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Resolution requirements for smoothed particle hydrodynamics simulations of self-gravitating accretion discs

Giuseppe Lodato[1★] and C. J. Clarke[2]

[1]*Dipartimento di Fisica, Università Degli Studi di Milano, Via Celoria, 16, Milano 20133, Italy*
[2]*Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*

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ABSTRACT

Stimulated by recent results by Meru & Bate, we revisit the issue of resolution requirements for simulating self-gravitating accretion discs with smoothed particle hydrodynamics (SPH). We show that all the results by Meru & Bate are actually consistent if they are interpreted as driven by resolution effects, therefore implying that the resolution criterion for cooling gaseous discs is a function of the imposed cooling rate. We discuss two possible numerical origins of such dependence, which are both consistent with the limited number of available data. Our results tentatively indicate that convergence for current simulations is being reached for a number of SPH particles approaching 10 million (for a disc mass of the order of 10 per cent of the central object mass), which would set the critical cooling time for fragmentation at about $15\Omega^{-1}$, roughly a factor of 2 larger than previously thought. More in general, we discuss the extent to which the large number of recent numerical results are reliable or not. We argue that those results that pertain to the dynamics associated with gravitational instabilities (such as the locality of angular momentum transport, and the relationship between density perturbation and induced stress) are robust, while those pertaining to the thermodynamics of the system (such as the determination of the critical cooling time for fragmentation) can be affected by poor resolution.

Key words: accretion, accretion discs – gravitation – instabilities – stars: formation – galaxies: active.

1 INTRODUCTION

Gravitational instabilities in accretion discs have been intensively studied in the last decade. Several analytical models of gravitationally unstable discs have been proposed over the years (Balbus & Papaloizou 1999; Bertin & Lodato 1999; Clarke 2009; Rafikov 2009; Krumholz & Burkert 2010; Rice, Mayo & Armitage 2010). However, a large number of important properties of self-gravitating discs have been determined through the use of numerical simulations (Gammie 2001; Boss 2004, 2006; Lodato & Rice 2004, 2005; Mejia et al. 2005; Rice, Lodato & Armitage 2005; Boley et al. 2006; Cai et al. 2006; Cossins, Lodato & Clarke 2009, 2010, just to mention a few). Such numerical studies have investigated several different aspects of self-gravitating disc dynamics. Some (e.g. Lodato & Rice 2004, 2005; Boley et al. 2006; Cossins et al. 2009) have considered the angular momentum transport in non-fragmenting discs, with the aim of determining whether the long-term disc evolution induced by the instability can be treated within a local framework. This is particularly relevant for constructing local analytical models of self-gravitating discs (Clarke 2009; Rafikov 2009). In some other cases (Gammie 2001; Boss 2004; Mayer et al. 2004, 2005; Rice et al. 2005; Boss 2006; Mayer et al. 2007; Cossins et al. 2010), the focus was on the conditions required for a self-gravitating disc to fragment into bound objects. This issue is extremely important in the context of planet formation theories, as this mechanism forms the basis of the so-called gravitational instability model for planet formation. More in general, such a mechanism might be responsible for the formation of stellar clusters in galactic centres, including our own (Nayakshin 2006; Nayakshin, Cuadra & Springel 2007). The mechanism leading to fragmentation is fairly well understood. In a cooling disc, thermal saturation of the instability occurs when the amplitude of the density perturbation is large enough that the associated shock heating provided by the instability balances the cooling rate (Cossins et al. 2009). If the required density perturbation exceeds a critical value, the disc fragments. Determining the critical value for the density perturbation is thus equivalent to determining a critical cooling time below which fragmentation occurs. Such critical cooling time has been variously determined over the years: Gammie (2001) sets it to $3\Omega^{-1}$ (where Ω is the local angular velocity of the disc), based on his two-dimensional shearing sheet simulations; Rice et al. (2005) set it to $6\Omega^{-1}$ for a specific heat ratio of $\gamma = 5/3$, based on their three-dimensional global SPH simulations, and note a dependency of the critical cooling time on γ.

★E-mail: giuseppe.lodato@unimi.it

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Other studies have found other dependencies either on the specific form of the cooling rate (Johnson & Gammie 2003; Cossins et al. 2010) or on the thermal history of the disc (Clarke, Harper-Clark & Lodato 2007). It is convenient to express the cooling time in units of the local dynamical time in the disc and introduce a 'cooling parameter', $\beta = \Omega t_{\rm cool}$.

