UNIVERSITÀ DEGLI STUDI DI MILANO
Facoltà di Scienze Matematiche, Fisiche e Naturali
Dipartimento di Matematica "F. Enriques"

Corso di Dottorato di Ricerca in Matematica, XXII ciclo

Tesi di Dottorato di Ricerca

CONFORMAL GEOMETRY IN THE FOUR-DIMENSIONAL MÖBIUS SPACE

MAT\03

Relatore
Prof. Marco Rigoli

Candidato
Dott. Marco Magliaro

Coordinatore di Dottorato
Prof. Alfredo Lorenzi

Anno Accademico 2008–2009

Contents

Introduction

In this thesis we study the geometry of submanifolds of the conformal sphere Q_n focusing, in particular, on surfaces immersed in Q_4.

In Chapter 1 we summarize the basic definitions and facts about the Möbius group $\text{Möb}(n)$, its Lie algebra and its Maurer-Cartan form Φ. We also define the conformal sphere, realized as a homogeneous space for the Möbius group.

Chapter 2 is devoted to the study of the general theory of submanifolds of the conformal sphere. Let $f : M \to Q_n$ be an immersion of an oriented surface in Q_n, let $e : U \subset M \to \text{Möb}(n)$ be a local Darboux frame field along f and $\phi = e^*\Phi$ the pull-back of the Maurer-Cartan form on M. Then ϕ satisfies the Maurer-Cartan structure equations

$$d\phi^a_b = -\phi^a_c \wedge \phi^c_b$$

and the following conditions:

(*i*) $\pi \circ e = f_{|U}$, $\quad \pi$ being the projection $\pi : \text{Möb}(n) \to Q_n$ sending a matrix in $\text{Möb}(n)$ to the projectivization of its first column;

(*ii*) $\phi^\alpha_0 = 0$;

(*iii*) $h^\alpha_{kk} = 0$, where h^α_{ij}, symmetric in the lower indices, are the coefficients of ϕ^α_i in the local coframe $\{\phi^i_0\}$, that is $\phi^\alpha_i = h^\alpha_{ij}\phi^j_0$.

Every immersion admits a Darboux frame, and such frames determine a splitting of the form ϕ, enabling us to define a natural Cartan connection on M and a Riemannian vector bundle over M, called the normal bundle N, locally spanned by the columns $\{e_\alpha\}$.
Moreover, a submanifold of Q_n is totally umbilical, i.e. $h^\alpha_{ij} = 0$ for every α, i, j, if and only if there exists $Q_m \subset Q_n$ such that $f(M) \subseteq Q_m$.

In Chapter 3 we introduce the conformal Grassmannian of s-planes in $\mathbb{R}^{n+2}$, $\mathcal{Q}_s(\mathbb{R}^{n+2})$, defined as the orbit of the point $O = [\varepsilon_\alpha]$ of the Grassmann bundle, with respect to the left action (by matrix multiplication) of the group $\text{Möb}(n)$, $\{\varepsilon_a\}$ being the standard basis of $\mathbb{R}^{n+2}$. $\mathcal{Q}_s(\mathbb{R}^{n+2})$ can be seen as a homogeneous space for $\text{Möb}(n)$ and can be endowed with a natural Kähler-Lorentzian structure. Particular attention is paid to the case $s = 2$, since in this case we provide a holomorphic embedding of $\mathcal{Q}_2(\mathbb{R}^{n+2})$ into a quadric in $\mathbb{P}^{n+1}_{\mathbb{C}}$.

Chapter 4 focuses on the study of surfaces in Q_4. With respect to a Darboux frame e, it is a natural, as well as useful, technique to consider the Hopf transform of the symmetric matrices (h^α_{ij}), denoted with L^α and defined as

$$L^\alpha = \frac{1}{2}(h^\alpha_{11} - h^\alpha_{22}) - ih^\alpha_{12}.$$

Although the Hopf transforms L^α strongly depend on the choice of the Darboux frame, through them we can define the following real 2-forms, which are independent of the choice of a particular Darboux frame, and therefore globally defined on M:

$$\begin{aligned}\omega_\pm &= \left|L^3 \pm iL^4\right|^2 \phi_0^1 \wedge \phi_0^2,\\ \mathrm{w} &= \left(|L^3|^2 + |L^4|^2\right)\phi_0^1 \wedge \phi_0^2,\\ \eta &= -i\Big(L^3\overline{L^4} - L^4\overline{L^3}\Big)\phi_0^1 \wedge \phi_0^2.\end{aligned}$$

These forms are related by the equality

$$\mathrm{w} = \omega_\pm \mp \eta,$$

and are significant in the study of the geometry of the immersed surface M.
We will say that f is $\pm$ isotropic if, respectively, $\omega_\pm = 0$.
As for the form w, it allows to define the Willmore functional of the immersion f as

$$W_K(f) = \int_K \mathrm{w}$$

where $K \subset M$ is compact. f is said to be a Willmore surface if, for any compact set $K \subset M$, it is a critical point of the Willmore functional W_K.
Lastly, if N indicates the normal bundle introduced in Chapter 2, then η is connected to the curvature K_N of such bundle, in that

$$\eta = K_N \phi_0^1 \wedge \phi_0^2.$$

As an immediate application of these formulae we obtain Theorem 4.1, which states that, in the case of M compact and denoting $W(f) := W_M(f)$,

$$W(f) = \int_M \omega_\pm \mp 2\pi\chi(N)$$

where $\chi(N)$ is the Euler number of the bundle N. In particular, if M is a compact $\pm$ isotropic surface, then the Willmore functional is quantized.
Another application of the above formulae is Corollary 4.2, which states that if M is compact, then

$$\int_M \omega_\pm \geq \pm 2\pi\chi(N)$$

equality holding if and only if $f(M)$ is a conformal 2-sphere $Q_2 \subset Q_4$.

In order to further investigate the geometry of surfaces in Q_4, we need to introduce the conformal equivalent of the Gauss map in the Riemannian setting. This and many other concepts and results studied here have been introduced in the study of minimal surfaces in the Riemannian four-sphere and even in oriented Riemannian four-manifolds in general. Two interesting papers in this direction are [10] and [5].
Given an immersed surface $f : M \to Q_4$, we can define its conformal Gauss map $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ as the map associating $p \in M$ to the 2-plane $[e_3, e_4]_p$, where e is any Darboux frame defined in a neighbourhood of p.
Since $\dim_{\mathbb{R}} M = 2$, it makes sense, even in the conformal setting, to ask if and when the conformal Gauss map γ_f is holomorphic, antiholomorphic or harmonic

and it was first proved in [11] that γ_f is harmonic if and only if f is Willmore. Here we also prove the remarkable fact, stated in Theorem 4.3, that γ_f is $\pm$ holomorphic (that is, holomorphic or antiholomorphic) if and only if f is $\mp$ isotropic.
In the proof of this latter result, certain important quantities are involved, namely

$$k^\alpha = \frac{1}{2}(p_1^\alpha - ip_2^\alpha),$$

where p_k^α are the coefficients of ϕ_α^0 with respect to ϕ_0^k, that is $\phi_\alpha^0 = p_k^\alpha \phi_0^k$.
Under the following condition on k^α:

$$\exists \gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \left|k^3 \pm ik^4\right| \leq \gamma\left|L^3 \pm iL^4\right| \qquad \text{a.e.} \tag{1}$$

for some $p > 2$, we prove that the function $\left|L^3 \pm iL^4\right|$ is of analytic type, i.e. it either vanishes identically or has isolated zeros. Therefore we have that if $f : M \to Q_4$ is an immersion satisfying (1), then either γ_f is $\pm$ holomorphic or the set $\mathcal{I}_\mp$ of $\mp$ isotropic points of M is discrete. This result is stated in Proposition 4.6.
Condition (1) is very natural in this setting because it allows to employ classical techniques such as Cauchy-Riemann inequalities and Carleman-type estimates. An advantage of these results is that they can be combined with classical index theorems for vector fields and, more generally, for sections of suitable vector bundles over M. Indeed, as a first result, the same technique, applied to a slightly more general context, provides, in the case of M compact and not $\pm$ isotropic, an upper bound on the Euler characteristic of M, as proved in Theorem 4.8.

Later on, we consider the notion of S-Willmore surface, first introduced by Ejiri in [7]. In our setting, with respect to a Darboux frame along f, the notion corresponds to the two following conditions being fulfilled on M

$$\begin{aligned} &K_N \neq 0, \\ &\alpha_1 \stackrel{\text{def}}{=} (k^3L^4 - k^4L^3)\varphi \otimes \varphi \otimes \varphi = 0, \end{aligned}$$

where $\varphi = \phi_0^1 + i\phi_0^2$ is the $(1,0)$-form defining the complex structure of M.
Ejiri proved that, in the Riemannian setting, an S-Willmore surface is a Willmore surface; this holds true also in our setting, as proved in Proposition 4.11. Moreover, in Proposition 4.12 we prove that if $f : M \to Q_4$ is a $\pm$ isotropic immersed surface, then f is S-Willmore if and only if $K_N \neq 0$ on M. In Proposition 4.13 we also deduce that if $f : M \to Q_4$ is an immersion which satisfies condition (1), without umbilical points and such that the set of $\pm$ isotropic points is not discrete, then f is S-Willmore.

Defining p^α_{ij} as follows,

$$p^\alpha_{ik}\phi_0^k = dp_i^\alpha - p_k^\alpha \phi_i^k + p_i^\beta \phi_\beta^\alpha + 2p_i^\alpha \phi_0^0 - h^\alpha_{ki}\phi_k^0,$$

we consider a condition similar to (1):

$$\exists \gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \left|p^3_{kk}L^4 - p^4_{kk}L^3\right| \leq \gamma\left|k^3L^4 - k^4L^3\right| \quad \text{a.e.} \tag{2}$$

for some $p > 2$ and in Theorem 4.14 we prove, using once again the aforementioned classical techniques, that if $f : M \to Q_4$ is an immersion such that (2) holds, then either $\alpha_1 \equiv 0$ or its zero set is discrete. In this latter case, for M compact we have

$$z(\alpha_1) = -3\chi(M),$$

where $z(\alpha_1)$ is the sum of the orders of the zeros of α_1.
In particular, if M is a Willmore surface, condition (2) is automatically satisfied, and in fact in this case we prove in Proposition 4.16 that α_1 is a holomorphic section of the vector bundle $\bigotimes^3 T^*M^{(1,0)}$. Moreover, if M is a topological 2-sphere, then $\alpha_1 \equiv 0$.

The last part of the thesis deals with the following problem: instead of considering immersions of M in Q_4, and associating to them their conformal Gauss maps, we start from a map $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ and, under certain suitable conditions, we retrieve a Q_4-valued map, called J_γ, whose conformal Gauss map, where defined, is exactly the original map γ. The map J_γ is not necessarily an immersion, but it is a weakly conformal branched immersion, and its conformal Gauss map can be continuously extended at the branch points.
In this way we establish a bijection between $-$ isotropic, non totally umbilical, weakly conformal branched immersions $f : M \to Q_4$, whose conformal Gauss maps can be continuously extended at the branch points, and non constant, holomorphic, totally isotropic maps $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ with non constant associated map J_γ. The correspondence is realized via the conformal Gauss map and the result is stated in Theorem 4.17.
This result is further extended so as to include the totally umbilical surfaces. To this end we introduce an appropriate Grassmann bundle, called $\mathcal{Q}_2(Q_4)$, defined as the orbit of a fixed point of the product manifold $Q_4 \times \mathcal{Q}_2(\mathbb{R}^6)$ with respect to the natural left action (defined componentwise) of the group Möb(4). $\mathcal{Q}_2(Q_4)$ can again be seen as a homogeneous space for Möb(4) and has a natural integrable complex structure. The result we obtain, stated in Theorem 4.20, is that there is a bijection between $-$ isotropic, weakly conformal branched immersions $f : M \to Q_4$, whose conformal Gauss maps can be continuously extended at the branch points, and holomorphic maps $\Gamma : M \to \mathcal{Q}_2(Q_4)$, solutions of a suitable Pfaffian system.

Chapter 1

The Möbius group and the conformal sphere

1.1 The Möbius group

We start by giving an outline of the construction of the conformal sphere Q_n and of the Möbius group. For further details see [14], [13].
Throughout this chapter we shall use the following index convention:

$$0 \leq a, b, \ldots \leq n+1, \qquad 1 \leq A, B, \ldots \leq n$$

On $\mathbb{R}^{n+2}$, consider the standard basis $\{\varepsilon_a\}$ and the Lorentzian inner product defined in the usual way: if $v = v^a \varepsilon_a$ and $w = w^a \varepsilon_a$, then

$$\langle v, w \rangle = \left(v^0, v^A, v^{n+1}\right) \begin{pmatrix} -1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w^0 \\ w^A \\ w^{n+1} \end{pmatrix}.$$

We perform the following change of basis: we set

$$\eta_0 = \frac{1}{\sqrt{2}}(\varepsilon_0 - \varepsilon_{n+1}), \qquad \eta_A = \varepsilon_A, \qquad \eta_{n+1} = \frac{1}{\sqrt{2}}(\varepsilon_0 + \varepsilon_{n+1}).$$

In this way the vectors η_0 and η_{n+1} are light-like and, with respect to this new basis, the Lorentzian inner product has the following representative matrix:

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

We denote by $\mathcal{L}$ the light cone, the set of light-like vectors, that is

$$\mathcal{L} = \left\{x \in \mathbb{R}^{n+2} \mid \langle x, x \rangle = 0\right\};$$

setting, as before, $v = x^a \eta_a$ we can write

$$\mathcal{L} = \left\{v \in \mathbb{R}^{n+2} \mid -2x^0 x^{n+1} + x^A x^A = 0\right\},$$

while the positive light cone is

$$\mathcal{L}^+ = \{v \in \mathcal{L} \mid x^0 + x^{n+1} > 0\}.$$

We consider the canonical projectivization $p : \mathbb{R}^{n+2} \to \mathbb{P}^{n+1}_{\mathbb{R}}$ with homogeneous coordinates defined with respect to the new basis $\{\eta_a\}$, i.e. the homogeneous coordinates of the projective class of the vector $v = x^a\eta_a$ are $(x^0 : x^A : x^{n+1})$. We let Q_n denote the Darboux hyperquadric, namely the projectivization of the positive light cone:

$$Q_n := \mathbb{P}\mathcal{L}^+ \subset \mathbb{P}^{n+1}_{\mathbb{R}};$$

it is trivial to see that the projective hyperquadric Q_n is diffeomorphic to S^n, an explicit diffeomorphism being given by $\delta : Q_n \to S^n$, defined as

$$\delta : [x] \mapsto {}^t\left(\frac{2x^{n+1} - x^0}{2x^{n+1} + x^0}, \frac{2x^A}{2x^{n+1} + x^0}\right), \tag{1.2}$$

where the square brackets indicate the projective class identified by $x \in \mathbb{R}^{n+2}$. A standard way of immersing $\mathbb{R}^n$ in Q_n is through the Dirac-Weyl chart $\chi : \mathbb{R}^n \to Q_n$ defined as

$$\chi : x \mapsto \left(1 : x^A : \frac{1}{2}|x|^2\right).$$

Note that $[\eta_0]$ and $[\eta_{n+1}]$ belong to Q_n, while none of the $[\eta_A]$ does. Furthermore, the origin of $\mathbb{R}^n$ is sent by χ to $[\eta_0]$, while $\delta([\eta_0]) = {}^t(-1, 0, \ldots, 0) = S$, the south pole of S^n, and $\delta([\eta_{n+1}]) = {}^t(1, 0, \ldots, 0) = N$, the north pole of S^n. Moreover, it is immediate to see that $\chi(\mathbb{R}^n) = Q_n \setminus \{[\eta_{n+1}]\}$ and $\delta \circ \chi = \sigma_N^{-1}$, the inverse of the stereographic projection from N, so that $[\eta_{n+1}]$ can be regarded as the point at infinity of $\mathbb{R}^n$.

Now, the group of projective transformations that fix Q_n is the projectivization of the group

$$\Gamma = \{G \in GL(n+2) \mid {}^tGSG = \lambda S \text{ for some } \lambda > 0\},$$

that is, the quotient of Γ with respect to the center $\mathbb{R}^*I$ of $GL(n+2)$. The quotient $\mathbb{P}\Gamma$ is trivially isomorphic to the group

$$O(n+1,1) = \{G \in \mathrm{GL}(n+2) \mid {}^tGSG = S\}.$$

This group has four connected components, according to the sign of the determinant and whether the positive light cone is sent onto itself or not. We are only interested in the identity component, that is

$$\{G \in O(n+1,1) \mid \det G = 1,\ G\mathcal{L}^+ = \mathcal{L}^+\},$$

because these transformations can be verified to be the ones that correspond exactly to the orientation preserving conformal diffeomorphisms of S^n, endowed with its standard metric, the one induced by the inclusion in $\mathbb{R}^{n+1}$. We can therefore set

Definition 1.1. *The **Möbius group** is the Lie subgroup of $GL(n+2)$ defined by*

$$\text{Möb}(n) = \{G \in O(n+1,1) \mid \det G = 1,\ G\mathcal{L}^+ = \mathcal{L}^+\}.$$

1.2 The Lie algebra of the Möbius group

We will now briefly study the structure of Möb(n) as a Lie group. The Lie algebra of Möb(n), which we shall denote by $\mathfrak{m\ddot{o}b}(n)$ is trivially the vector space of the matrices that satisfy

$$ {}^tAS + SA = 0, $$

that is

$$ \mathfrak{m\ddot{o}b}(n) = \left\{ \begin{pmatrix} -a & {}^tw & 0 \\ v & D & w \\ 0 & {}^tv & a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}, \\ D \in \mathfrak{o}(n) \\ v, w \in \mathbb{R}^n \end{array} \right\}; $$

thus it can be split as $\mathfrak{m\ddot{o}b}(n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & {}^tv & 0 \end{pmatrix} \middle| v \in \mathbb{R}^n \right\} \qquad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & {}^tw & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \middle| w \in \mathbb{R}^n \right\} $$

$$ \mathfrak{g}_0 = \left\{ \begin{pmatrix} -a & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}, \\ D \in \mathfrak{o}(n) \end{array} \right\}. $$

The vector space $\mathfrak{m\ddot{o}b}(n)$ is obviously $\frac{1}{2}(n+1)(n+2)$-dimensional and a basis is given by the following matrices

$$ P_{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad P_{(A,0)} = \begin{pmatrix} 0 & 0 & 0 \\ e_A & 0 & 0 \\ 0 & {}^te_A & 0 \end{pmatrix} $$

$$ P_{(0,A)} = \begin{pmatrix} 0 & {}^te_A & 0 \\ 0 & 0 & e_A \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{(A,B)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_{A,B} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A > B, \tag{1.3} $$

where e_A is the A-th vector of the canonical basis of $\mathbb{R}^n$ and $\{D_{A,B}\}_{A>B}$ is the standard basis for the vector space of skew-symmetric matrices, i.e. $D_{A,B}$ is defined as

$$ (D_{A,B})^a_b = \begin{cases} 1 & \text{if } a = A,\ b = B \\ -1 & \text{if } a = B,\ b = A \\ 0 & \text{otherwise.} \end{cases} $$

The Maurer-Cartan form of the Möbius group is the $\mathfrak{m\ddot{o}b}(n)$-valued 1-form defined as follows: given any $G \in$ Möb(n) and denoting by L_G the left translation on Möb(n), then, for every $X \in T_G$Möb(n),

$$ \Phi_G(X) = L_{G^{-1}*G} X. $$

From its very definition, one can easily check that Φ is left invariant and that its expression with respect to the basis (1.3) is the following

$$ \Phi = \widetilde{P}^*_{(0)} \otimes P_{(0)} + \sum_{B<A} \widetilde{P}^*_{(A,B)} \otimes P_{(A,B)} + \sum_A \left[\widetilde{P}^*_{(A,0)} \otimes P_{(A,0)} + \widetilde{P}^*_{(0,A)} \otimes P_{(0,A)} \right], $$

where, if $X \in \mathfrak{m\ddot{o}b}(n)$, $\widetilde{X}$ indicates the vector field on Möb(n) defined by left translation of X, and

$$\widetilde{P}^*_{(0)}, \quad \left\{\widetilde{P}^*_{(A,B)}\right\}_{B<A}, \quad \widetilde{P}^*_{(A,0)}, \quad \widetilde{P}^*_{(0,A)}$$

are the real-valued 1-forms forming the dual coframe of the frame

$$\widetilde{P}_{(0)}, \quad \left\{\widetilde{P}_{(A,B)}\right\}_{B<A}, \quad \widetilde{P}_{(A,0)}, \quad \widetilde{P}_{(0,A)}.$$

But the $\mathfrak{m\ddot{o}b}(n)$-valued 1-form Φ can also be seen as a $(n+2)\times(n+2)$ matrix of left invariant real-valued 1-forms in the following way:

$$\Phi = \begin{pmatrix} \Phi^0_0 & \Phi^0_B & \Phi^0_{n+1} \\ \Phi^A_0 & \Phi^A_B & \Phi^A_{n+1} \\ \Phi^{n+1}_0 & \Phi^{n+1}_B & \Phi^{n+1}_{n+1} \end{pmatrix},$$

with the obvious symmetry relations

$$\begin{array}{lll} \Phi^0_0 = -\Phi^{n+1}_{n+1}, & \Phi^A_B = -\Phi^B_A, & \Phi^0_{n+1} = \Phi^{n+1}_0 = 0 \\ \Phi^0_B = \Phi^B_{n+1}, & \Phi^A_0 = \Phi^{n+1}_A. & \end{array} \tag{1.4}$$

We point out that, at each point $G \in$ Möb(n), the 1-forms Φ^0_0, $\{\Phi^A_0\}$, $\{\Phi^0_A\}$ and $\{\Phi^A_B\}_{B<A}$ form a basis of T^*_GMöb(n). Moreover, the structure equation writes as

$$d\Phi^a_b = -\Phi^a_c \wedge \Phi^c_b,$$

which, using the symmetries (1.4), can be simplified to

$$\begin{cases} d\Phi^0_0 = -\Phi^0_A \wedge \Phi^A_0 \\ d\Phi^A_0 = -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0 \\ d\Phi^0_A = -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A \\ d\Phi^A_B = -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0 \end{cases} \tag{1.5}$$

Finally, since Möb(n) is a subgroup of $GL(n+2)$ and the left translation is a linear map, we can express the Maurer-Cartan form as

$$\Phi_G = G^{-1} dG, \tag{1.6}$$

meaning that, for every $X \in T_G$Möb(n),

$$\Phi_G(X) = G^{-1}X,$$

where the product $G^{-1}X$ is just the ordinary matrix product. This makes sense since T_GMöb(n) can be canonically included in the algebra of $(n+2)\times(n+2)$ matrices. This expression for Φ, although apparently carrying a slight abuse of notation, will turn out to be quite useful later on.
For more details on the Möbius group and other classical Lie groups, see [3], [1].

