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On the Generalized Harmonic Polylogarithms of One Complex Variable

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Abstract

We describe how to compute numerically in the complex plane a set of Generalized Harmonic Polylogarithms (GHPLs) with square roots in the weights, using the `C++/GiNaC` numerical routines of Vollinga and Weinzierl. As an example, we provide the numerical values of the NLO electroweak light-fermion corrections to the Higgs boson production in gluon fusion in the case of complex W and Z masses.

Key words: Harmonic Polylogarithms, Feynman diagrams, Multi-loop calculations, Higgs physics

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1 Introduction

In recent years, the need of accurate theoretical predictions for scattering amplitudes in collider physics requested a strong effort in the development of methods and strategies for the calculation of multi-loop Feynman diagrams. In particular, it was recently possible to afford many analytic calculations unthinkable up to fifteen years ago.

This success is largely due to reliable and powerful algorithms, as for instance the so-called "Laporta algorithm" [1]. The calculation of a physical observable in perturbation theory requires the (numerical or analytical) evaluation of a large number of regularized scalar integrals. The Laporta algorithm allows to reduce this large number of scalar integrals to a linear combination of a small set of independent scalar integrals, called the "Master Integrals" (MIs) of the problem under consideration. The Laporta algorithm is based on the Integration-by-Parts Identities (IBPs) [2], a set of relations that link scalar integrals with a different power of the propagators and of scalar products in the numerator among each other[1]. The interplay between the "Differential Equations Method" [4] and the techniques based on Mellin-Barnes representations of the integrals [5] provides, then, a powerful tool for the analytic calculation of the MIs.

Another important ingredient for the analytic calculation of the higher-order corrections to a physical observable is the identification of the base of functions in terms of which the MIs can be expressed. This base of functions is strictly related to the structure of the thresholds of the Feynman diagrams under consideration.

The connection between Feynman diagrams with a simple structure of thresholds and the functional base of the Harmonic Polylogarithms[2] (HPLs) [6] is completely clear. Feynman diagrams with a richer structure of thresholds require the introduction of one- [10, 11] and two-dimensional [12, 13, 14, 15, 16] extensions of the HPLs, that will be generically referred to as Generalized Harmonic Polylogarithms (GHPLs).

We briefly recall the advantages of using the (G)HPLs, that constitute a well suited functional base in which to express the analytic results. *i)* The structure of the (G)HPLs, is strictly connected with the solution, with the Euler method, of the first-order linear differential equations satisfied by the Feynman amplitudes. *ii)* The (G)HPLs constitute a base of linearly independent functions. *iii)* The base of (G)HPLs provides a perfect control on the analytical properties of the MIs and, therefore, on the physical observable that we aim to calculate. *iv)* Finally, there are available numerical routines, that allow a precise evaluation of the (G)HPLs in FORTRAN [17], Mathematica [18], `C++` [19].

In this paper, we provide a detailed analysis of the GHPLs of a single variable, with weights containing square roots. These GHPLs were introduced in [11] for the analytic expression of the MIs concerning the electroweak form factor [20, 11]. In [21], ad hoc numerical routines were made for the evaluation of a subset of functions of this class occurring in the calculation of the electroweak NLO corrections to the production of a Higgs boson in gluon fusion and its decay in two photons. The purpose of the present analysis is to give a general framework for the evaluation of GHPLs with weights containing square roots and to show that the evaluation of all of the GHPLs introduced in [11] can be performed using already existing numerical routines.

[1]Public implementations of the algorithm are available in [3].

[2]Together with the related harmonic [7], nested [8] and binomial sums [9].

In section 2, we recall the definition and the basic properties of the 1-dimensional HPLs and GHPLs with square roots in the weights. We start on the case of real variable x and on the subset of GHPLs that have the following possible weights: $\{-r, -4, -1, 0\}$. Subsequently we discuss their analytic continuation. In section 3, we illustrate how to move from the set of GHPLs with square roots in the weights to a set of generalized polylogarithms, with linear weights. This is the remark that allows a numerical evaluation of the GHPLs with square roots, using the `C++` routines of Vollinga and Weinzierl [19] (in the following they will be referred to as "VW routines"). In section 4, we consider the case in which the variable x is complex. We provide a demonstration of the GHPLs scale invariance in the complex plane, justifying the use of the VW routines also in this case. In section 5, we introduce additional weights and discuss the transformations of section 3 applied to this new extended set. In section 6, we apply the results of this paper to the numerical evaluation of the GHPLs involved in the NLO light-fermion electroweak corrections to the Higgs boson production in gluon fusion, in the case of complex W and Z masses. Finally, in the appendices, we provide the analytic expressions of the linearized GHPLs involved in the calculation shown in section 6.

2 HPLs and GHPLs of a Real Variable

In this section, we recall the definition and the properties of the one-dimensional Harmonic Polylogarithms (HPLs) of a real variable and their generalization (GHPLs), with square roots and linear weights, introduced in [11].

2.1 Harmonic Polylogarithms

The set of functions denominated Harmonic Polylogarithms (HPLs) [6] is defined as repeated integrations of the following three fundamental[3] "weight functions":

$$f(-1;t) = \frac{1}{t+1}\,, \qquad f(0;t) = \frac{1}{t}\,, \qquad f(1;t) = \frac{1}{t-1}\,. \tag{1}$$

Note that the functions in Eq. (1) have a non-integrable singularity in $t=-1$, $t=0$, and $t=1$ respectively. The related HPLs of weight 1 are

$$H(-1;x) = \int_0^x \frac{dt}{t+1} = \log(x+1)\,, \tag{2}$$

$$H(0;x) = \int_1^x \frac{dt}{t} = \log(x)\,, \tag{3}$$

$$H(1;x) = \int_0^x \frac{dt}{t-1} = \log(1-x)\,, \tag{4}$$

where x is a real variable ($x \in \mathbb{R}$). Since the logarithms have branch cuts on the real axis for $x \leq -1$, $x \leq 0$, and $x \geq 1$, respectively, the three HPLs in Eqs. (2,3,4) are real and uniquely defined only for $x > -1$, $x > 0$, and $x < 1$, respectively. Outside these intervals,

[3] Note a minus sign in the weight +1 with respect to the Remiddi-Vermaseren definition of [6]

the logarithms become complex, and a prescription for the approach to the branch cut has to be chosen (see section 2.1.1).

An HPL with weight 2 or bigger is defined through a repeated integration of the weight functions of Eq. (1). If $\mathbf{w}$ is a vector with w components consisting of a sequence of -1, 0, and $+1$, we define the HPL of weight $w+1$ as follows:

$$H(a, \mathbf{w}; x) = \int_0^x dt\, f(a; t)\, H(\mathbf{w}; t)\,, \quad a = -1, 0, 1\,, \tag{5}$$

with the exception of the case in which the weights are only zeroes, defined as

$$H(\mathbf{0}_{w+1}; x) = \int_1^x dt\, f(0; t)\, H(\mathbf{0}_w; t) = \frac{1}{(w+1)!} \log^{w+1} x\,. \tag{6}$$

The singularity structure and analyticity properties of the HPLs derive from the properties of the logarithms.

The HPLs satisfy a shuffle algebra according to which a product of two HPLs of weights n_1 and n_2 is a combination of HPLs of weight $n = n_1 + n_2$. We refer the interested reader to [6] for more details.

2.1.1 Analytic Continuation

The HPLs are, in general, complex, depending on the value of the real variable x. In many relevant physical cases, the calculation of the Feynman integrals involved in some observable is done in a restricted range of x. For instance, if x is related to the squared center of mass energy s through the relation $x = -s/m^2$, with m a mass scale of the problem, the Feynman integrals are usually solved in the so-called Euclidean region: $x \geq 0$[4].

Let us suppose, then, $x \geq 0$. In the range $0 \leq x \leq 1$ all the HPLs are real. For $x > 1$, instead, we have a possible cut, corresponding to the HPLs with a $+1$ in the right-most weight, and therefore, ultimately, to the $\log(1-x)$, that has an imaginary part in this region.

In the case of HPLs of weight 1, depending on the prescription adopted, this imaginary part is $\pm i\pi$:

$$H(1; x) = \log(1-x) = \log|1-x| \pm i\pi\, \theta(x-1)\,, \tag{7}$$

while $H(0; x)$ and $H(-1; x)$ are real for positive x.

