NON-VANISHING ELEMENTS OF FINITE GROUPS

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Abstract. Let G be a finite group, and let $\mathrm{Irr}(G)$ denote the set of irreducible complex characters of G. An element x of G is **non-vanishing** if, for every χ in $\mathrm{Irr}(G)$, we have $\chi(x) \neq 0$. We prove that, if x is a non-vanishing element of G and the order of x is coprime to 6, then x lies in the Fitting subgroup of G.

1. Introduction

The concept of **non-vanishing element** of a finite group G was introduced by M. Isaacs, the second author and T. Wolf in [10]: an element $x \in G$ is non-vanishing if $\chi(x) \neq 0$ for every irreducible complex character χ of G. It is a classical theorem of W. Burnside that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on some element of G. In other words, looking at the character table of G, the rows which do not contain the value 0 are precisely those corresponding to linear characters. Somehow violating the standard duality between characters and conjugacy classes, it is in general not true that the columns not containing the value 0 are precisely those corresponding to conjugacy classes of central elements, as there are finite groups having non-central non-vanishing elements. (See [10] for general hypotheses guaranteeing the existence of this type of elements.) In fact, a non-vanishing element of G can even fail to lie in an abelian normal subgroup of G (Theorem (5.1) in [10] provides a family of solvable examples). However, the main result of [10] was to prove that the non-vanishing odd order elements of a solvable group G all lie in a nilpotent normal subgroup of G, i.e. they lie in the Fitting subgroup $\mathbf{F}(G)$. (It remains an open problem to determine whether the odd order hypothesis is really necessary.) Although the authors in [10] were aware of the existence of many non-solvable groups G having non-vanishing elements outside $\mathbf{F}(G)$, now we realize, however, that all these are in fact $\{2,3\}$-elements.

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Theorem A. *Let G be a finite group, and let $x \in G$ be non-vanishing. If the order of x is coprime to* 6*, then $x \in \mathbf{F}(G)$.*

The groups A_6, A_7, A_{11}, A_{13}, $2.\mathsf{A}_{13}$, M_{22}, $2.M_{22}$ all are examples of groups having non-vanishing elements (all of them of order dividing 6). It is tempting to think that a non-vanishing element $x \in G$ should lie in the generalized Fitting subgroup $\mathbf{F}^*(G)$. But this is not true (at least not for even order elements): the group $G = 2^{11} : M_{24}$ has non-vanishing elements of order 2 and 4 outside $\mathbf{F}^*(G) = \mathbf{F}(G)$.

2. Theorem A

If $x \in G$ is non-vanishing and $N \triangleleft G$, it is then clear that $xN \in G/N$ is also non-vanishing. On the other hand, the hypothesis of being non-vanishing does not behave well when restricted to normal subgroups. This is compensated with the following elementary lemma which uses the character induction formula. Recall that $I_G(\psi)$ is the stabilizer of the character ψ of $M \triangleleft G$. In general, we use the notaton of [9].

Lemma 2.1. *Let M be a normal subgroup of G and let x be a non-vanishing element of G. Then x fixes an element in every G-orbit on $\mathrm{Irr}(M)$. In other words, for every $\psi \in \mathrm{Irr}(M)$ there exists $g \in G$ such that $x \in I_G(\psi^g)$.*

Proof. This is Lemma (2.3) of [10]. □

Theorem 2.2. *Let G act faithfully and irreducibly on a finite vector space V. Let $x \in \mathbf{F}(G)$ fix an element in each orbit of G on V. Then $x^2 = 1$.*

Proof. This is Theorem (4.2) of [10]. □

To prove Theorem A we need a new result on almost simple groups whose proof we defer until the next section.