Recently, some new results have cast doubts on the reliability of the determination of such a critical cooling time, and in particular on those obtained by SPH simulations. In particular, we consider here two new results. First, Meru & Bate (2011a) have run a number of simulations with different slopes of the disc surface density profile, different disc masses, and have considered the location at which the fragment appeared in their fragmenting runs. Their conclusion is that the critical value of β at a given radius R is a function of the 'local disc mass', $m(R) = \Sigma(R)R^2/M_\star$, where Σ is the surface density and $M_\star$ is the mass of the central object. In particular, they find that the critical cooling parameter for fragmentation increases roughly as $m^{1/2}$. Secondly, Meru & Bate (2011b) have noted that, for a given disc set-up (in terms of total disc mass, surface density profile, etc.) the critical β for fragmentation is an increasing function of the total number of SPH particles used, and – more importantly – they find no hint of convergence.

In this paper, we reconsider the issue of resolution requirements in order to simulate fragmentation in a cooling gaseous discs. We introduce a new resolution requirement that relates the number of SPH particles used to the externally imposed cooling rate. We then show that the whole set of results by Meru & Bate (2011a,b) can be naturally explained as a consequence of failing to satisfy this condition. Since the Meru & Bate (2011a,b) simulations are among the highest resolution SPH simulations of fragmenting discs to date, this failure is shared also by all previous SPH simulations. We will then discuss which of the previous results are reliable and which should be considered with more care.

2 RESOLUTION REQUIREMENTS FOR SELF-GRAVITATING COOLING DISCS

Several papers in the past have investigated the resolution requirements for resolving gas fragmentation in numerical simulations. In particular, Bate & Burkert (1997) require that the mass of SPH particles within a smoothing sphere be less than the Jeans mass. A similar condition, applicable also to grid-based simulations (Truelove et al. 1997), is that the minimum resolvable length (i.e. the mesh size or the smoothing length h for grid-based and SPH simulations, respectively) be less than the Jeans length, although note that failure to satisfy the Truelove condition in grid-based simulations generally induces spurious fragmentation, while in SPH poor resolution may well lead to suppression of fragmentation. Nelson (2006) lists three different conditions: the first is that adaptive gravitational softening is used (this is actually usually done in most modern SPH simulations, including those discussed here), the second is a variant of the Bate & Burkert (1997) and Truelove et al. (1997) conditions, and the third is that the smoothing length h be less than the disc thickness H. However, this last condition is also equivalent to the Bate & Burkert (1997) and Truelove et al. (1997) condition for a marginally stable disc, for which the Jeans length is actually equal to the disc thickness. All these conditions can therefore be cast in the simple requirement

$$\frac{h}{H} \lesssim f, \tag{1}$$

where f is a factor of order unity.

The natural measure of resolution for an SPH simulation of an accretion disc is thus the quantity h/H. In order to discuss the outcome of gravitationally unstable discs we are interested in evaluating this quantity at the onset of the instability, that is in a condition where the disc is marginally stable. Let us then estimate the ratio h/H for a marginally stable disc. We first recall than marginal stability for a Keplerian disc implies that

$$Q = \frac{c_{\rm s}\Omega}{\pi G\Sigma} \approx 1, \tag{2}$$

where $c_{\rm s}$ is the sound speed. This is equivalent to

$$\frac{H}{R} = \pi Q \frac{\Sigma R^2}{M_\star} = \pi Q m(R), \tag{3}$$

where $m(R) = \Sigma\, R^2/M_\star$.

