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Effective Stability of Hamiltonian Planetary Systems

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Ai miei genitori
e a Silvia.

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Introduction

"The whole of science is nothing more than the refinement of everyday thinking."
— Albert Einstein

1.1 State of the Art

The thesis is concerned with the old fashioned problem of studying the dynamics of the Solar System, paying particular attention to the problem of the stability of the planetary motions. Leaving for the technical part of the thesis the discussion about the exact meaning of the word "stability" when applied to the planets, we may ask, in rough terms: "Will the orbits of the planets remain essentially unchanged forever or, at least, for a time comparable with the age of the Universe? May a collision between two planets, or falling of a planet onto the Sun, or ejection of a planet occur?".

The answer to such questions is still unknown. The present thesis is devoted to a study of this problem in the light of two recent theorems, the first one is due to Kolmogorov (1954), the second one to Nekhoroshev (1978).

Since antiquity, the Solar System has fascinated humanity, both due to the intrinsic beauty of the sky and to the evident impact of astronomical phenomena on the common life, in particular on agriculture and navigation. It was soon observed that among the stars that appear to be fixed on a sphere surrounding the Earth, there were also a few wandering stars then called planets (i.e., wandering bodies). Moreover, strange objects like comets did appear from time to time, and solar and lunar eclipses were observed. The first efforts made by astronomers in order to predict the celestial phenomena introduced very clever geometrical methods. The underlying idea was that the planetary motions exhibit some periods that can be experimentally determined by accumulating enough observations. This kind of motion is known today as a quasi-periodic one. Among the beautiful geometric tools we may quote eccentrics, equates and epicycles, the latter representing the classical version of the Fourier series. The ancient astronomy, in particular the Greek one, has been remarkably successful in predicting the motions of the planets and the eclipses. The methods based on epicycles expansion represented the main tools of astronomy up to Copernicus and Kepler.

In 1609 Kepler, following Copernicus who had resurrected an old idea of Anassimandros, placed the Sun at the center of the Universe and, trying to reconstruct the orbit of Mars using the observations of Tycho Brahe, showed that the planets described ellipses around the Sun and stated the first two laws — the elliptic form of the orbit and the law of the areas. Ten years later, in 1618, he completed his work by adding the third law, namely the proportionality between the cube of the semi-major axis of the orbit and the periods of revolution of a planet — thus concluding his twenty years

long research on the harmony of the Universe. In this model, after every revolution, each planet returns at the starting point and so retraces the same ellipse. This idea of a perfectly stable Solar System in which all orbits were periodic would not remain unchanged for long. Actually, it was Kepler himself who realized that there are slow deviations from the elliptic motion. By comparing his calculations with ancient reports he remarked that the positions of the two biggest planets exhibit systematic deviations that had become observable after one and half a century only: Jupiter appeared to approach the Sun, while Saturn appeared to recede from it. Thus, he concluded that *secular motions* — Kepler named them so — should be added to the elliptic one, in some sense attempting at introducing new periods, but he was unable to identify the actual corrections. By the way, the deviations observed by him are mainly due to what is now called the *great inequality* of Jupiter and Saturn, which has a period of about 920 years. The first attempt to calculate the secular variations on an empirical basis is due to Halley around 1700. He just made a linear interpolation, adding a linear term to the mean motion. Such a solution is clearly unsatisfactory, because a straightforward calculation based on his corrections leads to the conclusion that about two millions of years ago Jupiter and Saturn were on the same orbit. However it was enough for practical purposes.

In 1687 Newton announced the law of universal gravitation. Kepler results were recovered by restricting this law to the interactions of planets with the Sun alone, but Newton's law includes all the interactions: Jupiter is attracted by the Sun, as Saturn, but also Jupiter and Saturn attract each other, and the same for the other planets and in general for all celestial bodies. There are no more reasons to assume that the orbits of the planets are fixed invariant ellipses, and Kepler beautiful regularity is destroyed. Newton was well aware of this fact, and indeed in his treatise on optics he claims that the motion of the planets can continue for long time obeying the law of gravitation, but it may happen that the orbits change considerably, and the action of God may be necessary in order to restore the initial configuration of the Solar System.

The problem whether the Newton theory may be able to account for the observed deviations from the elliptic motion has been more and more emphasized during the first half of the XVIII century, in particular by Lalande: the considerable accumulation of observations and the improvement of the precision had made evident not only that the problem raised by Kepler was a real one, but also that the empirical determination of the secular terms introduced by Halley was not sufficient. The Academy of France

announced some prizes for solving this problem, and this marked the beginning of perturbation theory with the works of Euler, Lagrange, and Laplace.

We do not enter the long development of the investigations concerned with the stability of the planetary system. We just recall that the common idea that the motions of the planets are an excellent example of perfect order has been questioned after the understanding of the effect of the resonances and of the role of the so called *small divisors*, and in particular by the discovery of the existence of chaotic orbits in the problem of three bodies, due to Poincaré.

The stability problem for the planetary motions was in some sense resurrected by the announcement by Kolmogorov, in 1954, of the theorem on persistence under small perturbations of quasi-periodic motions on invariant tori. If this is the case for the Solar System, then the motion of the planets can be described essentially with the method of epicycles (in the modern form of Fourier series). The more relevant differences are: (a) the number of frequencies was unknown to the Greek astronomers up to Copernicus, and new frequencies had to be introduced in order to make the calculus to agree with observations, while in Kolmogorov theory the number of basic frequencies for a system with N planets is fixed to be $3N$, and (b) the actual frequencies can be *calculated* as integer combination of the $3N$ basic frequencies on the basis of Newton's law of gravitation (possibly including relativistic corrections).

The announcement of Kolmogorov was immediately considered as the solution of the problem of stability for the planetary system, at least in probabilistic sense: the motion is predicted to be quasi-periodic for the majority of the initial conditions, with the exception of a set of small measure in phase space. However, the statement of the theorem contains the condition that the perturbation should be small enough, and it was soon realized that one should prove that this is the case for our Solar System, for which the actual perturbation is roughly evaluated as the ratio between the masses of Jupiter and the Sun, i.e., about 10^{-3}. A rough estimate, made by Hénon on the basis of Moser and Arnold proofs of the theorem of Kolmogorov, lead to the conclusion that Jupiter should be smaller than a proton! Thus, hard work must be done in order to prove that the theorem applies at least to a model of the planetary system that includes the biggest planets.

A different approach was suggested by Littlewood in 1959 for the case of the Lagrangian triangular equilibria of the restricted problem of three bodies, and stated in a more general form by Nekhoroshev in 1979. In the latter work the concept of *ex-*

ponential stability is introduced: one renounces to perpetual invariance of the orbits, just trying to prove that the critical orbital elements of the planets (i.e., semi-major axes, eccentricities and inclinations) remain close to the current ones for a time which grows as the exponential of a power of the inverse of the perturbation, and so it may be much larger than the lifetime of the Solar System itself, e.g., the estimated age of the Universe.

Besides the analytical results connected with the theorems of Kolmogorov and Nekhoroshev, the availability of fast computers has opened the possibility of investigating the planetary dynamics by numerically simulating the evolution of the system. The first long-time integrations are due to Carpino, Milani and Nobili [7][8], who could integrate the Newton equations for an interval of 200 millions of years.

The considerable increase of the computational power allows now to reach integration times larger that the estimated age of the Universe. Long term calculations have been worked out by some authors, e.g., by Sussman and Wisdom[85], by Laskar[44][47] and more recently by Hayes[86][87]. Among the results given by such long term integration the actual existence of chaotic motions in the Solar System is the relevant one for the purpose of the present thesis. According to the numerical indications, one may hope to be able to apply the Kolmogorov and Nekhoroshev theorems to the problem of three bodies including the Sun, Jupiter and Saturn.

Before proceeding, let me remark that the word "stability" has been used here without giving it a precise meaning. I just mentioned that, in rough terms, it means that the semi-major axes, the eccentricities and the inclinations of the planetary orbits should not change too much. Actually, different definitions have been used in the past. Later on I will introduce the concepts of *stability time* and *escape time* in a precise way, to be used in formal statements.

1.2 Contributions

The present thesis is devoted to the study of three main problems, namely:

(i) the applicability of Kolmogorov and Nekhoroshev theories to the problem of three bodies (see [25]);

(ii) the stability of the secular evolution of the planar Sun–Jupiter–Saturn–Uranus system (see [79]);

(iii) the explicit construction of the normal form for elliptic tori in planetary systems (paper in preparation).

This work gives an original contribution on the methods for studying the stability of planetary systems. It contains an analytical contribution, namely the proof of the existence of lower dimensional elliptic tori for the planetary systems obtained via an algorithmic constructive method, and the explicit calculation of the stability time for the Sun–Jupiter–Saturn system and for the planar secular Sun–Jupiter–Saturn–Uranus system. We emphasize that we obtain the first realistic estimates for these problems based on a well established theoretical framework.

The thesis work has been initiated in the framework of a research project on the stability of planetary systems, in collaboration with A. Giorgilli and U. Locatelli. I include here the results that I have obtained in the last three years, during my PhD work. We revisit the problem of the Solar System stability in the light of Kolmogorov and Nekhoroshev theorems with the aim of proving that they apply at least to realistic approximations of some part of our Solar System.

The celebrated theorem of Kolmogorov[37], announced in 1954, is concerned with the dynamics of nearly integrable Hamiltonian systems. The claim is that if the perturbation is small enough, then there exists a set of invariant tori of large measure. Let us recall that the smallness condition on the perturbation strongly depends on the non-degenerate properties of the unperturbed Hamiltonian.

The relevance of that result for the problem of the Solar System stability was pointed out by Kolmogorov himself, and later emphasized in the subsequent papers of Moser[63] and Arnold[1]. The three papers mentioned above marked the beginning of the so called KAM theory.

However the actual applicability of Kolmogorov's theorem to a planetary system encounters two major difficulties, namely: (i) the degeneracy of the Keplerian motion, and (ii) the extremely restrictive assumptions on the smallness of the perturbation.

The former difficulty is related to the elliptic form of the Keplerian orbits. Indeed a system including a central body (a star) and $n > 1$ planets, after the elimination of the known first integrals, has $3n - 2$ degrees of freedom, while only n actions appear in the Keplerian part of the Hamiltonian. The way out proposed by Arnold, and inspired by the approach of Lagrange and Laplace, was to introduce in the proof two separate time-scales for the orbital motion and for the secular evolution of the perihelia and of the nodes (see [2] and its recent extension in [15]). Such an approach has been successfully

extended to the $(n+1)$-body planetary systems thanks to the work done by Herman and Féjoz[20].

About the second difficulty, it was soon remarked by Hénon that the application of the Kolmogorov's theorem to the planetary motions is not straightforward, due to the condition that the masses of the planets should be small enough. Indeed, as mentioned on the first section, very crude estimates based on the proofs of Arnold and Moser gave exceedingly small values; e.g., the masses of the planets should not exceed that of a proton.

The reason for such unrealistic results — and even ridiculous — lies in the analytical methods. The classical perturbation methods are based on series expansions the aim of which is to calculate the quasi-periodic corrections of the semi-major axes, eccentricities and inclinations of the planets due to the mutual gravitational interaction. Essentially, what one does is to approximate the true orbit of the planets by adding epicycles to the orbital elements. Only a few of these terms have been calculated by hand by astronomers, and have produced spectacular results such as calculating the secular motions of the perihelia and the nodes of the planets and predicting the position of the unknown planet Neptune. However, the question is whether the *complete* expansion, which includes of course infinitely many terms, gives a convergent series or not. Such a calculation can not be performed by hand, of course. The proof of, e.g., the theorem of Kolmogorov is based on a sequence of quantitative estimates of the corrections produced by perturbation expansions, followed by the proof that such a sequence converges to an analytic function. However, the analytical estimates are certainly pessimistic, because one must always consider the worst situation, without being able to profit of algebraic cancellations that often occur. Attacking the second difficulty, concerned with the unrealistic requirements on the smallness of masses, eccentricities and inclinations of the planets, with purely analytical methods seems to be an hopeless task for such a complicated model as the planetary system.

A way out is offered by computer algebra. Using the power of computers we can explicitly perform some perturbation steps, thus calculating a huge number of terms (e.g., several millions). This allows us to know explicitly the largest part of the perturbation, without approximations — apart from round-off errors that may be controlled by implementing interval arithmetic. Thus, still referring to the case of a Kolmogorov torus, one gets an approximation which is good enough to allow us to apply analytical methods to the remaining part, which is hopefully very small. Such

an approach allowed some authors to rigorously prove the existence of KAM tori for some interesting problems in celestial mechanics (see, e.g., [11], [13], [14], [56] and [22]). However, all these works consider models having just two degrees of freedom. This because increasing the number of independent variables makes the explicit calculation of perturbation steps a big challenge, due to the dramatic increase of the number of coefficients to be calculated, so that obtaining a sufficiently good initial approximation of an invariant torus reveals to be a hard task. For what concerns problems with more than two degrees of freedom, in a few cases only the availability of an algorithmic version of Kolmogorov's theorem (see [3], [29] and [30]) allowed us to obtain a good approximation of the invariant tori, although this approach is not yet sufficient for a fully rigorous application of the theory. For instance, the constructed solution on a KAM torus has been successfully compared with the real motion of the Sun–Jupiter–Saturn system, which can be represented by a model with 4 degrees of freedom (see [58] for all the details). The first original contribution of this thesis (see [25]) focuses on the long time stability in a neighborhood of such KAM torus.

On the other hand, as I have already mentioned, numerical integrations of the full Solar System over a time span of billions of years have shown that the orbits of the inner planets exhibit a chaotic evolution. These facts are incompatible with the existence of Kolmogorov invariant tori.

A different and more physical approach was suggested by Moser and Littlewood, and fully stated by Nekhoroshev[68][69], with his celebrated theorem on exponential stability. According to this theorem the time evolution of the actions of the system remains bounded for a time exponentially increasing with a power of the inverse of the perturbation parameter. Thus, although the possibility of a chaotic motion is not excluded, nevertheless a dramatic change of the orbits should not occur for a very long time, and it may be conjectured that such a time exceeds the age of the Solar System itself. But also in this case the problem of the applicability of the theorem still persists, since the analytic estimates give again ridiculous values for the size of the masses of the planets.

In the last years, also the estimates for the applicability of Nekhoroshev's theorem to realistic models of some part of the Solar System have been improved by some authors; for example, concerning the stability of the Trojan asteroids in the vicinity of the triangular Lagrangian points see, e.g., [24], [80], [10] and [27].

The papers above are concerned with the study of the stability in the neighborhood

of an elliptic equilibrium. This may be considered as the simplest case to be handled via algebraic manipulation, since one has to deal only with polynomials. The application to the case of the planetary system is a major step, since it requires both an analytical study and the implementation of algebraic manipulation for more complex expressions, including, e.g., Fourier series. The analytic part consists in finding a good first approximation of an invariant object, typically an invariant torus, which is suitable for the study of stability in its neighborhood. The corresponding algorithms must be justified by giving analytical proofs of convergence, and then translated into a computer program which performs the expansions.

In the rest of this chapter I provide an informal discussion of the original methods and results of the thesis.

1.2.1 Kolmogorov and Nekhoroshev Theories for the Problem of Three Bodies

Let us consider the general problem of three bodies paying particular attention to the very interesting case of the Sun–Jupiter–Saturn system (hereafter, SJS for shortness), with real values of the orbital parameters and of the initial data. The starting point of my work is a previous paper by Locatelli and Giorgilli[58], where the existence of a Kolmogorov invariant torus is established for the SJS system, close to the actual orbit of the planets.

The result of Locatelli and Giorgilli gives a positive answer to the question whether the theorem of Kolmogorov applies at least to the two biggest planets, but does not solve the question of stability, because the unavoidable error in determining the initial conditions does not allow to assure that the actual orbit lies on the torus. Thus we exploit a previous result by Morbidelli and Giorgilli[60] on the Nekhoroshev stability in the neighborhood of an invariant torus. Our goal is to prove that the orbits of the bigger planets will not significantly change for a time as long as the estimated age of the Universe, e.g., about 10^9 periods of Jupiter.

In order to establish the stability of the system we will perform three steps. (i) We find a good approximation of the invariant torus for the SJS system, by explicitly constructing the corresponding Kolmogorov normal form; this gives the invariant object which should be proved to be stable in Nekhoroshev sense. (ii) We transform the previous Hamiltonian so to lead it in Birkhoff normal form up to a finite order; in view of Birkhoff theory on complete stability, the normal form furnishes an estimate of the stability time. (iii) We optimize the estimated stability time with respect to the initial

conditions and to the normalization order; this is the step inspired by Nekhoroshev theory on exponential stability.

We use an explicit expansion of the normal forms of Kolmogorov, in order to approximate the invariant torus, and of Birkhoff, in order to evaluate the stability time. To this end we proceed in two steps: first, we produce a constructive algorithm for normal forms; then we use algebraic manipulations on a computer in order to perform the actual expansions. A significant part of the work has been devoted to implementing the corresponding programs in the C programming language.

We give a few details about the procedure. Using the method of Locatelli and Giorgilli (see, e.g., [58]) we calculate the expansion of the Kolmogorov normal form up to a quite high order. Next, we proceed with the calculation of the stability in Nekhoroshev sense of orbits with initial conditions in the neighborhood of the torus. The computational complexity here is a major one; thus we must introduce some strong truncations in the expansion of the Hamiltonian. This allows us to get an estimated stability time comparable with the age of the Universe in a neighborhood of the torus big enough to contain the actual initial data of the SJS system.

The natural question is whether such results will remain valid if we add more and more terms in the Hamiltonian, thus making our approximation better. Answering such a question is presently beyond our limits, but in our opinion deserves to be investigated. Some improvements may be obtained by using more computational power, e.g., by performing our calculations on a cluster of computers but, at the present time, our algebraic manipulator does not include any parallelization capability. We plan to include a parallelization library using an MPI implementation and some GPU computing techniques in a near future. However, substantial improvements also require a further refinement of our analytical techniques in order to be able to evaluate the error induced by our truncations, thus allowing us to introduce better computational schemes and to evaluate the reliability of our approximations. An analytic result in this direction is given in the third part of the thesis.

1.2.2 The Stability of the Secular Evolution of the Planar SJSU System

The second step would be to consider the system of the so called Jovian planets, i.e. Jupiter, Saturn, Uranus and Neptune, hereafter referred to as the SJSUN problem. However, there are additional difficulties. The first one has a technical character: the computational power required exceeds our current limits. The second problem is that

the applicability of Kolmogorov theory to the four major bodies of our Solar System is doubtful. Indeed, the motion of such planetary subsystem has been shown by Sussman and Wisdom[85] to be exhibit a chaotic behavior, although very small. Murray and Holman[66] provided an enlightening explanation of this phenomenon. However, let us also recall that no chaotic motions are detected in the planar system including the Sun, Jupiter, Saturn and Uranus (hereafter, SJSU for shortness).

Moreover in two more recent works, Hayes[86][87] shows that the scenario is much more complicated and challenging. Indeed, calculating the Lyapunov exponent, Hayes numerically shows that the system of the Sun and the Jovian planets, integrated for 200 Myr as an isolated five body system using many sets of initial conditions all within the uncertainty bounds of their currently known positions, can display both chaos and near-integrability. The relative difference in the initial conditions is less than 10^{-8}, smaller than the observational error.

Let us remark that even if there are narrow regions with positive Lyapunov exponents, the diffusive effects could be very tiny due to the relationship between the Lyapunov times and the macroscopic diffusion times of Hamiltonian systems (see, e.g., [61]).

Our aim is to investigate whether the SJSU system may remain close to its current conditions for a time that exceeds the lifetime of the system itself; e.g., in our case the age of the Universe, which is estimated to be approximately 1.4×10^{10} years, could be enough. In the SJSUN system, we expect that a combination of both the KAM and the Nekhoroshev theories could prove that the motion remains close to an invariant torus for a very long time (see [60] and [25]).

In our work, in order to avoid the huge computational difficulties connected with the expansion of the Hamiltonian, we restrict our attention to the SJSU planar system. Indeed, a rather long preliminary work is necessary in order to give the Hamiltonian a convenient form for starting more standard perturbation methods (see [56], [57] and [58]). Furthermore, following the lines of Lagrange theory, we focus only on the secular part of the Hamiltonian.

Our calculation does not yet allow us to prove the stability for a time comparable to the age of our planetary system, even restricting ourselves to consider just the secular part of a planar approximation including the Sun, Jupiter, Saturn and Uranus, we get a stability time of about 10^8 years. Nevertheless, we think that our result is meaningful in that it indicates that the phenomenon of exponential stability in Nekhoroshev sense

may play an effective role for the Solar System stability, at least for the biggest planets. Indeed, our result is not dramatically far from the goal of proving stability over the age of the Solar System: we can ensure the stability for such a long time in a domain of radius 0.7 times the radius containing the real initial conditions for the planets. By the way, it may be worth to note that a similar result, with the same value of the radius, has been found in [26] where the spatial problem for the Sun–Jupiter–Saturn system is considered. We remark that such a restriction is not dramatic, especially in comparison with similar works where substantial changes of the orbital parameters and of the masses were introduced.

In the framework provided by the Nekhoroshev's theorem, the present work describes the first attempt to study the stability of a realistic model with more than two planets of our Solar System. Our results suggest that a better approximation of the true orbit could help a lot. This can be obtained, e.g., by following the procedure described in the previous paragraph for the SJS system, namely by first establishing the existence of a KAM torus close to the initial conditions of the planets, and then proving the stability in Nekhoroshev sense in a neighborhood of the torus that contains the initial data. However, as we have already remarked, this requires a computational power not yet available to us. A different method, which is a natural extension of Lagrange theory, may be based on a better approximation of the averaged planetary problem (i.e., the problem with zero inclinations and eccentricities), as we describe in the next subsection.

1.2.3 Algorithmic Construction of Elliptic Tori in Planetary Systems

In its original formulation the theorem of Kolmogorov applies to quasi-integrable *non-degenerate* Hamiltonian systems. This is not the case for the planetary problem, due to the degeneration of Kepler motions on elliptic obits with n frequencies (the mean motions). The extension of the theory to the planetary case, or more generally to properly degenerate systems, is due to Arnold, and is essentially based on a non linear reformulation of Lagrange and Laplace theory on the secular motions of the perihelia and the nodes of the planets, which introduces the $2n$ frequencies that are missing in Kepler approximation.

However, it may be expected that in the limit case of small circular orbits also n-dimensional invariant tori should exists. The proof of such a statement requires new ideas and analytical tools. The case of the planetary system has been recently treated by Biasco, Chierchia and Valdinoci for the spatial three-body planetary problem and

for a planar system with a central star and n planets (see [4] and [5], respectively). Their deep theoretical approach is related to a theorem due to Pöschel[74] assuring the existence of lower dimensional elliptic tori. However, their method is not suitable for explicit applications, even if one is just interested in finding the location of the elliptic invariant tori. In particular, the method is not constructive, which makes it unsuitable for an explicit implementation.

Our remark is that the original proof scheme invented by Kolmogorov is in a much better position both concerning the construction of full dimensional invariant tori (see [37], [3] and [58]) and the formulation of a constructive algorithm for elliptic tori, which is one of the subjects of the present thesis. Moreover, our algorithmic construction allows us to explicitly calculate the dynamics on the invariant surfaces, by using a procedure that we can characterize as semi-analytic.

Our work on this subject contains a pure theoretical part and an actual application to the planetary problem. On the one hand we construct a suitable normal form for lower dimensional elliptic tori through an explicit algorithm and give a formal proof of the convergence of the normal form so produced, thus proving the existence of such tori. On the other hand we check the effectiveness of our semi-analytic procedure for the case of a planetary system. To this end we calculate a finite number of steps of the algorithm via algebraic manipulation, thus checking that it can be effectively applied.

Let us remark that one of the main technical difficulties, that we do not discuss in this thesis, is the development of an efficient algebraic manipulator. In fact, despite the widespread availability of commercial general-purpose packages, scientists have often preferred to develop and employ specialized *ad-hoc* software. The reason of this choice lies mainly in the higher performance that can be obtained with them. The performance increase concerns both the computing time, with a speedup factor that may be bigger than thousand, and the amount of physical memory required, which is dramatically decreased by implementing specialized methods of storing of the coefficient which make the handling of the huge expansions needed in celestial mechanics possible.

We emphasize that all the calculations made in this thesis were done with the aid of a self-made algebraic manipulator written in C language that is specifically designed for the manipulation of power series expansions, i.e. polynomial type, and trigonometric expansions, i.e. Fourier series, the functions commonly used in perturbation methods of celestial mechanics. The implementation was started several years ago by A. Giorgilli,

and has then been continued by U. Locatelli an by the author of the present thesis in order to add more an more functionality.

2

Kolmogorov and Nekhoroshev Theories for the Three–Body Problem

"Le uniche cose serie, sono le serie!"
— Antonio Giorgilli

In this chapter we study the stability, in Nekhoroshev sense, of the neighborhood of an invariant torus for the Sun–Jupiter–Saturn system. The aim is to give evidence, with help of a computer-assisted calculation, that the size of the neighborhood of the invariant torus, for which the exponential stability holds for a time interval as long as the age of the Universe, is big enough to contain the *actual initial data* of Jupiter and Saturn.

Achieving such an ambitious program requires on the one hand the development of refined perturbation methods to be implemented with the help of algebraic manipulation on a computer and, on the other hand, a significant amount of computational power in order to deal with perturbation expansions containing several millions of terms. We should say that while the algorithmic part is well developed and has proven to be effective, the amount of computational power available to us revealed to be still insufficient. For, we need both a huge amount of computer memory in order to store all the intermediate expression of our expansions and a huge CPU time in order to perform all the algebra. We lack in particular the second part.

As a matter of fact our algorithm proves to be effective in implementing the optimization of the stability estimates in the spirit of Nekhoroshev theory, but a calculation of some further perturbation order is necessary. This is mainly matter of computational power, that will likely be available in some not far future. The possibility of a further improvement of the analytical tools is not excluded, of course.

2.1 Overview

Our aim is to study the dynamics in the neighborhood of an invariant torus for the SJS system. Thus, we first need to prove the existence and to construct a good approximation of such an invariant object. This part has been performed by Locatelli and Giorgilli[58] using a Kolmogorov normal form in a neighborhood of the actual orbits in the phase space. Thus in this thesis we report only the key points of the procedure, referring to the quoted article for all the details.

Having constructed the invariant torus, we start the original part of the present work, namely the study of long time stability of orbits in the neighborhood of the invariant torus. Starting from the Hamiltonian in Kolmogorov normal form, we perform a Birkhoff normalization and then we investigate the long time stability in Nekhoroshev

sense for the Sun–Jupiter–Saturn system.

All the explicit calculations described here have been done with the aid of a specially devised algebraic manipulator. We emphasize that the existing commercial packages, although excellent in many applications, are unsuitable for perturbation expansions up to high orders. The implementation of our analytical algorithms through computer algebra is an essential part of our work.

2.2 Theoretical Framework

The basis of our approach is the investigation of the stability of a neighborhood of a Kolmogorov invariant torus. To this end let us briefly recall the statement of Kolmogorov's theorem.

Theorem 2.1: *Consider a canonical system with Hamiltonian*

$$H(p,q) = h(p) + \varepsilon f(p,q)\,. \tag{2.1}$$

Let us assume that the unperturbed part of the Hamiltonian is non-degenerate, i.e., $\det\left(\frac{\partial^2 h}{\partial p_j \partial p_k}\right) \neq 0$, *and that* $p^* \in \mathbf{R}^n$ *is such that the corresponding frequencies* $\omega = \frac{\partial h}{\partial p}(p^*)$ *satisfy a Diophantine condition, i.e.,*

$$\left|\langle k,\omega\rangle\right| \geq \gamma |k|^{-\tau} \quad \forall\, 0 \neq k \in \mathbf{Z}^n\ ,$$

with some constants $\gamma \geq 0$ *and* $\tau \geq n-1$. *Then for* ε *small enough the Hamiltonian* (2.1) *possesses an invariant torus carrying quasi-periodic motions with frequencies* ω. *The invariant torus lies in a* ε*-neighborhood of the unperturbed torus* $\{(p,q)\,:\, p = p^*,\, q \in \mathbf{T}^n\}$.

By the way, let us recall some information about the threshold value ε^* such that for $\varepsilon < \varepsilon^*$ the Kolmogorov's theorem holds. The analytical estimates claim that $\varepsilon^* = \mathcal{O}(\gamma^2)$ for $\gamma \to 0$ (see, e.g., [73]), $\varepsilon^* = \mathcal{O}(m)$, where m is a constant such that $m\|v\| \leq \|\frac{\partial^2 h}{\partial p_j \partial p_k} v\|$ for every $v \in \mathbf{R}^n$ (see, e.g., [30] formula (22)) and $\varepsilon^* = \mathcal{O}(2^{-2\tau})$ for $\tau \to \infty$ (see, e.g., [30] formula (22)).

The question is about the dynamics in the neighborhood of the invariant torus. In order to discuss this point we need a few technical details about the Kolmogorov's proof method. The key points, clearly outlined in the original short note[37], are the following. First, one picks an unperturbed invariant torus p^* for the Hamiltonian (2.1) characterized by Diophantine frequencies ω, and expands the Hamiltonian in power

series of the actions p in the neighborhood of p^*. Thus (with a translation moving p^* to the origin of the actions space) one gives the initial Hamiltonian the form

$$H(p,q) = \langle \omega, p\rangle + \varepsilon A(q) + \varepsilon\big\langle B(q), p\big\rangle + \frac{1}{2}\big\langle Cp, p\big\rangle + \mathcal{O}(p^2)\,, \tag{2.2}$$

where $C = \left[\frac{\partial^2 h}{\partial p_j \partial p_k}(p^*)\right]$ is a symmetric matrix, and $A(q)$ and $\big\langle B(q), p\big\rangle$ are the terms independent of p and linear in p in the power expansion of the perturbation $f(p,q)$, respectively. The quadratic part in $\mathcal{O}(p^2)$ is of order ε, too. The next step consists in performing a near the identity canonical transformation which gives the Hamiltonian the Kolmogorov normal form

$$H'(p',q') = \langle \omega, p'\rangle + \mathcal{O}({p'}^2)\,. \tag{2.3}$$

As Kolmogorov points out, the invariance of the torus $p' = 0$ is evident, due to the particular form of the normalized Hamiltonian. The whole process requires a composition of an infinite sequence of transformations, and the most difficult part is to prove the convergence of such a sequence. The point which is of interest to us is that *the transformed Hamiltonian (2.3) is analytic in a neighborhood of the invariant torus* $p' = 0$.

Let us emphasize that the analytical form of the Hamiltonian (2.3) is quite similar to that of a Hamiltonian in the neighborhood of an elliptic equilibrium, namely

$$H(x,y) = \frac{1}{2}\sum_{j=1}^{n} \omega_j\left(x_j^2 + y_j^2\right) + \dots\,,$$

where the dots stand for terms of degree larger than 2 in the Taylor expansion. For, introducing the action-angle variables p, q through the usual canonical transformation $x_j = \sqrt{2p_j}\cos q_j$, $y_j = \sqrt{2p_j}\sin q_j$, the latter Hamiltonian takes essentially the form (2.3). Thus the exponential stability of the invariant torus $p' = 0$ may be proved using the theoretical scheme that works fine in the case of an elliptic equilibrium, e.g., in the case the triangular Lagrangian points.

As a matter of fact, a much stronger result holds true, namely that the invariant torus is superexponentially stable, as stated in [60] and [28]. However a computer-assisted method for the theory of superexponential stability seems not be currently available, so we limit our study to the exponential stability in Nekhoroshev sense. To this end let us briefly recall the statement of Nekhoroshev's theorem.

Theorem 2.2: *Consider a canonical system with Hamiltonian*

$$H(p,q) = h(p) + \varepsilon f(p,q)\,, \tag{2.4}$$

that satisfy the hypotheses

(i) both h and f are assumed to be holomorphic bounded function on the complex extension $\mathcal{D}_{\varrho,2\sigma} = \mathcal{G}_\varrho \times \mathbf{T}^n_{2\sigma}$ of the real domain $\mathcal{G} \times \mathbf{T}^n$;

(ii) for every $p \in \mathcal{G}_\varrho$,

$$\|C(p)v\| \le M\|v\|\,, \quad |\langle C(p)v, v\rangle| \ge m\|v\|^2 \qquad \text{for all } v \in \mathbf{R}^n\ ,$$

with positive constants $m \le M$, where C is the Hessian matrix of the unperturbed Hamiltonian $h(p)$.

Then there exist positive constants μ_* and $\mathcal{T}$ depending on ϱ, σ, $|f|$, m, M and the number n of degrees of freedom such that the following statement holds true: if

$$3^4\mu_*\varepsilon < 1\,,$$

then for every orbit $p(t)$, $q(t)$ satisfying $p(0) \in \mathcal{G}$ one has the estimate

$$\mathrm{dist}(p(t) - p(0)) \le (\mu_*\varepsilon)^{1/4}\varrho\,,$$

for all times t satisfying

$$|t| \le \frac{\mathcal{T}}{\varepsilon}\exp\left[\left(\frac{1}{\mu_*\varepsilon}\right)^{1/2a}\right]\,,$$

where $a = n^2 + n$.

Let us remark that the Nekhoroshev estimate are clearly pessimistic, due to analytic estimates; but in our work we apply an optimization procedure that leads to exponential stability in Nekhoroshev sense.

2.3 Technical Tools

Let us now come to the improvement of the estimates for the applicability of Kolmogorov and Nekhoroshev theorems. The key point is to use an explicit construction of the normal form up to a finite order with algebraic manipulation in order to reduce the size of the perturbation, and then apply a suitable reformulation of the theorems.

Let us explain this point by making reference to the theorem of Kolmogorov. Starting with the Hamiltonian (2.2) we perform a finite number, say r, of normalization steps

in order to give the Hamiltonian the normal form up to order r

$$H^{(r)}(p,q) = \langle \omega, p\rangle + \frac{1}{2}\langle Cp, p\rangle + \varepsilon^r A^{(r)}(q) + \varepsilon^r \langle B^{(r)}(q), p\rangle + \mathcal{R}^{(r)}(p,q)\,, \tag{2.5}$$

with $\mathcal{R}^{(r)}(p,q) = \mathcal{O}(|p|^2)$, so that the perturbation is now of order ε^r.

To this end we implement the normalization algorithm for the normal form of Kolmogorov step-by-step in powers of ε, as in the traditional expansions in celestial mechanics. The full justification of such a procedure, including the convergence proof, is given, e.g., in [29] and [30]. The resulting Hamiltonian has still the form (2.2), with, however, ε replaced by ε^r. Thus, a straightforward application of the theorem reads, in rough terms: *if $\varepsilon^r < \varepsilon_*$, then an invariant torus exists.* The power r may considerably improve the estimate of the threshold for the applicability of the theorem. This approach has been translated in a computer-assisted rigorous proof, which has been successfully applied to a few simple models, see, e.g., [12], [56] and [22].

Let us now come to the part concerning the estimate of the stability time which is the main contribution of the present chapter. To this end we remove from the Hamiltonian (2.5) all the contributions which are independent of or linear in the actions p, namely the terms $\varepsilon^r A^{(r)}(q) + \varepsilon^r \langle B^{(r)}(q), p\rangle$, which are small, thus obtaining a reduced Hamiltonian in Kolmogorov normal form. Moreover, we expand the perturbation $\mathcal{R}^{(r)}(p,q)$ in power series of p and Fourier series of q, thus getting a Hamiltonian in the form

$$H(p,q) = \langle \omega, p\rangle + H_1(p,q) + H_2(p,q) + \dots\,, \tag{2.6}$$

where $H_s(p,q)$ is a homogeneous polynomial of degree $s+1$ in the actions p and a trigonometric series in the angles q. Here, the upper index r of H has been removed because it is now meaningless, since we use the latter Hamiltonian as an approximation of the Kolmogorov normal form.

On this Hamiltonian we perform a Birkhoff normalization up to a finite order, that we denote again by r although it has no relation with the order of the Kolmogorov normalization used above. Thus we get a Birkhoff normalized Hamiltonian

$$H^{(r)} = \langle \omega, p\rangle + Z_1(p) + \ldots + Z_r(p) + \mathcal{F}_r(p,q)\,,$$

with $\mathcal{F}_r(p,q)$ a power series in p starting with terms of degree $r+2$. This part of the calculation may be performed using one of the well known formal algorithms that do the job. We actually used a method based on composition of Lie series. In short, assuming that the Hamiltonian $H^{(r-1)}$ has a normal form up to order $r-1$, we determine a

generating function χ_r and the part Z_r of the normal form by solving the homological equation $L_{\langle\omega,p\rangle}\chi_r + Z_r = F_r$, where $L_{\langle\omega,p\rangle}$ denotes the Lie derivative with respect to the Hamiltonian vector field of the unperturbed Hamiltonian $\langle\omega, p\rangle$, and F_r is the part of order r of the not yet normalized remainder $\mathcal{F}_{r-1}$. Then we construct the new Hamiltonian normalized up to order r by calculating

$$H^{(r)} = \exp\bigl(L_{\chi_r}\bigr)H^{(r-1)} \,.$$

We omit further details on the formal implementation of the calculation, because the Lie series method is now well known and widely used in perturbation expansions. A detailed exposition may be found, e.g., in [31]. We pay instead a particular attention to the quantitative estimates.