Theorem 2.3. *Suppose that $S \le G \le \mathrm{Aut}(S)$, where S is a nonabelian simple group. Let x be an element of odd order of G fixing an element in every G-orbit on $\mathrm{Irr}(S)$. Then $x \in S$.*

There are several remarks concerning this theorem. First of all, the hypothesis of $o(x)$ being odd is necessary: a counterexample is $S = \Omega_8^+(2)$ with $x \in \mathrm{Aut}(S) \setminus S$ having order 2 in $\mathrm{Out}(S)$. But, again breaking the symmetry between conjugacy classes and characters, we stress that the corresponding result holds when we replace the action of G on the set $\mathrm{Irr}(S)$ with that on the set $\mathrm{Cl}(S)$ of conjugacy classes of S. In fact, the proof of Theorem C of [4] can be adapted to prove that if $S \le G \le \mathrm{Aut}(S)$ and $x \in G$ is such that for every $y \in S$ there is a $g \in G$ such that x fixes (setwise) the conjugacy class of y^g in S, then $x \in S$. Therefore, $S = \Omega_8^+(2)$ is an example where the actions of $\mathrm{Aut}(S)$ on $\mathrm{Irr}(S)$ and $\mathrm{Cl}(S)$ are not permutation isomorphic.

Using Theorem 2.3 we can now prove Theorem A, which was stated in the Introduction.

Proof of Theorem A. Working by induction on $|G|$, we have that $xN \in \mathbf{F}(G/N)$ for every nontrivial normal subgroup N of G.

Assume that M_1, M_2 are minimal normal subgroups of G, with $M_1 \neq M_2$. Then $M_1 \cap M_2 = 1$ and the function $\varphi : G \to \hat{G} = G/M_1 \times G/M_2$, defined by $\varphi(g) = (gM_1, gM_2)$ for $g \in G$, is an injective homomorphism. Now,

$$\varphi(x) \in \mathbf{F}(G/M_1) \times \mathbf{F}(G/M_2) = \mathbf{F}(\hat{G})$$

and then $\varphi(x) \in \varphi(G) \cap \mathbf{F}(\hat{G}) \leq \mathbf{F}(\varphi(G))$. Since φ induces an isomorphism between G and $\varphi(G)$, we see that $x \in \mathbf{F}(G)$.

We can hence assume that G has an unique minimal normal subgroup M.

Assume first that M is abelian. Observe that we can also suppose that the Frattini subgroup $\mathbf{\Phi}(G)$ of G is trivial, because $\mathbf{F}(G/\mathbf{\Phi}(G)) = \mathbf{F}(G)/\mathbf{\Phi}(G)$. So, by Lemma 4.4 of [8, III], the abelian normal subgroup M has a complement H in G. Observing that $\mathbf{C}_H(M)$ is normal in G, it hence follows $\mathbf{C}_H(M) = 1$ and then $\mathbf{C}_G(M) = M$.

Let now V be the group of the irreducible characters of M. Then V is a faithful and irreducible G/M-module. Moreover, by Lemma 2.1 the element xM fixes some element of each orbit of G/M on V. Recalling that $xM \in \mathbf{F}(G/M)$, by Theorem 2.2, it follows that $x^2 \in M$ and, as x is an element of odd order, we conclude that $x \in M \leq \mathbf{F}(G)$.

Let us consider the case when M is nonabelian. To finish the proof, we shall show that $x = 1$.

Write $M = S_1 \times S_2 \times \cdots \times S_n$, where $S_i = S_1^{g_i}$ for suitable $g_i \in G$, $1 \leq i \leq n$ (set $g_1 = 1$), and S_1 is a nonabelian simple group. Let

