SIAM J. FINANCIAL MATH.
Vol. 2, pp. 357–382

Dual Representation of Quasi-convex Conditional Maps*

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Abstract. We provide a dual representation of quasi-convex maps $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$, between two locally convex lattices of random variables, in terms of conditional expectations. This generalizes the dual representation of quasi-convex real valued functions $\pi : L_{\mathcal{F}} \to \mathbb{R}$ and the dual representation of conditional convex maps $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$. These results were inspired by the theory of dynamic measurements of risk and are applied in this context.

Key words. quasi-convex functions, dual representation, quasi-convex optimization, dynamic risk measures, conditional certainty equivalent

AMS subject classifications. Primary, 46N10, 91G99, 60H99; Secondary, 46A20, 46E30

DOI. 10.1137/09078033X

1. Introduction.

Quasi-convex analysis has important applications in several optimization problems in science, economics, and finance, where convexity may be lost due to absence of global risk aversion, as, for example, in prospect theory [KT92].

The first relevant mathematical findings on quasi-convex functions were provided by De Finetti [DF49], and since then, many authors, such as those of [Fe49], [Cr80], [PP84], and [PV90], to mention just a few, contributed significantly to the subject. More recently, a decision theory complete duality involving quasi-convex real valued functions has been proposed in [CMM09b]. For a review of quasi-convex analysis and its applications and for an exhaustive list of references on this topic, we refer the reader to Penot [Pe07].

A function $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ defined on a vector space L is quasi-convex if for all $c \in \mathbb{R}$ the lower level sets $\{X \in L \mid f(X) \leq c\}$ are convex. In a general setting, the dual representation of such functions was shown by Penot and Volle [PV90]. The following theorem, reformulated in order to be compared to our results, was proved by Volle [Vo98, Thm. 3.4]. As shown in the section 5.2, its proof relies on a straightforward application of the Hahn–Banach theorem.

Theorem 1.1 (see [Vo98]). *Let L be a locally convex topological vector space, L' be its dual space, and $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be quasi-convex and lower semicontinuous. Then*

$$f(X) = \sup_{X' \in L'} R(X'(X), X'), \tag{1.1}$$

where $R : \mathbb{R} \times L' \to \overline{\mathbb{R}}$ is defined by

$$R(t, X') := \inf_{\xi \in L} \left\{ f(\xi) \mid X'(\xi) \geq t \right\}.$$

*Received by the editors December 16, 2009; accepted for publication (in revised form) February 5, 2011; published electronically May 24, 2011.
http://www.siam.org/journals/sifin/2/78033.html

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The generality of this theorem rests on the very weak assumptions made on the domain of the function f, i.e., on the space L. On the other hand, the fact that only *real valued* maps are admitted limits its potential applications considerably, especially in a dynamic framework.

To the best of our knowledge, a *conditional* version of this representation is lacking in the literature. When $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, many problems having dynamic features lead to the analysis of maps $\pi : L_t \to L_s$ between the subspaces $L_t \subseteq L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ and $L_s \subseteq L^0(\Omega, \mathcal{F}_s, \mathbb{P})$, $0 \leq s < t$ (see section 1.1 for some examples).

In this paper we consider quasi-convex maps of this form and analyze their dual representation. We provide (see Theorems 2.9 and 2.10 for the exact statements) a conditional version of (1.1):

$$\pi(X) = ess \sup_{Q \in L_t^* \cap \mathcal{P}} R(E_Q[X|\mathcal{F}_s], Q), \tag{1.2}$$

where

$$R(Y,Q) := ess \inf_{\xi \in L_t} \{\pi(\xi) \mid E_Q[\xi|\mathcal{F}_s] \geq_Q Y\},$$

$L_t^* \hookrightarrow L_t^1$ is the order continuous dual space of L_t, and

$$\mathcal{P} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \ll \mathbb{P} \text{ and } Q \text{ probability} \right\} = \left\{ \xi' \in L_+^1 \mid E_{\mathbb{P}}[\xi'] = 1 \right\}.$$

With a slight abuse of notation, we write $Q \in L_t^* \cap \mathcal{P}$ instead of $\frac{dQ}{d\mathbb{P}} \in L_t^* \cap \mathcal{P}$.

Furthermore, we prove in Proposition 2.13 that π is quasi-convex, monotone, continuous from below, and regular if and only if (1.2) holds with R belonging to the class $\mathcal{R}$ of maps $S : L_s^0 \times L_t^* \to \bar{L}_s^0$ such that $S(\cdot, \xi')$ is quasi-convex, monotone, continuous from below, and regular.

If the map π is quasi-convex, monotone, and cash additive, then we derive from (1.2) the well-known representation of a convex conditional risk measure, as in [DS05]. Of course, this is of no surprise since cash additivity and quasi convexity imply convexity, but it supports the correctness of our dual representation.

The formula (1.2) is obtained under quite weak assumptions on the space L_t which allow us to consider maps π defined on the typical spaces used in the literature: $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$, $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, and the Orlicz spaces $L^\Psi(\Omega, \mathcal{F}_t, \mathbb{P})$.

In Theorem 2.9 we assume that π is lower semicontinuous, with respect to the weak topology $\sigma(L_t, L_t^*)$. As shown in Proposition 2.5 this condition is equivalent to continuity from below, a natural requirement in this context. In Theorem 2.10 instead we provide the dual representation under a strong upper semicontinuity assumption.

The proofs of our main theorems (Theorems 2.9 and 2.10) are neither based on techniques similar to those applied in the quasi-convex real valued case [Vo98] nor to those used for convex conditional maps [DS05]. Indeed, the so-called scalarization of π via the real valued map $X \to E_{\mathbb{P}}[\pi(X)]$ does not work, since this scalarization preserves convexity but not quasi convexity. The idea of our proof is to apply (1.1) to the real valued quasi-convex map $\pi_A : L_t \to \overline{\mathbb{R}}$ defined by $\pi_A(X) := ess\sup_{\omega \in A} \pi(X)(\omega)$, $A \in \mathcal{F}_s$, and to approximate $\pi(X)$ with $\pi^\Gamma(X) := \sum_{A \in \Gamma} \pi_A(X) \mathbf{1}_A$, where Γ is a finite partition of Ω of $\mathcal{F}_s$-measurable sets $A \in \Gamma$. As explained in section 4.1, some delicate issues arise when one tries to apply this simple and natural idea to prove that

$$(1.3)\quad ess \sup_{Q\in L_t^*\cap\mathcal{P}} ess \inf_{\xi\in L_t} \{\pi(\xi) \mid E_Q[\xi|\mathcal{F}_s] \geq_Q E_Q[X|\mathcal{F}_s]\}$$
$$= ess \inf_{\Gamma}\; ess \sup_{Q\in L_t^*\cap\mathcal{P}} ess \inf_{\xi\in L_t} \left\{\pi^{\Gamma}(\xi) \mid E_Q[\xi|\mathcal{F}_s] \geq_Q E_Q[X|\mathcal{F}_s]\right\}.$$

The uniform approximation here needed is stated in the key result (Lemma 4.3), and section 5.1 is devoted to proving it.

It has recently been shown[1] in [FM10] that results of the same nature, but technically quite different, of those of Theorems 2.9 and 2.10 hold for maps defined on modules of L^p type (see [FKV09] for details on this setting). The module approach used in [FM10] permits one to prove that the maximum in (2.5) is attained, under the same assumptions of Theorem 2.10, when the maps are defined on modules of L^p type.

In the present paper we limit ourselves to considering conditional maps $\pi : L_t \to L_s$, and we defer to a forthcoming paper the study of the temporal consistency of the family of maps $(\pi_s)_{s\in[0,t]}$, $\pi_s : L_t \to L_s$.

The paper is organized as follows. In section 2 we introduce the key definitions in order to have all the ingredients to state, in section 2.1, our main results. Section 3 is a collection of a priori properties about the maps we use to obtain the dual representation. Theorems 2.9 and 2.10 are proved in section 4, and a brief outline of the proof is reported there to facilitate its understanding. The technically important lemmas are left to the appendix, where we also report the proof of Theorem 1.1.

1.1. Applications to finance. As a further motivation for our findings, we give some examples of quasi-convex (quasi-concave) conditional maps arising in economics and finance, which will also be analyzed in detail in a future paper.

Certainty equivalent in dynamic settings. Consider a stochastic dynamic utility $u : \mathbb{R}\times [0,\infty) \times \Omega \to \mathbb{R}$ where the function $x \to u(x,t,\omega)$ is strictly increasing and concave on $\mathbb{R}$ for almost any $\omega \in \Omega$ and for $t \in [0,\infty)$; the function $u(x,t,\cdot)$ is $\mathcal{F}_t$-measurable for all $(x,t) \in \mathbb{R}\times[0,\infty)$. These functions have recently been considered in [MZ06] to develop the theory of forward utility.

In [FM11] we defined the *conditional certainty equivalent* (CCE) of a random variable $X \in L_t$ as the random variable $\pi(X) \in L_s$ that is the solution of the equation

$$u(\pi(X), s) = E_{\mathbb{P}}\left[u(X,t)|\mathcal{F}_s\right].$$

Thus the CCE defines the *valuation* operator

$$\pi : L_t \to L_s,\ \pi(X) = u^{-1}\left(E_{\mathbb{P}}\left[u(X,t)|\mathcal{F}_s\right]\right), s).$$

We showed in [FM11] that the CCE as a map $\pi : L_t \to L_s$ is monotone, quasi-concave, and regular and that it admits the (concave version) representation as in (1.2).

[1] This was added in the revision.

Static risk measures. Our interest in quasi-convex analysis was triggered by the recent paper [CMM09a] on quasi-convex risk measures, where the authors show that it is reasonable to weaken the convexity axiom in the theory of convex risk measures, introduced in [FS02] and [FR02]. In fact when one replaces cash additivity with cash subadditivity (as explained in [ER09]), quasi-convexity and convexity are no longer equivalent. But the quasi convexity property is the literal translation of the principle “diversification should not increase the risk.”

The recent interest in quasi-convex static risk measures is also testified to by a second paper [DK10] on this subject that was inspired by [CMM09a] and disclosed after the first version of the present paper.

Dynamic risk measures. As already mentioned the dual representation of a conditional *convex* risk measure can be found in [DS05]. The findings of the present paper show the dual representation of conditional *quasi-convex* risk measures when cash additivity does not hold true.

For a better understanding we give a concrete example: consider a nonempty convex set $C \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $C + L^\infty_+ \subseteq C$. The set C represents the future positions considered acceptable by the supervising agency. Let $s \in [0, t]$. For all $m \in \mathbb{R}$ denote by $v_s(m, \omega)$ the price at time s of m euros at time t. The function $v_s(m, \cdot)$ will, in general, be $\mathcal{F}_s$ measurable, as in the case of a stochastic discount factor where $v_s(m, \omega) = D_s(\omega)m$. By adapting the definitions in the static framework of [ADEH99] and [CMM09a] we set

$$\rho_{s,v_s}(X)(\omega) = ess \inf_{Y \in L^0_{\mathcal{F}_s}} \{v_s(Y, \omega) \mid X + Y \in C\}.$$

When v_s is linear, ρ_{s,v_s} is a convex monetary dynamic risk measure, but the linearity of v_s may fail when zero coupon bonds with maturity t are illiquid. It seems reasonable to assume that $v_s(\cdot, \omega)$ is increasing and upper semicontinuous and $v_s(0, \omega) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. In this case

$$\rho_{s,v_s}(X)(\omega) = v_s\left(ess \inf_{Y \in L^0_{\mathcal{F}_s}} \{Y \mid X + Y \in C\}, \omega\right) = v_s(\rho_s(X), \omega),$$

where $\rho_s(X)$ is the convex monetary dynamic risk measure induced by the set C. Thus, in general, ρ_{s,v_s} is neither convex nor cash additive, but it is quasi-convex and eventually cash subadditive (under further assumptions on v_s).

Acceptability indices. As studied in [CM09] the index of acceptability is a map α from a space of random variables $L(\Omega, \mathcal{F}, \mathbb{P})$ to $[0, +\infty)$ which measures the performance or quality of the random X which may be the terminal cash flow from a trading strategy. Associated with each level x of the index there is a collection of terminal cash flows $\mathcal{A}_x = \{X \in L | \alpha(X) \geq x\}$ that are acceptable at this level. The authors in [CM09] suggest four axioms as the stronghold for an acceptability index in the static case: quasi concavity (i.e., the set $\mathcal{A}_x$ is convex for every $x \in [0, +\infty)$), monotonicity, scale invariance, and the Fatou property. It appears natural (see also the recent paper [BCZ10])[2] to generalize these kinds of indices to the conditional case, and to this aim we propose a couple of basic examples:

[2]This was added in the revision.