1.3 The Möbius space

Let us now turn back to the study of the action of Möb(n) on the Darboux hyperquadric Q_n. It can be proved that such action is transitive, therefore we can realize Q_n as a homogeneous space for Möb(n). In other words we can fix any point of Q_n, for instance the point $O = [\eta_0]$, and consider the isotropy subgroup of such point, that is the subgroup

$$G_0 = \{G \in \text{Möb}(n) \mid [G\eta_0] = [\eta_0]\}.$$

Then Q_n is diffeomorphic to the manifold whose points are the left cosets of G_0 in Möb(n), which we shall denote by Möb(n)$/G_0$.
The explicit expression of G_0 can be computed:

$$G_0 = \left\{ \begin{pmatrix} r^{-1} & {}^t x A & \frac{1}{2} r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix} \middle| \begin{array}{c} A \in SO(n), \\ r \in \mathbb{R}^+,\ x \in \mathbb{R}^n \end{array} \right\} \tag{1.7}$$

and the natural projection

$$\pi : \text{Möb}(n) \to Q_n$$

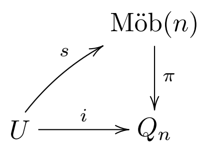
is defined, for $G \in \text{Möb}(n)$, by

$$\pi(G) = [G\eta_0],$$

that is, the projectivization of the first column of G. This projection makes $\pi : \text{Möb}(n) \to Q_n$ a principal G_0-bundle.
Let us now consider a local section of this bundle, i.e. a smooth map

$$s : U \to \text{Möb}(n)$$

defined on an open subset U of Q_n, such that the following diagram commutes



that is, $\pi \circ s = i$, the inclusion map of U into Q_n; then we can pull back the Maurer-Cartan form Φ to $U \subset Q_n$ and define the local, $\mathfrak{m\ddot{o}b}(n)$-valued 1-form

$$\psi = s^*\Phi.$$

Of course, since the pull-back commutes with both the exterior derivative and the wedge product, ψ satisfies the structure equation as well:

$$d\psi^a_b = -\psi^a_c \wedge \psi^c_b.$$

Moreover, using (1.6), we can write

$$\psi = s^*\Phi = s^{-1}ds. \tag{1.8}$$

Let $\widetilde{s} : V \to \text{Möb}(n)$ be another local section and set $\widetilde{\psi} = \widetilde{s}^*\Phi$; assuming $U \cap V \neq \varnothing$, s and $\widetilde{s}$ are related by

$$\widetilde{s} = sK,$$

for some smooth map $K : U \cap V \to G_0$. In particular, applying the last equality to (1.8), we get that the pull-back of the Maurer-Cartan form changes according to the following formula.

$$\widetilde{\psi} = K^{-1}\psi K + K^{-1}dK. \tag{1.9}$$

Let us focus on the 1-forms $\{\psi_0^A\}$. It is immediate to see that they locally span the cotangent space of Q_n. Indeed, from the very definition of Φ, the forms $\{\Phi_0^A\}$ span the semibasic forms of the bundle $G_0 \to \text{Möb}(n) \xrightarrow{\pi} Q_n$, that is, the forms that vanish on the vectors tangent to the fiber. Now if ω is a 1-form on Q_n, then $\pi^*\omega$ is a semibasic form on $\text{Möb}(n)$ and therefore $\pi^*\omega = \alpha_A \Phi_0^A$ for some suitable smooth functions α_A. It follows that

$$\omega = (\pi \circ s)^*\omega = s^*\pi^*\omega = (\alpha_A \circ s)\psi_0^A,$$

and thus $\{\psi_0^A\}$ is a local basis for the cotangent space of Q_n.
This fact allows us to define a local metric in the following way:

$$\sum_A \psi_0^A \otimes \psi_0^A \tag{1.10}$$

and a nowhere vanishing local n-form as follows:

$$\psi_0^1 \wedge \ldots \wedge \psi_0^n. \tag{1.11}$$

Under a change of section, the $(A, 0)$-component of (1.9) gives, in particular,

$$\widetilde{\psi}_0^A = r^{-1} A_A^B \psi_0^B,$$

so that

$$\sum_A \widetilde{\psi}_0^A \otimes \widetilde{\psi}_0^A = r^{-2} \sum_A \psi_0^A \otimes \psi_0^A$$

and

$$\widetilde{\psi}_0^1 \wedge \ldots \wedge \widetilde{\psi}_0^n = r^{-n} \psi_0^1 \wedge \ldots \wedge \psi_0^n.$$

Therefore (1.10) and (1.11) define a conformal class of metrics and an orientation on Q_n, namely a conformal structure, and we can set

Definition 1.2. *The **Möbius space**, or the **conformal sphere**, is the Darboux hyperquadric Q_n endowed with its structure of homogeneous space for* $\text{Möb}(n)$ *and the conformal structure given by* (1.10) *and* (1.11).

Finally, it is not hard to prove that the diffeomorphism $\delta : Q_n \to S^n$, defined in (1.2), becomes a conformal map once Q_n and S^n are equipped with their standard conformal structures.

Chapter 2

The conformal structure of a submanifold and its Darboux framing: the general case

2.1 The frame reduction procedure

In what follows M will always be assumed to be m-dimensional and oriented. We fix the index ranges

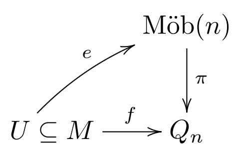
$$1 \leq A, B, \ldots \leq n, \qquad 1 \leq i, j, \ldots \leq m, \qquad m+1 \leq \alpha, \beta, \ldots n.$$

Let $f : M \to Q_n$ be an immersion. We recall that Q_n is realized as the homogeneous space $\text{Möb}(n)/G_0$ where G_0 is the isotropy subgroup at O given in (1.7). The corresponding principal G_0-bundle is

$$\pi : \text{Möb}(n) \to \text{Möb}(n)/G_0$$

where π acts on a matrix of $\text{Möb}(n)$ by projectivizing its first column.

Definition 2.1. *A* ***zeroth order frame field along*** f *is a smooth map* e *defined on an open set* $U \subseteq M$ *with values in* $\text{Möb}(n)$ *such that* $\pi \circ e = f_{|U}$*, that is, the following diagram commutes:*



From now on, dealing with frames along f, we will omit specifying their domains of definition since no possible confusion will arise.

Any two zeroth order frame fields e, $\widetilde{e}$ on the intersection of their domains of definition, if not empty, are related by

$$\widetilde{e} = eK \tag{2.1}$$

where $K : \widetilde{U} \cap U \to G_0$ is a smooth function. Setting

$$\phi = e^* \Phi$$

we obtain again equations (1.5) to be interpreted this time on M. Under the change of frames (2.1), $\widetilde{\phi} = \widetilde{e}^* \Phi$ expresses in terms of ϕ as in (1.9). In particular, for

$$K = \begin{pmatrix} r^{-1} & {}^t xA & \frac{1}{2} r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix}$$

$A \in SO(n)$, $x \in \mathbb{R}^n$, $r \in \mathbb{R}^+$,

$$\widetilde{\phi}_0^A = r^{-1}\, {}^t A(\phi_0^A) = r^{-1} A_A^B \phi_0^B,$$

As a consequence, at any point $p \in M$ we can choose a zeroth order frame such that

$$\phi_0^\alpha = 0. \tag{2.2}$$

The isotropy subgroup at this point is given by

$$G_1 = \left\{ \begin{pmatrix} r^{-1} & {}^t xA & {}^t yB & \frac{1}{2} r\big(|x|^2 + |y|^2\big) \\ 0 & A & 0 & rx \\ 0 & 0 & B & ry \\ 0 & 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} r \in \mathbb{R}^+, A \in SO(m), \\ B \in SO(n-m), \\ x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m} \end{array} \right\}. \tag{2.3}$$

and since G_1 is independent of p, smooth zeroth order frame fields such that (2.2) holds can be chosen in an appropriate neighbourhood of each point of M by general theory, see [14].

Definition 2.2. *A zeroth order frame field e such that* (2.2) *holds on its domain of definition is called* ***first order frame****.*

Any two such frame fields are related by (2.1) where now K takes values in G_1 defined in (2.3).

With the aid of first order frame fields we can define a conformal structure on M. Indeed, because of (1.9) and (2.3), under a change of first order frame fields the quadratic form $ds^2 = \sum_i \phi_0^i \otimes \phi_0^i$ transforms according to the law

$$\widetilde{ds^2} = r^{-2} ds^2$$

while the volume form $dV = \phi_0^1 \wedge \ldots \wedge \phi_0^m$ transforms according to

$$\widetilde{dV} = r^{-m} dV. \tag{2.4}$$

It is trivial to see that now the map f becomes a conformal immersion of the manifold M, endowed with this conformal structure, into Q_n, equipped with its standard conformal structure.

Differentiating (2.2) and using the structure equations of Möb(n) we obtain

$$0 = -\phi_i^\alpha \wedge \phi_0^i.$$

By Cartan's lemma there exist some (locally defined) functions h^α_{ij} such that

$$\phi^\alpha_i = h^\alpha_{ij}\phi^j_0, \qquad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.5}$$

We use (1.9) and (2.3) to obtain under a change of first order frame fields

$$\widetilde{\phi}^\alpha_i = B^\beta_\alpha A^k_i \phi^\beta_k - B^\beta_\alpha A^k_i y^\beta \phi^k_0. \tag{2.6}$$

Next, for first order frame fields

$$\widetilde{\phi}^j_0 = r^{-1} A^l_j \phi^l_0 \tag{2.7}$$

and using (2.6), (2.7) and the definition of h^α_{ij} given in (2.5) we finally obtain

$$\widetilde{h}^\alpha_{ij} = rB^\beta_\alpha A^l_j (A^k_i h^\beta_{lk} - A^l_i y^\beta), \tag{2.8}$$

where the meaning of A, B, y, r is given in (2.3). Taking the trace of (2.8) with respect to i and j we obtain

$$\widetilde{h}^\alpha_{ii} = rB^\beta_\alpha (h^\beta_{kk} - my^\beta).$$

The next step is therefore to consider at any point $p \in M$ a first order frame such that

$$h^\alpha_{kk} = 0. \tag{2.9}$$

The isotropy subgroup is given by

$$G_D = \left\{ \begin{pmatrix} r^{-1} & {}^t xA & 0 & \frac{1}{2}r|x|^2 \\ 0 & A & 0 & rx \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} A \in SO(m), \\ B \in SO(n-m), \\ r \in \mathbb{R}^+, x \in \mathbb{R}^m \end{array} \right\} \tag{2.10}$$

and is again independent of the point p considered, so that first order frames with the above property can be smoothly chosen in an appropriate neighbourhood of any point.

Definition 2.3. *A **Darboux frame field along** f is a first order frame field for which* (2.9) *holds.*

Any two Darboux frame fields are related by (2.1) where now K is a smooth function taking values in G_D.

We observe that for Darboux frames (2.8) becomes

$$\widetilde{h}^\alpha_{ij} = rB^\beta_\alpha A^l_j A^k_i h^\beta_{kl}. \tag{2.11}$$

For further details on the generality of the frame reduction procedure, we refer the reader to [13], [15], [14].

2.2 The geometry of submanifolds of the Möbius space

Differentiating (2.5), using the structure equations and Cartan's lemma with respect to a Darboux frame e we have

$$dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{kj}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^0_\alpha = h^\alpha_{ijk}\phi^k_0, \tag{2.12}$$

for some (locally defined) functions h^α_{ijk} symmetric in the lower indices. To see this latter fact, we observe that by Cartan's lemma

$$h^\alpha_{ijk} = h^\alpha_{ikj}.$$

On the other hand, by (2.12),

$$dh^\alpha_{ji} - h^\alpha_{jk}\phi^k_i - h^\alpha_{ki}\phi^k_j + h^\beta_{ji}\phi^\alpha_\beta + h^\alpha_{ji}\phi^0_0 + \delta_{ji}\phi^0_\alpha = h^\alpha_{jik}\phi^k_0;$$

however, $h^\alpha_{ij} = h^\alpha_{ji}$, so that, from the above equality we get

$$h^\alpha_{jik}\phi^k_0 = dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{kj}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^0_\alpha$$

and comparing with (2.12) we then deduce

$$h^\alpha_{ijk} = h^\alpha_{jik}$$

realizing the desired symmetries.

Taking the trace of (2.12) with respect to i and j and using (2.9) we obtain

$$\phi^0_\alpha = p^\alpha_k\phi^k_0 \tag{2.13}$$

where we have set

$$p^\alpha_k = \frac{1}{m}h^\alpha_{iik}. \tag{2.14}$$

With respect to a Darboux frame e defined on $U \subseteq M$ let us consider the matrix of 1-forms Ψ defined by

$$\Psi = \begin{pmatrix} \phi^0_0 & \phi^0_i & 0 \\ \phi^i_0 & \phi^i_j & \phi^0_i \\ 0 & \phi^i_0 & -\phi^0_0 \end{pmatrix}. \tag{2.15}$$

We can clearly think of Ψ as taking values in the Lie algebra of the $\frac{1}{2}(m+1)(m+2)$-dimensional Möbius group.
Under a change of Darboux frames $\widetilde{e} = eK$, where K takes values in G_D, we have

$$\widetilde{\Psi} = \bar{K}^{-1}\Psi\bar{K} + \bar{K}^{-1}d\bar{K},$$

with

$$\bar{K} = \begin{pmatrix} r^{-1} & {}^t xA & \frac{r}{2}|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix},$$

$x \in \mathbb{R}^m$, $A \in SO(m)$, $r \in \mathbb{R}^+$.
We therefore conclude that Ψ defines a Cartan connection on M, whose curvature forms are, as usual, given by the structure equations that we write in this case as:

$$\begin{cases} d\phi^0_0 = -\phi^0_i \wedge \phi^i_0 + \Omega^0_0 \\ d\phi^i_0 = -\phi^i_0 \wedge \phi^0_0 - \phi^i_j \wedge \phi^j_0 + \Omega^i_0 \\ d\phi^0_i = -\phi^0_0 \wedge \phi^0_i - \phi^0_j \wedge \phi^j_i + \Omega^0_i \\ d\phi^i_j = -\phi^i_0 \wedge \phi^0_j - \phi^i_k \wedge \phi^k_j - \phi^0_i \wedge \phi^j_0 + \Omega^i_j. \end{cases} \tag{2.16}$$

Comparing (2.16) with the structure equations of the group Möb(n) we immediately deduce that

$$\Omega^0_0 = 0 = \Omega^i_0$$

$$\Omega^0_i = -\phi^0_\alpha \wedge \phi^\alpha_i, \qquad \Omega^i_j = -\phi^i_\alpha \wedge \phi^\alpha_j$$

so that using (2.5) and (2.13) we obtain

$$\Omega^0_i = \frac{1}{2}\big(p^\alpha_k h^\alpha_{ij} - p^\alpha_j h^\alpha_{ik}\big)\phi^j_0 \wedge \phi^k_0 \tag{2.17}$$

$$\Omega^i_j = \frac{1}{2}\big(h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}\big)\phi^k_0 \wedge \phi^l_0 \tag{2.18}$$

We begin by analysing (2.18). We set

$$\tau^i_{jkl} = h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk} \tag{2.19}$$

so that (2.18) can be expressed as

$$\Omega^i_j = \frac{1}{2}\tau^i_{jkl}\phi^k_0 \wedge \phi^l_0.$$

Note that the coefficients τ^i_{jkl} satisfy the usual algebraic symmetries of the Riemann curvature tensor, including the first Bianchi identity.

Making use of (2.11) we observe that, with respect to Darboux frames $\widetilde{e}$, e we have

$$\widetilde{\tau}^i_{jkl} = r^2 A^s_i A^t_j A^u_k A^v_l \tau^s_{tuv}. \tag{2.20}$$

We denote by E_j the frame on M dual to ϕ^j_0, that is, characterized by the request

$$\phi^j_0(E_k) = \delta^j_k.$$

Using (2.7) we deduce

$$\widetilde{E}_k = r A^j_k E_j \tag{2.21}$$

(already at the level of first order frames). Therefore, using (2.20), (2.7) and (2.21) we define a global tensor τ on M by locally setting

$$\tau = \tau^i_{jkl}\phi^k_0 \otimes \phi^l_0 \otimes \phi^j_0 \otimes E_i \tag{2.22}$$

for a Darboux frame along f. We will call τ the generalized Weyl tensor. Observe that, unlike the usual Weyl tensor, τ is not traceless. Indeed, we have

$$\mathcal{N}_{jl} = -\tau^i_{jil} = h^\alpha_{il} h^\alpha_{ij}. \tag{2.23}$$

From (2.20) and (2.7) the above components define a global symmetric tensor $\mathcal{N}$, locally given by

$$\mathcal{N} = \mathcal{N}_{jk}\phi^j_0 \otimes \phi^k_0. \tag{2.24}$$

We observe that

$$\mathcal{N}_{jj} = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$$

and

$$\widetilde{\mathcal{N}}_{jj} = r^2 \mathcal{N}_{jj}.$$

Thus, the trace of $\mathcal{N}$ does not define a scalar and $\mathcal{N} = 0$ if and only if $h^\alpha_{ij} = 0$ for each α, i, j, if and only if $\Omega^i_j = 0$ $\forall i, j$. Furthermore, if $\mathcal{N} = 0$, then $\Omega^0_i = 0$ $\forall i$.

According to Cartan, we set the following

Definition 2.4. *The Cartan connection* Ψ *on* M *is* ***normal*** *if* $\mathcal{N} = 0$.

The normality condition in the present case is easily analysed. Indeed we have

Proposition 2.1. *Let* $f : M \to Q_n$ *be an immersion,* M *oriented,* $m = \dim M \geq 2$*, for which* $\mathcal{N} \equiv 0$*. Then, there exists* $Q_m \subset Q_n$ *such that* $f(M) \subseteq Q_m$*. Furthermore, if* M *is compact,* f *is a diffeomorphism onto* Q_m*.*

Proof. We use a standard technique in the method of the moving frame and therefore we give here only a sketch of the proof. Let e be any Darboux frame along f. Then, by assumption $h^\alpha_{ij} = 0$ and therefore, with respect to e we have

$$\phi^\alpha_0 \equiv 0 \equiv \phi^\alpha_i.$$

Differentiating $\phi^\alpha_i = 0$ and using the structure equations we obtain

$$0 = \phi^0_\alpha \wedge \phi^k_0 \qquad \text{for all } k$$

and, since $m \geq 2$, we deduce

$$\phi^0_\alpha = 0.$$

Consider now on Möb(n) the ideal $\mathcal{I}$ generated by the forms Φ^α_0, Φ^α_i, Φ^0_α. Using the Maurer-Cartan structure equations (1.5) and the symmetries (1.4) we see that $\mathcal{I}$ is a differential ideal, that is, $d\mathcal{I} \subseteq \mathcal{I}$. The distribution Δ defined by $\mathcal{I}$ is, at the identity

$$\Delta_I = \left\{ \begin{pmatrix} a & {}^t x & 0 & 0 \\ y & D & 0 & x \\ 0 & 0 & E & 0 \\ 0 & {}^t y & 0 & -a \end{pmatrix} \middle| \begin{array}{c} D \in \mathfrak{o}(m) \\ E \in \mathfrak{o}(n-m) \\ x, y \in \mathbb{R}^m \\ a \in \mathbb{R} \end{array} \right\}$$

and is obtained at any other point by left translation because of the left invariance of Φ. In particular, its maximal integral submanifold passing through the identity is the subgroup of Möb(n)

$$\left\{ \begin{pmatrix} a & {}^t Z & 0 & b \\ X & A & 0 & Y \\ 0 & 0 & B & 0 \\ c & {}^t W & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & {}^t Z & b \\ X & A & Y \\ c & {}^t W & d \end{pmatrix} \in \text{Möb}(m), B \in SO(n-m) \right\} \simeq$$

$$\simeq \text{Möb}(m) \times SO(n-m)$$

The image of this subgroup in the quotient Möb$(n)/G_0 \simeq Q_n$ is therefore an m-dimensional sphere Q_m, and so is the image of any other maximal integral submanifold. Now, it is trivial to see that $e_* TM \subseteq \Delta$, so that $f_* TM = \pi_* e_* TM \subseteq \pi_* \Delta$ and the connectedness of M grants that $f(M) \subseteq Q_m$. Assuming M compact, $f(M)$ is open and closed in Q_m, thus $f(M) = Q_m$ and f is a homeomorphism onto Q_m. □

Definition 2.5. *We say that a point* $p \in M$ *is an* ***umbilical point*** *if and only if* $\mathcal{N}(p) = 0$.

Equivalently, $p \in M$ is an umbilical point if and only if for some (hence any) Darboux frame

$$h^\alpha_{ij} = 0 \qquad \text{at } p.$$

The form (2.10) of the isotropy subgroup G_D of Darboux frames along f suggests the possibility of defining a suitable vector bundle N over M whose role should parallel that of the normal bundle of an isometric immersion into a Riemannian manifold. Indeed, let $p \in M$ and choose a Darboux frame e along f with p belonging to its domain of definition. Define the fiber N_p to be the $(n-m)$-dimensional vector space generated by $\{e_\alpha\}$. Because of (2.10), a change of Darboux frames as in (2.1) gives rise to a new basis $\{\widetilde{e}_\alpha\}$ such that

$$\widetilde{e}_\alpha = e_\beta B^\beta_\alpha \tag{2.25}$$

with $B \in SO(n-m)$. It follows that the bundle N is well defined and on it there is a naturally defined inner product $(\ ,\ )$ for which $\{e_\alpha\}$ is an orthonormal basis at p.With respect to this inner product we define a metric connection

$$D : \Gamma(N) \to \Gamma(T^*M \otimes N)$$

by setting

$$De_\alpha = \phi^\beta_\alpha \otimes e_\beta.$$

D is well defined, indeed if λ represents the matrix of the connection forms, then under a change of Darboux frames, according to (1.9), we have

$$\widetilde{\lambda} = {}^tB\lambda B + {}^tBdB.$$

On the other hand, D is clearly metric since $\phi^\alpha_\beta + \phi^\beta_\alpha = 0$.

As usual the curvature forms Λ^α_β are defined via the structure equations

$$d\phi^\alpha_\beta = -\phi^\alpha_\gamma \wedge \phi^\gamma_\beta + \Lambda^\alpha_\beta.$$

Using the structure equations of the group Möb(n) and (2.5), setting

$${}^\perp\tau^\alpha_{\beta ij} = h^\alpha_{ki}h^\beta_{kj} - h^\alpha_{kj}h^\beta_{ki} \tag{2.26}$$

we obtain

$$\Lambda^\alpha_\beta = \frac{1}{2}{}^\perp\tau^\alpha_{\beta ij}\phi^i_0 \wedge \phi^j_0.$$

Observe that we have the symmetry relations

$${}^\perp\tau^\alpha_{\beta ij} = -{}^\perp\tau^\alpha_{\beta ji} = -{}^\perp\tau^\beta_{\alpha ij}$$

Moreover, with respect to Darboux frames $\widetilde{e}$, e

$${}^\perp\widetilde{\tau}^\alpha_{\beta ij} = r^2 B^\gamma_\alpha B^\rho_\beta A^t_i A^v_j {}^\perp\tau^\gamma_{\rho tv}$$

It follows that we can define a tensor ${}^\perp\tau$ by locally setting

$${}^\perp\tau = {}^\perp\tau^\alpha_{\beta ij}\phi^i_0 \otimes \phi^j_0 \otimes e_\alpha \otimes e_\beta.$$

We will call ${}^\perp\tau$ the normal curvature tensor.

Chapter 3

The conformal Grassmannian

3.1 The conformal Grassmann bundle as a homogeneous space

The aim of this chapter is to introduce an appropriate conformal Grassmannian as an orbit of the Grassmann manifold of oriented s-planes in $\mathbb{R}^{n+2}$ under the action of the Möbius group Möb(n).

Its description and structure is given as follows. Set $s = n - m \geq 1$ and let $\{\varepsilon_0, \dots, \varepsilon_m, \varepsilon_{m+1}, \dots, \varepsilon_n, \varepsilon_{n+1}\}$ be the standard basis of $\mathbb{R}^{n+2}$. Fix as an origin in $G_s\big(\mathbb{R}^{n+2}\big)$ the point $O = [\varepsilon_{m+1}, \dots, \varepsilon_n]$ and consider the orbit $\mathcal{Q}_s\big(\mathbb{R}^{n+2}\big)$ of the point O under the left action (by matrix multiplication) of the group Möb(n) onto $G_s\big(\mathbb{R}^{n+2}\big)$. Then the isotropy subgroup of the action on the orbit at the point O is given by

$$H_0 = \left\{ \begin{pmatrix} a & {}^t z & 0 & b \\ x & A & 0 & y \\ 0 & 0 & B & 0 \\ c & {}^t w & 0 & d \end{pmatrix} \;\middle|\; \begin{array}{c} \begin{pmatrix} a & {}^t z & b \\ x & A & y \\ c & {}^t w & d \end{pmatrix} \in \text{Möb}(m), \\ B \in SO(s) \end{array} \right\} \subseteq \text{Möb}(n). \tag{3.1}$$

Note that, since $H_0 \subseteq$ Möb(n), $z, w, x, y, a, b, c, d, A$ cannot be chosen arbitrarily but have to satisfy certain compatibility relations between them that will be essential in determining that certain quantities are globally well defined.

