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Markovianity and non-Markovianity in quantum and classical systems

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Abstract. We discuss the conceptually different definitions used for the non-Markovianity of classical and quantum processes. The well-established definition of non-Markovianity of a classical stochastic process represents a condition on the Kolmogorov hierarchy of the n-point joint probability distributions. Since this definition cannot be transferred to the quantum regime, quantum non-Markovianity has recently been defined and quantified in terms of the underlying quantum dynamical map, using either its divisibility properties or the behavior of the trace distance between pairs of initial states. Here, we investigate and compare these definitions and their relations to the classical notion of non-Markovianity by employing a large class of non-Markovian processes, known as semi-Markov processes, which admit a natural extension to the quantum case. A number of specific physical examples are constructed that allow us to study the basic features of the classical and the quantum definitions and to evaluate explicitly the measures of quantum non-Markovianity. Our results clearly demonstrate several fundamental differences between the classical and the quantum notion of non-Markovianity, as well as between the various quantum measures of non-Markovianity. In particular, we show that the divisibility property in the classical case does not coincide with Markovianity

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and that the non-Markovianity measure based on divisibility assigns equal infinite values to different dynamics, which can be distinguished by exploiting the trace distance measure. A simple exact expression for the latter is also obtained in a special case.

Contents

1. Introduction

A great deal of effort has been put into the study and understanding of non-Markovian effects and dynamics within the description of open quantum systems [1]. These efforts have faced quite a large number of problems because of practical as well as fundamental reasons. Indeed, on the one hand, the treatment of non-Markovian systems is particularly demanding because one cannot rely on simplifying assumptions such as weak coupling, separation of time scales between the system and the environment and the factorization of the system–environment state. On the other, in the non-Markovian case there is no general characterization of the equations of motion, such as that in the Markovian setting thanks to the theory of quantum dynamical semigroups, and the very notion of non-Markovianity in the quantum case remains to be clarified. Over the last few years, this topic has experienced a significant revival, leading to important improvements and to a deeper understanding of quite a few issues in the theory of open quantum systems (see e.g. [2–9] and references therein).

Apart from the explicit detailed treatment of many specific quantum systems where memory effects show up, efforts have been made to obtain general classes of non-Markovian equations, leading to well-defined completely positive time evolutions [5, 10]. At the same time, advances have been made in order to actually define what is meant by a non-Markovian quantum dynamics [11–13]. This work has raised all sorts of questions regarding the connection between the different approaches, as well as the relationship between the notion of non-Markovianity used in classical and quantum settings, together with the quest for clear-cut signatures of non-Markovian behavior.

This paper is devoted to addressing some of these questions, focusing in particular on the connection between the very definition of the non-Markovian process used in classical probability theory, and the Markovian or non-Markovian behavior in the dynamics of a physical system. It will naturally appear that these two notions are quite different. Starting within the classical framework, we will analyze how the non-Markovianity of a process reflects itself in the behavior of its one-point probability density, which naturally leads to criteria for the characterization of non-Markovian behavior in the dynamics. These criteria can be extended to the quantum setting, thus providing natural tools to assess the non-Markovianity of a quantum time evolution. They are based on the possibility of connecting the probability vectors giving the state of the system at different times through well-defined transition matrices, and on the behavior of solutions corresponding to different initial states with respect to the Kolmogorov distance.

Despite the abstract framework, the whole presentation is built with reference to explicit examples. These examples find their common root in being related to realizations of a class of non-Markovian processes for which an explicit characterization is available, namely semi-Markov processes. This is indeed an exceptional situation, since the characterization of non-Markovian processes, and even more of classes of them, is actually in the general case a formidable task. Semi-Markov processes include, in particular, renewal processes, and the relevance in considering this extension with respect to the standard Markovian situation has already been proven to be of interest for the field of continuous time random walks [14]. This approach will enable us to construct different non-trivial dynamics in both the classical and quantum cases, showing quite different behaviors and amenable to an explicit analysis, both in terms of the previously introduced criteria and in terms of the recently proposed measures of non-Markovianity, thus allowing for a concrete comparison of these measures. In particular, building on this class of examples, we will show in section 4.2.3 how the two measures of non-Markovianity can behave in a completely different way for the same processes.

This paper is organized as follows. In section 2, we introduce the notion of classical and quantum maps for the dynamics of an open system, introducing the notions of P-divisibility, and CP-divisibility, which will turn out to be useful for the comparison of different behaviors. In section 3, we recall the notion of non-Markovianity for a classical process, showing its relation to the behavior of the one-point probability density. We further consider examples of classical semi-Markov processes characterized by different stochastic matrices and waiting time distributions, studying their P-divisibility and the behavior with respect to the Kolmogorov distance. In section 4, we perform a similar analysis in the quantum setting, considering classes of dynamics that can be related to semi-Markov processes. These dynamics still allow for an exact determination of their divisibility properties and of their quantum measure of non-Markovianity according to the recent proposals. In particular, we provide exact expressions for the value of these measures, thus allowing for their explicit comparison. In section 5 we further comment on our results.

2. Dynamical maps

If the initial state of the system and the environment factorizes, the dynamics of an open quantum system can be described by a trace preserving completely positive (CPT) map, so that the state of the system $\rho(t)$ evolving from an initial state $\rho(t_0)$ is given by

$$\rho(t) = \Phi(t, t_0)\rho(t_0). \tag{1}$$

The most general structure of such time-evolution maps is not known, apart from the important subclass of maps that obey a semigroup composition law, which can be characterized via a suitable generator [15, 16]. In order to characterize and actually define Markovianity or non-Markovianity in this setting, one can follow essentially two approaches: either one assumes that the map $\Phi(t, t_0)$ is known and therefore relies on looking at certain mathematical properties of the map itself, which is essentially the path followed, in different ways, in [11, 13]; or one studies the behavior in time of the solutions $\rho(t)$ allowing the initial condition to vary over the possible set of states, which is the approach elaborated in [12, 17], relying on a suitable notion of distinguishability of quantum states [18], an approach that captures the idea of information flow between the system and the environment. Note that Markovianity or non-Markovianity should be a property of the map or equivalently of the time-evolved states, not of the equations admitting such states as solutions. Indeed, quite a different form of the equations, e.g. integrodifferential or local in time, might admit the very same solutions, as we shall also see in the examples of section 4. Of course, since in a concrete physical setting one is faced with the equations of motion rather than with their general solution, it is of great interest to assess possible links between the equations themselves and the Markovian or non-Markovian behavior of their solutions.

To better understand and compare these two approaches, besides the notion of complete positivity and trace preservation, other finer characterizations of the time-evolution map $\Phi(t, 0)$ turn out to be useful, where we have set for the sake of simplicity $t_0 = 0$. If the map $\Phi(t, 0)$ can be split according to

$$\Phi(t, 0) = \Phi(t, s)\Phi(s, 0) \tag{2}$$

for any $t \geqslant s \geqslant 0$, with $\Phi(t, s)$ itself being a CPT map, we say that the map $\Phi(t, 0)$ is CP-divisible, which implies that $\Phi(t, s)$ is itself a well-defined time evolution having as the domain the whole set of states. Note that in contrast with [19] focusing on quantum channels, the notion of divisibility considered here refers to families of time-dependent dynamical maps. The existence of $\Phi(t, s)$ as a linear map is granted if $\Phi(s, 0)$ is invertible, which is typically the case away from isolated points of time, so that

$$\Phi(t, s) = \Phi(t, 0)\Phi^{-1}(s, 0). \tag{3}$$

The existence of $\Phi(t, s)$ as a linear map, however, does not entail its complete positivity. Indeed we will state that the map $\Phi(t, 0)$ is P-divisible if $\Phi(t, s)$ sends states into states but is only positive, and that it is indivisible if neither P-divisibility nor CP-divisibility holds. Examples of such maps together with their physical interpretation will be provided in section 4.

The notion of P-divisibility can also be considered in the classical setting. Suppose we consider a finite-dimensional classical system, described by a probability vector $\boldsymbol{p}(t)$. Its time evolution can be described by a time-dependent collection of stochastic matrices $\Lambda(t, 0)$ according to

$$\boldsymbol{p}(t) = \Lambda(t, 0)\boldsymbol{p}(0). \tag{4}$$

Similarly to before, we say that the classical map $\Lambda(t, 0)$ is P-divisible provided, that for any $t \geqslant s \geqslant 0$, one can write

$$\Lambda(t, 0) = \Lambda(t, s)\Lambda(s, 0), \tag{5}$$

where each of the $\Lambda(t, s)$ is itself a stochastic matrix. Its matrix elements therefore satisfy $(\Lambda(t, s))_{ij} \geqslant 0$ and $\sum_k (\Lambda(t, s))_{kj} = 1$, which provide the necessary and sufficient conditions

ensuring that probability vectors are sent into probability vectors [20]. Once again this need not generally be true even if the map $\Lambda(s, 0)$ is invertible as a linear operator. Note that here we are only considering the one-point probabilities $\boldsymbol{p}(t)$, which are certainly not sufficient to assess the Markovianity or non-Markovianity of a process according to the mathematically precise definition used in classical probability theory.

3. Classical non-Markovian processes

Let us now recall what is the very definition of a non-Markovian process in the classical probabilistic setting. Indeed, the analysis of classical processes is a natural starting point, also adopted in [21–25]. Suppose we are considering a stochastic process taking values in a numerable set $\{x_i\}_{i\in\mathbb{N}}$. The process is said to be Markovian if the conditional transition probabilities satisfy

$$p_{1|n}(x_n, t_n|x_{n-1}, t_{n-1}; \ldots; x_0, t_0) = p_{1|1}(x_n, t_n|x_{n-1}, t_{n-1}), \tag{6}$$

with $t_n \geqslant t_{n-1} \geqslant \ldots \geqslant t_1 \geqslant t_0$, so that the probability that the random variable assumes the value x_n at time t_n, given that it has assumed given values x_i at previous times t_i, actually depends only on the last assumed value and not on previous ones. In this sense, the process is said to lack memory. This statement obviously involves all n-time probabilities, so that the non-Markovianity of the process cannot be assessed by looking at the one-time probabilities alone [26, 27]. It immediately appears from equation (6) that such a definition cannot be transferred to the quantum realm, since the very notion of conditional probability does depend on the measurement carried out in order to ascertain the previous value of the random variable and on how it transforms the state for the subsequent time evolution.