The SPH smoothing length is set by the condition

$$\rho h^3 = \eta^3 m_{\rm p}, \tag{4}$$

where ρ is the local SPH density; η is a numerical parameter generally set to 1.2; $m_{\rm p} = M_{\rm disc}/N$ is the SPH particle mass, where $M_{\rm disc}$ is the total disc mass; and N is the total number of SPH particles. With this prescription, a smoothing sphere contains roughly 60 SPH 'neighbours'. We evaluate the smoothing length at the disc mid-plane. The mid-plane density $\rho_{\rm c}$ is related to the surface density by $\rho_{\rm c} = \Sigma/2H$, which, inserted in equation (4), and using equation (3) and the definition of $m(R)$, gives

$$\frac{h}{H} = \frac{\eta}{m(R)}\left(\frac{2q}{\pi^2 Q^2 N}\right)^{1/3} \approx \frac{5\times10^{-3}}{m(R)}\left(\frac{q}{0.1}\right)^{1/3}\left(\frac{N}{2.5\times10^5}\right)^{-1/3}, \tag{5}$$

where $q = M_{\rm disc}/M_\star$. Actually, if the disc is very poorly resolved, that is if $h/H > 1$, the SPH method underestimates the mid-plane density by a factor of H/h, and we have $\rho_{\rm c} \approx \rho(h/H)$. This corresponds to an equivalent 2D version of equation (4):

$$\Sigma h^2 \approx 2\eta^3 m_{\rm p}. \tag{6}$$

In this poorly resolved case, one readily gets that h/H is given by

$$\frac{h}{H} \approx \left(\frac{\eta}{m(R)}\right)^{3/2}\left(\frac{2q}{\pi^2 Q^2 N}\right)^{1/2} \approx \frac{3.4\times10^{-4}}{[m(R)]^{3/2}}\left(\frac{q}{0.1}\right)^{1/2}\left(\frac{N}{2.5\times10^5}\right)^{-1/2}. \tag{7}$$

It is also useful to evaluate h/H at the outer edge of the disc, $R_{\rm out}$. If the surface density profile is a power-law with respect to R with index $p < 2$, we have that $q = M_{\rm disc}/M_\star = 2\pi\, m(R_{\rm out})/(2-p)$, from which we get (for $h/H < 1$)

$$\left.\frac{h}{H}\right|_{\rm out} = \frac{2\pi\eta}{2-p}\left(\frac{2}{q^2\pi^2 Q^2 N}\right)^{1/3} \approx 0.3\left(\frac{q}{0.1}\right)^{-2/3}\left(\frac{N}{2.5\times10^5}\right)^{-1/3}, \tag{8}$$

where the last equality holds for $p = 1$. The equivalent of (8) when $h/H > 1$ is

$$\left.\frac{h}{H}\right|_{\rm out} = \left(\frac{2\pi\eta}{2-p}\right)^{3/2}\left(\frac{2}{q^2\pi^2 Q^2 N}\right)^{1/2} \approx 0.16\left(\frac{q}{0.1}\right)^{-1}\left(\frac{N}{2.5\times10^5}\right)^{-1/2}. \tag{9}$$

Note, however, that equation (9) is of little practical use, since in order to have $h/H > 1$ at the disc outer edge, the number of SPH particles needs to be very small, $N \lesssim 6000$.

We can now reconsider the results of Meru & Bate (2011a,b) in the light of equations (5) and (8) [or (7) and (9)]. In fact, on the one hand Meru & Bate (2011a) find that, for a given number of particles, there is a relation between the imposed value of β and the quantity $m(R)$, evaluated at the location of the first fragment appearing in their simulations. On the other hand, Meru & Bate (2011b) demonstrate that, for a given disc set-up, the critical value of β for fragmentation is an increasing function of N. Both results can be thus explained if (a) fragmentation is suppressed for low resolution and (b) the resolution requirement, as measured by the parameter h/H, is a function of the imposed cooling time. In this picture, Meru & Bate (2011a) thus determine the location at which, for a given N, the resolution requirement is satisfied, while Meru & Bate (2011b) determine the minimum N at which, for a given set-up, the requirement is satisfied.

In fact, it would be puzzling to interpret the results of Meru & Bate (2011a) otherwise. By construction, such simulations are self-similar and, as long as global effects do not play a role [which is the case when $m(R) \lesssim 0.1$], there should be no preferred location in the disc. The only possible mechanism that breaks the self-similarity is in fact resolution.