Let us introduce a norm for a function $f(p,q) = \sum_{|l|=s, k\in\mathbf{Z}^n} f_{l,k}\, p^l e^{i\langle k,q\rangle}$ which is a homogeneous polynomial of degree s in the actions p. Precisely define

$$\|f\| = \sum_{|l|=s\,,\,k\in\mathbf{Z}^n} |f_{l,k}| \,. \tag{2.7}$$

Moreover consider the domain

$$\Delta_\varrho = \{p \in \mathbf{R}^n \,,\ |p_j| \le \varrho \,,\ j = 1\,,\,\ldots\,,\,n\} \,. \tag{2.8}$$

Then we have

$$|f(p,q)| \le \|f\|\varrho^s \quad \text{for } p \in \Delta_\varrho \,,\ q \in \mathbf{T}^n \,.$$

Let now $p(0) \in \Delta_{\varrho_0}$ with $\varrho_0 < \varrho$. Then we have $p(t) \in \Delta_\varrho$ for $|t| < T$, where T is the escape time from the domain Δ_ϱ. This is the quantity that we want to evaluate. To this end we use the elementary estimate

$$|p(t) - p(0)| \le |t| \cdot \sup_{|p|<\varrho} |\dot{p}| < |t| \cdot \bigl\|\{p, \mathcal{F}_r\}\bigr\|\varrho^{r+2} \,. \tag{2.9}$$

The latter formula allows us to find a lower bound for the escape time from the domain Δ_ϱ, namely

$$\tau(\varrho_0, \varrho, r) = \frac{\varrho - \varrho_0}{\bigl\|\{p, \mathcal{F}_r\}\bigr\|\varrho^{r+2}} \,, \tag{2.10}$$

which however depends on ϱ_0, ϱ and r. We emphasize that in a practical application, e.g., to the SJS system, ϱ_0 is fixed by the initial data, while ϱ and r are left arbitrary. Thus we try to find an estimate of the escape time $T(\varrho_0)$ depending only on the physical parameter ϱ_0. To this end we optimize $\tau(\varrho_0, \varrho, r)$ with respect to ϱ and r, proceeding as

follows. First we keep r fixed, and remark that the function $\tau(\varrho_0, \varrho, r)$ has a maximum for

$$\varrho = \frac{r+2}{r+1}\varrho_0 \,.$$

This gives an optimal value of ϱ as a function of ϱ_0 and r, and so a new function

$$\widetilde{\tau}(\varrho_0, r) = \sup_{\varrho \geq \varrho_0} \tau(\varrho_0, \varrho, r) \,,$$

which is actually computed by putting the optimal value $\varrho = \varrho_0(r+2)/(r+1)$ in the expression above for $\tau(\varrho_0, \varrho, r)$. Next we look for the optimal value $r_{\rm opt}$ of r, which maximizes $\widetilde{\tau}(\varrho_0, r)$ when r is allowed to change. That is, we look for the quantity

$$T(\varrho_0) = \max_{r \geq 1} \widetilde{\tau}(\varrho_0, r) \,,$$

which is our best estimate of the escape time, depending only on the initial data. We define the latter quantity as the estimated stability time. We remark that the maximum in the r.h.s. of the latter formula actually exists. This follows from the asymptotic properties of the Birkhoff normal form. For, according to the available analytical estimates based on Diophantine inequalities for the frequencies, the norm $\big\|\{p, \mathcal{F}_r\}\big\|$ in the denominator of (2.10) is expected to grow as $(r!)^n$, n being the number of degrees of freedom. Thus, for ϱ_0 small enough the denominator $\big\|\{p, \mathcal{F}_r\}\big\|\varrho_0^r$ reaches a minimum for some $r^n \sim 1/\varrho_0$, which means that the wanted maximum actually exists, thus providing the optimal value $r_{\rm opt}$. We also remark that although no proof exists that the analytical estimates are optimal, accurate numerical investigations based on explicit expansions show that the $r!$ growth of the norms actually shows up (see, e.g., [16] and [17]). Working out an analytical evaluation of the stability time on the basis of these considerations leads to an exponential estimate of Nekhoroshev type for $T(\varrho_0)$ (see, e.g., [24]). Here we replace the analytical estimates with an explicit numerical optimization of $\widetilde{\tau}(\varrho_0, r)$ by just calculating it for increasing values of r until the maximum is reached.

Our aim is to perform the procedure above by using computer algebra. Thus some truncation of the functions must be introduced in order to implement the actual calculations. The most straightforward approach is the following. First we truncate the Hamiltonian (2.6) at a finite polynomial order in the actions. This is legitimate if the radius ϱ of the domain is small, due to the well known properties of Taylor series. However, the Fourier expansion of every term H_s still contains infinitely many contributions. Here we take advantage of the exponential decay of the Fourier coefficients of

analytic functions and of some algebraic properties of the Poisson brackets. Precisely, let $f(q) = \sum_k f_k\, e^{i\langle k,q\rangle}$ (the dependence of the coefficients f_k on the actions is irrelevant here), then the exponential decay of the coefficients means that $|f_k| \leq Ce^{-|k|\sigma}$ with some positive constants C and σ. Thus, having fixed a positive integer K we truncate the Fourier expansion as $f(q) = \sum_{|k|\leq K} f_k e^{i\langle k,q\rangle}$, i.e., we remove all Fourier modes $|k| > K$. This is allowed because the exponential decay assures that the neglected part is small. A more detailed discussion about this method of splitting the Hamiltonian can be found in [31].

Coming back to our problem, we include in $H_s(p,q)$ all Fourier coefficients with $|k| \leq sK$, so that $H_s(p,q)$ is a homogeneous polynomial of degree $s+1$ in the actions p and a trigonometric polynomial of degree sK in the angles q. The algebraic property mentioned above is that such a splitting of the Hamiltonian is preserved by the Lie series algorithm that we apply through all our calculations. This in view of the elementary fact that the Poisson bracket between two functions, f_r and f_s say, which are homogeneous polynomial of degree $r+1$ and $s+1$, respectively, in p and trigonometric polynomials of degree rK and sK, respectively, in q produces a new function of degree $r+s+1$ in p and $(r+s)K$ in q.

A final remark concerns the estimate of the remainder $\mathcal{F}_r$ in (2.9), which is an infinite series, too. Here we just calculate the first term of the remainder, namely the term of degree $r+1$, and multiply its norm by a factor 2. This factor is justified in view of the fact that the analytical estimates of the same quantities involve a sum of a geometric series which, for ϱ small enough, decreases with a ratio less than $1/2$. Here a natural objection could be that for some strange reason the norm of the remainder at some finite order could be smaller than predicted by the analytical estimates. However, it is a common experience that after a few perturbation steps the norms of the functions take a rather regular behavior consistent with the geometric decrease predicted by the theory. Thus, our choice appears to be justified by experience.

As a final remark we note that our way of dealing with the truncation is the most straightforward one, but it is not the sole possible. Other more refined criteria may be invented, of course, which may take into account the most important contribution while substantially reducing the number of coefficients to be calculated. In this sense our direct approach should be considered as a first attempt to check if the concept of Nekhoroshev stability may be expected to apply to our Solar System. Although being unable to produce rigorous results in a strict mathematical sense, we believe that our

method gives interesting results in the spirit of classical perturbation methods.

2.4 Application to the Planetary Problem of Three Bodies

As already said, applying the theories of Kolmogorov and Nekhoroshev to the planetary problem is not straightforward, due to the degeneration of the Keplerian motion. In order to remove such a degeneration, a lengthy procedure is needed; this essentially requires a suitable adaptation of the canonical coordinates, paying a very particular care to the secular ones (to appreciate some deep point of view about this problem, see, e.g., [2] and [68]).

In our approach the difficulty shows up in the part concerning the application of Kolmogorov theory. Once a Kolmogorov torus has been constructed, then there is no extra difficulty in applying the method of sect. 2.3, due to the fact that the method is local. In the present section we give a brief sketch of the procedure for the construction of a Kolmogorov torus. The complete procedure is described in [29] and [30], to which we refer for details.

Following a traditional approach, we first reduce the integrals of motion (i.e. the linear and angular momenta); therefore, we separate the fast variables (essentially the semi-major axes and the mean anomalies) from the slow ones (the eccentricities and the inclinations with the conjugated longitudes of the perihelia and of the nodes). This is usually done in Poincaré variables by writing a reduced Hamiltonian of the form

$$H^R(\Lambda, \lambda, \xi, \eta) = F^{(0)}(\Lambda) + \mu F^{(1)}(\Lambda, \lambda, \xi, \eta)\,, \tag{2.11}$$

with

$$\mu = \max\{m_1 \,/\, m_0\,,\, m_2 \,/\, m_0\,\}\,,$$

where m_0 is the mass of the star, m_1 and m_2 are the masses of the planets, Λ_j, λ_j are the fast variables and ξ_j, η_j are the slow (Cartesian-like) variables. Here, obviously, the values of the index $j = 1\,,\,2$ correspond to the internal planet and to the external one, respectively. A brief introduction to the method used for the explicit expansions is contained in appendix A.

On this Hamiltonian we perform a procedure which is the natural extension of the one devised by Lagrange and Laplace in order to calculate the secular motion of the eccentricities and the inclinations and the conjugated angles.

The first step is the identification of a good unperturbed invariant torus for the fast angles λ, setting for a moment the slow variables ξ, η to zero. Here is a short description.

(i) Having fixed a frequency vector $n^* \in \mathbf{R}^2$, we determine the corresponding action values Λ^* corresponding to a torus which is invariant for an integrable approximation of the system, where the dependency on both the fast angles λ and on the secular coordinates ξ, η is dropped. This can be done by solving the equation

$$\left.\frac{\partial \langle H^R \rangle_\lambda}{\partial \Lambda_j}\right|_{\substack{\Lambda=\Lambda^* \\ \xi\,,\,\eta=0}} = n_j^*\,, \quad j = 1,\, 2\,.$$

Here $\langle H^R \rangle_\lambda = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} H^R \, d\lambda_1 \, d\lambda_2$ is the average of the Hamiltonian H^R with respect to the fast angles. The explicit value of n^* is chosen so that it reflects the true mean motion frequencies of the planets (see next section for our values). Having solved the previous equation with respect to the unknown vector Λ^*, we expand H^R in power series of $\Lambda - \Lambda^*$. With a little abuse of notation we denote again by Λ the new variables.

(ii) We perform two further canonical transformations which make the torus $\Lambda = \xi = \eta = 0$ to be invariant up to order 2 in the masses. Indeed, these changes of coordinates are borrowed from the Kolmogorov normalization algorithm, but we look for a Kolmogorov normal form with respect to the fast variables only, considering the slow ones essentially as parameters, although they are changed too. More precisely, we determine generating functions of the form $\chi_j(\Lambda, \lambda, \xi, \eta) = \Lambda^{j-1} g_j(\lambda, \xi, \eta)$ for $j = 1, 2$, where $g_j(\lambda, \xi, \eta)$ includes a finite order expansion both in Fourier modes with respect to the fast angles λ and in polynomial terms of the slow variables ξ, η. The aim of this step is to reduce the size of terms independent of or linear in the fast actions so that it is of the same order as the rest of the perturbation. We denote by H^T the resulting Hamiltonian, which is still trigonometric in the fast angles λ and polynomial in Λ, ξ, η.

The next goal is to determine a good invariant torus for the slow variables ξ, η. To this end we combine the classical Lagrange calculation of the secular frequencies with a Birkhoff procedure that takes into account the non-linearity.

(iii) We consider the secular system, namely the average $\langle H^T \rangle$ of the Hamiltonian H^T resulting from the step (ii) above. Acting only on the quadratic part of the Taylor expansion of $\langle H^T \rangle$ in ξ, η we determine a first approximation of the secular fre-

quencies, and transform the Hamiltonian so that its quadratic part has a diagonal form. This part of the calculation follows the lines of Lagrange theory, but the calculation is worked out at the second order approximation in the masses. The diagonalization of the quadratic part requires a linear canonical transformation, which is a standard matter. Thus the quadratic part in ξ, η of the resulting Hamiltonian has the form $\frac{1}{2}\sum_j \nu_j(\xi_j^2 + \eta_j^2)$, where ν are the secular frequencies and we denote again by ξ, η the slow variables.

(iv) We perform a Birkhoff normalization up to order 6 in ξ, η. This gives a normalized secular Hamiltonian H^B which in action-angle variables $\xi_j = \sqrt{2I_j}\cos\varphi_j$, $\eta_j = \sqrt{2I_j}\sin\varphi_j$ takes the form

$$H^B = \nu \cdot I + h^{(4)}(I) + h^{(6)}(I) + F(\Lambda, I, \varphi)\,,$$

where $h^{(4)}$ and $h^{(6)}$ are polynomials of degree 2 and 3 in I, respectively. This step removes the degeneration of the secular motion, thus allowing us to take into account the non-linearity of the secular part of the problem.

(v) Having fixed the slow frequencies g^* so that they reflect the true frequencies of the system, we determine a secular torus I^* corresponding to these frequencies, by using the integrable approximation of H^B. This is done by solving for I the equation

$$\frac{\partial\, h^{(4)}}{\partial I_j}(I) + \frac{\partial\, h^{(6)}}{\partial I_j}(I) = g_j^* - \nu_j^*\ , \qquad j = 1,\, 2\,.$$

The values Λ^* and I^* so determined provide the first approximation of the Kolmogorov invariant torus. Reintroducing the fast angles and performing on the original Hamiltonian H^R all the transformations that we have done throughout our procedure (i)–(v) we get a Hamiltonian of the form (2.2) which is the starting point for Kolmogorov normalization algorithm. After a number of Kolmogorov steps the Hamiltonian takes the form (2.5), thus giving a good approximation of an invariant torus with frequencies n^* and g^*. The latter form is precisely the output of the calculation illustrated in [30], and by removing all terms which are independent of or linear in the actions p it provides a Hamiltonian as that in (2.6). This is the starting point for our algorithm evaluating the stability time in the neighborhood of the invariant torus.

Table 2.1. Physical parameters for the Sun–Jupiter–Saturn system taken from JPL at the Julian Date 2451220.5 .

		Jupiter ($j = 1$)	Saturn ($j = 2$)
mass	m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$
semi-major axis	a_j	5.20092253448245	9.55716977296997
mean anomaly	M_j	6.14053316064644	5.37386251998842
eccentricity	e_j	0.04814707261917873	0.05381979488308911
perihelion argument	ω_j	1.18977636117073	5.65165124779163
inclination	i_j	0.006301433258242599	0.01552738031933247
longitude of the node	Ω_j	3.51164756250381	0.370054908914043

Table 2.2. The frequencies of the unperturbed torus in the SJS system corresponding to the initial data and physical parameters in table 2.1. The values are calculated via frequency analysis on the orbits obtained by direct integration of the equations for the problem of three bodies.

	Jupiter	Saturn
fast frequencies	$n_1^* = 0.52989041594442$	$n_2^* = 0.21345444291052$
secular frequencies	$g_1^* = -0.00014577520419$	$g_2^* = -0.00026201915143$

2.5 Application to the Sun–Jupiter–Saturn System

We come now to the application of our procedure to the SJS system. Let us first recall the model. We consider the general problem of three bodies with the Newtonian potential. Thus, the contribution due to the other planets of the Solar System is not taken into account in our approximation. Here too we simplify the exposition by omitting many technical details. We report in appendix A all technicalities concerned with the expansion of the Hamiltonian, just recalling once more that all the expansions have been done via algebraic manipulation, using our package developed on purpose.

The choice of the model plays a crucial role in determining the frequencies of the torus, that we calculate by integrating the Newton equations for the problem of three bodies and applying the frequency analysis (see, e.g., [48]) to the computed orbit. As initial data we take the orbital elements of Jupiter and Saturn as given by JPL* for

* The data about the planetary motions provided by the Jet Propulsion Laboratory are publicly available starting from the web page `http://www.jpl.nasa.gov/`

Table 2.3. Estimates of the uncertainties on the initial values of the canonical coordinates $(\Lambda, \lambda, \xi, \eta)$. These evaluations are derived from the comparison of different sets of JPL's DE.

	$\Delta\Lambda_j$	$\Delta\lambda_j$	$\Delta\xi_j$	$\Delta\eta_j$
Jupiter $(j = 1)$	1.8×10^{-6}	6.6×10^{-5}	1.1×10^{-5}	2.8×10^{-6}
Saturn $(j = 2)$	1.7×10^{-6}	3.0×10^{-5}	3.3×10^{-6}	3.2×10^{-6}

the Julian Date 2451220.5. This is the point where the connection with the physical parameters of our Solar System is made. The physical parameters and the orbital elements are reported in table 2.1. The calculated frequencies are given in table 2.2 and are calculated using the frequency analysis on the orbits obtained by direct integration of the equations for the problem of three bodies. Such an integration has been performed by using the symplectic integrator $\mathcal{SBAB}_{C3}$ (see [51]) in quadruple precision with a time-step of 0.08 years.

The choice of the Julian Date 2451220.5 in order to set the initial data is completely arbitrary, of course, its sole justification being that such data are directly available from JPL. Choosing different dates or different determinations of the planets' elements could lead to a slightly different determination of the frequencies, and so also of the invariant torus. However, we emphasize that the aim of this study is precisely to give a long time stability result which applies to a neighborhood of the invariant torus. The size of such a neighborhood should be large enough to cover the unavoidable uncertainty in determining the initial data for the SJS system. This is a delicate matter, of course, because the JPL data reflect the dynamics of the full solar system, while our study is concerned only with the model of three bodies. However, we may get some hint on the size of the uncertainty precisely by looking at the JPL data.

As everybody knows, the initial positions and velocities of the planets are usually taken from the Development Ephemeris of the Jet Propulsion Laboratory (for short, JPL's DE). There are several sets of these ephemerides, each version of them being based on more and more observational data, which take benefit from the improvement of the techniques. Thus, each new version of the JPL's DE is expected to improve the precision of the data with respect to the older ones and, then, one can approximately evaluate the error of the older versions by comparison with the most recent one[83]. The positions and velocities of the planets given by five different sets of JPL's DE are listed in table 15 of Standish's paper[82] that we report here in table 2.4 for completeness.

Table 2.4. This is the Table 15 of Standish's paper[82]: the initial conditions of the Ephemerides at JD 2440400.5 in [AU] and [AUd^{-1}]. Given are heliocentric coordinates for Q the planets, geocentric for the Moon and solar system barycentric for the Sun.

DE 96						
Mercury	0.3557922580285	-0.0955355879504	-0.0877134178162	0.0037087808508	0.0248497847800	0.0129268670834
Venus	0.6033503910337	-0.3559062740335	-0.1984879214940	0.0111569232028	0.0154889017564	0.0062751524204
E-Mbary	0.1036934360992	-0.9278340545910	-0.4023396941889	0.0168335341142	0.0015550298980	0.0006742293017
Mars	-0.1324779851142	1.3269849183759	-0.6055541641333	0.0144822028777	0.0000753531675	-0.0003541207275
Jupiter	-5.3941958936061	0.7709913063093	-0.1989103464060	0.0010056321327	-0.0065350248379	-0.0028281072475
Saturn	7.9482565922231	4.5071645178408	1.5199515065662	-0.0031591864994	0.0043662806909	0.0019419084081
Uranus	-18.2827495322406	-0.9583802950893	-0.1615738451072	0.0001713682705	-0.0037698523267	-0.0016541983147
Neptune	-16.3716953023856	23.7616428640022	9.3216250128726	0.0026228085969	-0.0015329315014	0.0006940632445
Pluto	-30.4523499642980	-0.5324450661161	9.0594896804909	0.0002820486838	0.0031521332523	0.0010816447761
Moon	-0.0008357023040	-0.0019854403334	0.0010832674180	0.0005987522064	-0.0001741533929	0.0000884772995
Sun	0.0045277782503	0.0007233492806	0.0002437317358	-0.0000002839374	0.0000051830225	0.0000022313609
DE 102						
Mercury	0.3557925832924	-0.0955345485155	-0.0877132191455	0.0037087022239	0.0248497897092	0.0129268806051
Venus	0.6033515261042	-0.3559044873174	-0.1984876695292	0.0111568753206	0.0154889307388	0.0062751661931
E-Mbary	0.1036963194747	-0.9278335982481	-0.4023400078909	0.0168335292669	0.0015550772558	0.0006742408524
Mars	-0.1324738219138	1.3269850541589	-0.6055547646852	0.0144822029825	0.0000753943351	0.0003541110573
Jupiter	-5.3941932915539	-0.7710081200473	-0.1989150590738	0.0010056547485	-0.0065350206821	-0.0028281090999
Saturn	7.9482436367196	4.5071870232271	1.5199589946549	-0.0031592001390	0.0043662705224	0.0019419077867
Uranus	-18.2827508557988	-0.9584340318843	-0.1615867736876	0.0001713802650	-0.0037698507754	-0.0016541994628
Neptune	-16.3716335288293	-23.7617037839296	-9.3216533685651	0.0026228120284	-0.0015329244509	0.0006940623979
Pluto	-30.4523559663809	-0.5325496136974	9.0594633601116	0.0002820600485	-0.0031521318753	0.0010816458256
Moon	-0.0008356961442	-0.0019854419428	-0.0010832692002	0.0005987527434	-0.0001741517754	0.0000884768645
Sun	0.0045277754174	0.0007233658692	0.0002437367327	-0.0000002839606	0.0000051830181	0.0000022313623
DE 108						
Mercury	0.3557922619520	-0.0955356148809	-0.0877133662244	0.0037087807034	0.0248497853751	0.0129268662331
Venus	0.6033503963396	-0.3559063243655	-0.1984878145912	0.0111569232817	0.0154889016095	0.0062751527033
E-Mbary	0.1036934336616	-0.9278340956795	-0.4023396051242	0.168335341028	0.0015550290754	0.0006742311449
Mars	-0.1324779672058	-1.3269849745181	-0.6055540351789	0.0144822029859	0.0000753526453	-0.0003541193151
Jupiter	-5.3941974664785	0.7709876266693	-0.1989094810490	0.0010056275852	-0.0065350242925	-0.0028281069334
Saturn	7.9482629546463	4.5071580092641	1.5199489513674	-0.0031591823909	0.0043662826537	0.0019419085676
Uranus	-18.2827542886466	-0.9583689678224	-0.1615693323556	0.0001713662494	-0.0037698516271	-0.0016541979767
Neptune	16.3717257216743	-23.7616519595999	-9.3216311374360	0.0026228066676	-0.0015329326646	-0.0006940637179
Pluto	-30.4523499642980	-0.5324450661161	9.0594896804909	0.0002820486838	-0.0031521332523	-0.0010816447761
Moon	-0.0008357023040	-0.0019854403334	-0.0010832674180	0.0005987522064	-0.0001741533929	-0.0000884772995
Sun	0.0045346842384	0.0007373522487	0.0002492669043	-0.0000002853662	0.0000051832214	0.0000022314609
DE 111						
Mercury	0.3557922056845	-0.0955357755691	-0.0877134189685	0.0037087951557	0.0248497768577	0.0129268784786
Venus	0.6033501891138	-0.3559065723936	0.1984879999160	0.0111569322626	0.0154888921082	0.0062751601868
E-Mbary	0.1036928952888	-0.9278339564488	-0.4023400657643	0.0168335350008	0.0015550190939	0.0006742316402
Mars	-0.1324787391533	-1.3269846026524	-0.6055546820455	0.0144822030225	0.0000753444863	-0.0003541194740
Jupiter	-5.3941967321350	-0.7709883161678	-0.1989105167583	0.0010056287126	-0.0065350248876	0.0028281071512
Saturn	7.9482628711392	4.5071579954748	1.5199497085039	-0.0031591826199	0.0043662827477	0.0019419076667
Uranus	-18.2827490039608	-0.9583689566581	-0.1615708365945	0.0001713683275	-0.0037698514721	-0.0016541989939
Neptune	-16.3717137327784	23.7616447706433	-9.3216265451851	0.0026228074066	-0.0015329318513	0.0006940632309
Pluto	-30.4523499642980	0.5324450661161	9.0594896804909	0.0002820486838	-0.0031521332523	-0.0010816447761
Moon	-0.0008357035210	-0.0019854390373	-0.0010832688286	0.0005987521000	-0.0001741538297	-0.0000884771802
Sun	0.0045228034250	0.0007351654860	0.0002514542756	-0.0000002850583	0.0000051819763	0.0000022310117
DE 118						
Mercury	0.3557922218183	-0.0955357051533	-0.0877134311222	0.0037087906743	0.0248497775730	0.0129268783558
Venus	0.6033502530459	-0.3559064530483	-0.1984880201035	0.0111569294057	0.0154888943192	0.0062751597958
E-Mbary	0.1036930656779	-0.9278339358058	-0.4023400695644	0.0168335347145	0.0015550224233	0.0006742310813
Mars	-0.1324784979851	-1.3269846289449	-0.6055546775302	0.0144822029927	0.0000753473440	-0.0003541199691
Jupiter	-5.3941968663323	-0.7709883235977	-0.1989109026640	0.0010056289515	-0.0065350237136	-0.0028281094121
Saturn	7.9482624411543	4.5071593899891	1.5199489343134	-0.0031591825559	0.0043662822334	0.0019419088581
Uranus	-18.2827509982246	-0.9583685572263	-0.1615684949776	0.0001713668001	-0.0037698516627	0.0016541985199
Neptune	-16.3717057706423	23.7616195537846	-9.3216332089547	0.0026228079709	-0.0015329285068	0.0006940626345
Pluto	-30.4521946670408	-0.5325021780141	9.0594118976226	0.0002820612754	-0.0031521402749	-0.0010816544638
Moon	-0.0008357031642	-0.0019854391577	-0.0010832687705	0.0005987521184	-0.0001741537135	-0.0000884771962
Sun	0.0045144118714	0.0007228284115	0.0002465910049	-0.0000002836945	0.0000051811944	0.0000022306588

For each kind of these data we can determine a narrow interval containing all of them and we can calculate the Keplerian orbital elements corresponding to the extrema

Table 2.5. Maximal discrepancies about the orbital elements of the SJS system between a numerical integration and the semi-analytic one, that is based on the construction of the invariant torus corresponding to the frequencies values given in table 2.2. The maximal relative errors on the semi-major axis a_j and on the eccentricities e_j are reported here for both Jupiter and Saturn; the same is made also for the maximal absolute errors on the "fast angle" $\lambda_j = M_j + \omega_j$ and on the perihelion argument ω_j . In the present case, the comparisons are made starting from the initial conditions given in table 2.1 and for a time span of 100 Myr.

	$\mathrm{Max}_t \left\{ \left\lvert \frac{\Delta a_j(t)}{a_j(t)} \right\rvert \right\}$	$\mathrm{Max}_t \{\lvert\Delta\lambda_j(t)\rvert\}$	$\mathrm{Max}_t \left\{ \left\lvert \frac{\Delta e_j(t)}{e_j(t)} \right\rvert \right\}$	$\mathrm{Max}_t \{\lvert\Delta\omega_j(t)\rvert\}$
Jupiter	1.5×10^{-6}	5.0×10^{-4}	1.3×10^{-3}	1.3×10^{-3}
Saturn	6.8×10^{-6}	1.1×10^{-3}	4.3×10^{-3}	7.3×10^{-3}

of such intervals. By applying all the necessary transformations we translate these data into uncertainties for the Poincaré canonical coordinates $(\Lambda, \lambda, \xi, \eta)$ that have been used in order to write the Hamiltonian (2.11). These uncertainties are reported in table 2.3. This provides us with a first approximation of the neighborhood of our initial data that contains all JPL's DE reported in Standish's paper. We should now apply all the canonical transformations needed in order to construct an invariant torus close to the SJS orbit. However we remark that all such transformations are very smooth, being analytic, volume preserving, and most of them are close to identity, so that they add just a small correction with respect to the data in table 2.3. Thus we may confidently expect that at some time the phase space point representing the position of the SJS system lies in a neighborhood of our approximated invariant torus the size of which is evaluated to be $\mathcal{O}(10^{-6})$ for the fast actions and $\mathcal{O}(10^{-5})$ for the secular coordinates.

Let us now come back to the actual calculation. The Kolmogorov normal form has been computed up to order 17, with the generating function exhibiting a good geometric decay. Furthermore, we have compared the orbit on the approximate invariant torus with that produced by a direct numerical integration of the equations of motion, thus finding a quite good agreement between them, as shown in table 2.5. Here, we omit the details about these lengthy calculations, since a complete report has been already given in [30].

The calculation of Kolmogorov normal form produces a Hamiltonian which is analytic in the neighborhood of the approximated invariant torus. Our program performs

the calculation of this Hamiltonian with the polynomial series in the actions truncated at order 3 and the trigonometric series truncated at order 34 (see [30] for more details). On this Hamiltonian we would like to apply the procedure of sect. 2.3. However, a major obstacle is raised: the number of coefficients in the series that we have calculated is more than 7 100 000. Such a huge number of coefficients can not be handled in a Birkhoff normalization procedure. For, referring to the discussion at the end of sect 2.3 we should set the parameter K for the truncation of trigonometric series to 34, thus getting a truncation at trigonometric degree 68, 102, ..., $34r$, ... at successive order. A rough estimate of the number of generated coefficients shows that we shall soon run out of memory and of time on any available computer. Thus, we must introduce some further approximations. By the way, this is a part of the calculation where a substantial increase of the computational power combined with a clever selection of the coefficients to be handled would help in order to improve our results.

In view of the considerations above we decided, as a first approach, to strictly follow the truncation scheme illustrated at the end of sect. 2.3 by just lowering the value of K. We report the results of this first attempt, which in our opinion appear already to be interesting. Thus we expand the Hamiltonian in the form

$$H(p,q) = \langle \omega, p \rangle + H_1(p,q) + H_2(p,q)\,,$$

by keeping in H_1 all terms of degree 2 in the actions p and K in the angles q, and in H_2 all terms of degree 3 in the actions p and $2K$ in the angles q. The Birkhoff normalization produces a Hamiltonian of the form

$$H = \langle \omega, p \rangle + Z_1(p) + \ldots + Z_r(p) + \mathcal{F}_{r+1}(p,q)\,,$$

where $\mathcal{F}_{r+1}$ denotes the term of degree $r+2$ in the actions p and $(r+1)K$ in the angles, i.e., the first term of the remainder. With a suitable choice of K, this considerably reduces the number of coefficients in the expansions thus enabling us to perform the calculation on a workstation. We emphasize however that the algorithm is a general one so that in principle it can be applied to the full Hamiltonian or, better to a Hamiltonian obtained by removing all coefficients which are very small and will likely not produce big coefficients (due to the action of small denominators) during the calculation of the Birkhoff normal form. The rest of the calculation closely follows the discussion in section 2.3, so we come to illustrating the results. We performed the calculation with

two different values of K, as given in the following table.

K	r	# of coefficients
4	5	2 494 000
6	4	3 380 000

This shows in particular the dramatic increase of the number of coefficient in the remainder $\mathcal{F}_{r+1}$ (third column), which imposes strong constraints on the choice of the normalization order r.

A quite natural objection could be raised here. Since the most celebrated resonance of the SJS system (i.e., the mean motion resonance $5:2$) has trigonometrical degree 7, it seems that some of the main resonant terms are neglected because of our choice of K. This is actually not the case, due to a technical element that we have omitted in the previous section in order to make the discussion simpler. Our sequence of transformations includes a uni-modular linear transformation on the angles, and so also on the frequencies. The action on the frequencies changes the resonance $5:2$ into a $3:1$ one, which is of order 4. Thus, setting $K \geq 4$ as we did throughout all our calculations is enough in order to include the main resonant terms. A detailed discussion of this point can be found in sect. 3 of [30].

Let us now come to the results. In panel (a) of fig. 2.1 we report the results for $K = 4$. The crosses give the estimated stability time for the Birkhoff normal form at order 5; the dashed line gives the estimated time when the Birkhoff normal form is truncated at order 4, thus showing how relevant is the improvement when a single normalization order is added.

We can now come back to the estimate about the escape time $T = T(\varrho_0)$. Looking at panel (a) of fig. 2.1, one can remark that we have an estimated stability time of 10^{10} years, that is approximately equal to the age of the Universe, for a neighborhood of initial conditions of a radius that is about 10^{-5} in actions.

It may be noted that the stability curves exhibit a sharp change of slope around $\varrho_0 \sim 10^{-4}$. This is because the optimal normalization order increases when the radius is decreased. Actually, further changes of slope should be expected for smaller values of ϱ_0, but due to computational limits such changes can not appear in our figure, because the optimal order exceeds the actual order of our calculation. Thus, our estimate of the stability time should be considered as a very pessimistic lower bound.

Moreover, the behavior of the plots in fig. 2.1, clearly shows that the estimate of the escape time can be substantially improved if smaller values of the radius ϱ_0 can

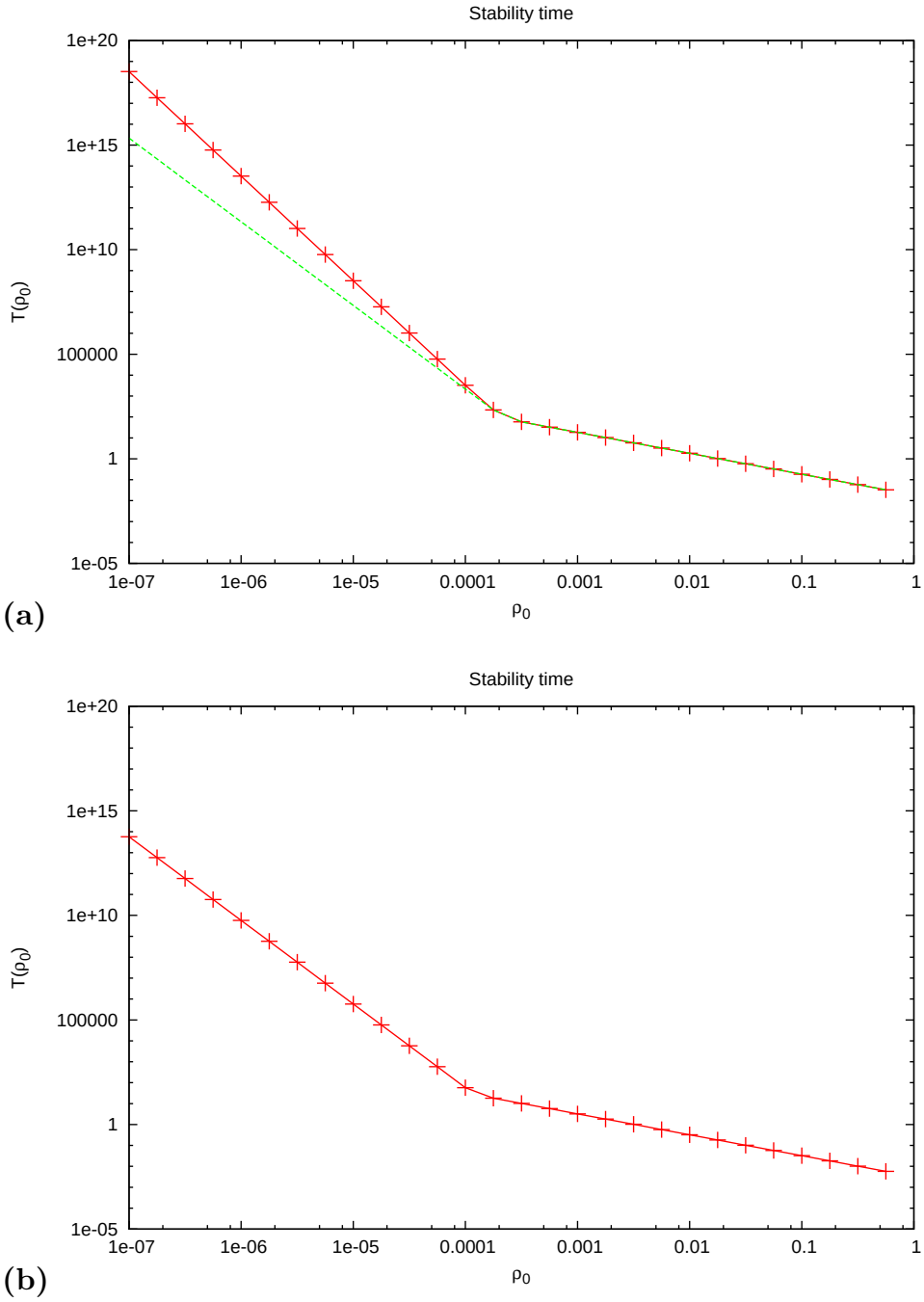

Figure 2.1. The estimated stability time, in years. (a) results for $K = 4$ and Birkhoff normalization order 5 (crosses) and 4 (dashed curve). (b) results for $K = 6$ and Birkhoff normalization order 4.

be considered. Recalling that our estimate of the size of the neighborhood in action variables is calculated from the discrepancy among different sets of JPL's DE data, we

may affirm that our neighborhood roughly covers such a width, which is tabulated in Standish's paper quoted above.

If we try a better approximation of the Hamiltonian, setting $K = 6$, then we are forced to stop the Birkhoff normalization at order 4, thus making the results definitely worse. The data for the estimated stability time are plotted in panel (b) of fig. 2.1. One sees that the estimate becomes comparable with the age of the Universe only in a neighborhood of initial conditions slightly larger than 10^{-6}. However, if we compare the curve in panel (b) with the dashed curve in panel (a) we see that we shall likely get substantially better results if we could compute the normal form at order 5.

Thus, our rough approximation gives results which apply to a set of initial data for the SJS system which is of the same order of magnitude as the uncertainty in JPL's data. We also emphasize that our evaluation of ϱ_0 is based on observational data which are presently older than 25 years; we expect that this is quite pessimistic with respect to the features of more recent JPL's DE.

2.6 Comments

We have developed an effective method to compute the Kolmogorov normal form for the problem of three bodies, and have successfully applied it to the SJS problem, using the data of our Solar System. Next, we have shown that a calculation of the stability in Nekhoroshev sense of orbits with initial point in the neighborhood of the torus is possible, at least if one accepts to make some strong truncation in the expansion of the Hamiltonian. A rather strong truncation allows us to get an estimated stability time comparable with the age of the Universe in a neighborhood of the torus that will likely contain the actual initial data of the SJS system.

The natural question is whether such results will remain valid if we add more and more terms in the Hamiltonian, thus making our approximation better. Answering such a question is presently beyond our limits, but in our opinion deserves to be investigated. Some improvement may be obtained by using more computational power, e.g., by performing our calculation on a cluster of computers. However, substantial improvements require also a refinement of our analytical techniques in order to be able to evaluate the error induced by our truncations, thus allowing us to introduce better computational schemes and to evaluate the reliability of our approximations.

3

The Planar Sun Jupiter Saturn Uranus System

"Perturbation theory is not a sport for young ladies!"
— Ugo Locatelli

In this chapter we continue our investigation on the long time stability of the Solar System by considering the planar secular model for Sun, Jupiter, Saturn and Uranus system. Some aspects are also related to the theory of Lagrange and Laplace on the secular motions of the perihelia and of the nodes of the planetary orbits. Indeed, concerning the planetary orbital revolutions, we improve the classical circular approximation by replacing it with a torus which is invariant up to order two in the masses; therefore, we investigate the stability of the elliptic equilibrium point of the secular system for small values of the eccentricities. In our work, for the initial data corresponding to a real set of astronomical observations, we find an estimated stability time of 10^7 years, which is not extremely smaller than the lifetime of the Solar System, which is approximately 5 Gyr.

3.1 Overview

One of our main aims is to point out the major dynamical and computational difficulties that arise in the application of Kolmogorov's theorem. In view of this, we attempt to apply the Nekhoroshev theory by trying essentially an extension of Lagrange theory. Although the final results appear to be interesting, our conclusion will be that further and more refined investigations are needed. We consider indeed this work as the beginning of a more comprehensive study of systems with more than two planets in the framework of perturbation methods related to the theories above.

As we said in the previous chapter, the actual applicability of Kolmogorov's theorem to the planetary system encounters two major difficulties, namely: (i) the degeneracy of the Keplerian motion, and (ii) the extremely restrictive assumptions on the smallness of the perturbation.

Besides the technical difficulty, the results of the numerical explorations have raised some doubts concerning the applicability of Kolmogorov theory to the major bodies of our planetary system, namely the Sun and the so called Jovian planets, i.e. Jupiter, Saturn, Uranus and Neptune, hereafter we will refer to this model as the SJSUN problem. Indeed, the motion of such planetary subsystem has been shown to be chaotic by Sussman and Wisdom[85]). Murray and Holman provided such an enlightening explanation of this phenomenon, that we think it is helpful to briefly summarize some of their results as follows (see [67] for completeness).