(i) conditional gain loss ratio: let $\mathcal{F}_s \subseteq \mathcal{F}_t$, and let

$$CGLR(X|\mathcal{F}_s) = \frac{E_{\mathbb{P}}[X|\mathcal{F}_s]}{E_{\mathbb{P}}[X^-|\mathcal{F}_s]}\mathbf{1}_{\{E_{\mathbb{P}}[X|\mathcal{F}_s]>0\}}.$$

This measure is clearly monotone, scale invariant, and well defined on $L^1(\Omega, \mathcal{F}_t, \mathbb{P})$. It can be proved that it is continuous from below and quasi-concave.

(ii) conditional coherent risk-adjusted return on capital: let $\mathcal{F}_s \in \mathcal{F}_t$, and suppose a coherent conditional risk measure $\rho : L(\Omega, \mathcal{F}_t, \mathbb{P}) \to L^0(\Omega, \mathcal{F}_s, \mathbb{P})$ is given, where $L(\Omega, \mathcal{F}_t, \mathbb{P}) \subseteq L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ is any vector space. We define

$$CRARoC(X|\mathcal{F}_s) = \frac{E_{\mathbb{P}}[X|\mathcal{F}_s]}{\rho(X)}\mathbf{1}_{\{E[X|\mathcal{F}_s]>0\}}.$$

We use the convention that $CRARoC(X|\mathcal{F}_s) = +\infty$ on the $\mathcal{F}_s$-measurable set where $\rho(X) \leq 0$. Again $CRARoC(\cdot|\mathcal{F}_s)$ is well defined on the space $L(\Omega, \mathcal{F}_t, \mathbb{P})$ and takes values in the space of extended random variables; moreover, it is monotone, quasi-concave, scale invariant, and continuous from below whenever ρ is continuous from above.

2. The dual representation.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout the paper, and $\mathcal{G} \subseteq \mathcal{F}$ is any sigma algebra contained in $\mathcal{F}$. As usual we denote with $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of $\mathcal{F}$ measurable random variables that are $\mathbb{P}$-a.s. finite and by $\bar{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of extended real valued random variables.

The $L^p(\Omega, \mathcal{F}, \mathbb{P})$ spaces, $p \in [0, \infty]$, will simply be denoted by L^p, unless it is necessary to specify the sigma algebra, in which case we write $L^p_{\mathcal{F}}$. In the presence of an arbitrary probability measure Q, if confusion may arise, we will explicitly write $=_Q$ (resp., $\geq_Q$), meaning Q-a.s. Otherwise, all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s. Moreover, the essential ($\mathbb{P}$-a.s.) *supremum* $ess\sup_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be denoted simply by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum* (see [FS04, section A.5] for reference). Here we notice only that $1_A \sup_\lambda(X_\lambda) = \sup_\lambda(1_A X_\lambda)$ for any $\mathcal{F}$ measurable set A. Hereafter the symbol $\hookrightarrow$ denotes inclusion and lattice embedding between two lattices; $\vee$ (resp., $\wedge$) denotes the essential ($\mathbb{P}$-a.s.) *maximum* (resp., the essential *minimum*) between two random variables, which are the usual lattice operations.

We consider a lattice $L_{\mathcal{F}} := L(\Omega, \mathcal{F}, \mathbb{P}) \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ and a lattice $L_{\mathcal{G}} := L(\Omega, \mathcal{G}, \mathbb{P}) \subseteq \bar{L}^0(\Omega, \mathcal{G}, \mathbb{P})$ of $\mathcal{F}$ (resp., $\mathcal{G}$) measurable random variables. Therefore, the range of a map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ includes extended real valued random variables.

Definition 2.1. *A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be*

(MON) *monotone increasing if for every $X, Y \in L_{\mathcal{F}}$*

$$X \leq Y \quad \Rightarrow \quad \pi(X) \leq \pi(Y);$$

(QCO) *quasi-convex if for every $X, Y \in L_{\mathcal{F}}$, $\Lambda \in L^0_{\mathcal{G}}$, and $0 \leq \Lambda \leq 1$*

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \pi(X) \vee \pi(Y);$$

(LSC) *τ-lower semicontinuous if the set $\{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$ is closed for every $Y \in L_{\mathcal{G}}$ with respect to a topology τ on $L_{\mathcal{F}}$;*

(CFB) *continuous from below if*

$$X_n \uparrow X \quad \mathbb{P}\text{-}a.s. \quad \Rightarrow \quad \pi(X_n) \uparrow \pi(X) \quad \mathbb{P}\text{-}a.s;$$

(USC)* *τ-strong upper semicontinuous if the set $\{X \in L_{\mathcal{F}} \mid \pi(X) < Y\}$ is open for every $Y \in L_{\mathcal{G}}$ with respect to a topology τ on $L_{\mathcal{F}}$ and there exists $\theta \in L_{\mathcal{F}}$ such that $\pi(\theta) < +\infty$.*

Remark 2.2 (on quasi convexity). As it happens for real valued maps, the definition of (QCO) is equivalent to the fact that for all $Y \in L_{\mathcal{G}}$ the lower level sets $\mathcal{A}(Y) = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$ are conditionally convex; i.e., for all $X_1, X_2 \in \mathcal{A}(Y)$ and for all $\mathcal{G}$-measurable random variables Λ, $0 \leq \Lambda \leq 1$, one has that $\Lambda X_1 + (1-\Lambda)X_2 \in \mathcal{A}(Y)$.

Remark 2.3 (on upper semicontinuity). When the map π is *real valued*, (USC)* is equivalent to

$$\{X \in L_{\mathcal{F}} \mid \pi(X) \geq Y\} \text{ is closed} \quad \forall Y \in \mathbb{R}.$$

But when the range of π is $L_{\mathcal{G}}$ (a space of random variables), this equivalence does not hold true. Our strong definition (USC)* implies that if a net $\{X_\alpha\}_\alpha \subset L_{\mathcal{F}}$ satisfies $X_\alpha \xrightarrow{\tau} X$, then $\limsup_\alpha \pi(X_\alpha) \leq \pi(X)$. Furthermore, this last condition implies that the set $\{X \in L_{\mathcal{F}} \mid \pi(X) \geq Y\}$ is closed, i.e., the usual upper semicontinuity (USC) condition, so that (USC)* $\Rightarrow$(USC).

We are assuming that there exists at least one $\theta \in L_{\mathcal{F}}$ such $\pi(\theta) < +\infty$; otherwise the set $\{X \in L_{\mathcal{F}} \mid \pi(X) < Y\}$ is always empty (and then open) and the condition (USC)* loses any meaning.

In [BF09] the equivalence between (CFB) and $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC) for monotone convex real valued functions is proved. In the next proposition, we state that this equivalence remains true for monotone quasi-convex conditional maps, under the same assumption on the topology adopted in [BF09]. Therefore, in Theorem 2.9 the $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC) condition can be replaced by (CFB), which is often easy to check.

Definition 2.4 (see [BF09]). *A linear topology τ on a Riesz space has the C-property if $X_\alpha \xrightarrow{\tau} X$ implies the existence of of a sequence $\{X_{\alpha_n}\}_n$ and a convex combination $Z_n \in conv(X_{\alpha_n}, \dots)$ such that $Z_n \xrightarrow{o} X$.*

As explained in [BF09], the assumption that $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$ has the C-property is very weak and is satisfied in all cases of interest.

Proposition 2.5. *Suppose that $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$ satisfies the C-property and $L_{\mathcal{F}}$ is order complete. If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is* (MON) *and* (QCO)*, then π is $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-*(LSC) *if and only if π is* (CFB).

We omit the proof since it is a simple extension of the one written in the cited reference.

Definition 2.6. *A vector space $L_{\mathcal{F}} \subseteq L^0_{\mathcal{F}}$ satisfies the property $1_{\mathcal{F}}$ if*

$$(1_F) \qquad X \in L_{\mathcal{F}} \text{ and } A \in \mathcal{F} \quad \Longrightarrow \quad (X\mathbf{1}_A) \in L_{\mathcal{F}}.$$

Suppose $L_{\mathcal{F}}$ (resp., $L_{\mathcal{G}}$) satisfies property $(1_{\mathcal{F}})$ (resp $1_{\mathcal{G}}$). A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is

(REG) *regular if for every $X, Y \in L_{\mathcal{F}}$ and $A \in \mathcal{G}$*

$$\pi(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \pi(X)\mathbf{1}_A + \pi(Y)\mathbf{1}_{A^C}$$

or, equivalently, if $\pi(X\mathbf{1}_A)\mathbf{1}_A = \pi(X)\mathbf{1}_A$.

2.1. The representation theorems and their consequences.

Standing assumptions.

From now on, the following are assumed

(a) $\mathcal{G} \subseteq \mathcal{F}$ and the lattice $L_{\mathcal{F}}$ (resp., $L_{\mathcal{G}}$) satisfies the property $(1_{\mathcal{F}})$ (resp., $1_{\mathcal{G}}$). Both $L_{\mathcal{G}}$ and $L_{\mathcal{F}}$ contain the constants as a vector subspace.

(b) The order continuous dual of $(L_{\mathcal{F}}, \geq)$, denoted by $L^*_{\mathcal{F}} = (L_{\mathcal{F}}, \geq)^*$, is a lattice (see [AB05, Thm. 8.28]) that satisfies $L^*_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$ and property $(1_{\mathcal{F}})$.

(c) The space $L_{\mathcal{F}}$ endowed with the weak topology $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$ is a locally convex Riesz space.

The condition (c) requires that the order continuous dual $L^*_{\mathcal{F}}$ be rich enough to separate the points of $L_{\mathcal{F}}$, so that $(L_{\mathcal{F}},\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}}))$ becomes a locally convex TVS and Proposition 5.5 can be applied.

Remark 2.7. Many important classes of spaces satisfy these conditions, such as

- the L^p-spaces, $p \in [1, \infty]$: $L_{\mathcal{F}} = L^p_{\mathcal{F}}$, $L^*_{\mathcal{F}} = L^q_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$;
- the Orlicz spaces L^Ψ for any Young function Ψ: $L_{\mathcal{F}} = L^\Psi_{\mathcal{F}}$, $L^*_{\mathcal{F}} = L^{\Psi^*}_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$, where Ψ^* denotes the conjugate function of Ψ;
- the Morse subspace M^Ψ of the Orlicz space L^Ψ for any continuous Young function Ψ: $L_{\mathcal{F}} = M^\Psi_{\mathcal{F}}$, $L^*_{\mathcal{F}} = L^{\Psi^*}_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$.

Define $K : L_{\mathcal{F}} \times (L^*_{\mathcal{F}} \cap \mathcal{P}) \to \bar{L}^0_{\mathcal{G}}$ and $R : L^0_{\mathcal{G}} \times L^*_{\mathcal{F}}$ as

$$K(X, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}, \tag{2.1}$$

$$R(Y, \xi') := \inf_{\xi \in L_{\mathcal{F}}} \left\{\pi(\xi) \mid E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq Y\right\}. \tag{2.2}$$

The function K is well defined on $L_{\mathcal{F}} \times (L^*_{\mathcal{F}} \cap \mathcal{P})$, while the actual domain of R is

$$\Sigma := \{(Y, \xi') \in L^0_{\mathcal{G}} \times L^*_{\mathcal{F}} | \, \exists \xi \in L_{\mathcal{F}} \text{ such that } E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq Y\}. \tag{2.3}$$

Obviously $(E_{\mathbb{P}}[\xi' X|\mathcal{G}], \xi') \in \Sigma$ for every $X \in L_{\mathcal{F}}$, $\xi' \in L^*_{\mathcal{F}}$. Notice that $K(X, Q)$ depends on X only through $E_Q[X|\mathcal{G}]$. Moreover, for every $\lambda > 0$, $R(E_{\mathbb{P}}[\xi' X|\mathcal{G}], \xi') = R(E_{\mathbb{P}}[\lambda\xi' X|\mathcal{G}], \lambda\xi')$. Thus we can consider $R(E_{\mathbb{P}}[\xi' X|\mathcal{G}], \xi')$, $\xi' \geq 0$, $\xi' \neq 0$, always defined on the normalized elements $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$. It is easy to check that

$$E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}\xi \mid \mathcal{G}\right] \geq E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X \mid \mathcal{G}\right] \Longleftrightarrow E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}],$$

and for $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$ we deduce that

$$K(X, Q) = R\left(E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X \mid \mathcal{G}\right], Q\right).$$

Remark 2.8. Since the order continuous functional on $L_{\mathcal{F}}$ is contained in L^1, then $Q(\xi) := E_Q[\xi]$ is well defined and finite for every $\xi \in L_{\mathcal{F}}$ and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$. In particular this and property $(1_{\mathcal{F}})$ imply that $E_Q[\xi|\mathcal{G}]$ is well defined. Moreover, $(1_{\mathcal{F}})$ guarantees that $\frac{dQ}{d\mathbb{P}} 1_A \in L^*_{\mathcal{F}}$ whenever $Q \in L^*_{\mathcal{F}}$ and $A \in \mathcal{F}$.