If we know that the process is Markovian, the n-point probabilities can be easily obtained in terms of the initial probability density and the conditional transition probability $p_{1|1}$ according to

$$p_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_0, t_0) = \prod_{i=1}^{n} p_{1|1}(x_i, t_i|x_{i-1}, t_{i-1}) p_1(x_0, t_0). \tag{7}$$

The Markov condition equation (6) in turn implies that the conditional transition probability $p_{1|1}$ obeys the Chapman–Kolmogorov equation

$$p_{1|1}(x, t|y, s) = \sum_z p_{1|1}(x, t|z, \tau) p_{1|1}(z, \tau|y, s), \tag{8}$$

with $t \geqslant \tau \geqslant s$, whose possible solutions characterize the transition probabilities of a Markov process. A Markov process is therefore uniquely characterized by its conditional transition probability and the initial distribution.

If $\boldsymbol{p}(t)$ denotes the vector giving the one-point probability of a Markov process taking values in a finite space, and if $\Lambda(t, s)$ is its transition probability expressed in matrix form, the probability vectors at different times are related according to

$$\boldsymbol{p}(t) = \Lambda(t, s)\boldsymbol{p}(s), \tag{9}$$

with $t \geqslant s \geqslant 0$ and $\Lambda(t, s)$ a stochastic matrix obeying the Chapman–Kolmogorov equation, which in this case is equivalent to the requirement of P-divisibility in the sense of equation (5). However, the validity of the Chapman–Kolmogorov equation and P-divisibility obviously do

not coincide, since the P-divisibility does not tell us anything about the higher-point conditional probabilities. Indeed, for a given process one might find a matrix $\Lambda(t, s)$ satisfying equation (9) even if the process is non-Markovian; however, in this case the matrix is not the conditional transition probability of the process [28, 29]. Similarly to equation (3), if $\Lambda(t, 0)$ is invertible one can obtain a matrix such as

$$\Lambda(t, s) = \Lambda(t, 0)\Lambda^{-1}(s, 0), \tag{10}$$

which warrants independence of the initial probability vector. In contrast, the transition probability of a non-Markovian process does depend on the initial probability vector [28].

3.1. Classical semi-Markov processes

To spell out this situation in detail, let us consider an example. The main difficulty lies in the fact that generally very little is known about non-Markovian processes. We will, however, consider a class of non-Markovian processes allowing for a compact characterization, that is to say, semi-Markov processes [30]. To this end, we consider a system having a finite set of states, which can jump from one state to the other according to certain jump probabilities π_{mn}, waiting a random time in each state before jumping to the next. Such a process turns out to be Markovian if and only if the waiting time distributions $f_n(\tau)$ characterizing the random sojourn time spent in each state are given by exponential probability distributions, and is non-Markovian otherwise [31]. Each such process is fixed by its semi-Markov matrix

$$(Q)_{mn}(\tau) = \pi_{mn} f_n(\tau), \tag{11}$$

where π_{mn} are the elements of a stochastic matrix. Assuming for the sake of simplicity a two-dimensional system, a site-independent waiting time distribution and the stochastic matrix to be actually bistochastic [20], the semi-Markov matrix is determined according to

$$Q(\tau) = \begin{pmatrix} 1-\pi & \pi \\ \pi & 1-\pi \end{pmatrix} f(\tau),$$
$$\equiv \Pi f(\tau) \tag{12}$$

with π being a positive number between zero and one giving the probability to jump from one site to the other and $f(\tau)$ an arbitrary waiting time distribution with the associated survival probability

$$g(t) = 1 - \int_0^t \mathrm{d}\tau f(\tau). \tag{13}$$

3.2. Transition probability

The transition probability for such a process can be determined exploiting the fact that it is known to obey the following integrodifferential Kolmogorov forward equation [32]:

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t, 0) = \int_0^t \mathrm{d}\tau W(\tau) T(t - \tau, 0), \tag{14}$$

here expressed in matrix form with

$$W(\tau) = (\Pi - \mathbb{1})k(\tau), \tag{15}$$

where the memory kernel $k(\tau)$ relates the waiting time distribution and survival probability according to

$$f(\tau) = \int_0^\tau \mathrm{d}t\, k(\tau - t) g(t). \tag{16}$$

Denoting by $\hat{v}(u)$ the Laplace transform of a function or matrix $v(\tau)$ according to

$$\hat{v}(u) = \int_0^{+\infty} \mathrm{d}\tau\, v(\tau) \mathrm{e}^{-u\tau}, \tag{17}$$

the solution of this equation with the initial condition $T(0, 0) = \mathbb{1}$ can be expressed as

$$\hat{T}(u) = \frac{\hat{g}(u)}{\mathbb{1} - \Pi \hat{f}(u)}. \tag{18}$$

3.3. Explicit examples

The explicit solution of equation (18) for $\pi = 1/2$, so that at each step the system has equal probability to remain in the same site or change, is given by

$$T(t, 0) = \frac{1}{2} \begin{pmatrix} 1 + g(t) & 1 - g(t) \\ 1 - g(t) & 1 + g(t) \end{pmatrix}, \tag{19}$$

so that according to equation (10), we can introduce the matrices

$$T(t, s) = T(t, 0) T^{-1}(s, 0) = \frac{1}{2} \begin{pmatrix} 1 + g(t)/g(s) & 1 - g(t)/g(s) \\ 1 - g(t)/g(s) & 1 + g(t)/g(s) \end{pmatrix}, \tag{20}$$

which indeed connect the probability vectors at different times according to

$$\boldsymbol{p}(t) = T(t, s) \boldsymbol{p}(s). \tag{21}$$

Given the fact that for any non-vanishing waiting time distribution the survival probability is a strictly positive monotonic decreasing function, the matrices $T(t, s)$ are well-defined stochastic matrices for any pair of times $t \geqslant s$, so that the classical map $T(t, 0)$ is always P-divisible, irrespective of the fact that the underlying process is Markovian only for the special choice of an exponential waiting time distribution of the form

$$f(\tau) = \lambda\, \mathrm{e}^{-\lambda \tau}. \tag{22}$$

This result implies that the one-point probabilities of the considered non-Markovian semi-Markov process can be equally well obtained from a Markov process with conditional transition probability $p_{1|1}$ given by T, whose n-point probabilities can be obtained as in equation (7). The latter would, however, differ from those of the considered semi-Markov process.

As a complementary situation, let us consider the case $\pi = 1$, so that once in a state the system jumps with certainty to the other, thus obtaining as an explicit solution of equation (18) the expression

$$T(t, 0) = \frac{1}{2} \begin{pmatrix} 1 + q(t) & 1 - q(t) \\ 1 - q(t) & 1 + q(t) \end{pmatrix}, \tag{23}$$

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and therefore

$$T(t,s) = T(t,0)T^{-1}(s,0) = \frac{1}{2}\begin{pmatrix} 1+q(t)/q(s) & 1-q(t)/q(s) \\ 1-q(t)/q(s) & 1+q(t)/q(s) \end{pmatrix}. \tag{24}$$

The quantity $q(t)$ appearing in these matrices is the inverse Laplace transform of the function

$$\hat{q}(u) = \frac{1}{u}\frac{1-\hat{f}(u)}{1+\hat{f}(u)}. \tag{25}$$

Recalling that the probability for n jumps in a time t for a waiting time distribution $f(t)$ is given by

$$p_n(t) = \int_0^t d\tau f(t-\tau)p_{n-1}(\tau), \tag{26}$$

so that

$$\hat{p}_n(u) = \hat{p}_0(u)\hat{f}^n(u), \tag{27}$$

one has

$$q(t) = \sum_{n=0}^{\infty} p_{2n}(t) - \sum_{n=0}^{\infty} p_{2n+1}(t) = p_{\text{even}}(t) - p_{\text{odd}}(t). \tag{28}$$

The quantity $q(t)$ therefore expresses the difference between the probability to have an even or an odd number of jumps. In contrast with the previous situation, the quantity $q(t)$ depending on the waiting time distribution can assume quite different behaviors, showing oscillations and going through zero at isolated time points, so that at these time points, corresponding to a crossing of trajectories starting from different initial conditions, the transition matrix $T(t,s)$ is not defined. Moreover, due to the non-monotonicity of $q(t)$ the matrices $T(t,s)$ cannot always be interpreted as stochastic matrices. Of course in the Markovian case, corresponding to equation (22) and therefore to $q(t) = \exp(-2\lambda t)$, all these features are recovered.

The variety of possible behaviors is best clarified by considering explicit expressions for the waiting time distribution $f(t)$ that determines the process once the stochastic matrix Π is given. Quite general expressions for the waiting time distribution can be obtained by considering convex mixtures or convolutions of exponential waiting time distributions with equal or different parameters, whose Laplace transform is given by rational functions [33, 34]. To better understand the dynamics generated by the maps of equations (19) and (23) in the following examples note that for $\pi = 1/2$ the matrix Π is idempotent, sending each probability vector to the uniform distribution, while for $\pi = 1$ one has $\Pi^{2n} = \mathbb{1}$, and the action of the bistochastic matrix consists in swapping the two elements of the probability vector.