In the case of HPLs of weight 2, or bigger, an explicit expression for the imaginary part can be found, using the shuffle algebra properties to move the weights 1 from the right to the left in the sequence and then using the relation of Eq. (7). For instance, for $H(0, 1; x) = \mathrm{Li}_2(x)$, the Euler Dilogarithm, we have:

$$\begin{aligned} H(0, 1; x) &= H(0; x) H(1; x) - H(1, 0; x)\,, \\ &= \log(x) \log|1-x| - H(1, 0; x) \pm i\pi\, \theta(x-1) \log(x)\,. \end{aligned} \tag{8}$$

The HPL $H(1, 0; x)$ is real for $x > 1$ and the imaginary part of $H(0, 1; x)$ is explicitly given by the last term in Eq. (8).

[4] If, moreover, x is related to s through a quadratic relation, as the transformation of variable in Eq. (27), the Euclidean region is even more restricted: $0 \leq x < 1$.

Once the searched analytic expression is known in the range of $x > 0$, one has to do an analytic continuation to move back to the Minkowski region ($s > 0$ and then $x < 0$). Because of causality, the Mandelstam invariant s has to be assigned a positive vanishing imaginary part, $s + i0^+$. Therefore, if $x = -s/m^2$, the case $s > 0$ is recovered using

$$x \to -x' - i0^+ \,, \tag{9}$$

where now $x' = s/m^2 > 0$.

If $0 < x' \leq 1$ ($-1 \leq x < 0$), we have to take into account the branch cut connected to the weight 0. For HPLs of weight 1 we have

$$\begin{aligned} H(0;x) &\to H(0;-x'-i0^+) = H(0;x') - i\pi \,, & (10)\\ H(1;x) &\to H(1;-x'-i0^+) = H(-1;x') \,, & (11)\\ H(-1;x) &\to H(-1;-x'-i0^+) = H(1;x') \,. & (12) \end{aligned}$$

Therefore, only the $\log(x)$ gets the imaginary part. The case of HPLs of weight 2 or bigger has to be treated extracting, with the help of the shuffle algebra, the right-most zeroes (trailing zeroes). The HPLs that have no zeroes on the right of the sequence of the weights do not get imaginary parts. Moreover, moving from x to x', the weights flip in sign. The $\log^n x$ extracted with the algebra are then transformed according to Eq. (10).

2.1.2 Transformations of Variables and Numerical Evaluation

A fast and precise numerical evaluation of the HPLs, for all the values of the real variable x, can be done using an appropriate Taylor expansion in the vicinity of a point of analyticity of the functions. The strategy is the following (see for instance [17]):

1. We focus on the point $x = 0$. We extract the possible logarithmic behaviour, $\log^n(x)$, of the HPL using the shuffle algebra (we move the rightmost zeroes to the left). The HPLs with no zeroes on the right are analytic in $x = 0$. Correspondingly, each HPL takes the form $\sum_{n,m} P_m(x)\log^n(x)$, where P_m is a polynomial of degree m. In the case $x \to 0^-$, the imaginary part comes from $\log^n(x)$, using the prescription of Eq. (7).

2. With an appropriate number of terms in the Taylor expansion, one is able to evaluate numerically the HPL in an interval around $x = 0$ with a given precision. In [17] the interval $-(\sqrt{2}-1) \leq x \leq (\sqrt{2}-1)$ is taken as the central region and using Bernoulli numbers and Chebyshev polynomials, the authors evaluate the HPLs in double precision using only few terms in the expansion.

3. Using the properties of the HPLs, one can find suitable transformation formulas for the argument in order to map different domains of the real axis back to the central value $-(\sqrt{2}-1) \leq x \leq (\sqrt{2}-1)$. In so doing, using the formulas found for that region, one is able to cover all the possible values of the variable $x \in \mathbb{R}$.

2.2 Generalized Harmonic Polylogarithms

In some physically relevant cases, it can happen that the weight functions defined in Eq. (1) are not sufficient to describe the analytic structure of the result. Therefore, additional weights (and/or additional structures) have to be taken into account, together with the ones introduced in the last section. This gives rise to an enlarged set of functions, called Generalized Harmonic Polylogarithms (GHPLs), which maintain the structure and properties of the HPLs.

Let us focus, for the moment, on the GHPLs that are involved in the calculation of the NLO light-fermion electroweak corrections to the cross section of production of a Higgs boson in gluon fusion and its decay in two photons[5], as considered in [21]. This is a restricted set, that contains only four weights, denominated as follows:

$$G(w_1, w_2, ..., w_n; x)\,, \quad \text{with}\, w_i \in \{-r, -4, -1, 0\}\,, \tag{13}$$

according to the definitions given below. Let furthermore restrict the analysis to the case of real variable $x \in \mathbb{R}$ (the case of complex x will be treated in section 4). Therefore, we consider the following set of weight functions:

$$g(-r; x) = \frac{1}{\sqrt{x(x+4)}}\,, \tag{14}$$

$$g(w; x) = \frac{1}{x-w}\,, \quad \text{with}\, w \in \{-4, -1, 0\}\,. \tag{15}$$

These functions have an integrable singularity in $x = 0$ and $x = -4$, and a non-integrable singularity in $x = w$, respectively. The related GHPLs of weight 1 are

$$G(0; x) = \log(x)\,, \tag{16}$$

$$G(-r; x) = \int_0^x \frac{dt}{\sqrt{t(t+4)}} = -\log\left(\frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{x+4}+\sqrt{x}}\right)\,, \tag{17}$$

$$G(w; x) = \int_0^x \frac{dt}{t-w} = \log(x-w) - \log(-w)\,, \quad \text{with}\, w \in \{-4, -1\}\,, \tag{18}$$

and they have at most a logarithmic singularity in $x = 0$, $x = -1, -4$. $G(0; x)$ and $G(w; x)$ have a branch cut for $x \le 0$ and $x \le w$, respectively. For these negative values of x, they become complex, with imaginary part depending on the prescription of approach to the cut. $G(-r; x)$ has a branch cut for $x \le 0$; it is purely imaginary in the range $-4 \le x < 0$ and it is a complex number, with non vanishing real part, for $x < -4$.

The GHPLs with weight 2 or bigger are defined as repeated integrations of the weight functions in Eqs. (14,15):

$$G(a, \mathbf{w}; x) = \int_0^x dt\, g(a; t)\, G(\mathbf{w}; t)\,, \tag{19}$$

with the exception of $G(\mathbf{0}_w; x)$, defined as:

$$G(\mathbf{0}_w; x) = \frac{1}{w!} \log^w(x)\,. \tag{20}$$

Such a set of functions obeys (by construction) the shuffle algebra and all the important properties of the HPLs.

The analytic properties of the functions defined in Eqs. (19,20) derive from the properties of the logarithm and of the square root.

[5] The more extended set introduced in [11] will be discussed in the following sections.

2.2.1 Analytic Continuation

When $x \geq 0$, every GHPL belonging to the set considered in the previous section is real and the only possible divergence is a logarithmic divergence in $x = 0$.

For negative x, since the logarithm and the square root have a branch cut for negative argument, we must choose how to approach the cut. In order to do that, we give a vanishing imaginary part to the variable x. Let us choose the following prescription:

$$x \to -x' - i\,0^+ \,. \tag{21}$$

The region $x < 0$ is divided in three sets, depending on the value of x'. The analytic continuation has to be done in each region differently.