Theorem 2.9. *Suppose that* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$ *has the C-property and* $L_{\mathcal{F}}$ *is order complete. If* $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is* (MON), (QCO), (REG), *and* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC), *then*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} K(X, Q). \tag{2.4}$$

Theorem 2.10. *If* $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is* (MON), (QCO), (REG), *and* τ-(USC)*, *then*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} K(X, Q). \tag{2.5}$$

Notice that in (2.4) and (2.5) the *supremum* is taken over the set $L^*_{\mathcal{F}} \cap \mathcal{P}$. In the next corollary, proved in section 4.2, we show that we match the conditional convex dual representation by restricting our optimization problem on the set

$$\mathcal{P}_{\mathcal{G}} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in \mathcal{P} \text{ and } Q = \mathbb{P} \text{ on } \mathcal{G} \right\}.$$

Clearly, when $Q \in \mathcal{P}_{\mathcal{G}}$, then $\bar{L}^0(\Omega, \mathcal{G}, \mathbb{P}) = \bar{L}^0(\Omega, \mathcal{G}, Q)$ and comparison of $\mathcal{G}$ measurable random variables is understood to hold indifferently for $\mathbb{P}$ or Q.

Corollary 2.11. *Under the same hypothesis of Theorem* 2.9 *or Theorem* 2.10, *suppose that for* $X \in L_{\mathcal{F}}$ *there exist* $\eta \in L_{\mathcal{F}}$ *and* $\delta > 0$ *such that* $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$. *Then*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}} K(X, Q). \tag{2.6}$$

If $S : \Sigma \to \bar{L}^0_{\mathcal{G}}$, then $S(\cdot, \xi')$ is (REG) if $S(Y\mathbf{1}_A, Q)\mathbf{1}_A = S(Y, Q)\mathbf{1}_A$ for all $A \in \mathcal{G}$; $S(\cdot, \xi')$ is (CFB) if $Y_n \uparrow Y$, $\mathbb{P}$-a.s.; and $(Y_n, \xi') \in \Sigma, (Y, \xi') \in \Sigma$ imply $S(Y_n, \xi') \uparrow S(Y, \xi')$. As shown in the next proposition, the dual characterization of maps $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ that are (MON), (REG), (QCO), and (CFB) is obtained through the class

$$\mathcal{R} := \left\{ S : \Sigma \to \bar{L}^0_{\mathcal{G}} \text{ such that } S(\cdot, \xi') \text{ is (MON), (REG), and (CFB)} \right\}.$$

Remark 2.12. Any map $S : \Sigma \to \bar{L}^0_{\mathcal{G}}$ such that $S(\cdot, \xi')$ is (MON) and (REG) is automatically (QCO) in the first component. Indeed, let $Y_1, Y_2, \Lambda \in L^0_{\mathcal{G}}$, $0 \leq \Lambda \leq 1$, and define $B = \{Y_1 \leq Y_2\}$, $S(\cdot, Q) = S(\cdot)$. Then $S(Y_1\mathbf{1}_B) \leq S(Y_2\mathbf{1}_B)$ and $S(Y_2\mathbf{1}_{B^C}) \leq S(Y_1\mathbf{1}_{B^C})$ so that from (MON) and (REG)

$$S(\Lambda Y_1 + (1 - \Lambda)Y_2) \leq S(Y_2\mathbf{1}_B + Y_1\mathbf{1}_{B^C}) = S(Y_2)\mathbf{1}_B + S(Y_1)\mathbf{1}_{B^C} \leq S(Y_1) \vee S(Y_2).$$

Proposition 2.13. *Suppose that* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$ *satisfies the C-property and* $L_{\mathcal{F}}$ *is order complete. The map* $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is* (MON), (QCO), (REG), *and* (CFB) *if and only if there exists* $S \in \mathcal{R}$ *such that*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} S\left(E\left[\frac{dQ}{d\mathbb{P}} X | \mathcal{G} \right], Q \right). \tag{2.7}$$

The proof is based on Theorem 2.9 and is postponed to section 4.2.

In [CMM09b] the authors provide a complete duality for *real valued* quasi-convex (either USC or LSC) functionals when the space $L_{\mathcal{F}}$ is an M-space (such as L^∞): the idea is to prove a one to one relationship between quasi-convex monotone functionals π and the function R in the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. A similar result has been recently obtained in [DK10] for general TVS in the LSC real valued case. In the conditional case, uniqueness is a very delicate issue which has only recently been addressed (see [FM10]),[3] but only in the module framework developed in [FKV09]. In the setting of the present paper, a partial result can be easily derived, from the static case, when $\mathcal{G}$ is countably generated by a partition $\{A_n\}_{n\in\mathbb{N}}$, and thus the map π is constant on each atom A_n.

In the following corollary (proved in section 4.2) we show that the (MON) property implies that the constraint $E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$, in the definition of $K(X,Q)$, may be restricted to $E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]$. We also show that we may recover, in agreement with [DS05], the dual representation of a dynamic risk measure when π also satisfies the following property: A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be

(CAS) cash additive if for all $X \in L_{\mathcal{F}}$ and $\Lambda \in L_{\mathcal{G}} \cap L_{\mathcal{F}}$

$$\pi(X+\Lambda) = \pi(X) + \Lambda.$$

Corollary 2.14. *Suppose that $E_Q[\xi|\mathcal{G}] \in L_{\mathcal{F}}$ for all $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$ and for all $\xi \in L_{\mathcal{F}}$.*

(i) *If $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$ and if $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is* (MON) *and* (REG)*, then*

$$K(X,Q) = \inf_{\xi\in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] = E_Q[X|\mathcal{G}]\}\,; \tag{2.8}$$

if, in addition, π is (CAS)*, then*

$$K(X,Q) = E_Q[X|\mathcal{G}] - \pi^*(Q), \tag{2.9}$$

where $\pi^*(Q) := \sup_{\xi\in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi)\}$.

(ii) *Under the same hypotheses of Theorem 2.9 or of Theorem 2.10 and if π is also* (CAS)*, then*

$$\pi(X) = \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}_{\mathcal{G}}} \{E_Q[X|\mathcal{G}] - \pi^*(Q)\}\,. \tag{2.10}$$

3. Preliminary results.

Hereafter we will always assume that $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is (REG).

3.1. Properties of $R(Y,\xi')$.

We recall that Σ is the actual domain of R as given in (2.3). For any $(Y,\xi') \in \Sigma$, we have $R(Y,\xi') = \inf \mathcal{A}(Y,\xi')$, where

$$\mathcal{A}(Y,\xi') := \{\pi(\xi) \,|\, \xi \in L_{\mathcal{F}},\ E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq Y\}.$$

By convention, $R(Y,\xi') = +\infty$ for every $(Y,\xi') \in (L^0_{\mathcal{G}} \times L^*_{\mathcal{F}}) \setminus \Sigma$.

Lemma 3.1. *For every $(Y,\xi') \in \Sigma$ the set $\mathcal{A}(Y,\xi')$ is downward directed, and therefore there exists a sequence $\{\eta_m\}_{m=1}^{\infty} \in L_{\mathcal{F}}$ such that $E_{\mathbb{P}}[\xi'\eta_m|\mathcal{G}] \geq Y$ and as $m \uparrow \infty$, $\pi(\eta_m) \downarrow R(Y,\xi')$.*

Proof. We have to prove that for every $\pi(\xi_1), \pi(\xi_2) \in \mathcal{A}(Y,\xi')$ there exists $\pi(\xi^*) \in \mathcal{A}(Y,\xi')$ such that $\pi(\xi^*) \leq \min\{\pi(\xi_1), \pi(\xi_2)\}$. Consider the $\mathcal{G}$-measurable set $G = \{\pi(\xi_1) \leq \pi(\xi_2)\}$; then

$$\min\{\pi(\xi_1), \pi(\xi_2)\} = \pi(\xi_1)\mathbf{1}_G + \pi(\xi_2)\mathbf{1}_{G^C} = \pi(\xi_1\mathbf{1}_G + \xi_2\mathbf{1}_{G^C}) = \pi(\xi^*),$$

[3] This was added in the revision.

where $\xi^* = \xi_1 \mathbf{1}_G + \xi_2 \mathbf{1}_{G^C}$. Hence $E_{\mathbb{P}}[\xi'\xi^*|\mathcal{G}] = E_{\mathbb{P}}[\xi'\xi_1|\mathcal{G}]\mathbf{1}_G + E_{\mathbb{P}}[\xi'\xi_2|\mathcal{G}]\mathbf{1}_{G^C} \geq Y$ so that we can deduce $\pi(\xi^*) \in \mathcal{A}(Y, \xi')$. ■

Lemma 3.2. *$R(Y, \xi')$ satisfies the following properties:*

(i) *$R(\cdot, \xi')$ is monotone for every $\xi' \in L^*_{\mathcal{F}}$.*

(ii) *$R(\lambda Y, \lambda \xi') = R(Y, \xi')$ for any $\lambda > 0$, $Y \in L^0_{\mathcal{G}}$, and $\xi' \in L^*_{\mathcal{F}}$.*

(iii) *For every $A \in \mathcal{G}$, $(Y, \xi') \in \Sigma$*

$$R(Y,\xi')\mathbf{1}_A = \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi(\xi)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq Y \right\} \tag{3.1}$$

$$= \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi(\xi)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A \right\} = R(Y\mathbf{1}_A, \xi')\mathbf{1}_A. \tag{3.2}$$

(iv) *For every $Y_1, Y_2 \in L^0_{\mathcal{G}}$*

(a) *$R(Y_1, \xi') \wedge R(Y_2, \xi') = R(Y_1 \wedge Y_2, \xi')$,*

(b) *$R(Y_1, \xi') \vee R(Y_2, \xi') = R(Y_1 \vee Y_2, \xi')$.*

(v) *The map $R(\cdot, \xi')$ is quasi-affine; i.e., for every $Y_1, Y_2, \Lambda \in L_{\mathcal{G}}$ and $0 \leq \Lambda \leq 1$,*
$R(\Lambda Y_1 + (1-\Lambda)Y_2, \xi') \geq R(Y_1, \xi') \wedge R(Y_2, \xi')$ (quasi concavity),
$R(\Lambda Y_1 + (1-\Lambda)Y_2, \xi') \leq R(Y_1, \xi') \vee R(Y_2, \xi')$ (quasi convexity).

(vi) *$\inf_{Y \in L^0_{\mathcal{G}}} R(Y, \xi'_1) = \inf_{Y \in L^0_{\mathcal{G}}} R(Y, \xi'_2)$ for every $\xi'_1, \xi'_2 \in L^*_{\mathcal{F}}$.*

Proof. (i) and (ii) are trivial.

(iii) By definition of the essential infimum one easily deduces (3.1). To prove (3.2), for every $\xi \in L_{\mathcal{F}}$ such that $E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A$ we define the random variable $\eta = \xi\mathbf{1}_A + \zeta\mathbf{1}_{A^C}$, where $E_{\mathbb{P}}[\xi'\zeta|\mathcal{G}] \geq Y$. Then $E_{\mathbb{P}}[\xi'\eta|\mathcal{G}] \geq Y$ and we can conclude that

$$\left\{ \eta\mathbf{1}_A \mid \eta \in L_{\mathcal{F}},\, E_{\mathbb{P}}[\xi'\eta|\mathcal{G}] \geq Y \right\} = \left\{ \xi\mathbf{1}_A \mid \xi \in L_{\mathcal{F}},\, E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A \right\}.$$

Hence from (3.1) and (REG)

$$\begin{aligned}
\mathbf{1}_A R(Y,\xi') &= \inf_{\eta \in L_{\mathcal{F}}} \left\{ \pi(\eta\mathbf{1}_A)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\eta|\mathcal{G}] \geq Y \right\} \\
&= \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi(\xi\mathbf{1}_A)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A \right\} \\
&= \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi(\xi)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A \right\}.
\end{aligned}$$

The second equality in (3.2) follows in a similar way.