3.3.1. Convolution of exponential waiting time distributions. The behavior of the quantity $q(t)$ can be explicitly assessed for the case of a waiting time distribution f(t) given by the convolution of two exponential waiting time distributions. Let us first consider the case when the waiting time distributions share the same parameter λ, so that f $= f * f$ with f as in equation (22), corresponding to

$$f(t) = \lambda^2 t e^{-\lambda t} \tag{29}$$
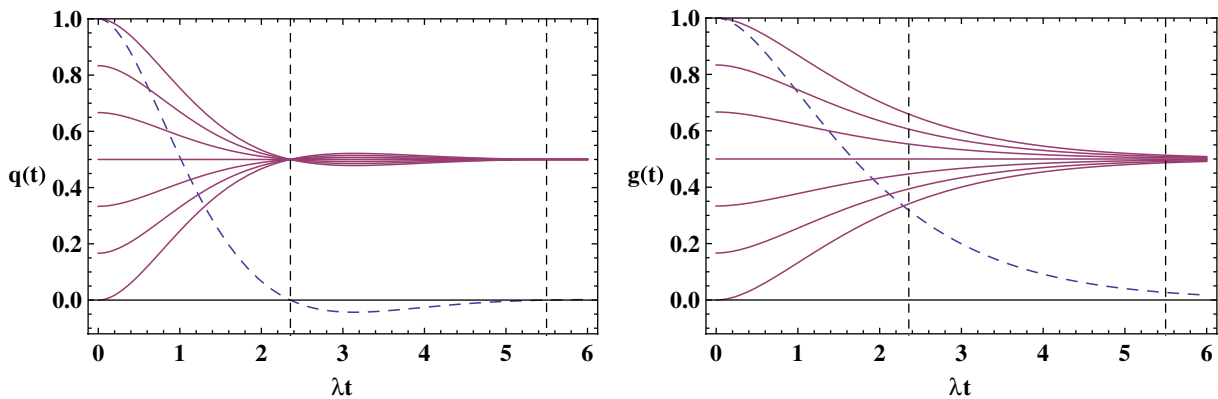
Figure 1. Left: plot of the function q (dashed line) given by equation (28) as a function of λt for the convolution of two equal exponential waiting time distributions, for the case of a semi-Markov process with $\pi = 1$, together with a few trajectories (continuous lines) for the upper element $w(t)$ of the one-point probability according to equation (32). The initial data $w(0)$ are uniformly taken between 0 and 1. When $q(t)$ goes through zero the trajectories cross. At these points $|q(t)|$ starts growing, which indicates the failure of P-divisibility of the time evolution as defined in equation (5). Right: plot of the function g given by equation (13) as a function of λt, for the same waiting time distribution, together with a few trajectories, corresponding to a semi-Markov process with $\pi = 1/2$. As it appears, despite sharing the same waiting time distribution the trajectories for this semi-Markov process never cross, and P-divisibility holds even if the process is non-Markovian.

and therefore

$$g(t) = (1 + \lambda t)e^{-\lambda t}. \tag{30}$$

This is a special case of the Erlang distribution [34], leading to

$$q(t) = e^{-\lambda t}[\cos(\lambda t) + \sin(\lambda t)], \tag{31}$$

which oscillates and crosses zero at isolated points, so that the matrices $T(t, s)$ are not defined at these points and cannot be always interpreted as stochastic matrices, since their entries can become negative. This behavior is exhibited in figure 1, where the quantity of equation (31) is plotted together with the trajectories of the probability vector with different initial conditions. We also consider the behavior of $g(t)$ and of the trajectories for the same waiting time distribution but a semi-Markov process with stochastic matrix fixed by $\pi = 1/2$. The probability vector is of the form

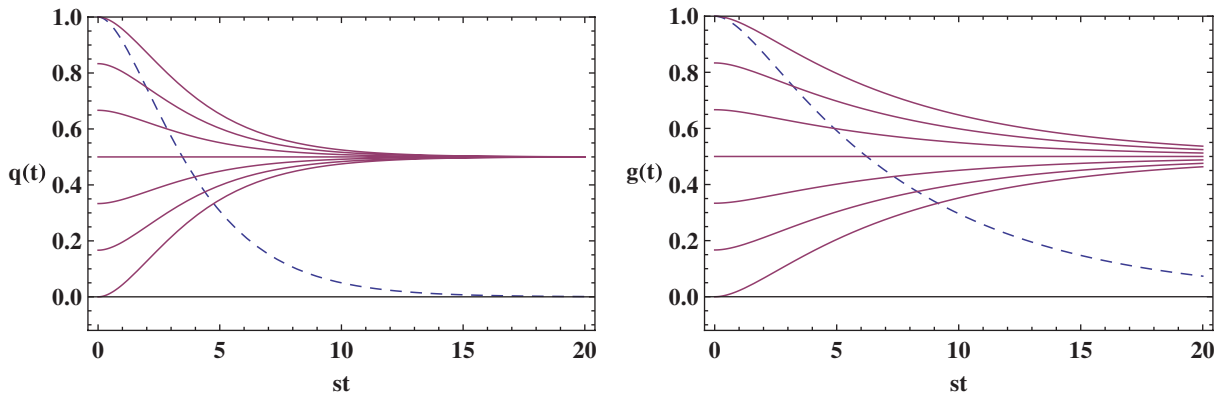$$\boldsymbol{p}(t) = \begin{pmatrix} w(t) \\ 1 - w(t) \end{pmatrix}, \tag{32}$$

so that its trajectories are displayed showing $w(t)$, where

$$w(t) = \tfrac{1}{2}[1 - q(t) + 2q(t)w(0)] \tag{33}$$

or

$$w(t) = \tfrac{1}{2}[1 - g(t) + 2g(t)w(0)] \tag{34}$$

Figure 2. Left: plot of the function $q(t)$ (dashed line) and of the trajectories (continuous lines) as in figure 1, but for the convolution of two different exponential waiting time distributions. We plot q as a function of st, taking $p/s^2 = 0.12$, with s and p the sum and product of the two parameters characterizing the exponential waiting time distributions as in equation (37). For this case, the quantity χ given by equation (39) is positive, ensuring monotonicity of $q(t)$. Right: plot of $g(t)$ and the corresponding trajectories for the same waiting time distribution.

in the two cases $\pi = 1$ and $\pi = 1/2$, respectively. Note how for $\pi = 1$ the different trajectories tend to group together and then separate again depending on the behavior of $q(t)$. A more general situation is given by $f = f_1 * f_2$, where each f_i is of the form equation (22) with parameter λ_i, so that one has

$$f(t) = 2\frac{p}{s}e^{-(1/2)st}\frac{1}{\xi}\mathrm{Sinh}\left(\frac{st}{2}\xi\right) \tag{35}$$

and correspondingly

$$g(t) = e^{-(1/2)st}\left[\mathrm{Cosh}\left(\frac{st}{2}\xi\right) + \frac{1}{\xi}\mathrm{Sinh}\left(\frac{st}{2}\xi\right)\right], \tag{36}$$

where we have set

$$s = \lambda_1 + \lambda_2, \quad p = \lambda_1\lambda_2, \quad \xi = \sqrt{1 - 4\frac{p}{s^2}}. \tag{37}$$

The function $q(t)$ determining the matrices $T(t, s)$ is now given by

$$q(t) = e^{-(1/2)st}\left[\mathrm{Cosh}\left(\frac{st}{2}\chi\right) + \frac{1}{\chi}\mathrm{Sinh}\left(\frac{st}{2}\chi\right)\right], \tag{38}$$

with

$$\chi = \sqrt{1 - 8\frac{p}{s^2}}. \tag{39}$$

The expression given by equation (38) shows oscillatory behavior if χ becomes imaginary, for $3 - 2\sqrt{2} \leqslant \lambda_1/\lambda_2 \leqslant 1/(3 - 2\sqrt{2})$, whereas it is a positive monotonic function of t otherwise. The latter situation is considered in figure 2.
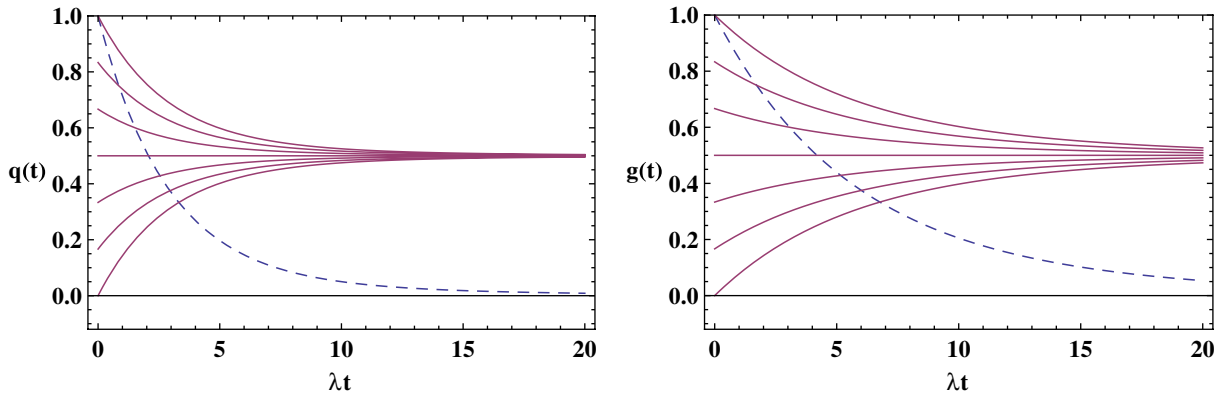
Figure 3. Left: plot of $q(t)$ (dashed line) and the trajectories (continuous lines) as in figure 1, but for the convex mixture of two different exponential waiting time distributions. We plot q as a function of λt, taking $\lambda_1 = a_1 \lambda$ and $\lambda_2 = a_2 \lambda$, with $a_1 = 0.1$ and $a_2 = 0.2$, together with the mixing parameter $\mu = 0.3$. Right: plot of $g(t)$ and the corresponding trajectories for the same waiting time distribution.