Thus $\mathcal{Q}_s\big(\mathbb{R}^{n+2}\big)$ is identified with the homogeneus space Möb$(n)/H_0$ with the canonical projection

$$\widehat{\pi} : \text{Möb}(n) \to \mathcal{Q}_s\big(\mathbb{R}^{n+2}\big)$$

given by

$$\widehat{\pi} : P \mapsto [P_{m+1}, \dots, P_n] \tag{3.2}$$

where P_0, P_A, P_{n+1} are the columns of the matrix P.

3.2 The Kähler-Lorentzian structure of the conformal Grassmannian

On their common domain of definition two local sections of the bundle $\widehat{\pi}$: Möb$(n) \to \mathcal{Q}_s(\mathbb{R}^{n+2})$ are related by $\widetilde{s} = sK$ where K is a function taking values in H_0. Considering the components Φ^0_α, Φ^i_α, Φ^α_0 of the Maurer-Cartan form of Möb(n) and setting $\varphi = s^*\Phi$, we find that their pull-backs under the sections s, $\widetilde{s}$ are related by the following transformation laws:

$$\begin{cases} \widetilde{\varphi}^0_\alpha = d\,\varphi^0_\beta B^\beta_\alpha - y^i \varphi^i_\beta B^\beta_\alpha + b\varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^i_\alpha = -w^i \varphi^0_\beta B^\beta_\alpha + A^k_i \varphi^k_\beta B^\beta_\alpha - z^i \varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^\alpha_0 = c\,\varphi^0_\beta B^\beta_\alpha - x^k \varphi^k_\beta B^\beta_\alpha + a\varphi^\beta_0 B^\beta_\alpha \end{cases} \tag{3.3}$$

where the meaning of $d, c, a, b, y, x, w, z, A, B$ is given in (3.1). From (3.3) and the relations defining the group Möb(n), it is not hard to deduce that the quadratic form dl^2 of signature $(s, s(m+1))$ given by

$$dl^2 = -\varphi^0_\alpha \otimes \varphi^\alpha_0 - \varphi^\alpha_0 \otimes \varphi^0_\alpha + \sum_{i,\alpha} \varphi^i_\alpha \otimes \varphi^i_\alpha \tag{3.4}$$

is well defined on $\mathcal{Q}_s(\mathbb{R}^{n+2})$ and determines a pseudo-metric on it. In particular the forms φ^0_α, φ^α_0, φ^i_α constitute a (local non orthonormal) coframe on $\mathcal{Q}_s(\mathbb{R}^{n+2})$ which thus turns out to be of dimension $s(m+2)$, $s = n - m$. It is convenient to set

$$\theta^{0,\alpha} = \varphi^\alpha_0, \qquad \theta^{\alpha,0} = \varphi^0_\alpha, \qquad \theta^{\alpha,i} = \varphi^i_\alpha \tag{3.5}$$

and to order the pairs $(\alpha, 0)$, (α, i), $(0, \alpha)$ as

$$\begin{aligned} &(\gamma, 0) < (\beta, i) < (0, \alpha) && \forall \alpha, \beta, \gamma, i \\ &(0, \beta) < (0, \alpha) && \text{iff } \beta < \alpha \\ &(\beta, j) < (\alpha, i) && \text{iff } \beta < \alpha \text{ or } \beta = \alpha \text{ and } j < i \\ &(\beta, 0) < (\alpha, 0) && \text{iff } \beta < \alpha. \end{aligned} \tag{3.6}$$

Thus, representing with the symbols $\widetilde{A}, \widetilde{B}, \ldots$ the $s(m+2)$ indices $(\alpha, 0)$, (α, i), $(0, \alpha)$, we can write dl^2 as

$$dl^2 = g_{\widetilde{A}\widetilde{B}} \theta^{\widetilde{A}} \otimes \theta^{\widetilde{B}} \tag{3.7}$$

with

$$\left(g_{\widetilde{A}\widetilde{B}}\right) = \begin{pmatrix} 0 & 0 & -I_s \\ 0 & I_{sm} & 0 \\ -I_s & 0 & 0 \end{pmatrix} \qquad s = n - m. \tag{3.8}$$

The Levi-Civita connection forms $\theta^{\widetilde{A}}_{\widetilde{B}}$ with respect to the previous coframe are therefore characterized by the equations

$$\begin{cases} d\theta^{\widetilde{A}} = -\theta^{\widetilde{A}}_{\widetilde{B}} \wedge \theta^{\widetilde{B}} \\ g_{\widetilde{A}\widetilde{C}} \theta^{\widetilde{C}}_{\widetilde{B}} + g_{\widetilde{B}\widetilde{C}} \theta^{\widetilde{C}}_{\widetilde{A}} = 0. \end{cases} \tag{3.9}$$

This allows us to determine the connection forms by simply taking exterior derivatives of (3.5) and using the structure equations of the group Möb(n). We obtain

$$\begin{cases} \theta^{\alpha,0}_{\beta,0} = \delta^\alpha_\beta \varphi^0_0 + \varphi^\alpha_\beta, & \theta^{\alpha,0}_{\beta,i} = \delta^\alpha_\beta \varphi^0_i, & \theta^{\alpha,0}_{0,\beta} = 0 \\ \theta^{\alpha,i}_{\beta,0} = \delta^\alpha_\beta \varphi^i_0, & \theta^{\alpha,i}_{\beta,k} = \delta^\alpha_\beta \varphi^i_k + \delta^i_k \varphi^\alpha_\beta, & \theta^{\alpha,i}_{0,\beta} = \delta^\alpha_\beta \varphi^0_i \\ \theta^{0,\alpha}_{\beta,0} = 0, & \theta^{0,\alpha}_{\beta,i} = \delta^\alpha_\beta \varphi^i_0, & \theta^{0,\alpha}_{0,\beta} = \varphi^\alpha_\beta - \delta^\alpha_\beta \varphi^0_0 \end{cases} \tag{3.10}$$

and, by a simple computation, one checks the validity of the skew-symmetry relations given by the second of (3.9).

It is worth considering the special case $s = 2$, that is $m = n - 2$. Indeed, starting from the $2n$ independent forms φ^0_α, φ^i_α, φ^α_0 we can construct the n independent forms over $\mathbb{C}$

$$\zeta^0 = \varphi^0_{n-1} + i\varphi^0_n, \qquad \zeta^k = \varphi^k_{n-1} + i\varphi^k_n, \qquad \zeta^{n-1} = \varphi^{n-1}_0 + i\varphi^n_0. \tag{3.11}$$

Using the structure equations it is immediate to verify that their differentials belong to the ideal they generate, showing that $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ is a complex manifold, in fact complex Lorentzian. Indeed the complex structure J induced by the forms (3.11) is determined by

$$\zeta^0(X + iJX) = \zeta^k(X + iJX) = \zeta^{n-1}(X + iJX) = 0 \qquad \forall X \in T\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big),$$

that is

$$\varphi^0_{n-1}(X) = \varphi^0_n(JX) \qquad \varphi^k_{n-1}(X) = \varphi^k_n(JX) \qquad \varphi^{n-1}_0(X) = \varphi^n_0(JX).$$

It is therefore trivial to verify that the metric dl^2 is Hermitian-Lorentzian.

$$\begin{aligned} dl^2(JX, JY) =& - \varphi^0_{n-1}(JX)\varphi^{n-1}_0(JY) - \varphi^0_n(JX)\varphi^n_0(JY) + \\ & - \varphi^{n-1}_0(JX)\varphi^0_{n-1}(JY) - \varphi^n_0(JX)\varphi^0_n(JY) + \\ & + \varphi^i_{n-1}(JX)\varphi^i_{n-1}(JY) + \varphi^i_n(JX)\varphi^i_n(JY) = \\ =& - \varphi^0_n(X)\varphi^n_0(Y) - \varphi^0_{n-1}(X)\varphi^{n-1}_0(Y) + \\ & - \varphi^n_0(X)\varphi^0_n(Y) - \varphi^{n-1}_0(X)\varphi^0_{n-1}(Y) + \\ & + \varphi^i_n(X)\varphi^i_n(Y) + \varphi^i_{n-1}(X)\varphi^i_{n-1}(Y) = \\ =& dl^2(X, Y). \end{aligned}$$

We verify that $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ is Kähler by showing that the differential of the Kähler form

$$\mathcal{K}(X, Y) = dl^2(JX, Y)$$

vanishes identically. This is a simple exercise using (3.11) and the Maurer-Cartan structure equations. Indeed

$$\begin{aligned} \mathcal{K}(X, Y) =& - \varphi^0_{n-1}(JX)\varphi^{n-1}_0(Y) - \varphi^0_n(JX)\varphi^n_0(Y) + \\ & - \varphi^{n-1}_0(JX)\varphi^0_{n-1}(Y) - \varphi^n_0(JX)\varphi^0_n(Y) + \\ & + \varphi^i_{n-1}(JX)\varphi^i_{n-1}(Y) + \varphi^i_n(JX)\varphi^i_n(Y) = \\ =& \varphi^0_n(X)\varphi^{n-1}_0(Y) - \varphi^0_{n-1}(X)\varphi^n_0(Y) + \\ & + \varphi^n_0(X)\varphi^0_{n-1}(Y) - \varphi^{n-1}_0(X)\varphi^0_n(Y) + \\ & - \varphi^i_n(X)\varphi^i_{n-1}(Y) + \varphi^i_{n-1}(X)\varphi^i_n(Y), \end{aligned}$$

that is,

$$\begin{aligned}\mathcal{K} =& -\varphi^0_{n-1}\wedge\varphi^n_0 - \varphi^{n-1}_0\wedge\varphi^0_n + \varphi^i_{n-1}\wedge\varphi^i_n = \\ =& \frac{i}{2}\Big(-\zeta^0\wedge\overline{\zeta^{n-1}} - \zeta^{n-1}\wedge\overline{\zeta^0} + \zeta^k\wedge\overline{\zeta^k}\Big).\end{aligned} \tag{3.12}$$

Therefore

$$\begin{aligned}d\mathcal{K} =& -d\varphi^0_{n-1}\wedge\varphi^n_0 + \varphi^0_{n-1}\wedge d\varphi^n_0 - d\varphi^{n-1}_0\wedge\varphi^0_n + \\ &+ \varphi^{n-1}_0\wedge d\varphi^0_n + d\varphi^i_{n-1}\wedge\varphi^i_n - \varphi^i_{n-1}\wedge d\varphi^i_n = \\ =&(\varphi^0_0\wedge\varphi^0_{n-1} + \varphi^0_k\wedge\varphi^k_{n-1} + \varphi^0_n\wedge\varphi^n_{n-1})\wedge\varphi^n_0 + \\ &-\varphi^0_{n-1}\wedge(\varphi^n_0\wedge\varphi^0_0 + \varphi^n_k\wedge\varphi^k_0 + \varphi^n_{n-1}\wedge\varphi^{n-1}_0) + \\ &+(\varphi^{n-1}_0\wedge\varphi^0_0 + \varphi^{n-1}_k\wedge\varphi^k_0 + \varphi^{n-1}_n\wedge\varphi^n_0)\wedge\varphi^0_n + \\ &-\varphi^{n-1}_0\wedge(\varphi^0_0\wedge\varphi^0_n + \varphi^0_k\wedge\varphi^k_n + \varphi^0_{n-1}\wedge\varphi^{n-1}_n) + \\ &-(\varphi^i_0\wedge\varphi^0_{n-1} + \varphi^i_k\wedge\varphi^k_{n-1} + \varphi^i_n\wedge\varphi^n_{n-1} + \varphi^0_i\wedge\varphi^{n-1}_0)\wedge\varphi^i_n + \\ &+\varphi^i_{n-1}\wedge(\varphi^i_0\wedge\varphi^0_n + \varphi^i_k\wedge\varphi^k_n + \varphi^i_{n-1}\wedge\varphi^{n-1}_n + \varphi^0_i\wedge\varphi^n_0) = \\ =&\varphi^0_0\wedge\varphi^0_{n-1}\wedge\varphi^n_0 + \varphi^0_k\wedge\varphi^k_{n-1}\wedge\varphi^n_0 + \varphi^0_n\wedge\varphi^n_{n-1}\wedge\varphi^n_0 + \\ &-\varphi^0_{n-1}\wedge\varphi^n_0\wedge\varphi^0_0 - \varphi^0_{n-1}\wedge\varphi^n_k\wedge\varphi^k_0 - \varphi^0_{n-1}\wedge\varphi^n_{n-1}\wedge\varphi^{n-1}_0 + \\ &+\varphi^{n-1}_0\wedge\varphi^0_0\wedge\varphi^0_n + \varphi^{n-1}_k\wedge\varphi^k_0\wedge\varphi^0_n + \varphi^{n-1}_n\wedge\varphi^n_0\wedge\varphi^0_n + \\ &-\varphi^{n-1}_0\wedge\varphi^0_0\wedge\varphi^0_n - \varphi^{n-1}_0\wedge\varphi^0_k\wedge\varphi^k_n - \varphi^{n-1}_0\wedge\varphi^0_{n-1}\wedge\varphi^{n-1}_n + \\ &-\varphi^i_0\wedge\varphi^0_{n-1}\wedge\varphi^i_n - \varphi^i_k\wedge\varphi^k_{n-1}\wedge\varphi^i_n - \varphi^0_i\wedge\varphi^{n-1}_0\wedge\varphi^i_n + \\ &+\varphi^i_{n-1}\wedge\varphi^i_0\wedge\varphi^0_n + \varphi^i_{n-1}\wedge\varphi^i_k\wedge\varphi^k_n + \varphi^i_{n-1}\wedge\varphi^0_i\wedge\varphi^n_0 = \\ =&0.\end{aligned}$$

3.3 The projective structure of the conformal Grassmannian

Finally we describe the complex projective structure of the conformal Grassmannian. There is a natural injection of $\mathcal{Q}_2(\mathbb{R}^{n+2})$ in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined as follows. Let $[G\varepsilon_{n-1}, G\varepsilon_n]$, with $G \in \text{Möb}(n)$, be a 2-plane of $\mathcal{Q}_2(\mathbb{R}^{n+2})$. The map sending $[G\varepsilon_{n-1}, G\varepsilon_n]$ to the projectivization of the complex, non-zero vector $G(\varepsilon_{n-1} + i\varepsilon_n)$ is well defined and injective, and thus provides a complex projective representation for the whole conformal Grassmannian of 2-planes in $\mathbb{R}^{n+2}$. Indeed, let $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ be two representatives for the same 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$, then G and G' must differ by an element of the isotropy subgroup H_0, namely $G' = GH$ for some $H \in H_0$. But H has an expression as in (3.1), with $B \in SO(2)$, that is

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in \mathbb{R}$, so we have

$$\begin{aligned}G'(\varepsilon_{n-1} + i\varepsilon_n) =& GH(\varepsilon_{n-1} + i\varepsilon_n) = \\ =& G(\cos\theta\varepsilon_{n-1} + \sin\theta\varepsilon_n - i\sin\theta\varepsilon_{n-1} + i\cos\theta\varepsilon_n) = \\ =& e^{-i\theta}G(\varepsilon_{n-1} + i\varepsilon_n)\end{aligned}$$

which projects to the same complex projective class as $G(\varepsilon_{n-1} + i\varepsilon_n)$. As for injectivity, if $G(\varepsilon_{n-1} + i\varepsilon_n)$ and $G'(\varepsilon_{n-1} + i\varepsilon_n)$ project to the same projective class, then there exists $\rho > 0$ and $\theta \in \mathbb{R}$ such that

$$G'(\varepsilon_{n-1} + i\varepsilon_n) = \rho e^{i\theta} G(\varepsilon_{n-1} + i\varepsilon_n) = \rho G H(\varepsilon_{n-1} + i\varepsilon_n),$$

where

$$H = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

clearly belongs to H_0. So $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ are in fact the same 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$.
We will show that, as a matter of fact, $\mathcal{Q}_2(\mathbb{R}^{n+2})$ can be identified with an open submanifold of the projective quadric of homogeneous equation

$$-2x^0 x^{n+1} + \sum_{A=1}^{n} (x^A)^2 = 0. \tag{3.13}$$

As we have explained above, the image in $\mathbb{P}^{n+1}_{\mathbb{C}}$ of a 2-plane of $\mathcal{Q}_2(\mathbb{R}^{n+2})$ is the projective class of a complex vector of the form $G(\varepsilon_{n-1} + i\varepsilon_n)$, for some $G \in \text{Möb}(n)$. Now, the vector $\varepsilon_{n-1} + i\varepsilon_n$ trivially satisfies equation (3.13), and therefore lies in the quadric. Note that the quadric (3.13) is represented by the matrix

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

introduced in (1.1) and, since $G \in \text{Möb}(n)$,

$$\begin{aligned} {}^t[G(\varepsilon_{n-1} + i\varepsilon_n)] S [G(\varepsilon_{n-1} + i\varepsilon_n)] &= {}^t(\varepsilon_{n-1} + i\varepsilon_n) {}^t G S G (\varepsilon_{n-1} + i\varepsilon_n) = \\ &= {}^t(\varepsilon_{n-1} + i\varepsilon_n) S (\varepsilon_{n-1} + i\varepsilon_n) = 0. \end{aligned}$$

Therefore $G(\varepsilon_{n-1} + i\varepsilon_n)$ lies in the quadric (3.13).
However, the conformal Grassmannian doesn't cover the whole quadric. Indeed the points of the quadric coming from a 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$ are those that have a representative $v + iw \in \mathbb{C}^{n+2}$ such that, with respect to the Lorentzian product in $\mathbb{R}^{n+2}$, $\|v\|^2 = \|w\|^2 > 0$. This leaves out the projective classes represented by vectors $v + iw$ where v and w are isotropic and non zero. All such vectors lie in the quadric but cannot be obtained from ε_{n-1} or ε_n through a matrix of $\text{Möb}(n)$, because such matrices preserve the Lorentzian norm defined through the matrix S.

Chapter 4

The geometry of surfaces in Q_4

4.1 Some conformal invariants

Let $f : M \to Q_4$ be an oriented immersed surface. Assume that M has been given the structure of a Riemann surface starting from an assigned metric g and assume that f is conformal in the sense that the conformal structure that it induces on M coincides with that of M as a Riemann surface.
We let $e : U \subset M \to \text{Möb}(n)$ be a local first order frame along f, so that, according to (2.2),

$$\phi_0^\alpha = 0 \qquad 3 \le \alpha \le 4$$

and the isotropy subgroup is given by (2.3). Then

$$\phi_i^\alpha = h_{ij}^\alpha \phi_0^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha \qquad 1 \le i, j \le 2 \tag{4.1}$$

and we have the transformation laws (2.7), (2.8).

Starting from first order frames, we are now going to introduce a number of geometric invariants. We let L^α denote the **Hopf transform** of the symmetric matrix (h_{ij}^α), that is

$$L^\alpha = \frac{1}{2}(h_{11}^\alpha - h_{22}^\alpha) - i h_{12}^\alpha. \tag{4.2}$$

Setting

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and recalling that $e^{it} = \cos t + i \sin t$, using (2.8) we compute

$$\begin{aligned}
\widetilde{h}_{11}^\alpha &= r B_\alpha^\beta A_1^l (A_1^k h_{kl}^\beta - A_1^l y^\beta) = \\
&= r B_\alpha^\beta (A_1^1 A_1^1 h_{11}^\beta + A_1^1 A_1^2 h_{21}^\beta + A_1^2 A_1^1 h_{12}^\beta + A_1^2 A_1^2 h_{22}^\beta - \delta_{11} y^\beta) = \\
&= r B_\alpha^\beta (\cos^2 t\, h_{11}^\beta + \cos t \sin t\, h_{21}^\beta + \sin t \cos t\, h_{12}^\beta + \sin^2 t\, h_{22}^\beta - y^\beta) = \\
&= r B_\alpha^\beta (\cos^2 t\, h_{11}^\beta + 2 \cos t \sin t\, h_{12}^\beta + \sin^2 t\, h_{22}^\beta - y^\beta).
\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{h}^{\alpha}_{22} &= rB^{\beta}_{\alpha}A^{l}_{2}(A^{k}_{2}h^{\beta}_{kl} - A^{l}_{2}y^{\beta}) = \\ &= rB^{\beta}_{\alpha}(A^{1}_{2}A^{1}_{2}h^{\beta}_{11} + A^{1}_{2}A^{2}_{2}h^{\beta}_{21} + A^{2}_{2}A^{1}_{2}h^{\beta}_{12} + A^{2}_{2}A^{2}_{2}h^{\beta}_{22} - \delta_{22}y^{\beta}) = \\ &= rB^{\beta}_{\alpha}(\sin^2 t\, h^{\beta}_{11} - 2\sin t\cos t\, h^{\beta}_{12} + \cos^2 t\, h^{\beta}_{22} - y^{\beta}),\end{aligned}$$

$$\begin{aligned}\widetilde{h}^{\alpha}_{12} &= rB^{\beta}_{\alpha}A^{l}_{2}(A^{k}_{1}h^{\beta}_{kl} - A^{l}_{1}y^{\beta}) = \\ &= rB^{\beta}_{\alpha}(A^{1}_{2}A^{1}_{1}h^{\beta}_{11} + A^{1}_{2}A^{2}_{1}h^{\beta}_{21} + A^{2}_{2}A^{1}_{1}h^{\beta}_{12} + A^{2}_{2}A^{2}_{1}h^{\beta}_{22} - \delta_{12}y^{\beta}) = \\ &= rB^{\beta}_{\alpha}(-\sin t\cos t\, h^{\beta}_{11} - \sin^2 t\, h^{\beta}_{21} + \cos^2 t\, h^{\beta}_{12} + \cos t\sin t\, h^{\beta}_{22}) = \\ &= rB^{\beta}_{\alpha}\Big(-\sin t\cos t(h^{\beta}_{11} - h^{\beta}_{22}) + (-\sin^2 t + \cos^2 t)h^{\beta}_{12}\Big).\end{aligned}$$

From the above formulae, we can deduce the one expressing the transformation of L^{α} under a change of first order frames

$$\begin{aligned}\widetilde{L}^{\alpha} =& \frac{1}{2}(\widetilde{h}^{\alpha}_{11} - \widetilde{h}^{\alpha}_{22} - 2i\widetilde{h}^{\alpha}_{12}) = \\ =& \frac{1}{2}rB^{\beta}_{\alpha}\Big(\cos^2 t\, h^{\beta}_{11} + 2\cos t\sin t\, h^{\beta}_{12} + \sin^2 t\, h^{\beta}_{22} - y^{\beta} + \\ & - \sin^2 t\, h^{\beta}_{11} + 2\sin t\cos t\, h^{\beta}_{12} - \cos^2 t\, h^{\beta}_{22} + y^{\beta} + \\ & - 2i\big(-\sin t\cos t(h^{\beta}_{11} - h^{\beta}_{22}) + (-\sin^2 t + \cos^2 t)h^{\beta}_{12}\big)\Big) = \\ =& \frac{1}{2}rB^{\beta}_{\alpha}\Big((\cos^2 t - \sin^2 t)(h^{\beta}_{11} - h^{\beta}_{22}) + 4\cos t\sin t\, h^{\beta}_{12} + \\ & + 2i\sin t\cos t(h^{\beta}_{11} - h^{\beta}_{22}) - 2i(-\sin^2 t + \cos^2 t)h^{\beta}_{12}\Big) = \\ =& \frac{1}{2}rB^{\beta}_{\alpha}\Big((\cos^2 t - \sin^2 t + 2i\sin t\cos t)(h^{\beta}_{11} - h^{\beta}_{22}) + \\ & - 2i(\cos^2 t - \sin^2 t + 2i\sin t\cos t)h^{\beta}_{12}\Big) = \\ =& \frac{1}{2}rB^{\beta}_{\alpha}(\cos^2 t - \sin^2 t + 2i\sin t\cos t)(h^{\beta}_{11} - h^{\beta}_{22} - 2ih^{\beta}_{12}),\end{aligned}$$

that is

$$\widetilde{L}^{\alpha} = re^{2it}B^{\beta}_{\alpha}L^{\beta}. \tag{4.3}$$

Therefore, setting

$$B = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

$$\begin{aligned}\widetilde{L}^{3} \pm i\widetilde{L}^{4} &= re^{2it}\Big(B^{\beta}_{3}L^{\beta} \pm iB^{\beta}_{4}L^{\beta}\Big) = \\ &= re^{2it}\big(\cos s\, L^{3} + \sin s\, L^{4} \mp i\sin s\, L^{3} \pm i\cos s\, L^{4}\big) = \\ &= re^{2it}(\cos s \mp i\sin s)(L^{3} \pm iL^{4}),\end{aligned}$$

that is

$$\widetilde{L}^{3} \pm i\widetilde{L}^{4} = re^{2it}e^{\mp is}\big(L^{3} \pm iL^{4}\big) \tag{4.4}$$

Using (2.4) and (4.4), we see that the real, locally defined 2-forms

$$\omega_{\pm} = \left|L^{3} \pm iL^{4}\right|^{2} \phi^{1}_{0} \wedge \phi^{2}_{0}, \tag{4.5}$$

are globally defined and smooth.