(a) The chaoticity of the Jovian planets appears to be due to the *overlap of some resonances involving three or four bodies.* An example is given by the resonances

$$3n_1 - 5n_2 - 7n_3 + [(3-j)g_1 + 6g_2 + jg_3] \ , \qquad \text{with } j = 0,\, 1,\, 2,\, 3\,,$$

where n_i stands for the mean motion frequency of the i-th planet, g_i means the (secular) frequency of its perihelion argument and the indexes 1, 2, 3 refer to Jupiter, Saturn and Uranus, respectively. In fact, during the planetary motion each angle corresponding to the resonances above moves from libration to rotation and viceversa. Many other resonances analogous to the previous ones are located in the vicinity of the real orbit of the SJSUN system, some of them involving also Neptune and the frequencies related to the longitudes of the nodes.

(b) The time needed by these resonances to eject Uranus from the Solar System is roughly evaluated to be about 10^{18} years.

(c) By moving the initial semi-major axis of Uranus in the range 19.18–19.35 AU one observes some regions that look filled by quasi-periodic ordered motions and other regions that are weakly chaotic, i.e., with a Lyapunov time ranging between 2×10^5 and 10^8 years. All the main resonances acting in this region involve the linear combination $3n_1 - 5n_2 - 7n_3$ among the mean motion frequencies of Jupiter, Saturn and Uranus.

(d) The result (c) qualitatively persists also for the *planar* SJSUN system or when the influence of Neptune is neglected.

(e) Conversely, no chaotic motions are detected in the *planar* system including the Sun, Jupiter, Saturn and Uranus (hereafter, SJSU for shortness) for the same initial values of the semi-major axis of Uranus considered at point (c). This suggests that the resonances described at point (a) affect observable regions only when combined with some effects induced by Neptune or by the mutual inclinations.

By the way, we note that the resonances involving the linear combination $3n_1 - 5n_2 - 7n_3$ are clearly related to the approximate ratio 5 : 2 and 7 : 1 between the orbital motion of Jupiter and Saturn and of Jupiter and Uranus, respectively. Similarly, the ratio 2 : 1 between Uranus and Neptune appears also to be relevant (historically, this helped Le Verrier to predict the existence and the location of Neptune). The low order of the latter resonance may explain why the influence of Neptune induces some chaotic behavior, as pointed out in (d) and (e) above.

As we already mentioned in the Introduction, recently Hayes[86] [87] shows that the

scenario is also much more intricate. Hayes shows that the system of Sun and Jovian planets, integrated for 200 Myr as an isolated five-body system using many sets of initial conditions all within the uncertainty bounds of their currently known positions, can display both chaos and near-integrability. The numerical results are based on the calculation of the Lyapunov time, and are checked using four different integrators, including several comparisons against integrations using quadruple precision.

In fact there were some discrepancies between different authors about the existence of chaos in the orbits of the Jovian planets. Hayes shows that this apparent dilemma has a simple solution: the boundary, in the phase space, between chaos and near-integrability is finer than previously recognized. Actually, we find that some initial condition lead to chaos while others do not. So, for example, drawing initial conditions from the same ephemeris at different times, one finds some solutions that are chaotic, and some that are near-integrable. Thus, different works taken initial conditions from the same ephemeris at different times can find well different Lyapunov time-scales.

Let us remark that the Lyapunov time is the time needed by a dynamical system to become chaotic and reflects the limits of the predictability of the system. A second and maybe more important characteristic time scale is the *escape time*, which is the time for a major change in the orbit, for example the mean value of the semi-axes.

By the way, even if there are narrow regions with positive Lyapunov exponents, it could happen that the diffusive effects are very tiny. A work in this direction was done by Morbidelli and Froeschlé[61] in 1995. They consider the relationship between Lyapunov times and the macroscopic diffusion times of Hamiltonian systems, and they find out that there are two regimes: the Nekhoroshev regime and the resonant overlapping regime. In the first case the diffusion time is exponentially long with respect to Lyapunov time, while in the second case, the relationship is polynomial although they do not find any theoretical reason for the existence of a universal power law. They show numerical evidences which confirm their theoretical considerations.

The (weak) chaos in the motion of the Jovian planets makes somehow hopeless the task of describing their long-term evolution by a quasi-periodic approximation, as it is provided by the KAM theory. Therefore it appears to be more natural to look for exponential stability as assured by Nekhoroshev theory[68][69]. Indeed the theorem of Nekhoroshev applies to an open set of initial conditions, and states that the stability time increases exponentially with the inverse of the perturbation parameter. Our aim is to investigate whether the SJSU system may remain close to its current conditions

for a time that exceeds the lifetime of the system itself; e.g., in our case the age of the Universe, which is estimated to be approximately 1.4×10^{10} years, could be enough. We stress that the rather long time reported in (b) concerning the possible dissolution of the SJSUN system seems to support our hope.

Here, we restrict our attention to the SJSU *planar* system, due to the huge computational difficulties one encounters during the expansion of the Hamiltonian. Indeed, a rather long preliminary work is necessary in order to give the Hamiltonian a convenient form for starting more standard perturbation methods (see [56], [57] and [58]). We devote sects. 3.2 and 3.3 to this part of the problem.

Furthermore, in the line of Lagrange theory, we focus only on the secular part of the Hamiltonian, which is derived in subsect. 3.3.2. Let us emphasize that all along both sects. 3.2 and 3.3 we pay a special attention to include all the relevant terms related to the three-body mean motion quasi-resonance $3n_1 - 5n_2 - 7n_3$ in view of the remarks reported at points (a) and (c) above.

The secular system turns out to have the form of a perturbed system of harmonic oscillators. It can be remarked that the reduction to the plane model makes it possible to investigate the stability of the equilibrium using the theorem of Dirichlet. However, this is enough if we restrict our attention to the planar secular model, but does not apply neither to the spatial case, due to the opposite signs in the secular frequencies of the perihelia and of the nodes, nor in the full non secular model. Thus, we think that it is also useful to proceed by investigating the stability in Nekhoroshev sense, since this may give indication about the possibility of extending our calculation to more refined models. This part is worked out in sect. 3.4.

3.2 Classical Expansion of the Planar Planetary Hamiltonian

In appendix A we discussed the classical expansion of the planetary Hamiltonian, so let us consider a Hamiltonian F, written in restricted canonical Poincaré variables, that reads

$$F(\Lambda, \lambda, \xi, \eta) = F^{(0)}(\Lambda) + \mu F^{(1)}(\Lambda, \lambda, \xi, \eta)\,, \tag{3.1}$$

where $F^{(0)}$ is the unperturbed part, while $\mu F^{(1)}$ is the perturbation. Here, the small dimensionless parameter $\mu = \max\{m_1 / m_0\,,\ m_2 / m_0\,,\ m_3 / m_0\}$ has been introduced in order to highlight the different size of the terms appearing in the Hamiltonian. Let

us remark that the time derivative of each coordinate is $\mathcal{O}(\mu)$ except in the case of the angles λ. Therefore, according to the common language in celestial mechanics, in the following we will refer to λ and to their conjugate actions Λ as the *fast variables*, while (ξ, η) will be called *secular variables*.

We proceed now by expanding the Hamiltonian (3.1) in order to construct the first basic approximation of Kolmogorov normal form. We pick a value Λ^* for the fast actions and perform a translation $\mathcal{T}_{\Lambda^*}$ defined as

$$L_j = \Lambda_j - \Lambda_j^* \,, \qquad \text{for } j = 1\,,\,2\,,\,3\,. \tag{3.2}$$

This is a canonical transformation that leaves the coordinates λ, ξ and η unchanged. The transformed Hamiltonian $\mathcal{H}^{(\mathcal{T})} = F \circ \mathcal{T}_{\Lambda^*}$ can be expanded in power series of L, ξ, η around the origin. Thus, forgetting an unessential constant we rearrange the Hamiltonian of the system as

$$\mathcal{H}^{(\mathcal{T})}(L, \lambda, \xi, \eta) = n^* \cdot L + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(L) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T})}(L, \lambda, \xi, \eta)\,, \tag{3.3}$$

where the functions $h_{j_1,j_2}^{(\mathcal{T})}$ are homogeneous polynomials of degree j_1 in the actions L and of degree j_2 in the secular variables (ξ, η). The coefficients of such homogeneous polynomials do depend analytically and periodically on the angles λ. The terms $h_{j_1,0}^{(\mathrm{Kep})}$ of the Keplerian part are homogeneous polynomials of degree j_1 in the actions L, the explicit expression of which can be determined in a straightforward manner. In the latter equation the term which is both linear in the actions and independent of all the other canonical variables (i.e., $n^* \cdot L$) has been separated in view of its relevance in perturbation theory, as it will be discussed in the next section. We also expand the coefficients of the power series $h_{j_1,j_2}^{(T_F)}$ in Fourier series of the angles λ. The expansion of the Hamiltonian is a traditional procedure in celestial mechanics. We work out these expansions for the case of the planar SJSU system slightly modifying the procedure described in the appendix A.

The reduction to the planar case is performed as follows. We pick from Table IV of [84] the initial conditions of the planets in terms of heliocentric positions and velocities at the Julian Date 2440400.5. Next, we calculate the corresponding orbital elements with respect to the invariant plane (that is perpendicular to the total angular momentum). Finally we include the longitudes of the nodes Ω_j (which are meaningless in the planar case) in the corresponding perihelion longitude ω_j and we eliminate the

Table 3.1. Masses m_j and initial conditions for Jupiter, Saturn and Uranus in our planar model. We adopt the AU as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The initial conditions are expressed by the usual heliocentric planar orbital elements: the semi-major axis a_j, the mean anomaly M_j, the eccentricity e_j and the perihelion longitude ω_j. The data are taken by JPL at the Julian Date 2440400.5.

	Jupiter ($j = 1$)	Saturn ($j = 2$)	Uranus ($j = 3$)
m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$	$(2\pi)^2/22902.98$
a_j	5.20463727204700266	9.54108529142232165	19.2231635458410572
M_j	3.04525729444853654	5.32199311882584869	0.19431922829271914
e_j	0.04785365972484999	0.05460848595674678	0.04858667407651962
ω_j	0.24927354029554571	1.61225062288036902	2.99374344439246487

inclinations by setting them equal to zero. The remaining initial values of the orbital elements are reported in Table 3.1.

Having determined the initial conditions we come to determining the average values (a_1^*, a_2^*, a_3^*) of the semi-major axes during the evolution. To this end we perform a long-term numerical integration of Newton equations starting from the initial conditions related to the data reported in Table 3.1. After having computed (a_1^*, a_2^*, a_3^*), we determine the corresponding values Λ^*. This allows us to perform the expansion (3.3) of the Hamiltonian as a function of the canonical coordinates (L, λ, ξ, η). In our calculations we truncate the expansion as follows. (a) The Keplerian part is expanded up to the quadratic terms. The terms $h^{(\mathcal{T})}_{j_1,j_2}$ include: (b1) the linear terms in the actions L, (b2) all terms up to degree 18 in the secular variables (ξ, η), (b3) all terms up to the trigonometric degree 16 with respect to the angles λ. Our choice of the limits will be motivated in the next section.

3.3 The Secular Model

We look now for a good description of the secular dynamics. A straightforward method would be to include in the unperturbed Hamiltonian also the average of the perturbation over the fast angles. However, it has been remarked by Robutel[78] that the frequencies

of the quasi-periodic flow given by this secular Hamiltonian (often called of order one in the masses) are quite different from the true ones. The reason lies in the effect of the mean motion quasi-resonance $5:2$. Therefore we look for an approximation of the secular Hamiltonian up to order two in the masses (see, e.g., [44], [46], [78], [56] and [53]). To this end we follow the approach in [58], carrying out two "Kolmogorov-like" normalization steps in order to eliminate the main perturbation terms depending on the fast angles λ. We concentrate our attention on the resonant angles $2\lambda_1 - 5\lambda_2$, $\lambda_1 - 7\lambda_3$ and $3\lambda_1 - 5\lambda_2 - 7\lambda_3$, which are the most relevant ones for the dynamics. Our aim is to replace the orbit with zero eccentricity with a quasi periodic one that takes into account the effect of such resonances up to the second order in the masses. The procedure is a little cumbersome, and requires two main steps that we describe in the next two subsections.

3.3.1 Partial Reduction of the Perturbation

We emphasize that the Fourier expansion of the Hamiltonian (3.3) is generated just by terms due to two-body interactions, and so harmonics including more than two fast angles cannot appear. Thus, at first order in the masses only harmonics with the resonant angles $2\lambda_1 - 5\lambda_2$ and $\lambda_1 - 7\lambda_3$ do occur. Actually, harmonics with the resonant angle $3\lambda_1 - 5\lambda_2 - 7\lambda_3$ are generated by the first Kolmogorov-like transformation, but are of second order in the masses, and shall be removed by the second Kolmogorov-like transformation described in the next section.

Let us go into details. We denote by $\lceil f \rceil_{\lambda;K_F}$ the Fourier expansion of a function f truncated so as to include only its harmonics $k\cdot\lambda$ satisfying the restriction $0 < |k| \leq K_F$. We also denote by $\langle\cdot\rangle_\lambda$ the average with respect to the angles λ_1, λ_2, λ_3. The canonical transformations are using the Lie series algorithm (see, e.g., [32]).

We set $K_F = 8$ and transform the translated Hamiltonian (3.3) as $\hat{\mathcal{H}}^{(\mathcal{O}2)} = \exp\left(\mathcal{L}_{\mu\,\chi_1^{(\mathcal{O}2)}}\right)\mathcal{H}^{(\mathcal{T})}$ with the generating function $\mu\,\chi_1^{(\mathcal{O}2)}(\lambda,\xi,\eta)$ determined by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial\,\chi_1^{(\mathcal{O}2)}}{\partial\lambda_j} + \sum_{j_2=0}^{6} \left\lceil h_{0,j_2}^{(\mathcal{T})} \right\rceil_{\lambda;8} (\lambda,\xi,\eta) = 0\,. \tag{3.4}$$

Notice that, by definition, $\left\langle \lceil f \rceil_{\lambda;K_F} \right\rangle_\lambda = 0$, which assures that equation (3.4) can be solved provided the frequencies (n_1^*, n_2^*, n_3^*) are not resonant up to order 8, as it actually occurs in our planar model of the SJSU system.

The Hamiltonian $\hat{\mathcal{H}}^{(\mathcal{O}2)}$ has the same form of $\mathcal{H}^{(\mathcal{T})}$ in (3.3), with the functions

$h^{(\mathcal{T})}_{j_1,j_2}$ replaced by new ones, that we denote by $\hat{h}^{(\mathcal{O}2)}_{j_1,j_2}$, generated by the expanding the Lie series $\exp\left(\mathcal{L}_{\mu\,\chi_1^{(\mathcal{O}2)}}\right)\mathcal{H}^{(\mathcal{T})}$ and by gathering all the terms having the same degree both in the fast actions and in the secular variables.

Now we perform a second canonical transformation $\mathcal{H}^{(\mathcal{O}2)} = \exp\left(\mathcal{L}_{\mu\,\chi_2^{(\mathcal{O}2)}}\right)\hat{\mathcal{H}}^{(\mathcal{O}2)}$, where the generating function $\mu\,\chi_2^{(\mathcal{O}2)}(L,\lambda,\xi,\eta)$ (which is linear with respect to L) is determined by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial\,\chi_2^{(\mathcal{O}2)}}{\partial\lambda_j} + \sum_{j_2=0}^{6}\left[\hat{h}^{(\mathcal{O}2)}_{1,j_2}\right]_{\lambda;8}(L,\lambda,\xi,\eta) = 0\,. \tag{3.5}$$

Again, the Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ can be written in a form similar to (3.3), namely

$$\mathcal{H}^{(\mathcal{O}2)}(L,\lambda,\xi,\eta) = n^*\cdot L + \sum_{j_1=2}^{\infty} h^{(\mathrm{Kep})}_{j_1,0}(L) + \mu\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} h^{(\mathcal{O}2)}_{j_1,j_2}(L,\lambda,\xi,\eta;\mu)\,. \tag{3.6}$$

where the new functions $h^{(\mathcal{O}2)}_{j_1,j_2}$ are calculated as previously explained for $\hat{h}^{(\mathcal{O}2)}_{j_1,j_2}$. Moreover, they still have the same dependence on their arguments as $h^{(\mathcal{T})}_{j_1,j_2}$ in (3.3).

If terms of second order in μ are neglected, then the Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ possesses the secular 3-dimensional invariant torus $L = 0$ and $\xi = \eta = 0$. Thus, in a small neighborhood of the origin of the fast actions and for small eccentricities the solutions of the system with Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ differ from those of its average $\langle\mathcal{H}^{(\mathcal{O}2)}\rangle_\lambda$ by a quantity $\mathcal{O}(\mu^2)$. In this sense the average of the Hamiltonian (3.6) approximates the real dynamics of the secular variables up to order two in the masses, and due to the choice $K_F = 8$ takes into account the resonances $5:2$ between Jupiter and Saturn and $7:1$ between Jupiter and Uranus.

In this part of the calculation we produce a truncated series which is represented as a sum of monomials

$$c_{j,k,r,s}\,L_1^{j_1}L_2^{j_2}L_3^{j_3}\,\xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3}\,\eta_1^{s_1}\eta_2^{s_2}\eta_3^{s_3}\,\begin{matrix}\sin\\ \cos\end{matrix}(k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3)\,.$$

The truncated expansion of $\mathcal{H}^{(\mathcal{O}2)}$ contains 94 109 751 such monomials. We truncate our expansion at degree 16 in the fast angles λ and at degree 18 in the slow variables ξ, η (we shall justify this choice at the end of the next section).

3.3.2 Second Approximation and Reduction to the Secular Hamiltonian

The huge number of coefficients determined till now does not allow us to continue by keeping all of them. Therefore, in view that we plan to consider the secular system,

we perform a partial average by keeping only the main terms that contain the resonant angle $3\lambda_1 - 5\lambda_2 - 7\lambda_3$. More precisely, we first consider the reduced Hamiltonian

$$\left\langle \mathcal{H}^{(\mathcal{O}2)}\big|_{L=0} \right\rangle_\lambda = \mu \sum_{j_2=0}^{\infty} \langle h_{0,j_2}^{(\mathcal{O}2)}(\xi,\eta;\mu)\rangle_\lambda \,, \tag{3.7}$$

namely we set $L = 0$, which results in replacing the orbit having zero eccentricity with a close invariant torus of the unperturbed Hamiltonian, and average $\mathcal{H}^{(\mathcal{O}2)}$ by removing all the Fourier harmonics depending on the angles. Next, we select in $\mathcal{H}^{(\mathcal{O}2)}$ the Fourier harmonics that contain the wanted resonant angle $3\lambda_1 - 5\lambda_2 - 7\lambda_3$ and add them to the Hamiltonian (3.7). Finally, we perform on the resulting Hamiltonian the second Kolmogorov-like step. With more detail, this is the procedure, for $(j_1, j_2) \in \mathbf{N}^2$ we select the resonant terms

$$\begin{aligned} \mu^2 h_{j_1,j_2}^{(\mathrm{res})}(L,\lambda,\xi,\eta) = \mu \left\langle h_{j_1,j_2}^{(\mathcal{O}2)}\, e^{-i(3\lambda_1-5\lambda_2-7\lambda_3)} \right\rangle_\lambda e^{i(3\lambda_1-5\lambda_2-7\lambda_3)} + \\ \mu \left\langle h_{j_1,j_2}^{(\mathcal{O}2)}\, e^{i(3\lambda_1-5\lambda_2-7\lambda_3)} \right\rangle_\lambda e^{-i(3\lambda_1-5\lambda_2-7\lambda_3)} \,. \end{aligned} \tag{3.8}$$

Actually, this means that in our expression we just remove all monomials but the ones containing the wanted resonant angle. Using the selected terms we determine a generating function $\mu^2 \chi_1^{(\mathrm{res})}(\lambda,\xi,\eta)$ by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial \chi_1^{(\mathrm{res})}}{\partial \lambda_j} + \sum_{j_2=0}^{9} h_{0,j_2}^{(\mathrm{res})}(\lambda,\xi,\eta) = 0 \,. \tag{3.9}$$

Here we make the calculation faster by keeping only terms up to degree 9 in (ξ,η), this allows us to keep the more relevant resonant contributions. Then, still following the procedure outlined in [56], we calculate only the interesting part of the transformed Hamiltonian $\exp\left(\mathcal{L}_{\mu^2 \chi_2^{(\mathrm{res})}}\right) \exp\left(\mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}}\right) \mathcal{H}^{(\mathcal{O}2)}$, namely we keep in the transformation only the part which is independent of all the fast variables (L,λ). This produces the secular Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, which satisfies the formal equation

$$\left\langle \exp\left(\mathcal{L}_{\mu^2 \chi_2^{(\mathrm{res})}}\right) \exp\left(\mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}}\right) \mathcal{H}^{(\mathcal{O}2)} \right\rangle_\lambda = \mathcal{H}^{(\mathrm{sec})} + \mathcal{O}(\|L\|) + o(\mu^4) \,,$$

where

$$\begin{aligned} \mathcal{H}^{(\mathrm{sec})}(\xi,\eta) = \mu \sum_{j_2=0}^{\infty} \langle h_{0,j_2}^{(\mathcal{O}2)}\rangle_\lambda + \mu^4 \Bigg\langle \frac{1}{2} \left\{ \chi_1^{(\mathrm{res})}, \mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}} h_{2,0}^{(\mathrm{Kep})} \right\}_{L,\lambda} + \\ \left\{ \chi_1^{(\mathrm{res})}, \sum_{j_2=0}^{\infty} h_{1,j_2}^{(\mathrm{res})} \right\}_{L,\lambda} + \frac{1}{2} \left\{ \chi_1^{(\mathrm{res})}, \sum_{j_2=0}^{\infty} h_{0,j_2}^{(\mathrm{res})} \right\}_{\xi,\eta} \Bigg\rangle_\lambda \,. \end{aligned} \tag{3.10}$$

Here, we denoted by $\{\cdot,\cdot\}_{L,\lambda}$ and $\{\cdot,\cdot\}_{\xi,\eta}$ the terms of the Poisson bracket involving only the derivatives with respect the variables (L,λ) and (ξ,η), respectively.

The Hamiltonian so constructed is the secular one, describing the slow motion of eccentricities and perihelia. In view of D'Alembert rules (see, e.g., [72]), it contains only terms of even degree and so the lowest order significant term has degree 2. We have determined the power series expansion of the Hamiltonian up to degree 18 in the slow variables. In order to allow a comparison with other expansions, we reported our results up to degree 4 in (ξ,η) in the appendix 3.A of the present chapter.

We close this section with a few remarks which justify our choice of the truncation orders. The limits on the expansions in the fast actions L have been illustrated at points (a) and (b1) at the end of section 3.3, and they are the smallest ones that are required in order to make the Kolmogorov-like normalization procedure significant. Since we want to keep the resonant angles $2\lambda_1 - 5\lambda_2$, $\lambda_1 - 7\lambda_3$ and $3\lambda_1 - 5\lambda_5 - 7\lambda_3$, we set the truncation order for Fourier series to 16, which is enough. The choice to truncate the expansion at degree 18 in the secular variables (ξ,η) is somehow subtler. In view of D'Alembert rules the harmonics $2\lambda_1 - 5\lambda_2$ and $\lambda_1 - 7\lambda_3$ have coefficients of degree at least 3 and 6, respectively, in the secular variables. Furthermore, the resonant angle $3\lambda_1 - 5\lambda_5 - 7\lambda_3$ does not appear initially in the Hamiltonian, but is generated by Poisson bracket between the harmonics $2\lambda_1 - 5\lambda_2$ and $\lambda_1 - 7\lambda_3$, which produces monomials of degree 9 in (ξ,η). Therefore, we decided to calculate the generating functions $\chi_1^{(\mathcal{O}2)}$ and $\chi_2^{(\mathcal{O}2)}$ up to degree 9 (recall equations (3.4) and (3.5)). Finally, in the second Kolmogorov-like step we want to keep the secular terms generated by the harmonic $3\lambda_1 - 5\lambda_5 - 7\lambda_3$, which are produced by Poisson bracket between monomials containing precisely this harmonic, and then the result has maximum degree 18 in (ξ,η). This explains the final truncation order for the slow variables.

3.4 Stability of the Secular Hamiltonian Model

The lowest order approximation of the secular Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, namely its quadratic term, is essentially the one considered in the theory first developed by Lagrange[38] and later improved by Laplace[41][42][43]) and by Lagrange himself[39][40]. In modern language, we say that the origin of the reduced phase space (i.e., $(\xi,\eta) = (0,0)$) is an elliptic equilibrium point (for a review using a modern formalism, see sect. 3 of [5],

Table 3.2. Angular velocities ω and initial conditions $(x(0), y(0))$ for our planar secular model about the motions of Jupiter, Saturn and Uranus. The frequency vector ω refer to the harmonic oscillators approximation of the Hamiltonian $H^{(0)}$ (written in (3.11)) and its values are given in $rad/year$.

	$j=1$	$j=2$	$j=3$
ω_j	$-1.1212724892 \times 10^{-4}$	$-1.9688444678 \times 10^{-5}$	$-1.1134564418 \times 10^{-5}$
$x_j(0)$	$1.5407573458 \times 10^{-2}$	$-3.0574059274 \times 10^{-2}$	$1.1186486403 \times 10^{-2}$
$y_j(0)$	$-2.5320810665 \times 10^{-2}$	$-5.2728862107 \times 10^{-3}$	$6.0669645406 \times 10^{-3}$

where a planar model of our Solar System is considered).

It is well known that (under mild assumptions on the quadratic part of the Hamiltonian which are satisfied in our case) one can find a linear canonical transformation $(\xi, \eta) = \mathcal{D}(x, y)$ which diagonalizes the quadratic part of the Hamiltonian, so that we may write $\mathcal{H}^{(\mathrm{sec})}$ in the new coordinates as

$$H^{(0)}(x, y) = \sum_{j=0}^{3} \frac{\omega_j}{2} \left(x_j^2 + y_j^2\right) + H_2^{(0)}(x, y) + H_4^{(0)}(x, y) + H_6^{(0)}(x, y) + \dots , \qquad (3.11)$$

where ω_j are the secular frequencies in the small oscillations limit and $H_{2s}^{(0)}$ is a homogeneous polynomial of degree $2s + 2$ in (x, y). The calculated values of $(\omega_1, \omega_2, \omega_3)$ in our case are reported in Table 3.2.

Thus, we are led to study the stability of the equilibrium for the Hamiltonian (3.11). As remarked in the introduction, perpetual stability in a neighborhood of the equilibrium is assured in our case by Dirichlet's theorem because all frequencies have the same sign, that is negative in our case. Actually, a very rough evaluation of the size of the stability neighborhood gives a value about 0.6 times the distance (from the origin) of the actual initial data of the planets. Such an estimate should certainly be improved by a more accurate calculation, i.e., by determining the stationary points of a function in 6 variables. However, we emphasize that our model is just a planar approximation of the true problem. If, for instance, one considers the spatial secular problem then the secular frequencies of the nodes have a positive sign, so that Dirichlet theory does not apply any more. Thus, we think it is more interesting to investigate the stability of the equilibrium in the light of Nekhoroshev theory.

3.4.1 Birkhoff Normal Form

Following a quite standard procedure we proceed to construct the Birkhoff normal

form for the Hamiltonian (3.11) (see [6]; for an application of Nekhoroshev theory see, e.g., [23]). This is a well known matter, thus we limit our exposition to a short sketch adapted to the present context.

The aim is to give the Hamiltonian the normal form at order r

$$H^{(r)}(x,y) = Z_0(\Phi) + \ldots + Z_r(\Phi) + \mathcal{F}^{(r)}_{r+1}(x,y) + \mathcal{F}^{(r)}_{r+2}(x,y) + \ldots\,, \tag{3.12}$$

where

$$\Phi_j = \frac{1}{2}\left(x_j^2 + y_j^2\right) \qquad \text{for } j = 1,\, 2,\, 3\,, \tag{3.13}$$

are the actions of the system, and Z_s for $s = 0, \ldots, r$ is a homogeneous polynomial of degree $s/2+1$ in Φ and in particular it is zero for odd s. The un-normalized remainder terms $\mathcal{F}^{(r)}_s$, where $s > r$, are homogeneous polynomials of degree $s+2$ in (x,y).

We proceed by induction. Assume that the Hamiltonian is in normal form up to a given order r, which is trivially true for $r = 0$, and determine a generating function $\chi^{(r+1)}$ and the normal form term Z_{r+1}, by solving the equation

$$\left\{\chi^{(r+1)}\,,\, \omega\cdot\Phi\right\} + \mathcal{F}^{(r)}_{r+1}(x,y) = Z_{r+1}(\Phi)\,. \tag{3.14}$$

Using the algorithm of Lie series transform, we can write the new Hamiltonian as $H^{(r+1)} = \exp \mathcal{L}_{\chi^{(r+1)}}\, H^{(r)}$. It is not difficult to show that $H^{(r+1)}$ has a form analogous to that written in (3.12) with new functions $\mathcal{F}^{(r+1)}_s$ of degree $s+2$ (where $s > r+1$) and the normal form part ending with Z_{r+1}, which is equal to zero if r is even (see, e.g., [33]). As usual when using the Lie series methods, we denote by (x,y) the new coordinates, so that the normal form $H^{(r)}$ possesses the approximate first integrals Φ given by (3.13). By the way, the algorithm can be iterated up to the step r provided that the non-resonance condition

$$k\cdot\omega \neq 0 \qquad \forall\, k \in \mathbf{Z}^3 \text{ such that } 0 < |k| \leq r+2\,, \tag{3.15}$$

is fulfilled.

3.4.2 Study of the Stability Time

It is well known that the composition of changes of variables to obtain the Birkhoff normal form, at any finite order r, are analytic functions in some neighborhood of the origin, but the analyticity radius shrinks to zero when $r \to \infty$. Thus, the best we can do is to look for stability for a finite but long time. We use the algorithm adopted in [26], that we describe here.

Let us pick three positive numbers R_1, R_2, R_3 and consider a polydisk $\Delta_{\varrho R}$ with center at the origin of $\mathbf{R}^6$ defined as

$$\Delta_{\varrho R} = \left\{(x, y) \in \mathbf{R}^6 : x_j^2 + y_j^2 \le \varrho^2 R_j^2 \,,\ j = 1,\, 2,\, 3\right\} \,,$$

$\varrho > 0$ being a parameter. Let $\varrho_0 = \varrho/2$, and let $(x_0, y_0) \in \Delta_{\varrho_0 R}$ be the initial point of an orbit, so that one has $\Phi_j(0) = (x_j^2 + y_j^2)/2 \le \varrho_0^2 R_j^2/2$. Therefore, there is $T(\varrho_0) > 0$ such that for $|t| \le T(\varrho_0)$ we have $\Phi(t) \le \varrho^2 R_j^2/2$, and so also $(x(t), y(t)) \in \Delta_{\varrho R}$. We call $T(\varrho_0)$ the *estimated stability time*, and our aim is to give a good estimate of it.

The key remark is that one has

$$\dot{\Phi}_j = \left\{\Phi_j \,,\, H^{(r)}\right\} = \sum_{s=r+1}^{\infty} \left\{\Phi_j \,,\, \mathcal{F}_s^{(r)}\right\} \simeq \left\{\Phi_j \,,\, \mathcal{F}_{r+1}^{(r)}\right\} \qquad \text{for } j = 1,\, 2,\, 3 \,, \tag{3.16}$$

which holds true for an arbitrary normalization order r. This means that the time derivative of $\Phi(t)$ is small, being $\mathcal{O}(\varrho^{r+3})$, so that the time $T(\varrho_0)$ may grow very large. The basis of Nekhoroshev theory is that one can choose an optimal value of r as a function of ϱ_0 letting it to get larger and larger when $\varrho_0 \to 0$, so that $T(\varrho_0)$ grows faster than any power of $1/\varrho_0$. Here we give this argument an algorithmic form, thus producing an explicit estimate of $T(\varrho_0)$.

Let us write a homogeneous polynomial $f(x, y)$ of degree s as

$$f(x, y) = \sum_{|j|+|k|=s} f_{j,k}\, x^j y^k \,,$$

where the multiindex notation $x^j y^k = x_1^{j_1} x_2^{j_2} x_3^{j_3} y_1^{k_1} y_2^{k_2} y_3^{k_3}$ has been used. We define the quantity $|f|_R$ as

$$|f|_R = \sum_{|j|+|k|=s} |f_{j,k}| R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \Theta_{j_1,k_1} \Theta_{j_2,k_2} \Theta_{j_3,k_3} \,, \qquad \Theta_{j,k} = \sqrt{\frac{j^j k^k}{(j+k)^{j+k}}} \,. \tag{3.17}$$

We claim that for $\varrho > 0$ one has

$$\sup_{(x,y)\in\Delta_{\varrho R}} \left|f(x, y)\right| < \varrho^s |f|_R \,. \tag{3.18}$$

The estimate is checked as follows. In the plane x_i, y_i consider a disk with radius R_i. Then inside the disk the inequality $|x_i^{j_i} y_i^{k_i}| \le R_i^{j_i+k_i} \Theta_{j_i,k_i}$ holds true. In fact, after having set $x_i = R_i \cos\vartheta$, $y_i = R_i \sin\vartheta$, one can easily check that $|\cos^{j_i}\vartheta \sin^{k_i}\vartheta| \le \Theta_{j_i,k_i}$. It is then straightforward to verify that for a monomial $x^j y^k$ of degree s one

has

$$\sup_{(x,y)\in\Delta_{\varrho R}} \left|x^j y^k\right| \le \varrho^s R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \Theta_{j_1,k_1}\Theta_{j_2,k_2}\Theta_{j_3,k_3}\,.$$

The wanted inequality is just the sum of the contributions of all monomials.

Using (3.18) and (3.16) we can estimate

$$\sup_{(x,y)\in\Delta_{\varrho R}} \left|\dot\Phi_j(x,y)\right| < C\varrho^{r+3}\left|\{\Phi_j, \mathcal{F}_{r+1}^{(r)}\}\right|_R\,, \tag{3.19}$$

for $j = 1, 2, 3$ and with some $C \ge 1$. In fact, after having set ϱ smaller than the convergence radius of the remainder series $\mathcal{F}_s^{(r)}$ (where $s > r$), the above inequality is true for some C. In our calculation we set $C = 2$.

We come now to the calculation of the estimated stability time. Since $\Phi_j = \varrho^2 R_j^2/2$, we have $\dot\Phi_j = R_j^2\varrho\dot\varrho$ and, in view of inequality (3.19), also

$$\dot\varrho \le \frac{B_{r,j}}{R_j^2}\varrho^{r+2}\,, \quad B_{r,j} = C\left|\{\Phi_j, \mathcal{F}_{r+1}^{(r)}\}\right|_R\,.$$

Thus a majorant of the function $\varrho(t)$ is given by the solution of the equation $\dot\varrho = B_{r,j}\varrho^{r+2}/R_j^2$. Setting ϱ_0 as the initial value we conclude that $\varrho(t) \le 2\varrho_0$ for all $|t| \le \tau(\varrho_0, r)$, where

$$\tau(\varrho_0, r) = \min_j \frac{R_j^2}{B_{r,j}} \int_{\varrho_0}^{2\varrho_0} \frac{d\sigma}{\sigma^{r+2}} = \min_j \left(1 - \frac{1}{2^{r+1}}\right) \frac{R_j^2}{(r+1)B_{r,j}\,\varrho_0^{r+1}}\,. \tag{3.20}$$

The latter estimate holds true for arbitrary normalization order r. Therefore we select an optimal order $r_{\rm opt}(\varrho_0)$ by looking for the maximum over r of $\tau(\varrho_0, r)$, thus getting

$$T(\varrho_0) = \max_r \tau(\varrho_0, 2\varrho_0, r)\,. \tag{3.21}$$

This is the best estimate of the stability time given by our algorithm.

3.4.3 Application to the SJSU System

We apply the algorithm of the previous section to the secular Hamiltonian $\mathcal{H}^{(\rm sec)}$ by explicitly performing the construction of Birkhoff normal form up to order 30. Meanwhile also the first term of the remainder has been stored, so that the estimate for $\dot\Phi$ is provided.

The calculation of the estimated stability time is performed by setting

$$R_1 = 2.5558203988 \times 10^{-2}\,, R_2 = 3.0601862602 \times 10^{-2}\,, R_3 = 1.1223294461 \times 10^{-2}\,.$$

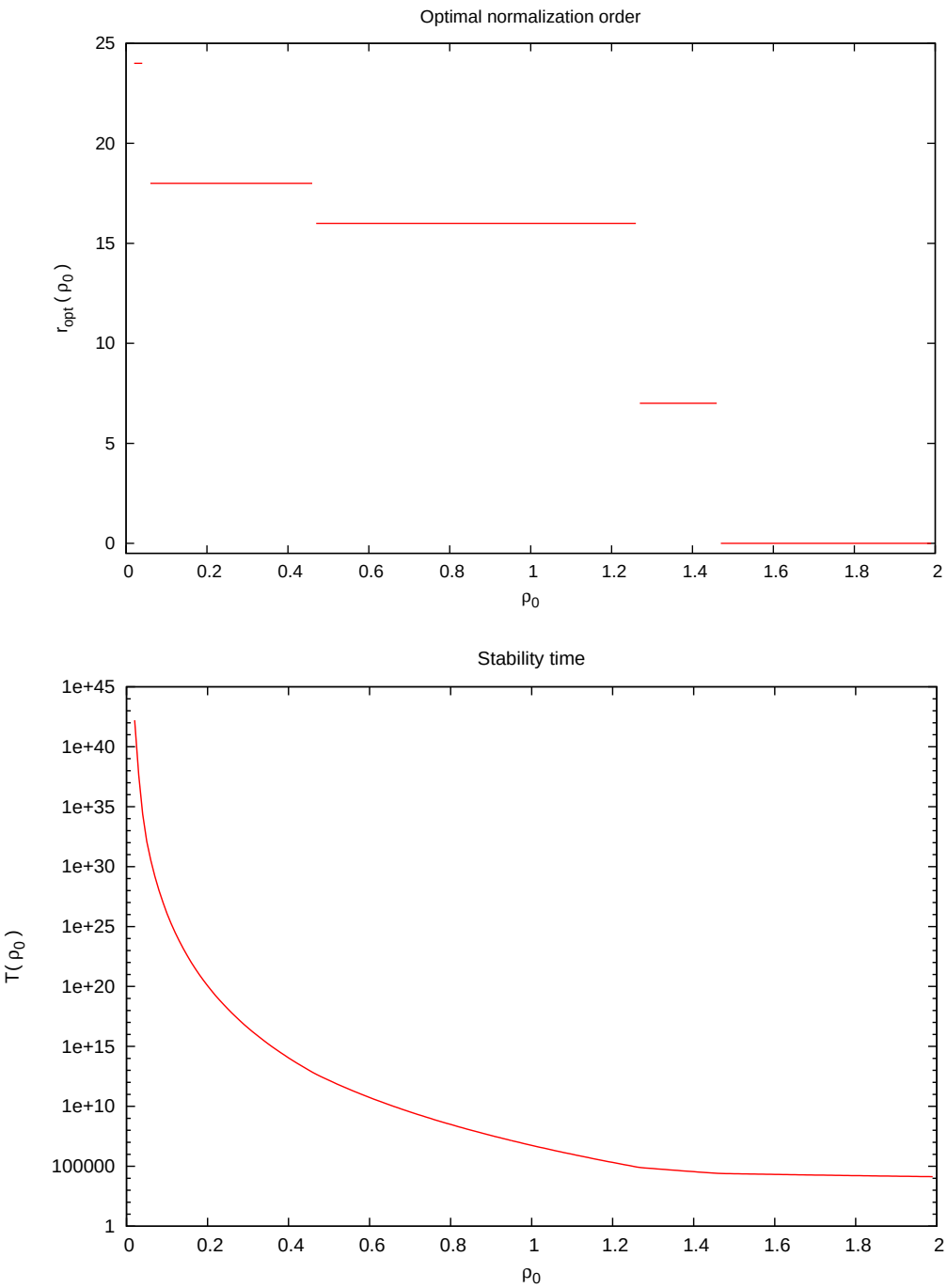

Figure 3.1. Optimal normalization order r_{opt} and estimated stability time $T(\varrho_0)$ evaluated according to the algorithm of sect. 3.4.2. The time unit is the year. See text for more details.