3.3.2. A mixture of exponential waiting time distributions. In contrast, for the case of a convex mixture of two exponential distributions, the trajectories never cross, and the matrices $T(t, s)$ are always well-defined stochastic matrices. Indeed, this can be seen from

$$\mathsf{f}(t) = \mu f_1(t) + (1 - \mu) f_2(t), \tag{40}$$

with $0 \leqslant \mu \leqslant 1$, so that

$$\mathsf{g}(t) = \mu \mathrm{e}^{-\lambda_1 t} + (1 - \mu) \mathrm{e}^{-\lambda_2 t} \tag{41}$$

and

$$q(t) = \mathrm{e}^{-(1/2)(\lambda_1 + \lambda_2 + \langle \lambda \rangle)t} \left[\mathrm{Cosh}\left(\frac{\bar{\lambda}t}{2}\right) + \frac{(\lambda_1 + \lambda_2 - 3\langle \lambda \rangle)}{\bar{\lambda}} \mathrm{Sinh}\left(\frac{\bar{\lambda}t}{2}\right) \right], \tag{42}$$

with

$$\langle \lambda \rangle = \mu \lambda_1 + (1 - \mu) \lambda_2 \tag{43}$$

being the mean rate and

$$\bar{\lambda} = \sqrt{(\lambda_1 + \lambda_2 + \langle \lambda \rangle)^2 - 8\lambda_1 \lambda_2}. \tag{44}$$

This case is considered in figure 3.

It should be noted that in all these situations the process is non-Markovian, but P-divisibility of the time evolution $\Lambda(t, 0)$ in the sense of equation (5) still holds in some cases. For a semi-Markov process with a semi-Markov matrix as in equation (12) and $\pi = 1/2$, P-divisibility always holds; in particular, as shown in the examples, the trajectories never cross and $|q(t)|$ never grows. In contrast, for $\pi = 1$ the behavior depends on the waiting time distribution, which determines whether or not the quantity $q(t)$ shows oscillating behavior, implying that the trajectories start getting closer till they cross and then move apart once again.

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3.4. Kolmogorov distance

It is interesting to relate this behavior to the time dependence of the Kolmogorov distance among probability distributions arising from different initial states. The Kolmogorov distance between two probability vectors can be written as [35, 36]

$$D_{\mathrm{K}}(\boldsymbol{p}^1(t), \boldsymbol{p}^2(t)) = \frac{1}{2} \sum_i \left| p_i^1(t) - p_i^2(t) \right|. \tag{45}$$

If the map $\Lambda(t, 0)$ is P-divisible in the sense of equation (5), then the Kolmogorov distance is a monotonic decreasing function of time. Indeed, by the two basic properties of a stochastic matrix, namely the positivity of its entries and the fact that each row sums up to one, one has for $t \geqslant s \geqslant 0$

$$
\begin{aligned}
D_{\mathrm{K}}(\boldsymbol{p}^1(t), \boldsymbol{p}^2(t)) &= \frac{1}{2} \sum_i \left| \sum_k \Lambda(t, s)_{ik} \left(\boldsymbol{p}^1(s) - \boldsymbol{p}^2(s) \right)_k \right| \\
&\leqslant \frac{1}{2} \sum_i \sum_k \Lambda(t, s)_{ik} \left| \left(\boldsymbol{p}^1(s) - \boldsymbol{p}^2(s) \right)_k \right| \\
&= \frac{1}{2} \sum_k \left| \left(\boldsymbol{p}^1(s) - \boldsymbol{p}^2(s) \right)_k \right| = D_{\mathrm{K}} \left(\boldsymbol{p}^1(s), \boldsymbol{p}^2(s) \right).
\end{aligned}
\tag{46}
$$

This holds true independently of whether the underlying classical process is Markovian or not; it only depends on the fact the one-point probabilities can be related at different times via stochastic matrices.

In a generic non-Markovian situation the Kolmogorov distance can show monotonic decreasing behavior as well as revival. Indeed, focusing on the examples considered above, for a semi-Markov matrix as in equation (12) and $\pi = 1/2$, one has

$$D_{\mathrm{K}}(\boldsymbol{p}^1(t), \boldsymbol{p}^2(t)) = g(t) D_{\mathrm{K}}(\boldsymbol{p}^1(0), \boldsymbol{p}^2(0)), \tag{47}$$

whereas for $\pi = 1$, one has

$$D_{\mathrm{K}}(\boldsymbol{p}^1(t), \boldsymbol{p}^2(t)) = |q(t)| \, D_{\mathrm{K}}(\boldsymbol{p}^1(0), \boldsymbol{p}^2(0)). \tag{48}$$

Thus, while for $\pi = 1/2$ the Kolmogorov distance is a monotonic contraction for any waiting time distribution, thanks to the fact that $g(t)$ is a survival probability, for $\pi = 1$ the distance among distributions can show revivals depending on the explicit expression of $q(t)$, as can be seen from figure 1 for the case of convolution of two exponential waiting time distributions with the same parameter.

We have thus studied, by means of explicit examples, the behavior of the probability vector or one-point probability $\boldsymbol{p}(t)$ of a classical process. In particular, we have seen that while for a Markovian process P-divisibility is always granted and the Kolmogorov distance is a monotonic contraction, non-Markovianity can spoil these features, even though neither the lack of P-divisibility nor the growth of the Kolmogorov distance can be taken as necessary signatures of non-Markovianity. This substantiates the fact that the non-Markovianity of a classical process cannot be traced back to the behavior of the one-point probabilities only, since it involves all n-point probabilities.

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4. Quantum non-Markovian processes

We now come back to the quantum realm, studying a class of quantum dynamics that has a clear-cut physical meaning, allowing both for the evaluation and comparison of two recently introduced measures of non-Markovianity for the quantum case and for a direct connection with the classical situation analyzed in section 3. Indeed, while in the classical case one has a well-settled definition of non-Markovianity for a stochastic process, which can be used to speak of Markovianity or non-Markovianity of the time evolution of the associated probability distributions, equivalent characterizations for the quantum case were proposed only recently. As follows from the discussion in section 3, such approaches cope, by necessity, with the behavior of the one-point probabilities only, which can be obtained from the statistical operator $\rho(t)$, since a definition involving the whole hierarchy of n-point probabilities cannot be introduced without explicit reference to a particular choice of the measurement scheme. Note that in the study of quantum dynamics one speaks of measures of non-Markovianity, since apart from clarifying what is the signature of non-Markovianity, so as to define it and therefore make it detectable, one would like to quantify the degree of non-Markovianity of a given dynamics (for recent applications, see e.g. [37, 38] and references therein).

The two measures we will consider here [12, 13] actually, respectively, rely on the violation of the quantum analogue of the classical properties of P-divisibility and monotonic decrease in time of the Kolmogorov distance, which we have considered in section 3 and which holds true for the Markovian case, whereas other approaches have also been considered [11]. Note that the violation of these properties in the classical case provide a sufficient but not necessary condition for detecting a non-Markovian process, as clarified in the examples considered in section 3.1.

4.1. Quantum semi-Markov processes

As in the classical case, in order to study the Markovian or non-Markovian features of a quantum dynamics, we consider a class of time evolutions that allows for an explicit treatment and the connection to a classical counterpart. Let us study the dynamics given by master equations of the form [10]

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho(t) = \int_0^t \mathrm{d}\tau\, k(t-\tau)\,[\mathcal{E} - \mathbb{1}]\,\rho(\tau), \tag{49}$$

where $\mathcal{E}$ is a CPT map and $k(t)$ a memory kernel associated with a waiting time distribution $f(t)$ according to equation (16). Such master equations generate, by construction, a completely positive dynamics, which provide a quantum counterpart of classical semi-Markov processes [5, 31]. Indeed, as can be checked, the solution of equation (49) can be written as

$$\rho(t) = \Phi(t, 0)\rho(0) = \sum_{n=0}^{\infty} p_n(t)\mathcal{E}^n \rho(0), \tag{50}$$

with $p_n(t)$ fixed by $f(t)$ according to equation (26), while $\mathcal{E}^n$ denotes the n-fold composition of the map $\mathcal{E}$. The map $\Phi(t, 0)$ is thus completely positive by construction and can be interpreted as the repeated action of the CPT map $\mathcal{E}$, implementing a quantum operation, distributed in time according to the waiting time distribution $f(t)$. For the case of an exponential waiting time distribution, the memory kernel is a delta function and one goes back to a semigroup dynamics. In section 3, we considered semi-Markov processes with a semi-Markov matrix of the form

of equation (12), with arbitrary $f(t)$ and Π, a bistochastic matrix. The Markovianity or non-Markovianity of the process depends solely on the choice of $f(t)$, whereas P-divisibility and the behavior of the Kolmogorov distance depend on both $f(t)$ and Π. In the quantum setting, we also leave $f(t)$ arbitrary and consider bistochastic CPT maps, in the sense that $\mathcal{E}[\mathbb{1}] = \mathbb{1}$, so that $\Phi(t, 0)$ is also bistochastic, preserving both the trace and identity.