Stated otherwise, we can say that in order to detect fragmentation in a cooling disc, the resolution requirement must depend on β, in such a way that discs with longer cooling times require a better resolution (that is, a smaller h/H). In this case, fragmentation would then first occur at the minimum radius for which such resolution requirement is satisfied, since the dynamical and cooling times are shorter than at larger radius. We can express such a condition in the form

$$\beta < \beta_{\rm res} \equiv \beta_{\rm res}(h/H). \quad (10)$$

What is the actual functional relation between $\beta_{\rm res}$ and h/H? We explore this by plotting in Fig. 1 the values of β in the fragmenting simulations of Meru & Bate (2011a) versus H/h at the point of fragmentation, and assuming that the correlation in these two quantities is purely driven by resolution, that is $\beta = \beta_{\rm res}$. The quantity H/h is computed using equation (5) or (7), and using the data provided by Meru & Bate (2010a) in their table 2. We also indicate with a leftward-pointing error bar the cases where our estimate of $m(R)$ (which is based on the initial disc surface density profile) is an overestimate: in these cases fragmentation occurs close to the disc inner edge and the surface density has been significantly depleted by boundary effects by the stage of fragmentation.

There are evidently a variety of functional forms that will pass through the data – we are hampered both by the fact that there is considerable scatter and by the fact that we have no simulation points with $H/h > 3$. One possibility is that there is a linear relationship between β and H/h (see Section 3 for a discussion of this form in terms of the role of artificial viscosity). We have fitted the data with a least-squares fit (shown with the solid line in Fig. 1) and have obtained

$$\beta_{\rm res} \approx 2\left(\frac{h}{H}\right)^{-1}. \quad (11)$$

It is worth stressing that if the trend $\beta = \beta_{\rm res}$, given by equation (11), extended to an indefinitely high H/h then it would mean that there would be no *physical* limit to the cooling rate required for fragmentation. Instead it would imply that – however slow the cooling rate in the disc – it would always fragment if modelled at a high enough resolution. In contrast, if there is indeed a physical limit to the cooling rate required for fragmentation then one would expect that at high H/h the data would start deviating from the relation $\beta = \beta_{\rm res}$ and will saturate at a finite value of β.

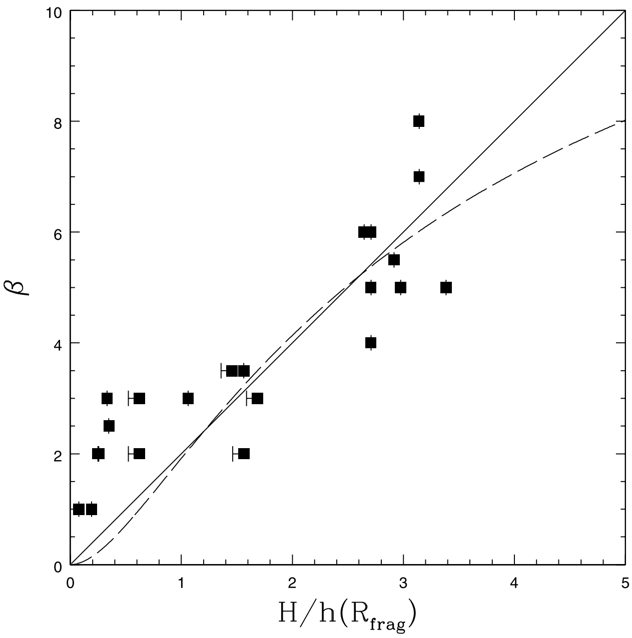

Figure 1. Correlation between linear resolution at the location of fragmentation, as measured by h/H, and the imposed cooling time, as measured by the parameter β. Data points are taken from table 2 of Meru & Bate (2010a). A leftward-pointing error bar indicates cases where fragmentation occurred near the disc inner boundary. The solid line shows a simple least-squares fit to the data, assuming a linear relation (equation 11). The best-fitting model in this case is $\beta = \beta_{\rm res} = 2(H/h)$. The dashed line shows a least-squares fit to the data assuming equation (12). The best-fitting parameters are $\beta_0 = 14.7$ and $a = 1.77$.

Alternatively, the dashed line shows a fit to the data of the form

$$\beta_{\rm res} = \frac{\beta_0}{(1 + ah/H)^2} \quad (12)$$

(see Section 3 for a discussion of the numerical effects that could give rise to such a relation). The important difference between this functional form and that given by equation (11) is that it implies that $\beta_{\rm res}$ converges to a finite value (β_0) at a high H/h. The best-fitting values of the parameters are $\beta_0 = 14.7$ and $a = 1.77$. We have excluded from the fit the data points that correspond to completely unresolved situations, that is $h/H > 1$.