(iv)(a) Since $R(\cdot, \xi')$ is monotone, the inequality $R(Y_1, \xi') \wedge R(Y_2, \xi') \geq R(Y_1 \wedge Y_2, \xi')$ holds true. To show the opposite inequality, define the $\mathcal{G}$-measurable sets $B := \{R(Y_1, \xi') \leq R(Y_2, \xi')\}$ and $A := \{Y_1 \leq Y_2\}$ so that

$$R(Y_1,\xi') \wedge R(Y_2,\xi') = R(Y_1,\xi')\mathbf{1}_B + R(Y_2,\xi')\mathbf{1}_{B^C} \leq R(Y_1,\xi')\mathbf{1}_A + R(Y_2,\xi')\mathbf{1}_{A^C}. \tag{3.3}$$

Set $D(A, Y) = \{\xi\mathbf{1}_A \mid \xi \in L_{\mathcal{F}},\, E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq Y\mathbf{1}_A\}$ and check that

$$D(A, Y_1) + D(A^C, Y_2) = \left\{ \xi \in L_{\mathcal{F}} \mid E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq Y_1\mathbf{1}_A + Y_2\mathbf{1}_{A^C} \right\} := D.$$

From (3.3) and using (3.2) we get,

$$\begin{aligned}
R(Y_1,\xi')\wedge R(Y_2,\xi') \leq & R(Y_1,\xi')\mathbf{1}_A + R(Y_2,\xi')\mathbf{1}_{A^C} \\
= & \inf_{\xi\mathbf{1}_A\in D(A,Y_1)} \{\pi(\xi\mathbf{1}_A)\mathbf{1}_A\} + \inf_{\eta\mathbf{1}_{A^C}\in D(A^C,Y_2)} \{\pi(\eta\mathbf{1}_{A^C})\mathbf{1}_{A^C}\} \\
= & \inf_{\substack{\xi\mathbf{1}_A\in D(A,Y_1)\\ \eta\mathbf{1}_{A^C}\in D(A^C,Y_2)}} \{\pi(\xi\mathbf{1}_A)\mathbf{1}_A + \pi(\eta\mathbf{1}_{A^C})\mathbf{1}_{A^C}\} \\
= & \inf_{(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\in D(A,Y_1)+D(A^C,Y_2)} \{\pi(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\} \\
= & \inf_{\xi\in D}\{\pi(\xi)\} = R(Y_1\mathbf{1}_A + Y_2\mathbf{1}_{A^C},\xi') = R(Y_1\wedge Y_2,\xi').
\end{aligned}$$

(iv)(b) follows in a similar way.
(v) This follows from item (iv) and Remark 2.12.
(vi) This is a trivial generalization of Theorem 2 (H2) in [CMM09b]. ■

Consider the map $R^+ : \Sigma \to \bar{L}^0_{\mathcal{G}}$ defined by

$$R^+(Y,\xi') = ess \sup_{Y'<Y} R(Y',\xi'). \tag{3.4}$$

Lemma 3.3. *If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is* (REG) *and* (MON), *then $R^+ \in \mathcal{R}$.*

Proof. Clearly $R^+(\cdot,Q)$ inherits from $R(\cdot,Q)$ the properties (REG) and (MON). From Remark 2.12 we then know that $R^+(\cdot,Q)$ is (QCO). We show that it is also (CFB). Let $Y_n \uparrow Y$. It is easy to check that (MON) of $R(\cdot,\xi')$ implies that the set $\{R(\eta,\xi')|\eta < Y\}$ is upward directed. Then for any $\varepsilon,\delta > 0$ we can find $\eta_\varepsilon < Y$ such that

$$\mathbb{P}(R^+(Y,\xi') - R(\eta_\varepsilon,\xi') < \varepsilon) > 1-\delta. \tag{3.5}$$

There exists an n_ε such that $\mathbb{P}(Y_n > \eta_\varepsilon) > 1-\delta$ for every $n > n_\varepsilon$. Denote $A_n = \{Y_n > \eta_\varepsilon\}$ so that from (REG) we have $R^+(Y_n,\xi')\mathbf{1}_{A_n} \geq R(\eta_\varepsilon,\xi')\mathbf{1}_{A_n}$. This last inequality together with (3.5) implies

$$\mathbb{P}(R^+(Y,\xi') - R^+(Y_n,\xi') < \varepsilon) > 1-2\delta \quad \forall\, n > n_\varepsilon,$$

i.e., $R^+(Y_n,Q) \xrightarrow{\mathbb{P}} R^+(Y,Q)$. From (MON), $R^+(Y_n,Q) \uparrow R^+(Y,Q)$ $\mathbb{P}$-a.s. ■

Proposition 3.4 is in the spirit of [CMM09b]: as a consequence of the dual representation the map π induces on R (resp., R^+) its characteristic properties, and so does R (resp., R^+) on π. Recall that $(E_{\mathbb{P}}[\xi' X|\mathcal{G}],\xi') \in \Sigma$ for every $X \in L_{\mathcal{F}}$, $\xi' \in L^*_{\mathcal{F}}$.

Proposition 3.4. *Consider a map $S : \Sigma \to L_{\mathcal{G}}$.*

(a) *Let $\chi \subseteq L^*_{\mathcal{F}}$, $X \in L_{\mathcal{F}}$, and*

$$\pi(X) = \sup_{\xi'\in\chi} S(E_{\mathbb{P}}[X\xi'|\mathcal{G}],\xi').$$

Then for every $(Y,\xi') \in \Sigma$, $A \in \mathcal{G}$, $\lambda \in \mathbb{R}_+$, $\Lambda \in L_{\mathcal{G}} \cap L_{\mathcal{F}}$
(i) *$S(Y\mathbf{1}_A,\xi')\mathbf{1}_A = S(Y,\xi')\mathbf{1}_A \Longrightarrow \pi$ (REG);*
(ii) *$Y \mapsto S(Y,\xi')$ (MON) $\Longrightarrow \pi$ (MON);*
(iii) *$Y \mapsto S(Y,\xi')$ (QCO) $\Longrightarrow \pi$ (QCO);*

(iv) $Y \mapsto S(Y,\xi')$ (CFB) $\Longrightarrow \pi$ (CFB);
(v) $S(\lambda Y,\xi') = S(Y,\xi') \Longrightarrow \pi(\lambda X) = \pi(X)$;
(vi) $S(\lambda Y,\xi') = \lambda S(Y,\xi') \Longrightarrow \pi(\lambda X) = \lambda\pi(X)$;
(vii) $S(E_{\mathbb{P}}[(X+\Lambda)\xi'|\mathcal{G}],\xi') = S(E_{\mathbb{P}}[X\xi'|\mathcal{G}],\xi') + \Lambda \Longrightarrow \pi(X+\Lambda) = \pi(X)+\Lambda$;
(viii) $S(E_{\mathbb{P}}[(X+\Lambda)\xi'|\mathcal{G}],\xi') \ge S(E_{\mathbb{P}}[X\xi'|\mathcal{G}],\xi') + \Lambda \Longrightarrow \pi(X+\Lambda) \ge \pi(X)+\Lambda$.

(b) *When the map S is replaced by R defined in* (2.2), *all the above items, except* (iv), *hold true replacing "$\Longrightarrow$" by "$\Longleftrightarrow$".*

(c) *When the map S is replaced by R^+ defined in* (3.4), *all the above items* (i)–(viii) *hold true replacing "$\Longrightarrow$" by "$\Longleftrightarrow$".*

Proof. The proofs of all items in (a) are trivial.

(b) The implication "$\Longleftarrow$" in (i) and (ii) follow from Lemma 3.2. We omit the remaining elementary proofs.

(c) the implications "$\Longleftarrow$" in (i), (ii), (iii), and (iv) are proved in Lemma 3.3. The remaining items are easily proved from the corresponding properties of R. ∎

3.2. Properties of $K(X,Q)$ and $H(X)$.

Lemma 3.5. *Let $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$ and $X \in L_{\mathcal{F}}$:*

(i) *$K(\cdot,Q)$ is monotone and quasi-affine.*

(ii) *$K(X,\cdot)$ is scaling invariant: $K(X,\lambda Q) = K(X,Q)$ for every $\lambda > 0$.*

(iii) *$K(X,Q)\mathbf{1}_A = \inf_{\xi\in L_{\mathcal{F}}} \{\pi(\xi)\mathbf{1}_A \mid E_Q[\xi\mathbf{1}_A|\mathcal{G}] \ge_Q E_Q[X\mathbf{1}_A|\mathcal{G}]\}$ for all $A \in \mathcal{G}$.*

(iv) *There exists a sequence $\{\xi^Q_m\}_{m=1}^\infty \in L_{\mathcal{F}}$ such that*

$$E_Q[\xi^Q_m|\mathcal{G}] \ge_Q E_Q[X|\mathcal{G}] \quad \forall\, m \ge 1, \quad \pi(\xi^Q_m) \downarrow K(X,Q) \quad \text{as } m \uparrow \infty.$$

(v) *The set $\mathcal{K} = \{K(X,Q) \mid Q \in L^*_{\mathcal{F}} \cap \mathcal{P}\}$ is upward directed; i.e., for every $K(X,Q_1)$, $K(X,Q_2) \in \mathcal{K}$ there exists $K(X,\widehat{Q}) \in \mathcal{K}$ such that $K(X,\widehat{Q}) \ge K(X,Q_1) \vee K(X,Q_2)$.*

(vi) *Let Q_1 and Q_2 be elements of $L^*_{\mathcal{F}}\cap\mathcal{P}$ and $B \in \mathcal{G}$. If $\frac{dQ_1}{d\mathbb{P}}\mathbf{1}_B = \frac{dQ_2}{d\mathbb{P}}\mathbf{1}_B$, then $K(X,Q_1)\mathbf{1}_B = K(X,Q_2)\mathbf{1}_B$.*

Proof. The monotonicity properties in (i), (ii), and (iii) are trivial; from Lemma 3.2(v) it follows that $K(\cdot,Q)$ is quasi-affine; (iv) is a consequence of Lemma 3.1.

(v) Define $F = \{K(X,Q_1) \ge K(X,Q_2)\}$, and let $\widehat{Q}$ given by $\frac{d\widehat{Q}}{d\mathbb{P}} := \mathbf{1}_F\frac{dQ_1}{d\mathbb{P}} + \mathbf{1}_{F^C}\frac{dQ_2}{d\mathbb{P}}$; up to a normalization factor (from property (ii)) we may suppose $\widehat{Q} \in L^*_{\mathcal{F}} \cap \mathcal{P}$. We need to show that

$$K(X,\widehat{Q}) = K(X,Q_1) \vee K(X,Q_2) = K(X,Q_1)\mathbf{1}_F + K(X,Q_2)\mathbf{1}_{F^C}.$$

The equality $E_{\widehat{Q}}[\xi|\mathcal{G}] =_{\widehat{Q}} E_{Q_1}[\xi|\mathcal{G}]\mathbf{1}_F + E_{Q_2}[\xi|\mathcal{G}]\mathbf{1}_{F^C}$ implies that $E_{\widehat{Q}}[\xi|\mathcal{G}]\mathbf{1}_F =_{Q_1} E_{Q_1}[\xi|\mathcal{G}]\mathbf{1}_F$ and $E_{\widehat{Q}}[\xi|\mathcal{G}]\mathbf{1}_{F^C} =_{Q_2} E_{Q_2}[\xi|\mathcal{G}]\mathbf{1}_{F^C}$. For $i = 1,2$, consider the sets

$$\widehat{A} = \{\xi \in L_{\mathcal{F}} \mid E_{\widehat{Q}}[\xi|\mathcal{G}] \ge_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\} \quad A_i = \{\xi \in L_{\mathcal{F}} \mid E_{Q_i}[\xi|\mathcal{G}] \ge_{Q_i} E_{Q_i}[X|\mathcal{G}]\}.$$

For every $\xi \in A_1$ define $\eta = \xi\mathbf{1}_F + X\mathbf{1}_{F^C}$. Notice that

$$\begin{aligned} Q_1 \ll \mathbb{P} \;&\Rightarrow\; \eta\mathbf{1}_F =_{Q_1} \xi\mathbf{1}_F \;\Rightarrow\; E_{\widehat{Q}}[\eta|\mathcal{G}]\mathbf{1}_F \ge_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\mathbf{1}_F,\\ Q_2 \ll \mathbb{P} \;&\Rightarrow\; \eta\mathbf{1}_{F^C} =_{Q_2} X\mathbf{1}_{F^C} \;\Rightarrow\; E_{\widehat{Q}}[\eta|\mathcal{G}]\mathbf{1}_{F^C} =_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\mathbf{1}_{F^C}.\end{aligned}$$

Then $\eta \in \widehat{A}$ and $\pi(\xi)\mathbf{1}_F = \pi(\xi\mathbf{1}_F)\mathbf{1}_F = \pi(\eta\mathbf{1}_F)\mathbf{1}_F = \pi(\eta)\mathbf{1}_F$. Vice versa, for every $\eta \in \widehat{A}$ define $\xi = \eta\mathbf{1}_F + X\mathbf{1}_{F^C}$. Then $\xi \in A_1$ and again $\pi(\xi)\mathbf{1}_F = \pi(\eta)\mathbf{1}_F$. Hence

$$\inf_{\xi\in A_1} \pi(\xi)\mathbf{1}_F = \inf_{\eta\in\widehat{A}} \pi(\eta)\mathbf{1}_F.$$

In a similar way $\inf_{\xi\in A_2} \pi(\xi)\mathbf{1}_{F^C} = \inf_{\eta\in\widehat{A}} \pi(\eta)\mathbf{1}_{F^C}$, and we can finally deduce $K(X,Q_1) \vee K(X,Q_2) = K(X,\widehat{Q})$.