$$K = \bigcap_{i=1}^{n} \mathbf{N}_G(S_i)$$

be the kernel of the permutation action of G on the set $\Omega = \{S_1, S_2, \ldots, S_n\}$. So, $M \leq K \lhd G$. Let now L/K be the 2-complement of $\mathbf{F}(G/K)$. By induction, $xK \in L/K$. Recall that by [5] (or [13]), there exists a subset Δ of Ω such that the (setwise) stabilizer of Δ in L/K is trivial. We can assume that $\Delta = \{S_1, \ldots, S_m\}$, for some $m < n$. Let θ_1 be a non-principal irreducible character of S_1 and let $\theta_i = \theta_1^{g_i}$ be the corresponding characters of S_i, $i = 2, \ldots, m$ (recall that $\theta_1^{g_i}$ is defined by $\theta_1^{g_i}(s_1^{g_i}) = \theta_1(s_1)$, for all s_1 in S_1). Consider the irreducible character of M

$$\psi = \theta_1 \times \cdots \times \theta_m \times 1_{S_{m+1}} \times \cdots \times 1_{S_n}\,.$$

By Lemma 2.1, there exists some $g \in G$ such that $x \in I_G(\psi^g)$. So, $y = x^{g^{-1}} \in I_G(\psi)$ and then yK stabilizes the subset Δ. Since $yK \in L/K$ as $L/K \lhd G/K$, by the choice of Δ it follows that $y \in K$ and hence that $x \in K$.

We shall next prove that $x \in M$. As the first step, we show that x lies in $S_i\mathbf{C}_G(S_i)$ for all $i \in \{1, ..., n\}$. Without loss of generality, we show this for $i = 1$. Let θ_1 be in

$\mathrm{Irr}(S_1)$, and let $\psi = \theta_1 \times \cdots \times \theta_n$, where $\theta_i = \theta_1^{g_i}$. By Lemma 2.1, we have that x fixes ψ^g for some $g \in G$. Write

$$S_i^{g^{-1}} = S_{\sigma(i)} = S_1^{g_{\sigma(i)}}.$$

Hence

$$S_1^{g_{\sigma(i)}g} = S_i.$$

Then

$$\psi^g = \theta_1^{g_{\sigma(1)}g} \times \cdots \times \theta_1^{g_{\sigma(m)}g}.$$

(This is easily seen by evaluating both sides on an arbitrary element of S_i, for all $i \in \{1, ..., n\}$.) Since $x \in K$ fixes ψ^g, we have that it fixes each of the factors of ψ^g. Hence

$$\theta_1^{g_{\sigma(1)}gx} = \theta_1^{g_{\sigma(1)}g}$$

and therefore $\theta_1^{uxu^{-1}} = \theta_1$, where $u = g_{\sigma(1)}g \in \mathbf{N}_G(S_1)$. We are now in a position to apply Theorem 2.3 with $\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)$ in place of G, and $S_1\mathbf{C}_G(S_1)/\mathbf{C}_G(S_1)$ in place of S, to conclude that x lies in $S_1\mathbf{C}_G(S_1)$.

Now, for all $i \in \{1, ..., n\}$, write $x = s_i c_i$ with $s_i \in S_i$ and $c_i \in \mathbf{C}_G(S_i)$. On the other hand, we can certainly write $x = s_1 s_2 \cdots s_n \cdot y$ for some $y \in G$, and we work to show that $y = 1$. We get $s_1 c_1 = s_1(s_2 \cdots s_n) \cdot y$, whence $y = c_1(s_2 \cdots s_n)^{-1} \in \mathbf{C}_G(S_1)$. Similarly, we see that y lies in $\mathbf{C}_G(S_i)$ also for every $i \in \{2, ..., n\}$, and therefore y is in $\mathbf{C}_G(M) = 1$.

We conclude that $x = s_1 \cdots s_n$ lies in M.