1. For $-1 \leq x < 0$, the imaginary parts come from $\log(x)$ and from the square root, that becomes purely imaginary:

$$\log(x) \quad \to \quad \log(-x' - i\,0^+) = \log(x') - i\pi \,, \tag{22}$$

$$\frac{1}{\sqrt{x(x+4)}} \quad \to \quad \frac{1}{\sqrt{(-x' - i\,0^+)(-x' - i\,0^+ + 4)}} = \frac{i}{\sqrt{x'(4-x')}} \,. \tag{23}$$

2. For $-4 \leq x < -1$, also the $\log(x+1)$ gives an imaginary part:

$$\log(x+1) \to \log(-x' - i\,0^+ + 1) = \log(x' - 1) - i\pi \,. \tag{24}$$

3. For $-\infty \leq x < -4$, an additional imaginary part comes from the logarithm $\log(x+4)$, while the square root becomes real again:

$$\log(x+4) \quad \to \quad \log(-x' - i\,0^+ + 4) = \log(x' - 4) - i\pi \,, \tag{25}$$

$$\frac{i}{\sqrt{x'(4-x')}} \quad \to \quad \frac{i}{\sqrt{(x' + i\,0^+)(-x' - i\,0^+ + 4)}} = -\frac{1}{\sqrt{x'(x'-4)}} \,. \tag{26}$$

2.2.2 Numerical evaluation

The numerical evaluation of the GHPLs considered in the previous section can be done in principle using basically the same strategy of the HPLs. One focuses on $x = 0$, extracts the logarithmic behaviour using the shuffle algebra and then expands the remaining analytic functions (with no zeroes in the rightmost weight). Then, using suitable transformations, one relates the basic interval around $x = 0$ to the rest of the real axis.

However, the actual implementation of this strategy is quite cumbersome. Instead, it turns out to be convenient to transform from the beginning the GHPLs with square roots in the weights into a combination of GHPLs with linear weights using a set of variable transformations that will be discussed in the following section. The advantage of doing so lies in the fact that there exist fast and precise public numerical routines, that allow for the evaluation of generalized polylogarithms with generic linear weights [19]. The latter can be used to evaluate the GHPLs belonging to the set discussed in the last section, or, more in general, to the wider set introduced in [11].

3 Linearization

The presence, in an analytic result, of GHPLs with square roots together with linear weights, is due to the structure of the thresholds and pseudo-thresholds of the corresponding Feynman diagrams.

Let us consider, for instance, the QED corrections to the vertex diagrams representing the decay of a photon into an electron-positron pair. The particle content reduces to massless photons and massive electrons/positrons. The threshold for the production of the electron-positron pair is at $s = 4m^2$, while the pseudo-threshold lies at $s = 0$. Let us look at the differential equations with respect to s, for the solution of the corresponding MIs. The structure of the thresholds and pseudo-thresholds emerges in the homogeneous part with terms such as $1/s$ and $1/(s-4m^2)$, that are also present in the non-homogeneous part (see for instance [22]). The solution of the homogeneous equation contains the inverse square root $1/\sqrt{s(s-4m^2)}$. The particular solution, then, comes from repeated integrations of $1/\sqrt{s(s-4m^2)}$, $1/s$, and $1/(s-4m^2)$ terms. In the Euclidean region ($p^2 = -s > 0$) the solution can be expressed in terms of GHPLs of the variable $x = p^2/m^2 = -s/m^2$, with weights $-r$, -4, and 0.

We can get rid of the square root (the weight $-r$) using the following quadratic transformation of variable:

$$x = \frac{(1-\xi)^2}{\xi}\,, \quad \xi = \frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{x+4}+\sqrt{x}}\,, \tag{27}$$

where $\xi \in \mathbb{C}$, $|\xi| \leq 1$, while $x \in \mathbb{R}$. In fact, we have:

$$\frac{1}{(x+4)} = \frac{\xi}{(\xi+1)^2}\,, \tag{28}$$

$$\frac{1}{\sqrt{x(x+4)}} = -\frac{\xi}{(\xi+1)(\xi-1)}\,. \tag{29}$$

Moving from x to ξ, the integration measure changes as follows:

$$\int_0^x dt = \int_1^\xi \frac{(\eta+1)(\eta-1)}{\eta^2}\, d\eta\,, \tag{30}$$

and every GHPLs reduces to a combination of repeated integrations of the simpler weight functions defined in Eq. (1). As a consequence of that, the set of weights $\{-r,-4,0\}$ is transformed into the set $\{-1,0,1\}$, and the GHPLs are transformed into the usual HPLs defined in section 2.1.

Let us consider, now, a more complicated problem, in which zero- and multiple-mass cuts are present at the same time. This is, for instance, the case of the electroweak corrections to leptonic or hadronic processes in which the lepton and quark masses are neglected and only the vector boson masses are considered different from zero. In this case, the homogeneous part of the differential equations for the corresponding MIs contain terms as $1/s$, $1/(s-4m^2)$, $1/\sqrt{s(s-4m^2)}$, together with terms as $1/(s-m^2)$. Therefore, the weights $-r$, -4, -1, and 0 are present at the same time. In this situation, it is more difficult to get rid of the square root. If we require, for instance, that the weights belong always to the set of real numbers, $w_i \in \mathbb{R}$, there is no transformation of variable that could linearize all the weights

$\{-r, -4, -1, 0\}$ at the same time. However, if we relax this constraint, we can move from the set with square roots and linear weights to a set of only linear weights, using the change of variable (27).

Using Eq. (27), the old weight functions are transformed into:

$$g(-r;t) = \frac{1}{\sqrt{t(t+4)}} = -\frac{\eta}{(\eta+1)(\eta-1)}, \tag{31}$$

$$g(-4;t) = \frac{1}{t+4} = \frac{\eta}{(\eta+1)^2}, \tag{32}$$

$$g(-1;t) = \frac{1}{t+1} = \frac{\eta}{(\eta-c)(\eta-\bar{c})}, \tag{33}$$

$$g(0;t) = \frac{1}{t} = \frac{\eta}{(\eta-1)^2}, \tag{34}$$

with

$$c = \frac{1+i\sqrt{3}}{2} = e^{i\frac{\pi}{3}}, \quad \bar{c} = \frac{1-i\sqrt{3}}{2} = e^{-i\frac{\pi}{3}}, \tag{35}$$

where c and $\bar{c}$ are the two primitive sixth roots of the unity. Then, combining the integration measure, Eq. (30), with the Eqs. (31–34), the original GHPLs with square root in the weight are transformed into

$$G(-r, \mathbf{w}; x) = \int_0^x dt\, g(-r;t)\, G(\mathbf{w};t) = -\int_1^\xi d\eta\, \frac{1}{\eta}\, G(\mathbf{w};t(\eta)), \tag{36}$$

$$G(-4, \mathbf{w}; x) = \int_0^x dt\, g(-4;t)\, G(\mathbf{w};t) = \int_1^\xi d\eta\, \left(-\frac{1}{\eta} + \frac{2}{\eta+1}\right) G(\mathbf{w};t(\eta)), \tag{37}$$

$$\begin{aligned} G(-1, \mathbf{w}; x) &= \int_0^x dt\, g(-1;t)\, G(\mathbf{w};t) = \\ &= \int_1^\xi d\eta\, \left(-\frac{1}{\eta} + \frac{1}{\eta-c} + \frac{1}{\eta-\bar{c}}\right) G(\mathbf{w};t(\eta)), \end{aligned} \tag{38}$$

$$G(0, \mathbf{w}; x) = \int_0^x dt\, g(0;t)\, G(\mathbf{w};t) = \int_1^\xi d\eta\, \left(-\frac{1}{\eta} + \frac{2}{\eta-1}\right) G(\mathbf{w};t(\eta)). \tag{39}$$

Therefore, the set $\{-r, -4, -1, 0\}$ of weights with square roots, has been transformed into a new set, with only linear weights: $\{-1, 0, 1, c, \bar{c}\}$. This new set, contains the original HPLs, discussed in section 2.1, and new GHPLs with complex weights c and $\bar{c}$. The latter, have branch cuts in the complex x plane, starting at $x = c, \bar{c}$ respectively. At weight 1, they are:

$$G(c;x) = \int_0^x \frac{dt}{t-c} = \log(x-c) - \log(-c), \tag{40}$$

$$G(\bar{c};x) = \int_0^x \frac{dt}{t-\bar{c}} = \log(x-\bar{c}) - \log(-\bar{c}). \tag{41}$$

We can summarize the linearization procedure as follows:

1. We base our analysis in the region in which $x \geq 0$ (the formulas will be afterwards analytically continued in the region $x < 0$, if necessary).

2. We transform the integration variable in the new variable η, on which we integrate from 1 to ξ. Troubles with the integration in $\eta = 1$ can occur, due to possible singular behaviours. Since such singularities can occur only from the weights 0 in the variable x, we avoid the possible logarithmic divergence in $\eta = 1$ using the shuffle algebra and extracting the trailing zeroes in x. The logarithms so found, $G(\mathbf{0}_n; x)$, are directly rewritten as $1/n! \log^n(x)$ and, then, straightforwardly transformed in the variable ξ using the relation $\log(x) = 2\log(1-\xi) - \log(\xi)$.

