Asymptotic Stability of Solitons.

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Outline

NLS

- Model problem

$$\mathrm{i}\psi_t = -\Delta\psi - \beta'(|\psi|^2)\psi \ , \quad x \in \mathbb{R}^3 \ ,$$

$$|\beta(u)(k)| \le C_k \langle x\rangle^{\tilde{p}-k} \ , \quad \tilde{p} \le 1 + \frac{2}{d-2} \ , \quad d = 3$$

- Symmetries:
    - translations $(\psi, q_i) \mapsto \psi(. - q_i \mathbf{e}^i)$, generated by $-\mathrm{i}\partial_{x_j}$
    - Gauge $(\psi, q_4) \mapsto e^{\mathrm{i}q_4}\psi$, generated by i.
- Conservation laws:

$$\mathcal{P}^j(\psi) := \int \frac{\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi}{2}$$

$$\mathcal{P}^4(\psi) := \int |\psi|^2$$

Ground States

- Look for special solutions

$$\psi(x,t) = e^{-\mathrm{i}(\omega_4 t + q_4)} \eta_p(x - (\omega_j t + q_j)\mathbf{e}^j)$$

- η_p is a critical point of

$$H := \int |\nabla\psi|^2 - \beta(|\psi|^2)$$

restricted to

$$\mathcal{S}_p := \{\psi \ : \mathcal{P}^j(\psi) = p^j \ , \ j = 1, ..., 4\}$$

- Ground state. A ground state is the minimum.

Theorem

Assume

- Assumptions on the linearized operator.
- Fermi golden rule (probably generic generic: work in progress)
- $\inf_{p,q} \left\| \psi_0 - e^{-\mathrm{i}q_4} \eta_p(. - q_i \mathbf{e}^j) \right\|_{H^1} \ll 1$

Theorem

There exist functions $\omega(t)$, $p(t)$ $q(t)$ having a limit as $t \to +\infty$ and a state ψ_∞ such that, writing

$$\psi(x,t) = e^{-\mathrm{i}y_4(t))} \eta_{p(t)}(x - y(t)) + \chi(x,t) \ ,$$
$$y_j(t) = \omega_j(t)t + q_j(t) \ , \quad |q_j(t)| \ll 1 \ , \quad j = 1, ..., 4$$

one has

$$\lim_{t \to +\infty} \left\| \chi(t) - e^{\mathrm{i}t\Delta} \psi_\infty \right\|_{L^6} = 0 \ .$$

Comments

- What's new? Old results when the Floquet spectrum has at most 1 eigenvalue: Weinstein, Soffer-Weinstein, Buslaev-Perelman, Cuccagna, Perelman.
- Ideas from D.B.-Cuccagna (on Klein Gordon), Cuccagna (case with potential), Perelman (no eigenvalues, energy space).
- Key difficulty: the generators of the symmetries are unbounded.
  - Development of reduction theory, Darboux theory and Normal form theory with only continuous transformations.
  - Validity of Strichartz estimates for the relevant operators.

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Model probelm

- The equation: $i\psi_t = -\Delta\psi + V\psi + \frac{\delta H_P}{\delta\bar{\psi}(x)}$.
  $\mathcal{H}_0 := -\Delta + V$ with one eigenvalue $\mathcal{H}_0\mathbf{e} = \omega\mathbf{e}$
- Spectral coordinates $\psi = \xi\mathbf{e} + f$:
  $H_0 = \langle\bar{f}; Bf\rangle + \omega|\xi|^2$
- Model nonlinearity

$$H_P := \bar{\xi}^\nu\langle\bar{\Phi}; f\rangle + \xi^\nu\langle\Phi; \bar{f}\rangle$$

- Equations

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\nu\bar{\xi}^\nu\langle\bar{\Phi}; f\rangle$$
$$\dot{f} = -\mathrm{i}(Bf + \xi^\nu\Phi)$$