These values have been calculated as $R_j = \sqrt{x_j^2(0) + y_j^2(0)}$ where $x_j(0)$, $y_j(0)$ are the

initial data reported in table 3.2, so that the initial point is on the border of the polydisk $\Delta_{\varrho R}$ with $\varrho = 1$.

Finally we proceed to calculating the optimal normalization order $r_{\rm opt}(\varrho_0)$ and the estimated stability time $T(\varrho_0)$ as functions of ϱ_0 in an interval such that the optimal normalization order produced by our algorithm is less than 30. The results are reported in fig. 3.1. The fast increase of the time when ϱ decreases is evident from the graph. We also remark that for $\varrho_0 = 1$, which corresponds to the initial data for the planets, the normalization order is already $r_{\rm opt} = 16$. This shows that the mechanism of long time stability is already active. The estimated time with our algorithm is about 10^7 years for $\varrho_0 = 1$. This seems to be quite short both with respect to the age of the Solar System (which is estimated to be approximately 5×10^9 years) and with respect to the numerical indications (10^{18} years). We shall comment on this point in the next section.

3.5 Comments

In the framework provided by the Nekhoroshev's theorem, the present work describes the first attempt to study the stability of a realistic model with more than two planets of our Solar System. As remarked at the end of the previous section we are not yet able to prove the stability for a time comparable to the age of our planetary system, even restricting ourselves to consider just the secular part of a planar approximation including the Sun, Jupiter, Saturn and Uranus. Nevertheless, we think that our results is meaningful in that it indicates that the phenomenon of exponential stability in Nekhoroshev sense may play an effective role for the Solar System, at least for the biggest planets. On the other hand, we stress that our result is not dramatically far from the goal of proving stability for the age of the Solar System: such a time is reached for a domain with radius ϱ_0 about 0.7 times the one containing the initial data. By the way, it may be worth to note that a similar result, with the same value of the radius, has been found in [26] where the spatial problem for the Sun–Jupiter–Saturn system is considered. Such a value of ϱ_0 appears to be not so small, especially if one recalls the rough estimates based on the first purely analytical proofs of the KAM theorem: in order to apply them to some model of our planetary system, the Jupiter mass should be smaller than that of a proton. Improvements are surely possible, and the relatively short history of the applications of the Nekhoroshev type estimates to celestial me-

chanics has shown definitely more remarkable improvements than the one required here (e.g., compare [24] with [27]).

Some drawbacks are immediately evident. The most relevant one is that the estimate in (3.20) actually assumes that the perturbation constantly forces the worst possible evolution. This is clearly pessimistic, and justifies the striking difference with respect to the indication given by the numerical integrations. On the other hand, general perturbation method are essentially based on estimates that are often very crude. The explicit calculation of normal forms and related quantities allows us to significantly improve our results, but the price is either a bigger and bigger computer power or more and more refined methods.

The natural question is whether there is a way to improve the present result. Our approach suggests that a better approximation of the true orbit could help a lot. This can be obtained, e.g., by first establishing the existence of a KAM torus close to the initial conditions of the planets, and then proving the stability in Nekhoroshev sense in a neighborhood of the torus that contains the initial data. Such an approach has been attempted in [25] for the Sun–Jupiter–Saturn case considering the full system, i.e., avoiding the approximation of the secular model. In that case the number of coefficients to be handled is so huge that the calculation can actually be performed only by introducing strong truncations on the expansions; this might artificially improve the results. Thus, some new idea is necessary, and this will be work for the future.

3.A Expansion of the Secular Hamiltonian of the Planar SJSU System

Our secular model is represented by the Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, which is defined in (3.10). Here, we limit ourselves to report the expansion of $\mathcal{H}^{(\mathrm{sec})}$ up to degree 4 in (ξ, η). Therefore, as a consequence of the D'Alembert rules, the terms related to the quasi-resonance $3\lambda_1 - 5\lambda_2 - 7\lambda_3$ do not give any contribution to the coefficients listed below. Thus, the following expansion of the rhs of (3.10) actually takes into account just $\mu\langle h_{0,2}^{(\mathcal{O}2)}\rangle_\lambda + \mu\langle h_{0,4}^{(\mathcal{O}2)}\rangle_\lambda$ (recall that $\mathcal{H}^{(\mathrm{sec})}$ contains just terms of even degree in its variables (ξ, η)). The calculation of the functions $h_{0,2}^{(\mathcal{O}2)}$ and $h_{0,4}^{(\mathcal{O}2)}$ is performed how it has been explained in subsect. 3.3.1.

$$
\begin{aligned}
&\mathcal{H}^{(\mathrm{sec})}(\xi,\eta) = \\
&-2.0438249530856989 \times 10^{-05}\,\xi_1^2 && +3.9042681895470743 \times 10^{-05}\,\xi_1^1\,\xi_2^1 \\
&+4.5005164146422330 \times 10^{-07}\,\xi_1^1\,\xi_3^1 && -4.5352294644578622 \times 10^{-05}\,\xi_2^2 \\
&+1.9490388069796070 \times 10^{-06}\,\xi_2^1\,\xi_3^1 && -5.6845848333331483 \times 10^{-06}\,\xi_3^2 \\
&-2.0438249530856989 \times 10^{-05}\,\eta_1^2 && +3.9042681895470675 \times 10^{-05}\,\eta_1^1\,\eta_2^1 \\
&+4.5005164146422409 \times 10^{-07}\,\eta_1^1\,\eta_3^1 && -4.5352294644578622 \times 10^{-05}\,\eta_2^2 \\
&+1.9490388069796070 \times 10^{-06}\,\eta_2^1\,\eta_3^1 && -5.6845848333331441 \times 10^{-06}\,\eta_3^2 \\
&-1.0838003720922759 \times 10^{-04}\,\xi_1^4 && +1.2014175808584642 \times 10^{-03}\,\xi_1^3\,\xi_2^1 \\
&+6.2045352476790196 \times 10^{-07}\,\xi_1^3\,\xi_3^1 && -4.5563232782076350 \times 10^{-03}\,\xi_1^2\,\xi_2^2 \\
&+8.8406443127175810 \times 10^{-07}\,\xi_1^2\,\xi_2^1\,\xi_3^1 && -9.7678628300067324 \times 10^{-06}\,\xi_1^2\,\xi_3^2 \\
&-2.1676479523871672 \times 10^{-04}\,\xi_1^2\,\eta_1^2 && +1.2014125316196400 \times 10^{-03}\,\xi_1^2\,\eta_1^1\,\eta_2^1 \\
&+6.2157409102827665 \times 10^{-07}\,\xi_1^2\,\eta_1^1\,\eta_3^1 && -1.5832006427474584 \times 10^{-03}\,\xi_1^2\,\eta_2^2 \\
&+3.0033462029049336 \times 10^{-07}\,\xi_1^2\,\eta_2^1\,\eta_3^1 && -7.4173186653205456 \times 10^{-06}\,\xi_1^2\,\eta_3^2 \\
&+7.6046689202847869 \times 10^{-03}\,\xi_1^1\,\xi_2^3 && -2.4429460187142667 \times 10^{-06}\,\xi_1^1\,\xi_2^2\,\xi_3^1 \\
&+3.9912387029285291 \times 10^{-07}\,\xi_1^1\,\xi_2^1\,\xi_3^2 && +1.2014125316196422 \times 10^{-03}\,\xi_1^1\,\xi_2^1\,\eta_1^2 \\
&-5.9464179266765730 \times 10^{-03}\,\xi_1^1\,\xi_2^1\,\eta_1^1\,\eta_2^1 && +5.8365071555190281 \times 10^{-07}\,\xi_1^1\,\xi_2^1\,\eta_1^1\,\eta_3^1 \\
&+7.6047082339419673 \times 10^{-03}\,\xi_1^1\,\xi_2^1\,\eta_2^2 && -1.6360484568891480 \times 10^{-06}\,\xi_1^1\,\xi_2^1\,\eta_2^1\,\eta_3^1 \\
&+2.2482538047243290 \times 10^{-07}\,\xi_1^1\,\xi_2^1\,\eta_3^2 && +2.6233130055605185 \times 10^{-05}\,\xi_1^1\,\xi_3^3 \\
&+6.2157409102827644 \times 10^{-07}\,\xi_1^1\,\xi_3^1\,\eta_1^2 && +5.8365071555190228 \times 10^{-07}\,\xi_1^1\,\xi_3^1\,\eta_1^1\,\eta_2^1 \\
&-4.6671904570366227 \times 10^{-06}\,\xi_1^1\,\xi_3^1\,\eta_1^1\,\eta_3^1 && -8.0739065076924997 \times 10^{-07}\,\xi_1^1\,\xi_3^1\,\eta_2^2 \\
&+5.4429327654203341 \times 10^{-08}\,\xi_1^1\,\xi_3^1\,\eta_2^1\,\eta_3^1 && +2.6230652324380928 \times 10^{-05}\,\xi_1^1\,\xi_3^1\,\eta_3^2 \\
&-4.8323841400859345 \times 10^{-03}\,\xi_2^4 && +2.9298658121783215 \times 10^{-05}\,\xi_2^3\,\xi_3^1 \\
&-1.3020117317952433 \times 10^{-04}\,\xi_2^2\,\xi_3^2 && -1.5832006427474452 \times 10^{-03}\,\xi_2^2\,\eta_1^2 \\
&+7.6047082339419534 \times 10^{-03}\,\xi_2^2\,\eta_1^1\,\eta_2^1 && -8.0739065076924796 \times 10^{-07}\,\xi_2^2\,\eta_1^1\,\eta_3^1 \\
&-9.6647220999081795 \times 10^{-03}\,\xi_2^2\,\eta_2^2 && +2.9299286278904711 \times 10^{-05}\,\xi_2^2\,\eta_2^1\,\eta_3^1 \\
&-7.7905487286026464 \times 10^{-05}\,\xi_2^2\,\eta_3^2 && +1.9476359726545943 \times 10^{-04}\,\xi_2^1\,\xi_3^3 \\
&+3.0033462029049320 \times 10^{-07}\,\xi_2^1\,\xi_3^1\,\eta_1^2 && -1.6360484568891460 \times 10^{-06}\,\xi_2^1\,\xi_3^1\,\eta_1^1\,\eta_2^1 \\
&+5.4429327654202779 \times 10^{-08}\,\xi_2^1\,\xi_3^1\,\eta_1^1\,\eta_3^1 && +2.9299286278906453 \times 10^{-05}\,\xi_2^1\,\xi_3^1\,\eta_2^2
\end{aligned}
$$

$$
\begin{aligned}
&-1.0427780546265602 \times 10^{-04}\, \xi_2^1\, \xi_3^1\, \eta_2^1\, \eta_3^1 && +1.9476665827159131 \times 10^{-04}\, \xi_2^1\, \xi_3^1\, \eta_3^2 \\
&-2.0277494194124600 \times 10^{-04}\, \xi_3^4 && -7.4173186653203601 \times 10^{-06}\, \xi_3^2\, \eta_1^2 \\
&+2.2482538047243565 \times 10^{-07}\, \xi_3^2\, \eta_1^1\, \eta_2^1 && +2.6230652324380681 \times 10^{-05}\, \xi_3^2\, \eta_1^1\, \eta_3^1 \\
&-7.7905487286035477 \times 10^{-05}\, \xi_3^2\, \eta_2^2 && +1.9476665827159768 \times 10^{-04}\, \xi_3^2\, \eta_2^1\, \eta_3^1 \\
&-4.0555535091919988 \times 10^{-04}\, \xi_3^2\, \eta_3^2 && -1.0838003720922736 \times 10^{-04}\, \eta_1^4 \\
&+1.2014175808584629 \times 10^{-03}\, \eta_1^3\, \eta_2^1 && +6.2045352476790196 \times 10^{-07}\, \eta_1^3\, \eta_3^1 \\
&-4.5563232782075760 \times 10^{-03}\, \eta_1^2\, \eta_2^2 && +8.8406443127175704 \times 10^{-07}\, \eta_1^2\, \eta_2^1\, \eta_3^1 \\
&-9.7678628300066206 \times 10^{-06}\, \eta_1^2\, \eta_3^2 && +7.6046689202847939 \times 10^{-03}\, \eta_1^1\, \eta_2^3 \\
&-2.4429460187142612 \times 10^{-06}\, \eta_1^1\, \eta_2^2\, \eta_3^1 && +3.9912387029285359 \times 10^{-07}\, \eta_1^1\, \eta_2^1\, \eta_3^2 \\
&+2.6233130055604931 \times 10^{-05}\, \eta_1^1\, \eta_3^3 && -4.8323841400860802 \times 10^{-03}\, \eta_2^4 \\
&+2.9298658121781443 \times 10^{-05}\, \eta_2^3\, \eta_3^1 && -1.3020117317952618 \times 10^{-04}\, \eta_2^2\, \eta_3^2 \\
&+1.9476359726546422 \times 10^{-04}\, \eta_2^1\, \eta_3^3 && -2.0277494194122486 \times 10^{-04}\, \eta_3^4 \\
&+o\big(\|(\xi,\eta)\|^4\big)\,. &&
\end{aligned}
$$

4

Construction of the Normal Form for Elliptic Tori in Planetary Systems

[Semi-Analytical Part]

"Be a man, make your own programs!"
— Carles Simó

In this chapter we improve Lagrange theory by looking for an unperturbed elliptic torus. The theory developed in the previous chapters is based on the secular problem, namely, the fast variables are removed by setting the corresponding actions to a constant value. This corresponds to using circular orbits as an initial approximation, as was done in Lagrange theory. However, we should remark that circular orbits are not solutions of Newton equations.

A better approximation may be found by looking for orbits which are actual solutions of Newton equations, and are close to elliptic ones. These orbits lie on invariant elliptic tori. In fact, an elliptic torus can be represented as the Cartesian product of the origin of the secular variables and of the origin of the fast actions, while the fast angles perform a quasi-periodic motion.

We develop an explicit algorithm to construct the normal form of a planetary Hamiltonian related to elliptic tori. In this chapter we focus our efforts on a direct application to a planetary system that is an approximation of the planar Sun–Jupiter–Saturn–Uranus system. Thus, we will check the effectiveness of our semi-analytic procedure, by calculating a finite number of steps of the algorithm by algebraic manipulations on a computer. The theoretical study of the convergence of our algorithm is deferred to the next chapter, where we translate our semi-analytic approach into a rigorous proof ensuring the existence of the elliptic tori.

4.1 Overview

Since the birth of the KAM theory (see [37], [63] and [1]), the invariant tori are expected to be the key dynamical object which explains the (nearly perfect) quasi-periodicity of the planetary motions of our Solar System.

Among the consequences of the KAM theory, which concern the tori of maximal dimension, the following one looks natural. One expects that the persistence under small perturbations should hold also for the n-dimensional invariant tori related to the limit case of small circular orbits (in the integrable approximation, these surfaces are such that p is constant, $q \in \mathbf{T}^n$ and $(x, y) = (0, 0)$). However, a separate proof is needed in order to ensure the existence of these lower dimensional tori which are said to be elliptic, because they correspond to stable equilibrium points of the secular motions. Such a theorem has been recently proved by Biasco, Chierchia and Valdinoci in two different

cases: for the spatial three-body planetary problem and for a planar system with a central star and n planets (see [4] and [5], respectively). In our opinion, their approach is deep from a theoretical point of view, but is not suitable for explicit applications, even if one is interested just in finding the locations of the elliptic invariant tori. In order to clarify this point, let us roughly summarize the scheme of their proofs as follows: first, they carry out all the preliminary canonical transformations that are necessary to bring the Hamiltonian in a particular form, to which they can subsequently apply a theorem due to Pöschel (see [74] and [75]), so to ensure the existence of elliptic lower dimensional tori. Moreover, Pöschel versions of this theorem are based on a careful adaptation of the usual Arnold's proof scheme for non-degenerate systems: the perturbation is removed by a sequence of canonical transformations which are defined on a subset of the phase space excluding the "resonant regions" (see [2]). Since resonances are everywhere dense (but the width of the regions eliminated around them is suitably decreased, when the order of the resonances increases), therefore the change of coordinates giving the shape of the invariant elliptic tori is defined on a Cantor set which *does not contain any open subset*. The efficiency of an eventual explicit application based of such an approach is highly questionable and, as far as we know, it has never been used to calculate an orbit of a celestial mechanics problem.

The original proof scheme of the KAM theorem, introduced by Kolmogorov himself, is in a much better position for what concerns the translation into an explicit algorithm constructing invariant tori (see [37], [3], [29] and [30]). In fact, this approach has been successfully used to calculate the orbits for some interesting problems in celestial mechanics (see [56], [57], [58] and [22]). The present work aims to adapt the Kolmogorov algorithm, in order to construct a suitable normal form related to the elliptic tori. Moreover, this will allow us to explicitly integrate the equations of motion on those invariant surfaces, by using a so called semi-analytic procedure.

When one is interested in showing the long term stability of a planetary system, the construction of a normal form related to some fixed elliptic torus could be a relevant milestone. In fact, as we see in chapter 2, it is possible to ensure the effective stability in the neighborhood of such an invariant surface by implementing a partial construction of the Birkhoff normal form (see, e.g., [35] and [25], where this approach is used in order to study the stability nearby an invariant KAM torus having maximal dimension). For what concerns our Solar System, such an approach might be applied to some asteroids with small orbital eccentricities and inclinations. However, as explained in the previous

chapter, this same approach cannot yet succeed in proving the long-time stability of the major planets of our Solar System.

The location of the elliptic tori can be useful also for practical purposes. In fact, the regions close to them are exceptionally stable, being mainly filled by invariant tori of maximal dimension. Therefore, they can be of interest for spatial missions aiming, for instance, to observe asteroids not far from the elliptic tori. Moreover, our technique should adapt quite easily also to the construction of hyperbolic tori that can be used in the design of spacecraft missions with transfers requiring low energy. Also in view of this kind of applications, lower dimensional tori of elliptic, hyperbolic and mixed type have been studied in the vicinity of the Lagrangian points for both the restricted three-body problem and the bicircular restricted four-body problem (see, e.g., [34], [36], [9] and [21]).

The present chapter is organized as follows. The search for elliptic tori is applied just to a model not far from the SJSU *planar* system (let us recall that the real orbits of the planets of our Solar System *are not* lying on lower dimensional tori). Therefore, sect. 4.2 is devoted to the introduction of our Hamiltonian model and to the description of its expansion in canonical coordinates. This will allow us to write down the form of the Hamiltonian to which our approach can be applied. By the way, we think that with some minor modifications our procedure should adapt also to the more general spatial case, after having performed the reduction of the angular momentum, which is not considered here in order to shorten the description of all the preliminary expansions (for an introduction to some methods performing both the partial and the total reduction, see [18], [59] and [70]).

Our algorithm constructing a normal form for elliptic tori is presented in a purely formal way in sect. 4.3. Let us recall that our procedure is mainly a reformulation of the classical Kolmogorov normalization algorithm, that is modified in a suitable way for our purposes. The theoretical background necessary to understand when our algorithm can converge is informally discussed in subsect. 4.3.4 and fully stated in the next chapter.

Sect. 4.4 is devoted to explain an application which is also a test of our procedure. First, in subsect. 4.4.2 we describe the way to implement our algorithm, by using an algebraic manipulator on a computer so to produce both the normal form and the semi-analytic integration of the motion on an invariant elliptic torus. The Fourier spectrum of the motions on elliptic tori is strongly characteristic: just the mean-motion frequencies and their linear combinations can show up. This simple remark allows us to check the

accuracy of our results by using frequency analysis, as it will be described in sect. 4.4.3.

4.2 Classical Expansion of the Planar Planetary Hamiltonian

As claimed in the introduction, in order to fix the ideas, we think it is convenient to focus on a concrete planetary model, to which we will apply our algorithm constructing the elliptic tori in the next sections.

As in the previous chapter, we study an approximation of the SJSU planar system. Let us stress that the four considered point bodies have the same masses as Sun, Jupiter, Saturn and Uranus, but the orbits studied here are significantly different with respect to the real ones.

We proceed now by expanding the Hamiltonian in order to construct the first basic approximation of the normal form for elliptic tori. For completeness we report once again only the key points of this scheme and the values of the parameters used, referring to the previous chapter for all the details. After having chosen a center value Λ^* for the Taylor expansions with respect to the fast actions (in a way we will explain later), we perform a translation $\mathcal{T}_{\Lambda^*}$ defined as

$$L_j = \Lambda_j - \Lambda_j^* \,, \qquad \text{for } j = 1\,,\,2\,,\,3\,.$$

This is a canonical transformation that leaves the coordinates λ, ξ and η unchanged. The transformed Hamiltonian $\mathcal{H}^{(\mathcal{T})} = F \circ \mathcal{T}_{\Lambda^*}$ can be expanded in power series of L, ξ, η around the origin. Thus, forgetting an unessential constant we rearrange the Hamiltonian of the system as

$$\mathcal{H}^{(\mathcal{T})}(L,\lambda,\xi,\eta) = n^* \cdot L + \sum_{j_1=2}^{\infty} h^{(\mathrm{Kep})}_{j_1\,,\,0}(L) + \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} h^{(\mathcal{T})}_{j_1\,,\,j_2}(L,\lambda,\xi,\eta)\,, \tag{4.1}$$

where again the functions $h^{(\mathcal{T})}_{j_1\,,\,j_2}$ are homogeneous polynomials of degree j_1 in the actions L and of degree j_2 in the secular variables (ξ,η).

The expansion of the Hamiltonian (4.1) is performed in exactly the same way as for (3.3), so we skip all the details and we focus only on the interesting part: the new limits in the expansions. In our calculations we truncate this initial expansion as follows. (a) The Keplerian part is expanded up to the quartic terms. The series where the general summand $h^{(\mathcal{T})}_{j_1\,,\,j_2}$ appears are truncated so to include: (b1) the terms having degree j_1 in the actions L with $j_1 \leq 3$, (b2) all terms having degree j_2 in the secular

Table 4.1. Masses m_j and initial conditions for Jupiter, Saturn and Uranus in our planar model. We adopt the AU as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The initial conditions are expressed by the usual heliocentric planar orbital elements: the semi-major axis a_j, the mean anomaly M_j, the eccentricity e_j and the perihelion longitude ω_j. The data are taken by JPL at the Julian Date 2440400.5.

	Jupiter ($j = 1$)	Saturn ($j = 2$)	Uranus ($j = 3$)
m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$	$(2\pi)^2/22902.98$
a_j	5.20463727204700266	9.54108529142232165	19.2231635458410572
M_j	3.04525729444853654	5.32199311882584869	0.19431922829271914
e_j	0.04785365972484999	0.05460848595674678	0.04858667407651962
ω_j	0.24927354029554571	1.61225062288036902	2.99374344439246487

variables (ξ, η), with j_2 such that $2j_1 + j_2 \leq 8$, (b3) all terms up to the trigonometric degree 18 with respect to the angles λ. Let us remark that with respect to the analogous initial expansion we performed in [79] that we describe in the previous chapter, here we preferred to considerably reduce the maximal degree in the secular coordinates, in order to increase those related to the fast ones. This choice is motivated by the fact that the orbits on elliptic tori experience smaller values of the eccentricities (let us recall that both $\xi_j = \mathcal{O}(e_j)$ and $\eta_j = \mathcal{O}(e_j)$ for $j = 1, 2, 3$) than those related to the real motions; moreover, larger limits on the fast coordinates are needed, in order to give a sharp enough numerical evidence of the convergence of the algorithm described in the next section.

Let us now focus on the average with respect to the fast angles of the Hamiltonian, i.e. $\langle \mathcal{H}^{(\mathcal{T})} \rangle_\lambda$. The fast actions L are obviously invariant with respect to the flow of $\langle \mathcal{H}^{(\mathcal{T})} \rangle_\lambda$, thus, they can be disregarded. The remaining most significant term is given by the lowest order approximation of the secular Hamiltonian, namely its quadratic term $\langle h_{0,2}^{(\mathcal{T})} \rangle_\lambda$, which is essentially the one considered in the theory first developed by Lagrange (see [38]) and further improved by Laplace (see [41], [42] and [43]) and by Lagrange himself (see [39], [40]). In modern language, we can say that the origin of the secular coordinates phase space (i.e., $(\xi, \eta) = (0, 0)$) is an elliptic equilibrium point. In fact, under mild assumptions on the quadratic part of the Hamiltonian which are

satisfied in our case (see sect. 3 of [5], where such hypotheses are shown to be generically fulfilled for a planar model of our Solar System), it is well known that one can find a canonical transformation $(L, \lambda, \xi, \eta) = \mathcal{D}(p, q, x, y)$ owning the following properties: (i) $L = p$ and $\lambda = q$, (ii) the map $(\xi, \eta) = \big(\xi(x), \eta(y)\big)$ is linear, (iii) $\mathcal{D}$ diagonalizes the quadratic part of the Hamiltonian, so that we can write $\big\langle h_{0,2}^{(\mathcal{T})} \big\rangle_\lambda$ in the new coordinates as $\sum_{j=1}^{3} \nu_j^{(0)} (x_j^2 + y_j^2)/2$, where all the entries of the vector $\nu^{(0)}$ have the same sign. Our algorithm constructing a suitable normal form for elliptic tori can be started from the Hamiltonian $H^{(0)} = \mathcal{H}^{(\mathcal{T})} \circ \mathcal{D}$, i.e.

$$H^{(0)}(p, q, x, y) = \mathcal{H}^{(\mathcal{T})}\big(\mathcal{D}(p, q, x, y)\big) . \tag{4.2}$$

4.3 Formal Algorithm

In the present section, let us more generically assume that the number of degrees of freedom of our system is $n_1 + n_2$, where the canonical coordinates (p, q, x, y) can naturally be split in two parts, that are $(p, q) \in \mathbf{R}^{n_1} \times \mathbf{T}^{n_1}$ and $(x, y) \in \mathbf{R}^{n_2} \times \mathbf{R}^{n_2}$.

In order to better understand our whole procedure, we think it is convenient to immediately state our final goal. We want to determine a canonical transformation $(p, q, x, y) = \mathcal{K}^{(\infty)}(P, Q, X, Y)$ such that the Hamiltonian $H^{(\infty)} = H^{(0)} \circ \mathcal{K}^{(\infty)}$ is brought to the following normal form[1]:

$$\begin{aligned} H^{(\infty)}(P, Q, X, Y) = \omega^{(\infty)} \cdot P + \sum_{j=1}^{n_2} \frac{\Omega_j^{(\infty)} \left(X_j^2 + Y_j^2\right)}{2} + \\ \mathcal{O}\big(\|P\|^2\big) + \mathcal{O}\big(\|P\| \|(X, Y)\|\big) + \mathcal{O}\big(\|(X, Y)\|^3\big) , \end{aligned} \tag{4.3}$$

where the notation means that we want to remove all terms which are linear in P and independent of X, Y, or quadratic in X, Y and independent of P.

When initial conditions of the type $(P, Q, X, Y) = (0, Q_0, 0, 0)$ (with $Q_0 \in \mathbf{T}^{n_1}$) are considered, the normal form (4.3) allows us to easily calculate the solution of the Hamilton equations, i.e.

$$\big(P(t), Q(t), X(t), Y(t)\big) = \big(0, Q_0 + \omega^{(\infty)} t, 0, 0\big) .$$

[1] Let us here stress a little abuse of notation. Hereafter, the symbol ω will mean the frequencies vector related to the motion on a torus (as it is usual in KAM theory), while in the previous sections it was used to represent the perihelion longitudes (according to the classical notation in celestial mechanics). Analogously, hereafter, Ω will denote the oscillation frequencies transverse to an elliptic torus, while before it was used for the longitudes of the nodes.

This clearly means that the n_1-dimensional (elliptic) torus corresponding to $P = X = Y = 0$ is invariant, and that the orbits are quasi periodic on it.

Let us start the description of the generic r-th step of our algorithm constructing the normal form. We begin with a Hamiltonian of the following type:

$$H^{(r-1)}(p,q,x,y) = \omega^{(r-1)} \cdot p + \Omega^{(r-1)} \cdot J + \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \sum_{\substack{2j_1+j_2=l \\ j_1 \geq 0\,,\, j_2 \geq 0}} f^{(r-1,s)}_{j_1\,,\,j_2}(p,q,x,y)\,, \quad (4.4)$$

where $J_j = (x_j^2 + y_j^2)/2$ is the action which is usually related to the j-th pair of secular canonical coordinates (x_j, y_j) for $j = 1\,,\ldots,n_2$. Moreover, there is a fixed integer value $K > 0$ such that the terms $f^{(r-1,s)}_{j_1\,,\,j_2}$ satisfy the following hypotheses:

(A) $f^{(r-1,s)}_{j_1\,,\,j_2} \in \mathcal{P}^{(sK)}_{j_1\,,\,j_2}$, where $\mathcal{P}^{(sK)}_{j_1\,,\,j_2}$ is the class of functions such that (a1) they are homogeneous polynomials of degree j_1 in the actions p, (a2) they are homogeneous polynomials of degree j_2 in the secular variables (x,y), (a3) their Fourier expansion is finite with maximal trigonometric degree equal to sK;

(B) the terms $f^{(r-1,s)}_{j_1\,,\,j_2}$ are "well Fourier-ordered"; this nonstandard definition means that $\forall\, j_1 \geq 0\,,\, j_2 \geq 0\,,\, s \geq 1$ every Fourier harmonic k appearing in the expansion of $f^{(r-1,s)}_{j_1\,,\,j_2}$ is such that its corresponding trigonometric degree is $|k| = |k_1| + \ldots + |k_{j_1}| > (s-1)K$.

By using the properties (i)–(iii) of the canonical transformation $\mathcal{D}$, one easily sees that the Hamiltonian $H^{(0)}$ (that is defined in (4.2) can be expanded in the form written in (4.4), after having suitably reordered its Fourier expansion so to satisfy the above requirements (A) and (B). Therefore, our constructive algorithm can be concretely applied to the Hamiltonian $H^{(0)}$ by starting with $r = 1$.

The comparison of the expansion in (4.4) with the normal form in (4.3) clearly shows that we have to eliminate all the terms $f^{(0,s)}_{j_1\,,\,j_2}$ where the index $l = 2j_1 + j_2$ is such that $0 \leq l \leq 2$. Thus, the r-th step of our algorithm can be naturally divided in three stages, each of ones aims to reduce the perturbation terms with $l = 0, 1, 2$, respectively.

4.3.1 First Stage of the r-th Normalization Step — Removing the Terms Depending only on q

By making use of the classical Lie series algorithm to calculate canonical transformations (see, e.g., [32] for an introduction), we first introduce the new Hamiltonian $H^{(\mathrm{I};r)} = \exp \mathcal{L}_{\chi_0^{(r)}} H^{(r-1)}$, where the generating function $\chi_0^{(r)}(q) \in \mathcal{P}^{(rK)}_{0,0}$ is determined as the

solution of the equation

$$\left\{\chi_0^{(r)}\,,\,\omega^{(r-1)}\cdot p\right\}+\sum_{s=1}^{r} f_{0,0}^{(r-1,s)}(q)=0\,, \tag{4.5}$$

where we used the classical symbol $\{\cdot,\cdot\}$ to represent the Poisson brackets. The previous equation (that is usually said to be of homological type) admits a solution provided the frequency vector $\omega^{(r-1)}$ is non-resonant up to order rK, i.e.

$$\min_{0<|k|\le rK}\left|k\cdot\omega^{(r-1)}\right|\ge\alpha_r \qquad \text{with}\ \ \alpha_r>0\,, \tag{4.6}$$

where, for the time being, $\{\alpha_r\}_{r>0}$ is nothing but a sequence of real positive numbers and $|k|$ denotes the l^1-norm of the integer vector k, i.e. $k=|k_1|+\ldots+|k_{n_1}|$. The solution of the homological equation (4.5) can be easily recovered by looking at the little more complicate case of $X_2^{(r)}$, which is discussed in the third stage of the r-th normalization step (see formulas (4.14)–(4.16)).

In order to avoid the proliferation of too many symbols, let us make a common abuse of notation so to still denote with (p,q,x,y) the new canonical coordinates $\exp\mathcal{L}_{\chi_0^{(r)}}(p,q,x,y)$. The expansion of the new Hamiltonian can be written as follows:

$$H^{(\mathrm{I};r)}(p,q,x,y)=\omega^{(r-1)}\cdot p+\Omega^{(r-1)}\cdot J+\sum_{s=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{2j_1+j_2=l\\ j_1\ge0\,,\,j_2\ge0}} f_{j_1\,,\,j_2}^{(\mathrm{I};r,s)}(p,q,x,y)\,. \tag{4.7}$$

The mathematical recursive definitions of the terms $f_{j_1\,,\,j_2}^{(\mathrm{I};r,s)}$ are lengthy, but it is rather easy to understand how to deal with them when they are translated in a programming language, moreover we will describe all the details in the next chapter, see (5.11). The main remark is concerned with the classes of functions, i.e.

$$\frac{1}{i!}\mathcal{L}^i_{\chi_0^{(r)}}f_{j_1\,,\,j_2}^{(r-1,s)}\in\mathcal{P}_{j_1-i\,,\,j_2}^{((s+i)K)} \qquad \forall\ 0\le i\le j_1\,,\ j_2\ge0\,,\ s\ge0\,.$$

Therefore, after having calculated all the Poisson brackets needed by $\frac{1}{i!}\mathcal{L}^i_{\chi_0^{(r)}}f_{j_1\,,\,j_2}^{(r-1,s)}$, it is enough to know that it contributes to the sum $\sum_{j=0}^{s+i}f_{j_1-i\,,\,j_2}^{(\mathrm{I};r,j)}$. A suitable "reordering of the Fourier series" will allow us to ensure that also the expansion (4.7) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section.

4.3.2 Second Stage of the r-th Normalization Step — Removing the Terms Linear in x, y and Independent of p

Let us now introduce the new Hamiltonian $H^{(\mathrm{II};r)}=\exp\mathcal{L}_{\chi_1^{(r)}}H^{(\mathrm{I};r)}$, where the gener-

ating function $\chi_1^{(r)}(q,x,y) \in \mathcal{P}_{0,1}^{(rK)}$ is determined as the solution of the equation

$$\left\{ \chi_1^{(r)} \,,\, \omega^{(r-1)} \cdot p + \sum_{j=1}^{n_2} \frac{\Omega_j^{(r-1)}}{2} \left(x_j^2 + y_j^2\right) \right\} + \sum_{s=0}^{r} f_{0,1}^{(\mathrm{I};r,s)}(q,x,y) = 0 \,. \tag{4.8}$$

In order to explicitly write down the solution of the previous equation, it is convenient to temporarily introduce action-angle coordinates so to replace the secular pairs (x,y) by putting $x_j = \sqrt{2J_j}\cos\varphi_j$ and $y_j = \sqrt{2J_j}\sin\varphi_j$ for $j = 1, \ldots, n_2$; therefore, let us assume that the expansion of the known terms appearing in equation (4.8) is the following one:

$$\sum_{s=0}^{r} f_{0,1}^{(\mathrm{I};r,s)}(q,J,\varphi) = \sum_{0\le|k|\le rK} \sum_{j=1}^{n_2} \sqrt{2J_j} \left[c_{k\,,\,j}^{(\pm)} \cos\left(k\cdot q \pm \varphi_j\right) + d_{k\,,\,j}^{(\pm)} \sin\left(k\cdot q \pm \varphi_j\right) \right] , \tag{4.9}$$

with suitable real coefficients $c_{k\,,\,j}^{(\pm)}$ and $d_{k\,,\,j}^{(\pm)}$. Thus, one can easily check that

$$\chi_1^{(r)}(q,J,\varphi) = \sum_{0\le|k|\le rK} \sum_{j=1}^{n_2} \sqrt{2J_j} \left[-\frac{c_{k\,,\,j}^{(\pm)} \sin\left(k\cdot q \pm \varphi_j\right)}{k\cdot\omega^{(r-1)} \pm \Omega_j^{(r-1)}} + \frac{d_{k\,,\,j}^{(\pm)} \cos\left(k\cdot q \pm \varphi_j\right)}{k\cdot\omega^{(r-1)} \pm \Omega_j^{(r-1)}} \right] , \tag{4.10}$$

is a solution of the homological equation (4.8) and it exists provided the frequency vector $\omega^{(r-1)}$ satisfies the so-called *first Melnikov non-resonance condition* up to order rK, i.e.

$$\min_{\substack{0<|k|\le rK \\ j=1\,,\,\ldots\,,\,n_2}} \left| k\cdot\omega^{(r-1)} \pm \Omega_j^{(r-1)} \right| \ge \alpha_r \qquad \text{with} \quad \alpha_r > 0 \,, \tag{4.11}$$

and all the entries of the frequency vector $\Omega^{(r-1)}$ are far enough from the origin, i.e.

$$\min_{j=1\,,\,\ldots\,,\,n_2} \left| \Omega_j^{(r-1)} \right| \ge \beta \qquad \text{with} \quad \beta > 0 \,. \tag{4.12}$$

For what concerns planetary Hamiltonians where the D'Alembert rules hold true, let us remark that all the coefficients $c_{k\,,\,j}^{(\pm)}$ and $d_{k\,,\,j}^{(\pm)}$ appearing in (4.9) and having even values of $|k|$ are equal to zero. In order to solve the equation (4.8), therefore, we do not need the condition (4.12), which however is substantially included in another one (i.e., (4.23)) that we will be forced to introduce later.