Evidently, either of the lines shown in Fig. 1 provide an adequate fit to the data over the limited dynamic range of H/h available.[1]

We now turn our attention to the results of Meru & Bate (2011b), who had found an increase in the critical β for fragmentation at any radius with N. These authors consider the case $p = 1$ and $q = 0.1$. In the case where we are simply interested in fragmentation and not in the exact location within the disc where this happens, we only

[1] We also note that the data are adequately fitted by the suggested parametrization of Meru & Bate (2011a), in which $\beta_{\rm res}$ [in scaling with $m(R)^{0.5}$] effectively scales with $(H/h)^{0.5}$. The slightly different dependence of our coordinate H/h on stellar mass compared with that of Meru & Bate is also worth noting: we note that all disc quantities depend on the central object mass only via the Keplerian angular velocity Ω and that consequently $\beta_{\rm res}$ must depend on $m(R)$ and q in the combination $m(R)/q^{1/3}$.

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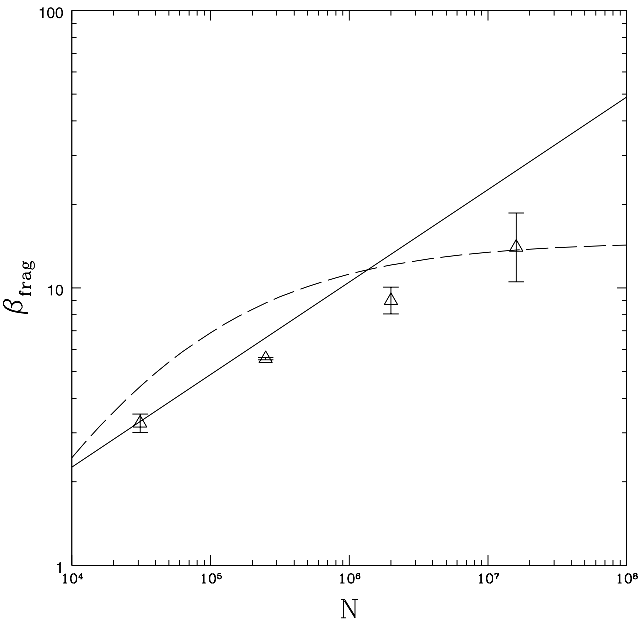

Figure 2. Correlation between the critical cooling time for fragmentation and the number of SPH particles used, for the simulations described in Meru & Bate (2010b). The threshold is obtained averaging the highest β for fragmenting simulations and the lowest β for non-fragmenting ones. The solid and dashed lines show equations (11) and (12), respectively, evaluated at the disc outer edge, and where the free parameters are set by the fit shown in Fig. 1.

need to make sure that the resolution criterion is satisfied at the disc outer edge, that is

$$\beta < \beta_{\mathrm{res}}(h/H)_{\mathrm{out}} \approx 6.6 \left(\frac{q}{0.1}\right)^{2/3} \left(\frac{N}{2.5 \times 10^5}\right)^{1/3}, \tag{13}$$

where the last approximate equality holds in the case that we adopt the linear fit (equation 11), and where we have used equations (8) and (10).

The data points of Meru & Bate (2011b) are plotted in Fig. 2, along with our prediction for β_{res} based on equation (13) (solid line) and also the corresponding relationship based on equation (12) (dashed line). We obtain the threshold for fragmentation β_{frag} from the data of Meru & Bate (2011b) by averaging the highest value of their fragmenting runs with the lowest non-fragmenting run. For this purpose, we consider as non-fragmenting those runs that were classified as 'borderline' by Meru & Bate (2011b).

We immediately see that the results of Meru & Bate (2011a) and Meru & Bate (2011b) are mutually consistent if both sets of results are explained in terms of resolution effects (that is, if $\beta_{\mathrm{frag}} = \beta_{\mathrm{res}}$), regardless of the exact parametrization that is used to describe how the cooling rate for fragmentation depends on H/h. We also note that, for the highest number of particles adopted by Meru & Bate (2011b), the results appear to deviate from the $N^{1/3}$ (solid) line implied by the linear fit (equation 11). In these cases, we thus have non-fragmenting simulations that are well resolved according to this criterion.