(vi) By the same argument used in (v), $\inf_{\xi\in A_1} \pi(\xi)\mathbf{1}_B = \inf_{\xi\in A_2} \pi(\xi)\mathbf{1}_B$ and the thesis follows. ∎

For $X \in L_{\mathcal{F}}$ we define

$$H(X) := \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} K(X,Q) = \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} \inf_{\xi\in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}$$

and notice that for all $A \in \mathcal{G}$

$$H(X)\mathbf{1}_A = \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} \inf_{\xi\in L_{\mathcal{F}}} \{\pi(\xi)\mathbf{1}_A \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}.$$

Applying Proposition 3.4 we already know the one to one correspondence between the properties of π and those of H. In particular, H is (MON) and the regularity of π implies that H is (REG) (i.e., $H(X\mathbf{1}_A)\mathbf{1}_A = H(X)\mathbf{1}_A$ for any $A \in \mathcal{G}$). As an immediate consequence of Lemma 3.5(iv) and (v) we deduce the next result.

Lemma 3.6. *Let $X \in L_{\mathcal{F}}$. There exist a sequence $\{Q^k\}_{k\geq1} \in L^*_{\mathcal{F}}\cap\mathcal{P}$ and, for all $k \geq 1$, a sequence $\{\xi_m^{Q^k}\}_{m\geq1} \in L_{\mathcal{F}}$ satisfying $E_{Q^k}[\xi_m^{Q^k} \mid \mathcal{G}] \geq_{Q^k} E_{Q^k}[X|\mathcal{G}]$ and*

$$\pi(\xi_m^{Q^k}) \downarrow K(X,Q^k) \quad \text{as} \quad m\uparrow\infty,\ K(X,Q^k)\uparrow H(X) \quad \text{as } k\uparrow\infty, \tag{3.6}$$

$$H(X) = \lim_{k\to\infty}\lim_{m\to\infty} \pi(\xi_m^{Q^k}). \tag{3.7}$$

3.3. On the map π_A.

Given $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ we define for every $A \in \mathcal{G}$ the map

$$\pi_A : L_{\mathcal{F}} \to \overline{\mathbb{R}} \ \text{ by } \ \pi_A(X) := ess\sup_{\omega\in A} \pi(X)(\omega).$$

Proposition 3.7. *Under the assumptions of Theorem 2.9 or of Theorem 2.10 and for any $A \in \mathcal{G}$*

$$\pi_A(X) = \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} \inf_{\xi\in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}. \tag{3.8}$$

Proof. Notice that the map π_A inherits from π the properties (MON) and (QCO). Under the assumptions of Theorem 2.9 we get, from Proposition 2.5, that π is (CFB), and this obviously implies that π_A is (CFB). Applying to π_A Proposition 2.5, which also holds for real valued maps, we deduce that π_A is $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC).

Under the assumptions of Theorem 2.10 we prove that π_A is τ-(USC) by showing that for all $c \in \mathbb{R}$ the set $\mathcal{B}_c := \{\xi \in L_{\mathcal{F}} | \pi_A(\xi) < c\}$ is τ open. Without loss of generality $\mathcal{B}_c \neq \emptyset$. If we fix an arbitrary $\eta \in \mathcal{B}_c$, we may find $\delta > 0$ such that $\pi_A(\eta) < c - \delta$. Define

$$\mathcal{B} := \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) < (c-\delta)\mathbf{1}_A + (\pi(\eta)+\delta)\mathbf{1}_{A^C}\}.$$

Since $(c-\delta)\mathbf{1}_A + (\pi(\eta)+\delta)\mathbf{1}_{A^C} \in L_{\mathcal{G}}$ and π is (USC) we deduce that $\mathcal{B}$ is τ open. Moreover, $\pi_A(\xi) \leq c-\delta$ for every $\xi \in \mathcal{B}$, i.e., $\mathcal{B} \subseteq \mathcal{B}_c$, and $\eta \in \mathcal{B}$ since $\pi(\eta) < c-\delta$ on A and $\pi(\eta) < \pi(\eta)+\delta$ on A^C.

Applying Theorem 1.1 in the (LSC) case and Theorem 5.4 in the (USC) one then deduces the representation of π_A:

$$\begin{aligned}\pi_A(X) &= \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi] \geq E_Q[X]\} \\ &\leq \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\} \leq \pi_A(X).\end{aligned}$$

Notice that in case π_A is (USC) the sup can be replaced by a max. ■

4. Proofs of the main results. Notation: In the following, we will consider only *finite* partitions $\Gamma = \{A^\Gamma\}$ of $\mathcal{G}$ measurable sets $A^\Gamma \in \Gamma$ and we set

$$\begin{aligned}\pi^\Gamma(X) &:= \sum_{A^\Gamma \in \Gamma} \pi_{A^\Gamma}(X)\mathbf{1}_{A^\Gamma}, \\ K^\Gamma(X,Q) &:= \inf_{\xi \in L_{\mathcal{F}}} \left\{\pi^\Gamma(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\right\}, \\ H^\Gamma(X) &:= \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} K^\Gamma(X,Q).\end{aligned}$$

4.1. Outline of the proof. We anticipate an heuristic sketch of the proof of Theorems 2.9 and 2.10, pointing out the essential arguments involved in it, and we defer to the following section the details and the rigorous statements.

Unfortunately, we cannot prove directly that for all $\varepsilon > 0$, there exists $Q_\varepsilon \in L^*_{\mathcal{F}} \cap \mathcal{P}$ such that

$$\{\xi \in L_{\mathcal{F}} \mid E_{Q_\varepsilon}[\xi|\mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathcal{G}]\} \subseteq \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) > \pi(X) - \varepsilon\} \tag{4.1}$$

relying on the Hahn–Banach theorem, as happened in the real case (see (5.26) in the proof of Theorem 1.1, in the appendix). Indeed, the complement of the set in the right-hand side of (4.1) is no longer a convex set, unless π is real valued, regardless of the continuity assumption made on π.

Also the idea applied in the conditional convex case [DS05] cannot be used here since the map $X \to E_{\mathbb{P}}[\pi(X)]$ adopted there preserves convexity but not quasi convexity.

Then our method is to apply an approximation argument, and the choice of approximating $\pi(\cdot)$ by $\pi^\Gamma(\cdot)$ is forced by the need to preserve quasi convexity.

- I The first step is to prove (see Proposition 4.4) that $H^\Gamma(X) = \pi^\Gamma(X)$. This is based on the representation of the *real valued* quasi-convex map π_A in Proposition 3.7. Therefore, the assumptions (MON), (REG), (QCO), and (LSC) or (USC)* on π are all needed here.
- II Then it is a simple matter to deduce that $\pi(X) = \inf_\Gamma \pi^\Gamma(X) = \inf_\Gamma H^\Gamma(X)$, where the inf is taken with respect to all finite partitions.
- III As anticipated in (1.3), the last step, i.e., proving that $\inf_\Gamma H^\Gamma(X) = H(X)$, is more delicate. It can be shown easily that is possible to approximate $H(X)$ with $K(X, Q_\varepsilon)$

on a set A_ε of probability arbitrarily close to 1. However, we need the following *uniform* approximation: For any $\varepsilon > 0$ there exists $Q_\varepsilon \in L^*_{\mathcal{F}} \cap \mathcal{P}$ such that for any finite partition Γ we have $H^\Gamma(X) - K^\Gamma(X, Q_\varepsilon) < \varepsilon$ on the same set A_ε. This key approximation result, based on Lemma 4.3, shows that the element Q_ε does not depend on the partition and allows us (see (4.7)) to conclude the proof.

4.2. Details. The following two lemmas are straightforward applications of measure theory; the third is the already mentioned key result and is proved in the appendix, for it needs a pretty long argument.

Lemma 4.1. *For every $Y \in L^0_{\mathcal{G}}$ there exists a sequence $\Gamma(n)$ of finite partitions such that $\sum_{\Gamma(n)} (\sup_{A^{\Gamma(n)}} Y) \mathbf{1}_{A^{\Gamma(n)}}$ converges in probability, and $\mathbb{P}$-a.s., to Y, so that for any $\varepsilon, \delta > 0$ we may find N such that for $\Gamma = \Gamma(N)$ we have*

$$\mathbb{P}\left\{\omega \in \Omega \mid \sum_{A^\Gamma \in \Gamma} \left(\sup_{A^\Gamma} Y\right) \mathbf{1}_{A^\Gamma}(\omega) - Y(\omega) < \delta \right\} > 1 - \varepsilon. \tag{4.2}$$

Lemma 4.2. *For each $X \in L_{\mathcal{F}}$ and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$*

$$\inf_\Gamma K^\Gamma(X, Q) = K(X, Q),$$

where the infimum *is taken with respect to all finite partitions Γ.*

Lemma 4.3. *Let $X \in L_{\mathcal{F}}$, and let P and Q be arbitrary elements of $L^*_{\mathcal{F}} \cap \mathcal{P}$. Suppose there exist $\varepsilon \geq 0$ and $B \in \mathcal{G}$ such that $K(X, P)\mathbf{1}_B > -\infty$, $\pi_B(X) < +\infty$, and*

$$K(X, Q)\mathbf{1}_B \leq K(X, P)\mathbf{1}_B + \varepsilon \mathbf{1}_B.$$

Then for every partition $\Gamma = \{B^C, \widetilde{\Gamma}\}$, where $\widetilde{\Gamma}$ is a partition of B, we have

$$K^\Gamma(X, Q)\mathbf{1}_B \leq K^\Gamma(X, P)\mathbf{1}_B + \varepsilon \mathbf{1}_B.$$

Since π^Γ assumes only a finite number of values, we may apply Proposition 3.7 and deduce the dual representation of π^Γ.

Proposition 4.4. *Suppose that the assumptions of Theorem 2.9 or of Theorem 2.10 hold true and Γ is a finite partition. If for every $X \in L_{\mathcal{F}}$, $|\pi(X)| < c$ with $c \in [0, +\infty)$, then*

$$H^\Gamma(X) = \pi^\Gamma(X) \geq \pi(X) \tag{4.3}$$

and therefore $\inf_\Gamma H^\Gamma(X) = \pi(X)$.

Proof. First notice that $K^\Gamma(X, Q) \leq H^\Gamma(X) \leq \pi^\Gamma(X) < +\infty$ for all $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$. Consider the sigma algebra $\mathcal{G}^\Gamma := \sigma(\Gamma) \subseteq \mathcal{G}$, generated by the finite partition Γ. From Proposition 3.7 we then have for every $A^\Gamma \in \Gamma$

$$\pi_{A^\Gamma}(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi_{A^\Gamma}(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \right\}. \tag{4.4}$$

Moreover, $H^\Gamma(X)$ is constant on A^Γ since it is $\mathcal{G}^\Gamma$-measurable as well. Using the fact that $\pi^\Gamma(\cdot)$ is constant on each A^Γ, for every $A^\Gamma \in \Gamma$ we then have

$$\begin{aligned} H^\Gamma(X)\mathbf{1}_{A^\Gamma} &= \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} \inf_{\xi\in L_{\mathcal{F}}} \left\{\pi^\Gamma(\xi)\mathbf{1}_{A^\Gamma} \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\right\} \\ &= \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} \inf_{\xi\in L_{\mathcal{F}}} \{\pi_{A^\Gamma}(\xi)\mathbf{1}_{A^\Gamma} \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\} \\ &= \pi_{A^\Gamma}(X)\mathbf{1}_{A^\Gamma} = \pi^\Gamma(X)\mathbf{1}_{A^\Gamma}, \end{aligned} \tag{4.5}$$

where the first equality in (4.5) follows from (4.4). The remaining statement is a consequence of (4.3) and Lemma 4.1. ■

Proofs of Theorems 2.9 *and* 2.10. Obviously $\pi(X) \geq H(X)$ since X satisfies the constraints in the definition of $H(X)$.

Step 1. First we assume that π is uniformly bounded; i.e., there exists $c > 0$ such that for all $X \in L_{\mathcal{F}}$, $|\pi(X)| \leq c$. Then $H(X) > -\infty$.