- Further decoupling $g = f + \xi^\nu\Psi$: if Ψ is such that

$$(B - \nu\omega)\Psi = \Phi$$

  then $\dot{g} = \mathrm{i}Bg + O(|\xi|^\nu|f| + |\xi|^{2\nu-1})$

Dissipation

- Define

$$R_\nu^{\mp} := \lim_{\epsilon\to 0^+} (B - \omega\nu \pm i\epsilon)^{-1} \ , \quad \Psi = R_\nu^- \Phi$$

- then $\Psi(x) \sim \langle x\rangle^{-1}$. Plug in the equation for ξ:

$$\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}\,|\xi|^{2\nu-1}\langle\bar{\Phi}; R_\nu^+\Phi\rangle\xi + O(\xi^{\nu-1}|\langle\Phi; g\rangle|)$$

- Plemelji formula:

$$R_\nu^- \equiv (B - \nu\omega - i0)^{-1} = PV(B-\nu\omega) - i\pi\delta(B-\nu\omega) \ .$$

  implies $\langle\bar{\Phi}; R_\nu^-\Phi\rangle = a - \mathrm{i}b, \ b \geq 0$

- $\dot{\xi} = -\mathrm{i}\omega\xi - \mathrm{i}a|\xi|^{2\nu-1}\xi - b|\xi|^{2\nu-1}\xi$+h.o.t.
- $\frac{d}{dt}|\xi|^2 = -2b|\xi|^{2\nu} \quad \implies |\xi|^\nu \in L^2_t$
- Use normal form to reduce to the model problem.

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Outline

On the wave equation

- The equation.

$$u_{tt} - \Delta u + Vu + m^2 u = -\beta'(u) \ , \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^3$$

  with $-\Delta + V(x) + m^2$ a positive Schrödinger operator (V smooth and fast decaying),
  β' a smooth function function fulfilling $|\beta(u)| \leq Cu^4$

- Consequence. $-\Delta + V$ has finitely many eigenvalues

$$-\lambda_1^2 \leq \cdots \leq -\lambda_n^2 \leq 0$$

  and $\sigma_c(-\Delta + V) = [0, +\infty)$

The theorem of BC09

- Denote $K_0(t) := \frac{\sin(t\sqrt{-\Delta + m^2})}{\sqrt{-\Delta + m^2}}$
- Remark $u(t) := K_0'(t)u_0 + K_0(t)v_0$ solves

$$\begin{aligned} & u_{tt} - \Delta u + m^2 u = 0 \\ & u(x,0) = u_0(x) \qquad \dot{u}(x,0) = v_0(x) \ . \end{aligned}$$

Theorem

There exists $\epsilon_0 > 0$ such that, if

$$\|(u_0, v_0)\|_{H^1 \times L^2} < \epsilon_0$$

then there exist $(u_\pm, v_\pm) \in H^1 \times L^2$ such that

$$\lim_{t \to \pm\infty} \|u(t) - K_0'(t)u_\pm + K_0(t)v_\pm\|_{H^1} = 0$$

Outline

Marsden-Weinstein reduction - generel facts

Connection between the two problems: Marsden-Weinstein reduction.

- Symplectic manifold: $(\mathcal{M}, \omega)$, J Poisson tensor
- Symmetry group: $(q, u) \mapsto e^{qJA}u$, generated by

$$\mathcal{P}(u) := \frac{1}{2}\langle u; Au\rangle \ ,$$

- Invariant Hamiltonian: H, s.t. $H(u) = H(e^{qJA}u)$,
- Reduced system: $\mathcal{S}_p := \{u \in \mathcal{M} \ : \ \mathcal{P}(u) = p\}$ and

$$\mathcal{M}_p := \mathcal{S}_p / \simeq \ , \quad \left(u \sim u' \iff u' = e^{qJA}u\right)$$

- Explicit construction: see the blackboard!
- $\Omega := i^*\omega$, and $H_r := i^*H = H \circ i$.
- Ground state, η_p: minimum of H_r!