3. We linearize the GHPLs of weight 1.

4. Weight-by-weight we proceed to the linearization of the GHPLs with weight 2 and bigger, integrating over the new integration measure the corresponding linearized GHPL times the corresponding linearized weight function.

As an example, we give here the expressions of the linearized GHPLs with weight 1. $G(0; x)$ can be converted directly in the new variable ξ, since[6]:

$$G(0;x) = \log(x) = 2\log(1-\xi) - \log(\xi) = 2G(1;\xi) - G(0;\xi)\,. \tag{42}$$

Using the relations in Eqs. (36–39), we have:

$$G(-r;x) = -\int_1^\xi \frac{d\eta}{\eta} = -G(0;\xi)\,, \tag{43}$$

$$G(-4;x) = \int_1^\xi d\eta \left(-\frac{1}{\eta} + \frac{2}{\eta+1}\right) = -2\log(2) + 2G(-1;\xi) - G(0;\xi)\,, \tag{44}$$

$$\begin{aligned} G(-1;x) &= \int_1^\xi d\eta \left(-\frac{1}{\eta} + \frac{1}{\eta - c} + \frac{1}{\eta - \bar{c}}\right)\,, \\ &= -G(c;1) - G(\bar{c};1) + G(c;\xi) + G(\bar{c};\xi) - G(0;\xi)\,. \end{aligned} \tag{45}$$

It is worth to notice that the linearization algorithm generates some constants, *i.e.* the linearized GHPLs evaluated in $\xi = 1$. In many cases, these constants have a representation in terms of known transcendental constants. In general, however, to find such a representation could be very difficult. For the purpose of the numerical evaluation of the GHPLs with square roots in the weights using existing `C++` routines, these constants can be left as they are. In fact, the routines provide a fast and accurate numerical evaluation in every point, and then also in $\xi = 1$. In our particular case we have[7]:

$$G(c;1) + G(\bar{c};1) = 0\,, \tag{46}$$

such that

$$G(-1;x) = G(c;\xi) + G(\bar{c};\xi) - G(0;\xi)\,. \tag{47}$$

[6]Note again the different sign with respect to the weight +1 in the Remiddi-Vermaseren notation [6].

[7]The constants (and also the GHPLs with complex weights in the actual expressions for the MIs) should appear always in a way such that their sum is real, as it has to be, since we are in the Euclidean region.

Knowing the expressions of the linearized GHPLs with weight 1 and linearized weight functions, we can proceed with the linearization of the GHPLs at weight 2. If we consider, for instance, the function $G(0,-1;x)$ we have:

$$\begin{aligned} G(0,-1;x) &= \int_0^x \frac{dt}{t} G(-1;t)\,, \\ &= \int_1^\xi d\eta \left(-\frac{1}{\eta} + \frac{2}{\eta-1}\right) \left[G(c;\eta) + G(\bar{c};\eta) - G(0;\eta)\right]\,, \\ &= \zeta(2) - G(0,\bar{c};\xi) - G(0,c;\xi) + \frac{1}{2}G(0;\xi)^2 - 2G(1;\xi)G(0;\xi) \\ & \quad +2G(0,1;\xi) + 2G(1,\bar{c};\xi) + 2G(1,c;\xi)\,. \end{aligned} \tag{48}$$

In the same way one can proceed for higher weights. Explicit formulas for the weight-2 and weight-3 GHPLs involved in the NLO electroweak corrections for the production of a Higgs boson in gluon fusion are provided in appendix A and appendix B, respectively.

3.1 Analytic Continuation of the Linearized GHPLs and their Numerical Evaluation

The analytic continuation of the linearized GHPLs is less complicated than the one concerning their original form. In fact, while the variable x ranges from ∞ to 0, the corresponding variable ξ is real and positive and it ranges from 0 to 1. When x becomes negative, but in the range $-4 \leq x < 0$,

$$x \to -x' - i\,0^+\,, \quad 0 < x' \leq 4\,, \tag{49}$$

ξ becomes imaginary:

$$\xi = \frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{x+4}+\sqrt{x}} \;\to\; \zeta = \frac{\sqrt{4-x'}+i\sqrt{x'}}{\sqrt{4-x'}-i\sqrt{x'}} = e^{i\,2\phi}\,, \tag{50}$$

where

$$\phi = \arctan\sqrt{\frac{x'}{4-x'}}\,, \quad 0 < \phi \leq \frac{\pi}{2}\,. \tag{51}$$

Finally, when x' ranges from 4 to ∞, ξ becomes real again:

$$\zeta \;\to\; \xi' = \frac{\sqrt{x'}-\sqrt{x'-4}}{\sqrt{x'}+\sqrt{x'-4}}\,, \tag{52}$$

and it ranges from 1 to 0. We must, therefore, discuss three regions.

1. For $0 \leq x < \infty$, we have $0 < \xi < 1$. The original GHPLs are real. The linearized GHPLs contain functions that are manifestly real, as the ones with weights $-1, 0, 1$, but also functions that are complex: those that contain the weights c and $\bar{c}$. However, the GHPLs containing the weights c and $\bar{c}$ appear in the formulas always in pairs (for instance $G(-1,c;\xi) + G(-1,\bar{c};\xi)$), in such a way that, although the single GHPLs of the pair are complex, their sum is real, since the imaginary parts are equal and opposite in sign. The numerical evaluation of such GHPLs can be done straightforwardly using the VW routines presented in [19].

2. For $-4 \leq x < 0$, ξ is a pure phase, $\xi = e^{i\,2\phi}$. In order to evaluate numerically the GHPLs in this region, we have to notice that the VW routines, while allowing the use of complex weights, do not provide the possibility of evaluation of GHPLs with complex argument. However, for the GHPLs with non-trailing zeroes the following general formula holds:

$$G(w_1, w_2, ..., w_n; x) = G(\lambda w_1, \lambda w_2, ..., \lambda w_n; \lambda x)\,, \quad \lambda \in \mathbb{C}\,, \tag{53}$$

as it will be discussed in the next section. Extracting the trailing zeroes and then choosing[8]

$$\lambda = \frac{1}{x}\,, \tag{54}$$

the GHPLs under consideration are transformed in GHPLs of real argument, $\xi = 1$, and complex weights, $\{\pm e^{-i\,2\phi}, 0, e^{-i\left(2\phi \pm \frac{\pi}{3}\right)}\}$, that can be evaluated using again the VW routines.

3. For $-\infty < x < -4$, ξ is again real and we are back to the case explained in the first point.

4 GHPLs of a Complex Variable

In this section, we consider the case in which the GHPLs have to be evaluated in the complex plane. Therefore, x is complex from the beginning[9]. We are particularly interested in the following situation. Let us suppose that the dimensionless variable x is indeed a ratio between two physically meaningful variables: a squared momentum and a squared mass,

$$x = \frac{p^2}{m^2} = -\frac{s}{m^2}\,, \tag{55}$$

with $\sqrt{s}$ the c.m. energy of a certain process. This is, for instance, the case of the corrections presented in [21], but it is a quite general assumption. If the particle to which the mass m belongs is an unstable particle, its width Γ is going to play an active role in the determination of the corresponding physical observable. Consequently, the parameter x becomes complex, since we should now consider

$$x = -\frac{s}{(m - i\Gamma/2)^2} = -\frac{s}{M^2} e^{i\phi}\,, \tag{56}$$

where

$$M^2 = m^2 + \frac{\Gamma^2}{4}\,, \quad \text{and} \quad \phi = \arctan\left\{\frac{m\Gamma}{m^2 - \frac{\Gamma^2}{4}}\right\}. \tag{57}$$

In the non-physical region, in which $s < 0$, we have from Eq. (56) that

$$\mathcal{R}e(x) > 0\,, \quad \text{and} \quad \mathcal{I}m(x) > 0\,. \tag{58}$$

[8]Actually, it is sufficient to choose $\lambda = e^{-i\,arg(x)}$.

[9]The case in which x is real, but the corresponding reduced variable ξ is complex, was already discussed in the previous section.