From (3.6), there exists a sequence $Q_k \in L^*_{\mathcal{F}} \cap \mathcal{P}$ such that

$$K(X, Q_k) \uparrow H(X) \quad \text{as } k \uparrow \infty.$$

Therefore, for any $\varepsilon > 0$, we may find $Q_\varepsilon \in L^*_{\mathcal{F}} \cap \mathcal{P}$ and $A_\varepsilon \in \mathcal{G}$, $\mathbb{P}(A_\varepsilon) > 1 - \varepsilon$ such that

$$H(X)\mathbf{1}_{A_\varepsilon} - K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon} \leq \varepsilon\mathbf{1}_{A_\varepsilon}.$$

Since $H(X) \geq K(X, Q)$ for all $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$,

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq K(X, Q)\mathbf{1}_{A_\varepsilon} \quad \forall Q \in L^*_{\mathcal{F}} \cap \mathcal{P}.$$

This is the basic inequality that enables us to apply Lemma 4.3, replacing there P with Q_ε and B with A_ε. Notice only that $\sup_\Omega \pi(X) \leq c$ and $K(X, Q) > -\infty$ for every $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$. This lemma assures that for every partition Γ of Ω

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq K^\Gamma(X, Q)\mathbf{1}_{A_\varepsilon} \quad \forall Q \in L^*_{\mathcal{F}} \cap \mathcal{P}. \tag{4.6}$$

From the definition of *essential supremum* of a class of random variables, (4.6) implies that for every Γ

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \sup_{Q\in L^*_{\mathcal{F}}\cap\mathcal{P}} K^\Gamma(X, Q)\mathbf{1}_{A_\varepsilon} = H^\Gamma(X)\mathbf{1}_{A_\varepsilon}. \tag{4.7}$$

Since $\pi^\Gamma \leq c$, applying Proposition 4.4, (4.3), we get

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}.$$

Taking the *infimum* over all possible partitions, as in Lemma 4.2, we deduce that

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}. \tag{4.8}$$

Hence, for any $\varepsilon > 0$,

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon} \geq H(X)\mathbf{1}_{A_\varepsilon} \geq K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon},$$

which implies $\pi(X) = H(X)$, since $\mathbb{P}(A_\varepsilon) \to 1$ as $\varepsilon \to 0$.

Step 2. Now we consider the case when π is not necessarily bounded. We define the new map $\psi(\cdot) := \arctan(\pi(\cdot))$ and notice that $\psi(X)$ is a $\mathcal{G}$-measurable random variable satisfying $|\psi(X)| \leq \frac{\Pi}{2}$ for every $X \in L_{\mathcal{F}}$. Moreover, ψ is (MON), (QCO) and $\psi(X\mathbf{1}_G) = \psi(X)\mathbf{1}_G$ for every $G \in \mathcal{G}$. In addition, ψ inherits the (LSC) (resp., the (USC)*) property from π. The first is a simple consequence of (CFB) of π. For the second we may apply Lemma 4.5.

The map ψ is uniformly bounded, and by Step 1 we may conclude that

$$\psi(X) = H_\psi(X) := \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} K_\psi(X,Q), \quad \text{where}$$
$$K_\psi(X,Q) := \inf_{\xi \in L_{\mathcal{F}}} \{\psi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}.$$

Applying again (3.6), there exists $Q^k \in L^*_{\mathcal{F}}$ such that

$$H_\psi(X) = \lim_k K_\psi(X, Q^k).$$

We will show below that

$$K_\psi(X,Q^k) = \arctan K(X,Q^k). \tag{4.9}$$

Admitting this, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{aligned}\arctan(\pi(X)(\omega)) = \psi(X)(\omega) = H_\psi(X)(\omega) &= \lim_k K_\psi(X,Q^k)(\omega)\\ &= \lim_k \arctan K(X,Q^k)(\omega)) = \arctan(\lim_k K(X,Q^k)(\omega)),\end{aligned}$$

where we used the continuity of the function arctan. This implies $\pi(X) = \lim_k K(X,Q^k)$, and we conclude that

$$\pi(X) = \lim_k K(X,Q^k) \leq H(X) \leq \pi(X).$$

It remains only to show (4.9). We prove that for every fixed $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$

$$K_\psi(X,Q) = \arctan\left(K(X,Q)\right).$$

Since π and ψ are regular, from Lemma 3.5(iv), there exist $\xi_h^Q \in L_{\mathcal{F}}$ and $\eta_h^Q \in L_{\mathcal{F}}$ such that

$$E_Q[\xi_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}], \quad E_Q[\eta_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \quad \forall h \geq 1, \tag{4.10}$$

$\psi(\xi_h^Q) \downarrow K_\psi(X,Q)$, and $\pi(\eta_h^Q) \downarrow K(X,Q)$ as $h \uparrow \infty$. From (4.10) and the definitions of $K(X,Q)$, $K_\psi(X,Q)$ and by the continuity and monotonicity of arctan we get

$$\begin{aligned}K_\psi(X,Q) &\leq \lim_h \psi(\eta_h^Q) = \lim_h \arctan \pi(\eta_h^Q) = \arctan \lim_h \pi(\eta_h^Q)\\ &= \arctan K(X,Q) \leq \arctan \lim_h \pi(\xi_h^Q) = \lim_h \psi(\xi_h^Q) = K_\psi(X,Q).\end{aligned}$$ ■

Let Y be $\mathcal{G}$-measurable, and define

$$A := \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) < \tan(Y)\}, \quad B := \{\xi \in L_{\mathcal{F}} \mid \arctan(\pi(\xi)) < Y\},$$

where $\tan(Y) = +\infty$ on $\{Y \geq \frac{\pi}{2}\}$ and $\tan(Y) = -\infty$ on $\{Y \leq -\frac{\pi}{2}\}$. Notice that $A = \{\xi \in B \mid \pi(\xi) < \infty \text{ on } \{Y > \frac{\pi}{2}\}\}$.

Lemma 4.5. *Suppose that π is regular and there exists $\theta \in L_{\mathcal{F}}$ such that $\pi(\theta) < +\infty$. For any $\mathcal{G}$-measurable random variable Y, if A is open, then B is also open.*

Proof. We may assume $Y \geq -\frac{\pi}{2}$; otherwise $B = \emptyset$. Let $\xi \in B$, $\theta \in L_{\mathcal{F}}$ such that $\pi(\theta) < +\infty$. Define $\xi_0 := \xi \mathbf{1}_{\{Y \leq \frac{\pi}{2}\}} + \theta \mathbf{1}_{\{Y > \frac{\pi}{2}\}}$. Then $\xi_0 \in A$. Since A is open, we may find a neighborhood U of 0 such that $\xi_0 + U \subseteq A$. Define

$$V := (\xi_0 + U)\mathbf{1}_{\{Y \leq \frac{\pi}{2}\}} + (\xi + U)\mathbf{1}_{\{Y > \frac{\pi}{2}\}} = \xi + U\mathbf{1}_{\{Y \leq \frac{\pi}{2}\}} + U\mathbf{1}_{\{Y > \frac{\pi}{2}\}}.$$

It is easy to show that $\xi \in V$ and $V \subseteq B$. Finally V is a neighborhood of ξ, since the set $U\mathbf{1}_{\{Y \leq \frac{\pi}{2}\}} + U\mathbf{1}_{\{Y > \frac{\pi}{2}\}}$ contains U, and therefore it is a neighborhood of 0. ■

Proof of Proposition 2.13. The "if" part is trivial, as the various properties are easy to check. For the "only if" part we already know from Theorem 2.9 that

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R\left(E\left[\frac{dQ}{d\mathbb{P}}X|\mathcal{G}\right], Q\right),$$

where R is defined in (2.2). For every $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$ we consider $R^+(\cdot, Q) \leq R(\cdot, Q)$ and denote $X^Q = E[\frac{dQ}{d\mathbb{P}}X|\mathcal{G}]$. We observe that

$$\begin{aligned}
\pi(X) &\geq \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R^+(X^Q, Q) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \sup_{Y' < X^Q} R(Y', Q) \\
&\overset{\delta > 0}{\geq} \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \sup_{X^Q - \delta < X^Q} R(X^Q - \delta, Q) \\
&= \sup_{\delta > 0} \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R(E_{\mathbb{P}}[(X - \delta) \cdot dQ/d\mathbb{P}|\mathcal{G}], Q) = \sup_{\delta > 0} \pi(X - \delta) \overset{(CFB)}{=} \pi(X),
\end{aligned}$$

and so we have the representation

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R^+(E_Q[X|\mathcal{G}], Q),$$

and we already know from Lemma 3.3 that $R^+ \in \mathcal{R}$. ■

Remark 4.6. Take $Q \in \mathcal{P}$ such that $Q \sim \mathbb{P}$ on $\mathcal{G}$, and define the probability

$$\widetilde{Q}(F) := E_Q\left[\frac{d\mathbb{P}}{dQ}^{\mathcal{G}} \mathbf{1}_F\right], \quad \text{where} \quad \frac{d\mathbb{P}}{dQ}^{\mathcal{G}} := E_Q\left[\frac{d\mathbb{P}}{dQ} | \mathcal{G}\right], \ F \in \mathcal{F}.$$

Then $\widetilde{Q}(G) = \mathbb{P}(G)$ for all $G \in \mathcal{G}$, and so $\widetilde{Q} \in \mathcal{P}_{\mathcal{G}}$. Moreover, it is easy to check that for all $X \in L_{\mathcal{F}}$ and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$ such that $Q \sim \mathbb{P}$ on $\mathcal{G}$ we have

$$E_{\widetilde{Q}}[X|\mathcal{G}] = E_Q[X|\mathcal{G}],$$

which implies $K(X, \widetilde{Q}) = K(X, Q)$.

Proof of Corollary 2.11. Consider the probability $Q_\varepsilon \in L^*_{\mathcal{F}} \cap \mathcal{P}$ built up in Theorem 2.9, satisfying (4.8). We claim that Q_ε is equivalent to $\mathbb{P}$ on A_ε. By contradiction, there exist $B \in \mathcal{G}$, $B \subseteq A_\varepsilon$ such that $\mathbb{P}(B) > 0$ but $Q_\varepsilon(B) = 0$. Consider $\eta \in L_{\mathcal{F}}$, $\delta > 0$ such that $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$, and define $\xi = X\mathbf{1}_{B^C} + \eta\mathbf{1}_B$ so that $E_{Q_\varepsilon}[\xi|\mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathcal{G}]$. By regularity, $\pi(\xi) = \pi(X)\mathbf{1}_{B^C} + \pi(\eta)\mathbf{1}_B$, which implies for $\mathbb{P}$-a.e. $\omega \in B$ that

$$\pi(\xi)(\omega) + \delta = \pi(\eta)(\omega) + \delta < \pi(X)(\omega) \leq K(X, Q_\varepsilon)(\omega) + \varepsilon \leq \pi(\xi)(\omega) + \varepsilon,$$

which is impossible for $\varepsilon \leq \delta$. So $Q_\varepsilon \sim \mathbb{P}$ on A_ε for all small $\varepsilon \leq \delta$.

Consider $\widehat{Q}_\varepsilon$ such that $\frac{d\widehat{Q}_\varepsilon}{d\mathbb{P}} = \frac{dQ_\varepsilon}{d\mathbb{P}}\mathbf{1}_{A_\varepsilon} + \frac{d\mathbb{P}}{d\mathbb{P}}\mathbf{1}_{(A_\varepsilon)^C}$. Up to a normalization factor, $\widehat{Q}_\varepsilon \in L^*_{\mathcal{F}} \cap \mathcal{P}$ and is equivalent to $\mathbb{P}$. Moreover, from Lemma 3.5(vi), $K(X, \widehat{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$, and from Remark 4.6 we may define $\widetilde{Q}_\varepsilon \in \mathcal{P}_{\mathcal{G}}$ such that $K(X, \widetilde{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, \widehat{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$. From (4.8) we finally deduce that $K(X, \widetilde{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} + \varepsilon\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}$, and the thesis then follows from $\widetilde{Q}_\varepsilon \in \mathcal{P}_{\mathcal{G}}$. ■

Proof of Corollary 2.14. First we prove (2.8): let us denote with $k(X, Q)$ the right-hand side of (2.8), and notice that $K(X, Q) \leq k(X, Q)$. By contradiction, suppose that $\mathbb{P}(A) > 0$, where $A =: \{K(X, Q) < k(X, Q)\}$. As shown in Lemma 3.5(iv), there exists a random variable $\xi \in L_{\mathcal{F}}$ such that the following hold:

· $E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$ and $Q(E_Q[\xi|\mathcal{G}] > E_Q[X|\mathcal{G}]) > 0$.