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Difficulty

Only continuous group actions:
$u(.) \mapsto u(. - q)$, generated by $\partial_t u = -\partial_x u$.

(1) Does reduction theory holds, and in particular
- Is the reduced manifold a manifold?
- Can one define the reduced system?

(2) Canonical coordinates are needed: is it possible to prove Darboux theorem?

(3) develop transformation theory with unbounded generators

(4) Dispersive estimates: do Strichartz estimates persist under unbounded perturbation?

Outline

Framework

- Phase space:
  - H^k, $k \in \mathbb{Z}$ Scale of Hilbert spaces, $\langle .; . \rangle$ scalar prod of H^0
  - $J : H^k \mapsto H^{k-d}$ Poisson tensor
  - $E := J^{-1}$, $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$
  - $X_H := J\nabla H$ Hamiltonian vector field
- The system: $H := \frac{1}{2}\left\langle A^0 u; u \right\rangle + H_P(u)$; $A^0 : H^k \to H^{k-d_0}$
- Symmetries: $\mathcal{P}^j(u) := \frac{1}{2}\left\langle A^j u; u \right\rangle$;

  $A^j : H^k \to H^{k-d_j}$, JA^j generate a flow: $e^{qJA^j} : H^k \mapsto H^k$, $\forall k$.

Ground state and decomposition of the space

$$A^0\eta_p + \nabla H_P(\eta_p) - \sum_j \lambda_j A^j \eta_p = 0$$

- Assumptions:
  (1) $\mathbb{R}^n \supset I \ni p \mapsto \eta_p \in H^\infty$ is smooth.
  Normalization condition $\mathcal{P}^j(\eta_p) = p^j$.
  (2) $\bigcup_{p\in I}\{\eta_p\}$ is isotropic
- Key remark: $q \mapsto e^{qJA^j}\eta_p$ is smooth!
- Consequence: $\mathcal{T} := \bigcup_{q,p} e^{\sum q_j JA^j}\eta_p \simeq I \times (\mathbb{T}^l \times \mathbb{R}^{n-l})$,
- Natural decomposition: $H^0 \equiv T_{\eta_p}H^0 \simeq T_{\eta_p}\mathcal{T} \oplus T^\omega_{\eta_p}\mathcal{T}$
  with $T^\omega_{\eta_p}\mathcal{T} := \left\{U \ : \ \omega(U;X) = 0\ ,\ \forall X \in T_{\eta_p}\mathcal{T}\right\}$.

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A nonlinear construction

- Level surface of $\mathcal{P}$: $\mathcal{S}_p := \left\{ u \ : \mathcal{P}^j(u) = p^j \ , \ \forall j \right\} \ni \eta_p$
- Invariance: $e^{\sum q_j JA^j} \eta_p \subset \mathcal{S}_p$.
- Transversality: if $\mathcal{M}_p \subset \mathcal{S}_p$ is such that $T_{\eta_p}\mathcal{M}_p = T^{\omega}_{\eta_p}\mathcal{T}$ then it should be a eqivalent (locally) to the reduced system.
- Explicit construction of the reduced system
  Fix $p_0 \in I$.
- Local model: $\mathcal{V} := T^{\omega}_{\eta_p}\mathcal{T}$, $\mathcal{V}^k := H^k \cap \mathcal{V}$, $(k \geq 0)$, $\mathcal{V}^{-k}$ dual of $\mathcal{V}^k$.
- Look for $\mathcal{V}^k \ni \phi \mapsto p(\phi) \in I$ s.t.

$$u := \eta_{p(\phi)} + \Pi_{p(\phi)}\phi \in \mathcal{S}_{p_0} \ .$$

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Reduction

- Define: $i(\phi)$ by

$$\mathcal{V}^k \ni \phi \mapsto \eta_{p(\mathcal{P}(\phi),\phi)} + \Pi_{p(\mathcal{P}(\phi),\phi)}\phi =: i(\phi) \in \mathcal{S}_{p_0}$$

$$H_r := i^* H \ , \quad \Omega := i^*\omega \ .$$

- Vector field: H_r defines a Hamiltonian vector field in $\mathcal{V}$.
- Assumptions
  - X_H defines a local flow on H^{k_0} which leaves invariant H^k $\forall k \geq k_0$.
  - The same for X_{H_r}.