The variable ξ defined in Eq. (27), correspondingly, is also complex. The definition of the GHPLs does not change, except from the fact that now the integration is over a curve in the complex plane. Since the functions are analytic in the region defined by Eq. (58), the value of the GHPL does not depend on the path. In this region we have:

$$\xi = \frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{x+4}+\sqrt{x}} = \frac{\sqrt{r_1+x_1+4}-\sqrt{r_2+x_1}+i\left(\sqrt{r_1-x_1-4}-\sqrt{r_2-x_1}\right)}{\sqrt{r_1+x_1+4}+\sqrt{r_2+x_1}+i\left(\sqrt{r_1-x_1-4}+\sqrt{r_2-x_1}\right)}\,, \tag{59}$$

where:

$$x_1 = \mathcal{R}e(x) > 0\,, \tag{60}$$

$$x_2 = \mathcal{I}m(x) > 0\,, \tag{61}$$

$$r_1 = \sqrt{(x_1+4)^2+x_2^2}\,, \tag{62}$$

$$r_2 = \sqrt{x_1^2+x_2^2}\,. \tag{63}$$

Therefore, we are in the situation in which we have to evaluate GHPLs with linear complex weights as functions of a complex variable ξ.

Let us consider a generic GHPL, $G(w_1, w_2, ..., w_n; x)$ in the case in which $w_i, x \in \mathbb{R}$. If no trailing zeroes are present, we can define a non-vanishing real parameter $\lambda \in \mathbb{R}$, such that the following scale invariance holds:

$$G(w_1, w_2, ..., w_n; x) = G(\lambda w_1, \lambda w_2, ..., \lambda w_n; \lambda x)\,. \tag{64}$$

The demonstration of Eq. (64) can be done by induction. It is trivially verified for $n = 1$ ($\lambda, w_1 \neq 0$). In fact:

$$G(\lambda w_1; \lambda x) = \int_0^{\lambda x} dt\, g(\lambda w_1; t)\,, \tag{65}$$

and moving to the new integration variable $r = t/\lambda$, we have:

$$G(\lambda w_1; \lambda x) = \int_0^{x} \lambda dr\, g(\lambda w_1; \lambda r) = \int_0^{x} dr\, g(w_1; r) = G(w_1; x)\,. \tag{66}$$

Let us suppose it is verified for $n = i$. For $n = i + 1$ we have:

$$\begin{aligned} G(\lambda w_{i+1}, \lambda \mathbf{w}; \lambda x) &= \int_0^{\lambda x} dt\, g(\lambda w_{i+1}; t) G(\lambda \mathbf{w}; t) = \int_0^{x} \lambda dr\, g(\lambda w_1; \lambda r) G(\lambda \mathbf{w}; \lambda r)\,, \\ &= \int_0^{x} dr\, g(w_1; r) G(\mathbf{w}; r) = G(w_{i+1}, \mathbf{w}; x)\,. \end{aligned} \tag{67}$$

Let us suppose, now, that $w_i, \lambda, x \in \mathbb{C}$. For the weight 1 we have (remember that we are considering the case in which $|\lambda|, |w_1| \neq 0$):

$$G(\lambda w_1; \lambda x) = \int_{0,\gamma}^{\lambda x} dz\, g(\lambda w_1; z) = \int_{0,\gamma}^{\lambda x} \frac{dz}{z - \lambda w_1}\,, \tag{68}$$

where γ is a path in the complex plane connecting the origin, $z = 0$, to the point $\lambda x = |\lambda||x|e^{i(arg(\lambda)+arg(x))} = |\lambda||x|e^{i(\Lambda+X)}$. If we rescale the integration variable by the real number $|\lambda||x|$, we have

$$G(\lambda w_1; \lambda x) = \int_{0,\gamma'}^{e^{i(\Lambda+X)}} \frac{dz'}{z' - \xi e^{i(\Lambda+W_1)}}\,, \tag{69}$$

where $\xi = |w_1|/|x|$ and $W_1 = arg(w_1)$. The path γ' connects the origin and the point on the circle of radius 1 with argument $(\Lambda + X)$. Let us define γ_1 the path along the radius from the origin to $e^{i(\Lambda+X)}$. $\Gamma = \gamma' - \gamma_1$ is a closed path that we suppose not to include the pole $z' = \xi e^{i(\Lambda+W_1)}$. The integral along the path Γ vanishes for the Cauchy's theorem. The integral over the radius can be rewritten as a one-dimensional integral of real variable with the substitution $t = z' \exp(-i(\Lambda + X))$. Therefore:

$$G(\lambda w_1; \lambda x) = \int_{0,\gamma_1}^{e^{i(\Lambda+X)}} \frac{dz'}{z' - \xi e^{i(\Lambda+W_1)}} = \int_0^1 \frac{dt}{t - \xi e^{i(W_1-X)}} = G(w_1/x; 1)\,. \tag{70}$$

On the other hand, we have also:

$$G(w_1; x) = \int_{0,\gamma_1}^{e^{i(X)}} \frac{dz'}{z' - \xi e^{i(W_1)}} = \int_0^1 \frac{dt}{t - \xi e^{i(W_1-X)}} = G(w_1/x; 1)\,, \tag{71}$$

thus,

$$G(\lambda w_1; \lambda x) = G(w_1; x)\,. \tag{72}$$

Note that, in the end, for our purposes, we can just use Eq. (71).

Let us suppose, now, that the rescaling is verified for $n = i$. For $n = i + 1$ we have:

$$\begin{aligned} G(\lambda w_{i+1}, \lambda \mathbf{w}; \lambda x) &= \int_{0,\gamma_1}^{\lambda x} dz\, g(\lambda w_{i+1}; z)\, G(\lambda \mathbf{w}; z) = \int_{0,\gamma_1}^{e^{i(\Lambda+X)}} \frac{dz'}{z' - \xi e^{i(\Lambda+W_{i+1})}} G(\lambda \mathbf{w}; |\lambda||x|z')\,, \\ &= \int_0^1 \frac{dt}{t - \xi e^{i(W_{i+1}-X)}}\, G(\lambda \mathbf{w}; \lambda x\, t)\,, \\ &= \int_0^1 \frac{dt}{t - \xi e^{i(W_{i+1}-X)}}\, G(\mathbf{w}/x; t) \ = G(w_{i+1}/x, \mathbf{w}/x; 1)\,. \end{aligned} \tag{73}$$

Choosing $\lambda = 1$ in Eq. (73), we can demonstrate that

$$G(w_{i+1}, \mathbf{w}; x) = G(w_{i+1}/x, \mathbf{w}/x; 1)\,, \tag{74}$$

and, therefore

$$G(\lambda w_{i+1}, \lambda \mathbf{w}; \lambda x) = G(w_{i+1}, \mathbf{w}; x)\,. \tag{75}$$

Using Eq. (74), we can employ the numerical routines provided in [19] for the evaluation of the GHPLs. In fact, now the GHPLs have complex weights (ratios of the original weights w_i and the variable x), but real variable[10], equal to 1.

Let us consider again our set $\{-r, -4, -1, 0\}$, and see what happens in the different regions. The analytic continuation from the non-physical $s < 0$ region to the physical

[10]Note that it is sufficient to divide by $e^{i\,arg(x)}$

region in which $p^2 \to -s - i\,0^+$, with $s > 0$, corresponds to the transformation $x \to -x'$, where, now, $x' \in \mathbb{C}$ and it is defined as follows:

$$x' = x'_1 + i\,x'_2 = \frac{s}{M^2} e^{i\phi}\,. \tag{76}$$

Correspondingly, the variable ξ becomes $\xi \to \zeta$, with $\zeta \in \mathbb{C}$ defined as follows:

$$\zeta = \frac{\sqrt{4-x'} - \sqrt{-x'}}{\sqrt{4-x'} + \sqrt{-x'}} = \frac{\sqrt{r'_1 - x'_1 + 4} - \sqrt{r_2 - x'_1} - i\left(\sqrt{r'_1 + x'_1 - 4} - \sqrt{r_2 + x_1}\right)}{\sqrt{r'_1 - x'_1 + 4} + \sqrt{r_2 - x'_1} - i\left(\sqrt{r'_1 + x'_1 - 4} + \sqrt{r_2 + x_1}\right)}\,, \tag{77}$$

where now

$$r'_1 = \sqrt{(4-x_1)^2 + x_2^2}\,. \tag{78}$$

Note that ζ does not have anymore modulus 1, as it was the case of real x shown in Eq. (35).