Definition 4.1. *We will say that* $f : M \to Q_4$ *is* $+$ ***or*** $-$ ***isotropic*** *respectively if* $\omega_+ \equiv 0$ *or* $\omega_- \equiv 0$.

Note that, when f is at the same time $+$ and $-$ isotropic then

$$h^\alpha_{12} = 0, \qquad h^\alpha_{11} = h^\alpha_{22}.$$

Thus passing to a Darboux frame $h^\alpha_{ij} = 0$ for every α, i, j, and $f(M) \subseteq Q_2 \subset Q_4$ according to Proposition 2.1.

We underline the fact that the forms $\omega_\pm$ are invariant with respect to first order frames.

It is easy to see, using (2.7) and (2.8), that the 2-form

$$\mathrm{w} = \frac{1}{4}\left\{\sum_\alpha (h^\alpha_{11} - h^\alpha_{22})^2 + 4(h^\alpha_{12})^2\right\}\phi^1_0 \wedge \phi^2_0 = \left(|L^3|^2 + |L^4|^2\right)\phi^1_0 \wedge \phi^2_0 \tag{4.6}$$

is globally defined. In particular the form η is globally defined, which satisfies

$$\mathrm{w} = \omega_\pm \mp \eta. \tag{4.7}$$

We now identify η. A simple computation, using the definitions of w and $\omega_\pm$ yields

$$\eta = -i\left(L^3\overline{L^4} - L^4\overline{L^3}\right)\phi^1_0 \wedge \phi^2_0. \tag{4.8}$$

Expressing it in terms of the h^α_{ij}'s we obtain

$$-i\left(L^3\overline{L^4} - L^4\overline{L^3}\right) = h^3_{11}h^4_{12} - h^3_{22}h^4_{12} - h^3_{12}h^4_{11} + h^3_{12}h^4_{22}.$$

If we specialise to a Darboux frame e along f, since $h^\alpha_{11} + h^\alpha_{22} = 0$ we obtain

$$-i\left(L^3\overline{L^4} - L^4\overline{L^3}\right) = 2(h^3_{11}h^4_{12} - h^3_{12}h^4_{11}).$$

We go back to the bundle N introduced in section 2.2 through (2.25). The curvature K_N of this bundle is now given by

$$\Lambda^3_4 = \frac{1}{2}{}^\perp\tau^3_{4ij}\phi^i_0 \wedge \phi^j_0 = K_N\phi^1_0 \wedge \phi^2_0$$

and using (2.26) we deduce that

$$K_N = -i\left(L^3\overline{L}^4 - L^4\overline{L}^3\right) \tag{4.9}$$

or, in other words

$$d\phi^3_4 = K_N\phi^1_0 \wedge \phi^2_0 = \eta. \tag{4.10}$$

Using (4.7), (4.10) and the generalized Gauss-Bonnet theorem, having set

$$W(f) = \int_M \mathrm{w} \tag{4.11}$$

in the case of M compact, we obtain

Theorem 4.1. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface; then*

$$W(f) = \int_M \omega_{\pm} \mp 2\pi\chi(N) \tag{4.12}$$

where $\chi(N)$ is the Euler number of the bundle N introduced above.

The functional $W(f)$ defined in (4.11) for M compact or, more generally on compact domains of M, is called the **Willmore functional**.

Corollary 4.2. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface. Then*

$$\int_M \omega_{\pm} \geq \pm 2\pi\chi(N)$$

equality holding if and only if $f(M) = Q_2 \subset Q_4$.

Proof. An easy computation shows that

$$\mathrm{w} = \frac{1}{2}\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\phi^1_0 \wedge \phi^2_0,$$

so clearly $W(f) \geq 0$ and $W(f) = 0$ if and only if $f(M) = Q_2 \subset Q_4$ by Proposition 2.1. □

Suppose that M is compact and orientable; (4.12) implies that, if M is either $+$ or $-$ isotropic, then the values of $W(f)$ are "quantized".

4.2 The conformal Gauss map of a surface in Q_4

Our next goal is to give a geometric interpretation to $+$ and $-$ isotropic immersions. Towards this aim we introduce the conformal Gauss map. We let $\mathcal{Q}_2(\mathbb{R}^6)$ be the conformal Grassmannian of 2-planes introduced in Chapter 3. As we have seen, $\mathcal{Q}_2(\mathbb{R}^6)$ has the structure of a complex, Kähler-Lorentzian manifold with a local basis of $(1,0)$-type forms given by

$$\zeta^0 = \varsigma^*\Phi^0_3 + i\varsigma^*\Phi^0_4, \qquad \zeta^k = \varsigma^*\Phi^k_3 + i\varsigma^*\Phi^k_4, \qquad \zeta^3 = \varsigma^*\Phi^3_0 + i\varsigma^*\Phi^4_0, \tag{4.13}$$

where ς is any local section of $\widehat{\pi}$.
Given a Riemann surface M, a map $h : M \to \mathcal{Q}_2(\mathbb{R}^6)$ is respectively $\pm$ holomorphic if the pull-back of the forms ζ^0, ζ^k, ζ^3 in (4.13) is respectively of type $(1,0)$ or $(0,1)$

Definition 4.2. *Let $f : M \to Q_4$ be an immersed oriented surface and let e be a (local) Darboux frame along f. The **conformal Gauss map** $\gamma_f : M \to \mathcal{Q}_2(\mathbb{R}^6)$ is defined by setting*

$$\gamma_f : p \mapsto [e_3, e_4]_p$$

where with $[e_3, e_4]_p$ we denote the oriented 2-plane generated by the vectors e_3, e_4 at the point p.

Note that, because of the transformation law (2.25) under a change of Darboux frames, γ_f is globally well defined, and the orientation of the 2-plane $[e_3, e_4]$ is also preserved.

4.3 Isotropic surfaces and the holomorphicity of the conformal Gauss map

We introduce some notation. We recall, see (2.13) and (2.14), that under a Darboux frame e

$$\phi^0_\alpha = p^\alpha_k \phi^k_0 \qquad \text{with} \qquad p^\alpha_k = \frac{1}{2} h^\alpha_{iik}. \tag{4.14}$$

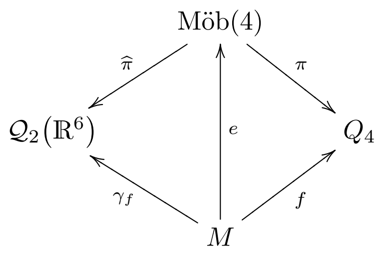
We define

$$k^\alpha = \frac{1}{2}(p^\alpha_1 - i p^\alpha_2). \tag{4.15}$$

We are now ready to prove the next

Theorem 4.3. *Let $f : M \to Q_4$ be an immersed oriented Riemann surface. Then f is $\pm$ isotropic if and only if $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is $\mp$ holomorphic.*

Proof. We begin by observing that if e is any Darboux frame along f, then the following diagram is commutative.



This fact enables us to compute in a simple way $\gamma_f^*\zeta^0$, $\gamma_f^*\zeta^k$, $\gamma_f^*\zeta^3$. Indeed, setting

$$\theta^{0,\alpha} = \varsigma^*\Phi^\alpha_0, \qquad \theta^{\alpha,0} = \varsigma^*\Phi^0_\alpha, \qquad \theta^{\alpha,i} = \varsigma^*\Phi^i_\alpha \tag{4.16}$$

and using (4.14), (4.15) and (4.1) we have:

$$\begin{cases} \gamma_f^*\theta^{\alpha,0} = p^\alpha_k \phi^k_0 \\ \gamma_f^*\theta^{\alpha,i} = -h^\alpha_{ik}\phi^k_0 \\ \gamma_f^*\theta^{0,\alpha} = 0. \end{cases} \tag{4.17}$$

In order to see this, we observe that

$$\gamma_f^*\varsigma^*\Phi = (\widehat{\pi} \circ e)^*\varsigma^*\Phi = e^*(\varsigma \circ \widehat{\pi})^*\Phi.$$

And since $\widehat{\pi} \circ (\varsigma \circ \widehat{\pi}) = \widehat{\pi}$, then for every g in the inverse image through $\widehat{\pi}$ of the domain of definition of ς, it holds

$$\varsigma(\widehat{\pi}(g)) = g\widetilde{K}(g),$$

where $\widetilde{K}$ is an H_0-valued function. Therefore

$$(\varsigma \circ \widehat{\pi})^*\Phi_g = \widetilde{K}(g)^{-1}g^{-1}dg\widetilde{K}(g) + \widetilde{K}(g)^{-1}d\widetilde{K}_g,$$

and since $\widetilde{K}(g)^{-1}d\widetilde{K}_g$ has values in the Lie algebra of H_0, we deduce that

$$(\varsigma \circ \widehat{\pi})^* \Phi^{\alpha}_{0\,g_0} = \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^{\alpha}_{0}$$
$$(\varsigma \circ \widehat{\pi})^* \Phi^{0}_{\alpha\,g_0} = \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^{0}_{\alpha}$$
$$(\varsigma \circ \widehat{\pi})^* \Phi^{\alpha}_{i\,g_0} = \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^{\alpha}_{i}.$$

If for a fixed $\widetilde{g}$ we replace the section ς with the section $\varsigma\widetilde{K}(\widetilde{g})^{-1}$ obtained multiplying ς by a constant matrix, we will have defined a new section $\widetilde{\varsigma}$ which satisfies, at the point $\widetilde{g}$ (and in general only there), the equality $\widetilde{\varsigma}(\widehat{\pi}(\widetilde{g})) = \widetilde{g}$, and therefore

$$(\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^{\alpha}_{0\,\widetilde{g}} = \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^{\alpha}_{0} = \Phi^{\alpha}_{0\,\widetilde{g}}$$
$$(\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^{0}_{\alpha\widetilde{g}} = \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^{0}_{\alpha} = \Phi^{0}_{\alpha\widetilde{g}}$$
$$(\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^{\alpha}_{i\,\widetilde{g}} = \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^{\alpha}_{i} = \Phi^{\alpha}_{i\,\widetilde{g}}.$$

Now let us fix $p_0 \in M$ and set $\widetilde{g} = e(p_0)$. Given a section ς defined in a neighbourhood of $\gamma_f(p_0)$, and possibly replacing it with the section $\varsigma\widetilde{K}(e(p_0))^{-1}$, which we shall still call ς, we have at the point p_0

$$\varsigma(\widehat{\pi}(e(p_0))) = e(p_0),$$

and thus

$$\big(\gamma_f^* \varsigma^* \Phi^{\alpha}_{0}\big)_{p_0} = \big(e^*(\varsigma \circ \widehat{\pi})^* \Phi^{\alpha}_{0}\big)_{p_0} = \big(e^* \Phi^{\alpha}_{0}\big)_{p_0} = \phi^{\alpha}_{0\,p_0} = 0$$
$$\big(\gamma_f^* \varsigma^* \Phi^{0}_{\alpha}\big)_{p_0} = \phi^{0}_{\alpha p_0} = p^{\alpha}_{k}(p_0)\phi^{k}_{0 p_0}$$
$$\big(\gamma_f^* \varsigma^* \Phi^{\alpha}_{i}\big)_{p_0} = \phi^{\alpha}_{i\,p_0} = h^{\alpha}_{ik}(p_0)\phi^{k}_{0 p_0}.$$

Hence, setting $\varphi = \phi^1_0 + i\phi^2_0$ and observing that, if α_k, β_k are real-valued functions, one has

$$(\alpha_k + i\beta_k)\phi^k_0 = \left\{\frac{\alpha_1 + \beta_2}{2} + i\frac{\beta_1 - \alpha_2}{2}\right\}\varphi + \left\{\frac{\alpha_1 - \beta_2}{2} + i\frac{\beta_1 + \alpha_2}{2}\right\}\overline{\varphi}, \quad (4.18)$$

we get, at the point p_0,

$$\begin{aligned}\gamma_f^* \zeta^0 =& \big(p^3_k + ip^4_k\big)\phi^k_0 = \\ =& \frac{1}{2}\big\{\big(p^3_1 + p^4_2\big) + i\big(p^4_1 - p^3_2\big)\big\}\varphi + \frac{1}{2}\big\{\big(p^3_1 - p^4_2\big) + i\big(p^4_1 + p^3_2\big)\big\}\overline{\varphi} = \\ =& \big(k^3 + ik^4\big)\varphi + \Big(\overline{k^3 - ik^4}\Big)\overline{\varphi}\end{aligned}$$

and similarly, using (4.2),

$$\begin{aligned}\gamma_f^* \zeta^1 &= -\frac{1}{2}\big\{\big(h^3_{11} - ih^3_{12}\big) + i\big(h^4_{11} - ih^4_{12}\big)\big\}\varphi - \frac{1}{2}\big\{\big(h^3_{11} + ih^3_{12}\big) + i\big(h^4_{11} + ih^4_{12}\big)\big\}\overline{\varphi} = \\ &= -\frac{1}{2}\big(L^3 + iL^4\big)\varphi - \frac{1}{2}\Big(\overline{L^3 - iL^4}\Big)\overline{\varphi}\end{aligned}$$

and

$$\begin{aligned}\gamma_f^*\zeta^2 =& -\frac{1}{2}\{(h_{21}^3 - ih_{22}^3) + i(h_{21}^4 - ih_{22}^4)\}\varphi - \frac{1}{2}\{(h_{21}^3 + ih_{22}^3) + i(h_{21}^4 + ih_{22}^4)\}\overline{\varphi} = \\ =& -\frac{1}{2}\{(h_{21}^3 + ih_{11}^3) + i(h_{21}^4 + ih_{11}^4)\}\varphi - \frac{1}{2}\{(h_{21}^3 - ih_{11}^3) + i(h_{21}^4 - ih_{11}^4)\}\overline{\varphi} = \\ =& -\frac{i}{2}(L^3 + iL^4)\varphi + \frac{i}{2}\left(\overline{L^3 - iL^4}\right)\overline{\varphi}.\end{aligned}$$

Namely, at the point p_0,

$$\begin{aligned}\gamma_f^*\zeta^0 &= (k^3 + ik^4)\varphi + \left(\overline{k^3 - ik^4}\right)\overline{\varphi} \\ \gamma_f^*\zeta^1 &= -\frac{1}{2}(L^3 + iL^4)\varphi - \frac{1}{2}\left(\overline{L^3 - iL^4}\right)\overline{\varphi} \\ \gamma_f^*\zeta^2 &= -\frac{i}{2}(L^3 + iL^4)\varphi + \frac{i}{2}\left(\overline{L^3 - iL^4}\right)\overline{\varphi} \\ \gamma_f^*\zeta^3 &= 0.\end{aligned}$$

It is therefore clear, using (4.5), that if γ_f is $\mp$ holomorphic then f is $\pm$ isotropic. To prove the converse we need to show that

$$L^3 \pm iL^4 = 0 \qquad \text{implies} \qquad k^3 \pm ik^4 = 0. \tag{4.19}$$

Towards this aim we differentiate the first of (4.19) and we use (2.12) to perform the computations. Note that, since we are using Darboux frames along f,

$$L^\alpha = h_{11}^\alpha - ih_{12}^\alpha.$$

A simple check yields

$$\begin{aligned}dh_{11}^3 =& h_{11k}^3\phi_0^k + 2h_{12}^3\phi_1^2 + h_{11}^4\phi_3^4 - h_{11}^3\phi_0^0 - p_k^3\phi_0^k \\ dh_{12}^3 =& h_{12k}^3\phi_0^k - 2h_{11}^3\phi_1^2 + h_{12}^4\phi_3^4 - h_{12}^3\phi_0^0 \\ dh_{11}^4 =& h_{11k}^4\phi_0^k + 2h_{12}^4\phi_1^2 - h_{11}^3\phi_3^4 - h_{11}^4\phi_0^0 - p_k^4\phi_0^k \\ dh_{12}^4 =& h_{12k}^4\phi_0^k - 2h_{11}^4\phi_1^2 - h_{12}^3\phi_3^4 - h_{12}^4\phi_0^0,\end{aligned}$$

and we can compute

$$\begin{aligned}d(L^3 \pm iL^4) =& d((h_{11}^3 \pm ih_{11}^4) - i(h_{12}^3 \pm ih_{12}^4)) = \\ =& dh_{11}^3 \pm idh_{11}^4 - idh_{12}^3 \pm dh_{12}^4 = \\ =& h_{11k}^3\phi_0^k + 2h_{12}^3\phi_1^2 + h_{11}^4\phi_3^4 - h_{11}^3\phi_0^0 - p_k^3\phi_0^k + \\ & \pm i(h_{11k}^4\phi_0^k + 2h_{12}^4\phi_1^2 - h_{11}^3\phi_3^4 - h_{11}^4\phi_0^0 - p_k^4\phi_0^k) + \\ & - i(h_{12k}^3\phi_0^k - 2h_{11}^3\phi_1^2 + h_{12}^4\phi_3^4 - h_{12}^3\phi_0^0) + \\ & \pm (h_{12k}^4\phi_0^k - 2h_{11}^4\phi_1^2 - h_{12}^3\phi_3^4 - h_{12}^4\phi_0^0) = \\ =& (h_{11k}^3 - p_k^3 \pm h_{12k}^4)\phi_0^k \pm i(h_{11k}^4 - p_k^4 \mp h_{12k}^3)\phi_0^k + \\ & + 2h_{12}^3\phi_1^2 + h_{11}^4\phi_3^4 - h_{11}^3\phi_0^0 \pm 2ih_{12}^4\phi_1^2 \mp ih_{11}^3\phi_3^4 \mp ih_{11}^4\phi_0^0 + \\ & + i2h_{11}^3\phi_1^2 - ih_{12}^4\phi_3^4 + ih_{12}^3\phi_0^0 \mp 2h_{11}^4\phi_1^2 \mp h_{12}^3\phi_3^4 \mp h_{12}^4\phi_0^0 =\end{aligned}$$

$$
\begin{aligned}
=&\big(h^3_{11k}-p^3_k\pm h^4_{12k}\big)\phi^k_0\pm i\big(h^4_{11k}-p^4_k\mp h^3_{12k}\big)\phi^k_0+\\
&+2\big(h^3_{12}\pm ih^4_{12}+ih^3_{11}\mp h^4_{11}\big)\phi^2_1+\big(h^4_{11}\mp ih^3_{11}-ih^4_{12}\mp h^3_{12}\big)\phi^4_3+\\
&+\big(-h^3_{11}\mp ih^4_{11}+ih^3_{12}\mp h^4_{12}\big)\phi^0_0=\\
=&\big(h^3_{11k}-p^3_k\pm h^4_{12k}\big)\phi^k_0\pm i\big(h^4_{11k}-p^4_k\mp h^3_{12k}\big)\phi^k_0+\\
&+2i\big(h^3_{11}-ih^3_{12}\pm ih^4_{11}\pm h^4_{12}\big)\phi^2_1\mp i\big(h^3_{11}-ih^3_{12}\pm ih^4_{11}\pm h^4_{12}\big)\phi^4_3+\\
&-\big(h^3_{11}-ih^3_{12}\pm ih^4_{11}\pm h^4_{12}\big)\phi^0_0=\\
=&(h^3_{11k}-p^3_k\pm h^4_{12k})\phi^k_0\pm i(h^4_{11k}-p^4_k\mp h^3_{12k})\phi^k_0+\\
&+(L^3\pm iL^4)\big[i(2\phi^2_1\mp\phi^4_3)-\phi^0_0\big],
\end{aligned}
$$

namely

$$
\begin{aligned}
d\big(L^3\pm iL^4\big)=&(h^3_{11k}-p^3_k\pm h^4_{12k})\phi^k_0\pm i(h^4_{11k}-p^4_k\mp h^3_{12k})\phi^k_0+\\
&+(L^3\pm iL^4)\big[i(2\phi^2_1\mp\phi^4_3)-\phi^0_0\big].
\end{aligned}
\tag{4.20}
$$

Hence, if the first of (4.19) holds, we have

$$
0=\big\{(h^3_{11k}-p^3_k\pm h^4_{12k})\pm i(h^4_{11k}-p^4_k\mp h^3_{12k})\big\}\phi^k_0,
$$

that is,

$$
h^3_{11k}-p^3_k\pm h^4_{12k}=0=h^4_{11k}-p^4_k\mp h^3_{12k}.
$$

Therefore, using the symmetries of the h^α_{ijk}'s and the definitions of p^α_k,

$$
0=(h^3_{111}-p^3_1\pm h^4_{121})\mp(h^4_{112}-p^4_2\mp h^3_{122})=h^3_{111}+h^3_{122}-p^3_1\pm p^4_2=p^3_1\pm p^4_2
$$

and similarly

$$
0=(h^3_{112}-p^3_2\pm h^4_{122})\pm(h^4_{111}-p^4_1\mp h^3_{121})=\pm(h^4_{122}+h^4_{111})-p^3_2\mp p^4_1=-p^3_2\pm p^4_1.
$$

Hence

$$
2(k^3\pm ik^4)=\big(p^3_1-ip^3_2\pm i(p^4_1-ip^4_2)\big)=(p^3_1\pm p^4_2)+i(-p^3_2\pm p^4_1)=0.
$$

□

Let us now further analyze the quantities k^α as defined in (4.15). It is not hard to show that under a change of Darboux frames

$$
\widetilde{k}^3\pm i\widetilde{k}^4=r^2e^{it}e^{\mp is}\Big\{k^3\pm ik^4+\frac{1}{2}(x^1+ix^2)(L^3\pm iL^4)\Big\}.
\tag{4.21}
$$