Theorem

$\exists C > 0$ s.t., if

$$d_{k_0}(u_0, \mathcal{T}) < C \ , \quad u_0 \in \mathcal{S}_{p_0} \ .$$

then $\exists q(t)$ s.t. $u(t) = e^{q_j(t)JA^j} i(\phi(t))$.
The converse is also true.

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Outline

Search for canonical coordinates

$$\Omega_\phi(\Phi;\Phi) = \langle \mathcal{E}(\phi)\Phi;\Phi\rangle \ , \qquad \mathcal{E}(\phi) = E + O(\phi^2) \ .$$

Darboux theorem

There exists a map of the form

$$\phi = \mathcal{F}(\phi') = e^{\sum_j q_j J A^j}(\phi' + S(N,\phi')) \ , \quad N^j := \mathcal{P}^j(\phi') \ , \tag{1}$$

where the following following properties hold

1. $q_i(N,\phi)$ is defined on $\mathbb{R}^n \times \mathcal{V}^{-\infty}$
2. $S : \mathbb{R}^n \times \mathcal{V}^{-k} \mapsto S(N,\phi) \in \mathcal{V}^l$ is smoothing.
3. in terms of the variables ϕ' the symplectic form is given by

$$\Omega(\Phi_1';\Phi_2') = \langle E\Phi_1';\Phi_2'\rangle \ . \tag{2}$$

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Recovering smoothness

- The differential of $e^{q_j(\phi)JA^j}\phi$ involves

$$e^{q_j(\phi)JA^j}JA^j\phi dq_j$$

- Smoothness: If H is invariant: $H(e^{q_j(\phi)JA^j}u) = H(u)$ then:

$$H(\mathcal{F}(\phi)) = H(\phi + smoothing\ \ terms)$$

- Explicitly

$$H \circ \mathcal{F} = H_L + H_P + \text{small corrections}$$

$$H_L(\phi) = H(\eta_{p_0} + \phi) - H(\eta_{p_0})$$

Restriction of the linearization at the soliton to the symplectic orthogonal!

Outline

Spectrum

- Write $H_L(\phi) = \frac{1}{2}\langle \phi; L\phi \rangle$
  One can prove that L is selfadjoint and
- Spectrum $\sigma(L) = \{\omega_1^2, ..., \omega_N^2\} \cup [\Omega^2, +\infty)$
- Coordinates There exist coordinates such that
  $H_L = \sum_l \omega_l |\xi_l|^2 + \langle \bar{f}; Bf \rangle$
- One can start with perturbation theory.

Normal form.

- Definition: Normal form A function $Z(M,\phi)$ is in normal form up to order N, if the following holds

$$\{\omega\cdot(\mu-\nu)\neq 0 \ \& \ |\mu|+|\nu|\leq N+2\} \Longrightarrow \frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^\mu\partial^\nu\bar{\xi}}(M,0)=0$$

$$\{\omega\cdot(\mu-\nu)<\Omega \ \& \ |\mu|+|\nu|\leq N+1\} \Longrightarrow d_f\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^\mu\partial^\nu\bar{\xi}}(M,0)=0$$

$$\{-\omega\cdot(\mu-\nu)>\Omega \ \& \ |\mu|+|\nu|\leq N+1\} \Longrightarrow d_{\bar{f}}\frac{\partial^{|\mu|+|\nu|}Z}{\partial\xi^\mu\partial^\nu\bar{\xi}}(M,0)=0$$

Theorem

For any $N\geq 0$ there exists a canonical transformation T_N of the form (1) such that $H_r\circ T_N$ is in normal form up to order N.

THE END

THANK YOU