5 Generalizations and Additional Weights

In this section, we enlarge the set of possible weights in order to cover the GHPLs needed for the analytic expressions of the MIs in [11]. The goal is to be able to describe the following set:

$$\{-1-r, -r, -4, -1, 0, 1, 4, r, 1+r, c, \bar{c}\}\,, \tag{79}$$

where the additional weight functions (not introduced in the previous sections) are defined as follows[11]:

$$g(4;x) = \frac{1}{x-4}\,, \tag{80}$$

$$g(r;x) = \frac{1}{\sqrt{x(x-4)}}\,, \tag{81}$$

$$g(1+r;x) = \frac{1}{\sqrt{x(x-4)}(x-1)}\,, \tag{82}$$

$$g(-1-r;x) = \frac{1}{\sqrt{x(x+4)}(x+1)}\,. \tag{83}$$

The guidelines sketched in this section can be used for other, more complicated, sets.

It is first worth to notice that the possible weights listed in Eq. (79) do not appear all together at the same time. The appearance of a particular weight in a GHPL depends on the cut structure of the relative Feynman diagram. In the MIs presented in [11] we cannot have, for instance, the weights r and $-r$ at the same time in the same GHPL. The same happens for the pair $(c, \bar{c})$ with the square roots r or $-r$. Actually, the structure of the MIs in [11] is such that we are concerned effectively with three different subsets, that form each a closed base. They are:

$$\{-1, 0, 1, c, \bar{c}\}\,, \quad \{-1-r, -r, -4, -1, 0\}\,, \quad \{0, 1, 4, r, 1+r\}\,. \tag{84}$$

The three subsets do not mix with each other and they can be linearized (once and for all) using different variable transformations.

[11]Note the difference in sign in the definition of $g(4;x)$, $g(r;x)$, and $g(1+r;x)$ with respect to [11].

5.1 The set $\{-1, 0, 1, c, \bar{c}\}$

The GHPLs belonging to this set can be evaluated straightforwardly with the help of the routines in [19] without any further variable transformation. In the case in which the variable x is complex, we just have to use the scale invariance of the GHPLs, as explained in section 4.

5.2 The set $\{-1-r, -r, -4, -1, 0\}$

This set contains the weights treated in section 3, $\{-r, -4, -1, 0\}$, with a small enlargement due to the weight $(-1-r)$. Note that this enlargement is totally painless, since the new weight $(-1-r)$ transforms in the same set of linearized weights $\{-1, 0, 1, c, \bar{c}\}$. In fact,

$$
\begin{aligned}
G(-1-r, \mathbf{w}; x) &= \int_0^x dt\, g(-1-r; t)\, G(\mathbf{w}; t) = \int_0^x \frac{dt}{\sqrt{t(t+4)}(t+1)} G(\mathbf{w}; t)\,, \\
&= i\frac{\sqrt{3}}{3} \int_1^\xi d\eta \left(\frac{1}{\eta - c} - \frac{1}{\eta - \bar{c}} \right) G(\mathbf{w}; t(\eta))\,. \qquad (85)
\end{aligned}
$$

The GHPL $G(-1-r, \mathbf{w}; x)$, which is real for $x \geq 0$, is written as a difference of the two complex GHPLs: $G(c, ...; \eta)$ and $G(\bar{c}, ...; \eta)$. This difference is indeed complex, since the two GHPLs have the same real part but opposite imaginary parts. The factorized i in Eq. (85) makes in such a way that the combination is real.

5.3 The set $\{0, 1, 4, r, 1+r\}$

These positive weights cannot be linearized with the change of variable in Eq. (27). Instead, we must use the change of variable that was used in [23]:

$$
x = \frac{(1+\omega)^2}{\omega}\,, \quad \omega = \frac{\sqrt{x} - \sqrt{x-4}}{\sqrt{x} + \sqrt{x-4}}\,. \qquad (86)
$$

When x is positive and ranges from ∞ to 4, the corresponding variable ω ranges between 0 and 1. When $0 \leq x < 4$, ω becomes imaginary. Giving to x a negative vanishing imaginary part (anticipating the prescription for the continuation to the Minkowski region), we have:

$$
\omega = \frac{\sqrt{x} - \sqrt{x-4}}{\sqrt{x} + \sqrt{x-4}} \to \omega' = \frac{\sqrt{x} - \sqrt{x-4-i0^+}}{\sqrt{x} + \sqrt{x-4-i0^+}} = \frac{\sqrt{x} + i\sqrt{4-x}}{\sqrt{x} - i\sqrt{4-x}} = e^{i\,2\phi}\,, \qquad (87)
$$

where

$$
\phi = \arctan\sqrt{\frac{4-x}{x}}\,, \quad 0 \leq \phi < \frac{\pi}{2}\,. \qquad (88)
$$

Finally, when x becomes negative,

$$
x \to -x' - i0^+\,, \quad x' > 0\,, \qquad (89)
$$

we have

$$
\omega' \to \omega'' = \frac{\sqrt{x'+4} - \sqrt{x'}}{\sqrt{x'+4} + \sqrt{x'}}\,, \qquad (90)
$$

and ω'' ranges between 1 and 0 when x' ranges from 0 to ∞.

Moving from x to ω, the integration measure changes as follows:

$$\int_0^x dt = \int_{-1}^{\omega} \frac{(\eta+1)(\eta-1)}{\eta^2}\, d\eta \,. \tag{91}$$

Using Eq. (86) the old weight functions are transformed into:

$$\begin{aligned} g(0;t) &= \frac{1}{t} = \frac{\eta}{(\eta+1)^2}\,, & (92)\\ g(1;t) &= \frac{1}{t-1} = \frac{\eta}{(\eta+c)(\eta+\bar{c})}\,, & (93)\\ g(4;t) &= \frac{1}{t-4} = \frac{\eta}{(\eta-1)^2}\,, & (94)\\ g(r;t) &= \frac{1}{\sqrt{t(t-4)}} = -\frac{\eta}{(\eta+1)(\eta-1)}\,, & (95)\\ g(1+r;t) &= \frac{1}{\sqrt{t(t-4)}(t-1)} = -\frac{\eta^2}{(\eta+1)(\eta-1)(\eta+c)(\eta+\bar{c})}\,, & (96) \end{aligned}$$

where the complex numbers c and $\bar{c}$ were defined in section 3.

Combining Eq. (91) with Eqs. (92–96), we have the following transformation formulas for the definition of the GHPLs:

$$\begin{aligned} G(0,\mathbf{w};x) &= \int_0^x dt\, g(0;t)\, G(\mathbf{w};t) = \int_{-1}^{\omega} d\eta \left(-\frac{1}{\eta} + \frac{2}{\eta+1}\right) G(\mathbf{w};t(\eta))\,, & (97)\\ G(1,\mathbf{w};x) &= \int_0^x dt\, g(1;t)\, G(\mathbf{w};t) = \int_{-1}^{\omega} d\eta \left(-\frac{1}{\eta} + \frac{1}{\eta+c} + \frac{1}{\eta+\bar{c}}\right) G(\mathbf{w};t(\eta))\,, & (98)\\ G(4,\mathbf{w};x) &= \int_0^x dt\, g(4;t)\, G(\mathbf{w};t) = \int_{-1}^{\omega} d\eta \left(-\frac{1}{\eta} + \frac{2}{\eta-1}\right) G(\mathbf{w};t(\eta))\,, & (99)\\ G(r,\mathbf{w};x) &= \int_0^x dt\, g(r;t)\, G(\mathbf{w};t) = -\int_{-1}^{\omega} d\eta\, \frac{1}{\eta}\, G(\mathbf{w};t(\eta))\,, & (100)\\ G(1+r,\mathbf{w};x) &= \int_0^x dt\, g(1+r;t)\, G(\mathbf{w};t) = i\frac{\sqrt{3}}{3}\int_{-1}^{\omega} d\eta \left(\frac{1}{\eta+c} - \frac{1}{\eta+\bar{c}}\right) G(\mathbf{w};t(\eta)). & (101) \end{aligned}$$

The integration in η deserves a further discussion. As in the case already presented in section 3, the point $\eta = -1$ can be source of a non integrable singularity. However, the possible divergence in $\eta = -1$ is connected to the original point $x = 0$, and then, ultimately, to the right-most weight 0 in the GHPLs of x. It is sufficient, therefore, to extract the right-most trailing zeroes in x before the change of variable (86) is applied, using the shuffle algebra. The functions $G(\mathbf{0}_n; x) = 1/n!\, \log^n(x)$ can be directly transformed in the new variable ω using the relation $\log(x) = 2\log(\omega+1) - \log(\omega)$. The GHPLs that do not contain trailing zeroes in the right-most weights are regular in $\eta = -1$ after the variable transformation.