Starting from the expansion (4.10) of $\chi_1^{(r)}(q,J,\varphi)$, one can immediately recover the expression of $\chi_1^{(r)}(q,x,y)$ as a function of the original polynomial variables. We can then explicitly calculate the expansion of the new Hamiltonian, which can be written

as follows:

$$H^{(\mathrm{II};r)}(p,q,x,y) = \omega^{(r-1)} \cdot p + \Omega^{(r-1)} \cdot J + \sum_{s=0}^{\infty}\sum_{l=0}^{\infty} \sum_{\substack{2j_1+j_2=l \\ j_1\geq 0\,,\, j_2\geq 0}} f^{(\mathrm{II};r,s)}_{j_1\,,\,j_2}(p,q,x,y)\,. \quad (4.13)$$

Also in this case, providing mathematical recursive definitions of the terms $f^{(\mathrm{II};r,s)}_{j_1\,,\,j_2}$ is a quite annoying task. Thus, we think it is better to just describe how to deal with them when they are translated in a programming language, deferring all the details to the next chapter, see (5.15). Let us remark that the following relations about the classes of functions hold true:

$$\frac{1}{i!}\mathcal{L}^i_{\chi_1^{(r)}} \sum_{2j_1+j_2=l} f^{(\mathrm{I};r,s)}_{j_1\,,\,j_2} \in \bigcup_{2j_1+j_2=l-i} \mathcal{P}^{((s+i)K)}_{j_1\,,\,j_2} \qquad \forall\, 0\leq i\leq l\,,\ s\geq 0\,.$$

Therefore, after having calculated all the Poisson brackets appearing in the expression of the term $\frac{1}{i!}\mathcal{L}^i_{\chi_1^{(r)}} \sum_{2j_1+j_2=l} f^{(\mathrm{I};r,s)}_{j_1\,,\,j_2}$, it is enough to know that it contributes to the sum $\sum_{j=0}^{s+i}\sum_{2j_1+j_2=l-i} f^{(\mathrm{II};r,j)}_{j_1\,,\,j_2}$. Again a suitable "reordering of the Taylor-Fourier series" will allow us to ensure that also the expansion (4.13) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section.

4.3.3 Third Stage of the r-th Normalization Step — Removing the Linear Term in p Independent of x, y, or Quadratic in x, y and Independent of p

The Hamiltonian produced at the end of the r-th normalization step is provided by the composition of three canonical transformations which can be given in terms of Lie series, i.e. $H^{(r)} = \exp\mathcal{L}_{\mathcal{D}_2^{(r)}} \circ \exp\mathcal{L}_{Y_2^{(r)}} \circ \exp\mathcal{L}_{X_2^{(r)}} H^{(\mathrm{II};r)}$, where the generating functions belong to three different classes: $X_2^{(r)}(p,q) \in \mathcal{P}^{(rK)}_{1,0}$, $Y_2^{(r)}(q,x,y) \in \mathcal{P}^{(rK)}_{0,2}$ and $\mathcal{D}_2^{(r)}(x,y) \in \mathcal{P}^{(0)}_{0,2}$. The explicit expressions of these generating functions are given below, in formulas (4.14)), (4.17) and (4.21), respectively.

Let us anticipate that, when one focuses on the estimates needed to prove the convergence of the algorithm, it is certainly simpler to introduce the generating function $\chi_2^{(r)}(p,q,x,y) = X_2^{(r)}(p,q) + Y_2^{(r)}(q,x,y)$ and to consider the new Hamiltonian $\exp\mathcal{L}_{\mathcal{D}_2^{(r)}} \circ \exp\mathcal{L}_{\chi_2^{(r)}} H^{(\mathrm{II};r)}$ which slightly differs from $H^{(r)}$, because $X_2^{(r)}$, and $Y_2^{(r)}$ do not commute with respect to the Poisson brackets. Since in the present section we do not want to theoretically study the problem of the convergence of our algorithm, we think here is better to distinguish the present third stage of the r-th normalization step in other three parts, so to highlight their different roles. Moreover, this choice

looks more natural to us, when one implements the constructive algorithm by algebraic manipulations on a computer.

We start with $X_2^{(r)}(p,q) \in \mathcal{P}_{1,0}^{(rK)}$, which is determined as the solution of the equation

$$\left\{X_2^{(r)}\,,\,\omega^{(r-1)}\cdot p\right\} + \sum_{s=1}^{r} f_{1,0}^{(\mathrm{II};r,s)}(p,q) = 0\,. \tag{4.14}$$

This implies that

$$X_2^{(r)}(p,q) = \sum_{0<|k|\le rK}\ \sum_{j=1}^{n_1} p_j \left[-\frac{c_{k\,,\,j}\sin\big(k\cdot q\big)}{k\cdot\omega^{(r-1)}} + \frac{d_{k\,,\,j}\cos\big(k\cdot q\big)}{k\cdot\omega^{(r-1)}}\right]\,, \tag{4.15}$$

where we preliminarily assumed that the expansion of the known terms appearing in equation (4.14)) has the form

$$\sum_{s=1}^{r} f_{1,0}^{(\mathrm{II};r,s)}(p,q) = \sum_{0<|k|\le rK}\ \sum_{j=1}^{n_1} p_j\left[c_{k\,,\,j}\cos\big(k\cdot q\big) + d_{k\,,\,j}\sin\big(k\cdot q\big)\right]\,, \tag{4.16}$$

with suitable real coefficients $c_{k\,,\,j}$ and $d_{k\,,\,j}$. Let us here recall that the solution (4.15) for the equation (4.14) exists provided the frequency vector $\omega^{(r-1)}$ satisfies the non-resonance condition (4.6).

Let us now consider $Y_2^{(r)}(q,x,y) \in \mathcal{P}_{0,2}^{(rK)}$, which is a solution of the equation

$$\left\{Y_2^{(r)}\,,\,\omega^{(r-1)}\cdot p + \sum_{j=1}^{n_2}\frac{\Omega_j^{(r-1)}}{2}\big(x_j^2+y_j^2\big)\right\} + \sum_{s=1}^{r} f_{0,2}^{(\mathrm{II};r,s)}(q,x,y) = 0\,. \tag{4.17}$$

In order to explicitly write down the expansion of $Y_2^{(r)}$, it is convenient to temporarily reintroduce the action-angle coordinates (J,φ) so to replace the secular pairs (x,y); therefore, let us assume that the expansion of the known terms appearing in equation (4.17) has the form:

$$\begin{aligned}\sum_{s=1}^{r} f_{0,2}^{(\mathrm{II};r,s)}(q,J,\varphi) = \sum_{0<|k|\le rK}\ \sum_{i\,,\,j\,=\,1}^{n_2} 2\sqrt{J_iJ_j}\Big[&c_{k\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\cos\big(k\cdot q\pm\varphi_i\pm\varphi_j\big)+\\ &d_{k\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\sin\big(k\cdot q\pm\varphi_i\pm\varphi_j\big)\Big]\,,\end{aligned} \tag{4.18}$$

with suitable real coefficients $c^{(\pm\,,\,\pm)}_{k\,,\,i\,,\,j}$ and $d^{(\pm\,,\,\pm)}_{k\,,\,i\,,\,j}$. Thus, one can easily check that

$$Y_2^{(r)}(q,J,\varphi) = \sum_{0<|k|\le rK}\ \sum_{i\,,\,j\,=\,1}^{n_2} 2\sqrt{J_iJ_j}\Bigg[-\frac{c^{(\pm\,,\,\pm)}_{k\,,\,i\,,\,j}\sin\big(k\cdot q\pm\varphi_i\pm\varphi_j\big)}{k\cdot\omega^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}} + \frac{d^{(\pm\,,\,\pm)}_{k\,,\,i\,,\,j}\cos\big(k\cdot q\pm\varphi_i\pm\varphi_j\big)}{k\cdot\omega^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}}\Bigg], \tag{4.19}$$

is a solution of equation (4.17) and it exists provided the frequency vector $\omega^{(r-1)}$ satisfies the so-called *second Melnikov non-resonance condition* up to order rK , i.e.

$$\min_{\substack{0<|k|\le rK\\ i\,,\,j\,=1\,,\,\dots\,,\,n_2}} \big|k\cdot\omega^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}\big| \ge \alpha_r \qquad \text{with}\ \ \alpha_r>0\,. \tag{4.20}$$

Let us here remark that the previous assumption includes also the non-resonance condition (4.6) as a special case, i.e. when $i=j$ and the signs appearing in the expression $\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}$ are opposite.

Also for what concerns the generating function $\mathcal{D}_2^{(r)}$, once again it is convenient to replace the secular pairs (x,y) with the action-angle coordinates (J,φ) . Let us here remark that $\Omega^{(r-1)}\cdot J$ and $f^{(\mathrm{II};r,0)}_{0\,,\,2}(x,y)$ are the only terms appearing in expansion (4.13), quadratic in (x,y) (so they also are $\mathcal{O}(J)$) and not depending on p and q . The canonical transformation induced by the Lie series $\exp\mathcal{L}_{\mathcal{D}_2^{(r)}}$ aims to eliminate the part of $f^{(\mathrm{II};r,0)}_{0\,,\,2}$ depending on the secular angles φ [1]. Therefore, the generating function $\mathcal{D}_2^{(r)}$ is defined so to solve the following equation:

$$\Big\{\mathcal{D}_2^{(r)}\,,\,\Omega^{(r-1)}\cdot J\Big\} + f^{(\mathrm{II};r,0)}_{0\,,\,2}(J,\varphi) - \big\langle f^{(\mathrm{II};r,0)}_{0\,,\,2}\big\rangle_\varphi = 0\,, \tag{4.21}$$

where $\langle\cdot\rangle_\varphi$ denotes the average with respect to the angles φ . This implies that

$$\mathcal{D}_2^{(r)}(J,\varphi) = \sum_{i\,,\,j\,=\,1}^{n_2}\ \sum_{\substack{s_i\,,\,s_j\,=\pm1\\ s_i\cdot i+s_j\cdot j\,\neq\,0}} 2\sqrt{J_iJ_j}\Bigg[-\frac{c_{i\,,\,j\,,\,s_i\,,\,s_j}\sin\big(s_i\varphi_i+s_j\varphi_j\big)}{s_i\Omega_i^{(r-1)}+s_j\Omega_j^{(r-1)}} + \frac{d_{i\,,\,j\,,\,s_i\,,\,s_j}\cos\big(s_i\varphi_i+s_j\varphi_j\big)}{s_i\Omega_i^{(r-1)}+s_j\Omega_j^{(r-1)}}\Bigg], \tag{4.22}$$

where we preliminarily assumed that the expansion of the known terms appearing in

[1] This transformation, as we will see in the next chapter where the complex canonical variables are adopted, corresponds to a diagonalization

equation (4.21) is the following one:

$$f_{0,2}^{(\mathrm{II};r,0)}(J,\varphi) = \sum_{i\,,\,j\,=\,1}^{n_2} \sum_{\substack{s_i=\pm 1\\ s_j=\pm 1}} 2\sqrt{J_i J_j}\Big[c_{i\,,\,j\,,\,s_i\,,\,s_j}\cos\big(s_i\varphi_i + s_j\varphi_j\big) + \\ d_{i\,,\,j\,,\,s_i\,,\,s_j}\sin\big(s_i\varphi_i + s_j\varphi_j\big)\Big]\,,$$

with suitable real coefficients $c_{i\,,\,j\,,\,s_i\,,\,s_j}$ and $d_{i\,,\,j\,,\,s_i\,,\,s_j}$. Let us remark that the solution (4.15) for the equation (4.14) exists provided the frequency vector $\Omega^{(r-1)}$ satisfies the following finite non-resonance condition:

$$\min_{|l|=2}\left|l\cdot\Omega^{(r-1)}\right| \geq \beta \qquad \text{with} \quad \beta > 0\,. \tag{4.23}$$

At this point of the algorithm, it is convenient to slightly modify the frequencies ω^{r-1} and Ω^{r-1}, so to include the terms which are linear with respect to the actions and that do not depend on the angles and, then, cannot be eliminated by our normalization procedure. More precisely, we define $\omega^{(r)}$ and $\Omega^{(r)}$, so that

$$\omega^{(r)}\cdot p = \omega^{(r-1)}\cdot p + f_{1,0}^{(\mathrm{II};r,0)}(p)\,, \qquad \Omega^{(r)}\cdot J = \Omega^{(r-1)}\cdot J + \big\langle f_{0\,,\,2}^{(\mathrm{II};r,0)}\big\rangle_{\varphi}\,. \tag{4.24}$$

Standard utilities provided by any computer algebra system should allow everyone to get the expansions of $Y_2^{(r)}(q,x,y)$ and $\mathcal{D}_2^{(r)}(x,y)$, starting from those of $Y_2^{(r)}(q,J,\varphi)$ and $\mathcal{D}_2^{(r)}(J,\varphi)$, which are written in (4.19)) and (4.22)), respectively. We are now able to explicitly produce the expansion of the new Hamiltonian, which can be written as follows:

$$H^{(r)}(p,q,x,y) = \omega^{(r)}\cdot p + \Omega^{(r)}\cdot J + \sum_{s=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{2j_1+j_2=l\\ j_1\geq 0\,,\,j_2\geq 0}} f_{j_1\,,\,j_2}^{(r,s)}(p,q,x,y)\,. \tag{4.25}$$

Let us remark that this expansion of $H^{(r)}$ has exactly the same form of that written for $H^{(r-1)}$ in (4.4), but we stress that the algorithm is arranged so to make smaller and smaller the contribution of the terms $f_{j_1\,,\,j_2}^{(r,s)}$, when the value of r is increased, for $s \geq 0$ and $2j_1 + j_2 = 0,\ 1,\ 2$.

In this case too, we avoid to write down the lengthy mathematical recursive definitions of the terms $f_{j_1\,,\,j_2}^{(r,s)}$. Instead, we provide some relations about the classes of functions, which are useful to understand how to translate this third stage of the r-th normalization step in a programming language. For what concerns the generating

function $X_2^{(r)}$, the following relations about the classes of the functions hold true:

$$\frac{1}{i!}\mathcal{L}^i_{X_2^{(r)}} f^{(\mathrm{II};r,s)}_{j_1\,,\,j_2} \in \mathcal{P}^{((s+i)K)}_{j_1\,,\,j_2} \qquad \forall\ i \geq 0\,,\ j_1 \geq 0\,,\ \ j_2 \geq 0\,,\ s \geq 0\,. \tag{4.26}$$

The relations involving the generating function $Y_2^{(r)}$ are a little more complicated:

$$\frac{1}{i!}\mathcal{L}^i_{Y_2^{(r)}} \sum_{2j_1+j_2=l} f^{(\mathrm{II};r,s)}_{j_1\,,\,j_2} \in \bigcup_{2j_1+j_2=l} \mathcal{P}^{((s+i)K)}_{j_1\,,\,j_2} \qquad \forall\ i \geq 0\,,\ l \geq 0\,,\ s \geq 0\,.$$

Finally, one can easily remark that each class of function is invariant with respect to a Poisson bracket with the generating function $\mathcal{D}_2^{(r)}$, therefore:

$$\frac{1}{i!}\mathcal{L}^i_{\mathcal{D}_2^{(r)}} f^{(\mathrm{II};r,s)}_{j_1\,,\,j_2} \in \mathcal{P}^{(sK)}_{j_1\,,\,j_2} \qquad \forall\ i \geq 0\,,\ j_1 \geq 0\,,\ \ j_2 \geq 0\,,\ s \geq 0\,. \tag{4.27}$$

By taking into account the relations (4.26)–(4.27) about the classes of functions, the definition (4.24) of the new frequencies vectors and by suitably "reordering" the Taylor-Fourier series, it is possible to ensure that also the expansion (4.25) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section about the equation (4.4). Therefore, the whole normalization procedure, that has been here described for the r-th step can be iteratively repeated.

4.3.4 Some Remarks on the Normalization Algorithm

We devote this section to an informal discussion of the relations between the normalization procedure for an elliptic torus, which is the subject of the present chapter, and the algorithm of Kolmogorov for a torus of full dimensions. Our aim is to bring into evidence, on one hand, the differences that make the case of an elliptic low dimensional torus definitely more difficult, and the impact that these differences have on the explicit calculation. We hope that this informal discussion will also enlighten the main points of the detailed proof of existence of elliptic tori that will be the subject of the next chapter.

The main hypotheses of Kolmogorov's theorem are (a) that the perturbation should be small enough and (b) that a strong non-resonance condition must be satisfied by the frequencies of the unperturbed torus. Both these conditions appear also in the proof of existence of elliptic tori, but the condition of non resonance presents some critical peculiarities.

Concerning the smallness of the perturbation, the main problem remains that the analytical estimates are unrealistic: we could repeat here the remarks that we have made for the theorem of Kolmogorov. Thus, also in this case we can obtain realistic

results by using algebraic manipulation in order to implement a computer assisted proof. In our case, by comparing the Hamiltonian normal form (4.3) with the expansion (4.25) of $H^{(r)}$, one easily realizes that the initial expression of the perturbation (making part of the Hamiltonian $H^{(0)}$, written in (4.2)) is given by

$$\sum_{s=0}^{\infty}\sum_{l=0}^{2}\sum_{\substack{2j_1+j_2=l\\ j_1\geq 0\,,\,j_2\geq 0}} f^{(0,s)}_{j_1\,,\,j_2}\,. \tag{4.28}$$

Looking at all the preliminary expansions described in the appendix A and recalled in the previous chapters to get the Hamiltonian $H^{(0)}$, one immediately sees that all the perturbing terms appearing in (4.28) are proportional to μ. Let us also recall that the small parameter μ is equal to the mass ratio between the biggest planet and the central star. In this respect, the explicit application of the algorithm is matter of programming the method described in the previous sections.

The problem is concerned with the conditions on non resonance for the frequencies. In the case of a full dimensional torus one must choose the n frequencies $\omega_1, \ldots, \omega_n$ so as to satisfy a strong non-resonance condition. A typical request is that they obey a Diophantine condition, i.e., that the sequence $\{\alpha_r\}_{r\geq 1}$ appearing in the inequalities (4.6), (4.11) and (4.20) must be such that $\alpha_r \geq \gamma/(rK)^\tau$ with suitable positive values of the constant γ and τ. This choice must be made at the very beginning of the procedure, and the perturbed invariant torus that is found at the end has *the same* frequencies as the unperturbed one. The reason is that at every step a small translation is introduced in order to keep the frequencies constant.

In the case of the elliptic low-dimensional torus one deals instead with two separate set of frequencies, namely $\omega^{(0)} \in \mathbf{R}^{n_1}$ which characterize the orbits on the torus, and the transverse frequencies $\Omega^{(0)}\,\mathbf{R}^{(n_2)}$ that characterize the oscillation of orbits close to but not lying on the torus. By the way, this justifies the adjective "transverse" that is commonly used. Now, the frequencies $\omega^{(0)}$ on the torus can be chosen in an arbitrary manner, but the transverse frequencies $\Omega^{(0)}$ are functions of $\omega^{(0)}$, being given by the Hamiltonian. This is easily understood by considering the case of a periodic orbit, i.e., $n_1 = 1$, since in that case the transverse frequencies are related to the eigenvalues of the monodromy matrix.

The striking fact is that, due precisely to the dependence of the transverse frequencies $\Omega^{(0)}$ on $\omega^{(0)}$, the algorithm forces us to change these frequencies at every step. That is, one actually deals with infinite sequences $\omega^{(r)}$ and $\Omega^{(r)}$, all required to satisfy

at every order a non-resonance condition of the form (4.20). Moreover, both sequences should converge to a final set of frequencies $\omega^{(\infty)} = \omega^{(\infty)}\big(\omega^{(0)}\big)$ and $\Omega^{(\infty)} = \Omega^{(\infty)}\big(\omega^{(0)}\big)$ which must be non-resonant (e.g., Diophantine). Thus, we are forced to conclude that, depending on the initial choice of $\omega^{(0)}$, it may happen that the algorithm stops at some step because the frequencies fail to satisfy the non resonance conditions (4.6), (4.11), (4.12), (4.20) and (4.23). This is indeed one of the main difficulties in working out the proof of existence of an elliptic torus.

Let us first consider the analytical aspect, anticipating the ideas that will be exploited in detail in the next chapter. One considers initially an open ball $\mathcal{B} \subset \mathbf{R}^{n_1}$ such that the Diophantine condition *at finite order* required for the first step is satisfied by every $\omega^{(0)} \in \mathcal{B} \subset \mathbf{R}^{n_1}$ and by the corresponding transverse frequencies $\Omega^{(0)}$. This can be done, because only a finite number of non resonance relations are considered. Then one shows that at every step there exists a subset of frequencies in $\mathcal{B}$ which satisfies the non resonance conditions (still at finite but increasing order) required in order to perform the next step, together with the corresponding transverse frequencies. This is obtained by a procedure which is reminiscent of Arnold scheme of proof of Kolmogorov's theorem: at every step one removes from $\mathcal{B}$ a finite number of intersections of $\mathcal{B}$ with a small strip around a resonant plane in $\mathbf{R}^{n_1}$, assuring that the width of the strip decreases fast enough so that an open set always remains. By the way, this is strongly reminiscent of the process of construction of a Cantor set. The final goal is to prove precisely that one is left with a Cantor set on non resonant frequencies which satisfy the required resonance conditions and has positive Lebesgue measure. Moreover, the relative measure with respect to $\mathcal{B}$ is close to 1. This is the idea underlying the proof that will be expanded in the next chapter. We emphasize that the procedure outlined here is strongly inspired by the scheme of proof of Kolmogorov's theorem introduced by Arnold[1], which is quite different form Kolmogorov one.

Let us now come to the numerical aspect. At first sight the formal algorithm seems to require a cumbersome trial and error procedure in order to find the good frequencies: when the non resonance conditions fail to be satisfied at a given step one has to change the initial frequencies and restart the whole process. Moreover, since the non resonance condition must be satisfied by the final frequencies, which obviously can not be calculated, the whole process seem to be unsuitable for a computer-assisted proof. We explain here in which sense the computer-assisted proofs can help to improve the results also in this context. We make two remarks.

The first remark is connected with the use of interval arithmetic while performing the actual construction. Following the suggestion of the analytic scheme of proof, we look for uniform estimates on a small open ball $\mathcal{B}$, such that $\forall\ \omega^{(0)} \in \mathcal{B}$ we explicitly perform R normalization steps, with R as large as possible. Essentially, we may reproduce numerically the process of eliminating step by step the unwanted resonant frequencies by suitably determining the intervals. Once R steps have been explicitly performed we may apply to the partially normalized Hamiltonian $H^{(R)}$ a suitable formulation of the KAM theorem for elliptic tori. This means that we recover the scheme that we have already applied to the case of full-dimension tori. That is, we can take advantage of the fact that the perturbing terms are much smaller than the corresponding ones for the initial Hamiltonian $H^{(0)}$; thus, in principle we could ensure that for realistic values of μ the relative measure of the invariant tori is so large that the set of those $\omega^{(0)}$ for which the algorithm can not work (i.e., $\mathcal{B} \setminus \mathcal{S}$) is so small that can be neglected when we are dealing with a practical application.

Taking a more practical attitude, we may rely on the fact that the set of good frequencies, according to the theory, has Lebesgue measure close to one, so that the case of frequencies which are resonant at some finite order occurs with very low probability. Thus, we just make a choice of the initial frequencies and proceed with the construction, checking at every order that the non-resonance conditions that we need at that order are fulfilled. We emphasize that the most extended resonant regions are those of low order, so that it is not very difficult to check initially that the chosen frequencies will likely be good enough. It may happen, of course, that the whole procedure must be restarted with different frequencies, but we expect that this will rarely occur. However, since the size of the perturbation is expected to decrease geometrically, we may confidently expect that the probability of failure will decrease, too. This is confirmed by the actual calculations.

When R steps have been made, we apply the theorem to a small neighborhood of the calculated frequencies by choosing a suitable initial ball around the frequencies approximated at that step.

4.4 Elliptic Tori for the SJSU System

We come now to the application of the formal algorithm for the construction of an

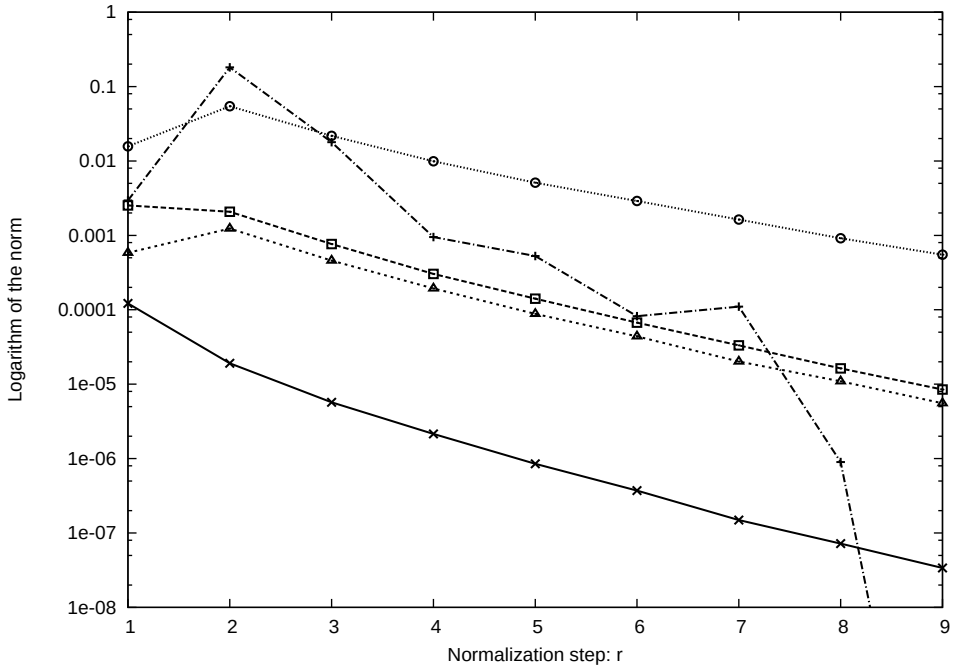

Figure 4.1. Algorithm constructing the normal form related to an elliptic torus for the planar SJSU system: plot of the norm of the generating functions as a function of the normalization step r; more precisely, the symbols $\times$, $\square$, $\triangle$, $\circ$ and $+$ refer to the norm of the generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $X_2^{(r)}$, $Y_2^{(r)}$ and $\mathcal{D}_2^{(r)}$, respectively, which are defined during the normalization algorithm, as described in the present sect. 4.3. The norm is calculated by simply adding up the absolute values of all the coefficients appearing in the expansion of each generating function.

elliptic torus to the planar SJSU system.

The initial Hamiltonian is (4.1), with a suitable rearrangement of terms so that it is given the form (4.4) with $r = 1$. This requires also a diagonalization of the quadratic part in the secular variables, which is performed as in the previous chapters.

We explicitly construct the normal form at a finite order checking that the norms of the generating function decrease as predicted by the theory. Then we perform a numerical check by comparing the orbit obtained via the normal form with the numerically integrated one.

4.4.1 Constructing the Elliptic Torus by Using Computer Algebra

We applied the algorithm constructing elliptic tori (which has been widely described in sect. 4.3) to the Hamiltonian $H^{(0)}$ (that is defined in (4.2) and has been obtained as

described in sect. 4.2). The parameters have been fixed according to the specific values of the planar SJSU system, which are reported in Table 4.1. Our software package for computer algebra allowed us to explicitly calculate all the expansions (4.2) of $H^{(r)}$ with index r ranging between 0 and 9, so to include: (c1) the terms having degree j_1 in the actions p with $j_1 \leq 3$, (c2) all terms having degree j_2 in the variables (x, y), with j_2 such that $2j_1 + j_2 \leq 8$, (c3) all terms up to the trigonometric degree 18 with respect to the angles q. Let us recall that the truncation rules (c1)–(c3) are in agreement with those prescribed about the expansion (4.1) in sect. 4.2 at points (b1)–(b3). Let us remark that both the truncation rules (c1) and (c2) are preserved by all the canonical transformations included in our algorithm. Since the maximal trigonometric degree of the generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $X_2^{(r)}$ and $Y_2^{(r)}$ is equal to rK, the choice to set $K = 2$ and the rule (c3) explain why we stopped the algorithm after having ended the normalization step with $r = 9$.

The behavior of the norms of the generating functions is reported in Fig. 4.1. Let us make a few comments. The theoretical estimates assure the convergence of the normal form provided the norms decrease geometrically with the order. The figure shows that this is indeed the behavior in our case. The behavior of the $\mathcal{D}_2^{(r)}$ functions appears to be quite irregular, but we should recall that these functions do not play the role of *normalization* function, since they represent the *diagonalization* of the quadratic part related to the secular variables. We emphasize that the presence of a dangerous resonance would be reflected in a sudden increase of the coefficients; thus, the plot may be considered as a practical confirmation that the frequencies are well chosen.

Concerning the computational effort, performing the construction of the normal form up to order $r = 9$ has taken about 10 hours of CPU time on an Intel Core i7, using near 6 Gb of memory.

4.4.2 Explicit Calculation of the Orbits on the Elliptic Torus

We now perform a check on the approximation of the elliptic torus. To this end, we calculate the orbit on the torus using the analytic expression and we compare it with a numerical integration of Hamilton equations. In this section we explain how the calculation of the orbit via normal form is performed.

According to the theory of Lie series, the canonical transformation $(p, q, x, y) =$

$\mathcal{K}^{(r)}\big(p^{(r)}, q^{(r)}, x^{(r)}, y^{(r)}\big)$ inducing the normalization up to the order r is given by

$$\begin{aligned}\mathcal{K}^{(r)}\big(p^{(r)}, q^{(r)}, x^{(r)}, y^{(r)}\big) = {} & \exp \mathcal{L}_{\mathcal{D}_2^{(r)}} \circ \exp \mathcal{L}_{Y_2^{(r)}} \circ \exp \mathcal{L}_{X_2^{(r)}} \circ \\ & \exp \mathcal{L}_{\chi_1^{(r)}} \circ \exp \mathcal{L}_{\chi_0^{(r)}} \circ \ldots \circ \exp \mathcal{L}_{\mathcal{D}_2^{(1)}} \circ \exp \mathcal{L}_{Y_2^{(1)}} \circ \\ & \exp \mathcal{L}_{X_2^{(1)}} \circ \exp \mathcal{L}_{\chi_1^{(1)}} \circ \exp \mathcal{L}_{\chi_0^{(1)}} \big(p^{(r)}, q^{(r)}, x^{(r)}, y^{(r)}\big)\,,\end{aligned}$$

where $\big(p^{(r)}, q^{(r)}, x^{(r)}, y^{(r)}\big)$ are meant to be the new coordinates. Thus, the canonical transformation $(p, q, x, y) = \mathcal{K}^{(\infty)}(P, Q, X, Y)$ brings $H^{(0)}$ in the normal form $H^{(\infty)} = H^{(0)} \circ \mathcal{K}^{(\infty)}$, which is written in (4.3), with $\mathcal{K}^{(\infty)} = \lim_{r\to\infty} \mathcal{K}^{(r)}$. Let us introduce a new symbol to denote the composition of all the canonical change of coordinates defined in sects. 4.2 and 4.3, i.e.

$$\mathcal{C}^{(r)} = \mathcal{E} \circ \mathcal{T}_{\Lambda^*} \circ \mathcal{D} \circ \mathcal{K}^{(r)}\,,$$

where $(\tilde{r}, r) = \mathcal{E}(\Lambda, \lambda, \xi, \eta)$ is the canonical transformation giving the heliocentric positions r and their conjugated momenta $\tilde{r}$ as a function of the Poincaré variables. If $\big(\tilde{r}(0), r(0)\big)$ is an initial condition on an invariant elliptic torus, in principle we might use the following calculation scheme to integrate the equation of motion:

$$\begin{array}{ccc} \big(\tilde{r}(0),\, r(0)\big) & \xrightarrow{(\mathcal{C}^{(\infty)})^{-1}} & \big(P(0) = 0,\, Q(0)\,,\, X(0) = 0,\, Y(0) = 0\big) \\ & & \Big\downarrow \Phi^t_{\omega^{(\infty)}\cdot P} \\ \big(\tilde{r}(t),\, r(t)\big) & \xleftarrow{\mathcal{C}^{(\infty)}} & \big(P(t) = 0,\, Q(t) = Q(0) + \omega^{(\infty)} t,\, X(t) = 0,\, Y(t) = 0\big) \end{array} \tag{4.29}$$

where $\Phi^t_{\omega^{(\infty)}\cdot P}$ induces the quasi-periodic flow related to the frequencies vector $\omega^{(\infty)}$. Of course, the previous scheme requires an unlimited computing power; from a practical point of view, we can just approximate it, by replacing $\mathcal{C}^{(\infty)}$ with $\mathcal{C}^{(R)}$, where R is as large as possible. Thus, the integration via normal form actually reduces to a transformation of the coordinates of the initial point to the coordinates of the normal form, the calculation of the flow at time t in the latter coordinates, which is a trivial matter since the flow is exactly quasi-periodic with known frequencies, followed y a transformation back to the original coordinates.

Such an approximated semi-analytic calculation scheme can be directly compared with the results provided by a numerical integrator. As it has been shown in [56], [57], [58] and [22], this kind of comparisons provide a very stressing test for the accuracy of the whole algorithm constructing the normal form.

4.4.3 *Validation of the Results by Using Frequency Analysis*

The ideal calculation scheme (4.29) highlights that the Fourier spectrum of each component of the motion law $t \mapsto \left(\tilde{r}(t), r(t)\right)$ is the very peculiar one

$$\sum_{j=0}^{\infty} c_j \exp\left(\mathrm{i}\zeta_j t\right), \quad \text{where, } \forall\ j \geq 0\,,\ \ c_j \in \mathbf{C} \text{ and } \exists\ k_j \in \mathbf{Z}^{n_1} \text{ such that } \zeta_j = k_j \cdot \omega^{(\infty)}\,. \tag{4.30}$$

In other words, the Fourier spectrum of the planetary motions on elliptic tori is so characteristic, because all its frequencies are given by linear combinations of the fast frequencies. From a strictly mathematical point of view, let us recall that the previous formula for the Fourier spectrum can be deduced by the scheme (4.29), because of the analyticity of the so called conjugacy function $Q \mapsto \mathcal{C}^{(\infty)}\,(0, Q, 0, 0)$ and this will be ensured as a byproduct of the theoretical study of the convergence of the constructive algorithm that we present in the next chapter.

In the present section we aim to check the peculiar quasi-periodicity of the motions on our approximation of an elliptic torus, by using the frequency map analysis (see, e.g., [48] and [50] for an introduction). We focus on the following initial conditions:

$$\left(\mathcal{C}^{(9)}\right)^{-1}(0, 0, 0, 0)\ ; \tag{4.31}$$

according to the expansions described in the previous subsect. 4.4.2, this should be an accurate approximation of a point on an elliptic torus. Therefore, we preliminarily integrated the motion of the planar SJSU system over a time interval of 2^{24} years, by using the symplectic method $\mathcal{SBAB}_3$ (see [51]) with a time-step of 0.04 years.

Here we should add a remark concerning the precision. In order to have a signal clean enough to be analyzed a particular care about the precision is mandatory. After some trials tuning the parameters of the numerical integration we found that the 80 bits floating point numbers provided by the current AMD and INTEL fits our needs. Technically this is obtained by using the `long double` types of the GNU C compiler under Linux operating system.

The orbits have been sampled with a time interval of 1 year. The resulting signals $\xi_l(t) + \mathrm{i}\eta_l(t)$, with $l = 1,\ 2,\ 3$ have been submitted to the frequency analysis method using the so-called Hanning filter.