Assume now, working by contradiction, that $x \neq 1$. Since x is a $\{2, 3\}'$-element, there exists a prime divisor $p \geq 5$ of the order of x. So, there exists a character $\theta_1 \in \mathrm{Irr}(S_1)$ of p-defect zero (see [7, Corollary 1]). Let $\theta_i = \theta_1^{g_i} \in \mathrm{Irr}(S_i)$, and consider $\psi = \theta_1 \times \cdots \times \theta_n \in \mathrm{Irr}(M)$. Observe that ψ is a character of p-defect zero of M. Let now $\chi \in \mathrm{Irr}(G)$ be a constituent of the induced character ψ^G. By Frobenius reciprocity and Clifford's theorem, we see that χ_M is sum of characters $\psi_j = \psi^{y_j}$, for suitable elements $y_j \in G$. Since all the ψ_j are characters of p-defect zero of M and $x \in M$ is an element of order multiple of p, by a classical result of R. Brauer ([9, (8.17)]) we have that all the characters ψ_j vanish on x. We conclude that $\chi(x) = 0$, against the assumption on x. This final contradiction yields $x = 1$, and the proof is complete. □

3. Almost Simple Groups

The aim of this section is to prove Theorem 2.3:

Proof of Theorem 2.3. We will assume that $x \notin S$ and aim to produce a G-orbit $\mathcal{O}$ on $\mathrm{Irr}(S)$ such that x **moves** every character in $\mathcal{O}$. Consider the subgroup $J := \langle xS \rangle$ in $A := G/S \leq \mathrm{Out}(S)$.

1) First we show that the theorem holds in the case $J \lhd A$. Indeed, by Theorem C of [4], we have that in the action of J on the conjugacy classes of S there is some

orbit of length > 1. Since J is cyclic, this action of J is permutation isomorphic to its action on $\mathrm{Irr}(S)$. In particular, J has an orbit $\mathcal{O}_1$ of length > 1 on $\mathrm{Irr}(S)$. Now let $\mathcal{O}$ be the G-orbit on $\mathrm{Irr}(S)$ that contains $\mathcal{O}_1$. Since $J \lhd A$, J acts semi-transitively on $\mathcal{O}$, i.e. all J-orbits on $\mathcal{O}$ have the same length. Hence we are done as $|\mathcal{O}_1| > 1$.

2) The structure of the outer automorphism group $\mathrm{Out}(S)$ is described for instance in [6]. By the result of 1), we are done if $\mathrm{Out}(S)$ is abelian. Thus we are left with the cases, where $S = PSL^{\epsilon}_n(q)$ with $n \geq 3$, $P\Omega^{\epsilon}_{2n}(q)$ with odd q and $n \geq 4$, or $E^{\epsilon}_6(q)$. Here, $q = p^f$, and $\epsilon = +$ in the untwisted case and $\epsilon = -$ in the twisted case.

Next we consider the case where, modulo the inner-diagonal and field automorphisms of S, x induces a graph automorphism of order $t > 1$. Since $o(x)$ is odd, this implies that $t = 3$ and $S = P\Omega^+_8(q)$. In this case, [12, Theorem 2.5] explicitly describes two subsets of $\mathrm{Irr}(S)$, each containing three irreducible unipotent characters of S such that they are permuted cyclically by graph automorphisms of order 3 of S, but every diagonal or field automorphism of S acts trivially on each of these two sets. Now we can just choose $\mathcal{O}$ to be any of these two sets.

3) Here we consider the case where x induces an inner-diagonal automorphism of S: $xS \in I := \mathrm{Outdiag}(S)$ in the notation of [6]. Thus x belongs to $\mathbf{O}_{2'}(I)$. Notice that I is either cyclic, or elementary abelian of order 4; in particular, $\mathbf{O}_{2'}(I)$ is cyclic. It follows that J char $\mathbf{O}_{2'}(I)$ char $I \lhd \mathrm{Out}(S)$ and so we are done again.