5.4 Mixed Weights

Although the weights belonging to the different sets described above do not mix in the expressions of the MIs of [11], we can further extend the analysis and try variable transformations that linearize wider sets of weights. This can be done provided that we do not mix the square roots with different signs. For instance, it can be shown that the weights belonging to the set $\{-1-r, -r, -4, -1, 0, 1, 4\}$ can be linearized at the same time, using the variable transformation in Eq. (27). Analogously, the set $\{-4, -1, 0, 1, 4, r, 1+r\}$ can be linearized with the help of the change of variable of Eq. (86).

6 Two-loop Light-Fermion contributions to the Higgs Production in Gluon Fusion

In this section, we revisit the calculation of the NLO light-fermion electroweak corrections to the Higgs boson production in gluon fusion.

In [21] these corrections were evaluated analytically, and the results were expressed in terms of GHPLs with square root in the weights. The numerical evaluation was done using real W and Z masses and with FORTRAN routines written *ad hoc*[12]. The electroweak corrections appear to be very peaked at $m_H \sim 2m_W$ and $m_H \sim 2m_Z$ because of the opening of the two corresponding thresholds. In this section we recompute the NLO-EW corrections using the VW routines employing complex values for the W and Z masses. As a result, the finite W and Z widths smear the peaks at the thresholds and resize the relative importance of the corrections in the region $m_H \sim 2m_W, 2m_Z$.

Neglecting QCD corrections, the partonic production cross section, up to two-loop level, has the following form:

$$\sigma(gg \to H) = \frac{G_F \alpha_S^2}{512\sqrt{2}\,\pi} \left|\mathcal{G}^{1l} + \alpha\, \mathcal{G}^{2l}_{EW}\right|^2 , \tag{102}$$

where G_F is the Fermi constant, α_S the strong coupling constant, and α the fine structure constant.

The lowest order, $\mathcal{G}^{1l}$, is due to one-loop diagrams with heavy quarks running in the loop. The dominant contribution comes from a loop of top, while the contribution of a b-quark loop is of the order of some percents of the previous one. The analytic expression of $\mathcal{G}^{1l}$ is:

$$\mathcal{G}^{1l} = \sum_{q=t,b} \frac{4}{x_q} \left[2 - \left(1 + \frac{4}{x_q} \right) G(-r, -r; x_q) \right] , \tag{103}$$

where $x_q = -m_H^2/m_q^2$, m_q is the heavy-quark mass (top or bottom mass), and the analytic continuation has to be taken considering a positive vanishing m_H imaginary part: $x_q \to -x'_q - i\,0^+$, where $x'_q = m_H^2/m_q^2$. $G(-r, -r; x_q)$ can be immediately transformed into a square logarithm of the variable ξ, defined in Eq. (27), as for instance in Eq. (111).

[12] In [24], the remaining electroweak corrections due to the top quark were calculated as a Taylor expansion in $m_H^2/(4m_W^2)$. Finally, in [25] a numerical calculation with complex W and Z masses was done for the complete set of NLO electroweak corrections.

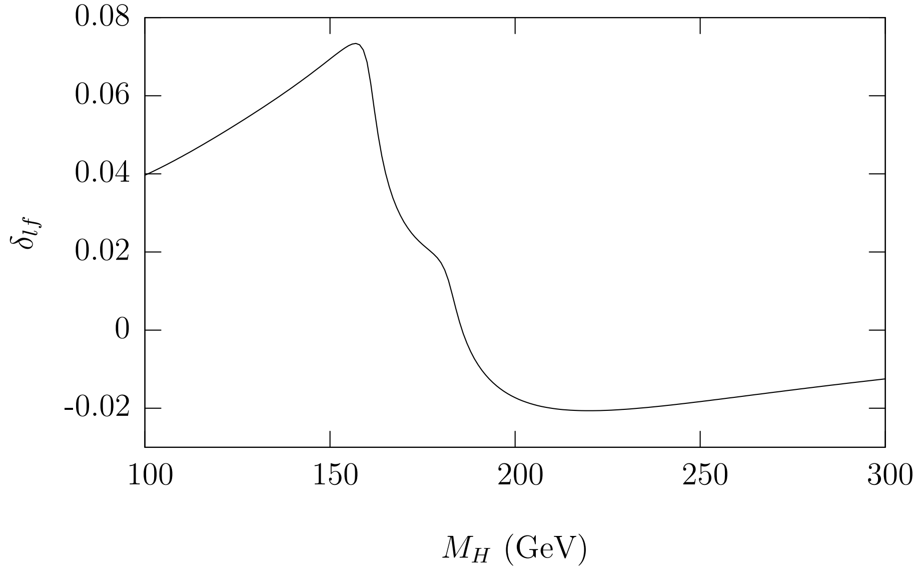

Figure 1: δ_{lf} as defined in Eq. (109).

The two-loop electroweak light-fermion contributions, $\mathcal{G}^{2l}_{lf}$, to $\mathcal{G}^{2l}_{EW}$ can be expressed as [21]:

$$\mathcal{G}^{2l}_{lf} = \frac{(m_W - i\Gamma_W/2)^2}{2\pi s^2\, m_H^2}\left[\frac{2}{c^4}\left(\frac{5}{4} - \frac{7}{3}\,s^2 + \frac{22}{9}s^4\right) A_1(x_Z) + 4\,A_1(x_W)\right], \tag{104}$$

where $s^2 = \sin^2\theta_W$, $c^2 = 1 - s^2$,

$$x_W = -\frac{m_H^2}{(m_W - i\Gamma_W/2)^2}\,, \quad x_Z = -\frac{m_H^2}{(m_Z - i\Gamma_Z/2)^2} \tag{105}$$

and

$$\begin{aligned} A_1(x) \;=\;& -4 + 2\left(1 + \frac{1}{x}\right) G(-1;x) + \frac{2}{x} G(0,-1;x) + 2\left(1 + \frac{3}{x}\right) G(0,0,-1;x) \\ & + \left(1 + \frac{2}{x}\right)\left[2G(0,-r,-r;x) - 3G(-r,-r,-1;x)\right] - \sqrt{x(x+4)}\Bigg\{\frac{2}{x} G(-r;x) \\ & + \frac{x+2}{x^2}\Big[2G(-r,-r,-r;x) + 2G(-r,0,-1;x) - 3G(-4,-r,-1;x)\Big]\Bigg\}. \end{aligned} \tag{106}$$

The GHPLs with square root in the weights involved in Eq. (106) are listed in section 3 and in appendix B.

Writing

$$\sigma(gg \to H) = \sigma_0\,(1 + \delta_{lf})\,, \tag{107}$$

where σ_0 is:

$$\sigma_0 = \frac{G_F\alpha_S^2}{512\sqrt{2}\,\pi}\,|\mathcal{G}^{1l}|^2\,, \qquad \mathcal{G}^{1l} = \mathcal{G}^{1l}_t + \mathcal{G}^{1l}_b\,, \tag{108}$$

m_H	δ_{lf}	m_H	δ_{lf}	m_H	δ_{lf}	m_H	δ_{lf}	m_H	δ_{lf}
110	0.04445	180	0.01723	250	-0.01830	320	-0.01047	390	-0.00462
120	0.04992	190	-0.00888	260	-0.01711	330	-0.00952	400	-0.00419
130	0.05585	200	-0.01729	270	-0.01590	340	-0.00860	410	-0.00382
140	0.06227	210	-0.02003	280	-0.01472	350	-0.00754	420	-0.00350
150	0.06939	220	-0.02063	290	-0.01358	360	-0.00655	430	-0.00322
160	0.06862	230	-0.02025	300	-0.01249	370	-0.00576	440	-0.00297
170	0.02764	240	-0.01939	310	-0.01145	380	-0.00514	450	-0.00275