In Table 4.2 we report our numerical results about the first 25 summands of the decomposition (4.30) for the Uranus secular signal $\xi_3(t) + \mathrm{i}\eta_3(t)$. Let us point out that the values of the fast frequencies vector $\omega^{(\infty)}$ have been preliminarily calculated by

Table 4.2. Decomposition of the Fourier spectrum of the signal $\xi_3(t) + \mathrm{i}\eta_3(t)$, which is related to the Uranus secular motion. The following numerical values have been obtained by applying the frequency analysis method. See the text for more details.

j	ζ_j	k_j	$\lvert\zeta_j - k_j \cdot \omega^{(\infty)}\rvert$	$\lvert c_j\rvert$
0	$-7.48019221455542005 \times 10^{-2}$	$(0, 0, -1)$	$0.0 \times 10^{+00}$	2.9770×10^{-4}
1	$3.80127210702886631 \times 10^{-1}$	$(1, 0, -2)$	5.6×10^{-17}	5.5428×10^{-5}
2	$6.37064849761184715 \times 10^{-2}$	$(0, 1, -2)$	5.6×10^{-17}	1.8199×10^{-5}
3	$2.02214892097791255 \times 10^{-1}$	$(0, 2, -3)$	$0.0 \times 10^{+00}$	1.7410×10^{-5}
4	$3.40723299219463982 \times 10^{-1}$	$(0, 3, -4)$	$0.0 \times 10^{+00}$	6.1013×10^{-6}
5	$8.35056343551327518 \times 10^{-1}$	$(2, 0, -3)$	5.6×10^{-17}	3.7452×10^{-6}
6	$4.79231706341136876 \times 10^{-1}$	$(0, 4, -5)$	1.7×10^{-16}	2.4485×10^{-6}
7	$-2.13310329267227178 \times 10^{-1}$	$(0, -1, 0)$	2.5×10^{-16}	1.4521×10^{-6}
8	$-3.51818736388899878 \times 10^{-1}$	$(0, -2, 1)$	2.2×10^{-16}	1.0175×10^{-6}
9	$6.17740113462809326 \times 10^{-1}$	$(0, 5, -6)$	1.1×10^{-16}	1.0447×10^{-6}
10	$1.28998547639976824 \times 10^{+0}$	$(3, 0, -4)$	2.2×10^{-16}	7.8098×10^{-7}
11	$-4.90327143510572494 \times 10^{-1}$	$(0, -3, 2)$	1.1×10^{-16}	7.1175×10^{-7}
12	$7.56248520584482442 \times 10^{-1}$	$(0, 6, -7)$	2.2×10^{-16}	4.6141×10^{-7}
13	$-9.84660187842435808 \times 10^{-1}$	$(-2, 0, 1)$	1.7×10^{-16}	4.1885×10^{-7}
14	$-6.28835550632244611 \times 10^{-1}$	$(0, -4, 3)$	5.0×10^{-16}	3.8157×10^{-7}
15	$-5.29731054993994532 \times 10^{-1}$	$(-1, 0, 0)$	5.6×10^{-16}	2.9840×10^{-7}
16	$-1.11363990387936461 \times 10^{-5}$	$(0, 0, 0)$	1.1×10^{-05}	2.2654×10^{-7}
17	$8.94756927706155003 \times 10^{-1}$	$(0, 7, -8)$	1.1×10^{-16}	2.0832×10^{-7}
18	$-7.67343957753918837 \times 10^{-1}$	$(0, -5, 4)$	1.0×10^{-15}	1.9160×10^{-7}
19	$1.74491460924820907 \times 10^{+0}$	$(4, 0, -5)$	2.8×10^{-16}	1.7777×10^{-7}
20	$-1.43958932069087675 \times 10^{+0}$	$(-3, 0, 2)$	1.1×10^{-16}	1.3450×10^{-7}
21	$2.43025734926846093 \times 10^{-2}$	$(-1, 4, -4)$	1.1×10^{-14}	1.1086×10^{-7}
22	$-9.05852364875589622 \times 10^{-1}$	$(0, -6, 5)$	9.4×10^{-16}	9.3984×10^{-8}
23	$1.03326533482782756 \times 10^{+0}$	$(0, 8, -9)$	$0.0 \times 10^{+00}$	9.5496×10^{-8}
24	$-1.96924221667578817 \times 10^{-5}$	$(0, 0, 0)$	2.0×10^{-05}	5.2900×10^{-8}

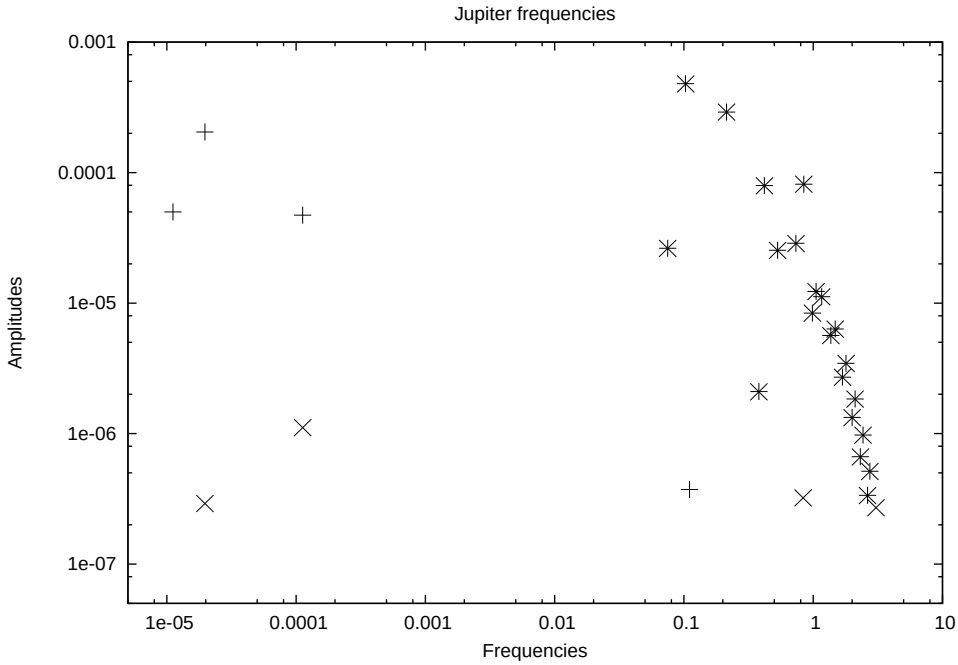

Figure 4.2. Frequency analysis of the secular signal related to the secular Jupiter motion: $\xi_1(t) + \mathrm{i}\eta_1(t) = \sum_{j=0}^{\infty} c_j \exp\left(\mathrm{i}\zeta_j t\right)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The symbol $\times$ [+, resp.] refer to the signal related to the motion starting from the initial conditions (4.31) [(4.32), resp.], i.e. the approximation of a point on a elliptic torus after having performed 9 [0, resp.] steps of the algorithm constructing the corresponding normal form. In both cases, the results for just the first 25 components have been reported in the figure above.

looking at the main components of the Fourier spectrum of the signals $\Lambda_l(t) \exp\left(\mathrm{i}\lambda_l(t)\right)$, with $l = 1,\ 2,\ 3$. Moreover, we stress that the vectors $k_j \in \mathbf{Z}^{n_1}$ listed in the third column are determined so to minimize the absolute difference $|\zeta_j - k_j \cdot \omega^{(\infty)}|$ with $|k_j| \leq 20$; indeed, one has to fix some limits on the absolute value of k_j, in order to make consistent its calculation, and our choice is motivated by the fact that the Fourier decay of the analytic conjugacy function $\mathcal{C}^{(\infty)}(0, Q, 0, 0)$ is such that the main contributions to the spectrum are related to low order harmonics.

If the initial conditions (4.31) were exactly on an elliptic torus, each value $|\zeta_j - k_j \cdot \omega^{(\infty)}|$ reported in the fourth column of Table 4.2 should be equal to zero. We see that all of them, except for the cases corresponding to $j = 16,\ 24$, are actually small enough to be considered as generated by round-off errors. On the other hand, we can

definitely say that $\zeta_{16} \simeq -1.1 \times 10^{-5}$ and $\zeta_{24} \simeq -1.9 \times 10^{-5}$ are "secular frequencies", because their values are $\mathcal{O}(\mu)$. Indeed, let us recall that $\mu \simeq 10^{-3}$, but the mass ratio for Uranus, i.e. $m_3/m_0 \simeq 4.4 \times 10^{-5}$, is even smaller.

Let us say that the occurrence of secular frequencies in the Fourier decomposition of the signal should be expected. Indeed, they could be completely avoided only in a very ideal situation, namely: (i) all the calculations described in sects. 4.2 and 4.3 should be carried out without performing any truncations on the expansions, (ii) the initial conditions (4.31) should be replaced with $\left(\mathcal{C}^{(\infty)}\right)^{-1}(0,0,0,0)$, (iii) no numerical errors should be there. In a practical calculation the orbit can not be exactly placed on an elliptic torus, so the presence of secular frequencies just means that we are just close to it. Nevertheless, it is very remarkable that the amplitude of the first found secular frequency is three orders of magnitude smaller than the main component of the spectrum. In our opinion, this is a first clear indication that our algorithm is properly working.

Other components corresponding to secular frequencies are expected to be even smaller than those found with $j = 16, 24$. In fact, let us recall that the frequency analysis method detects the summands $c_j \exp\left(\mathrm{i}\zeta_j t\right)$ appearing in (4.30) in a nearly decreasing order with respect to the amplitude $|c_j|$ (for instance, one can easily see that just two exchanges are needed in order to rewrite Table 4.2 in the correct decreasing order); moreover, we calculated that the discrepancy $\left|\xi_3(t) + \mathrm{i}\eta_3(t) - \sum_{j=0}^{24} c_j \exp\left(\mathrm{i}\zeta_j t\right)\right|$ is smaller than about $\simeq 3.7 \times 10^{-7}$ for all the time values t for which we sampled the signal. Let us emphasize that such an upper bound on the maximal discrepancy is just a little larger than the amplitude $|c_{16}|$.

A similar decomposition has been calculated for both the signals $\xi_1(t) + \mathrm{i}\eta_1(t)$ and $\xi_2(t) + \mathrm{i}\eta_2(t)$ (which are related to the secular motions of Jupiter and Saturn. The behavior is very similar to that of Table 4.2, so we omit the corresponding tables because the results are more evident from the figures that we are going to present.

The most relevant information about such decompositions of the secular motions of the three planets is summarized in the plots done with the $\times$ symbol appearing in Figs. 4.2–4.4.

Those figures contain also a comparison with the results provided by a, say, trivial approximation of an orbit on an elliptic torus. In fact, the dots marked with the $+$ symbol appearing in Figs. 4.2–4.4 refer to a frequency analysis which is performed exactly in the same way as that corresponding to the $\times$ symbol, except the fact that

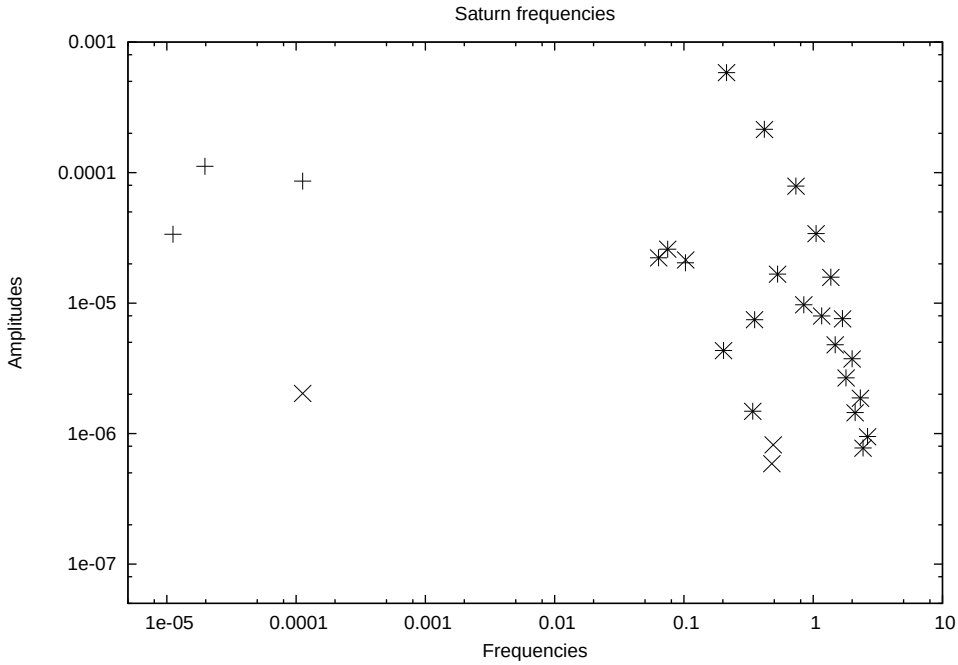

Figure 4.3. Frequency analysis of the secular signal related to the secular Saturn motion: $\xi_2(t) + \mathrm{i}\eta_2(t) = \sum_{j=0}^{\infty} c_j \exp\left(\mathrm{i}\zeta_j t\right)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The meaning of the symbols $\times$ and $+$ is the same as in Fig. 4.2.

the numerical integration of the equations of motion is started from the following initial conditions

$$\left(\mathcal{C}^{(0)}\right)^{-1}(0,0,0,0)\ , \tag{4.32}$$

instead of that reported in formula (4.31). Let us remark that $\left(\mathcal{C}^{(0)}\right)^{-1}(0,0,0,0) = \mathcal{E} \circ \mathcal{T}_{\Lambda^*} \circ \mathcal{D}\,(0,0,0,0)$ is a sort of trivial approximation of a point on the elliptic torus as it is provided by simply avoiding to apply the part of our algorithm constructing the normal form, as it is described in sect. 4.3. In order to discuss in a more definite way, we have already assumed that the secular frequencies are $\mathcal{O}(\mu)$, thus, let us separate them from the fast ones, when they are smaller than 10^{-3}. By looking at the right side of Figs. 4.2–4.4, one can immediately remark that the parts of the spectra related to the fast frequencies are nearly indistinguishable when the initial conditions (4.31) or (4.32) are considered, because the dots marked with the symbols $\times$ and $+$ superpose each other in a nearly exact way for what concerns all the main components. On the other hand, the secular parts of the spectra (that are in the left side of Figs. 4.2–4.4

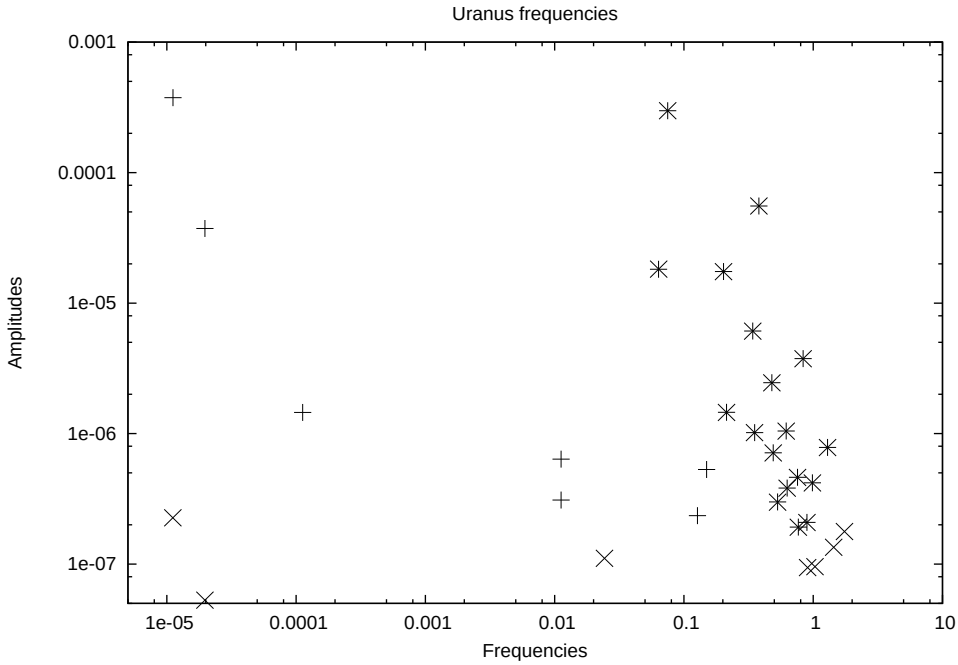

Figure 4.4. Frequency analysis of the secular signal related to the secular Uranus motion: $\xi_3(t) + \mathrm{i}\eta_3(t) = \sum_{j=0}^{\infty} c_j \exp\left(\mathrm{i}\zeta_j t\right)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The meaning of the symbols $\times$ and $+$ is the same as in Fig. 4.2.

strongly differ. In fact, when the initial conditions (4.32) (that trivially approximate a point on the elliptic torus) are considered, three secular frequencies are detected; while at most two are found in the case of the more accurate initial data (4.31). Moreover, by comparing the amplitudes, one can see that the secular components detected by both the frequency analysis are decreased by at least two orders of magnitude when our algorithm is applied. In our opinion, this comparison makes evident the effectiveness of our procedure constructing the normal form for an elliptic torus.

5

Construction of the Normal Form for Elliptic Tori in Planetary Systems

[Analytical Part]

"Science is what we understand well enough to explain to a computer. Art is everything else we do."
— Donald Knuth

5.1 Overview

In chapter 4 we described an algorithm for the explicit construction of the normal form related to elliptic tori in planetary systems, focusing our attention on the direct application to an approximation of the SJSU system.

Here we translate our algorithm into a rigorous proof and we ensure, under some smallness conditions on the size of the perturbation, the existence of elliptic tori.

This chapter contains the complete proof of our theorem about the existence of elliptic tori for planetary systems. It is basically divided in two parts: (i) the analytical part, where we work out the recursive estimates, obtaining a geometrically increasing bound on the growth of the norms; (ii) the measure theory part, where we define a set of boxed domains in which we can perform the complete normalization procedure, this completes the proof.

5.2 Technical Tools

In this section we introduce the technical background needed in the proof of the theorem. In order to simplify the calculations, we slightly change the analytical setting introduced in the previous chapter, using a set of complex conjugated variables.

5.2.1 Domains and Functions

Consider the complex domain $\mathcal{D}_{\varrho,R,\sigma}$,

$$\mathcal{D}_{\varrho,R,\sigma} = \Delta_\varrho(0) \times \mathbf{T}^{n_1}_\sigma \times \Delta_R(0)\,,$$

where ϱ, R and σ are positive parameters and

$$\begin{aligned}\Delta_\varrho(0) &= \{p \in \mathbf{C}^{n_1} : |p_j| < \varrho\}\ ,\\ \mathbf{T}^{n_1}_\sigma &= \{q \in \mathbf{C}^{n_1} : |\mathrm{Im}(q_j)| < \sigma\}\ ,\\ \Delta_R(0) &= \left\{z \in \mathbf{C}^{2n_2} : |z_j| < R\right\}\ .\end{aligned}$$

We choose the fixed centers of the poly-discs equal to zero in order to simplify the notation, but let us remark that it is not essential.

We basically consider analytic functions $f : \mathcal{D}_{\varrho,R,\sigma} \to \mathbf{R}$ of the form

$$f(p,q,z) = \sum_l \sum_m \sum_k f_{l,m,k} p^l z^m \exp(ik \cdot q)\,, \tag{5.1}$$

with $l \in \mathbf{N}^{n_1}$, $m \in \mathbf{N}^{2n_2}$ and $k \in \mathbf{Z}^{n_1}$, and we define two classes of functions:

$$\begin{aligned} \mathcal{P}_{L,M,sK} &= \{ f \text{ of the form (5.1)} : |l| = L,\ |m| = M,\ |k| \le sK \} \ , \\ \mathcal{P}_{\ell,sK} &= \bigcup_{2L+M=\ell} \mathcal{P}_{L,M,sK} \ . \end{aligned} \tag{5.2}$$

Considering again functions of the form (5.1), we introduce a norm depending on the three positive parameters ϱ, R and σ, as

$$\|f(p,q,z)\|_{\varrho,R,\sigma} = \sum_l \sum_m \sum_k |f_{l,m,k}| \varrho^{|l|} R^{|m|} \exp(|k|\sigma) \ ,$$

or, with a slightly different notation,

$$\|f(p,q,z)\|_{\varrho,R,\sigma} = \sum_k |f_k(p,z)|_{\varrho,R} \exp(|k|\sigma) \ ,$$

where

$$|f_k(p,z)|_{\varrho,R} = \sum_l \sum_m |f_{l,m,k}| \varrho^{|l|} R^{|m|} \ .$$

In the following we use the shorter notation $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{\alpha\varrho,\alpha R,\alpha\sigma}$, where α is any real positive number. Incidentally, according to convenience, we also use the notation $z = (x, y)$, where x, $y \in \mathbf{C}^{n_2}$ are a pair of complex conjugated variables.

5.2.2 Estimates about Derivatives and Lie Series

Using the Cauchy estimates we can give an upper bound of the norms of the derivatives (see [31]), but we have to pay with a restriction of the domain:

$$\left\| \frac{\partial f}{\partial p_j} \right\|_{1-d} \le \frac{1}{d\varrho} \|f\|_1 \ , \qquad \left\| \frac{\partial f}{\partial q_j} \right\|_{1-d} \le \frac{1}{ed\sigma} \|f\|_1 \ , \qquad \left\| \frac{\partial f}{\partial z_j} \right\|_{1-d} \le \frac{1}{dR} \|f\|_1 \ .$$

In order to estimate the norm of the Poisson bracket between two functions, let us remark that we can split the terms related to the derivatives of different pairs of conjugate canonical variables, writing

$$\{f,g\} = \{f,g\}_{p,q} + \{f,g\}_{x,y} \ ,$$

and we can bound separately the two summands.

Concerning the term related to the p and q variables, we have

$$\begin{aligned}\|\{f,g\}_{p,q}\|_{1-d-\delta} &\le \sum_{k,k'}\sum_{j} |k_j|\,|f_k|_{1-d-\delta}\, e^{|k|(1-d-\delta)\sigma}\frac{1}{(d+\delta)\varrho}\,|g_{k'}|_1\, e^{|k'|(1-d-\delta)\sigma} \\ &\qquad + \frac{1}{\delta\varrho}\,|f_k|_{1-d}\, e^{|k|(1-d-\delta)\sigma}|k'_j|\,|g_{k'}|_{1-d-\delta}\, e^{|k|(1-d-\delta)\sigma} \\ &\le \frac{2}{e\varrho\sigma}\frac{1}{(d+\delta)\delta}\,\|f\|_{1-d}\,\|g\|_1\end{aligned} \tag{5.3}$$

The estimate of $\{f,g\}_{x,y}$ requires more work. The function f, generates a vector field

$$F(p,q,x,y) = \left(0,0,-\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right),$$

and with this field, we can define an auxiliary function

$$G_{(x,y)}(t) = g\Big((0,0,x,y) + tF(0,0,x,y)\Big).$$

We want $(0,0,x,y)+tF(0,0,x,y) \in \mathcal{D}_{\varrho,R,\sigma}$, if $(x,y)\in\Delta_{1-d-\delta}(0)$ which is true provided

$$|t||F(0,0,x,y)| \le (d+\delta)R\,,$$

so that $G_{(x,y)}(t)$ is analytic in a disk. It follows that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}G_{(x,y)}(t)\,\Big|_{t=0}\right\|_{1-d-\delta} \le \frac{|F(0,0,x,y)|}{(d+\delta)R}\|g\|_1 \le \frac{1}{R^2}\frac{1}{(d+\delta)\delta}\|f\|_{1-d}\|g\|_1\,. \tag{5.4}$$

Using the inequalities (5.3) and (5.4), we get the estimate for the norm of the Poisson bracket,

$$\|\{f,g\}\|_{1-d-\delta} \le \left(\frac{2}{e\varrho\sigma}+\frac{1}{R^2}\right)\frac{1}{(d+\delta)\delta}\|f\|_{1-d}\|g\|_1\,. \tag{5.5}$$

Using this inequality, it's an easy matter to give a bound for the Lie series (see [31]). To this end consider the generic term $\mathcal{L}_f^j g$ of the Lie series and fix the final restriction of the domain, say d. Using a smaller step-size restriction, e.g. $\delta = d/j$, we get

$$\begin{aligned}\left\|\mathcal{L}_f^j g\right\|_{1-d} &= \left\|\mathcal{L}_f(\mathcal{L}_f^{j-1}g)\right\|_{1-(j-1)\delta-\delta} \\ &\le \left(\frac{2}{e\varrho\sigma}+\frac{1}{R^2}\right)\frac{1}{j\delta^2}\|f\|_1\left\|\mathcal{L}_f^{j-1}g\right\|_{1-d-\delta} \\ &\le \dots \\ &\le \frac{j!}{e^2}\left(\frac{2}{e\varrho\sigma}+\frac{1}{R^2}\right)^j\frac{e^{2j}}{(d^2)^j}\|f\|_1^j\|g\|_1\,,\end{aligned} \tag{5.6}$$

where we used the trivial inequality $j^j \le j!e^{j-1}$.

5.2.3 The Hamiltonian

We consider a Hamiltonian of the form

$$H(p,q,x,y) = \omega \cdot p + i \sum_j \varepsilon \Omega_j x_j y_j + \sum_{l \geq 0} \sum_{s \geq 0} \varepsilon^s f_l^{(s)}(p,q,x,y)\,,$$

where the functions $f_l^{(s)} \in \mathcal{P}_{l,sK}$ satisfy the parity condition[1] and the following equations,

$$f_0^{(0)} = f_1^{(0)} = f_2^{(0)} = 0, \qquad \langle f_0^{(1)} \rangle_q = \langle f_1^{(1)} \rangle_q = \langle f_2^{(1)} \rangle_q = 0\,.$$

As noted in the previous chapter, in order to construct the normal form related to the elliptic tori, we need to kill the terms $f_0^{(s)}$, $f_1^{(s)}$ and $f_2^{(s)}$, using a sequence of canonical transformations. In order to develop a perturbative algorithm we add an upper index to the functions, denoting the normalization order, so we start with a Hamiltonian $H^{(0)}$ that reads,

$$\begin{aligned} H^{(0)} = \omega^{(0)} \cdot p + i \sum_j \varepsilon \Omega_j^{(0)} x_j y_j \\ + \sum_{s \geq 1} \varepsilon^s f_0^{(0,s)} + \sum_{s \geq 1} \varepsilon^s f_1^{(0,s)} + \sum_{s \geq 1} \varepsilon^s f_2^{(0,s)} \\ + \sum_{l > 2} \sum_{s \geq 0} \varepsilon^s f_l^{(0,s)}\,, \end{aligned}$$

In the next section we explain the normalization procedure, assuming that the Hamiltonian is already in normal form up to some finite order, say $r-1$, we describe the r-th normalization step.

5.3 Scheme of the Proof

In the spirit of Kolmogorov scheme, we construct an infinite sequence of Hamiltonians, $\{H^{(r)}\}_{r \geq 0}$, with the request that each $H^{(r)}$ is in normal form up to order r. To this end, we perform an infinite sequence of normalization steps, each of which consists of a canonical transformation close to identity, which transforms the Hamiltonian $H^{(r-1)}$ to $H^{(r)}$. As explained in the previous chapter, the canonical transformation at order r

[1] Here parity condition means that all the coefficients of the expansion of $f_l^{(s)}$ having even (odd) trigonometrical degree in the q variables and odd (even) degree in p, are identically zero.

is generated via a composition of four Lie series of the form

$$\exp\left(\mathcal{L}_{\mathcal{D}_2^{(r)}}\right) \circ \exp\left(\mathcal{L}_{\chi_2^{(r)}}\right) \circ \exp\left(\mathcal{L}_{\chi_1^{(r)}}\right) \circ \exp\left(\mathcal{L}_{\chi_0^{(r)}}\right)$$

where $\chi_0^{(r)}(q) \in \mathcal{P}_{0,rK}$, $\chi_1^{(r)}(q,x,y) \in \mathcal{P}_{1,rK}$, $\chi_2^{(r)}(p,q,x,y) \in \mathcal{P}_{2,rK}$ and $\mathcal{D}_2^{(r)}(x,y) \in \mathcal{P}_{2,rK}$.

As usual, the generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$ and $\mathcal{D}_2^{(r)}$ are unknowns to be determined so that $H^{(r)}$ is in normal form up to order r.

In chapter 4, although we have considered a Hamiltonian written in real variables, we gave all details about the solutions of the homological equations for the generating functions. We refer to those formulas, referring to the semi-analytic part of our work. Instead we describe in detail how the Hamiltonian is transformed during the four stages of the r-th normalization step.

5.3.1 First Stage of the Normalization Step

The Hamiltonian $H^{(r-1)}$, which is in normal form up to order $r-1$, can be written as

$$\begin{aligned} H^{(r-1)} = \omega^{(r-1)} \cdot p + i \sum_j \varepsilon \Omega_j^{(r-1)} x_j y_j \\ + \sum_{s \geq r} \varepsilon^s f_0^{(r-1,s)} + \sum_{s \geq r} \varepsilon^s f_1^{(r-1,s)} + \sum_{s \geq r} \varepsilon^s f_2^{(r-1,s)} \\ + \sum_{l>2} \sum_{s \geq 0} \varepsilon^s f_l^{(r-1,s)} . \end{aligned} \tag{5.7}$$

In order to eliminate the term $f_0^{(r-1,r)}$, we have to solve the homological equation

$$\mathcal{L}_{\chi_0^r}\left(\omega^{r-1} \cdot p\right) + f_0^{(r-1,r)} = 0 , \tag{5.8}$$

as discussed in subsect. 4.3.1. We only stress that (5.8) admits a solution provided the frequency vector $\omega^{(r-1)}$ is non-resonant up to order rK, i.e.

$$\min_{0<|k|\leq rK} |k \cdot \omega^{(r-1)}| = \alpha_{r,0} > 0 . \tag{5.9}$$

We also remark that the generating function $\chi_0^{(r)}(q)$ depends only on the angles q, and by definition belongs to the family $\mathcal{P}_{0,rK}$.

Applying the Lie series algorithm, we get the transformed Hamiltonian,

$$H^{(\mathrm{I};r)} = \exp\left(\varepsilon^r \mathcal{L}_{\chi_0^{(r)}}\right) H^{(r-1)} ,$$

which can be written in a form similar to (5.7), with $f^{(\mathrm{I};r,s)}$ in place of $f^{(r-1,s)}$ and without the term $f_0^{(r-1,r)}$ which has just been eliminated. Thus we have

$$\begin{aligned} H^{(\mathrm{I};r)} &= \omega^{(r-1)} \cdot p + i \sum_j \varepsilon \Omega_j^{(r-1)} x_j y_j \\ &\quad + \sum_{s>r} \varepsilon^s f_0^{(\mathrm{I};r,s)} + \sum_{s\geq r} \varepsilon^s f_1^{(\mathrm{I};r,s)} + \sum_{s\geq r} \varepsilon^s f_2^{(\mathrm{I};r,s)} \\ &\quad + \sum_{l>2} \sum_{s\geq 0} \varepsilon^s f_l^{(\mathrm{I};r,s)} , \end{aligned} \tag{5.10}$$

where the functions $f_l^{(\mathrm{I};r,s)}$ are recursively defined by the equations

$$\begin{aligned} f_l^{(\mathrm{I};r,m)} &= 0 && \text{for } 0 < m < r\,,\ l = 0,1,2\,, \\ f_0^{(\mathrm{I};r,r)} &= 0\,, \\ f_0^{(\mathrm{I};r,r+m)} &= f_0^{(r-1,r+m)} && \text{for } 0 < m < r\,, \\ f_l^{(\mathrm{I};r,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} \mathcal{L}^j_{\chi_0^r} f_{l+2j}^{(r-1,s-jr)} && \text{elsewhere.} \end{aligned} \tag{5.11}$$

It is straightforward that, by construction, $f_l^{(\mathrm{I};r,s)} \in \mathcal{P}_{l,sK}$, indeed taken two generic functions $f \in \mathcal{P}_{l,rK}$ and $g \in \mathcal{P}_{m,sK}$, we have $\{f,g\} \in \mathcal{P}_{l+m-2,(r+s)K}$.

5.3.2 Second Stage of the Normalization Step

In order to eliminate the term $f_1^{(\mathrm{I};r,r)}$, we have to solve the homological equation

$$\mathcal{L}_{\chi_1^{(r)}} \left(\omega^{(r-1)} \cdot p + i \sum_j \varepsilon \Omega_j^{(r-1)} x_j y_j \right) + f_1^{(\mathrm{I};r,r)} = 0\,. \tag{5.12}$$

as discussed in subsect. 4.3.2. The solution of (5.12) exists if the so-called first Melnikov non-resonant condition is satisfied up to order rK, i.e.

$$\min_{\substack{0<|k|<rK \\ |l|=1}} |k \cdot \omega^{(r-1)} + l \cdot \Omega^{(r-1)}| = \alpha_{r,1} > 0\,. \tag{5.13}$$

Let us stress that, due to the parity condition, $\langle f_1^{(\mathrm{I};r,r)} \rangle_q$ is equal to zero, thus we can consistently solve the homological equation. Moreover the generating function $\chi_1^{(r)}(q,x,y)$ depends on the angles q and on the (x,y) variables and, by definition, belongs to the family $\mathcal{P}_{1,rK}$.

Applying the Lie series algorithm, we get the transformed Hamiltonian,

$$H^{(\mathrm{II};r)} = \exp(\varepsilon^r \mathcal{L}_{\chi_1^{(r)}}) H^{(\mathrm{I};r)}\,,$$

which can be written again in a form similar to (5.10), with $f^{(\mathrm{II};r,s)}$ in place of $f^{(\mathrm{I};r,s)}$ and without the term $f_1^{(\mathrm{I};r,r)}$ just removed. Thus we have

$$\begin{aligned} H^{(\mathrm{II};r)} &= \omega^{(r-1)} \cdot p + i \sum_j \varepsilon \Omega_j^{(r-1)} x_j y_j \\ &\quad + \sum_{s>r} \varepsilon^s f_0^{(\mathrm{II};r,s)} + \sum_{s>r} \varepsilon^s f_1^{(\mathrm{II};r,s)} + \sum_{s\geq r} \varepsilon^s f_2^{(\mathrm{II};r,s)} \\ &\quad + \sum_{l>2} \sum_{s\geq 0} \varepsilon^s f_l^{(\mathrm{II};r,s)} , \end{aligned} \tag{5.14}$$

where the functions $f_l^{(\mathrm{II};r,s)}$ are recursively defined by the equations

$$\begin{aligned} f_l^{(\mathrm{II};r,m)} &= 0 && \text{for } 0 < m < r\,,\ l = 0,1,2\,, \\ f_l^{(\mathrm{II};r,r)} &= 0 && \text{for } l = 0,1\,, \\ f_l^{(\mathrm{II};r,r+m)} &= f_l^{(\mathrm{I};r,r+m)} && \text{for } 0 < m < r\,,\ l = 0,1\,, \\ f_0^{(\mathrm{II};r,2r)} &= f_0^{(\mathrm{I};r,2r)} + \frac{1}{2}\mathcal{L}_{\chi_1^{(r)}} f_1^{(\mathrm{I};r,r)}\,, \\ f_0^{(\mathrm{II};r,2r+m)} &= f_0^{(\mathrm{I};r,2r+m)} + \frac{1}{2}\mathcal{L}_{\chi_1^{(r)}} f_1^{(\mathrm{I};r,r+m)}\,, && \text{for } 0 < m < r\,, \\ f_l^{(\mathrm{II};r,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} \mathcal{L}^j_{\chi_0^{(r)}} f_{l+j}^{(\mathrm{I};r,s-jr)} && \text{elsewhere.} \end{aligned} \tag{5.15}$$

As we remark in the previous subsection, also in this case, $f_l^{(\mathrm{II};r,s)} \in \mathcal{P}_{l,sK}$.

5.3.3 *Third Stage of the Normalization Step*

In order to complete the normalization step, we need to eliminate $f_2^{(\mathrm{II};r,r)}$ and so to solve the homological equations

$$\begin{aligned} &\mathcal{L}_{X_2^{(r)}} \left(\omega^{r-1} \cdot p\right) + f_2^{(\mathrm{II};r,r)} \Big|_{z=0} - \langle f_2^{(\mathrm{II};r,r)} \rangle_q \Big|_{z=0} = 0\,, \\ &\mathcal{L}_{Y_2^{(r)}} \left(\omega^{r-1} \cdot p + i \sum_j \varepsilon \Omega_j^{r-1} x_j y_j\right) + f_2^{(\mathrm{II};r,r)} \Big|_{p=0} - \langle f_2^{\mathrm{II};r,r} \rangle_q \Big|_{p=0} = 0\,, \\ &\mathcal{L}_{\mathcal{D}_2^{(r)}} \left(i \sum_j \varepsilon \Omega_j^{r-1} x_j y_j\right) + \langle f_2^{(\mathrm{II};r,r)} \rangle_q^{ND} \Big|_{p=0} = 0\,. \end{aligned} \tag{5.16}$$

In the last equation of (5.16), we use the notation $\langle f_2^{(\mathrm{II};r,r)} \rangle_q^{ND}$ to denote the terms of $f_2^{(\mathrm{II};r,r)}$ that do not depend only on the product of the two corresponding complex

conjugated variables x and y, i.e., that are not of the type[1] $c\,x_j y_j$.

The solutions of these equations have already been discussed in subsect. 4.3.3. The solutions of (5.16) exists provided again the non-resonance condition (5.9), the so-called second Melnikov non-resonance condition is satisfied up to order rK, i.e.

$$\min_{\substack{0<|k|<rK \\ |l|=2}} |k\cdot\omega^{(r-1)} + l\cdot\Omega^{(r-1)}| = \alpha_{r,2} > 0\,. \tag{5.17}$$

and the condition

$$\min_{|l|=2} |l\cdot\Omega^{(r-1)}| = \beta_{r,2} > 0\,, \tag{5.18}$$

are satisfied.

We collect the first two generating functions into a single one by adding them together, $\chi_2^{(r)} = X_2^{(r)} + Y_2^{(r)}$, while we keep separate the $\mathcal{D}_2^{(r)}$ function, that has the role of a diagonalization. Let us remark that by hypothesis we have $\langle f_0^{(0,1)}\rangle_q = \langle f_1^{(0,1)}\rangle_q = \langle f_2^{(0,1)}\rangle_q = 0$, and by construction we also know that $f_2^{(\mathrm{II};1,1)}$ contains no terms with zero average on the angle q, so we do not have to consider the $\mathcal{D}_2^{(1)}$ function, because we do not have anything to remove.

Applying the Lie series with the generating function $\chi_2^{(r)}$, we get the Hamiltonian

$$H^{(\mathrm{III};r)} = \exp(\varepsilon^r \mathcal{L}_{\chi_2^{(r)}}) H^{(\mathrm{II};r)}$$

which can be written in a form similar to (5.14), with $f^{(\mathrm{II};r,s)}$ in place of $f^{(\mathrm{I};r-1,s)}$ and with the new fast frequencies $\omega^{(r)}$ instead of $\omega^{(r-1)}$. Thus we have

$$\begin{aligned} H^{(\mathrm{III};r)} &= \omega^{(r)}\cdot p + i\sum_j \varepsilon\Omega_j^{(r-1)} x_j y_j \\ &\quad + \sum_{s>r} \varepsilon^s f_0^{(\mathrm{III};r,s)} + \sum_{s>r} \varepsilon^s f_1^{(\mathrm{III};r,s)} + \sum_{s\geq r} \varepsilon^s f_2^{(\mathrm{III};r,s)} \\ &\quad + \sum_{l>2}\sum_{s\geq 0} \varepsilon^s f_l^{(\mathrm{III};r,s)}\,, \end{aligned} \tag{5.19}$$

where the fast frequencies $\omega^{(r)}$ change as

$$\omega_j^{(r)} = \omega_j^{(r-1)} + \varepsilon^r \frac{\partial\langle f_2^{(\mathrm{II};r,r)}\rangle_q}{\partial p_j}\,,$$

[1] The corresponding term used in the previous chapter is $\langle f_{0\,,\,2}^{(\mathrm{II};r,0)}\rangle_\varphi$. As anticipated this notation reflects that, with a little abuse of notation, we call *diagonal* the secular term $i\sum_j \varepsilon\Omega_j^{r-1} x_j y_j$, and with ND we indicate the *non diagonal* terms.

and the functions $f_l^{(\mathrm{III};r,s)}$ are recursively defined by the equations

$$
\begin{aligned}
f_l^{(\mathrm{III};r,m)} &= 0 && \text{for } 0<m<r\,,\\
& && \quad l=0,1,2\,,\\
f_l^{(\mathrm{III};r,r)} &= 0 && \text{for } l=0,1\,,\\
f_2^{(\mathrm{III};r,r)} &= \langle f_2^{(\mathrm{II};r,r)}\rangle_q^{ND}\,\Big|_{p=0}\,, && \\
f_l^{(\mathrm{III};r,r+m)} &= f_l^{(\mathrm{II};r,r+m)} && \text{for } 0<m<r\,,\\
& && \quad l=0,1,2\,,\\
f_l^{(\mathrm{III};r,s)} &= \sum_{j=0}^{\lfloor (s-r-1)/r\rfloor} \frac{1}{j!}\mathcal{L}^j_{\chi_2^r} f_l^{(\mathrm{II};r,s-jr)} && \text{for } l=0,1\,,\\
& && \quad s\geq 2r\,,\\
f_2^{(\mathrm{III};r,kr)} &= \frac{k-1}{k!}\mathcal{L}^{k-1}_{\chi_2^r} f_2^{(\mathrm{II};r,r)} + \sum_{j=0}^{k-2}\frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r,kr-jr)} && \text{for } k\geq 2\,,\\
f_2^{(\mathrm{III};r,kr+m)} &= \sum_{j=0}^{k-1}\frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r,s-jr)} && \text{for } 0<m<r\,,\\
& && \quad k\geq 2\,,\\
f_l^{(\mathrm{III};r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor}\frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}} f_l^{(\mathrm{II};r,s-jr)} && \text{elsewhere.}
\end{aligned}
$$

Again we remark that $f_l^{(\mathrm{II};r,s)} \in \mathcal{P}_{l,sK}$.