Indeed

$$
\begin{aligned}
2(\widetilde{k}^3\pm i\widetilde{k}^4)=&(\widetilde{p}^3_1-i\widetilde{p}^3_2)\pm i(\widetilde{p}^4_1-i\widetilde{p}^4_2)=\\
=&r^2B^\beta_3\Big(A^k_1\Big(p^\beta_k+h^\beta_{kj}x^j\Big)-iA^k_2\Big(p^\beta_k+h^\beta_{kj}x^j\Big)\Big)+\\
&\pm ir^2B^\beta_4\Big(A^k_1\Big(p^\beta_k+h^\beta_{kj}x^j\Big)-iA^k_2\Big(p^\beta_k+h^\beta_{kj}x^j\Big)\Big)=\\
=&r^2\Big(B^\beta_3\pm iB^\beta_4\Big)\big(A^k_1-iA^k_2\big)\Big(p^\beta_k+h^\beta_{kj}x^j\Big)=
\end{aligned}
$$

$$\begin{aligned}
=&r^2\big(B_3^3 \pm iB_4^3\big)\big(A_1^1 - iA_2^1\big)\big(p_1^3 + h_{1j}^3x^j\big)+\\
&+ r^2\big(B_3^3 \pm iB_4^3\big)\big(A_1^2 - iA_2^2\big)\big(p_2^3 + h_{2j}^3x^j\big)+\\
&+ r^2\big(B_3^4 \pm iB_4^4\big)\big(A_1^1 - iA_2^1\big)\big(p_1^4 + h_{4j}^4x^j\big)+\\
&+ r^2\big(B_3^4 \pm iB_4^4\big)\big(A_1^2 - iA_2^2\big)\big(p_2^4 + h_{2j}^4x^j\big) =\\
=&r^2(\cos s \mp i\sin s)(\cos t + i\sin t)\big(p_1^3 + h_{1j}^3x^j\big)+\\
&+ r^2(\cos s \mp i\sin s)(\sin t - i\cos t)\big(p_2^3 + h_{2j}^3x^j\big)+\\
&+ r^2(\sin s \pm i\cos s)(\cos t + i\sin t)\big(p_1^4 + h_{1j}^4x^j\big)+\\
&+ r^2(\sin s \pm i\cos s)(\sin t - i\cos t)\big(p_2^4 + h_{2j}^4x^j\big) =\\
=&r^2\Big(e^{\mp is}e^{it}\big(p_1^3 + h_{1j}^3x^j\big) - ie^{\mp is}e^{it}\big(p_2^3 + h_{2j}^3x^j\big)+\\
&\pm ie^{\mp is}e^{it}\big(p_1^4 + h_{4j}^4x^j\big) \pm ie^{\mp is}\big(-ie^{it}\big)\big(p_2^4 + h_{2j}^4x^j\big)\Big) =\\
=&r^2e^{\mp is}e^{it}\Big(\big(p_1^3 - ip_2^3\big) \pm i\big(p_1^4 - ip_2^4\big)+\\
&+ \big(h_{1j}^3 - ih_{2j}^3\big) \pm i\big(h_{1j}^4 - ih_{2j}^4\big)x^j\Big) =\\
=&r^2e^{\mp is}e^{it}\Big(2\big(k^3 \pm ik^4\big) + \big(\big(h_{11}^3 - ih_{21}^3\big) \pm i\big(h_{11}^4 - ih_{21}^4\big)\big)x^1+\\
&+ \big(\big(h_{12}^3 - ih_{22}^3\big) \pm i\big(h_{12}^4 - ih_{22}^4\big)\big)x^2\Big) =\\
=&r^2e^{\mp is}e^{it}\big(2\big(k^3 \pm ik^4\big) + \big(L^3 \pm iL^4\big)x^1 + \big(iL^3 \mp L^4\big)x^2\big) =\\
=&r^2e^{it}e^{\mp is}\big\{2\big(k^3 \pm ik^4\big) + (L^3 \pm iL^4)(x^1 + ix^2)\big\}.
\end{aligned}$$

For $p > 2$, consider the condition

$$\exists\gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \big|k^3 \pm ik^4\big| \le \gamma\big|L^3 \pm iL^4\big| \qquad \text{a.e.} \tag{4.22}$$

Of course we have to check that this condition actually makes sense, since the quantities involved strongly depend on the choice of the Darboux frame. To this end we use (4.21) and (4.4) and observe that if condition (4.22) holds for some Darboux frame, then for any other Darboux frame we can estimate

$$\begin{aligned}
\Big|\widetilde{k}^3 \pm i\widetilde{k}^4\Big| &= r^2\Big|k^3 \pm ik^4 + \frac{1}{2}(x^1 + ix^2)(L^3 \pm iL^4)\Big| \le\\
&\le r^2\Big(\gamma + \frac{1}{2}\big|x^1 + ix^2\big|\Big)\big|L^3 \pm iL^4\big| = r\Big(\gamma + \frac{1}{2}\big|x^1 + ix^2\big|\Big)\Big|\widetilde{L}^3 \pm i\widetilde{L}^4\Big|.
\end{aligned}$$

Therefore condition (4.22) still holds provided we replace γ with another suitable function in $L^p_{\mathrm{loc}}(M)$. We recall the following result by Eschenburg and Tribuzy (see [8]).

Lemma 4.4. *Let $U \subset \mathbb{C}$ be an open domain containing 0 and $f : U \to \mathbb{C}^n$ a smooth function satisfying the Cauchy-Riemann condition*

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \le \gamma|f| \tag{4.23}$$

for some L^p-function γ with $p > 2$. Then, in a neighbourhood of the origin, either $f \equiv 0$ or

$$f(z) = z^k f_0(z)$$

for some nonnegative integer k and a continuous function f_0 such that $f_0(0) \ne 0$.

This result prompts us to set the following

Definition 4.3. *Let M be a Riemann surface and $E \to M$ a complex vector bundle. A smooth section s of E is said to be of* ***analytic type*** *if it either vanishes identically or near any zero p, we have*

$$s = z^k s_0$$

for some positive integer k and some continuous section s_0 with $s_0(p) \neq 0$, where z is any holomorphic chart centered at p.

Sections of analytic type, and particularly functions of analityc type, are quite useful in many different settings, and have therefore been studied thoroughly (e.g. see [2]).
In order to prove Lemma 4.4, we need the following

Lemma 4.5. *Let $g : U \setminus \{0\} \to \mathbb{C}^n$ be a C^1-function which is bounded near 0 and satisfies*

$$\left|\frac{\partial g}{\partial \bar{z}}\right| \leq \gamma |g|$$

for some L^p-function γ on U with $p > 2$. Then $\lim_{z\to 0} g(z)$ exists, and for a suitably small closed disc $D \subset U$ of radius R centered at 0, the L^q-norms on D and its boundary ∂D are related by

$$\frac{\|g\|_{q,D}}{\|g\|_{q,\partial D}} \leq C R^{\frac{1}{p}}$$

with $q^{-1} + p^{-1} = 1$ and for a constant C depending only on $\|\gamma\|_p$.

Proof of Lemma 4.5. Let $0 \neq \zeta \in \mathrm{Int}(D)$ and consider the 1-form

$$\eta = \frac{g(z) - g(\zeta)}{z - \zeta} dz$$

on $D_\varepsilon = D \setminus (B_\varepsilon(0) \cup B_\varepsilon(\zeta))$, ε small. Applying Stokes' theorem we get

$$\int_{D_\varepsilon} d\eta = \int_{D_\varepsilon} \frac{\partial g}{\partial \bar{z}} (z - \zeta)^{-1}\, d\bar{z} \wedge dz = \int_{\partial D} \eta - \int_{\partial B_\varepsilon(\zeta)} \eta - \int_{\partial B_\varepsilon(0)} \eta,$$

and

$$\int_{\partial D} \eta = \int_{\partial D} \frac{g(z)}{z - \zeta}\, dz - 2\pi i\, g(\zeta),$$

$$\int_{\partial B_\varepsilon(\zeta)} \eta = 0, \qquad \int_{\partial B_\varepsilon(0)} \eta \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, letting $\varepsilon \to 0$, we have

$$2\pi i\, g(\zeta) = \int_{\partial D} \frac{g(z)}{z - \zeta}\, dz - \int_D \frac{\partial g}{\partial \bar{z}} (z - \zeta)^{-1}\, d\bar{z} \wedge dz \tag{4.24}$$

for every $\zeta \in \mathrm{Int}(D) \setminus \{0\}$.
Since $\frac{\partial g}{\partial \bar{z}} \in L^p$, we can show that the right-hand side of (4.24) has a limit for

$\zeta \to 0$. In order to do this, we will show that, for $h \in L^p$, the function Ph defined by

$$Ph(\zeta) = \int_{\mathbb{C}} h(z) \left| \frac{1}{z-\zeta} - \frac{1}{z} \right| |d\bar{z} \wedge dz|$$

is continuous at $\zeta = 0$. Let us first observe that the integral defining Ph converges, since $h \in L^p$ and

$$\frac{1}{z-\zeta} - \frac{1}{z} = \frac{\zeta}{z(z-\zeta)} \in L^q.$$

Indeed $1 < q < 2$, so the integral of $|z(z-\zeta)|^{-q}$ converges at 0, at ζ and at infinity.
Therefore, by Holder's inequality,

$$|Ph(\zeta)| \leq |\zeta| \|h\|_p \left\| \frac{1}{z(z-\zeta)} \right\|_q$$

and, performing the change of variable $z = \zeta Z$, we have

$$\begin{aligned} \left\| \frac{1}{z(z-\zeta)} \right\|_q &= \left(\int_{\mathbb{C}} |z(z-\zeta)|^{-q} |d\bar{z} \wedge dz| \right)^{\frac{1}{q}} = \\ &= |\zeta|^{\frac{2-2q}{q}} \left(\int_{\mathbb{C}} |z(z-1)|^{-q} |d\bar{z} \wedge dz| \right)^{\frac{1}{q}} = |\zeta|^{-\frac{2}{p}} \left\| \frac{1}{z(z-1)} \right\|_q . \end{aligned}$$

Now, setting $K_p = \left\| \frac{1}{z(z-1)} \right\|_q$ and observing that K_p depends only on p, we get

$$|Ph(\zeta)| \leq K_p \|h\|_p |\zeta|^{1-\frac{2}{p}}$$

and so we have

$$\begin{aligned} \left| \int_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z-\zeta} \, d\bar{z} \wedge dz - \int_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z} \, d\bar{z} \wedge dz \right| &\leq \int_D \left| \frac{\partial g}{\partial \bar{z}} \right| \left| \frac{1}{z-\zeta} - \frac{1}{z} \right| |d\bar{z} \wedge dz| = \\ &= |Ph(\zeta)| \leq K_p \|h\|_p |\zeta|^{1-\frac{2}{p}} \end{aligned}$$

where h is defined as $\left| \frac{\partial g}{\partial \bar{z}} \right|$ on D, and is zero elsewhere.
This shows that $g(z)$ has limit as $z \to 0$.
Let us now estimate the L^q-norm of g. Since $q \leq 2$, the function

$$z \mapsto g(z)(z-\zeta)^{-1}$$

is L^q on D. Hence we can use Holder's inequality on the second term in the right-hand side of (4.24) and get

$$\begin{aligned} \left| \int_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z-\zeta} \, d\bar{z} \wedge dz \right| &\leq \int_D \gamma \, |g(z)| |z-\zeta|^{-1} |d\bar{z} \wedge dz| \leq \\ &\leq \|\gamma\|_p \left(\int_D |g(z)|^q |z-\zeta|^{-q} |d\bar{z} \wedge dz| \right)^{\frac{1}{q}} . \end{aligned}$$

Elevating (4.24) to the q (a convex operation, because $q > 1$),

$$\begin{aligned}(2\pi)^q|g(\zeta)|^q &= \left|\int_{\partial D} g(z)\,(z-\zeta)^{-1}\,dz - \int_D \frac{\partial g}{\partial \bar z}\,\frac{1}{z-\zeta}\,d\bar z\wedge dz\right|^q \le \\ &\le \left(\int_{\partial D}|g(z)|\,|z-\zeta|^{-1}\,|dz| + \|\gamma\|_p\left(\int_D |g(z)|^q|z-\zeta|^{-q}|d\bar z\wedge dz|\right)^{\frac{1}{q}}\right)^q \le \\ &\le 2^q\left(\frac{1}{2}\left(\int_{\partial D}|g(z)|\,|z-\zeta|^{-1}\,|dz|\right)^q + \frac{1}{2}\|\gamma\|_p^q\int_D |g(z)|^q|z-\zeta|^{-q}|d\bar z\wedge dz|\right).\end{aligned}$$

We use Holder's inequality to estimate

$$\int_{\partial D}|g(z)|\,|z-\zeta|^{-1}\,|dz| \le (2\pi R)^{\frac{1}{p}}\left(\int_{\partial D}|g(z)|^q|z-\zeta|^{-q}\,|dz|\right)^{\frac{1}{q}}$$

so that we have

$$\begin{aligned}(2\pi)^q|g(\zeta)|^q \le &(4\pi R)^{q-1}\int_{\partial D}|g(z)|^q|z-\zeta|^{-q}\,|dz| + \\ &+ 2^{q-1}\|\gamma\|_p^q\int_D |g(z)|^q|z-\zeta|^{-q}|d\bar z\wedge dz|.\end{aligned}$$

Integrating with respect to ζ over D, we get

$$\begin{aligned}(2\pi)^q\|g\|_{q,D}^q \le &(4\pi R)^{q-1}\int_D\left[\int_{\partial D}|g(z)|^q|z-\zeta|^{-q}\,|dz|\right]\left|d\bar\zeta\wedge d\zeta\right| + \\ &+ 2^{q-1}\|\gamma\|_p^q\int_D\left[\int_D |g(z)|^q|z-\zeta|^{-q}|d\bar z\wedge dz|\right]\left|d\bar\zeta\wedge d\zeta\right|.\end{aligned}$$

Applying Fubini's theorem and setting

$$\alpha = \sup_{z\in D}\int_D |z-\zeta|^{-q}\left|d\bar\zeta\wedge d\zeta\right|,$$

we can write

$$(2\pi)^q\|g\|_{q,D}^q \le (4\pi R)^{q-1}\alpha\|g\|_{q,\partial D}^q + 2^{q-1}\|\gamma\|_p^q\alpha\|g\|_{q,D}^q.$$

But $\alpha \to 0$ as $R \to 0$, so that, up to choosing R sufficiently small, we can assume

$$(2\pi)^q - 2^{q-1}\|\gamma\|_p^q\alpha > 0.$$

Therefore

$$\frac{\|g\|_{q,D}^q}{\|g\|_{q,\partial D}^q} \le \frac{(4\pi)^{q-1}\alpha}{(2\pi)^q - 2^{q-1}\|\gamma\|_p^q\alpha}R^{q-1},$$

which proves the claim. □

Proof of Lemma 4.4. Let us assume (4.23) is satisfied and, without loss of generality, let us take $z_0 = 0$. Assume $f \not\equiv 0$ in a neighbourhood of 0, i.e. for every disc D of radius R centered at 0 there exists $z_1 \in D$ such that $f(z_1) \neq 0$, and set $r := |z_1| < R$. First we shall show that in this case we cannot have

$$f(z) = o(|z|^k)$$

for every $k \in \mathbb{N}$. To this end, we set

$$g_k = \frac{f}{z^k}$$

and observe that, since $|f(z_1)| > 0$, there exists a constant a, independent of k, R and r, such that

$$\|g_k\|_{q,D} \geq a r^{-k}.$$

On the other hand

$$\|g_k\|_{q,\partial D} \leq b R^{-k}.$$

for a suitable constant b independent of k, R and r. Hence we have

$$\frac{\|g_k\|_{q,D}}{\|g_k\|_{q,\partial D}} \geq \frac{a}{b}\left(\frac{R}{r}\right)^k,$$

that goes to infinity as $k \to +\infty$. Since g_k obviously satisfies the hypotheses of Lemma 4.5 with the function γ of (4.23) depending only on f and not on k, the quotient $\frac{\|g_k\|_{q,D}}{\|g_k\|_{q,\partial D}}$ must be bounded as $k \to +\infty$, a contradiction.
Now let k be the degree of the first nonzero Taylor polynomial of f at 0 and set $g = f/z^k$. Then, by Lemma 4.5, g has a limit for $z \to 0$ and

$$\lim_{z\to 0} g(z) = a \neq 0$$

by the assumption on k. Hence we can write $g(z) = a + h(z)$ for some function $h = o(1)$ as $z \to 0$, that is

$$f(z) = z^k(a + h(z)).$$

By assumption, $a + h(z) \neq 0$ on a suitable neighbourhood of 0, so we have the desired claim. □

This result has many applications in this context, starting with the following

Proposition 4.6. *Let* $f : M \to Q_4$ *be an immersion satisfying* (4.22)*. Then either* γ_f *is* $\pm$ *holomorphic or the set* $\mathcal{I}_\mp$ *of* $\mp$ *isotropic points of* M *is discrete.*

Proof. From (4.20) and (4.18) we obtain

$$d\big(L^3 \pm iL^4\big) = (L^3 \pm iL^4)\big[i(2\phi_1^2 \mp \phi_3^4) - \phi_0^0\big] + (\zeta^3 \pm i\zeta^4)\varphi + (k^3 \pm ik^4)\overline{\varphi} \quad (4.25)$$

where we have set

$$\zeta^\alpha = \overline{k^\alpha} - i(h^\alpha_{112} - ih^\alpha_{122}). \quad (4.26)$$

Indeed

$$\begin{aligned}
&\big(h^3_{11k} - p^3_k \pm h^4_{12k} + i(\pm h^4_{11k} \mp p^4_k - h^3_{12k})\big)\phi_0^k = \\
&= \frac{1}{2}\Big\{h^3_{111} - p^3_1 \pm h^4_{121} \pm h^4_{112} \mp p^4_2 - h^3_{122} + \\
&\qquad \pm ih^4_{111} \mp ip^4_1 - ih^3_{121} - ih^3_{112} + ip^3_2 \mp ih^4_{122}\Big\}\varphi + \\
&+ \frac{1}{2}\Big\{h^3_{111} - p^3_1 \pm h^4_{121} \mp h^4_{112} \pm p^4_2 + h^3_{122} + \\
&\qquad \pm ih^4_{111} \mp ip^4_1 - ih^3_{121} + ih^3_{112} - ip^3_2 \pm ih^4_{122}\Big\}\overline{\varphi},
\end{aligned}$$

and we have

$$\begin{aligned}
&h^3_{111} - p^3_1 \pm h^4_{121} \pm h^4_{112} \mp p^4_2 - h^3_{122} + \\
&\pm i h^4_{111} \mp i p^4_1 - i h^3_{121} - i h^3_{112} + i p^3_2 \mp i h^4_{122} =
\end{aligned}$$

$$\begin{aligned}
=& - i\big(h^3_{112} - i h^3_{122}\big) + h^3_{111} - p^3_1 \pm h^4_{121} \pm h^4_{112} + \\
&\mp p^4_2 \pm i h^4_{111} \mp i p^4_1 + i p^3_2 \mp i h^4_{122} - i h^3_{121} = \\
=& - i\big(2h^3_{112} - i h^3_{122}\big) + \frac{1}{2} h^3_{111} - \frac{1}{2} h^3_{221} \pm h^4_{121} + \\
&\pm h^4_{112} \mp p^4_2 \pm i h^4_{111} \mp i p^4_1 + i p^3_2 \mp i h^4_{122} = \\
=& - i\big(2h^3_{112} - i h^3_{122}\big) + p^3_1 - h^3_{221} \pm h^4_{121} + \\
&\pm h^4_{112} \mp p^4_2 \pm i h^4_{111} \mp i p^4_1 + i p^3_2 \mp i h^4_{122} = \\
=& - 2i\big(h^3_{112} - i h^3_{122}\big) + (p^3_1 + i p^3_2) \pm h^4_{121} + \\
&\pm h^4_{112} \mp p^4_2 \pm i h^4_{111} \mp i p^4_1 \mp i h^4_{122} = \\
=& - 2i\big(h^3_{112} - i h^3_{122}\big) + (p^3_1 + i p^3_2) \pm h^4_{121} + \\
&\pm h^4_{112} \mp p^4_2 \pm i h^4_{111} \mp \frac{i}{2} h^4_{111} \mp \frac{i}{2} h^4_{221} \mp i h^4_{122} = \\
=& - 2i\big(h^3_{112} - i h^3_{122}\big) + (p^3_1 + i p^3_2) \pm i(p^4_1 + i p^4_2) \pm 2(h^4_{121} - i h^4_{122}) = \\
=& 2\big(\zeta^3 \pm i \zeta^4\big)
\end{aligned}$$

and

$$\begin{aligned}
&h^3_{111} - p^3_1 \pm h^4_{121} \mp h^4_{112} \pm p^4_2 + h^3_{122} + \\
&\pm i h^4_{111} \mp i p^4_1 - i h^3_{121} + i h^3_{112} - i p^3_2 \pm i h^4_{122} = \\
&\quad =(h^3_{111} + h^3_{122}) - p^3_1 + (\pm p^4_2 - i p^3_2) \mp i p^4_1 + \\
&\quad\quad \pm i(h^4_{111} + h^4_{122}) \pm h^4_{121} \mp h^4_{112} - i h^3_{121} + i h^3_{112} = \\
&\quad =p^3_1 + (\pm p^4_2 - i p^3_2) \pm i p^4_1 = \\
&\quad =2(k^3 \pm i k^4).
\end{aligned}$$

Now we use (4.22) in order to apply Lemma 4.4 to the functions $L^3 \pm i L^4$. □

Let us now consider the canonical projection $p : \mathbb{R}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{R}}$, sending x to its projective class $[x]$. Given two Darboux frames e and $\widetilde{e}$ along $f : M \to Q_4$, we have

$$p_{*\widetilde{e}_0} \widetilde{e}_\alpha = r B^\beta_\alpha p_{*e_0} e_\beta.$$

Indeed, since $p(\lambda x) = p(x)$ for every $\lambda \in \mathbb{R}^*$ and for every $x \in \mathbb{R}^6 \setminus \{0\}$, then $p_{*\lambda x} \lambda_{*x} v = p_{*x} v$, that is $p_{*\lambda x} \lambda v = p_{*x} v$. Therefore

$$p_{*\widetilde{e}_0} \widetilde{e}_\alpha = p_{*r^{-1} e_0} \widetilde{e}_\alpha = p_{*e_0}(r \widetilde{e}_\alpha) = r B^\beta_\alpha p_{*e_0} e_\beta$$

Hence, setting $E_\alpha = p_{*e_0} e_\alpha$, we get

$$\widetilde{E}_\alpha = r B^\beta_\alpha E_\beta. \tag{4.27}$$

It follows that the bundle P over M locally spanned by E_3, E_4 is globally well defined. Let P_c be its complexification and $P_c = P^{(1,0)}_c \oplus P^{(0,1)}_c$ the splitting of P_c into $(1,0)$ and $(0,1)$ parts, locally spanned by $E_3 - iE_4$ and $E_3 + iE_4$ respectively. Observe that under a change of Darboux frames, by virtue of (4.27) we have

$$\widetilde{E}_3 \pm i\widetilde{E}_4 = re^{\mp is}(E_3 \pm iE_4). \tag{4.28}$$

On the other hand, if $\varphi = \phi_0^1 + i\phi_0^2$ is the form that gives M its complex structure, from (2.7) we deduce that

$$\widetilde{\varphi} = r^{-1}e^{-it}\varphi. \tag{4.29}$$

From (4.4), (4.28) and (4.29) we conclude that

$$\mu_{\mp} = \big(L^3 \mp iL^4\big)(E_3 \pm iE_4) \otimes \varphi \otimes \varphi$$

are sections of the bundles

$$P_c^{(0,1)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)} \quad \text{and} \quad P_c^{(1,0)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)}$$

respectively, which are globally defined on M. Under assumption (4.22) we can deduce that these sections either vanish identically or have isolated zeros with positive integer multiplicities. Indeed, since φ is a holomorphic section of $T^*M^{(1,0)}$, then

$$D_{\frac{\partial}{\partial \bar z}}\mu_{\mp} = d(L^3 \mp iL^4)\left(\frac{\partial}{\partial \bar z}\right)(E^3 \pm iE^4) \otimes \varphi^2 + (L^3 \mp iL^4)D_{\frac{\partial}{\partial \bar z}}(E^3 \pm iE^4) \otimes \varphi^2$$

and now, using (4.25), assumption (4.22), and the fact that $P_c^{(1,0)}$ and $P_c^{(0,1)}$ are line bundles, we have

$$\left\|D_{\frac{\partial}{\partial \bar z}}\mu_{\mp}\right\| \le \gamma|L^3 \mp iL^4|\,\big\|E^3 \pm iE^4\big\| = \gamma\|\mu_{\mp}\|$$

for some $\gamma \in L^p_{\rm loc}(M)$. Thus the sections $\mu_{\mp}$ satisfy a Cauchy-Riemann type inequality; we can therefore apply Lemma 4.4 to their local trivializations and deduce that they are of analytic type.
Assume now M compact. By the Poincaré-Hopf index theorem (see, e.g. [8] and [9]) we have

Proposition 4.7. *Let M be a compact Riemann surface and L a complex line bundle over M. If $s \not\equiv 0$ is a section of L of analytic type, then the Euler number of L, $\chi(L)$, is equal to the sum of the orders of the zeros of s.*

By virtue of this result, assuming γ_f not $\pm$ holomorphic and letting $z(\mu_{\mp})$ be the sum of the orders of the zeros of $\mu_{\mp}$, then using the properties of the Chern classes of line bundles we obtain

$$\begin{cases} z(\mu_-) = -2\chi(M) + \chi\big(P_c^{(0,1)}\big) = -2\chi(M) - \chi(P) \\ z(\mu_+) = -2\chi(M) + \chi\big(P_c^{(1,0)}\big) = -2\chi(M) + \chi(P). \end{cases}$$

We have therefore proved the following

Theorem 4.8. *Let $f : M \to Q_4$ be an immersed compact surface satisfying (4.22). Then either $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is $\pm$ holomorphic or*

$$2\chi(M) \le -|\chi(P)|.$$

4.4 Willmore surfaces and the harmonicity of the conformal Gauss map

Besides $\pm$ holomorphicity of γ_f, since M is 2-dimensional, we can also consider the harmonicity of γ_f, which in this case only depends on the conformal class of the Riemann surface.