Table 1: δ_{lf} as a function of the Higgs boson mass (m_H in GeV).

we have for δ_{lf}:

$$\delta_{lf} = \frac{2\alpha}{|\mathcal{G}^{1l}|^2} \left[\mathcal{R}e(\mathcal{G}^{1l})\, \mathcal{R}e(\mathcal{G}^{2l}_{lf}) + \mathcal{I}m(\mathcal{G}^{1l})\, \mathcal{I}m(\mathcal{G}^{2l}_{lf}) \right] \, . \tag{109}$$

In Fig. 1 we plot δ_{lf} computed with the VW routines and some numerical values are collected in Table 1. The set of parameters used is the following:

$$\begin{aligned} & m_t = 173.1\,\text{GeV}\,, \quad m_b = 4.6\,\text{GeV}\,, \quad m_W = 80.398\,\text{GeV}\,, \quad \Gamma_W = 2.141\,\text{GeV}\,, \\ & m_Z = 91.1876\,\text{GeV}\,, \quad \Gamma_Z = 2.4952\,\text{GeV}\,, \\ & \alpha = 1/128\,, \quad G_F = 1.16637 \cdot 10^{-5}\,\text{GeV}^{-2}\,, \quad \sin^2\theta_W = 0.23149\,. \end{aligned} \tag{110}$$

7 Conclusions

In this paper we analyzed the set of GHPLs of a single variable containing square roots in the weights. After recalling the definition and basic properties of the HPLs, we introduced the GHPLs with weights belonging to the set $\{-1-r, -r, -4, -1, 0, 1, 4, r, 1+r, c, \bar{c}\}$. This specific set of GHPLs appears in the analytic expressions of the MIs that enter into the calculation of the electroweak form factor [20, 11].

One of the main observations of the paper lies in the fact that, once the weights are allowed to be complex, the GHPLs with square roots in the weights can be "linearized", *i.e.* expressed as a combination of GHPLs with linear weights. These linearized GHPLs are functions of a transformed variable, that is not unique, but can be properly chosen depending on the nature of the weights. The set $\{-1-r, -r, -4, -1, 0, 1, 4, r, 1+r, c, \bar{c}\}$ can be linearized, once and for all, with just two variable transformations.

The other observation concerns the possibility of a fast and precise numerical evaluation of the linearized GHPLs using already existing numerical routines. In particular, the `C++/GiNaC` routines by Vollinga and Weinzierl [19] offer a well suited tool for this goal.

Finally, the strategy for the numerical evaluation of GHPL presented in the paper is applied to the known case of electroweak light-fermion NLO corrections to the Higgs production in gluon fusion. We evaluate the GHPLs with square roots using the VW numerical routines. As a further refinement, while in [21] the corrections were evaluated neglecting the effects of the W and Z widths, we consider here the case of complex m_W and m_Z, getting a more realistic result. It is worth to notice that the GHPLs with square roots allow for a very compact analytic expression of the results, which would be extremely lengthy if expressed in terms of the linearized GHPLs.

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A Some Examples at Weight 2

As simple examples, in this appendix we apply the procedure outlined in the paper to the following GHPLs at weight 2: $G(-r,-r;x)$ and $G(-r,-1;x)$. We find:

$$\begin{aligned} G(-r,-r;x) &= \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(-r;t) = -\int_1^\xi \frac{d\eta}{\eta} \left[-G(0;\eta)\right] , \\ &= \frac{1}{2} G(0;\xi)^2 , \end{aligned} \tag{111}$$

$$\begin{aligned} G(-r,-1;x) &= \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(-1;t) = -\int_1^\xi \frac{d\eta}{\eta} \left[G(c;\eta) + G(\bar{c};\eta) - G(0;\eta)\right] , \\ &= -\frac{1}{3}\zeta(2) - G(0,c;\xi) - G(0,\bar{c};\xi) + \frac{1}{2} G(0;\xi)^2 . \end{aligned} \tag{112}$$

B Some Examples at Weight 3

In this appendix, we provide the expressions for the 5 GHPLs at weight 3 containing square roots in the weights, involved in the corrections of [21]. We have:

$$\begin{aligned} G(-r,-r,-r;x) &= \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(-r,-r;t) = -\int_1^\xi \frac{d\eta}{\eta} \frac{1}{2} G(0;\eta)^2 , \\ &= -\frac{1}{6} G(0;\xi)^3 , \end{aligned} \tag{113}$$

$$\begin{aligned} G(0,-r,-r;x) &= \int_0^x \frac{dt}{t} G(-r,-r;t) = \int_1^\xi d\eta \left(-\frac{1}{\eta} + \frac{2}{\eta - 1}\right) \frac{1}{2} G(0;\eta)^2 , \\ &= 2\zeta(3) - \frac{1}{6} G(0;\xi)^3 + G(0;\xi)^2 G(1;\xi) + 2G(0,0,1;\xi) \\ &\quad -2G(0;\xi)G(0,1;\xi) , \end{aligned} \tag{114}$$

$$G(-r,0,-1;x) = \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(0,-1;t) ,$$

$$
\begin{aligned}
&= -\int_1^{\xi}\frac{d\eta}{\eta}\Big[\zeta(2) - G(0,\bar{c};\eta) - G(0,c;\eta) + \frac{1}{2}G(0;\eta)^2 \\
&\quad -2G(1;\eta)G(0;\eta) + 2G(0,1;\eta) + 2G(1,\bar{c};\eta) + 2G(1,c;\eta)\Big]\,, \\
&= -\frac{10}{3}\zeta(3) + 2K_1 - \zeta(2)G(0;\xi) - \frac{1}{6}G(0;\xi)^3 + G(0,0,\bar{c};\xi) \\
&\quad +G(0,0,c;\xi) - 2G(0,1,\bar{c};\xi) - 2G(0,1,c;\xi) \\
&\quad +2G(0;\xi)G(0,1;\xi) - 2G(0,0,1;\xi)\,, \qquad (115)
\end{aligned}
$$

$$
\begin{aligned}
G(-r,-r,-1;x) &= \int_0^x \frac{dt}{\sqrt{t(t+4)}}G(-r,-1;t)\,, \\
&= \int_1^{\xi}\frac{d\eta}{\eta}\left[\frac{1}{3}\zeta(2) + G(0,c;\eta) + G(0,\bar{c};\eta) - G(0,0;\eta)\right], \\
&= \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2)G(0;\xi) - \frac{1}{6}G(0;\xi)^3 + G(0,0,c;\xi) \\
&\quad +G(0,0,\bar{c};\xi)\,, \qquad (116)
\end{aligned}
$$

$$
\begin{aligned}
G(-4,-r,-1;x) &= \int_0^x \frac{dt}{t+4}G(-r,-1;t)\,, \\
&= \int_1^{\xi} d\eta\left(\frac{1}{\eta} - \frac{2}{\eta+1}\right)\left[\frac{1}{3}\zeta(2) + G(0,c;\eta) + G(0,\bar{c};\eta) - G(0,0;\eta)\right], \\
&= -\frac{5}{6}\zeta(3) - 2K_2 - \frac{2}{3}\zeta(2)G(-1;\xi) - 2G(-1;\xi)G(0,\bar{c};\xi) \\
&\quad -2G(-1;\xi)G(0,c;\xi) + G(-1;\xi)G(0;\xi)^2 + 2G(0,\bar{c},-1;\xi) \\
&\quad +2G(0,c,-1;\xi) + \frac{1}{3}\zeta(2)G(0;\xi) - \frac{1}{6}G(0;\xi)^3 + 2G(0,-1,\bar{c};\xi) \\
&\quad +2G(0,-1,c;\xi) - 2G(0;\xi)G(0,-1;\xi) + G(0,0,\bar{c};\xi) \\
&\quad +G(0,0,c;\xi) + 2G(0,0,-1;\xi)\,. \qquad (117)
\end{aligned}
$$

In the formulas above, we introduced the two constants K_1 and K_2. They have a cumbersome expression in terms of known transcendental constants, that we omit here. Their numerical value is known with infinite precision and it is:

$$
\begin{aligned}
K_1 &= 0.27842507663972774844197359081 4..\,, \qquad (118) \\
K_2 &= -0.15222624822760754658910077827 8..\,. \qquad (119)
\end{aligned}
$$

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