4.2. Dephasing dynamics and the time-local equation

A purely quantum dynamics, only affecting coherences, is obtained considering the CPT map

$$\mathcal{E}_z\rho = \sigma_z\rho\sigma_z, \tag{51}$$

which describes dephasing. It is to be stressed that while a dephasing dynamics can be formally represented in terms of the action of local random unitary operators, e.g. due to random fluctuating fields [39], it arises in many physically interesting situations and the relevance of non-Markovian effects in this setting has been the object of both theoretical and experimental efforts in recent years (see e.g. [40] and [41]). The map $\mathcal{E}_z$ satisfies $\mathcal{E}_z^{2n} = \mathbb{1}$ and $\mathcal{E}_z^{2n+1} = \mathcal{E}_z$, so that one has

$$\rho(t) = p_{\text{even}}(t)\rho(0) + p_{\text{odd}}(t)\sigma_z\rho(0)\sigma_z = \begin{pmatrix} \rho_{11}(0) & q(t)\rho_{10}(0) \\ q(t)\rho_{01}(0) & \rho_{00}(0) \end{pmatrix}, \tag{52}$$

recalling the definition (28) of $q(t)$ and considering matrix elements in the basis of eigenvectors of σ_z. Before addressing the issue of characterization of these dynamics, it is of interest to recast the integrodifferential master equation (49) in a time-convolutionless form. Indeed, while Markovianity or non-Markovianity is a property of the solution $\rho(t)$, rather than of the equation, it is quite important to read the signatures of a non-Markovian behavior from the equations themselves, and this task turns out to be much easier when the equations are written in time-local form. To rewrite the master equation (49) in time-local form we follow the constructive approach used in [42]. We thus obtain the time-convolutionless generator, which is formally given by $\dot{\Phi}(t, 0)\Phi^{-1}(t, 0)$, by a representation of the map as a matrix with respect to a suitable operator basis in $\mathbb{C}^2$, given by $\{X_i\} = \{\frac{1}{\sqrt{2}}\mathbb{1}, \frac{1}{\sqrt{2}}\sigma_i\}_{i=1,2,3}$, orthonormal according to $\text{Tr}_S\{X_iX_j\} = \delta_{ij}$. The matrix $F = (F_{kl})$ associated with the map Φ is then determined to be

$$F_{kl} = \text{Tr}_S\{X_k^\dagger\Phi[X_l]\},$$

so that

$$\Phi[\rho] = \sum_{kl} F_{kl}\text{Tr}_S\{X_l^\dagger\rho\}X_k.$$

The matrix F for our map is given by

$$F(t, 0) = \text{diag}(1, q(t), q(t), 1) \tag{53}$$

and accordingly the time-convolutionless master equation simply reads

$$\frac{\text{d}}{\text{dt}}\rho(t) = \gamma(t)\mathcal{L}_z[\rho(t)], \tag{54}$$

where we have a single quantum channel given by

$$\mathcal{L}_z[\rho] = \sigma_z\rho\sigma_z - \rho \tag{55}$$

and the time-dependent rate $\gamma(t)$ is

$$\gamma(t) = -\frac{1}{2}\frac{\dot{q}(t)}{q(t)} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\log|q(t)|. \tag{56}$$

4.2.1. Divisibility. Relying on the matrix representation of the map $\Phi(t, 0)$ given by equation (53), we are now in a position to study its divisibility according to equation (2). In particular, the map $\Phi(t, 0)$ will turn out to be P-divisible if the matrices

$$F(t, s) = \mathrm{diag}\left(1, \frac{q(t)}{q(s)}, \frac{q(t)}{q(s)}, 1\right) \tag{57}$$

obtained as in equation (3) represent a positive map $\Phi(t, s)$ for any $t \geqslant s \geqslant 0$, which is the case if

$$\left|\frac{q(t)}{q(s)}\right| \leqslant 1. \tag{58}$$

This condition is satisfied if $|q(t)|$ is a monotonic decreasing function and therefore the time-dependent rate $\gamma(t)$ is always positive. In order to assess when CP-divisibility holds, one can consider positivity of the associated Choi matrix [43], which still leads to the constraint (58). It follows that for this model, CP-divisibility and P-divisibility are violated at the same time whenever $|q(t)|$ increases, so that $\gamma(t)$ becomes negative. Thus, as discussed in [17], for the case of a single quantum channel, positivity of the time-dependent rate ensures CP-divisibility of the time evolution, which is violated if $\gamma(t)$ becomes negative at some point.

4.2.2. Measures of non-Markovianity. We can now evaluate the measures of non-Markovianity for this model according to the two approaches devised in [12] and [13].

   The approach of Breuer *et al* [12] relies on the study of the behavior in time of the trace distance between different initial states. The trace distance between quantum states quantifies their distinguishability [35, 36] and is defined in terms of their distance with respect to the trace class norm, thus providing the natural quantum analogue of the Kolmogorov distance

$$D\left(\rho^1(t), \rho^2(t)\right) = \tfrac{1}{2}\|\rho^1(t) - \rho^2(t)\|_1. \tag{59}$$

For our case it reads

$$D\left(\rho^1(t), \rho^2(t)\right) = \sqrt{\Delta_p(0)^2 + |\Delta_c(0)|^2\, q^2(t)}, \tag{60}$$

where we have set

$$\Delta_p(0) = \rho_{11}^1(0) - \rho_{00}^2(0), \tag{61}$$

$$\Delta_c(0) = \rho_{10}^1(0) - \rho_{10}^2(0) \tag{62}$$

for the differences of the populations and the coherences between two given initial states $\rho^1(0)$ and $\rho^2(0)$, respectively. Its time derivative is then given by

$$\sigma(t, \rho^{1,2}(0)) = \frac{|\Delta_c(0)|^2}{\sqrt{\Delta_p(0)^2 + |\Delta_c(0)|^2\, q^2(t)}}q(t)\dot{q}(t), \tag{63}$$

so that the trace distance among states can indeed grow provided $q(t)$ and $\dot{q}(t)$ have the same sign, so that $|q(t)|$ does increase. Thus the map has a positive measure of non-Markovianity whenever P-divisibility or equivalently CP-divisibility is broken.

In order to obtain the measure of non-Markovianity for this time evolution according to [12], one has to integrate the quantity $\sigma(t, \rho^{1,2}(0))$ over the region of time Ω_+ in which it is positive, and then maximize the result over all possible initial pairs of states. The region Ω_+ now corresponds to the time intervals where $|q(t)|$ increases, and the maximum is obtained for initial states such that $\Delta_p(0) = 0$ and $\Delta_c(0) = 1$, so that we have the following explicit expression for the measure of non-Markovianity:

$$\mathcal{N}(\Phi) = \int_{\Omega_+} dt \frac{d}{dt} |q(t)| = \sum_i (|q(b_i)| - |q(a_i)|), \tag{64}$$

where we have set $\Omega_+ = \bigcup_i (a_i, b_i)$. The couple of states that maximizes the growth of the trace distance is given in this case by the pure states $\rho^{1,2}(0) = |\psi_\pm\rangle\langle\psi_\pm|$, with $|\psi_\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$.

The approach of Rivas *et al* [13] instead focuses on the CP-divisibility of the map. While for this model this requirement for non-Markovianity is satisfied at the same time as the growth of the trace distance, the effect is quantified in a different way. Indeed these authors quantify non-Markovianity as the integral

$$\mathcal{I}(\Phi) = \int_{\mathbb{R}_+} dt \, \mathfrak{g}(t), \tag{65}$$

where the quantity $\mathfrak{g}(t)$ is given by

$$\mathfrak{g}(t) = \lim_{\epsilon \to 0} \frac{\frac{1}{2} \|\Phi_{\mathrm{Choi}}(t, t+\epsilon)\|_1 - 1}{\epsilon}, \tag{66}$$

where Φ_{Choi} denotes the Choi matrix associated with the map Φ, and it is different from zero only when CP-divisibility is broken. For the case at hand, one has

$$\mathcal{I}(\Phi) = \int_{\Omega_+} dt \frac{d}{dt} \log |q(t)| = \sum_i (\log |q(b_i)| - \log |q(a_i)|) = -2 \int_{\Omega_+} dt \, \gamma(t). \tag{67}$$

For this model, the growth of $|q(t)|$ determines both the breaking of CP-divisibility and the growth of the trace distance, so that both the approaches detect non-Markovianity at the same time, even though they quantify it in different ways. This, however, is not generally true, as was observed in [17] and considered in [23, 44]. We will point out examples of the differing performance of the two measures later in this paper. Exploiting the results of section 3.1, it is now interesting to consider explicit choices of waiting time distributions so as to clarify the different possible behaviors.

4.2.3. Explicit example. For the case of a memoryless waiting time distribution of the form of equation (22), so that $k(t)$ is actually a delta function and $q(t) = \exp(-2\lambda t)$, according to equation (56) the function $\gamma(t)$ is simply given by the positive constant λ. Each non-Markovianity measure is easily assessed to be zero.

To consider non-trivial situations, non-Markovian in the classical case, let us first assume a waiting time distribution of the form equation (29), arising by convoluting two exponential memoryless distributions with the same parameter. The function $q(t)$ is then given by equation (31), so that $\gamma(t)$ reads

$$\gamma(t) = \lambda \frac{1}{1 + \mathrm{cotg}(\lambda t)}, \tag{68}$$
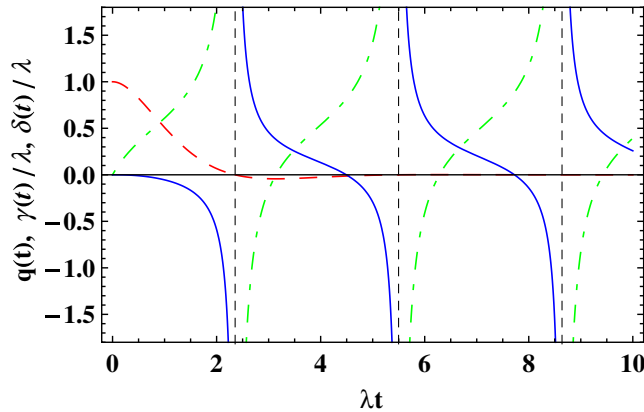
Figure 4. Plot of q (dashed line), γ (dot-dashed line) and δ (continuous line) defined in equations (28), (56) and (83), respectively, as functions of λt for the convolution of two equal exponential waiting time distributions. The vertical asymptotes denote the points where $q(t)$ goes through zero, so that $\gamma(t)$ diverges. The functions $\gamma(t)$ and $\delta(t)$ appear as time-dependent coefficients in front of the various quantum channels in the time local quantum master equations given by equations (54) and (80), so that their sign determines the P-divisibility and CP-divisibility of the corresponding quantum semi-Markov processes, as discussed in section 4.1.

