CORRIGENDA TO "REDUCIBLE VERONESE SURFACES"

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1. Introduction

In [A-B1] we claimed to give the complete list of reducible Veronese surfaces according to the following definition.

Definition 1. *For any positive integer* $n \geq 1$*, we will call reducible Veronese surface any algebraic surface* $X \subset \mathbb{P}^{n+4}(\mathbb{C})$ *such that:*

i) X *is a non degenerated, reduced, reducible surface of pure dimension* 2*;*

ii) $\deg(X) = n+3$, $cod(X) = n+2$, *so that* X *is a minimal degree surface*;

iii) $\dim[Sec(X)] \leq 4$, *so that it is possible to choose a generic linear space* $\mathcal{L}$ *of dimension* $n-1$ *in* $\mathbb{P}^{n+4}$ *such that* $\pi_{\mathcal{L}}(X)$ *is isomorphic to* X*, where* $\pi_{\mathcal{L}}$ *is the the rational projection* $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} - - - > \Lambda$*, from* $\mathcal{L}$ *to a generic target* $\Lambda \simeq \mathbb{P}^4$;

iv) X *is connected in codimension* 1*, i.e. if we drop any finite number (eventually* 0*) of points* $P_1, ..., P_r$ *from* X *we have that* $X \backslash \{P_1, ..., P_r\}$ *is connected;*

v) X *is a locally Cohen-Macaulay surface.*

Condition *iii)* deserves particular attention. When $\dim[Sec(X)] \leq 4$, for a generic linear $(n-1)$-dimensional linear space $\mathcal{L}$ we have that $\pi_{\mathcal{L}|X}$ is injective. However this condition, obviously necessary, is not sufficient to get that $\pi_{\mathcal{L}|X}$ is an isomorphism. The condition $\dim[Sec(X)] \leq 4$ is in fact equivalent to have that $\pi_{\mathcal{L}|X}$ is only a J-embedding according to the definition of Johnson (see [J], 1.2, and Proposition 1.5 of [Z], chapter II, page 37). To have that X is a reducible Veronese surface, i.e. to have that $\pi_{\mathcal{L}|X}$ is an isomorphism, instead of *iii)* **we need to use a different condition**:

iii)' $\dim[Sec(X)] \leq 4$ <u>and</u> $\dim[\bigcup_{x \in X} \langle T_x(X) \rangle] \leq 4$;

where $T_x(X)$ is the Zariski tangent space to X at x and $\langle V \rangle$ is the linear span of a variety V in a projective space. See [A-B2] for the proof of the equivalence. From now on a reducible Veronese surface will be a surface satisfying conditions *i)*, *ii)*, *iii)'*, *iv)* and *v)*.

Throughout [A-B1], to get condition *iii)* for the members of our list, we used the condition on $\dim[Sec(X)]$ and, independently, the fact that $\pi_{\mathcal{L}|X}$ has to be an isomorphism, see for instance the proof of Lemma 4. As the condition on $\dim[Sec(X)]$ is necessary for *iii)'*, it follows that to classify reducible Veronese surfaces, according to the above new definition, we have to check the list of [A-B1] and we have to exclude surfaces for which $\dim[\bigcup_{x \in X} \langle T_x(X) \rangle] \leq 4$ does not hold.

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In this note we perform this check and we also fix some mistakes occurred in the proof of Proposition 2 of [A-B1].

2. Refining and completing the list

The list in [A-B1] contained 3 types of surfaces X:

a_n) for any integer $n \geq 1$, a suitable union of $n+3$ planes which sits as a linearly normal scheme in $\mathbb{P}^{n+4}$ (see Definition 2 of [A-B1] for a precise description); these surfaces were introduced in [F].

b) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface Q in $\mathbb{P}^3$ and two planes X_1 and X_2 sitting as a linearly normal scheme in $\mathbb{P}^5$; X_1 and X_2 cut Q, respectively, along two lines L_1, L_2, intersecting at a point $P := X_1 \cap X_2$, and $L_1 = \langle Q \rangle \cap X_1$, $L_2 = \langle Q \cup X_1 \rangle \cap X_2$.

c) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface Q in $\mathbb{P}^3$ and two planes X_1 and X_2, sitting as a linearly normal scheme in $\mathbb{P}^5$; X_1, X_2 and Q intersect pairwise transversally along a unique line $L := Q \cap X_1 \cap X_2$ and $L = \langle Q \rangle \cap X_1 \cap X_2$.