4) Now we may assume that, modulo $\mathrm{Inndiag}(S)$, x induces a field automorphism σ of prime order $t > 2$. We can find a simple, simply connected, algebraic group $\mathcal{G}$ in characteristic p and a Frobenius endomorphism F on $\mathcal{G}$ such that $S = L/\mathbf{Z}(L)$ for $L := \mathcal{G}^F$. We will also consider the pair $(\mathcal{G}^*, F^*)$ dual to $(\mathcal{G}, F)$ and the dual group $H := (\mathcal{G}^*)^{F^*}$, cf. [1]. We will use the Deligne-Lusztig theory (cf. [11], [1], [3]) and aim to find a semisimple element $s \in H$ such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, $s \in [H, H]$, but the conjugacy class s^H of s in H is not σ-invariant. The first two conditions imply that the semisimple character $\chi = \chi_s$ of L is irreducible and trivial at $\mathbf{Z}(L)$, hence can be viewed as an irreducible character $\chi \in \mathrm{Irr}(S)$. Notice that, in the cases under consideration, the inner-diagonal automorphisms of S are induced by conjugation using elements in H (when we embed S in H), and so they preserve s^H; also, we may write $H = \mathrm{Inndiag}(S)$. As a result, I fixes χ, cf. [15, §2]. Since σ moves s^H, [15, Corollary 2.4] and the disjointness of Lusztig series imply that $\chi^\sigma \neq \chi$ and so $\chi^x \neq \chi$.

Now let $\mathcal{O}$ be the G-orbit of χ. Observe that, in our cases, $\mathrm{Out}(S)/I$ is either abelian, or $C_f \times \mathsf{S}_3$, where the latter case occurs only when $S = P\Omega^+_8(q)$ (and $q = p^f$). In either case, since x induces the field automorphism σ modulo H, we see that $\langle xH \rangle$ is a normal subgroup of $\mathrm{Out}(S)/I$, and so $\langle x(G \cap H) \rangle \lhd G/(G \cap H)$. Recall that $G \cap H$ fixes χ. Now arguing as in 1), we see that x moves every character in $\mathcal{O}$, and so we are done.

The rest of the proof is to construct the desired element s. This construction will follow some arguments given in [14]. In what follows, once the prime ℓ is chosen, we will fix $\alpha \in \overline{\mathbb{F}}_q^\times$ of order ℓ.

5) Let $S = PSL_n(q)$ with $n \geq 3$. Then $H = PGL_n(q)$. We may assume $q > 2$ as otherwise $\mathrm{Out}(S)$ is abelian and we are done. Hence, by [16] there is a **primitive prime divisor** (p.p.d. for short) ℓ of $p^{nf} - 1$, that is, a prime divisor of $p^{nf} - 1$ which does not divide $\prod_{j=1}^{nf-1}(p^j - 1)$. Next, choose $s \in GL_n(q)$ represented by the diagonal matrix $\mathrm{diag}(\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}})$ over $\overline{\mathbb{F}}_q$. Abusing the notation, we will denote the image of s in H also by s (and we will do the same in subsequent parts of the proof). Notice that $\ell \geq nf + 1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H, H]$ (as $o(s)$ is coprime to $|\mathbf{Z}(\mathcal{G})|$ and $|H/[H, H]|$). It remains to show that s and s^σ are not conjugate in H. We may assume that s^σ is represented by the diagonal matrix $\mathrm{diag}(\alpha^r, \alpha^{qr}, \ldots, \alpha^{q^{n-1}r})$ over $\overline{\mathbb{F}}_q$, with $r := p^{f/t}$. Hence it suffices to show that there is no $\lambda \in \mathbb{F}_q^\times$ and $0 \leq j \leq n-1$ such that $\alpha^r = \lambda\alpha^{q^j}$. Assume the contrary. Since $o(\alpha) = \ell$ is coprime to $q - 1$, we must have $\lambda = 1$. Next, if $j = 0$, then ℓ divides $p^{f/t} - 1$, and if $j > 0$, then ℓ divides $p^{jf-f/t} - 1$. In either case we get a contradiction, as ℓ is a p.p.d. of $p^{nf} - 1$.