which indeed takes on both positive and negative values, diverging for $\lambda t = (3/4)\pi \mod \pi$. Both functions are plotted in figure 4. In this case the region Ω_+ can be exactly determined and is given by

$$\Omega_+ = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{\lambda}(\pi + n\pi), \frac{1}{\lambda}\left(\frac{3}{4}\pi + n\pi\right) \right).$$

As already observed, both measures become non-zero when $|q(t)|$ grows. The measure proposed by Breuer *et al* according to equation (64) can now be exactly calculated and is given by

$$\mathcal{N}(\Phi) = \sum_{n=0}^{\infty} (-)^{n+1} \left[q\left(\frac{\pi + n\pi}{\lambda}\right) - q\left(\frac{3\pi/4 + n\pi}{\lambda}\right) \right] = \frac{1}{e^\pi - 1}, \tag{69}$$

which is finite and independent of λ. It is to be stressed that considering the convolution of a higher number of exponential waiting time distributions, one obtains a higher value for this measure, according to the fact that the overall waiting time distribution departs more and more from the memoryless exponential case [33]. The measure proposed by Rivas *et al* instead is equal to infinity, $\mathcal{I}(\Phi) = \infty$, due to the fact that $q(t)$ goes through zero and therefore $\gamma(t)$ diverges. Actually $\mathcal{I}(\Phi)$ is equal to infinity whenever the inverse time-evolution mapping fails to exist. It therefore quantifies in the same way non-Markovianity for quite different situations, e.g. in this case waiting time distributions given by the convolution of a different number of exponentials.

As a further example, we consider a convolution of two different exponential distributions, corresponding to equations (35) and (38), so that one now has

$$\gamma(t) = 2\frac{p}{s}\frac{1}{1 + \chi \mathrm{Coth}\left(\frac{st}{2}\chi\right)}. \tag{70}$$
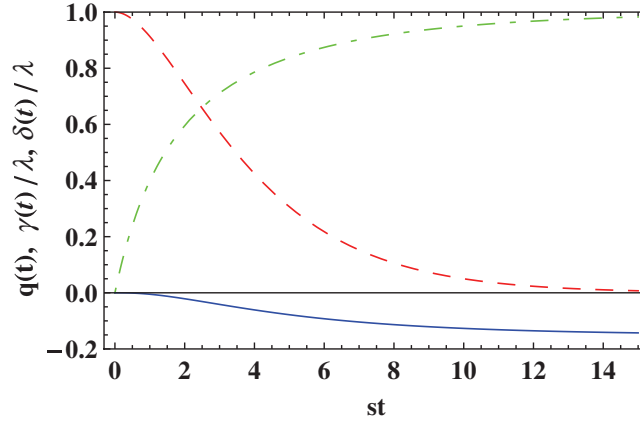
Figure 5. The same as figure 4, but for the convolution of two different exponential waiting time distributions. We plot the quantities as a function of st, taking $p/s^2 = 0.12$. Note that in this case $\gamma(t)$ (dot-dashed line) is always positive, while $\delta(t)$ (continuous line) is always negative. The function $q(t)$ (dashed line) decreases monotonically, reaching the value zero only at infinity.

Recalling equations (37) and (39), the argument of the hyperbolic cotangent is real, so that $\gamma(t)$ always stays positive, if $p \leqslant s^2/8$. In this case, despite the underlying non-Markovian classical process, both measures of non-Markovianity are equal to zero. The behavior of $q(t)$ and $\gamma(t)$ for this case is depicted in figure 5. When $p > s^2/8$, which includes the case $\lambda_1 = \lambda_2$, $q(t)$ again oscillates between positive and negative values, so that one has a similar behavior to before, with $\mathcal{N}(\Phi)$ assuming a finite value and $\mathcal{I}(\Phi) = \infty$.

Finally, let us consider a convex mixture of two memoryless distributions as given by equation (40), so that $q(t)$ is now given by equation (42) and one has

$$\gamma(t) = \langle\lambda\rangle \frac{1 + \frac{1}{4}\frac{(\lambda_1+\lambda_2-3\langle\lambda\rangle)(\lambda_1+\lambda_2+\langle\lambda\rangle)-\bar{\lambda}^2}{\bar{\lambda}\langle\lambda\rangle}\mathrm{Coth}\left(\frac{\bar{\lambda}t}{2}\right)}{1 + \frac{(\lambda_1+\lambda_2-3\langle\lambda\rangle)}{\bar{\lambda}}\mathrm{Coth}\left(\frac{\bar{\lambda}t}{2}\right)}, \tag{71}$$

which according to the definitions of $\langle\lambda\rangle$ and $\bar{\lambda}$ given in equations (43) and (44) can be checked to always take on positive values. Its behavior is given in figure 6. In this situation both measures are equal to zero.

4.3. Dephasing dynamics via projection

A quantum dynamics corresponding to pure dephasing is also obtained considering a CPT map $\mathcal{E}$ which is also a projection, that is,

$$\mathcal{E}_P\rho = \sigma_+\sigma_-\rho\sigma_+\sigma_- + \sigma_-\sigma_+\rho\sigma_-\sigma_+, \tag{72}$$

so that one has idempotency $\mathcal{E}_P^2 = \mathcal{E}_P$. For this case, the analysis closely follows that performed for $\mathcal{E}_z$, but the crucial quantity instead of $q(t)$ is the survival probability $g(t)$, similarly to the classical case with $\pi = 1/2$ dealt with in section 3.1. The matrix F is given by

$$F(t, 0) = \mathrm{diag}(1, g(t), g(t), 1) \tag{73}$$
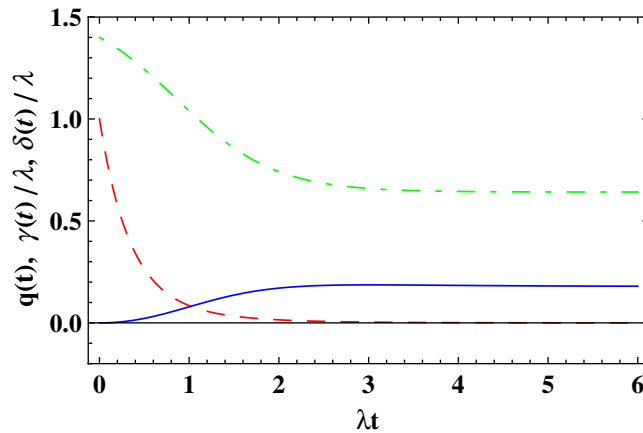
Figure 6. The same as figure 4, but for a convex mixture of two different exponential waiting time distributions. We plot q (dashed line), γ (dot-dashed line) and δ (continuous line) as a function of λt, taking $\lambda_1 = a_1\lambda$ and $\lambda_2 = a_2\lambda$, with $a_1 = 1$ and $a_2 = 6$, together with the mixing parameter $\mu = 0.6$. For this kind of waiting time distribution all functions always stay positive, quickly reaching an asymptotic constant value.

and the time-local master equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = h(t)\left(\mathcal{L}_{+-}\left[\rho(t)\right] + \mathcal{L}_{-+}\left[\rho(t)\right]\right), \tag{74}$$

with the Lindblad operators

$$\mathcal{L}_{+-}[\rho] = \sigma_+\sigma_-\rho\sigma_+\sigma_- - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho\} \tag{75}$$

and similarly for $\mathcal{L}_{-+}$. The quantity $h(t)$ is given by

$$h(t) = \frac{f(t)}{g(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}\log g(t), \tag{76}$$

which provides the so-called hazard rate function associated with the waiting time distribution $f(t)$, given by the ratio of waiting time distribution to survival probability. It gives information on the probability for the first jump to occur right after time t [45]. Note that the survival probability is a positive monotonically decreasing function, and the hazard rate function is always positive. As a result, CP-divisibility always holds, so that both non-Markovianity measures are equal to zero, whatever the waiting time distribution is.

4.4. Dissipative dynamics and the time-local equation

The choice of the CPT map considered above, corresponding to pure dephasing, shows how different probability densities for the waiting time, corresponding to different distributions of the action of the quantum operation in time, can lead to dynamics whose measures of non-Markovianity can be both positive or zero, irrespective of the fact that the only memoryless waiting time distribution is the exponential one. In this case, however, there is no direct

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connection with a classical dynamics, since only the coherences evolve in time. Another natural choice of CPT map that leads to a non-trivial dynamics for the populations is given by

$$\mathcal{E}_{\pm}\rho = \sigma_- \rho \sigma_+ + \sigma_+ \rho \sigma_-. \tag{77}$$