Finally, applying the Lie series with $\mathcal{D}_2^{(r)}$, we get the Hamiltonian in normal form up to order r and we complete the r-th normalization step, getting

$$H^{(r)} = \exp(\varepsilon^r \mathcal{L}_{\mathcal{D}_2^{(r)}}) H^{(\mathrm{III};r)}\,,$$

which can be written in form similar to (5.19), with $f^{(r,s)}$ in place of $f^{(\mathrm{III};r-1,s)}$, without the term $f_2^{(\mathrm{III};r,r)}$ and with the new fast frequencies $\Omega^{(r)}$ instead of $\Omega^{(r-1)}$. Thus we

have

$$
\begin{aligned}
H^{(r)} = \omega^{(r)} \cdot p + \mathrm{i} \sum_j \varepsilon \Omega_j^{(r)} x_j y_j \\
+ \sum_{s>r} \varepsilon^s f_0^{(r,s)} + \sum_{s>r} \varepsilon^s f_1^{(r,s)} + \sum_{s>r} \varepsilon^s f_2^{(r,s)} \\
+ \sum_{l>2} \sum_{s\geq 0} \varepsilon^s f_l^{(r,s)} ,
\end{aligned}
\tag{5.20}
$$

where the slow frequencies $\Omega^{(r)}$ change according to the following formula

$$
\mathrm{i}\, \varepsilon \Omega_j^{(r)} = \mathrm{i}\, \varepsilon \Omega_j^{(r-1)} + \varepsilon^r \frac{\partial^2 \langle f_2^{(\mathrm{III};r,r)} \rangle_q}{\partial x_j \partial y_j} ,
$$

and the functions $f_l^{r,s}$ are recursively defined by the equations

$$
\begin{aligned}
f_l^{(r,m)} &= 0 && \text{for } 0 < m < r\,,\ l = 0,1,2\,, \\
f_l^{(r,r)} &= 0 && \text{for } l = 0,1,2\,, \\
f_2^{(r,kr)} &= \frac{k-1}{k!} \mathcal{L}^{k-1}_{\mathcal{D}_2^{(r)}} f_2^{(\mathrm{III};r,r)} + \sum_{j\geq 0}^{k-1} \frac{1}{j!} \mathcal{L}^{j}_{\mathcal{D}_2^{(r)}} f_2^{(\mathrm{III};r,kr-jr)} && \text{for } k \geq 2\,, \\
f_l^{(r,s)} &= \sum_{j\geq 0} \frac{1}{j!} \mathcal{L}^{j}_{\mathcal{D}_2^{(r)}} f_l^{(\mathrm{III};r,s)} && \text{elsewhere.}
\end{aligned}
$$

This concludes the r-th step of normalization. Let us finally remark that, at the end of the normalization step, by construction, $f_l^{(r,s)} \in \mathcal{P}_{l,sK}$ and the parity condition is satisfied again. Thus, ensuring the validity of the small divisors conditions, we can apply recursively our algorithm step-by-step to get the normal form.

5.4 Estimates for the Normalization Algorithm

In this section, we translate our formal algorithm, as expressed in section 5.3, into a recursive scheme of estimates on the norms of the functions involved in the normalization algorithm.

Let us start with the estimates about the generating functions that we collect in the following lemma.

Lemma 5.1: *The generating functions, $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$ and $\mathcal{D}_2^{(r)}$, defined by equations (5.8), (5.12), (5.16), are recursively bounded by:*

$$
\left\| \chi_0^{(r)} \right\| \leq \frac{\| f_0^{(r-1,r)} \|}{\alpha_{r,0}} ,
\tag{5.21}
$$

$$\left\|\chi_1^{(r)}\right\| \leq \frac{\|f_1^{(\mathrm{I};r,r)}\|}{\alpha_{r,1}}\,, \tag{5.22}$$

$$\left\|\chi_2^{(r)}\right\| \leq \frac{\|f_2^{(\mathrm{II};r,r)}\|}{\min(\alpha_{r,0},\,\alpha_{r,2})}\,, \tag{5.23}$$

$$\left\|\mathcal{D}_2^{(r)}\right\| \leq \frac{\|f_2^{(\mathrm{II};r,r)}\|}{\beta_{r,2}}\,, \tag{5.24}$$

where the denominator $\alpha_{r,0}$, $\alpha_{r,1}$, $\alpha_{r,2}$ *and* $\beta_{r,2}$ *are defined as*

$$\begin{aligned}
\alpha_{r,0} &= \min_{0<|k|\leq rK} |k\cdot\omega^{(r-1)}|\,,\\
\alpha_{r,1} &= \min_{\substack{0<|k|<rK\\|l|=1}} |k\cdot\omega^{(r-1)} + l\cdot\Omega^{(r-1)}|\,,\\
\alpha_{r,2} &= \min_{\substack{0<|k|<rK\\|l|=2}} |k\cdot\omega^{(r-1)} + l\cdot\Omega^{(r-1)}|\,,\\
\beta_{r,2} &= \min_{|l|=2} |l\cdot\Omega^{(r-1)}|\,.
\end{aligned}$$

Proof. The estimates (5.21)–(5.24) follow from equations (5.8), (5.12) and (5.16), respectively. We analyze only the $\chi_0^{(r)}$ function, the others follow similarly. As is evident from the form (5.8) of the solution, every coefficient of $f_0^{(r-1,r)}$ is divided by a quantity $k\cdot\omega^{(r-1)}$ (see (4.6)), which is bounded from below in view of the non resonance condition. Thus the norm of the function is divided by the same quantity. *Q.E.D.*

The small divisors $\alpha_{r,0}$ have the same expression as that we find in the Kolmogorov's theorem, while $\alpha_{r,1}$ and $\alpha_{r,2}$ are related to the first and to the second Melnikov conditions, respectively.

It is now useful to introduce a common single value instead of the three small divisors, so we define the quantity α_r as,

$$\alpha_r = \min_{\substack{0<|k|\leq rK\\0<|l|\leq 2}} |k\cdot\omega^{r-1} + l\cdot\Omega^{r-1}|\,.$$

Now we need to translate the formulas defining the functions $f_l^{(r,s)}$, $f_l^{(\mathrm{I};r,s)}$, $f_l^{(\mathrm{II};r,s)}$ and $f_l^{(\mathrm{III};r,s)}$ into a recursive scheme of estimates.

5.4.1 The Sets of Indexes

Let us remark that, considering the function $f_l^{(\mathrm{I};r,s)}$, due to the estimate (5.6) about the Lie series terms, each small divisor, α_r, is multiplied by a factor, say δ_r^2, corresponding

to the restriction of the domain. The one-to-one correspondence between these quantities and the r index suggest to take care only of the indexes in order to check the accumulation of these quantities. To be more precise, at each normalization step, we (potentially) introduce a new small divisor factor. We call this process the *accumulation* of the small divisors.

Definition 5.1: *Let $\mathcal{I}$ be a set of positive indexes, without loss of generality we can denote by i the i-th element of this set, we define the evaluation operator, $\mathcal{E}(\mathcal{I})$, as*

$$\mathcal{E}(\mathcal{I}) = \begin{cases} \displaystyle\prod_{i=1}^{\#\mathcal{I}} (d_i^2 \alpha_i)^{-1} & \text{if } \#\mathcal{I} \neq 0\,, \\ 1 & \text{elsewhere}\,. \end{cases}$$

Definition 5.2: *Consider n sets of indexes $\mathcal{I}_1, \ldots, \mathcal{I}_n$, we define the operator $\mathcal{MAX}(\mathcal{I}_1, \ldots, \mathcal{I}_n)$, that selects the set that maximize the evaluation operator defined above,*

$$\mathcal{MAX}(\mathcal{I}_1, \ldots, \mathcal{I}_n) = \mathcal{I}_{\bar{\jmath}} \qquad \text{s.t. } \mathcal{E}(\mathcal{I}_j) \leq \mathcal{E}(\mathcal{I}_{\bar{\jmath}}) \quad \forall\ 1 \leq j \leq n\,,$$

Definition 5.3: *Let $\mathcal{I}$ be a set of positive indexes and $h \in \mathbf{N}$, we define the operator $\mathcal{N}_h$, that estimate the number of indexes belonging to the interval $[2^h, 2^{h+1})$,*

$$\mathcal{N}_h(\mathcal{I}) = \#\left\{ j \in \mathcal{I} : 2^h \leq j < 2^{h+1} \right\}\,.$$

The choice of such intervals allows us to introduce the Bruno condition in a very natural way, as we will see at the end of the discussion.

In order to translate the transformations of the functions described in the normalization procedure into operations on sets of indexes, we associate to every function a set of indexes using the notation below,

$$\chi_0^{(r)} \leftrightarrow \mathcal{G}_0^{(r)}\,, \qquad \chi_1^{(r)} \leftrightarrow \mathcal{G}_1^{(r)}\,, \qquad \chi_2^{(r)} \leftrightarrow \mathcal{G}_2^{(r)}\,,$$

$$f_l^{(\mathrm{I};r,s)} \leftrightarrow \mathcal{F}_l^{(\mathrm{I};r,s)}\,, \qquad f_l^{(\mathrm{II};r,s)} \leftrightarrow \mathcal{F}_l^{(\mathrm{II};r,s)}\,, \qquad f_l^{(r,s)} \leftrightarrow \mathcal{F}_l^{(r,s)}\,.$$

Remember that the transformation due to $\mathcal{D}_2^{(r)}$, does not involve any small divisor, so we do not introduce any set of indexes for this part of the transformation.

Now we define recursively these sets, taking into account how the functions are transformed during the normalization procedure with special care on how the small

divisors are propagated during the transformations.

$$
\begin{aligned}
\mathcal{F}_l^{(0,s)} &= \emptyset && \text{for } l, s \geq 0\,, \\
\mathcal{G}_0^{(r)} &= \mathcal{F}_0^{(r-1,r)} \cup \{r\} && \text{for } r \geq 1\,, \\
\mathcal{F}_l^{(\mathrm{I};r,m)} &= \emptyset && \text{for } 0 < m < r\,,\ l = 0,1,2\,, \\
\mathcal{F}_0^{(\mathrm{I};r,r)} &= \emptyset\,, && \\
\mathcal{F}_0^{(\mathrm{I};r,r+m)} &= \mathcal{F}_0^{(r-1,r+m)} && \text{for } 0 < m < r\,, \\
\mathcal{F}_l^{(\mathrm{I};r,s)} &= \underset{0 \leq j \leq \lfloor s/r \rfloor}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_0^{(r)} \cup \mathcal{F}_{l+2j}^{(r-1,s-jr)} \right) && \text{for } l > 2 \text{ or} \\
& && \text{or } l = 1,2\,,\ s \geq r \\
& && \text{or } l = 0\,,\ s \geq 2r\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_1^{(r)} &= \mathcal{F}_1^{(\mathrm{I};r,r)} \cup \{r\} && \text{for } r \geq 1\,, \\
\mathcal{F}_l^{(\mathrm{II};r,m)} &= \emptyset && \text{for } 0 < m < r\,,\ l = 0,1,2\,, \\
\mathcal{F}_l^{(\mathrm{II};r,r)} &= \emptyset\,, && \text{for } l = 0,1\,, \\
\mathcal{F}_l^{(\mathrm{II};r,r+m)} &= \mathcal{F}_l^{(\mathrm{I};r,r+m)} && \text{for } 0 < m < r\,,\ l = 0,1\,, \\
\mathcal{F}_0^{(\mathrm{II};r,2r)} &= \underset{0 \leq j \leq 1}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_1^{(r)} \cup \mathcal{F}_j^{(\mathrm{I};r,2r-jr)} \right)\,, && \\
\mathcal{F}_l^{(\mathrm{II};r,s)} &= \underset{0 \leq j \leq \lfloor s/r \rfloor}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_1^{(r)} \cup \mathcal{F}_{l+j}^{(\mathrm{I};r-1,s-jr)} \right) && \text{for } l > 2 \\
& && \text{or } l = 2\,,\ s \geq r \\
& && \text{or } l = 1\,,\ s \geq 2r \\
& && \text{or } l = 0\,,\ s > 2r\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_2^{(r)} &= \mathcal{F}_2^{(\mathrm{II};r,r)} \cup \{r\} && \text{for } r \geq 1\,,\\
\mathcal{F}_l^{(r,m)} &= \emptyset && \text{for } 0 < m < r\,,\ l = 0,1,2\,,\\
\mathcal{F}_l^{(r,r)} &= \emptyset\,, && \text{for } l = 0,1,2\,,\\
\mathcal{F}_l^{(r,r+m)} &= \mathcal{F}_l^{(\mathrm{II};r,r+m)} && \text{for } 0 < m < r\,,\ l = 0,1,2\,,\\
\mathcal{F}_l^{(r,s)} &= \underset{0\leq j\leq \lfloor (s-r-1)/r\rfloor}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_2^{(r)} \cup \mathcal{F}_l^{(\mathrm{II};r,s-jr)} \right)\,, && \text{for } l = 0,1\,,\ s \geq 2r\\
\mathcal{F}_2^{(r,s)} &= \underset{0\leq j\leq \lfloor (s-r)/r\rfloor}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_2^{(r)} \cup \mathcal{F}_2^{(\mathrm{II};r,s-jr)} \right)\,, && \text{for } s \geq 2r\\
\mathcal{F}_l^{(r,s)} &= \underset{0\leq j\leq \lfloor s/r\rfloor}{\mathcal{MAX}} \left(\bigcup_{i=1}^{j} \mathcal{G}_2^{(r)} \cup \mathcal{F}_l^{(\mathrm{II};r,s-jr)} \right) && \text{for } l > 2
\end{aligned}
$$

For these sets of indexes, as stated in the following lemma, we are able to give an upper bound to the operators $\mathcal{N}_h$, hence we can control the accumulation of the small divisors.

Lemma 5.2: *From the definition 5.3, taken a set of indexes $\mathcal{I}$, the operator $\mathcal{N}_h(\mathcal{I})$ is defined as*

$$\mathcal{N}_h(\mathcal{I}) = \#\left\{ j \in \mathcal{I} : 2^h \leq j < 2^{h+1} \right\}\,.$$

Consider the sets of indexes $\mathcal{F}_l^{(\mathrm{I};r,s)}$, $\mathcal{F}_l^{(\mathrm{II};r,s)}$, $\mathcal{F}_l^{(r,s)}$, $\mathcal{G}_0^{(r)}$, $\mathcal{G}_1^{(r)}$ and $\mathcal{G}_2^{(r)}$, defined above, then the following estimates are true,

$$\mathcal{N}_h\left(\mathcal{F}_l^{(r,s)}\right) \leq \begin{cases} 3\left\lfloor \frac{s}{2^h} \right\rfloor - (3-l)_+ & \text{if } h \leq \lfloor \log_2 r \rfloor,\\ 0 & \text{elsewhere,} \end{cases}$$

where $\mathcal{F}_l^{(r,s)}$ can be replaced by $\mathcal{F}_l^{(\mathrm{I};r,s)}$ or $\mathcal{F}_l^{(\mathrm{II};r,s)}$, and

$$\mathcal{N}_h\left(\mathcal{G}_l^{(r)}\right) \leq \begin{cases} 3\left\lfloor \frac{r}{2^h} \right\rfloor - (2-l)_+ & \text{if } h \leq \lfloor \log_2 r \rfloor,\\ 0 & \text{elsewhere.} \end{cases}$$

where in the last formula l ranges between 0 and 2.

Proof. The proof of this lemma requires only careful calculations and is done by induction. We use the notation $(a)_+$ meaning $\max(0,\, a)$.

For $r = 0$ is trivial, so assume that the relations are true up to $r - 1$.

Starting with the $\mathcal{G}_0^{(r)}$ set, if $r = 2^h$,

$$\mathcal{N}_h\left(\mathcal{G}_0^{(r)}\right) = 1 \leq 3\left\lfloor \frac{r}{2^h} \right\rfloor - 2\,,$$

and if $r > 2^h$,

$$\mathcal{N}_h\left(\mathcal{G}_0^{(r)}\right) \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - 3 + 1 \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - 2\,.$$

Considering the $\mathcal{F}_l^{(r,s)}$ set, if $r = 2^h$, using the fact that $\mathcal{N}_h\left(\mathcal{G}_0^{(r)}\right) = 1$, we get

$$\mathcal{N}_h\left(\mathcal{F}_l^{(r,s)}\right) = \left\lfloor \frac{s}{2^h} \right\rfloor \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - (3-l)_+\,,$$

we remark that the last inequality is clearly not true in general, but in our case we have:

1. if $l = 0$, then $s \ge 2r$;
2. if $l = 1, 2$, then $s \ge r$;
3. if $l \ge 3$, the inequality is true.

If $r > 2^h$,

$$\begin{aligned}\mathcal{N}_h\left(\mathcal{F}_l^{(r,s)}\right) &\le 3j\left\lfloor \frac{r}{2^h} \right\rfloor - 2j + 3\left\lfloor \frac{s-jr}{2^h} \right\rfloor - (3-l-2j)_+ \\ &\le 3\left\lfloor \frac{s}{2^h} \right\rfloor - (3-l)_+\,,\end{aligned}$$

where we use the trivial inequality

$$-b + (a-b)_+ \ge (a)_+\,, \qquad \text{for all } b \ge 0\,.$$

The estimate for $\mathcal{G}_1^{(r)}$ is trivial,

$$\mathcal{N}_h\left(\mathcal{G}_1^{(r)}\right) \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - 2 + 1 \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - 1\,,$$

and for $\mathcal{F}_l^{(r,s)}$ we have

$$\begin{aligned}\mathcal{N}_h\left(\mathcal{F}_l^{(r,s)}\right) &\le 3j\left\lfloor \frac{r}{2^h} \right\rfloor - j + 3\left\lfloor \frac{s-jr}{2^h} \right\rfloor - (3-l-j)_+ \\ &\le 3\left\lfloor \frac{s}{2^h} \right\rfloor - (3-l)_+\,.\end{aligned}$$

Finally for $\mathcal{G}_2^{(r)}$ we have,

$$\mathcal{N}_h\left(\mathcal{G}_2^{(r)}\right) \le 3\left\lfloor \frac{r}{2^h} \right\rfloor - 1 + 1 \le 3\left\lfloor \frac{r}{2^h} \right\rfloor,$$

and for $\mathcal{F}_l^{(r,s)}$

$$\begin{aligned}\mathcal{N}_h\left(\widehat{\mathcal{F}}_l^{(r,s)}\right) &\le 3j\left\lfloor \frac{r}{2^h} \right\rfloor + 3\left\lfloor \frac{s-jr}{2^h} \right\rfloor - (3-l)_+ \\ &\le 3\left\lfloor \frac{s}{2^h} \right\rfloor - (3-l)_+\,.\end{aligned}$$

Q.E.D.

To complete the scheme of estimates of the norms of the functions involved in the normalization procedure, we also have to control the number of terms in every summation. To this aim we define three sequences related to the normalization procedure.

Definition 5.4: *The three sequences of integer numbers* $\{\nu_{r,s}\}_{r,s\geq 0}$, $\{\nu^{(\mathrm{I})}_{r,s}\}_{r,s\geq 0}$ *and* $\{\nu^{(\mathrm{II})}_{r,s}\}_{r,s\geq 0}$, *are defined by*

$$\begin{aligned}
\nu_{0,s} &= 1\,,\\
\nu^{(\mathrm{I})}_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} \nu^{j}_{r-1,r}\nu_{r-1,s-jr}\,,\\
\nu^{(\mathrm{II})}_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} (\nu^{(\mathrm{I})}_{r,r})^{j}\nu^{(\mathrm{I})}_{r,s-jr}\,,\\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} (\nu^{(\mathrm{II})}_{r,r})^{j}\nu^{(\mathrm{II})}_{r,s-jr}\,,
\end{aligned}$$

With some calculation it's easy to write the sequence $\{\nu_{r,s}\}_{s\geq 0}$ with the sole dependence on the sequence $\{\nu_{r-1,s}\}_{s\geq 0}$, first can write

$$\begin{aligned}
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} \left(\nu^{(\mathrm{I})}_{r,r} + \nu^{(\mathrm{I})}_{r,r}\nu^{(\mathrm{I})}_{r,0}\right)^{j} \sum_{i=0}^{\lfloor s/r\rfloor - j} (\nu^{(\mathrm{I})}_{r,r})^{i}\nu^{(\mathrm{I})}_{r,s-jr-ir}\\
&= \sum_{j=0}^{\lfloor s/r\rfloor} 2^{j}(\nu^{(\mathrm{I})}_{r,r})^{j} \sum_{i=j}^{\lfloor s/r\rfloor} (\nu^{(\mathrm{I})}_{r,r})^{i-j}\nu^{(\mathrm{I})}_{r,s-ir}\\
&= \sum_{j=0}^{\lfloor s/r\rfloor} (\nu^{(\mathrm{I})}_{r,r})^{j}\nu^{(\mathrm{I})}_{r,s-jr} \sum_{i=0}^{j} 2^{i}\\
&= \sum_{j=0}^{\lfloor s/r\rfloor} \left(2^{j+1}-1\right)(\nu^{(\mathrm{I})}_{r,r})^{j}\nu^{(\mathrm{I})}_{r,s-jr}\,.
\end{aligned}$$

In this way we eliminate the $\nu^{(\mathrm{II})}$ dependency and, with the same procedure, we get

$$\begin{aligned}
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} \left(2^{j+1}-1\right)\left(\nu_{r-1,r} + \nu_{r-1,r}\nu_{r-1,0}\right)^{j} \sum_{i=0}^{\lfloor s/r\rfloor - j} \nu^{i}_{r-1,r}\nu_{r-1,s-jr-ir}\\
&= \sum_{j=0}^{\lfloor s/r\rfloor} 2^{j}\left(2^{j+1}-1\right)\nu^{j}_{r-1,r} \sum_{i=j}^{\lfloor s/r\rfloor} \nu^{i-j}_{r-1,r}\nu_{r-1,s-ir}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\lfloor s/r \rfloor} \nu_{r-1,r}^{j} \nu_{r-1,s-jr} \sum_{i=0}^{j} 2^{2i+1} - 2^{i} \\
&= \sum_{j=0}^{\lfloor s/r \rfloor} \left(\frac{2}{3} \left(2^{2(j+1)} - 1 \right) - 2^{j+1} + 1 \right) \nu_{r-,r}^{j} \nu_{r-,s-jr} \,.
\end{aligned}
$$

Let us introduce the shorter notation

$$
n_j = \frac{2}{3} \left(2^{2(j+1)} - 1 \right) - 2^{j+1} + 1 \,,
$$

and remark that

$$
n_0 = 1 \qquad \text{and} \qquad n_1 = 7 \,.
$$

It's easy to see that

$$
\frac{1}{3} 2^{2(j+1)} \leq n_j \leq \frac{2}{3} 2^{2(j+1)} \qquad \text{for } j \geq 1 \,,
$$

thus we can ensure the estimate

$$
n_{j+1} \leq 2^3 n_j \qquad \text{for } j \geq 0 \,. \tag{5.25}
$$

Lemma 5.3: *Consider the sequence $\{\nu_{r,s}\}_{r,s\geq 0}$ as in definition 5.4, the following properties are satisfied:*

(i) $\nu_{0,s} < \nu_{1,s} < \ldots < \nu_{s,s} = \nu_{s+1,s} = \ldots$

(ii) $\nu_{r,r} = 8\nu_{r-1,r}$

(iii) $\nu_{1,s} \leq \nu_{s-1,s-1}$

(iv) $\nu_{r,s} \leq \nu_{r-1,s} + \nu_{r,r}\nu_{s-r,s-r}$

Proof. (i) is trivial; (ii) as remarked above, $n_0 = 1$ and $n_1 = 7$, so

$$
\nu_{r,r} = n_0 \nu_{r-1,r} + n_1 \nu_{r-1,r} = 8\nu_{r-1,r} \,.
$$

(iv) requires some calculations,

$$
\begin{aligned}
\nu_{r,s} &= \nu_{r-1,s} + \sum_{j=1}^{\lfloor s/r \rfloor} n_j \nu_{r-1,r}^{j} \nu_{r-1,s-jr} \\
&= \nu_{r-1,s} + \nu_{r-1,r} \sum_{j=0}^{\lfloor s/r \rfloor - 1} n_{j+1} \nu_{r-1,r}^{j} \nu_{r-1,s-r-jr} \\
&\leq \nu_{r-1,s} + 2^3 \nu_{r-1,r} \sum_{j=0}^{\lfloor s/r \rfloor - 1} n_j \nu_{r-1,r}^{j} \nu_{r-1,s-r-jr} \\
&\leq \nu_{r-1,s} + 2^3 \nu_{r-1,r} \nu_{r-1,s-r} \leq \nu_{r-1,s} + \nu_{r,r} \nu_{s-r,s-r} \,,
\end{aligned}
$$

where we use the estimate (5.25) in the third line. Finally (iii) is very similar, being estimated as

$$\begin{aligned}
\nu_{1,s} &= \nu_{0,s} + \nu_{0,1} \sum_{j=0}^{s-1} n_{j+1} \nu_{0,1}^{j} \nu_{0,s-1-j} \\
&\leq \nu_{0,s} + n_1 \nu_{0,s-1} + 2^3 \sum_{j=1}^{s-1} n_j \nu_{0,1}^{j} \nu_{0,s-1-j} \\
&\leq 2^3 \nu_{0,s-1} \leq \nu_{s-1,s-1}
\end{aligned}$$

Q.E.D.

Lemma 5.4: *Consider the sequence $\{\nu_{r,s}\}_{r,s\geq 0}$ as in definition 5.4, the following estimate is true,*

$$\nu_{r,r} \leq \frac{256^r}{32}$$

Proof. In view of lemma 5.3, we have

$$\begin{aligned}
\nu_{r,r} &= 8\nu_{r-1,r} \\
&\leq 8\nu_{r-2,r} + 8\nu_{r-1,r-1}\nu_{1,1} \\
&\leq \dots \\
&\leq 8\nu_{1,r} + 8\left(\nu_{2,2}\nu_{r-2,r-2} + \dots + \nu_{r-1,r-1}\nu_{1,1}\right) \\
&\leq 8 \sum_{j=1}^{r-1} \nu_{j,j}\nu_{r-j,r-j}
\end{aligned}$$

Assume that the terms $\nu_{r,r}$ satisfy the following relation,

$$\nu_{r,r} \leq \frac{B^r}{8}\lambda_r$$

with a constant B to be determined and where the λ_r are defined recursively as

$$\lambda_1 = 1\,, \qquad \lambda_r = \sum_{j=1}^{r-1} \lambda_j \lambda_{r-j}\,.$$

This is the well known Catalan's sequence, which is bounded by

$$\lambda_r \leq 4^{r-1}\,,$$

thus we have to choose the constant B in order to satisfy

$$\nu_{1,1} \leq \frac{B}{8}\,, \qquad \nu_{2,2} \leq \frac{B^2}{8}4.$$

Setting $B = 64$ is sufficient and we get the bound

$$\nu_{r,r} \leq \frac{256^r}{32} ,$$

that concludes the proof. *Q.E.D.*

Let us remark that, in order to use the Cauchy formula to estimate the norm of the Lie series, we have to pay a restriction of the domain in which the estimates are valid. Having divided the formal algorithm in four steps, we need four successive restrictions, to this end we introduce a sequence $\{d_r\}_{r\geq 0}$, as

$$d_0 = 0 , \qquad d_r = d_{r-1} + 4\delta_r ,$$

where

$$\delta_r = \frac{1}{4}\frac{3}{2\pi^2 r^2} .$$

At each step we give a restriction equal to δ_r, and for every normalization order, we pay a restriction equal to $4\delta_r$. With this choice of the parameters, we remark that $\lim_{r\to\infty} d_r = 1/4$.

Lemma 5.5: *Assume that*

$$\|f_l^{0,s}\| \leq E \qquad \text{for all } l, s \geq 0 ,$$

define two constants, C and M, as

$$C = \left(\frac{2}{e\varrho\sigma} + \frac{1}{R^2}\right)\frac{64\pi^4}{9} , \qquad M = \max\{1, CE\} ,$$

and a sequence $\{\zeta_r\}$ as

$$\zeta_1 = 0 , \qquad \zeta_r = \zeta_{r-1} + \tau_r ,$$

with

$$\tau_r = \frac{r^4\varepsilon^{r-1}C\|\mathcal{D}_2^r\|}{1 - r^4\varepsilon^{r-1}C\|\mathcal{D}_2^r\|} ,$$

The following estimates are true:

$$C\|\chi_0^{(r)}\|_{1-d_{r-1}} \leq M^{3r-2}\mathcal{E}(\mathcal{G}_0^{(r)})\nu_{r-1,r}\exp(\zeta_r) ,$$

$$\begin{aligned} \|f_0^{(\mathrm{I};r,r+m)}\|_{1-d_{r-1}-\delta_r} &\leq M^{3(r+m)-3}E\mathcal{E}(\mathcal{F}_0^{(\mathrm{I};r,r+m)})\nu_{r,r+m}^{(\mathrm{I})}\exp(\zeta_r) && \text{for } 0 < m < r , \\ \|f_0^{(\mathrm{I};r,s)}\|_{1-d_{r-1}-\delta_r} &\leq M^{3s-2\lfloor s/r\rfloor}E\mathcal{E}(\mathcal{F}_0^{(\mathrm{I};r,s)})\nu_{r,s}^{(\mathrm{I})}\exp(\zeta_r) && \text{elsewhere.} \end{aligned}$$

$$C\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \leq M^{3r-1}\mathcal{E}(\mathcal{G}_1^{(r)})\nu_{r,r}^{(\mathrm{I})}\exp(\zeta_r) ,$$

$$\begin{aligned}
\|f_0^{(\mathrm{II};r,s)}\|_{1-d_{r-1}-2\delta_r} &\le M^{3s-3}E\mathcal{E}(\mathcal{F}_0^{(\mathrm{II};r,s)})\nu_{r,s}^{(\mathrm{II})}\exp(\zeta_r) && \text{for } r<s<3r\,,\\
\|f_1^{(\mathrm{II};r,s)}\|_{1-d_{r-1}-2\delta_r} &\le M^{3s-2}E\mathcal{E}(\mathcal{F}_1^{(\mathrm{II};r,s)})\nu_{r,s}^{(\mathrm{II})}\exp(\zeta_r) && \text{for } r<s<2r\,,\\
\|f_l^{(\mathrm{II};r,s)}\|_{1-d_{r-1}-2\delta_r} &\le M^{3s-\lfloor s/r\rfloor}E\mathcal{E}(\mathcal{F}_l^{(\mathrm{II};r,s)})\nu_{r,s}^{(\mathrm{II})}\exp(\zeta_r) && \text{elsewhere.}
\end{aligned}$$

$$C\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} \le M^{3r}\mathcal{E}(\mathcal{G}_2^{(r)})\nu_{r,r}^{(\mathrm{II})}\exp(\zeta_r)\,,$$

$$\begin{aligned}
\|f_0^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} &\le M^{3s-3}E\mathcal{E}(\mathcal{F}_0^{r,s})\nu_{r,s}\exp(\zeta_r)\,,\\
\|f_1^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} &\le M^{3s-2}E\mathcal{E}(\mathcal{F}_1^{r,s})\nu_{r,s}\exp(\zeta_r)\,,\\
\|f_2^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} &\le M^{3s-1}E\mathcal{E}(\mathcal{F}_2^{r,s})\nu_{r,s}\exp(\zeta_r)\,,\\
\|f_l^{(\mathrm{III};r,s}\|_{1-d_{r-1}-3\delta_r} &\le M^{3s}E\mathcal{E}(\mathcal{F}_l^{r,s})\nu_{r,s}\exp(\zeta_r) && \text{for } l>2
\end{aligned}$$

$$\begin{aligned}
\|f_0^{r,s}\|_{1-d_r} &\le M^{3s-3}E\mathcal{E}(\mathcal{F}_0^{r,s})\nu_{r,s}\exp(\zeta_{r+1})\,,\\
\|f_1^{r,s}\|_{1-d_r} &\le M^{3s-2}E\mathcal{E}(\mathcal{F}_1^{r,s})\nu_{r,s}\exp(\zeta_{r+1})\,,\\
\|f_2^{r,s}\|_{1-d_r} &\le M^{3s-1}E\mathcal{E}(\mathcal{F}_2^{r,s})\nu_{r,s}\exp(\zeta_{r+1})\,,\\
\|f_l^{r,s}\|_{1-d_r} &\le M^{3s}E\mathcal{E}(\mathcal{F}_l^{r,s})\nu_{r,s}\exp(\zeta_{r+1}) && \text{for } l>2
\end{aligned}$$

Proof. The proof of this lemma requires lengthy but trivial computation. Using the definitions of the indexes sets and of the sequences in definition 5.4, applying the lemma 5.1 we only need to get rid of the accumulation of the M and ζ_r factors.

The statements of the lemma are clearly true for $r = 0, 1$. Assuming that all the estimates are satisfied up to $r-1$, we proceed by induction.

Let us introduce the symbol $\preceq$, to indicate that in the following estimates we omit all factors but M.

$$C\|\chi_0^{(r)}\|_{1-d_{r-1}} \le C\|f_0^{(r-1,r)}\|_{1-d_{r-1}} \preceq CM^{3r-3}E \le M^{3r-2}$$

$$\begin{aligned}
\|f_0^{(\mathrm{I};r,r+m)}\|_{1-d_{r-1}-\delta_r} &\preceq M^{3(r+m)-3}E && \text{for } 0<m<r\,,\\
\|f_l^{(\mathrm{I};r,s)}\|_{1-d_{r-1}-\delta_r} &\preceq M^{(3r-2)j}M^{3(s-jr)}E \le M^{3s-2\lfloor s/r\rfloor}E && \text{elsewhere.}
\end{aligned}$$

$$C\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \preceq M^{3r-1}$$

$$
\begin{aligned}
\|f_0^{(\mathrm{II};r,r+m)}\|_{1-d_{r-1}-2\delta_r} &\preceq M^{3(r+m)-3}E && \text{for } r<s<2r,\\
\|f_0^{(\mathrm{II};r,2r)}\|_{1-d_{r-1}-2\delta_r} &\preceq M^{3r-1}M^{3r-2}E \le M^{3(2r)-3}E\,, && \\
\|f_0^{(\mathrm{II};r,2r+m)}\|_{1-d_{r-1}-2\delta_r} &\preceq M^{3r-1}M^{3(r+m)-2}E \le M^{3(2r+m)-3}E && \text{for } 2r<s<3r,\\
\|f_1^{(\mathrm{II};r,r+m)}\|_{1-d_{r-1}-2\delta_r} &\preceq M^{3(r+m)-2}E && \text{for } r<s<2r,\\
\|f_l^{(\mathrm{II};r,s)}\|_{1-d_{r-1}-2\delta_r} &\preceq M^{3s-\lfloor s/r\rfloor}E && \text{elsewhere.}
\end{aligned}
$$

$$C\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} \preceq M^{3r}$$

for the estimate of $\|f_0^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r}$ we have to consider separately the contribution to the sum in the case $s-jr\ge 3r$, where we get

$$M^{3rj}M^{3(s-jr)-\lfloor (s-jr)/r\rfloor}E \le M^{3s-3}E\,,$$

and the case $s-jr<3r$ where we have

$$M^{3rj}M^{3(s-jr)-3}E \le M^{3s-3}E\,.$$

Consider $\|f_1^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r}$, also in this case we have to split the sum considering the case $s-jr\ge 2r$

$$M^{3rj}M^{3(s-jr)-\lfloor (s-jr)/r\rfloor}E \le M^{3s-2}E\,,$$

and the case $s-jr<2r$ where we have

$$M^{3rj}M^{3(s-jr)-2}E \le M^{3s-2}E\,.$$

For $\|f_2^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r}$ we have,

$$M^{3rj}M^{3(s-jr)-\lfloor (s-jr)/r\rfloor}E \le M^{3s-1}E\,,$$

and for all the other cases we have

$$\|f_l^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} \preceq M^{3rj}M^{3(s-jr)-\lfloor (s-jr)/r\rfloor}E \le M^{3s}E\,.$$

To conclude we only need to take into account the effect of the $\mathcal{D}_2^{(r)}$ generating functions. By definition, we have

$$\|\mathcal{D}_2^{(r)}\| \sim \mathcal{O}(\varepsilon^{r-1})\,,$$

and the terms of the transformed Hamiltonian are

$$f_l^{(r,s)} = \sum_{j\ge 0}\frac{1}{j!}\mathcal{L}^j_{\mathcal{D}_2^r} f_l^{(\mathrm{III};r,s)}\,.$$

We remark that this transformation do not involve small divisors, so we have

$$\begin{aligned}
\|f_l^{r,s}\| &\leq \sum_{j\geq 0} C^j r^4 \varepsilon^{j(r-1)} \|\mathcal{D}_2^r\|^j \|f_l^{(\mathrm{III};r,s)}\| \\
&\leq \|f_l^{(\mathrm{III};r,s)}\| \sum_{j\geq 0} \left(r^4\varepsilon^{r-1} C\|\mathcal{D}_2\|^r\|\right)^j \\
&\leq \|f_l^{(\mathrm{III};r,s)}\| \frac{1}{1-r^4\varepsilon^{r-1}C\|\mathcal{D}_2\|^r} \leq 1 + \frac{r^4\varepsilon^{r-1}C\|\mathcal{D}_2\|^r}{1-r^4\varepsilon^{r-1}C\|\mathcal{D}_2\|^r} \\
&\leq \|f_l^{(\mathrm{III};r,s)}\| \exp\left(\frac{r^4\varepsilon^{r-1}C\|\mathcal{D}_2\|^r}{1-r^4\varepsilon^{r-1}C\|\mathcal{D}_2\|^r}\right),
\end{aligned}$$

and using the definition of the ζ_r, we only have to add a factor $\exp(\zeta_r)$ to every terms of the previous estimate in order to add the effect of the $\mathcal{D}_2^r$ functions. *Q.E.D.*

To conclude the analytic part of the proof, we need the estimates for the evaluation operators. Let us denote by $\mathcal{I}^s$ a generic set of indexes considered during the normalization algorithm, we need to give an upper bound for

$$\prod_{j=1}^{\#\mathcal{I}^s} \frac{j^4}{\alpha_j}.$$

From lemma 5.2 we have $\mathcal{N}_h(\mathcal{I}^s) \leq (3s/2^h)$, so the previous product is dominated by

$$\prod_{h\geq 0} \left(\frac{1}{\alpha_{2^{h+1}}}\right)^{3s/2^h} \left(2^{4(h+1)}\right)^{3s/2^h},$$

and we can bound geometrically the growth of this product using the Bruno condition. In fact by the Bruno condition

$$-\sum_{h\geq 0} \frac{\log_2 \alpha_{2^{h+1}}}{2^h} \leq \text{Б} < \infty,$$

and the trivial calculation

$$\sum_{h\geq 0} \frac{12(h+1)}{2^h} = 48,$$

we can bound the evaluation functions by

$$2^{s(3\text{Б}+48)}.$$

Collecting all the results, we are able to ensure that, during the normalization algorithm, the growth of the norm of the $f_l^{r,s}$ function is bounded by

$$\|f_l^{r,s}\| \leq \frac{1}{32} M^{3s} E 2^{s(3\text{Б}+8+48)} \exp(\zeta_r).$$

In view of the geometrical bound on the growth, to conclude we can use the standard arguments which consist essentially in bounding the norms of the functions with a geometric series, see, e.g., [30].