· $K(X, Q)(\omega) \leq \pi(\xi)(\omega) < k(X, Q)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in B \subseteq A$ and $\mathbb{P}(B) > 0$.

Set $Z =_Q E_Q[\xi - X|\mathcal{G}]$. By assumption, $Z \in L_{\mathcal{F}}$ and it satisfies $Z \geq_Q 0$ and, since $Q \in \mathcal{P}_{\mathcal{G}}$, $Z \geq 0$. Then, thanks to (MON), $\pi(\xi) \geq \pi(\xi - Z)$. From $E_Q[\xi - Z|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]$ we deduce the following contradiction:

$$K(X, Q)(\omega) \leq \pi(\xi)(\omega) < k(X, Q)(\omega) \leq \pi(\xi - Z)(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in B.$$

Second we show (2.9). From (2.8) we deduce that

$$\begin{aligned}
K(X, Q) &= \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\} \\
&= E_Q[X|\mathcal{G}] + \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) - E_Q[X|\mathcal{G}] \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\} \\
&= E_Q[X|\mathcal{G}] + \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) - E_Q[\xi|\mathcal{G}] \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\} \\
&= E_Q[X|\mathcal{G}] - \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\} \\
&= E_Q[X|\mathcal{G}] - \pi^*(Q),
\end{aligned}$$

where the last equality follows from $Q \in \mathcal{P}_{\mathcal{G}}$ and

$$\begin{aligned}
\pi^*(Q) &= \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi + E_Q[X - \xi|\mathcal{G}] \mid \mathcal{G}] - \pi(\xi + E_Q[X - \xi|\mathcal{G}])\} \\
&= \sup_{\eta \in L_{\mathcal{F}}} \{E_Q[\eta|\mathcal{G}] - \pi(\eta) \mid \eta = \xi + E_Q[X - \xi|\mathcal{G}]\} \\
&\leq \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\} \leq \pi^*(Q).
\end{aligned}$$

(ii) The (CAS) property implies that for every $X \in L_{\mathcal{F}}$ and $\delta > 0$, $\mathbb{P}(\pi(X - 2\delta) + \delta < \pi(X)) = 1$. So the hypothesis of Corollary 2.11 holds true and (2.10) is a consequence of

(2.9) and (2.6). As noticed in [DS05] and in [CDK06] when $L_{\mathcal{F}} = L^{\infty}_{\mathcal{F}}$, the assumption that π is (REG) is not necessary, as (CAS) and (MON) or (CAS) and convexity already imply regularity. ■

5. Appendix.

5.1. Proof of the key approximation result, Lemma 4.3.

We will adopt the following notation: If Γ_1 and Γ_2 are two finite partitions of $\mathcal{G}$-measurable sets, then $\Gamma_1 \cap \Gamma_2 := \{A_1 \cap A_2 \mid A_i \in \Gamma_i,\ i = 1, 2\}$, is a finite partition finer than each Γ_i.

Lemma 5.1 is the generalization of Lemma 3.1 to the approximated problem.

Lemma 5.1. *For every partition Γ, $X \in L_{\mathcal{F}}$ and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$, the set*

$$\mathcal{A}^{\Gamma}_{Q}(X) \doteq \{\pi^{\Gamma}(\xi) \mid \xi \in L_{\mathcal{F}} \text{ and } E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}$$

is downward directed. This implies that there exists a sequence $\{\eta^Q_m\}_{m=1}^{\infty} \in L_{\mathcal{F}}$, depending also on Γ, such that

$$E_Q[\eta^Q_m|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \quad \forall m \geq 1, \quad \pi^{\Gamma}(\eta^Q_m) \downarrow K^{\Gamma}(X, Q) \quad \text{as } m \uparrow \infty.$$

Proof. To show that the set $\mathcal{A}^{\Gamma}_{Q}(X)$ is downward directed we use the notation and the results in the proof of Lemma 3.1 and check that

$$\pi^{\Gamma}(\xi^*) = \pi^{\Gamma}(\xi_1 \mathbf{1}_G + \xi_2 \mathbf{1}_{G^C}) \leq \min\left\{\pi^{\Gamma}(\xi_1), \pi^{\Gamma}(\xi_2)\right\}. \qquad ■$$

For any given sequence of partition there exists one sequence that works for all.

Lemma 5.2. *For any fixed, at most countable, family of partitions $\{\Gamma(h)\}_{h\geq 1}$ and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}$, there exists a sequence $\{\xi^Q_m\}_{m=1}^{\infty} \in L_{\mathcal{F}}$ such that*

$$\begin{aligned} E_Q[\xi^Q_m|\mathcal{G}] &\geq_Q E_Q[X|\mathcal{G}] \quad \forall\, m \geq 1, \\ \pi(\xi^Q_m) &\downarrow K(X, Q) \quad \text{as } m \uparrow \infty, \\ \forall\, h \quad \pi^{\Gamma(h)}(\xi^Q_m) &\downarrow K^{\Gamma(h)}(X, Q) \quad \text{as } m \uparrow \infty. \end{aligned}$$

Proof. Apply Lemmas 3.1 and 5.1 and find $\{\varphi^0_m\}_m, \{\varphi^1_m\}_m, \dots, \{\varphi^h_m\}_m, \dots$ such that for every i and m we have $E_Q[\varphi^i_m \mid \mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$ and

$$\begin{aligned} \pi(\varphi^0_m) &\downarrow K(X, Q) \quad \text{as } m \uparrow \infty, \\ \forall\, h \quad \pi^{\Gamma(h)}(\varphi^h_m) &\downarrow K^{\Gamma(h)}(X, Q) \quad \text{as } m \uparrow \infty. \end{aligned}$$

For each $m \geq 1$ consider $\bigwedge_{i=0}^{m} \pi(\varphi^i_m)$: then there will exist a (nonunique) finite partition of Ω, $\{F^i_m\}_{i=1}^{m}$ such that

$$\bigwedge_{i=0}^{m} \pi(\varphi^i_m) = \sum_{i=0}^{m} \pi(\varphi^i_m) \mathbf{1}_{F^i_m}.$$

Denote $\xi^Q_m =: \sum_{i=0}^{m} \varphi^i_m \mathbf{1}_{F^i_m}$, and notice that $\sum_{i=0}^{m} \pi(\varphi^i_m)\mathbf{1}_{F^i_m} \overset{\text{(REG)}}{=} \pi(\xi^Q_m)$ and $E_Q[\xi^Q_m|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$ for every m. Moreover, $\pi(\xi^Q_m)$ is decreasing and $\pi(\xi^Q_m) \leq \pi(\varphi^0_m)$ implies $\pi(\xi^Q_m) \downarrow K(X, Q)$. For any fixed h, $\pi(\xi^Q_m) \leq \pi(\varphi^h_m)$ for all $h \leq m$ and

$$\pi^{\Gamma(h)}(\xi^Q_m) \leq \pi^{\Gamma(h)}(\varphi^h_m) \quad \text{implies} \quad \pi^{\Gamma(h)}(\xi^Q_m) \downarrow K^{\Gamma(h)}(X, Q) \quad \text{as } m \uparrow \infty. \qquad ■$$

Finally, we state the basic step used in the proof of Lemma 4.3.

Lemma 5.3. *Let $X \in L_{\mathcal{F}}$ and let P and Q be arbitrary elements of $L^*_{\mathcal{F}} \cap \mathcal{P}$. Suppose there exist $\varepsilon \geq 0$ and $B \in \mathcal{G}$ such that $K(X,P)\mathbf{1}_B > -\infty$, $\pi_B(X) < +\infty$, and*

$$K(X,Q)\mathbf{1}_B \leq K(X,P)\mathbf{1}_B + \varepsilon \mathbf{1}_B.$$

Then for any $\delta > 0$ and any partition Γ_0 there exists $\Gamma \supseteq \Gamma_0$ for which

$$K^{\Gamma}(X,Q)\mathbf{1}_B \leq K^{\Gamma}(X,P)\mathbf{1}_B + \varepsilon \mathbf{1}_B + \delta \mathbf{1}_B.$$

Proof. By our assumptions we have $-\infty < K(X,P)\mathbf{1}_B \leq \pi_B(X) < +\infty$ and $K(X,Q)\mathbf{1}_B \leq \pi_B(X) < +\infty$. Fix $\delta > 0$ and the partition Γ_0. Suppose by contradiction that for any $\Gamma \supseteq \Gamma_0$ we have $\mathbb{P}(C) > 0$, where

$$C = \{\omega \in B \mid K^{\Gamma}(X,Q)(\omega) > K^{\Gamma}(X,P)(\omega) + \varepsilon + \delta\}. \tag{5.1}$$

Notice that C is the union of a finite number of elements in the partition Γ.

Lemma 3.5 guarantees the existence of $\{\xi_h^Q\}_{h=1}^{\infty} \in L_{\mathcal{F}}$ satisfying

$$\pi(\xi_h^Q) \downarrow K(X,Q) \quad \text{as } h \uparrow \infty, \qquad E_Q[\xi_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \quad \forall h \geq 1. \tag{5.2}$$

Moreover, for each partition Γ and $h \geq 1$, define

$$D_h^{\Gamma} := \left\{\omega \in \Omega \mid \pi^{\Gamma}(\xi_h^Q)(\omega) - \pi(\xi_h^Q)(\omega) < \frac{\delta}{4}\right\} \in \mathcal{G},$$

and observe that $\pi^{\Gamma}(\xi_h^Q)$ decreases if we pass to finer partitions. From (4.2), we deduce that for each $h \geq 1$ there exists a partition $\widetilde{\Gamma}(h)$ such that $\mathbb{P}(D_h^{\widetilde{\Gamma}(h)}) \geq 1 - \frac{1}{2^h}$. For every $h \geq 1$ define the new partition $\Gamma(h) = (\bigcap_{j=1}^{h} \widetilde{\Gamma}(h)) \cap \Gamma_0$ so that for all $h \geq 1$ we have $\Gamma(h+1) \supseteq \Gamma(h) \supseteq \Gamma_0$, $\mathbb{P}(D_h^{\Gamma(h)}) \geq 1 - \frac{1}{2^h}$, and

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4}\right)\mathbf{1}_{D_h^{\Gamma(h)}} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{D_h^{\Gamma(h)}} \quad \forall\, h \geq 1. \tag{5.3}$$

Lemma 5.2 guarantees that for the fixed sequence of partitions $\{\Gamma(h)\}_{h\geq 1}$ there exists a sequence $\{\xi_m^P\}_{m=1}^{\infty} \in L_{\mathcal{F}}$, which does not depend on h, satisfying

$$E_P[\xi_m^P|\mathcal{G}] \geq {}_P E_P[X|\mathcal{G}] \quad \forall m \geq 1, \tag{5.4}$$

$$\pi^{\Gamma(h)}(\xi_m^P) \downarrow K^{\Gamma(h)}(X,P) \quad \text{as } m \uparrow \infty \quad \forall\, h \geq 1. \tag{5.5}$$

For each $m \geq 1$ and $\Gamma(h)$ define

$$C_m^{\Gamma(h)} := \left\{\omega \in C \mid \pi^{\Gamma(h)}(\xi_m^P)(\omega) - K^{\Gamma(h)}(X,P)(\omega) \leq \frac{\delta}{4}\right\} \in \mathcal{G}.$$

Since the expressions in the definition of $C_m^{\Gamma(h)}$ assume only a finite number of values, from (5.5) and from our assumptions, which imply that $K^{\Gamma(h)}(X,P) \geq K(X,P) > -\infty$ on B, we deduce that for each $\Gamma(h)$ there exists an index $m(\Gamma(h))$ such that $\mathbb{P}(C \setminus C^{\Gamma(h)}_{m(\Gamma(h))}) = 0$ and

$$K^{\Gamma(h)}(X,P)\mathbf{1}_{C^{\Gamma(h)}_{m(\Gamma(h))}} \geq \left(\pi^{\Gamma(h)}(\xi^P_{m(\Gamma(h))}) - \frac{\delta}{4}\right)\mathbf{1}_{C^{\Gamma(h)}_{m(\Gamma(h))}} \quad \forall\, h \geq 1. \tag{5.6}$$

Set $E_h = D_h^{\Gamma(h)} \cap C^{\Gamma(h)}_{m(\Gamma(h))} \in \mathcal{G}$, and observe that

$$\mathbf{1}_{E_h} \to \mathbf{1}_C \quad \mathbb{P}\text{-a.s.} \tag{5.7}$$

From (5.3) and (5.6) we then deduce that

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4}\right)\mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{E_h} \quad \forall h \geq 1, \tag{5.8}$$

$$K^{\Gamma(h)}(X,P)\mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi^P_{m(\Gamma(h))}) - \frac{\delta}{4}\right)\mathbf{1}_{E_h} \quad \forall h \geq 1. \tag{5.9}$$

We then have for any $h \geq 1$

$$\begin{aligned}
\pi(\xi_h^Q)\mathbf{1}_{E_h} + \frac{\delta}{4}\mathbf{1}_{E_h} &\geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{E_h} && (5.10)\\
&\geq K^{\Gamma(h)}(X,Q)\mathbf{1}_{E_h} && (5.11)\\
&\geq \left(K^{\Gamma(h)}(X,P) + \varepsilon + \delta\right)\mathbf{1}_{E_h} && (5.12)\\
&\geq \left(\pi^{\Gamma(h)}(\xi^P_{m(\Gamma(h))}) - \frac{\delta}{4} + \varepsilon + \delta\right)\mathbf{1}_{E_h} && (5.13)\\
&\geq \left(\pi(\xi^P_{m(\Gamma(h))}) + \varepsilon + \frac{3}{4}\delta\right)\mathbf{1}_{E_h}. && (5.14)
\end{aligned}$$

In the above chain of inequalities, (5.10) follows from (5.8); (5.11) follows from (5.2) and the definition of $K^{\Gamma(h)}(X,Q)$; (5.12) follows from (5.1); (5.13) follows from (5.9); (5.14) follows from the definition of the maps $\pi_{A^{\Gamma(h)}}$.