6) Consider the case $S = PSU_n(q)$ and $n \geq 3$, whence $H = PGU_n(q)$.

First assume that n is odd. Since $(n, q) \neq (3, 2)$, there is a p.p.d. ℓ of $p^{2nf} - 1$. Next, choose $s \in GU_n(q)$ represented by the matrix $\mathrm{diag}(\alpha, \alpha^{-q}, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}})$ over $\overline{\mathbb{F}}_q$. Notice that $\ell \geq 2nf + 1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H, H]$. It remains to show that s and s^σ are not conjugate in H. We may assume that s^σ is represented by the diagonal matrix $\mathrm{diag}(\alpha^r, \alpha^{-qr}, \ldots, \alpha^{q^{n-1}r})$ over $\overline{\mathbb{F}}_q$, with $r := p^{2f/t}$. Hence it suffices to show that there is no $\lambda \in \mathbb{F}_{q^2}^\times$ and $0 \leq i \leq n-1$ such that $\alpha^r = \lambda\alpha^{(-q)^i}$; equivalently, $\alpha^r = \lambda\alpha^{q^{2j}}$ for some $0 \leq j \leq n-1$. As above, this however is impossible as ℓ is a p.p.d. of $p^{2nf} - 1$.

Now assume that $n \geq 4$ is even. Since $\mathrm{Out}(PSU_4(2))$ is abelian, we may assume that $(n, q) \neq (4, 2)$, whence there exists a p.p.d. ℓ of $p^{2(n-1)f} - 1$. Next, choose $s \in GU_n(q)$ represented by the matrix $\mathrm{diag}(1, \alpha, \alpha^{-q}, \alpha^{q^2}, \ldots, \alpha^{q^{n-2}})$ over $\overline{\mathbb{F}}_q$. Then $\ell \geq (n+3)f + 1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H, H]$. Arguing as above, we see that s and s^σ are not conjugate in H.

7) Assume $S = P\Omega_{2n}^-(q)$ and $n \geq 4$. Here we choose ℓ to be a p.p.d. of $p^{2nf} - 1$. Next, choose $s \in GO_{2n}^-(q)$ represented by the matrix $\mathrm{diag}(\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{2n-1}})$ over $\overline{\mathbb{F}}_q$. Since ℓ is odd, $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H, H]$. It remains to show that s and s^σ are not conjugate in H. Here, if q is odd, we can choose γ to be a nonzero non-square element in $\mathbb{F}_{q^{1/t}}$ (whence γ is also a non-square in $\mathbb{F}_q$ as t is odd), and define $GO_{2n}^-(q)$ as the group of linear transformations of $\mathbb{F}_q^{2n}$ that preserve the quadratic form $\sum_{i=1}^{n-1}(x_i^2 - y_i^2) + (x_n^2 - \gamma y_n^2)$. If $2|q$, we choose $0 \neq \gamma \in \mathbb{F}_{q^{1/t}}$ such that the

polynomial $v^2+v+\gamma$ is irreducible in $\mathbb{F}_{q^{1/t}}[v]$ (and so in $\mathbb{F}_q[v]$ as t is odd), and define $GO^-_{2n}(q)$ as the group of linear transformations of $\mathbb{F}_q^{2n}$ that preserve the quadratic form $\sum_{i=1}^n x_iy_i + (x_n^2+\gamma y_n^2)$. Then we can define σ as induced by the field automorphism $\lambda \mapsto \lambda^r$, and so s^σ is represented by the diagonal matrix $\mathrm{diag}(\alpha^r, \alpha^{qr}, \ldots, \alpha^{q^{2n-1}r})$ over $\overline{\mathbb{F}}_q$, with $r := p^{f/t}$. Hence it suffices to show that there is no $\lambda \in \mathbb{F}_q^\times$ and $0 \leq i \leq 2n-1$ such that $\alpha^r = \lambda\alpha^{q^i}$. As above, this is impossible as ℓ is a p.p.d. of $p^{2nf}-1$.