The results of the analytic part are collected in the following proposition for the construction of the normal form related to elliptic tori.

Proposition 5.1: *Consider a Hamiltonian of the form*

$$H(p,q,x,y) = \omega \cdot p + i \sum_j \varepsilon \Omega_j x_j y_j + \sum_{l \geq 0} \sum_{s \geq 0} \varepsilon^s f_l^{(s)}(p,q,x,y)\,,$$

where the functions $f_l^{(s)} \in \mathcal{P}_{l,sK}$ *satisfy the parity condition and the following equations*

$$f_0^{(0)} = f_1^{(0)} = f_2^{(0)} = 0, \qquad \langle f_0^{(1)} \rangle_q = \langle f_1^{(1)} \rangle_q = \langle f_2^{(1)} \rangle_q = 0\,.$$

and assume that

$$\|f_l^{0,s}\| \leq E \qquad \text{for all } l, s \geq 0\,.$$

If the small parameter ε *is small enough and at every step the conditions*

$$\begin{aligned} &\min_{\substack{0<|k|<rK \\ 0<|l|\leq 2}} |k \cdot \omega^{(r-1)} + l \cdot \Omega^{(r-1)}| \geq \alpha_r\,, \\ &\min_{|l|=2} |l \cdot \Omega^{(r-1)}| \geq \beta_r\,, \end{aligned} \tag{5.26}$$

are satisfied, then there exists a near the identity canonical transformation which gives the Hamiltonian the normal form

$$\begin{aligned} H^{(\infty)}(p,q,x,y) = &\omega^{(\infty)} \cdot p + \sum_{j=1}^{n_2} \frac{\Omega_j^{(\infty)} \left(x_j^2 + y_j^2\right)}{2} + \\ &\mathcal{O}\big(\|p\|^2\big) + \mathcal{O}\big(\|p\|\|(x,y)\|\big) + \mathcal{O}\big(\|(x,y)\|^3\big)\,. \end{aligned}$$

The n_1*-dimensional (elliptic) torus corresponding to* $P = X = Y = 0$ *is invariant, and the orbits are quasi periodic on it with frequencies* $\omega^{(\infty)}$*.*

Let us stress this proposition left open a question, *how can we ensure the small divisors conditions, i.e. formula (5.26), at every step?* The answer to this question is the subject of the following section, where we prove the existence of a set of positive measure of non-resonant frequencies.

5.5 Measure Related to the Elliptic Tori

The final part of the proof is based on a measure theory argument. In the previous section we have considered a Hamiltonian in normal form up to order r, $H^{(r)}(p,q,x,y)$, with a fixed set of initial frequencies ω, but, on the other hand, we can consider a family of Hamiltonians

$$H^{(r)}(p,q,x,y;\omega) = \omega^{(r)}(\omega)\cdot p + i\sum_j \varepsilon\Omega^{(r)}(\omega)_j x_j y_j + \sum_{l\geq 0}\sum_{s\geq 0}\varepsilon^s f_l^{(s)}(p,q,x,y;\omega)\,,$$

parametrized by the initial frequency ω that takes value in an initial set, say Ξ_0.

In order to perform the full normalization, we must ensure that the non resonance condition is satisfied and the main difficulty, as already mentioned in subsection 4.3.4, is due to the fact that the frequencies $\omega^{(r)}(\omega)$ and $\Omega^{(r)}(\omega)$ are modified after each normalization step, making the check of the non resonant conditions a big challenge.

In a nutshell the question is: considering an open ball in which the initial frequencies ω ranges, how much of that will survive to the whole normalization procedure? The answer is given in the rest of this section. The proof uses the Diophantine condition on the frequencies, which is more restrictive than the Bruno condition.

Let us recall that the Hamiltonian is written in the form

$$\begin{aligned} H^{(r)}(\omega) = \omega^{(r)}\cdot p + \mathrm{i}\sum_j \varepsilon\Omega_j^{(r)} x_j y_j \\ + \sum_{s>r}\varepsilon^s f_0^{(r,s)} + \sum_{s>r}\varepsilon^s f_1^{(r,s)} + \sum_{s>r}\varepsilon^s f_2^{(r,s)} \\ + \sum_{l>2}\sum_{s\geq 0}\varepsilon^s f_l^{(r,s)}\,, \end{aligned}$$

where $\omega^{(r)} = \omega^{(r)}(\omega)$, $\Omega^{(r)} = \Omega^{(r)}(\omega)$ and $f_l^{(r,s)} = f_l^{(r,s)}(\omega)$, moreover we have

$$f_l^{(r,s)} = 0 \qquad \forall l = 0,\,1,\,2\,, \quad 0\leq s\leq r\,.$$

The estimate by lemma 5.5 ensures that the norm of $f_l^{(r,s)}$ is bounded by a sequence geometrically increasing with respect to s, i.e., in a synthetic form

$$\|f_l^{(r,s)}\|_{1-d_r} \leq \mathcal{A}\beta^s\,,$$

with $\mathcal{A}$ and β constants.

The change of the frequency at order r is given by the equations

$$\begin{aligned} \omega_j^{(r)}(\omega) &= \omega_j^{(r-1)}(\omega) + \varepsilon^r \frac{\partial \langle f_2^{(\mathrm{II};r,r)} \rangle_q}{\partial p_j} \,, \\ \mathrm{i}\,\varepsilon \Omega_j^{(r)}(\omega) &= \mathrm{i}\,\varepsilon \Omega_j^{(r-1)}(\omega) + \varepsilon^r \frac{\partial^2 \langle f_2^{(\mathrm{III};r,r)} \rangle_q}{\partial x_j \partial y_j} \,, \end{aligned} \tag{5.27}$$

and the Hamiltonians $H^{(r)}(\omega)$ are defined for a set of ω s.t.

$$\left| k \cdot \omega^{(s)}(\omega) + \varepsilon l \cdot \Omega^{(s)}(\omega) \right| \geq \frac{\gamma}{(sK)^\tau} \,, \tag{5.28}$$

for $0 < s \leq r$, $0 < |k| \leq sK$ and $|l| = 0,\, 1,\, 2$; moreover we also need

$$|l \cdot \Omega^{(s)}(\omega)| \geq \Xi \,, \tag{5.29}$$

where Ξ is a positive constant, $|l| = 1,\, 2$ and $0 < s \leq r$.

As mentioned above, we have an upper bound for the norm of the $f_2^{(r,r)}$ function involved in the change of coordinates,

$$\|f_2^{(r,r)}\|_{1-d_r} \leq \mathcal{A}\beta^r \,.$$

Let us remember that $f_2^{(r,r)}$ is either linear in p, or quadratic *diagonal* in x, y precisely of the type $c\, x_j y_j$. In view of this, we can estimate the terms in (5.27) as

$$\left\| \frac{\partial f_2^{(r,r)}}{\partial p_j} \right\|_{1-d_r} \leq \frac{\mathcal{A}}{\varrho} \beta^r \,, \qquad \left\| \frac{\partial^2 f_2^{(r,r)}}{\partial x_y \partial y_j} \right\|_{1-d_r} \leq \frac{\mathcal{A}}{R^2} \beta^r \,,$$

and with a little abuse of notation we define again $\mathcal{A}$ as the maximum between $\mathcal{A}/\varrho$ and $\mathcal{A}/R^2$, so we have again an estimate of the type $\mathcal{A}\beta^r$ for the norms.

In order to satisfy (5.29), let us assume that $|l \cdot \Omega^{(0)}(\omega)| = \Xi_0 > 0$. Then we have

$$\begin{aligned} |l \cdot \Omega^{(s)}(\omega)| &= \left| l \cdot \left(\Omega^{(0)}(\omega) + \sum_{j=2}^{s} \varepsilon^{j-1} \frac{\partial f_2^{(r,r)}}{\partial p} \right) \right| \\ &\geq \Xi_0 - |l| \sum_{j=2}^{s} \varepsilon^{j-1} \mathcal{A}\beta^j \\ &\geq \Xi_0 - 2\mathcal{A} \frac{\varepsilon\beta^2}{1-\varepsilon\beta} \,, \end{aligned}$$

and for ε small enough we get (5.29).

Let us introduce the condition

$$(rK + 2\varepsilon) \frac{\mathcal{A}(\varepsilon\beta)^r}{1-\varepsilon\beta} \leq \frac{\gamma}{rK} \qquad \forall r \geq 0$$

that is another smallness condition on the parameter ε. We also require that

$$|k \cdot \omega^{(r)}(\omega) + \varepsilon l \cdot \Omega^{(r)}(\omega)| \geq \frac{2\gamma}{(rK)^\tau} = \alpha_r \,.$$

Then we get

$$\begin{aligned} |k \cdot \omega^{(s)}(\omega) + \varepsilon l \cdot \Omega^{(s)}(\omega)| = \Bigg| & k \cdot \omega^{(r)}(\omega) + \varepsilon l \cdot \Omega^{(r)}(\omega) \\ & + k \cdot \sum_{j>r}^{s} \varepsilon^j \frac{\partial f_2^{(r,r)}}{\partial p_j} + k \cdot \sum_{j>r}^{s} \varepsilon^{j-1} \frac{\partial^2 f_2^{(r,r)}}{\partial x_y \partial y_j} \Bigg| \\ \geq & \frac{2\gamma}{(rK)^\tau} - \frac{\gamma}{(rK)^\tau} \end{aligned}$$

from which (5.28) follows.

Let us now introduce the following *pseudo* Lipschitz condition

(i) $\|\omega^{(r)}(\omega) - \omega^{(r)}(\tilde{\omega}) - \omega + \tilde{\omega}\| \leq \varepsilon L_r \|\omega - \tilde{\omega}\|$,

(ii) $\|\Omega^{(r)}(\omega) - \Omega^{(r)}(\tilde{\omega})\| \leq L_r \|\omega - \tilde{\omega}\|$,

(iii) $L_r \leq \bar{L}$.

For $r = 0$, condition (i) is trivially satisfied and considering (ii), we have to assume

$$\|\Omega^{(0)}(\omega) - \Omega^{(0)}(\tilde{\omega})\| \leq L_0 \|\omega - \tilde{\omega}\| \,,$$

that is a reasonable hypothesis. Starting from this, we will show that under a smallness condition in ε, we can control the whole sequence.

First we define two auxiliary functions

$$F_{2,j}^{(r)}(\omega) = \frac{\partial f_2^{(r,r)}}{\partial p_j}(\omega) \quad \text{and} \quad \hat{F}_{2,j}^{(r)}(\omega) = \frac{\partial f_2^{(r,r)}}{\partial x_j \partial y_j}(\omega) \,,$$

and remark that we have bounds for the norms of these functions

$$\|F_{2,j}^{(r)}\| \leq \mathcal{A}\beta^r \quad \text{and} \quad \|\hat{F}_{2,j}^{(r)}\| \leq \mathcal{A}\beta^r \,.$$

We proceed by induction. Assuming that (i), (ii) and (iii) are true for $r-1$, we

have

$$\begin{aligned}\|\omega^{(r)}(\omega)-\omega^{(r)}(\tilde{\omega})-\omega+\tilde{\omega}\| &= \Big\|\omega^{(r-1)}(\omega)-\omega^{(r-1)}(\tilde{\omega})-\omega+\tilde{\omega} \\ &\qquad +\varepsilon^r\left(F_{2,j}^{(r)}(\omega)-F_{2,j}^{(r)}(\tilde{\omega})\right)\Big\| \\ &\le \varepsilon L_{r-1}\|\omega-\tilde{\omega}\|+\varepsilon^r\|F_{2,j}^{(r)}(\omega)-F_{2,j}^{(r)}(\tilde{\omega})\| \\ &\le \varepsilon L_{r-1}\|\omega-\tilde{\omega}\|+\varepsilon^r\sup\left\|\frac{\partial F_{2,j}^{(r)}}{\partial\omega}\right\|\|\omega-\tilde{\omega}\| \\ &\le \varepsilon L_r\|\omega-\tilde{\omega}\|\,,\end{aligned}$$

where $L_r = L_{r-1}+\varepsilon^{r-1}\mathcal{A}(\beta/\mu)^r$ and μ^r is the restriction of the frequency domain in order to apply the Cauchy's estimates, so if ε is small enough then (i) is satisfied. In a similar way, we have

$$\begin{aligned}\|\Omega^{(r)}(\omega)-\Omega^{(r)}(\tilde{\omega})\| &= \Big\|\Omega^{(r-1)}(\omega)-\Omega^{(r-1)}(\tilde{\omega}) \\ &\qquad +\varepsilon^{r-1}\left(\hat{F}_{2,j}^{(r)}(\omega)-\hat{F}_{2,j}^{(r)}(\tilde{\omega})\right)\Big\| \\ &\le L_{r-1}\|\omega-\tilde{\omega}\|+\varepsilon^{r-1}\|\hat{F}_{2,j}^{(r)}(\omega)-\hat{F}_{2,j}^{(r)}(\tilde{\omega})\| \\ &\le \varepsilon L_{r-1}\|\omega-\tilde{\omega}\|+\varepsilon^{r-1}\sup\left\|\frac{\partial \hat{F}_{2,j}^{(r)}}{\partial\omega}\right\|\|\omega-\tilde{\omega}\| \\ &\le \varepsilon L_r\|\omega-\tilde{\omega}\|\,,\end{aligned}$$

where again $L_r = L_{r-1}+\varepsilon^{r-1}\mathcal{A}(\beta/\mu)^r$ and μ^r is the restriction of the domain in order to apply the Cauchy's estimates, so if ε is small enough then (ii) is satisfied, too.

5.6 Crossing the Resonant Region

We define the *resonant manifold* $\mathcal{V}_{k,l}^{(r)}$ as

$$\mathcal{V}_{k,l}^{(r)} = \left\{\tilde{\omega} : k\cdot\omega^{(r)}(\tilde{\omega})+\varepsilon l\cdot\Omega^{(r)}(\tilde{\omega})=0\right\}\,,$$

and remark that since $\omega^{(0)}\to\omega^{(r)}$ is ε close to the identity and $k\neq 0$, it's easy to see that $\mathcal{V}_{k,l}^{(r)}$ is a (n_1-1) dimensional manifold.

Let us introduce the *resonant region* $\mathcal{R}_{k,l}^{(r)}$ as

$$\mathcal{R}_{k,l}^{(r)} = \left\{\omega\in\Theta_0 : k\cdot\omega^{(r)}(\omega)+\varepsilon l\cdot\Omega^{(r)}(\omega)<\alpha_r\right\}\,,$$

where Θ_0 is a closed ball of radius R_0 and center C, containing the initial frequencies.

We want to estimate the width of the resonant region, To this end, considering $e_k = k/\|k\|$, the unit vector related to k, we want to calculate how far we must go in the k direction in order to go out of the resonant region.

Take $\tilde{\omega}$ in the resonant manifold $\mathcal{V}_{k,l}^{(r)}$ and define the vector

$$\omega = \tilde{\omega} + \zeta e_k \,,$$

we can write

$$\begin{aligned}
|k \cdot \omega^{(r)}(\omega) + \varepsilon l \cdot \Omega^{(r)}(\omega)| &= |k \cdot (\omega^{(r)}(\omega) - \omega^{(r)}(\tilde{\omega})) + \varepsilon l \cdot (\Omega^{(r)}(\omega) - \Omega^{(r)}(\tilde{\omega}))| \\
&\geq |k \cdot (\omega - \tilde{\omega})| - \varepsilon \|k\| L_r \|\omega - \tilde{\omega}\| - \varepsilon \|l\| L_r |\omega - \tilde{\omega}\| \\
&\geq \|k\| |\zeta| - \varepsilon (\|k\| + \|l\|) L_r |\zeta| \\
&\geq (1 + 3\varepsilon L_r) |\zeta| \,,
\end{aligned}$$

where we use the conditions (i), (ii) and (iii), and the bounds $\|k\| \geq 1$ and $\|l\| \leq 2$.

Requiring the smallness condition

$$1 + 3\varepsilon L_r \geq \frac{1}{2} \qquad \Rightarrow \qquad \varepsilon \leq \frac{1}{6\bar{L}} \,,$$

as we want to assure the following estimate

$$|k \cdot \omega^{(r)}(\omega) + \varepsilon l \cdot \Omega^{(r)}(\omega)| \geq \alpha_r \,,$$

we get the condition

$$\zeta \geq 2\alpha_r \,.$$

We conclude that the crossing of the resonant region has a maximum width of $4\alpha_r$.

From what we have stated above, we can estimate the measure of the resonant region as

$$\text{mes } \mathcal{R}_{k,l}^{(r)} \leq 4\alpha_r \int_{\mathcal{V}_{k,l}^{(r)} \cap \Theta_0} \mathrm{d}S \,,$$

where $\mathcal{V}_{k,l}^{(r)} \cap \Theta_0$ is the $(n_1 - 1)$ dimensional surface of the resonant manifold contained in the initial ball Θ_0.

As observed before, $\mathcal{V}_{k,l}^{(r)}$ is a $n_1 - 1$ dimensional manifold, so, without loss of generality, we can invert the resonance relation with respect to the variable ω_{n_1}, so that the following relation is satisfied

$$|k_{n_1}| \geq |k_j| \qquad \text{for } j = 1, \ldots, n_1 - 1 \,.$$

Then the integral can be written as

$$\int_{\mathcal{V}_{k,l}^{(r)}\cap\Theta_0} \mathrm{d}S = \int_{B_{R_0}(C)} \sqrt{1+\left(\frac{\partial\omega_{n_1}}{\partial\omega_1}\right)^2+\ldots+\left(\frac{\partial\omega_{n_1}}{\partial\omega_{n_1-1}}\right)^2}\mathrm{d}\omega_1\cdot\ldots\cdot\mathrm{d}\omega_{n_1-1}\,.$$

To estimate the last integral, we need give a bound for the terms $\partial\omega_{n_1}/\partial\omega_j$ and we can use the implicit function theorem. Indeed let us define $\hat{\omega}=(\omega_1,\,\ldots,\,\omega_{n_1-1})$ and the function

$$f(\omega)=k\cdot\omega^{(r)}(\omega)+\varepsilon l\cdot\Omega^{(r)}(\omega)\,,$$

the resonance relation can be written as

$$f(\omega)=f(\hat{\omega},\omega_{n_1})=0\,,$$

and as we said we can write ω_{n_1} as a function of the other variables, in the form

$$\omega_{n_1}=g(\hat{\omega})\,.$$

Using the implicit function theorem we get the equation

$$\frac{\partial g}{\partial\omega_j}(\hat{\omega})=-\left(\frac{\partial f}{\partial\omega_{n_1}}(\omega)\right)^{-1}\frac{\partial f}{\partial\omega_j}(\omega)\,,$$

that we use to estimate the integral. The numerator of the previous formula can be bounded by

$$\begin{aligned}\frac{\partial}{\partial\omega_j}\left(k\cdot\omega^{(r)}(\omega)+\varepsilon l\cdot\Omega^{(r)}(\omega)\right)\leq\frac{\partial}{\partial\omega_j}&\left(k\cdot\left(\omega^{(0)}(\omega)+\sum_{j\geq1}\varepsilon^j\frac{\partial f_2^{j,j}}{\partial p}\right)+\right.\\&\left.\varepsilon l\cdot\left(\Omega^{(r)}(\omega)+\sum_{j\geq2}\varepsilon^{j-1}\frac{\partial^2 f_2^{j,j}}{\partial x\partial y}\right)\right)\\&\leq k_j+\|k\|\mathcal{A}\frac{\varepsilon(\beta/\mu)}{1-\varepsilon(\beta/\mu)}\\&\quad+2\mathcal{A}\frac{\varepsilon(\beta/\mu)}{1-\varepsilon(\beta/\mu)}+2\varepsilon\left\|\frac{\partial\Omega^{(0)}}{\partial\omega_j}(\omega)\right\|\,,\end{aligned}$$

and if ε is small enough we can bound the last inequality with $2\|k\|$.

The numerator is quite similar, but remember that we have $k_{n_1}\geq\|k\|/n_1$ and this

is the key point, indeed

$$\frac{\partial}{\partial \omega_{n_1}}\left(k\cdot\omega^{(r)}(\omega)+\varepsilon l\cdot\Omega^{(r)}(\omega)\right)=\frac{\partial}{\partial \omega_{n_1}}\left(k\cdot\left(\omega^{(0)}(\omega)+\sum_{j\geq 1}\varepsilon^j\frac{\partial f_2^{j,j}}{\partial p}\right)+\right.$$
$$\left.\varepsilon l\cdot\left(\Omega^{(r)}(\omega)+\sum_{j\geq 2}\varepsilon^{j-1}\frac{\partial^2 f_2^{j,j}}{\partial x\partial y}\right)\right)$$
$$\geq k_{n_1}-\|k\|\mathcal{A}\frac{\varepsilon(\beta/\mu)}{1-\varepsilon(\beta/\mu)}$$
$$-2\mathcal{A}\frac{\varepsilon(\beta/\mu)}{1-\varepsilon(\beta/\mu)}-2\varepsilon\left\|\frac{\partial\Omega^{(0)}}{\partial\omega_j}(\omega)\right\|,$$

so if ε is small enough we can bound with $\|k\|/2n_1$. From this estimates, we have

$$\left\|\frac{\partial\omega_{n_1}}{\partial\omega_j}\right\|\leq 4n_1\,.$$

and we get the estimate

$$\int_{\mathcal{V}_{k,l}^{(r)}\cap\Theta_0}\mathrm{d}S\leq 4n_1^2 2^{n_1}R_0^{n_1-1}\,.$$

We obtain the estimate for the measure of the resonant region

$$\text{mes }\mathcal{R}_{k,l}^{(r)}\leq 4n_1^2 2^{n_1+2}R_0^{n_1-1}\alpha_r\,.$$

We start with an initial domain

$$\Theta_0=\overline{B_{R_0}(C)},\qquad \Theta_0\in\mathbf{R}^{n_1}\ ,$$

we describe the restriction of the frequency domain needed in order to perform the r-th step. Let us introduce an intermediate domain

$$\widetilde{\Theta}_r=\Theta_{r-1}-\bigcup_{\substack{(r-1)K<|k|\leq rK\\ 0\leq|l|\leq 2}}\mathcal{R}_{k,l}^{(r)}$$

on which we have to do a further restriction in order to apply the estimates (i), (ii) and (iii), to this end let us introduce

$$A_r=\left\{\omega\in\widetilde{\Theta}_r : B_{\mu^r}(\omega)\not\subset\widetilde{\Theta}_r\right\}\,.$$

We remark that we have to require $\varepsilon\beta/\mu<1$ and also $\mu<1$, otherwise the restrictions does not converge.

Finally we define the new domain

$$\Theta_r=\widetilde{\Theta}_r-A_r\,.$$

Let us calculate the measure of the domain we lost in the r-th step, first passing from Θ_{r-1} to $\widetilde{\Theta}_r$, we have

$$\begin{aligned}
\text{mes}\left(\widetilde{\Theta}_r - \Theta_{r-1}\right) &\leq \sum_{\substack{(r-1)K<|k|\leq rK \\ 0\leq|l|\leq 2}} 4n_1^2 2^{n_1+2} R_0^{n_1-1}\alpha_r \\
&\leq 4n_1^2 2^{n_1+2} c_{n_2} R_0^{n_1-1} \sum_{(r-1)K<|k|\leq rK} \frac{\gamma}{(rK)^\tau} \\
&\leq 4n_1^2 2^{2n_1+2} c_{n_2} R_0^{n_1-1} \frac{\gamma K}{(rK)^{\tau-n_1+1}}
\end{aligned}$$

where $c_{n_2} = ((n_2+1)+n_2)/2 + n_1 + 1$, and the last inequality follows from the fact that the number of vectors $k \in \mathbf{N}^n$ satisfying $|k| = s$ does not exceed $2^n s^{n-1}$. Then we also remove A_r, and we have

$$\begin{aligned}
\text{mes}\,(A_r) &\leq 2\pi^{n_1/2} R_0^{n_1-1}\mu_r + c_{n_2}\#\left\{k : 0 < |K| \geq rK\right\} 4n_1^2 2^{2n_1} c_{n_2} R_0^{n_1-1} \\
&\leq \left(2\pi^{n_1/2} + c_{n_2}(rK)^{n_1} 4n_1^2 4^{n_1}\right) R_0^{n_1-1}\mu^r\,.
\end{aligned}$$

Finally we can ensure that the elliptic tori exist if the following inequality is true

$$\begin{aligned}
&\sum_{r\geq 1} 4n_1^2 2^{2n_1+2} c_{n_2} R_0^{n_1-1} \frac{\gamma K}{(rK)^{\tau-n_1+1}} \\
&\qquad + \sum_{r\geq 1}\left(2\pi^{n_1/2} + c_{n_2}(rK)^{n_1} 4n_1^2 4^{n_1}\right) R_0^{n_1-1}\mu^r < \text{mes}\,\Theta_0\,.
\end{aligned}$$

The first series is convergent if $\tau > n_1$ and is of order $\mathcal{O}(\gamma)$, so we can make the sum small by decreasing γ, so decreasing ε in view of $\varepsilon = \mathcal{O}(\gamma^3)$. The second series converges because we have assumed $\mu < 1$, and we can decrease the sum decreasing μ, so decreasing ε, as $\varepsilon < \mu/\beta$.

In conclusion if γ and μ are small enough, and so if ε is small enough,

$$\text{mes}\,\Theta_\infty > 0.$$

Collecting the result of the last section, we get the following theorem.

Theorem 5.1: *Let the Hamiltonian be as in proposition 5.1, and assume that the conditions*

$$|l \cdot \Omega^{(0)}(\omega)\| \geq \Xi_0\,,$$

and

$$\|\Omega^{(0)}(\omega) - \Omega^{(0)}(\tilde{\omega})\| \leq L_0\|\omega - \tilde{\omega}\|\,,$$

are satisfied for some positive constants Ξ and L_0 and for $|l| = 1, 2$.

If ε is small enough, we can perform the complete normalization procedure as stated in proposition 5.1, and the result is valid in a domain of positive measure.

Then there is a set Θ_∞ of positive measure such that for $\omega \in \Theta_\infty$, the normalization procedure of proposition 5.1 can be applied.

A

The Planetary Problem and the Perturbation Function

"In the old days when people invented a new function they had something useful in mind."
— Henri Poincaré

A.1 Introduction

In order to study the stability of a planetary systems in a Hamiltonian framework, we need to write the planetary Hamiltonian in a convenient way. Moreover the improvement of the purely analytical estimates requires the explicit calculation of the functions' expansions.

The expansion of the planetary Hamiltonian is a classical topic of celestial mechanics and, for sake of completeness, let us briefly recall a method for the expansion in canonical Poincaré variables that we worked out following the scheme sketched in the two articles by Laskar and Robutel (see [49] and [76]), that is also described in section 3.3 of [56]. The algorithm implemented in our programs is discussed in details in these papers, so we omit here the details.

Let us emphasize that the aim of the present chapter is only to set up the common framework for the present thesis:

1. in chapter 2, the reduction of the angular momentum allows us to remove the inclinations and the longitudes of the nodes from the Hamiltonian. Therefore the actual number of degrees of freedom is reduced to 4, and the system is conveniently described by the reduced set of Poincarés canonical coordinates;
2. in the other chapters, we deal with planar planetary systems, so again we don't have to consider the inclinations and the longitudes of the nodes in the Hamiltonian, and we can use again the reduced set of Poincarés canonical coordinates.

For these reasons, we introduce a simplified version of the expansions, that make use of the reduced set of Poincarés canonical coordinates.

A.2 The Hamiltonian of the Planetary System

Let us consider $n+1$ point bodies P_0, P_1, $\ldots$, P_n, with masses m_0, m_1, $\ldots$, m_n, mutually interacting according to Newton's gravitational law. We now recall how the classical Poincaré variables can be introduced so to perform a first expansion of the Hamiltonian around circular orbits, i.e., having zero eccentricity. We basically follow the formalism introduced by Poincaré (see [71] and [72]; for a modern exposition, see, e.g., [45] and [49]). We remove the motion of the center of mass by using heliocentric coordinates $r_j = \overrightarrow{P_0 P_j}$, with $j = 1, \ldots, n$. Denoting by $\tilde{r}_j$ the momenta conjugated to

r_j, the Hamiltonian of the system has $2n$ degrees of freedom, and reads

$$F(\tilde{r}, r) = T^{(0)}(\tilde{r}) + U^{(0)}(r) + T^{(1)}(\tilde{r}) + U^{(1)}(r)\,,$$

where

$$T^{(0)}(\tilde{r}) = \frac{1}{2}\sum_{j=1}^{n} \frac{m_0 + m_j}{m_0 m_j}\,\|\tilde{r}_j\|^2\,, \qquad T^{(1)}(\tilde{r}) = \frac{1}{m_0}\sum_{0<i<j} \tilde{r}_i \cdot \tilde{r}_j\,,$$

$$U^{(0)}(r) = -\mathcal{G}\sum_{j=1}^{n} \frac{m_0\, m_j}{\|r_j\|}\,, \qquad U^{(1)}(r) = -\mathcal{G}\sum_{0<i<j} \frac{m_i\, m_j}{\|r_i - r_j\|}\,.$$

The plane set of Poincaré canonical variables is introduced as

$$\begin{aligned} \Lambda_j &= \frac{m_0\, m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j}\,, & \lambda_j &= M_j + \omega_j\,, \\ \xi_j &= \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\,\cos\omega_j\,, & \eta_j &= -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\,\sin\omega_j\,, \end{aligned} \tag{A.1}$$

for $j = 1, \ldots, n$, where a_j , e_j , M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and the perihelion argument, respectively, of the j-th planet. One immediately sees that both ξ_j and η_j are of the same order of magnitude as the eccentricity e_j .

Using the Poincaré variable (A.1), the Hamiltonian F can be rearranged so that one has

$$F(\Lambda, \lambda, \xi, \eta) = F^{(0)}(\Lambda) + F^{(1)}(\Lambda, \lambda, \xi, \eta)\,,$$

where $F^{(0)} = T^{(0)} + U^{(0)}$ has the form,

$$F^{(0)} = -\sum_{j=1}^{n} \frac{\mu_j^2 \beta_j^3}{2\Lambda_j^2}\,, \tag{A.2}$$

while $F^{(1)} = T^{(1)} + U^{(1)}$, where

$$\begin{aligned} U^{(1)} &= -\mathcal{G}\sum_{o<i<j} \frac{m_i m_j}{\|\boldsymbol{r}_i - \boldsymbol{r}_j\|}\,, \\ T^{(1)} &= \frac{1}{m_0}\sum_{0<i<j} \tilde{\boldsymbol{r}}_i \cdot \tilde{\boldsymbol{r}}_j\,. \end{aligned} \tag{A.3}$$

and, using the classical notations,

$$\mu_j = m_0 + m_j\,, \qquad \beta_j = \frac{m_0 m_j}{m_0 + m_j}\,.$$

Let us emphasize that $F^{(0)} = \mathcal{O}(1)$ and $F^{(1)} = \mathcal{O}(\mu)$, where the small dimensionless parameter $\mu = \max_{0<i\leq n}\{m_i \,/\, m_0\,\}$ highlights the different size of the terms appearing

in the Hamiltonian. Therefore, let us remark that the time derivative of each coordinate is $\mathcal{O}(\mu)$ but in the case of the angles λ. Thus, according to the common language in celestial mechanics, in the following we will refer to λ and to their conjugate actions Λ as the *fast variables*, while (ξ, η) will be called *secular (slow) variables*.

We now describe the expansions of the three terms in order to perform the expansion of the Hamiltonian in the Poincaré variables.

A.3 The Keplerian Part

The expansion of the Keplerian term (A.2) is straightforward, we have

$$F^{(0)} = -\sum_{j=1}^{n} \frac{\mu_j^2 \beta_j^3}{2\Lambda_j^2} ,$$

so, taking a fixed value for the fast actions, Λ^*, we can introduce the translated fast actions $L_j = \Lambda_j - \Lambda_j^*$ for $j = 1, \ldots, n$ and we can write

$$\begin{aligned} -\frac{\mu_j^2 \beta_j^3}{2\Lambda_j^2} &= -\frac{\mu_j^2 \beta_j^3}{2\left(L_j + \Lambda_j^*\right)^2} = -\frac{\mu_j^2 \beta_j^3}{2\Lambda_j^{*2}} \frac{1}{\left(1 + \frac{L_j}{\Lambda_j^*}\right)^2} \\ &= -\frac{m_0 m_j}{2a_j^*} \sum_{k=0}^{\infty} \left(-\frac{1}{\Lambda^*}\right)^k (k+1) L_j^k \qquad \text{for } j = 1, \ldots, n . \end{aligned}$$

In this way we obtain the expansion in Poincaré variables of the $F^{(0)}$ function, that can be easily implemented on an algebraic manipulator.

A.4 The Perturbation Function

We now focus our attention on the perturbation function $U^{(1)}$, and in particular let us consider only the term

$$\frac{1}{\|r_0 - r_1\|} ,$$

the other terms are treated in the same way. Let us introduce a new variable defined as

$$\Xi = \|r_0 - r_1\|^2 - \left(a_0^2 + a_1^2 - 2a_0 a_1 \cos(\lambda_0 - \lambda_1)\right) ,$$

and remark that this variable is small in eccentricity and/or inclination.

After some preliminary calculations, we can write the usually called *disturbing function* as

$$U_1 = \frac{Gm_s m_g}{a_1^*} \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k}{(1)_k} \left(\frac{\Xi}{a_1^{*2}}\right)^k \left(\frac{a_1^*}{a_1}\right)^{2k+1} D^{-(2k+1)} .$$

where the terms $D^{-(2k+1)}$ are defined starting with

$$D^2 = 1 + \left(\frac{a_0}{a_1}\right)^2 - 2\left(\frac{a_0}{a_1}\right)\cos(\lambda_0 - \lambda_1) ,$$

and the $(1/2)_k$ and $(1)_k$ are the generalized factorials, defined as

$$(m)_n = \begin{cases} 1 & \text{if } n < 1 ; \\ m(m+1)\cdots(m+n-1) & \text{if } n \geq 1 . \end{cases}$$

First we need to expand the ratio of the semi-major axis,

$$\begin{aligned} \frac{a_0}{a_1} &= \frac{\Lambda_0^2}{\beta_0^2\mu_0}\frac{\beta_1^2\mu_1}{\Lambda_1^2} = \frac{\beta_0^2\mu_0}{\beta_1^2\mu_1}\left(\frac{\Lambda_0}{\Lambda_1}\right)^2 \\ &= \frac{a_0^*}{a_1^*}\left(1 + \frac{2}{\Lambda_0^*}L_0 + \left(\frac{1}{\Lambda_0^*}\right)^2 L_0^2\right)\sum_{k=0}^{\infty}(k+1)\left(-\frac{1}{\Lambda_1^*}\right)^k L_1^k . \end{aligned}$$

and again we get an expansion in terms of the translated fast actions $L_1 = \Lambda_1 - \Lambda_1^*$.

We skip the rest of the procedure, that is quite tricky, but we remark that following the works by Laskar and Robutel, one can get the explicit expansion of the perturbation function in terms of the restricted Poincaré canonical variables.

A.5 The Complementary Term

Finally, let us consider the complementary term, $T^{(1)}$, we know that

$$T^{(1)} = \frac{\tilde{r}_i \cdot \tilde{r}_j}{m_0} ,$$

where we use the classical notations,

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \qquad \dot{r} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} , \qquad \tilde{r} = m\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} .$$

Let us introduce the new variables

$$\begin{aligned} X &= r\cos(v) , & \dot{X} &= -\frac{na}{\sqrt{1-e^2}}\sin(v) , \\ Y &= r\sin(v) , & \dot{Y} &= \frac{na}{\sqrt{1-e^2}}\cos(v) . \end{aligned}$$

that satisfies the equation,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = R_1(i) \times R_3(\omega) \times \begin{pmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{pmatrix} ,$$

where the matrix $R_1(i)$ and $R_3(\omega)$ are defined as,

$$R_1(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{pmatrix} , \qquad R_3(\omega) = \begin{pmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

With some elementary and tedious calculations, we get

$$\begin{aligned} T_1 = &- \frac{\beta_0 n_0 a_0 \beta_1 n_1 a_1}{m_0 \sqrt{1-e_0^2}\sqrt{1-e_1^2}} \Big((\sin(v_0+\omega_0) + e_0 \sin(\omega_0)) (\sin(v_1+\omega_1) + e_1 \sin(\omega_1)) \\ &+ (\cos(v_0+\omega_0) + e_0 \cos(\omega_0)) (\cos(v_1+\omega_1) + e_1 \cos(\omega_1)) \\ &- (\cos(v_0+\omega_0) + e_0 \cos(\omega_0)) (\cos(v_1+\omega_1) + e_1 \cos(\omega_1)) (1-\cos(J)) \Big) . \end{aligned}$$

We introduce the modulus of the elliptic Poincaré variables

$$|X_j| = \sqrt{2}\sqrt{1-\sqrt{1-e_j^2}} .$$

we can easily get the expansion of the eccentricity, e_j, as a function of $|X_j|$, that is related to the Poincareé variables,

$$\begin{aligned} e_j &= \sum_{i=0}^{\infty} \left(\prod_{k=0}^{i-1} \left(\frac{1}{2} - k \right) \left(\frac{-1}{4(k+1)} \right) \right) |X_j|^{2i+1} \\ &= \sum_{i=0}^{\infty} \left(\prod_{k=0}^{i-1} \frac{2k-1}{4(2k+2)} \right) |X_j|^{2i+1} \\ &= \sum_{i=0}^{\infty} \left(\prod_{k=0}^{i-1} \frac{(2k+1)-2}{4(2k+1)+4} \right) |X_j|^{2i+1} . \end{aligned}$$

Moreover, considering the quantity

$$n_j a_j ,$$

we can write

$$\begin{aligned} n_j a_j &= n_j a_j^{3/2} \frac{1}{\sqrt{a_j}} = \frac{1}{\sqrt{a_j}} \left(n_j^2 a_j^3 \right)^{1/2} \\ &= \sqrt{\frac{\mu_j}{a_j}} = \sqrt{\frac{\mu}{a_j^*}} \sqrt{\frac{a_j^*}{a_j}} = \sqrt{\frac{\mu_j}{a_j^*}} \frac{\Lambda^*}{\Lambda} = \sqrt{\frac{\mu_j}{a_j^*}} \left(1 + \frac{L_j}{\Lambda_j^*} \right)^{-1} . \end{aligned}$$

We skip again the last part of the algorithm, concerning the expansions of the angular component of the complementary term which is quite tricky, but following the

works by Laskar and Robutel, we can get the expansion of the complementary term in the Poincaré canonical variables.

Substituting the expansions of sect. A.3, A.4 and A.5, in the Hamiltonian F, we give the Hamiltonian the required form for applying our algorithms.

We emphasize once more that a non negligible part of the thesis work consisted precisely in implementing the expansions via computer algebra.

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