As we now show, this expression for the CPT map leads to a time-local master equation given by equation (80), where channels related to the operators σ_+, σ_- and σ_z appear with a particular expression of the time-dependent rates. In particular, while the operator structure is the most general one, the rates in front of the two dissipative channels are equal. As shown in [5, 31] within the framework of quantum semi-Markov processes, more general situations can also be considered, thus covering a wider range of physical applications. This example, however, already serves to highlight the differing behavior of the two distinct measures of non-Markovianity as discussed in section 4.4.4, and is amenable to a full analytical treatment. This makes the comparison with the classical case and the discussion of the various features more transparent. The map $\mathcal{E}_{\pm}$ satisfies $\mathcal{E}_{\pm}^{2n} = \mathcal{E}_{\pm}^2$ and $\mathcal{E}_{\pm}^{2n+1} = \mathcal{E}_{\pm}$, so that one can obtain the explicit representation

$$\rho(t) = \begin{pmatrix} p_{\text{even}}(t)\rho_{11}(0) + p_{\text{odd}}(t)\rho_{00}(0) & g(t)\rho_{10}(0) \\ g(t)\rho_{01}(0) & p_{\text{odd}}(t)\rho_{11}(0) + p_{\text{even}}(t)\rho_{00}(0) \end{pmatrix}, \tag{78}$$

where $g(t)$ denotes as usual the survival probability. As in the previous case, we can obtain the matrix F representing the action of the map with respect to the chosen basis of operators in $\mathbb{C}^2$, now given by

$$F(t, 0) = \text{diag}\,(1, g(t), g(t), q(t)), \tag{79}$$

and accordingly the time-convolutionless master equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \gamma(t)\,(\mathcal{L}_+\,[\rho(t)] + \mathcal{L}_-\,[\rho(t)]) + \delta(t)\mathcal{L}_z[\rho(t)], \tag{80}$$

where in addition to $\mathcal{L}_z$ as given by equation (55), the quantum channels

$$\mathcal{L}_+[\rho] = \sigma_+ \rho \sigma_- - \tfrac{1}{2}\{\sigma_- \sigma_+, \rho\} \tag{81}$$

and

$$\mathcal{L}_-[\rho] = \sigma_- \rho \sigma_+ - \tfrac{1}{2}\{\sigma_+ \sigma_-, \rho\} \tag{82}$$

appear. The time-dependent rate $\gamma(t)$ is still given by equation (56), while the function $\delta(t)$ is given by the difference

$$\delta(t) = \tfrac{1}{2}(h(t) - \gamma(t)), \tag{83}$$

where $h(t)$ is the so-called hazard rate function introduced in equation (76), which is always positive.

4.4.1. Divisibility. Also in this case we can consider the divisibility properties of the time evolution. According to the matrix representation of the map, equation (3) now leads to

$$F(t, s) = \text{diag}\left(1, \frac{g(t)}{g(s)}, \frac{g(t)}{g(s)}, \frac{q(t)}{q(s)}\right), \tag{84}$$

so that thanks to the property of the survival probability the only condition for P-divisibility is still given by equation (58). Therefore the map is P-divisible whenever $|q(t)|$ is a monotonic

decreasing function. Note that this condition implies positivity of $\gamma(t)$ and therefore of the time-dependent rate in front of the $\mathcal{L}_+$ and $\mathcal{L}_-$ channels, which affect the dynamics of the populations. In order to study CP-divisibility, one has to consider the associated Choi matrix, whose positivity is granted on the further condition

$$\frac{g(t)}{g(s)} \leqslant \frac{1}{2} \left(1 + \frac{q(t)}{q(s)} \right), \tag{85}$$

which sets a nontrivial requirement, implying positivity of the function $\delta(t)$ which provides the coefficient of the purely quantum channel $\mathcal{L}_z$. Thus CP-divisibility is violated if and only if at least one of the prefactors in the time-local form equation (80) becomes negative. Note, however, that in this case, due to the presence of different quantum channels, P-divisibility and CP-divisibility are not necessarily violated together, since it can well happen that $\gamma(t)$ stays positive, but $\delta(t)$ takes on negative values. As discussed in the examples below and shown in figure 5, for a suitable choice of waiting time distribution one can have a process that is P-divisible, but not CP-divisible.

4.4.2. Measures of non-Markovianity. Also for this model we can obtain the explicit expression for the measures of non-Markovianity according to [12] and [13]. The trace distance now reads

$$D\left(\rho^1(t), \rho^2(t)\right) = \sqrt{q^2(t)\Delta_p(0)^2 + |\Delta_c(0)|^2 \ g^2(t)}, \tag{86}$$

where we have used the same notation as in equations (61) and (62), so that the derivative is

$$\sigma\left(t, \rho^{1,2}(0)\right) = \frac{\Delta_p(0)^2 q(t)\dot{q}(t) - |\Delta_c(0)|^2 \ g(t)f(t)}{\sqrt{q^2(t)\Delta_p(0)^2 + |\Delta_c(0)|^2 \ g^2(t)}} \tag{87}$$

and can grow in the region Ω_+ where $q(t)$ and $\dot{q}(t)$ have the same sign. In this region, $|q(t)|$ does increase and P-divisibility—but in general, as we have seen, not CP-divisibility—is granted. The growth is maximal for $\Delta_c(0) = 0$ and $\Delta_p(0) = 1$, so that the couple of states that maximize it are given by the projectors on ground and excited states. As a result, similarly to before, we have, for the measure of non-Markovianity introduced by Breuer *et al*,

$$\mathcal{N}(\Phi) = \int_{\Omega_+} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} |q(t)| = \sum_i \left(|q(b_i)| - |q(a_i)| \right). \tag{88}$$

This result for the choice of the CPT map $\mathcal{E}_\pm$ is exactly the same as that for the CPT map $\mathcal{E}_z$. This measure becomes non-zero if and only if P-divisibility is broken. It can be shown that this is generally the case for a bistochastic map Φ in $\mathbb{C}^2$.

The criterion of Rivas *et al* instead assigns to the map a non-zero measure whenever one of the coefficients $\gamma(t)$ and $\delta(t)$ takes on negative values, so that CP-divisibility is broken. Since $h(t)$ is always positive, these two functions can take on negative values only on separate time intervals, as can also be seen from figure 4. The measure is then given by equation (65), where according to equation (66) we have $\mathfrak{g}(t) = 0$, whenever both $\gamma(t)$ and $\delta(t)$ are positive, while $\mathfrak{g}(t) = -2\gamma(t)$ whenever $\gamma(t)$ is negative, and $\mathfrak{g}(t) = -2\delta(t)$ in the complementary time

intervals in which $\delta(t)$ takes on negative values. Note that this measure $\mathcal{I}(\Phi)$ can become positive even when the measure $\mathcal{N}(\Phi)$ is zero. Indeed, the latter measure for this dynamics is related to P-divisibility rather than CP-divisibility.

4.4.3. Population dynamics.   For the dynamics described by equation (49), with the CPT map given by $\mathcal{E}_\pm$ as in equation (77), coherences and populations decouple, and the populations obey the same equation as the transition probabilities of the classical semi-Markov processes considered in section 3 for $\pi = 1$ and arbitrary waiting time distribution. This is immediately seen identifying the two components of the probability vector with the populations in excited and ground states. Setting $P_+(t) = \langle +|\rho(t)|+\rangle$ and $P_-(t) = \langle -|\rho(t)|-\rangle$ one has in fact from equation (49) with $\mathcal{E}_\pm$ the integrodifferential rate equations

$$\frac{\mathrm{d}}{\mathrm{d}t}P_\pm(t) = \int_0^t \mathrm{d}\tau \, k(t - \tau) \left[P_\mp(\tau) - P_\pm(\tau) \right] \tag{89}$$

corresponding to equation (14) for

$$W(\tau) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} k(\tau). \tag{90}$$

Also in the classical case the integrodifferential time-evolution equation can be generally recast in time-local form [46]. In this case, equation (89) would then simply correspond to the diagonal matrix elements of equation (80).

The Kolmogorov distance as in equation (48) is given by

$$D_K \left(\left\{ P_+^1(t), P_-^1(t) \right\}, \left\{ P_+^2(t), P_-^2(t) \right\} \right) = \left| \Delta_p(0) \right| \left| q(t) \right|,$$

so that $\mathcal{N}(\Phi)$, being obtained by considering as initial states the projections onto ground and excited states, is also given by taking the maximum over the possible initial classical states of the integral of the derivative σ_K of the Kolmogorov distance in the time intervals in which it is positive. Growth of the Kolmogorov distance again depends solely on the behavior of $|q(t)|$, which determines whether the map is P-divisible or not. In view of this connection it appears that one can have the non-Markovianity measure $\mathcal{N}(\Phi)$ for the quantum map Φ equal to zero even when the dynamics for the populations can be related to a non-Markovian classical process. Again this is not too surprising, since the one-point probabilities cannot really keep track of Markovianity or non-Markovianity in the classical sense, even though in the non-Markovian case they can show up different behaviors than those typical of the Markovian one.

4.4.4. Explicit examples.   At variance with the case of pure dephasing, the two measures of non-Markovianity do not agree for this model. The measure $\mathcal{N}(\Phi)$ becomes positive as soon as P-divisibility is broken, which depends on the sign of $\gamma(t)$ only, while $\mathcal{I}(\Phi)$ becomes positive even when only CP-divisibility does not hold, which also depends on the sign of the function $\delta(t)$ appearing in front of the purely quantum channel $\mathcal{L}_z$, which determines the dynamics of the coherences. To consider the behavior of the measures for this model we thus also have to consider the quantity $\delta(t)$, which is simply equal to zero for an exponential waiting time distribution, so that in the proper Markovian case this pure quantum channel is not available.

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For the case of the convolution of two equal exponential distributions exploiting equations (29) and (30) together with equation (68), we have

$$\delta(t) = \frac{\lambda}{2}\left(\frac{\lambda t}{1 + \lambda t} - \frac{1}{1 + \text{cotg}(\lambda t)}\right), \tag{91}$$

so that both $\gamma(t)$ and $\delta(t)$ oscillate in sign and diverge when $\text{cotg}(\lambda t)$ takes on the value -1, as shown in figure 4. In this case both measures are positive, while considering the convolution of two different exponential distributions one has thanks to equations (35), (36) and (70)

$$\delta(t) = \frac{p}{s}\left(\frac{1}{1 + \xi \text{Coth}\left(\frac{st}{2}\xi\right)} - \frac{1}{1 + \chi \text{Coth}\left(\frac{st}{2}\chi\right)}\right). \tag{92}$$

If the ratio λ_1/λ_2 is far enough from one, $\gamma(t)$ given by equation (70) as discussed above always stays positive, so that one has P-divisibility and the measure $\mathcal{N}(\Phi)$ is equal to zero. In contrast, the function $\delta(t)$ is negative, so that the coefficient in front of the quantum channel is always negative and CP-divisibility is violated, thus determining a positive measure $\mathcal{I}(\Phi)$. This situation is considered in figure 5.