Now assume that $S = P\Omega^+_{2n}(q)$ with $n \geq 4$. Since $\mathrm{Out}(\Omega^+_8(2)) \cong \mathsf{S}_3$ consists only of graph automorphisms, we may assume that $(n,q) \neq (4,2)$. Hence there exists a p.p.d. ℓ of $p^{2(n-1)f}-1$. Next, choose $s \in GO^+_{2n}(q)$ represented by the matrix $\mathrm{diag}(1,1,\alpha,\alpha^q,\alpha^{q^2},\ldots,\alpha^{q^{2n-3}})$ over $\overline{\mathbb{F}}_q$. Again, $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Arguing as above, we see that s and s^σ are not conjugate in H.

8) Finally, we consider the case $S = E_6^\epsilon(q)$. It is easy to see that $J \lhd A$ in the cases where $o(x)$ is coprime to 3 or $(3, q-\epsilon) = 1$, so we are done in these cases. Assume $3|(q-\epsilon)$. In the notation of [1], we have $H = E_6^\epsilon(q)_{ad}$ and $S = [H,H]$. Next, the proof of [4, Theorem 3.1] yields a maximal torus T of H (of order $(q^4-q^2+1)(q^2+\epsilon q+1)$), such that $T \cap S = \langle s \rangle$ is cyclic and $\mathbf{C}_H(s) = T = \mathbf{C}_{\mathrm{Aut}(S)}(s)$. Now assume that s^σ is conjugate to s in H: $s^\sigma = s^h$ for some $h \in H$. Then $h^{-1}\sigma \in \mathbf{C}_{\mathrm{Aut}(S)}(s) = T < H$ and so $\sigma \in H$, a contradiction.

To complete the proof, we need to show that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected. Under our hypotheses, $Z := \mathbf{Z}(\mathcal{G}) = \langle z \rangle$ has order 3, $\mathcal{G}^* = \mathcal{G}/Z$, and F acts trivially on Z. Abusing the notation, we will identify s with an inverse image of it in $L = \mathcal{G}^F$. Furthermore, we can find an F-stable maximal torus $\mathcal{T} \ni s$ of $\mathcal{G}$ such that $T = (\mathcal{T}/Z)^F$. Then $\mathbf{C}_\mathcal{G}(s) \geq \mathcal{T}$. Since $\mathcal{G}$ is simply connected, $\mathbf{C}_\mathcal{G}(s)$ is connected. Moreover, $\mathbf{C}_\mathcal{G}(s)^F/Z \leq T \cap S$ consists only of semisimple elements. It follows that $\mathbf{C}_\mathcal{G}(s) = \mathcal{T}$. Assume that there is some $x \in \mathcal{G}$ such that $xsx^{-1} = zs$. Then

$$xsx^{-1} = zs = F(zs) = F(xsx^{-1}) = F(x)sF(x)^{-1}$$

and so $x^{-1}F(x) \in \mathbf{C}_\mathcal{G}(s)$. Since $\mathbf{C}_\mathcal{G}(s)$ is connected and F-stable, by the Lang-Steinberg Theorem, there is some $c \in \mathbf{C}_\mathcal{G}(s)$ such that $x^{-1}F(x) = c^{-1}F(c)$. Setting $y := xc^{-1}$, we see that $ysy^{-1} = xsx^{-1} = zs$ and $F(y) = y$. Thus $yZ \in \mathbf{C}_{L/Z}(s) = \mathbf{C}_S(s) = T \cap S = \langle s \rangle$ and so $y \in \langle s, z \rangle$. But both s and z centralize s, so we obtain $ysy^{-1} = s$, a contradiction. We have shown that $\mathbf{C}_{\mathcal{G}/Z}(s) = \mathbf{C}_\mathcal{G}(s)/Z = \mathcal{T}/Z$, whence $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, as stated. □

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