The simulation results in Fig. 2 are also broadly consistent with the dashed line (constructed using equation 12) which is a parametrization that implies a converged value of the required cooling rate for simulations that are sufficiently well resolved.

Fig. 2, and especially the deviation from the $N^{1/3}$ dependence at high N, therefore contains the tentative evidence that convergence may be being reached at the largest number of particles, and that the 'true' value of the threshold cooling time for fragmentation would therefore lie between $\beta \approx 10$ and 15, which is roughly a factor of 2 larger than previously thought.

3 PHYSICAL ORIGIN OF THE RESOLUTION CONDITION

In the previous section, we have explained the available simulation data in terms of two alternative parametrizations of how the cooling rate required for fragmentation may depend on the ratio of the smoothing length to the disc scaleheight. We now turn our attention to possible physical explanations of such requirements.

The first possibility is related to artificial viscosity. Indeed, the artificial viscosity term in SPH is linearly proportional to h/H (Monaghan 1992), and its effect might in principle alter the thermal balance of the disc.

As discussed above, for cooling discs the saturation of the gravitational instability is due to the balance of internal heating due to shocks induced by the instability itself and the externally imposed cooling. Clearly, if any other form of external heating is present, the saturation amplitude of the instability will be decreased and fragmentation might therefore be inhibited even for cooling parameters that would lead to fragmentation in the absence of external heating. For actual astrophysically relevant discs, such external heating sources might well be an important physical ingredient to be taken into account. On the other hand, in numerical simulations, where one wishes to determine the critical cooling rate, one has to be sure that such external heating sources, and in particular numerical heating sources, are negligible.

It is well known (Pringle 1981) that the condition of thermal equilibrium for a viscous disc can be easily expressed in terms of a relation between the viscosity parameter α (Shakura & Sunyaev 1973) and the cooling parameter β:

$$\alpha = \frac{4}{9\gamma(\gamma-1)}\frac{1}{\beta}. \tag{14}$$

For SPH simulations, the most significant source of numerical heating is provided by artificial viscosity. In particular, it can be shown (Artymowicz & Lubow 1996; Murray 1996; Lodato & Price 2010) that the linear term in the standard artificial viscosity implementation of SPH leads to an equivalent Shakura–Sunyaev parameter α_{art} given by

$$\alpha_{\mathrm{art}} = \frac{1}{10}\alpha_{\mathrm{SPH}}\frac{h}{H}, \tag{15}$$

where α_{SPH} is an SPH input parameter (usually set to 1, but reduced to 0.1 in the simulations discussed here). Actually, the above relation holds only in the continuum limit and if artificial viscosity is applied in regions of both converging and diverging flow. Usually, a switch enabling artificial viscosity only for convergent flows is used, and sometimes additional switches (Balsara 1995) are used, so that the above relation is actually an overestimate of the resulting viscosity.

When α_{art} becomes comparable to the value implied by equation (14), artificial heating offsets the imposed cooling completely, and no gravitational perturbations will be able to grow. This occurs when

$$\beta \approx \frac{40}{9\gamma(\gamma-1)}\frac{1}{\alpha_{\mathrm{SPH}}}\left(\frac{h}{H}\right)^{-1} = 40\left(\frac{h}{H}\right)^{-1}, \tag{16}$$

where the last equality refers to the case of $\gamma = 5/3$ and $\alpha_{\mathrm{SPH}} = 0.1$. The resolution criterion proposed in the previous section (equations 10 and 11) thus corresponds to the case where artificial

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viscosity provides 5 per cent of the heating required for thermal equilibrium.

We draw attention to two aspects of this result. First, it would imply – rather surprisingly – that artificial viscosity effects can prevent fragmentation even when they contribute a tiny fraction (5 per cent) of the total thermal energy balance. Secondly, it should be stressed that this condition is a necessary (but not necessarily a sufficient) criterion for fragmentation. As noted in Section 2, there is tentative evidence from Fig. 2 that the points at the highest N fall below the solid line – in other words we have simulations that are not fragmenting even though they are in the regime where the artificial viscosity contribution is less than 5 per cent. We interpret this result as evidence that there is indeed a physical mechanism preventing fragmentation for small cooling rates.