It is easy to see that $\dim[\bigcup_{x \in X} \langle T_x(X) \rangle] \leq 4$ in both cases a_n) and *b*). On the contrary, if we consider points $x \in L$ in case *c*), we have that the tangent space at x to X is $\langle T_x(Q) \cup X_1 \cup X_2 \rangle \simeq \mathbb{P}^4$ and $\bigcup_{x \in L} \langle T_x(Q) \cup X_1 \cup X_2 \rangle = \mathbb{P}^5$, so that there is no point $\mathcal{L} \in \mathbb{P}^5$ such that $\pi_{\mathcal{L}|X}$ is an isomorphism.

Unfortunately, there exist two other surfaces to check, i.e. two surfaces satisfying conditions *i*), *ii*), *iii*), *iv*), *v*) but not considered in [A-B1]. These surfaces sit as linearly normal schemes, respectively, in $\mathbb{P}^5$ and $\mathbb{P}^6$:

d) $X = S \cup X_1$ where S is a smooth rational cubic scroll in $\mathbb{P}^4$ having a line L as fundamental section and X_1 is a plane such that $S \cap X_1 = \langle S \rangle \cap X_1 = L$;

e) $X = S \cup X_1 \cup X_2$ where $S \cup X_1$ is a surface as in *d*) and X_2 is a plane such that $S \cap X_1 \cap X_2 = \langle S \cup X_1 \rangle \cap X_2 = L$.

Obviously conditions *i*), *ii*) and *iv*) are satisfied. Condition *v*) is satisfied by arguing as in Lemma 1 of [A-B1]. For a surface X as in *d*) we have $\dim[Sec(X)] \leq 4$ by direct calculation with a computer algebra system or by considering that every line joining generic points of S and X_1 is contained in the 4-dimensional quadric cone having X_1 as vertex and the smooth conic Γ as base, where Γ is the smooth conic generating S with L. For a surface X as in *e*) we have $\dim[Sec(X)] \leq 4$ by looking at every pair of irreducible components of X.

A surface X as in *d*) can also be isomorphically projected in $\mathbb{P}^4$ because $\dim[\bigcup_{x \in X} \langle T_x(X) \rangle] \leq 4$. On the contrary, if we consider points $x \in L$ in case *e*), we have that the tangent space at x to X is $\langle T_x(S) \cup X_1 \cup X_2 \rangle \simeq \mathbb{P}^4$ and $\bigcup_{x \in L} \langle T_x(S) \cup X_1 \cup X_2 \rangle$ is a quadric cone in $\mathbb{P}^6$, so that its dimension is 5, hence, for any line $\mathcal{L} \in \mathbb{P}^6$, $\pi_{\mathcal{L}|X}$ cannot be an isomorphism.

Now we prove that there are no other reducible Veronese surfaces up to the above ones. In Proposition 2 of [A-B1] we claimed that every irreducible component of a reducible Veronese surface X can be only a plane, a smooth quadric in $\mathbb{P}^3$ or a quadric in $\mathbb{P}^3$ having rank 3. With this assumption we get only the surfaces a_n), *b*), *c*) as it is proved in [A-B1]. However there are other possibilities for the irreducible components of X: by Theorem 1 of [A-B1], they are reduced surfaces of minimal degree in their spans, and the classification of such surfaces is quoted in Theorem 0.1 of [E-G-H-P] where "rational normal scroll" for 2-dimensional varieties means:

a smooth rational normal scroll or a cone over a smooth rational normal curve. Not all these surfaces were well considered in Proposition 2 of [A-B1], so we have to fill this gap.