Recalling (5.4) we then get, for each $h \geq 1$,

$$\pi(\xi_h^Q)\mathbf{1}_{E_h} \geq \left(\pi(\xi^P_{m(\Gamma(h))}) + \varepsilon + \frac{\delta}{2}\right)\mathbf{1}_{E_h} \geq \left(K(X,P) + \varepsilon + \frac{\delta}{2}\right)\mathbf{1}_{E_h} > -\infty. \tag{5.15}$$

From (5.2) and (5.7) we have $\pi(\xi_h^Q)\mathbf{1}_{E_h} \to K(X,Q)\mathbf{1}_C$ $\mathbb{P}$-a.s. as $h \uparrow \infty$, and hence from (5.15)

$$\mathbf{1}_C K(X,Q) = \lim_h \pi(\xi_h^Q)\mathbf{1}_{E_h} \geq \lim_h \mathbf{1}_{E_h}\left(K(X,P) + \varepsilon + \frac{\delta}{2}\right) = \mathbf{1}_C\left(K(X,P) + \varepsilon + \frac{\delta}{2}\right),$$

which contradicts the assumption of the lemma, since $C \subseteq B$ and $\mathbb{P}(C) > 0$. ■

Proof of Lemma 4.3. First notice that the assumptions of this lemma are those of Lemma 5.3. Assume by contradiction that there exists $\Gamma_0 = \{B^C, \widetilde{\Gamma}_0\}$, where $\widetilde{\Gamma}_0$ is a partition of B, such that

$$\mathbb{P}(\omega \in B \mid K^{\Gamma_0}(X,Q)(\omega) > K^{\Gamma_0}(X,P)(\omega) + \varepsilon) > 0. \tag{5.16}$$

By our assumptions we have $K^{\Gamma_0}(X,P)\mathbf{1}_B \geq K(X,P)\mathbf{1}_B > -\infty$ and $K^{\Gamma_0}(X,Q)\mathbf{1}_B \leq \pi_B(X)\mathbf{1}_B < +\infty$. Since K^{Γ_0} is constant on every element $A^{\Gamma_0} \in \Gamma_0$, we denote with $K^{A^{\Gamma_0}}(X,Q)$ the value that the random variable $K^{\Gamma_0}(X,Q)$ assumes on A^{Γ_0}. From (5.16) we deduce that there exists $\widehat{A}^{\Gamma_0} \subseteq B$, $\widehat{A}^{\Gamma_0} \in \Gamma_0$, such that

$$+\infty > K^{\widehat{A}^{\Gamma_0}}(X,Q) > K^{\widehat{A}^{\Gamma_0}}(X,P) + \varepsilon > -\infty.$$

Then let $d > 0$ be defined by

$$d =: K^{\widehat{A}^{\Gamma_0}}(X,Q) - K^{\widehat{A}^{\Gamma_0}}(X,P) - \varepsilon. \tag{5.17}$$

Apply Lemma 5.3 with $\delta = \frac{d}{3}$: then there exists $\Gamma \supseteq \Gamma_0$ (without loss of generality $\Gamma = \{B^C, \widetilde{\Gamma}\}$, where $\widetilde{\Gamma} \supseteq \widetilde{\Gamma}_0$) such that

$$K^{\Gamma}(X,Q)\mathbf{1}_B \leq \left(K^{\Gamma}(X,P) + \varepsilon + \delta\right)\mathbf{1}_B. \tag{5.18}$$

Considering only the two partitions Γ and Γ_0, we may apply Lemma 5.2 and conclude that there exist two sequences $\{\xi_h^P\}_{h=1}^{\infty} \in L_{\mathcal{F}}$ and $\{\xi_h^Q\}_{h=1}^{\infty} \in L_{\mathcal{F}}$ satisfying as $h \uparrow \infty$

$$E_P[\xi_h^P|\mathcal{G}] \geq_P E_P[X|\mathcal{G}], \quad \pi^{\Gamma_0}(\xi_h^P) \downarrow K^{\Gamma_0}(X,P), \quad \pi^{\Gamma}(\xi_h^P) \downarrow K^{\Gamma}(X,P), \tag{5.19}$$

$$E_Q[\xi_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}], \quad \pi^{\Gamma_0}(\xi_h^Q) \downarrow K^{\Gamma_0}(X,Q), \quad \pi^{\Gamma}(\xi_h^Q) \downarrow K^{\Gamma}(X,Q). \tag{5.20}$$

Since $K^{\Gamma_0}(X,P)$ is constant and finite on $\widehat{A}^{\Gamma_0}$, from (5.19) we find $h_1 \geq 1$ such that

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) - K^{\widehat{A}^{\Gamma_0}}(X,P) < \frac{d}{2} \quad \forall\, h \geq h_1. \tag{5.21}$$

From (5.17) and (5.21) we deduce that

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) < K^{\widehat{A}^{\Gamma_0}}(X,P) + \frac{d}{2} = K^{\widehat{A}^{\Gamma_0}}(X,Q) - \varepsilon - d + \frac{d}{2} \quad \forall\, h \geq h_1,$$

and therefore, knowing from (5.20) that $K^{\widehat{A}^{\Gamma_0}}(X,Q) \leq \pi_{\widehat{A}^{\Gamma_0}}(\xi_h^Q)$,

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) + \frac{d}{2} < \pi_{\widehat{A}^{\Gamma_0}}(\xi_h^Q) - \varepsilon \quad \forall\, h \geq h_1. \tag{5.22}$$

We now take into account all the sets $A^{\Gamma} \subseteq \widehat{A}^{\Gamma_0} \subseteq B$. For the convergence of $\pi_{A^{\Gamma}}(\xi_h^Q)$ we distinguish two cases. On those sets A^{Γ} for which $K^{A^{\Gamma}}(X,Q) > -\infty$ we may find, from (5.20), $\overline{h} \geq 1$ such that

$$\pi_{A^{\Gamma}}(\xi_h^Q) - K^{A^{\Gamma}}(X,Q) < \frac{\delta}{2} \quad \forall\, h \geq \overline{h}.$$

Then using (5.18) and (5.19) we have

$$\pi_{A^\Gamma}(\xi_h^Q) < K^{A^\Gamma}(X,Q) + \frac{\delta}{2} \le K^{A^\Gamma}(X,P) + \varepsilon + \delta + \frac{\delta}{2} \le \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \delta + \frac{\delta}{2}$$

so that

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall\, h \ge \overline{h}.$$

On the other hand, on those sets A^Γ for which $K^{A^\Gamma}(X,Q) = -\infty$ the convergence (5.20) guarantees the existence of $\widehat{h} \ge 1$ for which we again obtain

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall\, h \ge \widehat{h}. \tag{5.23}$$

Notice that $K^\Gamma(X,P) \ge K(X,P)\mathbf{1}_B > -\infty$ and (5.19) imply that $\pi_{A^\Gamma}(\xi_h^P)$ converges to a finite value for $A^\Gamma \subseteq B$.

Since the partition Γ is finite, there exists $h_2 \ge 1$ such that (5.23) stands for every $A^\Gamma \subseteq \widehat{A}^{\Gamma_0}$ and for every $h \ge h_2$ and for our choice of $\delta = \frac{d}{3}$ (5.23) becomes

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{d}{2} \quad \forall\, h \ge h_2, \quad \forall\, A^\Gamma \subseteq \widehat{A}^{\Gamma_0}. \tag{5.24}$$

Fix $h^* > \max\{h_1, h_2\}$, and consider the value $\pi_{\widehat{A}^{\Gamma_0}}(\xi_{h^*}^Q)$. Then among all $A^\Gamma \subseteq \widehat{A}^{\Gamma_0}$ we may find $B^\Gamma \subseteq \widehat{A}^{\Gamma_0}$ such that $\pi_{B^\Gamma}(\xi_{h^*}^Q) = \pi_{\widehat{A}^{\Gamma_0}}(\xi_{h^*}^Q)$. Thus

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_{h^*}^Q) = \pi_{B^\Gamma}(\xi_{h^*}^Q) \overset{(5.24)}{<} \pi_{B^\Gamma}(\xi_{h^*}^P) + \varepsilon + \frac{d}{2} \le \pi_{\widehat{A}^{\Gamma_0}}(\xi_{h^*}^P) + \varepsilon + \frac{d}{2} \overset{(5.22)}{<} \pi_{\widehat{A}^{\Gamma_0}}(\xi_{h^*}^Q),$$

which is a contradiction. ∎

5.2. On quasi-convex real valued maps.

Proof of Theorem 1.1. By definition, for any $X' \in L'$, $R(X'(X), X') \le f(X)$ and therefore

$$\sup_{X' \in L'} R(X'(X), X') \le f(X), \;\; X \in L.$$

Fix any $X \in L$ and take $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then X does not belong to the closed convex set $\{\xi \in L : f(\xi) \le f(X) - \varepsilon\} := \mathcal{C}_\varepsilon$ (if $f(X) = +\infty$, replace the set $\mathcal{C}_\varepsilon$ with $\{\xi \in L : f(\xi) \le M\}$ for any M). By the Hahn–Banach theorem there exist $\alpha \in \mathbb{R}$ and $X'_\varepsilon \in L'$ such that

$$X'_\varepsilon(X) > \alpha > X'_\varepsilon(\xi) \forall\, \xi \in \mathcal{C}_\varepsilon. \tag{5.25}$$

$$\text{Hence } \left\{\xi \in L : X'_\varepsilon(\xi) \ge X'_\varepsilon(X)\right\} \subseteq (\mathcal{C}_\varepsilon)^C = \{\xi \in L : f(\xi) > f(X) - \varepsilon\} \tag{5.26}$$

$$\begin{aligned} \text{and } f(X) &\ge \sup_{X' \in L'} R(X'(X), X') \ge R(X'_\varepsilon(X), X'_\varepsilon) \\ &= \inf\left\{f(\xi) \mid \xi \in L \text{ such that } X'_\varepsilon(\xi) \ge X'_\varepsilon(X)\right\} \\ &\ge \inf\left\{f(\xi) \mid \xi \in L \text{ satisfying } f(\xi) > f(X) - \varepsilon\right\} \ge f(X) - \varepsilon. \quad ∎ \end{aligned}$$

We now state in Theorem 5.4 the upper semicontinuous variants of Theorem 1.1 and in Proposition 5.5 their versions under the monotonicity assumption, which are used in the proofs of the main theorems. These results are minor modifications of those appearing, for instance, in [CMM09b], and their proofs are standard and are omitted.

Theorem 5.4. *Let L be a locally convex topological vector space, L' be its dual space, $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be quasi-convex and upper semicontinuous, and R be defined as in Theorem 1.1. Then*

$$f(X) = \max_{X' \in L'} R(X'(X), X'). \tag{5.27}$$

Proposition 5.5. *Suppose L is a lattice, $L^* = (L, \geq)^*$ is the order continuous dual space satisfying $L^* \hookrightarrow L^1$, and $(L, \sigma(L, L^*))$ is a locally convex TVS. If $f : L \to \overline{\mathbb{R}}$ is quasi-convex, monotone increasing, and $\sigma(L, L^*)$-(LSC) or $\sigma(L, L^*)$-(USC), then*

$$f(X) = \sup_{Q \in L^*_+ | Q(\mathbf{1})=1} R(Q(X), Q).$$

Acknowledgment. We wish to thank Dott. G. Aletti for helpful discussion on this subject.

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