In addition to the thermal effects described above, artificial viscosity might also dynamically stabilize the disc. The specific effect of viscosity in this regard depends on how viscosity scales with surface density and might also lead to a secular instability (Schmit & Tscharnuter 1995). While this is beyond the scope of the present paper, such effects should be further investigated.

Since we find it rather surprising that artificial viscosity should suppress fragmentation when it contributes such a minor component to the disc's thermal balance, we also explore the hypothesis that the effect of under-resolution is one of artificially smoothing density enhancements over the SPH smoothing length h. In poorly resolved simulations, where h is not much smaller than the length-scale of density peaks (λ), this will suppress the peak amplitude of the resulting density fluctuations by a factor of $1 + h/\lambda$ (assuming that gravitational instabilities generate predominantly linear structures). According to Cossins et al. (2009), the gravitational heating rate associated with modes of rms fractional amplitude $\Delta\Sigma/\Sigma$ is proportional to $(\Delta\Sigma/\Sigma)^2$ and therefore in thermal equilibrium this quantity scales with the cooling rate ($\propto \beta^{-1}$). If we assume that in a perfectly resolved calculation the peak amplitude would scale with the rms amplitude (but be degraded by a factor of $1 + h/\lambda$ in the case of finite h) and if we furthermore associate fragmentation with the peak fluctuations achieving a critical value of $\Delta\Sigma/\Sigma$ (of order unity) then it follows that the cooling rate required for fragmentation is increased in the case of simulations of finite h. Specifically we have

$$\beta_{\mathrm{res}} = \beta_0(1 + h/\lambda)^{-2}, \tag{17}$$

where β_0 is the value required for fragmentation in the case of a well-resolved simulation (i.e. small h). In estimating λ we note that the Jeans length in a disc with Toomre Q parameter close to unity is just the vertical scaleheight H; it therefore seems logical that λ should scale with H and this motivates the form of equation (12).

We note that both our resolution criteria do not abandon the notion that (in a well-resolved simulation) there is a critical cooling rate associated with fragmentation but take into account the fact that more vigorous cooling (lower β) is required in the case of poorly resolved simulations.

4 DISCUSSION AND CONCLUSIONS

In this paper we have shown that the results of Meru & Bate (2011a) (concerning the location at which fragmentation occurs in SPH simulations of cooling self-gravitating discs) are quantitatively consistent (if interpreted as being driven by resolution effects) with the results of Meru & Bate (2011b) in which it is shown that the critical cooling rate for fragmentation depends on the number of particles N. Indeed the results of each paper imply the results of the other and both imply that the measured cooling rate for fragmentation is a function of h/H (where h is the SPH smoothing length and H is the disc scaleheight).

It is difficult to assign a precise functional form for the way that the critical cooling rate depends on resolution, given the scatter in the simulation data and the relatively small range in H/h for which simulations are currently available. We have explored two possibilities for effects that may explain the simulation results. In one case we explore the possibility that a necessary criterion for fragmentation to be detected is that the artificial viscosity contributes less than a certain fraction of the thermal energy input (and associated angular momentum transport) in the disc. We find that the results at low resolution can be explained in these terms but that the requirement on the contribution from artificial viscosity is surprisingly stringent (i.e. fragmentation occurs only where this contributes less than 5 per cent of the energy input to the disc). Alternatively, we consider the possibility that finite resolution just smooths out density peaks so that fluctuations that would collapse if well resolved do not achieve a critical amplitude at finite h.

Irrespective of how one explains how resolution affects the cooling requirements for fragmentation, an important issue – as stressed by Meru & Bate (2011b) – is whether there is indeed a convergence in the required cooling rate at high resolution. In other words, is there a level of resolution above which one recovers the result that has been assumed hitherto – i.e. that there is a *physical* condition on the cooling rate required for fragmentation? We evidently cannot answer this question with calculations that have not attained convergence (see Fig. 2 where β_{frag} rises mildly with N even at the highest N values studied). However, Fig. 2 contains hints of approaching convergence. One way to see this is to compare the simulation data with the solid line which corresponds to the requirement of a 5 per cent contribution from artificial viscosity. The fact that the right-hand points lie below this solid line (where β_{frag} scales with $N^{1/3}$) is evidence that this condition may be necessary but not sufficient for fragmentation. Similarly, the simulation data are consistent with the dashed line in which the actual β required for fragmentation converges to a value of $\beta_0 \approx 14.7$ at high resolution.