In order to do so, we introduce another geometric quantity. Consider equation (4.14), that is

$$\phi^0_\alpha = p^\alpha_k \phi^k_0$$

(note that in what follows we can consider arbitrary dimension $m \geq 2$ and codimension n). Taking exterior derivative of the above equation and using the Maurer-Cartan structure equations together with Cartan's lemma, we obtain

$$dp^\alpha_i - p^\alpha_k \phi^k_i + p^\beta_i \phi^\alpha_\beta + 2p^\alpha_i \phi^0_0 - h^\alpha_{ki}\phi^0_k = p^\alpha_{ik}\phi^k_0 \tag{4.30}$$

with

$$p^\alpha_{ik} = p^\alpha_{ki}. \tag{4.31}$$

With a simple but tedious computation, one verifies that under a change of Darboux frames we have

$$\begin{aligned}\widetilde{p}^\alpha_{ij} =& r^3 B^\beta_\alpha A^k_i A^t_j \left(p^\beta_{kt} + x^l h^\beta_{lkt} - x^t x^l h^\beta_{lk} - x^k x^l h^\beta_{lt} - \frac{1}{2} x^l x^l h^\beta_{kt} - 2x^t p^\beta_k - 2x^k p^\beta_t \right) + \\ &+ r^3 B^\beta_\alpha \delta_{ij} \left(x^l x^t h^\beta_{lt} + x^l p^\beta_l \right)\end{aligned} \tag{4.32}$$

so that, tracing with respect to i and j

$$\widetilde{p}^\alpha_{ii} = r^3 B^\beta_\alpha \Big(p^\beta_{tt} + (m-2)\Big(2x^l p^\beta_l + x^l x^t h^\beta_{lt}\Big)\Big) \tag{4.33}$$

showing that, when $m = 2$, the system of equations

$$p^\alpha_{ii} = 0 \tag{4.34}$$

is conformally invariant.

Given the oriented immersed Riemann surface $f : M \to Q_n$ we are now ready to compute a local version of the tension field $\tau(\gamma_f)$ of the conformal Gauss map $\gamma_f : M \to \mathcal{Q}_{n-2}(\mathbb{R}^{n+2})$. Towards this aim we observe that

$$ds^2 = \sum_i (\phi^i_0)^2$$

is in the conformal class of M and we can consider ds^2 as a local representative of the metric of M. We compute the Levi-Civita connection forms ρ^i_j corresponding to the local orthonormal coframe $\{\phi^i_0\}$. To that end we set

$$\phi^0_0 = \mu_k \phi^k_0. \tag{4.35}$$

Defining

$$\rho^i_k = \phi^i_k + \mu_k \phi^i_0 - \mu_i \phi^k_0 \tag{4.36}$$

we observe that

$$\rho^i_k + \rho^k_i = 0 \tag{4.37}$$

and, because of the Maurer-Cartan structure equations

$$d\phi_0^i = -\rho_k^i \wedge \phi_0^k.$$

Next, we recall that the forms $\theta^{0,\alpha} = \varsigma^*\Phi_0^\alpha$, $\theta^{\alpha,0} = \varsigma^*\Phi_\alpha^0$ and $\theta^{\alpha,i} = \varsigma^*\Phi_\alpha^i$ describe the real structure of $\mathcal{Q}_{n-2}(\mathbb{R}^{n+2})$. Considering a Darboux frame e along f and using (4.17) we have

$$\gamma_f^*\theta^{\alpha,0} = p_k^\alpha \phi_0^k, \qquad \gamma_f^*\theta^{\alpha,i} = -h_{ik}^\alpha \phi_0^k, \qquad \gamma_f^*\theta^{0,\alpha} = 0$$

so that

$$\gamma_f{}_k^{\alpha,0} = p_k^\alpha, \qquad \gamma_f{}_k^{\alpha,i} = -h_{ik}^\alpha, \qquad \gamma_f{}_k^{0,\alpha} = 0. \tag{4.38}$$

Using the connection forms given in (3.10) and (4.36) we compute

$$\begin{aligned}
\gamma_f{}_{k,t}^{\alpha,i}\phi_0^t =& d\gamma_f{}_k^{\alpha,i} - \gamma_f{}_j^{\alpha,i}\rho_k^j + \gamma_f{}_k^{\beta,0}\theta_{\beta,0}^{\alpha,i} + \gamma_f{}_k^{\beta,j}\theta_{\beta,j}^{\alpha,i} + \gamma_f{}_k^{0,\beta}\theta_{0,\beta}^{\alpha,i} = \\
=& - dh_{ik}^\alpha + h_{ij}^\alpha\phi_k^j + p_k^\alpha\phi_0^i - h_{jk}^\alpha\phi_j^i - h_{ik}^\beta\phi_\beta^\alpha + h_{it}^\alpha\mu_k\phi_0^t - \delta_{kt}h_{ij}^\alpha\mu_j\phi_0^t = \\
=& (-h_{ikt}^\alpha + \delta_{it}p_k^\alpha + \delta_{ik}p_t^\alpha + h_{it}^\alpha\mu_k - \delta_{kt}h_{ij}^\alpha\mu_j + h_{ik}^\alpha\mu_t)\phi_0^t
\end{aligned}$$

so that

$$\gamma_f{}_{k,t}^{\alpha,i} = -h_{ikt}^\alpha + \delta_{it}p_k^\alpha + \delta_{ik}p_t^\alpha + h_{it}^\alpha\mu_k - \delta_{kt}h_{ij}^\alpha\mu_j + h_{ik}^\alpha\mu_t.$$

Similarly,

$$\begin{aligned}
\gamma_f{}_{k,t}^{\alpha,0} =& p_{kt}^\alpha - p_k^\alpha\mu_t - p_t^\alpha\mu_k + p_i^\alpha\mu_i\delta_{kt} \\
\gamma_f{}_{k,t}^{0,\alpha} =& - h_{kt}^\alpha.
\end{aligned}$$

It follows that

$$\gamma_f{}_{k,k}^{\alpha,0} = p_{kk}^\alpha, \qquad \gamma_f{}_{k,k}^{\alpha,i} = 0 = \gamma_f{}_{k,k}^{0,\alpha}.$$

We have therefore proved the following

Theorem 4.9. *Let $f : M \to Q_n$ be an immersed oriented Riemann surface with conformal Gauss map $\gamma_f : M \to \mathcal{Q}_{n-2}(\mathbb{R}^{n+2})$. Then γ_f is harmonic if and only if* (4.34) *is satisfied.*

It was proved in [12] and [4] that condition (4.34) is also equivalent to f being a critical point of the Willmore functional, prompting us to set the following

Definition 4.4. *We will say that $f : M \to Q_n$ is a* ***Willmore surface*** *if, for any compact $K \subseteq M$ and any smooth variation $f_t : M \to Q_n$ with support in K, we have*

$$\left.\frac{d}{dt}\right|_{t=0} W_K(f_t) = 0,$$

where

$$W_K(f) = \int_K \mathrm{w}. \tag{4.39}$$

Theorem 4.10. *Let $f : M \to Q_n$ be an immersed oriented Riemann surface with conformal Gauss map $\gamma_f : M \to \mathcal{Q}_{n-2}(\mathbb{R}^{n+2})$. Then f is a Willmore surface if and only if γ_f is harmonic.*

Proof. By virtue of Theorem 4.9 we only need to compute the Euler-Lagrange equations of the Willmore functional. Let K be any compact domain in M and let $f_t : K \to Q_n$, for $t \in (-\varepsilon, \varepsilon)$ and for some $\varepsilon > 0$, be a smooth one-parameter family of immersions with compact support $C' \subset K \setminus \partial K$, i.e. $f_t(p) = f(p)$ for every $t \in (-\varepsilon, \varepsilon)$, $p \in K \setminus C'$ and such that f_0 coincides with the given immersion f. A simple computation shows that we can write (4.39) as

$$W_K(f) = \int_K \Omega_f,$$

where $\Omega_f = -\phi_1^\alpha \wedge \phi_2^\alpha$. Now since K is compact we may assume the variation to be normal.
To be more precise, let $v : K \times (-\varepsilon, \varepsilon) \to Q_n$ be the smooth variation, that is

$$f_t = v(\cdot, t).$$

Up to taking a smaller ε, we can consider a smooth frame along v, that is a map

$$e : U \times (-\varepsilon, \varepsilon) \to \text{Möb}(n),$$

where U is a neighbourhood of a given point $p_0 \in K$, such that $\pi \circ e = v$, the map $e_t = e(\cdot, t) : U \to \text{Möb}(n)$ is a Darboux frame along f_t for every $t \in (-\varepsilon, \varepsilon)$, and

$$e(p,t) = e(p,0) \qquad \forall p \in U \setminus C,\ t \in (-\varepsilon, \varepsilon), \tag{4.40}$$

C being a compact such that $C' \subset C \subset K \setminus \partial K$.
For such frames we define, as usual, $\phi = e^*\Phi = e^{-1}de$, so that the components ϕ_b^a satisfy the usual symmetry relations and the structure equations.
For each $t \in (-\varepsilon, \varepsilon)$ we denote by $\phi(t)$ the $\mathfrak{m\ddot{o}b}(n)$-valued 1-form on U

$$\phi(t) = e_t^*\Phi,$$

with components $\phi_b^a(t)$ which also satisfy the symmetry relations and the structure equations. Being ϕ a 1-form on $U \times (-\varepsilon, \varepsilon)$, at any point (p,t) it can be written as

$$\phi_{(p,t)} = \phi(t)_p + \Lambda(p,t)dt, \tag{4.41}$$

where $\Lambda : U \times (-\varepsilon, \varepsilon) \to \mathfrak{m\ddot{o}b}(n)$ is given by

$$\Lambda(p,t) = e^*\Phi\left(\frac{\partial}{\partial t}\Big|_{(p,t)}\right) = \Phi_{e_t(p)}\left(e_{*(p,t)}\frac{\partial}{\partial t}\Big|_{(p,t)}\right) = \Phi_{e_t(p)}\left(\frac{\partial e}{\partial t}(p,t)\right).$$

From (4.40) we know that $\Lambda(p,t) = 0$ and $\phi_{(p,t)} = \phi_{(p,0)}$ for every $t \in (-\varepsilon, \varepsilon)$ and $p \in U \setminus C$.
We set $\lambda_0^A = \Lambda_0^A$ and observe that, since e_t is a Darboux frame , then

$$\phi^\alpha_{0\,(p,t)} = \lambda_0^\alpha(p,t)dt, \tag{4.42}$$

and since $\phi_i^\alpha(t)_p = h_{ij}^\alpha(p,t)\phi_0^j(t)_p$, with $h_{ii}^\alpha = 0$, and $h_{ij}^\alpha(p,t) = h_{ij}^\alpha(p,0)$ for every $p \in U \setminus C$, $t \in (-\varepsilon, \varepsilon)$, we have

$$\phi^\alpha_{i\,(p,t)} = h_{ij}^\alpha(p,t)\phi_0^j(t)_p + \Lambda_i^\alpha(p,t)dt = h_{ij}^\alpha(p,t)\phi^j_{0(p,t)} + \lambda_i^\alpha(p,t)dt, \tag{4.43}$$

where $\lambda_i^\alpha = -h_{ij}^\alpha \Lambda_0^j + \Lambda_i^\alpha$ are smooth functions satisfying $\lambda_i^\alpha(p,t) = 0$ for $p \in U \setminus C$ and $t \in (-\varepsilon, \varepsilon)$.
Differentiating (4.42) and using the structure equations we find

$$\begin{aligned} d\phi_0^\alpha =& d\lambda_0^\alpha \wedge dt = \\ =& -\phi_0^\alpha \wedge \phi_0^0 - \phi_i^\alpha \wedge \phi_0^i - \phi_\beta^\alpha \wedge \phi_0^\beta = \\ =& \lambda_0^\alpha \phi_0^0 \wedge dt + \lambda_i^\alpha \phi_0^i \wedge dt + h_{ij}^\alpha \phi_0^i \wedge \phi_0^j - \lambda_0^\beta \phi_\beta^\alpha \wedge dt = \\ =& \lambda_0^\alpha \phi_0^0 \wedge dt + \lambda_i^\alpha \phi_0^i \wedge dt - \lambda_0^\beta \phi_\beta^\alpha \wedge dt, \end{aligned}$$

that is

$$(d\lambda_0^\alpha - \lambda_0^\alpha \phi_0^0 - \lambda_i^\alpha \phi_0^i + \lambda_0^\beta \phi_\beta^\alpha) \wedge dt = 0$$

and by Cartan's Lemma there exist smooth functions μ^α such that

$$d\lambda_0^\alpha = \lambda_0^\alpha \phi_0^0 + \lambda_i^\alpha \phi_0^i - \lambda_0^\beta \phi_\beta^\alpha + \mu^\alpha dt.$$

We remark that a variation v and a frame e with these properties can always be defined with assigned arbitrary $\lambda_0^\alpha(\cdot, 0)$, as long as $\operatorname{supp} \lambda_0^\alpha \subset C'$. For any $t \in (-\varepsilon, \varepsilon)$ we have

$$\Omega_{f_t} = -\phi_1^\alpha(t) \wedge \phi_2^\alpha(t),$$

which we can rewrite using (4.41) as

$$\begin{aligned} (\Omega_{f_t})_p =& -\Big(\phi_1^\alpha{}_{(p,t)} - \Lambda_1^\alpha dt\Big) \wedge \Big(\phi_2^\alpha{}_{(p,t)} - \Lambda_2^\alpha dt\Big) = \\ =& -\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)} + \phi_1^\alpha{}_{(p,t)} \wedge \Lambda_2^\alpha dt + \Lambda_1^\alpha dt \wedge \phi_2^\alpha{}_{(p,t)} = \\ =& -\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)} + dt \wedge \Big(\Lambda_1^\alpha \phi_2^\alpha{}_{(p,t)} - \Lambda_2^\alpha \phi_1^\alpha{}_{(p,t)}\Big) = \\ =& -\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)} + dt \wedge \Big[i_{\frac{\partial}{\partial t}}\Big(\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)}\Big)\Big]. \end{aligned}$$

Now we set

$$\omega_{(p,t)} = -\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)} + dt \wedge \Big[i_{\frac{\partial}{\partial t}}\Big(\phi_1^\alpha{}_{(p,t)} \wedge \phi_2^\alpha{}_{(p,t)}\Big)\Big]$$

and observe that obviously, for any $t_0 \in (-\varepsilon, \varepsilon)$,

$$\Omega_{f_{t_0}} = \omega_{(\cdot, t_0)},$$

and $i_{\frac{\partial}{\partial t}} \omega = 0$, indeed

$$i_{\frac{\partial}{\partial t}} \omega = -i_{\frac{\partial}{\partial t}} (\phi_1^\alpha \wedge \phi_2^\alpha) + i_{\frac{\partial}{\partial t}} (\phi_1^\alpha \wedge \phi_2^\alpha) - \Big[(\phi_1^\alpha \wedge \phi_2^\alpha)\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big)\Big] dt = 0.$$

We define

$$g(t_0) = W_K(f_{t_0}) = \int_K \omega_{|t=t_0}$$

and consider the variation

$$g'(0) = \int_K \Big(\mathcal{L}_{\frac{\partial}{\partial t}} \omega\Big)\Big|_{t=0} = \int_K \Big[d\Big(i_{\frac{\partial}{\partial t}} \omega\Big) + i_{\frac{\partial}{\partial t}} d\omega\Big]_{t=0} = \int_K \Big(i_{\frac{\partial}{\partial t}} d\omega\Big)\Big|_{t=0},$$

where $\mathcal{L}$ denotes the Lie derivative. This expands to

$$g'(0) = \int_K \left[i_{\frac{\partial}{\partial t}} (-d(\phi_1^\alpha \wedge \phi_2^\alpha)) - d(\Lambda_1^\alpha \phi_2^\alpha - \Lambda_2^\alpha \phi_1^\alpha) \right]_{t=0},$$

and, using the structure equations,

$$\begin{aligned} -d(\phi_1^\alpha \wedge \phi_2^\alpha) =& \phi_0^\alpha \wedge (\phi_1^0 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_2^0) + \phi_\alpha^0 \wedge (\phi_0^1 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_0^2) + \\ &+ \phi_\beta^\alpha \wedge \phi_1^\beta \wedge \phi_2^\alpha - \phi_1^\alpha \wedge \phi_\beta^\alpha \wedge \phi_2^\beta = \\ =& \phi_0^\alpha \wedge (\phi_1^0 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_2^0) + \phi_\alpha^0 \wedge (\phi_0^1 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_0^2) + \\ &+ \phi_\beta^\alpha \wedge (\phi_1^\beta \wedge \phi_2^\alpha - \phi_1^\beta \wedge \phi_2^\alpha) = \\ =& \phi_0^\alpha \wedge (\phi_1^0 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_2^0) + \phi_\alpha^0 \wedge (\phi_0^1 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_0^2). \end{aligned}$$

This allows us to compute

$$\begin{aligned} -i_{\frac{\partial}{\partial t}} (d(\phi_1^\alpha \wedge \phi_2^\alpha)) =& \lambda_0^\alpha (\phi_1^0 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_2^0) - \phi_0^\alpha \wedge i_{\frac{\partial}{\partial t}} (\phi_1^0 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_2^0) + \\ &+ \Lambda_\alpha^0 (\phi_0^1 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_0^2) - \phi_\alpha^0 \wedge \big(-\lambda_2^\alpha \phi_0^1(t) + \lambda_1^\alpha \phi_0^2(t)\big), \end{aligned}$$

where, for the last term, we computed

$$\begin{aligned} & i_{\frac{\partial}{\partial t}} \big(\phi_0^1 \wedge \phi_2^\alpha + \phi_1^\alpha \wedge \phi_0^2\big) = \\ =& \Lambda_0^1 \phi_2^\alpha - \Lambda_2^\alpha \phi_0^1 + \Lambda_1^\alpha \phi_0^2 - \Lambda_0^2 \phi_1^\alpha = \\ =& \Lambda_0^1 \Big(h_{2j}^\alpha \phi_0^j + \lambda_2^\alpha dt\Big) - \Lambda_2^\alpha \phi_0^1 + \Lambda_1^\alpha \phi_0^2 - \Lambda_0^2 \Big(h_{1j}^\alpha \phi_0^j + \lambda_1^\alpha dt\Big) = \\ =& \big(h_{12}^\alpha \Lambda_0^1 - h_{11}^\alpha \Lambda_0^2 - \Lambda_2^\alpha\big)\phi_0^1 + \big(h_{22}^\alpha \Lambda_0^1 - h_{21}^\alpha \Lambda_0^2 + \Lambda_1^\alpha\big)\phi_0^2 + \Lambda_0^1 \lambda_2^\alpha dt - \Lambda_0^2 \lambda_1^\alpha dt = \\ =& \lambda_1^\alpha \big(\phi_0^2 - \Lambda_0^2 dt\big) - \lambda_2^\alpha \big(\phi_0^1 - \Lambda_0^1 dt\big) = \\ =& -\lambda_2^\alpha \phi_0^1(t) + \lambda_1^\alpha \phi_0^2(t). \end{aligned}$$

Now, restricting to $t = 0$, we find that

$$\begin{aligned} -i_{\frac{\partial}{\partial t}} (d(\phi_1^\alpha \wedge \phi_2^\alpha))_{|t=t_0} =& \lambda_0^\alpha(0)\big(\phi_1^0(0) \wedge \phi_2^\alpha(0) + \phi_1^\alpha(0) \wedge \phi_2^0(0)\big) + \\ &+ \phi_\alpha^0(0) \wedge \big(\lambda_2^\alpha(0)\phi_0^1(0) - \lambda_1^\alpha(0)\phi_0^2(0)\big) \end{aligned}$$

since $\phi_0^\alpha(0) = 0$ and

$$\phi_0^1(0) \wedge \phi_2^\alpha(0) + \phi_1^\alpha(0) \wedge \phi_0^2(0) = (h_{11}^\alpha + h_{22}^\alpha)\phi_0^1(0) \wedge \phi_0^2(0) = 0.$$

Now a tedious but straightforward computation involving (4.30) shows that

$$-i_{\frac{\partial}{\partial t}} (d(\phi_1^\alpha \wedge \phi_2^\alpha))_{|t=t_0} = \lambda_0^\alpha(0)(p_{11}^\alpha + p_{22}^\alpha)\phi_0^1(0) \wedge \phi_0^2(0) - d\big[\lambda_0^\alpha \big(p_1^\alpha \phi_0^2 - p_2^\alpha \phi_0^1\big)\big]_{t=0}$$

and thus

$$g'(0) = \int_K \big\{\lambda_0^\alpha(0)(p_{11}^\alpha + p_{22}^\alpha)\phi_0^1(0) \wedge \phi_0^2(0) - d\chi\big\}$$

where

$$\chi = \Lambda_1^\alpha(0)\phi_2^\alpha(0) - \Lambda_2^\alpha(0)\phi_1^\alpha(0) + \lambda_0^\alpha(0)\big(p_1^\alpha \phi_0^2(0) - p_2^\alpha \phi_0^1(0)\big).$$

It can be easily checked that χ is a well defined, smooth 1-form on M, and since λ_0^α and λ_i^α are supported in $C \subset K$, so is χ. Therefore, applying Stoke's theorem, we obtain

$$g'(0) = \int_K \lambda_0^\alpha(0)(p_{11}^\alpha + p_{22}^\alpha)\phi_0^1(0) \wedge \phi_0^2(0)$$

and, by the arbitrariness of $\lambda_0^\alpha(\cdot, 0)$, the claim is proved. □

4.5 S-Willmore surfaces and Willmore surfaces

Let us go back to surfaces in Q_4. In this context the concepts of harmonicity and $\pm$ holomorphicity of the conformal Gauss map both make sense, and since $\pm$ holomorphicity implies harmonicity, we find that $\pm$ isotropic surfaces in Q_4 are in particular Willmore surfaces.

