate

$$\begin{aligned}
\|\widetilde{W}_{3,2}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\leq \rho_0^{-1/p_3}\frac{r_2K_{42}(\mathbf{r})}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2) \\
&\times\left[\left\|\frac{1}{a_0^{1/p_3}}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\big(1+K_1(T)r_7\big) + K_1(T)r_9\left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\right. \\
&\times\left.\Big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3 + M_1 + \widetilde{T}^{1/p_1}K_1(T)r_2M_2\Big)\right] \\
&\times\left(K_5(\mathbf{r}) + \tau_4K_6(\mathbf{r})\exp\left(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\right) + \rho_0^{-1/p_3}K_7(\mathbf{r})\right).
\end{aligned} \tag{3.76}$$

Reasoning as for the estimate of $\widetilde{W}_{3,2}$, taking into account (3.9), we get

$$\|\widetilde{W}_{3,3}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} \le \rho_0^{-1/p_3}\frac{r_2}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)$$
$$\times\Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3}+\tau_4K_{12}(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)\Big]$$
$$\times\Bigg[\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\big(1+K_1(T)r_7\big)+K_1(T)r_9\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$
$$\times\Big(\|\Phi\|_{X^*}T^{1/p_1}r_3+M_1+T^{1/p_1}K_1(T)r_2M_2\Big)\Bigg]. \tag{3.77}$$

Summing up (3.75), (3.76), (3.77), we conclude that

$$\|\widetilde{W}_3(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} \le \rho_0^{-1/p_3}\frac{1}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}$$
$$\times\Bigg[\|f_2-f_1\|_{W^{1,p_1}(0,\widetilde{T};\mathbb{R})}\Bigg(\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(1+K_1(T)r_7\Big)$$
$$+K_1(T)r_9\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\Big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3+M_1r_9+2r_2M_2\widetilde{T}^{1/p_1}K_1(T)\Big)\Bigg)$$
$$+\Big\|\frac{k_1'}{k_1}-\frac{k_2'}{k_2}\Big\|_{L^{p_1}(0,\widetilde{T};\mathbb{R})}K_1(T)r_2\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}+\Big\|\frac{1}{k_2}-\frac{1}{k_1}\Big\|_{L^\infty(0,\widetilde{T};\mathbb{R})}$$
$$\times\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}K_1(T)r_2\big(\|\Phi\|_{X^*}T^{1/p_1}r_3+M_1+M_2T^{1/p_1}K_1(T)r_2\big)\Bigg]$$
$$\times\Big[K_5(\mathbf{r})+\tau_4K_6(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)+\rho_0^{-1/p_3}K_7(\mathbf{r})\Big]$$
$$\times\exp(T^{1/p_1'}r_7+M_1r_8)+\rho_0^{-1/p_3}\frac{r_2K_{42}(\mathbf{r})}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)$$
$$\times\Bigg[\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\big(1+K_1(T)r_7\big)+K_1(T)r_9\Big\|a_0^{1/p_3'}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$
$$\times\Big(\|\Phi\|_{X^*}\widetilde{T}^{1/p_1}r_3+M_1+\widetilde{T}^{1/p_1}K_1(T)r_2M_2\Big)\Bigg]$$
$$\times\Big(K_5(\mathbf{r})+\tau_4K_6(\mathbf{r})\exp\Big(-\rho_0\int_0^{T_1}a_0(\sigma)d\sigma\Big)+\rho_0^{-1/p_3}K_7(\mathbf{r})\Big)$$
$$+\rho_0^{-1/p_3}\frac{r_2}{[p_3(1-K_2(r)/\rho_0)]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2)$$

$$\times \Big[K_{11}(\mathbf{r})\rho_0^{-1/p_3} + \tau_4 K_{12}(\mathbf{r}) \exp\Big(-\rho_0 \int_0^{T_1} a_0(\sigma)d\sigma\Big)\Big]$$

$$\times \Bigg[\Big\|\frac{1}{a_0^{1/p_3}}\Big\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}\big(1 + K_1(T)r_7\big) + K_1(T)r_9 \left\|a_0^{1/p_3'}\right\|_{L^{p_2}(0,\widetilde{T};\mathbb{R})}$$

$$\times \Big(\|\Phi\|_{X^*}T^{1/p_1}r_3 + M_1 + T^{1/p_1}K_1(T)r_2M_2\Big)\Bigg].$$