Clearly, the question of ultimate convergence will only be settled by further simulations at higher N. If convergence is *not* ultimately attained then it would involve a radical rethink of all our understanding of gravitationally unstable discs since it would imply that *any* self-gravitating disc (however slowly cooling) should in reality fragment. This would be surprising because, in the case of very slow cooling, the rms mode amplitude required to achieve thermal balance should be very low and it would be unexpected (though possible in principle) that such a disc nevertheless exhibited locally non-linear density fluctuations giving rise to collapsing fragments.

A more conservative conclusion (if convergence is ultimately attained) is that we have simply underestimated the critical β value hitherto. This would involve some quantitative adjustment of previous conclusions. For example, if the critical cooling time for fragmentation is roughly a factor of 2 higher than had been previously thought then this affects the location at which giant planets can form through gravitational instability (Meru & Bate 2011b): in the regime of ice cooling appropriate to the outer parts of protostellar discs, this change would bring in the minimum radius for fragmentation by a factor of $\sim 2^{2/9}$ (Clarke 2009; Cossins et al. 2010). Although this change is quite modest, it may be important to the discussion of whether recently imaged planets (e.g. around HR 8799, Marois et al. 2008, around β Pic, Lagrange et al. 2009 or around Fomahault, Kalas et al. 2008) could have been formed by gravitational instability. Note, however, that – for a given critical cooling time – the uncertainty in the determination of the fragmentation radius due to uncertainties in the relevant opacities, in the detailed vertical structure of the disc and due to the effects of magnetic fields are probably larger than the modest change discussed above.

More in general, we might ask which ones of the properties of non-fragmenting self-gravitating accretion discs are going to be significantly affected by poor resolution, in the sense expressed by equation (10). Probably the most important property, apart from fragmentation, is related to the stress induced by the instability, and in particular its magnitude and the locality of the associated dissipation (which are in turn related to the power spectrum of the modes excited by the instability).

As mentioned in Section 1, several studies (Lodato & Rice 2004, 2005; Cossins et al. 2009) have considered the issue of locality of the transport induced by the instability. Here the question is whether the typical wavelength of the disturbances is a significant fraction of the radius R. The above studies (and in particular Cossins et al. 2009) have shown that the spectrum of the disturbances peaks at the wavelength $\lambda \approx H$ and becomes negligible for $\lambda \gtrsim 3H$. In this respect, thus, the relevant resolution requirement is the generally less restrictive condition $h/H \lesssim 1$ already discussed in Nelson (2006). Furthermore, not resolving the smallest scales would only prevent the disc from developing small-scale structures, which would not contribute to global transport anyway.

Let us now discuss whether the evaluation of the stress from such simulations is affected by equation (10). Clearly, *for a given disturbance*, regardless of the way it has been excited, a correct evaluation of the resulting stress only requires that the typical wavelengths of the disturbance are resolved. However, here the question is whether the disc response to an external boundary condition (in this case, to an externally imposed cooling rate) depends on resolution. The condition of thermal equilibrium relates in a unique way the external cooling to the magnitude of the stress induced by the instability (equation 14), under the hypothesis that the only heating source in the disc is provided by the instability itself and is dissipated locally. Here, we have shown that the new empirically determined resolution condition corresponds to the case where artificial viscosity only provides 5 per cent of the heating necessary for thermal balance. Therefore, unless the effects of artificial viscosity are much larger than expected, one would not expect a change in the stress above the 5 per cent level. In this respect, the effect of poor resolution is just to suppress the development of small-scale disturbances, and thus reduce the peak value of the density perturbation, while keeping the overall average value of the stress essentially unchanged.

Finally, it is worth noting that the discussion above is focused on SPH simulations. Many previous results (Gammie 2001; Mejia et al. 2005; Boley et al. 2006), including several determinations of the fragmentation boundary (Gammie 2001; Boss 2004, 2006; Mejia et al. 2005), have been obtained using grid-based codes. Artificial viscosity is also present to various degrees is grid-based codes, and one might thus conclude that an analogous resolution requirement should also be formulated in this case. Whether grid-based simulations also do not converge in the determination of the critical time-scale for fragmentation and the exact form of the resolution requirement in this case are questions that still need to be worked on.

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