In [7], Ejiri has introduced the notion of S-Willmore surface. In our setting, with respect to a Darboux frame along f, the notion corresponds to the two following conditions

$$\begin{aligned} &(a) \quad L^\alpha e_\alpha \not\parallel \overline{L^\alpha} e_\alpha \\ &(b) \quad k^\alpha e_\alpha // L^\alpha e_\alpha \end{aligned} \tag{4.44}$$

whose conformal invariance is apparent once we recognize that, at $p \in M$, condition (4.11a) is equivalent to

$$\begin{vmatrix} L^3 & L^4 \\ \overline{L^3} & \overline{L^4} \end{vmatrix} \neq 0 \qquad \text{that is} \qquad L^3\overline{L^4} - \overline{L^3}L^4 \neq 0,$$

and by (4.9) this translates to

$$K_N(p) \neq 0.$$

On the other hand, condition (4.11b) can be expressed as

$$k^3L^4 - k^4L^3 = 0,$$

and the quantity on the left-hand side, under a change of Darboux frames, obeys the transformation law

$$\widetilde{k}^3\widetilde{L}^4 - \widetilde{k}^4\widetilde{L}^3 = r^3e^{3it}(k^3L^4 - k^4L^3).$$

Indeed, from (4.3) and

$$\widetilde{k}^\alpha = r^2 B^\beta_\alpha e^{it}\Big(k^\beta + \frac{1}{2}(x^1 + ix^2)L^\beta\Big),$$

we get

$$\begin{aligned} \widetilde{k}^3\widetilde{L}^4 - \widetilde{k}^4\widetilde{L}^3 =& r^3e^{3it}\Big(B^\beta_3 B^\gamma_4 - B^\beta_4 B^\gamma_3\Big)\Big(k^\beta + \frac{1}{2}(x^1+ix^2)L^\beta\Big)L^\gamma = \\ =& r^3e^{3it}\Bigg[\Big(k^3 + \frac{1}{2}(x^1+ix^2)L^3\Big)L^4 + \\ & - \Big(k^4 + \frac{1}{2}(x^1+ix^2)L^4\Big)L^3\Bigg] = \\ =& r^3e^{3it}(k^3L^4 - k^4L^3). \end{aligned}$$

Thus the element of $\bigotimes^3 T^*M^{(1,0)}$

$$\alpha_1 = (k^3L^4 - k^4L^3)\varphi \otimes \varphi \otimes \varphi \tag{4.45}$$

is globally defined on M and condition (4.11b) is satisfied at $p \in M$ if and only if

$$\alpha_1(p) = 0.$$

Ejiri proved that, in the Riemannian setting, an S-Willmore surface is a Willmore surface. This can be easily checked in our setting, too.

Proposition 4.11. *Let $f : M \to Q_4$ be an S-Willmore surface, namely an immersed oriented Riemann surface such that $K_N \neq 0$ and $\alpha_1 = 0$. Then f is a Willmore surface.*

Proof. Suppose f is S-Willmore. In particular $k^3L^4 - k^4L^3 = 0$ on M. Differentiating the left-hand side and using the structure equations we find

$$\begin{aligned} d(k^3L^4 - k^4L^3) = &-3(k^3L^4 - k^4L^3)(\phi_0^0 + i\phi_2^1) + \frac{1}{2}(Q^3L^4 - Q^4L^3)\varphi + \\ &+ (k^3\zeta^4 - k^4\zeta^3)\varphi + \frac{1}{4}(p^3_{kk}L^4 - p^4_{kk}L^3)\overline{\varphi}, \end{aligned} \tag{4.46}$$

where

$$Q^\alpha = \frac{1}{2}(p^\alpha_{11} - p^\alpha_{22}) - ip^\alpha_{12}$$

and ζ^α has been defined in (4.26). Indeed,

$$\begin{aligned} &2d(k^3L^4 - k^4L^3) = \\ &= d\big((p_1^3 - ip_2^3)(h^4_{11} - ih^4_{12}) - (p_1^4 - ip_2^4)(h^3_{11} - ih^3_{12})\big) = \\ &= d\big(p_1^3h^4_{11} - ip_1^3h^4_{12} - ip_2^3h^4_{11} - p_2^3h^4_{12} - p_1^4h^3_{11} + ip_1^4h^3_{12} + ip_2^4h^3_{11} + p_2^4h^3_{12}\big) = \\ &= h^4_{11}dp_1^3 + p_1^3dh^4_{11} - ih^4_{12}dp_1^3 - ip_1^3dh^4_{12} - ih^4_{11}dp_2^3 - ip_2^3dh^4_{11} - h^4_{12}dp_2^3 + \\ &\quad - p_2^3dh^4_{12} - h^3_{11}dp_1^4 - p_1^4dh^3_{11} + ih^3_{12}dp_1^4 + ip_1^4dh^3_{12} + ih^3_{11}dp_2^4 + \\ &\quad + ip_2^4dh^3_{11} + h^3_{12}dp_2^4 + p_2^4dh^3_{12} \end{aligned}$$

and recalling that

$$dp_i^\alpha = p_k^\alpha\phi_i^k + h^\alpha_{ki}\phi_k^0 - p_i^\beta\phi_\beta^\alpha - 2p_i^\alpha\phi_0^0 + p^\alpha_{ik}\phi_0^k$$

and

$$dh^\alpha_{ij} = h^\alpha_{ijk}\phi_0^k + h^\alpha_{ik}\phi_j^k + h^\alpha_{kj}\phi_i^k - h^\beta_{ij}\phi_\beta^\alpha - h^\alpha_{ij}\phi_0^0 - \delta_{ij}p_k^\alpha\phi_0^k,$$

we get

$$\begin{aligned} &2d(k^3L^4 - k^4L^3) = \\ &= h^4_{11}(p_2^3\phi_1^2 + h^3_{11}\phi_1^0 + h^3_{21}\phi_2^0 - p_1^4\phi_4^3 - 2p_1^3\phi_0^0 + p^3_{11}\phi_0^1 + p^3_{12}\phi_0^2) + \\ &\quad + p_1^3(h^4_{11k}\phi_0^k + h^4_{12}\phi_1^2 + h^4_{21}\phi_1^2 - h^3_{11}\phi_3^4 - h^4_{11}\phi_0^0 - p_k^4\phi_0^k) + \\ &\quad - ih^4_{12}(p_2^3\phi_1^2 + h^3_{11}\phi_1^0 + h^3_{21}\phi_2^0 - p_1^4\phi_4^3 - 2p_1^3\phi_0^0 + p^3_{11}\phi_0^1 + p^3_{12}\phi_0^2) + \\ &\quad - ip_1^3(h^4_{12k}\phi_0^k + h^4_{11}\phi_2^1 + h^4_{22}\phi_1^2 - h^3_{12}\phi_3^4 - h^4_{12}\phi_0^0) + \\ &\quad - ih^4_{11}(p_1^3\phi_2^1 + h^3_{12}\phi_1^0 + h^3_{22}\phi_2^0 - p_2^4\phi_4^3 - 2p_2^3\phi_0^0 + p^3_{21}\phi_0^1 + p^3_{22}\phi_0^2) + \\ &\quad - ip_2^3(h^4_{11k}\phi_0^k + h^4_{12}\phi_1^2 + h^4_{21}\phi_1^2 - h^3_{11}\phi_3^4 - h^4_{11}\phi_0^0 - p_k^4\phi_0^k) + \\ &\quad - h^4_{12}(p_1^3\phi_2^1 + h^3_{12}\phi_1^0 + h^3_{22}\phi_2^0 - p_2^4\phi_4^3 - 2p_2^3\phi_0^0 + p^3_{21}\phi_0^1 + p^3_{22}\phi_0^2) + \end{aligned}$$

$$
\begin{aligned}
&- p_2^3(h_{12k}^4\phi_0^k + h_{11}^4\phi_2^1 + h_{22}^4\phi_1^2 - h_{12}^3\phi_3^4 - h_{12}^4\phi_0^0)+\\
&- h_{11}^3(p_2^4\phi_1^2 + h_{11}^4\phi_1^0 + h_{21}^4\phi_2^0 - p_1^3\phi_3^4 - 2p_1^4\phi_0^0 + p_{11}^4\phi_0^1 + p_{12}^4\phi_0^2)+\\
&- p_1^4(h_{11k}^3\phi_0^k + h_{12}^3\phi_1^2 + h_{21}^3\phi_1^2 - h_{11}^4\phi_4^3 - h_{11}^3\phi_0^0 - p_k^3\phi_0^k)+\\
&+ ih_{12}^3(p_2^4\phi_1^2 + h_{11}^4\phi_1^0 + h_{21}^4\phi_2^0 - p_1^3\phi_3^4 - 2p_1^4\phi_0^0 + p_{11}^4\phi_0^1 + p_{12}^4\phi_0^2)+\\
&+ ip_1^4(h_{12k}^3\phi_0^k + h_{11}^3\phi_2^1 + h_{22}^3\phi_1^2 - h_{12}^4\phi_4^3 - h_{12}^3\phi_0^0)+\\
&+ ih_{11}^3(p_1^4\phi_2^1 + h_{12}^4\phi_1^0 + h_{22}^4\phi_2^0 - p_2^3\phi_3^4 - 2p_2^4\phi_0^0 + p_{21}^4\phi_0^1 + p_{22}^4\phi_0^2)+\\
&+ ip_2^4(h_{11k}^3\phi_0^k + h_{12}^3\phi_1^2 + h_{21}^3\phi_1^2 - h_{11}^4\phi_4^3 - h_{11}^3\phi_0^0 - p_k^3\phi_0^k)+\\
&+ h_{12}^3(p_1^4\phi_2^1 + h_{12}^4\phi_1^0 + h_{22}^4\phi_2^0 - p_2^3\phi_3^4 - 2p_2^4\phi_0^0 + p_{21}^4\phi_0^1 + p_{22}^4\phi_0^2)+\\
&+ p_2^4(h_{12k}^3\phi_0^k + h_{11}^3\phi_2^1 + h_{22}^3\phi_1^2 - h_{12}^4\phi_4^3 - h_{12}^3\phi_0^0).
\end{aligned}
$$

Then, taking into account that $h_{22}^\alpha = -h_{11}^\alpha$ and cancelling out a few terms,

$$
\begin{aligned}
&2d(k^3L^4 - k^4L^3) =\\
&= \big(-3p_1^3h_{11}^4 + 3ip_1^3h_{12}^4 + 3ip_2^3h_{11}^4 + 3p_2^3h_{12}^4 + 3p_1^4h_{11}^3+\\
&\quad -3ip_1^4h_{12}^3 - 3ip_2^4h_{11}^3 - 3p_2^4h_{12}^3\big)\phi_0^0+\\
&\quad +\big(3p_2^3h_{11}^4 + 3p_1^3h_{12}^4 - 3ip_2^3h_{12}^4 + 3ip_1^3h_{11}^4 - 3p_2^4h_{11}^3+\\
&\qquad - 3p_1^4h_{12}^3 + 3ip_2^4h_{12}^3 - 3ip_1^4h_{11}^3\big)\phi_1^2+\\
&\quad +\big((p_{11}^3h_{11}^4 + p_1^3h_{111}^4 - p_1^3p_1^4 - p_{21}^3h_{12}^4 - p_2^3h_{121}^4+\\
&\qquad - p_{11}^4h_{11}^3 - p_1^4h_{111}^3 + p_1^4p_1^3 + p_{21}^4h_{12}^3 + p_2^4h_{121}^3)+\\
&\qquad +i(-p_{11}^3h_{12}^4 - p_1^3h_{121}^4 - p_{21}^3h_{11}^4 - p_2^3h_{111}^4 + p_2^3p_1^4+\\
&\qquad + p_{11}^4h_{12}^3 + p_1^4h_{121}^3 + p_{21}^4h_{11}^3 + p_2^4h_{111}^3 - p_2^4p_1^3)\big)\phi_0^1+\\
&\quad +\big((p_{12}^3h_{11}^4 + p_1^3h_{112}^4 - p_1^3p_2^4 - p_{22}^3h_{12}^4 - p_2^3h_{122}^4+\\
&\qquad - p_{12}^4h_{11}^3 - p_1^4h_{112}^3 + p_1^4p_2^3 + p_{22}^4h_{12}^3 + p_2^4h_{122}^3)+\\
&\qquad +i(-p_{12}^3h_{12}^4 - p_1^3h_{122}^4 - p_{22}^3h_{11}^4 - p_2^3h_{112}^4 + p_2^3p_2^4+\\
&\qquad + p_{12}^4h_{12}^3 + p_1^4h_{122}^3 + p_{22}^4h_{11}^3 + p_2^4h_{112}^3 - p_2^4p_2^3)\big)\phi_0^2.
\end{aligned}
$$

Now, making once again use of (4.18) and cancelling out a few more terms,

$$
\begin{aligned}
&2d(k^3L^4 - k^4L^3) =\\
&\quad = -6\big(k^3L^4 - k^4L^3\big)\big(\phi_0^0 + i\phi_1^2\big)+\\
&\qquad +\frac{1}{2}\Big\{p_{11}^3h_{11}^4 + p_1^3h_{111}^4 - p_{21}^3h_{12}^4 - p_2^3h_{121}^4 - p_{11}^4h_{11}^3+\\
&\qquad\quad -p_1^4h_{111}^3 + p_{21}^4h_{12}^3 + p_2^4h_{121}^3 - p_{12}^3h_{12}^4 - p_1^3h_{122}^4 - p_{22}^3h_{11}^4+\\
&\qquad\quad -p_2^3h_{112}^4 + p_{12}^4h_{12}^3 + p_1^4h_{122}^3 + p_{22}^4h_{11}^3 + p_2^4h_{112}^3+\\
&\qquad\quad +i\big[-p_{11}^3h_{12}^4 - p_1^3h_{121}^4 - p_{21}^3h_{11}^4 - p_2^3h_{111}^4 + p_{11}^4h_{12}^3+\\
&\qquad\qquad + p_1^4h_{121}^3 + p_{21}^4h_{11}^3 + p_2^4h_{111}^3 - p_{12}^3h_{11}^4 - p_1^3h_{112}^4 + p_{22}^3h_{12}^4+\\
&\qquad\qquad + p_2^3h_{122}^4 + p_{12}^4h_{11}^3 + p_1^4h_{112}^3 - p_{22}^4h_{12}^3 - p_2^4h_{122}^3\big]\Big\}\varphi+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Big\{p^3_{11}h^4_{11}+p^3_1h^4_{111}-p^4_{11}h^3_{11}-p^4_1h^3_{111}+\\
&\quad +p^3_1h^4_{122}+p^3_{22}h^4_{11}-p^4_1h^3_{122}-p^4_{22}h^3_{11}+\\
&\quad +i\big[-p^3_{11}h^4_{12}-p^3_2h^4_{111}+2p^3_2p^4_1+p^4_{11}h^3_{12}+p^4_2h^3_{111}+\\
&\qquad -2p^3_1p^4_2-p^3_{22}h^4_{12}-p^3_2h^4_{122}+p^4_{22}h^3_{12}+p^4_2h^3_{122}\big]\Big\}\overline{\varphi}
\end{aligned}
$$

Recalling the definitions of $Q^3L^4-Q^4L^3$ and $p^3_{kk}L^4-p^4_{kk}L^3$,

$$
\begin{aligned}
2d(k^3L^4-k^4L^3)&=\\
&=-6\big(k^3L^4-k^4L^3\big)\big(\phi^0_0+i\phi^2_1\big)+\\
&\quad+\Big\{Q^3L^4-Q^4L^3+\frac{1}{2}\big[p^3_1h^4_{111}-p^3_2h^4_{121}-p^4_1h^3_{111}+\\
&\qquad+p^4_2h^3_{121}-p^3_1h^4_{122}-p^3_2h^4_{112}+p^4_1h^3_{122}+p^4_2h^3_{112}\big]+\\
&\qquad+\frac{i}{2}\big[-p^3_1h^4_{121}-p^3_2h^4_{111}+p^4_1h^3_{121}+p^4_2h^3_{111}+\\
&\qquad-p^3_1h^4_{112}+p^3_2h^4_{122}+p^4_1h^3_{112}-p^4_2h^3_{122}\big]\Big\}\varphi+\\
&\quad+\frac{1}{2}\Big\{p^3_{kk}L^4-p^4_{kk}L^3+\\
&\qquad+p^3_1h^4_{111}+p^3_1h^4_{122}-p^4_1h^3_{111}-p^4_1h^3_{122}+\\
&\qquad+i\big[-p^3_2h^4_{111}+2p^3_2p^4_1+p^4_2h^3_{111}+\\
&\qquad-2p^3_1p^4_2-p^3_2h^4_{122}+p^4_2h^3_{122}\big]\Big\}\overline{\varphi}=\\
&=-6\big(k^3L^4-k^4L^3\big)\big(\phi^0_0+i\phi^2_1\big)+\\
&\quad+\Big\{Q^3L^4-Q^4L^3+\frac{1}{2}\big[p^3_1h^4_{111}-p^3_2h^4_{121}-p^4_1h^3_{111}+\\
&\qquad+p^4_2h^3_{121}-p^3_1h^4_{122}-p^3_2h^4_{112}+p^4_1h^3_{122}+p^4_2h^3_{112}\big]+\\
&\qquad+\frac{i}{2}\big[-p^3_1h^4_{121}-p^3_2h^4_{111}+p^4_1h^3_{121}+p^4_2h^3_{111}+\\
&\qquad-p^3_1h^4_{112}+p^3_2h^4_{122}+p^4_1h^3_{112}-p^4_2h^3_{122}\big]\Big\}\varphi+\\
&\quad+\frac{1}{2}\big(p^3_{kk}L^4-p^4_{kk}L^3\big)\overline{\varphi}.
\end{aligned}
$$

Finally, using the definition of $k^3\zeta^4-k^4\zeta^3$ and manipulating the remaining terms,

$$
\begin{aligned}
2d(k^3L^4-k^4L^3)&=\\
&=-6\big(k^3L^4-k^4L^3\big)\big(\phi^0_0+i\phi^2_1\big)+\\
&\quad+\Big\{Q^3L^4-Q^4L^3+\big[-p^3_1h^4_{122}-p^3_2h^4_{112}+p^4_1h^3_{122}+\\
&\qquad+p^4_2h^3_{112}-ip^3_1h^4_{121}-ip^3_2p^4_1+ip^3_2h^4_{122}+ip^4_1h^3_{121}+\\
&\qquad+ip^4_2p^3_1-ip^4_2h^3_{122}\big]\Big\}\varphi+\frac{1}{2}\big(p^3_{kk}L^4-p^4_{kk}L^3\big)\overline{\varphi}=\\
&=-6\big(k^3L^4-k^4L^3\big)(\phi^0_0+i\phi^1_2)+(Q^3L^4-Q^4L^3)\varphi+\\
&\quad+2(k^3\zeta^4-k^4\zeta^3)\varphi+\frac{1}{2}(p^3_{kk}L^4-p^4_{kk}L^3)\overline{\varphi}.
\end{aligned}
$$

Now, setting $k^3L^4 - k^4L^3 = 0$ in (4.46), we can deduce that in particular

$$p^3_{kk}L^4 = p^4_{kk}L^3.$$

Assume by contradiction that f is not a Willmore surface, that is, either $p^3_{kk} \neq 0$ or $p^4_{kk} \neq 0$, say $p^3_{kk} \neq 0$. Then we have

$$iK_N = L^3\overline{L^4} - \overline{L^3}L^4 = \frac{p^4_{kk}}{p^3_{kk}}(L^3\overline{L^3} - \overline{L^3}L^3) = 0$$

which contradicts (4.11a). □

From the proof of Theorem 4.3, we have that γ_f is $\pm$ holomorphic if and only if $k^3 = \pm ik^4$ and $L^3 = \pm iL^4$, hence in this case we automatically have $\alpha_1 = 0$, so that

Proposition 4.12. *Let $f : M \to Q_4$ be a $\pm$ isotropic immersed surface. Then f is S-Willmore if and only if $K_N \neq 0$ on M.*

The next risult is another application of Lemma 4.4.

Proposition 4.13. *Let $f : M \to Q_4$ be an immersion without umbilical points and such that the set of $\pm$ isotropic points is not discrete. If f satisfies condition* (4.22)*, then f is S-Willmore.*

Proof. By Proposition 4.6, f must be $\pm$ isotropic. This implies $\alpha_1 = 0$ and

$$K_N = -i(L^3\overline{L^4} - \overline{L^3}L^4) = \mp 2\left|L^4\right|^2 = \mp 2\left|L^3\right|^2.$$

Therefore $K_N(p) = 0$ if and only if p is an umbilical point, and the result follows. □

Observe that under a change of Darboux frames we have

$$\widetilde{p}^3_{kk}\widetilde{L}^4 - \widetilde{p}^4_{kk}\widetilde{L}^3 = r^3e^{3it}\left(p^3_{kk}L^4 - p^4_{kk}L^3\right), \tag{4.47}$$

therefore, applying once more Lemma 4.4 we have the following

Theorem 4.14. *Let $f : M \to Q_4$ be an immersion such that*

$$\exists\gamma \in L^p_{\mathrm{loc}}(M) \quad \textit{such that} \quad \left|p^3_{kk}L^4 - p^4_{kk}L^3\right| \leq \gamma\left|k^3L^4 - k^4L^3\right| \quad \textit{a.e.} \tag{4.48}$$

for some $p > 2$. Then either $\alpha_1 \equiv 0$ or its zero set is discrete. In this latter case, for M compact we have

$$z(\alpha_1) = -3\chi(M),$$

where $z(\alpha_1)$ is the sum of the orders of the zeros of α_1.

Remark 4.15. *If M is a Willmore surface, condition* (4.48) *is automatically satisfied. Moreover, if M is a topological 2-sphere, then $\alpha_1 \equiv 0$.*

Proposition 4.16. *Let $f : M \to Q_4$ be a Willmore surface. Then α_1 is holomorphic.*

Proof. Let e be a Darboux frame along f and $g = \phi_0^1 \otimes \phi_0^1 + \phi_0^2 \otimes \phi_0^2$ be the local metric on M defined by e. There exists a local isothermal coordinate $z = x + iy$ on M such that $g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = r^2$; therefore $\widetilde{g} = r^{-2} g$ is a flat local metric, conformally related to g. Since $\left\{r^{-1}\frac{\partial}{\partial x}, r^{-1}\frac{\partial}{\partial y}\right\}$ is an orthonormal frame for g, we can consider the locally defined, $SO(2)$-valued function A given by $A_1^i = \phi_0^i\left(r^{-1}\frac{\partial}{\partial x}\right)$, $A_2^i = \phi_0^i\left(r^{-1}\frac{\partial}{\partial y}\right)$. If we set $\widetilde{e} = eK$, with K defined by

$$K = \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

then $\widetilde{e}$ is a Darboux frame, since K is obviously G_D-valued. Moreover, trivially $\widetilde{\phi}_0^1 \otimes \widetilde{\phi}_0^1 + \widetilde{\phi}_0^1 \otimes \widetilde{\phi}_0^2 = \widetilde{g}$ and, from (2.7), the dual frame to the coframe $\left\{\widetilde{\phi}_0^i\right\}$ is just $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. Indeed, for instance,

$$\widetilde{\phi}_0^1\left(\frac{\partial}{\partial x}\right) = r^{-1} A_1^j \phi_0^j\left(\frac{\partial}{\partial x}\right) = \phi_0^j\left(r^{-1}\frac{\partial}{\partial x}\right)\phi_0^j\left(r^{-1}\frac{\partial}{\partial x}\right) = 1.$$

Now, from the structure equations, we have $d\widetilde{\varphi} = (\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1) \wedge \widetilde{\varphi}$ and, denoting $Z = \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ and $W = \overline{Z}$ we get

$$d\widetilde{\varphi}(W, Z) = d(\widetilde{\varphi}(W))(Z) - d(\widetilde{\varphi}(Z))(W) + \widetilde{\varphi}([Z, W]) = 0$$

since $\widetilde{\varphi}(W) = 0$, $\widetilde{\varphi}(Z) = 1$ and $[Z, W] = 0$. On the other hand

$$[(\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1) \wedge \widetilde{\varphi}](Z, W) = -(\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1)(W),$$

proving that $\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1$ is of type $(1,0)$, and hence can be expressed as $\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1 = \mu\widetilde{\varphi}$, for some locally defined complex valued function μ.
Now, with respect to $\widetilde{e}$, (4.45) is the expression of α_1 in a local holomorphic trivialization of the bundle $\bigotimes^3 T^*M^{(1,0)}$ so, in order to check if α_1 is holomorphic (that is, if $\bar{\partial}\alpha_1 = 0$) we only need to check that the differential of its coefficient in such trivialization, $k^3 L^4 - k^4 L^3$, is a local form of type $(1,0)$. But, assuming that f is Willmore, (4.46) (with respect to the frame $\widetilde{e}$) shows that this is exactly the case. □

4.6 $\mathcal{Q}_2(\mathbb{R}^6)$-valued maps and surfaces in Q_4

So far we have considered immersions of oriented surfaces in the conformal sphere Q_4 and we have associated to them certain maps with values in the conformal Grassmannian $\mathcal{Q}_2(\mathbb{R}^6)$, i.e. the conformal Gauss map. This map has some remarkable properties, for instance it is holomorphic if and only if the original immersion is $-$ isotropic. Now we are going to do the converse: starting from a holomorphic map γ with values in $\mathcal{Q}_2(\mathbb{R}^6)$ we want to see if, and under what conditions, it is possible to retrieve a Q_4-valued map whose conformal Gauss map is exactly the map γ.