As the last example we consider a convex mixture of exponential distributions, leading to equation (71) as well as

$$h(t) = \frac{\mu \lambda_1 e^{-\lambda_1 t} + (1 - \mu)\lambda_2 e^{-\lambda_2 t}}{\mu e^{-\lambda_1 t} + (1 - \mu)e^{-\lambda_2 t}}. \tag{93}$$

For this case, independently of the value of the mixing parameter μ, one has that both $\gamma(t)$ and $\delta(t)$ stay positive. A dynamics of this kind for an open quantum system is often termed a time-dependent Markovian [17], since at any time the generator is in Lindblad form. Once again both measures $\mathcal{N}(\Phi)$ and $\mathcal{I}(\Phi)$ give a zero value of non-Markovianity, despite the fact that the underlying waiting time distribution is not memoryless, corresponding to a population dynamics described by a non-Markovian classical process.

5. Conclusions and outlook

In this paper, we have analyzed the notion of non-Markovianity for the dynamics of open quantum systems, starting from the classical setting and focusing on concrete examples. While knowledge of a non-Markovian classical process requires information on all the conditional probability densities, when studying the dynamics of an open system one only considers the evolution of the state, expressed by a probability vector in the classical case and a statistical operator in the quantum case. The notion of non-Markovianity for classical processes and for dynamical evolutions are therefore by necessity distinct concepts. One is then naturally led to the question of whether and how the non-Markovianity of a process is reflected in the behavior of the one-point probability. For processes that are Markovian according to the classical definition, both P-divisibility, in the sense of the existence of well-defined stochastic matrices connecting probability vectors at different time points, and a monotonic decrease, in time, of the Kolmogorov distance between states arising from different initial conditions are always obeyed. Therefore the lack of these properties can be interpreted as a signature of non-Markovianity and can be used to quantify it. Note, however, that due to the fact that the classical definition of non-Markovianity actually involves all n-point probability densities, these signatures indeed provide a different notion of non-Markovianity, which only gives a

sufficient condition in order to assess non-Markovianity in the original sense. This behavior is shown by means of examples relying on the study of certain semi-Markov processes. We stress, in particular, that in the classical case P-divisibility is not equivalent to Markovianity. Such signatures of non-Markovianity can be brought over to the quantum framework, considering the notion of CP-divisibility, corresponding to the fact that the quantum time-evolution map can be arbitrarily split into CPT maps corresponding to intermediate time intervals, and of the trace distance, which is the quantum version of the Kolmogorov distance. These two criteria are at the basis of two recently introduced measures of non-Markovianity for open quantum systems [12, 13], which we compare here considering a quantum counterpart of classical semi-Markov processes. Moreover, analyzing both the integrodifferential and the time-local form of the equations of motion, we show that one can point to possible signatures of non-Markovianity to be read directly at the level of the equation, despite the fact that Markovianity or non-Markovianity itself is a feature of the solution. In this respect it appears that the time-local form of the equations, despite isolated singularities, is certainly more convenient.

More specifically, the classes of examples discussed in this paper, in addition to clarifying the relationship between the distinct notions of non-Markovianity used for classical processes and for dynamical evolutions, allow an exact evaluation of the measure of non-Markovianity introduced for quantum dynamical maps; see e.g. the remarkable result given by equation (69). These physical examples further allow an explicit comparison of the two measures for whole classes of quantum dynamical maps. In particular, as shown by equations (64) and (67) and discussed thoroughly in section 4.2.3, we have demonstrated that the measure based on CP-divisibility [13] gives the same infinite value to quite different time evolutions, at variance with the measure based on the dynamics of the trace distance [12], which assigns them different weights.

In this paper, we have discussed the different definitions of non-Markovianity relevant for classical stochastic processes and dynamical evolutions. The latter are based on the divisibility of the time-evolution map and on the dynamic behavior of the distinguishability between different initial states. While these definitions of non-Markovianity can be considered in both the classical and the quantum case, as discussed in sections 3 and 4 the original definition of non-Markovianity for a classical process cannot be transferred to the quantum realm, because of basic principles of quantum mechanics. On the one hand, to make statements about the value of a certain observable at different times, a measurement scheme has to be specified which affects the subsequent time evolution; on the other, the statistical operator of a quantum system provides different, and generally incompatible, classical probability densities for different observables, as a typical feature of quantum probability with respect to classical probability [47, 48]. In this respect, one should be satisfied with different definitions and measures of non-Markovianity within the classical and the quantum framework.

It is clear that physical systems can provide us with much more complicated dynamics than those addressed in this paper and in the recent literature. The main aim of the examples discussed, however, was to consider realistic situations for which a thorough and exact analysis is feasible, so as to allow a clarification of the conceptual issues related to the very definition of Markovianity and non-Markovianity discussed and elucidated in the paper, pointing in particular to the connection between classical and quantum situations. We hope that this approach might open the way for an even deeper understanding and characterization of the very notion of Markovianity, spurred by the recent exciting developments.

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Acknowledgments

References

[1] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[2] Stockburger J T and Grabert H 2002 *Phys. Rev. Lett.* **88** 170407
[3] Daffer S, Wódkiewicz K, Cresser J D and McIver J K 2004 *Phys. Rev.* A **70** 010304
[4] Piilo J, Maniscalco S, Harkonen K and Suominen K-A 2008 *Phys. Rev. Lett.* **100** 180402
[5] Breuer H-P and Vacchini B 2008 *Phys. Rev. Lett.* **101** 140402
[6] Shabani A and Lidar D A 2009 *Phys. Rev. Lett.* **102** 100402
[7] Chruscinski D and Kossakowski A 2010 *Phys. Rev. Lett.* **104** 070406
[8] Barchielli A, Pellegrini C and Petruccione F 2010 *Europhys. Lett.* **91** 24001
[9] Laine E-M, Piilo J and Breuer H-P 2010 *Europhys. Lett.* **92** 60010
[10] Budini A A 2004 *Phys. Rev.* A **69** 042107
[11] Wolf M M, Eisert J, Cubitt T S and Cirac J I 2008 *Phys. Rev. Lett.* **101** 150402
[12] Breuer H-P, Laine E-M and Piilo J 2009 *Phys. Rev. Lett.* **103** 210401
[13] Rivas A, Huelga S F and Plenio M B 2010 *Phys. Rev. Lett.* **105** 050403
[14] Hughes B D 1995 *Random Walks and Random Environments* vol 1 (New York: Clarendon)
[15] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
[16] Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821
[17] Laine E-M, Piilo J and Breuer H-P 2010 *Phys. Rev.* A **81** 062115
[18] Hayashi M 2006 *Quantum Information* (Berlin: Springer)
[19] Wolf M M and Cirac J I 2008 *Commun. Math. Phys.* **279** 147
[20] Norris J R 1999 *Markov Chains* (Cambridge: Cambridge University Press)
[21] Chruscinski D, Kossakowski A, Aniello P, Marmo G and Ventriglia F 2010 *Open Syst. Inf. Dyn.* **17** 255
[22] Chruscinski D and Kossakowski A 2010 arXiv:1010:4745v1
[23] Chruscinski D, Kossakowski A and Rivas A 2011 *Phys. Rev.* A **83** 052128
[24] Rivas A and Huelga S F 2011 arXiv:1104:5242v1
[25] Karimipour V and Memarzadeh L 2011 arXiv:1105:2728v1
[26] van Kampen N 1998 *Braz. J. Phys.* **28** 90
[27] Gillespie D T 1998 *Am. J. Phys.* **66** 533
[28] Hänggi P and Thomas H 1977 *Z. Phys.* B **26** 85
[29] Hänggi P and Thomas H 1982 *Phys. Rep.* **88** 207
[30] Feller W 1971 *An Introduction to Probability Theory and its Applications* vol 2 (New York: Wiley)
[31] Breuer H-P and Vacchini B 2009 *Phys. Rev.* E **79** 041147
[32] Feller W 1964 *Proc. Natl Acad. Sci. USA* **51** 653
[33] Cox D R and Miller H D 1965 *The Theory of Stochastic Processes* (New York: Wiley)
[34] Medhi J 1994 *Stochastic Processes* (New York: Wiley)
[35] Fuchs C A and van de Graaf J 1999 *IEEE Trans. Inf. Theory* **45** 1216
[36] Nielsen M and Chuang I 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[37] Apollaro T J G, Di Franco C, Plastina F and Paternostro M 2011 *Phys. Rev.* A **83** 032103
[38] Haikka P, McEndoo S, de Chiara G Palma M and Maniscalco S 2011 arXiv:1105.4790v1
[39] Helm J and Strunz W T 2009 *Phys. Rev.* A **80** 042108
[40] Kaer P, Nielsen T R, Lodahl P, Jauho A-P and Mørk J 2010 *Phys. Rev. Lett.* **104** 157401

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[41] Xu J-S, Li C-F, Zhang C-J, Xu X-Y, Zhang Y-S and Guo G-C 2010 *Phys. Rev.* A **82** 042328
[42] Smirne A and Vacchini B 2010 *Phys. Rev.* A **82** 022110
[43] Choi M D 1975 *Linear Algebr. Appl.* **10** 285
[44] Haikka P, Cresser J D and Maniscalco S 2011 *Phys. Rev.* A **83** 012112
[45] Ross S M 2007 *Introduction to Probability Models* (Burlington, MA: Academic)
[46] Hänggi P, Thomas H, Grabert H and Talkner P 1978 *J. Stat. Phys.* **18** 155
[47] Holevo A S 2001 *Statistical Structure of Quantum Theory* (*Lecture Notes in Physics* vol 67) (Berlin: Springer)
[48] Strocchi F 2005 *An Introduction to the Mathematical Structure of Quantum Mechanics* (Singapore: World Scientific)