From (2.24) we easily get the difference

$$\begin{aligned}
&W_4(\overline{w},\overline{z},\mathbf{d}_2,t) - W_4(\overline{w},\overline{z},\mathbf{d}_1,t)\\
&\quad= \int_0^t a_0(s) \exp\Big(\int_s^t\Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big)\\
&\quad\times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)(g_2(s) - g_1(s))ds\\
&\quad+ \int_0^t a_0(s)\Big[\exp\Big(\int_s^t\Big(\overline{k}_2(\sigma) - \frac{a_0(\sigma)}{k_2(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_2(\sigma)}{k_2(\sigma)}\overline{z}\Big)d\sigma\Big)\\
&\quad- \exp\Big(\int_s^t\Big(\overline{k}_1(\sigma) - \frac{a_0(\sigma)}{k_1(\sigma)}\,\overline{w}(\sigma) - \frac{a_0(\sigma)f_1(\sigma)}{k_1(\sigma)}\overline{z}\Big)d\sigma\Big)\Big]\\
&\quad\times A\exp\Big(\int_s^t a_0(\sigma)d\sigma A\Big)g_1(s)ds =: \sum_{i=1}^{2}\widetilde{W}_{4,i}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2,t).
\end{aligned}$$

From (3.68) we deduce

$$\begin{aligned}
\|\widetilde{W}_{4,1}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\le \rho_0^{-1/p_3}\ C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\\
&\times \frac{\|a_0^{1/p_1}(g_2 - g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}}{[(1 - K_2(r)/\rho_0)p_3]^{1/p_3}}\ \exp(T^{1/p_1'}r_7 + M_1r_8). \qquad (3.78)
\end{aligned}$$

Analogously to (3.69) we have

$$\begin{aligned}
\|\widetilde{W}_{4,2}(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} &\le \rho_0^{-1/p_3}\ C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\\
&\times \frac{r_4K_{42}(\mathbf{r})}{[(1 - K_2(r)/\rho_0)p_3]^{1/p_3}}\ \mathbf{dist}(\mathbf{d}_1,\mathbf{d}_2). \qquad (3.79)
\end{aligned}$$

As a result, from (3.78) and (3.79), we get the estimate

$$\|\widetilde{W}_4(\overline{w},\overline{z},\mathbf{d}_1,\mathbf{d}_2)\|_{L^\infty(0,\widetilde{T};X)} \le \rho_0^{-1/p_3}\ C_{10}(\gamma,T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}$$

$$\times \frac{1}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}} \ \exp(T^{1/p_1'}r_7 + M_1 r_8)\|a_0^{1/p_1}(g_2 - g_1)\|_{L^{p_1}(0,\widetilde{T};D_A(\gamma,\infty))}$$

$$+ \rho_0^{-1/p_3} \ C_{10}(\gamma, T)\frac{\|a_0\|_{L^1(0,\widetilde{T};\mathbb{R})}^{(1-(1-\gamma)p_2)/p_2}}{(1-(1-\gamma)p_2)^{1/p_2}}\frac{r_4 K_{42}(\mathbf{r})}{[(1-K_2(r)/\rho_0)p_3]^{1/p_3}} \ \mathbf{dist}(\mathbf{d}_1, \mathbf{d}_2).$$

Summing estimates for all $\widetilde{W_j}, j = 1, 2, 3, 4,$ from (2.71) we get the desired form (3.27). ■

Thus, we have just proved Lemma 3.8, which was necessary to obtain the continuous dependence result in the Section 3.2.

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