Let us consider cones Y over smooth rational normal curves and let E be the vertex of a cone Y. The tangent space at E to Y, which is $\langle Y \rangle$, cannot have dimension bigger than 4 otherwise condition $iii)'$ would be not satisfied, so that $\deg(Y) \leq 3$. If $\deg(Y) = 2$ the other irreducible components of X must be planes (see the final part of the proof of Proposition 2 in [A-B1]) and the union of a rank 3 quadric cone in $\mathbb{P}^3$ and planes can be excluded by arguing as in case 1) of the proof of Theorem 3 in [A-B1]. It follows that here we have to consider only the case $\deg(Y) = 3$. By contradiction, let us assume that an irreducible component of a reducible Veronese surface X is a degree 3 cone Y as above, having vertex E. Let X_i another component of X. To satisfy condition $iii)'$ we must have $E \notin X_i$ so that $Y \cap X_i = \langle Y \rangle \cap \langle X_i \rangle$ is a single point $P \in Y$, $P \neq E$, by Corollary 2 of [A-B1]. If X_i is not a plane the join of Y and X_i has dimension 5 hence $\dim[Sec(X)] \geq 5$: contradiction. If X_i is a plane any projection $\pi_{\mathcal{L}}$ of $Y \cup X_i$ in $\mathbb{P}^4$ cannot be an isomorphism because $\pi_{\mathcal{L}}(Y) \cap \pi_{\mathcal{L}}(X_i)$ cannot be a single point.

Now let us consider smooth rational normal scrolls of dimension 2. As no smooth surface can be isomorphically projected in $\mathbb{P}^4$ but the Veronese surface, we have to consider only smooth rational cubic scrolls S in $\mathbb{P}^4$ (other than smooth quadrics in $\mathbb{P}^3$ examined in [A-B1]). In spite of what we said in the proof of Proposition 2 of [A-B1], (page 126, lines 13-18) also a smooth rational cubic scroll S in $\mathbb{P}^4$ can be an irreducible component of a reducible Veronese surface X. The correct part of the proof of Proposition 2 in [A-B1] shows that this is possible only when all other components of X are planes cutting $\langle S \rangle$ and S only along a line L which is its fundamental section. This line escaped to the analysis made in [A-B1], where only the fibres of the scroll were considered. All other possibilities, involving planes and quadrics, are considered and correctly excluded in Proposition 2 of [A-B1].

As we have seen, the union of a smooth cubic scroll S in $\mathbb{P}^4$ and one or two planes, cutting $\langle S \rangle$ and S along its fundamental section L, gives rise to two surfaces to be checked. No other plane can be admitted by Lemma 3 of [A-B1] and condition $iii)'$.

In conclusion: the surfaces $a_n)$, $b)$ and $d)$ can be isomorphically projected in $\mathbb{P}^4$, (but not $c)$ and $e)$). This is the complete list of reducuble Veronese surfaces with the correct condition $iii)'$ instead of $iii)$.

Remark 1. *This note is also a correction of the list of reducible Veronese surfaces quoted in Theorem 1 of* [A-B2] *and never used in that paper.*

References

[A-B1] A.Alzati-E.Ballico: "Reducible Veronese surfaces". Adv. in Geom. **10** (4) (2010), p. 719-735.

[A-B2] A.Alzati-E.Ballico: "Projectable Veronese varieties". Rev. Mat. Complut. **24** (1) (2011), p.219-249.

[E-G-H-P] D.Eisenbud-M.Green-K.Hulek-S.Popescu: "Small schemes and varieties of minimal degree". Amer. J. Math. **128** (2006), p. 1363-1389.

[F] G.Floystad: "Monads on projective spaces". Comm. Algebra **28** (12) (2000), p. 5503-5516.

[J] K.W.Johnson: "Immersion and embedding of projective varieties". Acta Math. **140** (1981), p. 49-74.

[Z] F.L.Zak: "Tangents and secants of algebraic varieties". Translations of Mathematical Monographs of A.M.S., **127**, Providence R.I., 1993.

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