First of all, let us observe that, given a $-$ isotropic immersion $f : M \to Q_4$, the conformal Gauss map γ_f is constant if and only if f is totally umbilical, namely $f(M) \subseteq Q_2$, or equivalently $W_K(f) = 0$ for any compact domain $K \subset M$.

Let M be a Riemann surface and $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ a non constant holomorphic map. Let φ be a (local) $(1,0)$-form defining the complex structure on M and let $s : U \subset \mathcal{Q}_2(\mathbb{R}^6) \to \text{Möb}(4)$ be a local section of $\widehat{\pi}$. Then

$$\gamma^*\zeta^0 = \Lambda^0\varphi, \qquad \gamma^*\zeta^k = \Lambda^k\varphi, \qquad \gamma^*\zeta^3 = \Lambda^3\varphi, \tag{4.49}$$

where ζ^0, ζ^k and ζ^3 are defined as in (3.11) with respect to the section s. The vector Λ of components Λ^0, Λ^k, Λ^3 is of analytic type, i.e. it either vanishes identically or has isolated zeros. Indeed, let ω be such that $d\varphi = i\omega \wedge \varphi$; then, differentiating (4.49) and using (3.11) and the structure equations, we have

$$\begin{aligned} d(\gamma^*\zeta^0) &= d\Lambda^0 \wedge \varphi + \Lambda^0 d\varphi = \big(d\Lambda^0 + i\Lambda^0\omega\big) \wedge \varphi = \\ &= \gamma^* d\zeta^0 = -\gamma^* s^*(\Phi^0_0 + i\Phi^4_3) \wedge \Lambda^0\varphi - \gamma^* s^*\Phi^0_k \wedge \Lambda^k\varphi. \end{aligned}$$

Hence

$$\big(d\Lambda^0 + i\Lambda^0\omega + \Lambda^0\gamma^* s^*\big(\Phi^0_0 + i\Phi^4_3\big) + \Lambda^k\gamma^* s^*\Phi^0_k\big) \wedge \varphi = 0$$

and similarly

$$\begin{aligned} d(\gamma^*\zeta^1) &= \big(d\Lambda^1 + i\Lambda^1\omega\big) \wedge \varphi = \\ &= \gamma^* d\zeta^1 = \gamma^*\big(-s^*\Phi^1_0 \wedge \zeta^0 - s^*\Phi^1_2 \wedge \zeta^2 - is^*\Phi^4_3 \wedge \zeta^1 - s^*\Phi^0_1 \wedge \zeta^3\big) = \\ &= \big(-\Lambda^0\gamma^* s^*\Phi^1_0 - \Lambda^2\gamma^* s^*\Phi^1_2 - i\Lambda^1\gamma^* s^*\Phi^4_3 - \Lambda^3\gamma^* s^*\Phi^0_1\big) \wedge \varphi, \end{aligned}$$

implying

$$\big(d\Lambda^1 + i\Lambda^1 + \Lambda^0\gamma^* s^*\Phi^1_0 + \Lambda^2\gamma^* s^*\Phi^1_2 + i\Lambda^1\gamma^* s^*\Phi^4_3 + \Lambda^3\gamma^* s^*\Phi^0_1\big) \wedge \varphi = 0;$$

$$\begin{aligned} d(\gamma^*\zeta^2) &= \big(d\Lambda^2 + i\Lambda^2\omega\big) \wedge \varphi = \\ &= \gamma^* d\zeta^2 = \gamma^*\big(-s^*\Phi^2_0 \wedge \zeta^0 - s^*\Phi^2_1 \wedge \zeta^1 - is^*\Phi^4_3 \wedge \zeta^2 - s^*\Phi^0_2 \wedge \zeta^3\big) = \\ &= \big(-\Lambda^0\gamma^* s^*\Phi^2_0 - \Lambda^1\gamma^* s^*\Phi^2_1 - i\Lambda^2\gamma^* s^*\Phi^4_3 - \Lambda^3\gamma^* s^*\Phi^0_2\big) \wedge \varphi, \end{aligned}$$

hence

$$\big(d\Lambda^2 + i\Lambda^2 + \Lambda^0\gamma^* s^*\Phi^2_0 + \Lambda^1\gamma^* s^*\Phi^2_1 + i\Lambda^2\gamma^* s^*\Phi^4_3 + \Lambda^3\gamma^* s^*\Phi^0_2\big) \wedge \varphi = 0.$$

Finally

$$\begin{aligned} d(\gamma^*\zeta^3) &= \big(d\Lambda^3 + i\Lambda^3\omega\big) \wedge \varphi = \\ &= \gamma^* d\zeta^3 = \gamma^*\big(s^*\big(\Phi^0_0 - i\Phi^4_3\big) \wedge \zeta^3 - s^*\Phi^1_0 \wedge \zeta^1 - s^*\Phi^2_0 \wedge \zeta^2\big) = \\ &= \big(\Lambda^3\gamma^* s^*\big(\Phi^0_0 - i\Phi^4_3\big) - \Lambda^1\gamma^* s^*\Phi^1_0 - \Lambda^2\gamma^* s^*\Phi^2_0\big) \wedge \varphi, \end{aligned}$$

which gives

$$\big(d\Lambda^3 + i\Lambda^3 - \Lambda^3\gamma^* s^*\big(\Phi^0_0 - i\Phi^4_3\big) + \Lambda^1\gamma^* s^*\Phi^1_0 + \Lambda^2\gamma^* s^*\Phi^2_0\big) \wedge \varphi = 0,$$

that is,

$$\begin{cases} d\Lambda^0 = -i\Lambda^0\big(\omega + \gamma^* s^*\Phi^4_3 - i\gamma^* s^*\Phi^0_0\big) - \Lambda^k\gamma^* s^*\Phi^0_k & \mod \varphi \\ d\Lambda^k = -i\Lambda^k\big(\omega + \gamma^* s^*\Phi^4_3\big) - \Lambda^j\gamma^* s^*\Phi^k_j - \Lambda^0\gamma^* s^*\Phi^k_0 - \Lambda^3\gamma^* s^*\Phi^0_k & \mod \varphi \\ d\Lambda^3 = -i\Lambda^3\big(\omega + \gamma^* s^*\Phi^4_3 + i\gamma^* s^*\Phi^0_0\big) - \Lambda^k\gamma^* s^*\Phi^k_0 & \mod \varphi. \end{cases}$$

Thus $d\Lambda^a = \Psi^a_b \Lambda^b$ modulo φ, for some $\mathfrak{gl}(4,\mathbb{C})$-valued one form $\Psi = (\Psi^a_b)$, namely the vector Λ is a solution of the system

$$\frac{\partial \Lambda}{\partial \bar{z}} = \Psi\left(\frac{\partial}{\partial \bar{z}}\right)\Lambda$$

and, by Lemma 4.4 (but see also [6] for a direct proof of this case), the claim follows.
Since we assumed γ to be non constant, it follows that the zeros of Λ are isolated, and in a neighbourhood of any zero, Λ factorizes as $\Lambda = z^t\widetilde{\Lambda}$, with $\widetilde{\Lambda} \neq 0$, z a local holomorphic chart centered at the zero and $t \in \mathbb{N}$.
Since $\mathcal{Q}_2(\mathbb{R}^6)$ can be identified with an open subset of a quadric in $\mathbb{P}^5_{\mathbb{C}}$ the map γ can be lifted to a smooth, $\mathbb{C}^6 \setminus \{0\}$-valued map $\{\gamma\} = e_3 + ie_4$, where $e = s \circ \gamma : U \subset M \to \text{Möb}(n)$ (note that e is not necessarily an immersion, because in general γ is not). Denoting $\phi = e^{-1}de$, we have

$$\begin{aligned}
&\Lambda^0\varphi = \gamma^*\zeta^0 = e^*\Phi^0_3 + ie^*\Phi^0_4 = \phi^0_3 + i\phi^0_4,\\
&\Lambda^k\varphi = \gamma^*\zeta^k = \phi^k_3 + i\phi^k_4,\\
&\Lambda^3\varphi = \gamma^*\zeta^3 = \phi^5_3 + i\phi^5_4,
\end{aligned}$$

and since $de = e\phi$,

$$\begin{aligned}
d\{\gamma\} &= i(e_3 + ie_4)\phi^3_4 + e_0(\phi^0_3 + i\phi^0_4) + e_k(\phi^k_3 + i\phi^k_4) + e_5(\phi^5_3 + i\phi^5_4) = \\
&= i\{\gamma\}\phi^3_4 + (\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)\varphi
\end{aligned}$$

If $p : \mathbb{C}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{C}}$ is the canonical projection, then $\gamma = p \circ \{\gamma\}$ and

$$d\gamma_x = \gamma_{*x} = p_{*\{\gamma\}(x)}\{\gamma\}_{*x} = \varphi\, p_{*\{\gamma\}(x)}\big(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5\big).$$

The complex tangent line to the curve $\gamma(M)$ at the point $\gamma(x)$ is therefore the vector space spanned over $\mathbb{C}$ by the non-zero vector $p_{*\{\gamma\}(x)}(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)$. This prompts us to define a new map, called the "derivative" of γ, $\gamma' : M \to \mathbb{P}^5_{\mathbb{C}}$ which associates to the point $x \in M$ the projectivization of the non-zero vector $\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5$. This map is trivially well defined and does not depend on the choice of the section s.
We will need to add the further assumption that γ' be valued in the quadric $\mathcal{Q}_2(\mathbb{R}^6)$; this happens if and only if the vector ${}^t(\Lambda^0, \Lambda^k, 0, 0, \Lambda^3)$ satisfies the equation

$$-2\Lambda^0\Lambda^3 + \Lambda^k\Lambda^k = 0.$$

Definition 4.5. *A map $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ will be called a* ***totally isotropic holomorphic map*** *if it is holomorphic, non constant, and if γ' is valued in $\mathcal{Q}_2(\mathbb{R}^6)$.*

Let $\widetilde{s}$ be another local section of the bundle $\widehat{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(\mathbb{R}^6)$, and $\widetilde{e} = \widetilde{s} \circ \gamma$. Then $\widetilde{e} = eK$ where K takes values in H_0 as defined in (3.1). At any point $p \in M$ we can therefore choose a section such that $\Lambda^3 = 0$, hence $\Lambda^0 = a$, $\Lambda^1 = \lambda$ and $\Lambda^2 = i\lambda$, for some $a, \lambda \in \mathbb{C}$. Since Λ is of analytic type, such sections can be locally smoothly chosen in a neighbourhood of p. The frame e corresponding to such section will be called an **isotropic frame**, and the isotropy subgroup for such frames is exactly G_D as defined in (2.10). With this choice of frame, (4.49) rewrites as

$$\gamma^*\zeta^0 = a\varphi, \quad \gamma^*\zeta^1 = \lambda\varphi, \quad \gamma^*\zeta^2 = i\lambda\varphi, \quad \gamma^*\zeta^3 = 0. \tag{4.50}$$

We can associate, to any totally isotropic holomorphic map γ, a map $J_\gamma : M \to Q_4$ defined as follows. Let e be any isotropic frame along γ and set $J_\gamma = [e_0]$. In this way J_γ is well defined, because isotropic frames change by matrices in G_D. Differentiating the second and third equalities of (4.50), we obtain

$$\begin{aligned} d(\gamma^*\zeta^1) =& -\phi_0^1 \wedge \gamma^*\zeta^0 - \phi_2^1 \wedge \gamma^*\zeta^2 - i\phi_3^4 \wedge \gamma^*\zeta^1 - \phi_1^0 \wedge \gamma^*\zeta^3 = \\ =& \big(-a\phi_0^1 - i\lambda\phi_2^1 - i\lambda\phi_3^4\big) \wedge \varphi, \\ d(\gamma^*\zeta^2) =& (-a\phi_0^2 - \lambda\phi_1^2 + \lambda\phi_3^4) \wedge \varphi, \end{aligned}$$

but on the other hand $\gamma^*\zeta^2 = i\gamma^*\zeta^1$, so we have

$$\big(-ia\phi_0^1 + \lambda\phi_2^1 + \lambda\phi_3^4\big) \wedge \varphi = \big(-a\phi_0^2 - \lambda\phi_1^2 + \lambda\phi_3^4\big) \wedge \varphi$$

that is, $ia\big(\phi_0^1 + i\phi_0^2\big) \wedge \varphi = 0$. Differentiating the last of (4.50) we get

$$0 = d(\gamma^*\zeta^3) = \big(-\lambda\phi_0^1 - i\lambda\phi_0^2\big) \wedge \varphi.$$

Therefore we have obtained

$$\begin{aligned} a\big(\phi_0^1 + i\phi_0^2\big) \wedge \varphi = 0 \\ \lambda\big(\phi_0^1 + i\phi_0^2\big) \wedge \varphi = 0 \end{aligned}$$

Since Λ is of analytic type, outside a discrete set (the set of zeros of a and λ), we must have

$$\phi_0^1 + i\phi_0^2 = \mu\varphi \tag{4.51}$$

for some locally defined complex function μ, whose vanishing is independent of the choice of the isotropic frame. Differentiating (4.51), we have

$$d\mu \wedge \varphi + i\mu\omega \wedge \varphi = d\phi_0^1 + id\phi_0^2 = \mu\phi_0^0 \wedge \varphi + i\mu\phi_2^1 \wedge \varphi,$$

that is

$$d\mu = -i\mu\big(\omega - \phi_2^1 + i\phi_0^0\big) \qquad \text{mod } \varphi.$$

Therefore μ is of analytic type, and so it either vanishes identically or has isolated zeros.

Let us now consider an open set $U \subset M$ where μ is nonzero and let e be an isotropic frame along γ defined on U. Then e is trivially a zeroth order frame along J_γ, since $\pi \circ e = J_\gamma$. Moreover, it is a first order frame, since from (4.50)

$$0 = \gamma^*\zeta^3 = \phi_0^3 + i\phi_0^4,$$

so $\phi_0^\alpha = 0$. Also, J_γ is a conformal immersion on U, since the only points where J_γ is not an immersion are the zeros of μ. In the case of μ vanishing identically, then J_γ is constant. Indeed in this case not only $\phi_0^\alpha = 0$, but also $\phi_0^1 = \phi_0^2 = 0$. So

$$dJ_\gamma = p_* de_0 = p_*(e_0\phi_0^0 + e_A\phi_0^A) = \phi_0^A p_* e_A = 0$$

where $p : \mathbb{R}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{R}}$ is the canonical projection.
Thus, either J_γ is constant on M or it is a weakly conformal branched immersion. Assume to be in this latter case; we will prove that an isotropic frame e along γ is a Darboux frame along J_γ.
To this end we use (4.50) to deduce that

$$\gamma^*\zeta^2 = i\,\gamma^*\zeta^1. \tag{4.52}$$

Now we set, as usual, $\phi^\alpha_i = h^\alpha_{ij}\phi^j_0$, $h^\alpha_{ij} = h^\alpha_{ji}$, and observe that

$$\gamma^*\zeta^k = e^*(\Phi^k_3 + i\Phi^k_4) = -\phi^3_k - i\phi^4_k = -(h^3_{kj} + ih^4_{kj})\phi^j_0$$

and equation (4.52) is equivalent to the following system

$$\begin{cases} h^3_{1j} = h^4_{2j} \\ h^3_{2j} = -h^4_{1j} \end{cases}$$

which gives

$$h^3_{11} = h^4_{21} = -h^3_{22}, \qquad h^4_{11} = -h^3_{21} = -h^4_{22}.$$

An alternative proof can be performed by exploiting the holomorphicity of γ. Consider on $U \subseteq M$ the local metric $ds^2 = (\phi^1_0)^2 + (\phi^2_0)^2$, which is in the conformal class of M. Since $\phi^\alpha_i = h^\alpha_{ij}\phi^j_0$, using the notations of (3.5), we have

$$\begin{aligned} \gamma^*\theta^{\alpha,i} &= -h^\alpha_{ik}\phi^k_0 \\ \gamma^*\theta^{0,\alpha} &= 0 \\ \gamma^*\theta^{\alpha,0} &= B^{\alpha,0}_k\phi^k_0, \end{aligned}$$

where at present we are not able to relate $B^{\alpha,0}_k$ with h^α_{ik} because e is just a first order frame. We set

$$B^{\alpha,i}_k = -h^\alpha_{ik}, \quad B^{0,\alpha}_k = 0.$$

We can also consider on U the tension field $\tau(\gamma)$ of the map γ, which, since we have not fixed a global metric on M, but only a class of conformally related local metrics, is defined up to a nonzero factor.
Since γ is holomorphic and both M and $\mathcal{Q}_2(\mathbb{R}^6)$ are Kähler manifolds, the tension field $\tau(\gamma)$ of γ must vanish. In particular its coefficients $B^{0,\alpha}_{kk}$ must vanish, and we are now going to compute their values.
To this end, let us denote by ∇ the covariant derivative of (M, ds^2), and by ∇' the one on $(\mathcal{Q}_2(\mathbb{R}^6), dl^2)$. Let $\{E_i\}$ be the orthonormal frame on M dual to the coframe $\{\phi^i_0\}$ and $\{Y_{\alpha,0}, Y_{\alpha,i}, Y_{0,\alpha}\}$ be the frame on $\mathcal{Q}_2(\mathbb{R}^6)$ dual to the coframe $\{\theta^{\alpha,0}, \theta^{\alpha,i}, \theta^{0,\alpha}\}$; then

$$\nabla d\gamma = \left(B^{\alpha,0}_{ij}Y_{\alpha,0} + B^{\alpha,k}_{ij}Y_{\alpha,k} + B^{0,\alpha}_{ij}Y_{0,\alpha}\right) \otimes \phi^i_0 \otimes \phi^j_0.$$

First of all observe that, if $v \in T\mathcal{Q}_2(\mathbb{R}^6)$, then

$$v = \theta^{\alpha,0}(v)Y_{\alpha,0} + \theta^{0,\alpha}(v)Y_{0,\alpha} + \theta^{\alpha,i}(v)Y_{\alpha,i};$$

in particular,

$$\begin{aligned} \gamma_{*x}E_j =& \theta^{\alpha,0}(\gamma_{*x}E_j)(Y_{\alpha,0})_{\gamma(x)} + \theta^{0,\alpha}(\gamma_{*x}E_j)(Y_{0,\alpha})_{\gamma(x)} + \theta^{\alpha,k}(\gamma_{*x}E_j)(Y_{\alpha,k})_{\gamma(x)} = \\ =& \gamma^*\theta^{\alpha,0}(E_j)(Y_{\alpha,0})_{\gamma(x)} + \gamma^*\theta^{0,\alpha}(E_j)(Y_{0,\alpha})_{\gamma(x)} + \gamma^*\theta^{\alpha,k}(E_j)(Y_{\alpha,k})_{\gamma(x)} = \\ =& B^{\alpha,0}_j(Y_{\alpha,0})_{\gamma(x)} + B^{\alpha,k}_j(Y_{\alpha,k})_{\gamma(x)}, \end{aligned}$$

that is

$$\gamma_* E_j = B_j^{\alpha,0} \gamma^{-1} Y_{\alpha,0} + B_j^{\alpha,i} \gamma^{-1} Y_{\alpha,i}.$$

Now, if X is a vector field on M,

$$\nabla d\gamma(E_j; X) = \nabla_X^{\gamma^{-1}} (\gamma_* E_j) - \gamma_* \nabla_X E_j$$

and $\nabla_X E_j = \rho_j^k(X) E_k$, where ρ_j^k, defined as in (4.36), are the connection forms relative to the orthonormal coframe $\{\phi_0^i\}$, while $\nabla' Y_{\widetilde{A}} = \theta_{\widetilde{A}}^{\widetilde{B}} Y_{\widetilde{B}}$, with $\theta_{\widetilde{A}}^{\widetilde{B}}$ defined in (3.10). Then we can compute

$$\begin{aligned}
\nabla_X^{\gamma^{-1}} (\gamma_* E_j) =& dB_j^{\alpha,0}(X) \gamma^{-1} Y_{\alpha,0} + B_j^{\alpha,0} \nabla'_{\gamma_* X} Y_{\alpha,0} + \\
& + dB_j^{\alpha,k}(X) \gamma^{-1} Y_{\alpha,k} + B_j^{\alpha,k} \nabla'_{\gamma_* X} Y_{\alpha,k} = \\
=& dB_j^{\alpha,0}(X) \gamma^{-1} Y_{\alpha,0} + B_j^{\alpha,0} \theta_{\alpha,0}^{\widetilde{B}} (\gamma_* X) \gamma^{-1} Y_{\widetilde{B}} + \\
& + dB_j^{\alpha,k}(X) \gamma^{-1} Y_{\alpha,k} + B_j^{\alpha,k} \theta_{\alpha,k}^{\widetilde{B}} (\gamma_* X) \gamma^{-1} Y_{\widetilde{B}} = \\
=& dB_j^{\alpha,0}(X) \gamma^{-1} Y_{\alpha,0} + B_j^{\alpha,0} \gamma^* \theta_{\alpha,0}^{\beta,0}(X) \gamma^{-1} Y_{\beta,0} + \\
& + B_j^{\alpha,0} \gamma^* \theta_{\alpha,0}^{\beta,i}(X) \gamma^{-1} Y_{\beta,i} + B_j^{\alpha,0} \gamma^* \theta_{\alpha,0}^{0,\beta}(X) \gamma^{-1} Y_{0,\beta} + \\
& + dB_j^{\alpha,k}(X) \gamma^{-1} Y_{\alpha,k} + B_j^{\alpha,k} \gamma^* \theta_{\alpha,k}^{\beta,0}(X) \gamma^{-1} Y_{\beta,0} + \\
& + B_j^{\alpha,k} \gamma^* \theta_{\alpha,k}^{\beta,i}(X) \gamma^{-1} Y_{\beta,i} + B_j^{\alpha,k} \gamma^* \theta_{\alpha,k}^{0,\beta}(X) \gamma^{-1} Y_{0,\beta}
\end{aligned}$$

and

$$\gamma_* \nabla_X E_j = \rho_j^k(X) \gamma_* E_k = \rho_j^k(X) \Big(B_k^{\alpha,0} \gamma^{-1} Y_{\alpha,0} + B_k^{\alpha,i} \gamma^{-1} Y_{\alpha,i} \Big).$$

Therefore,

$$\begin{aligned}
\nabla d\gamma(E_j; X) =& \Big(dB_j^{\alpha,0} + B_j^{\beta,0} \gamma^* \theta_{\beta,0}^{\alpha,0} + B_j^{\beta,k} \gamma^* \theta_{\beta,k}^{\alpha,0} - B_k^{\alpha,0} \rho_j^k \Big)(X) \gamma^{-1} Y_{\alpha,0} + \\
& + \Big(dB_j^{\alpha,k} + B_j^{\beta,0} \gamma^* \theta_{\beta,0}^{\alpha,k} + B_j^{\beta,i} \gamma^* \theta_{\beta,i}^{\alpha,k} - B_i^{\alpha,k} \rho_j^i \Big)(X) \gamma^{-1} Y_{\alpha,k} + \\
& + \Big(B_j^{\beta,0} \gamma^* \theta_{\beta,0}^{0,\alpha} + B_j^{\beta,k} \gamma^* \theta_{\beta,k}^{0,\alpha} \Big)(X) \gamma^{-1} Y_{0,\alpha}.
\end{aligned}$$

By (3.10), $\theta_{\beta,0}^{0,\alpha} = 0$, so that, taking the $Y_{0,\alpha}$ component, one has

$$B_{ij}^{0,\alpha} \phi_0^i(X) = B_j^{\beta,k} \gamma^* \theta_{\beta,k}^{0,\alpha}(X).$$

Now, since

$$\tau(\gamma) = \sum_j \nabla d\gamma(E_j; E_j)$$

vanishes by assumption and, again by (3.10), $\theta_{\beta,k}^{0,\alpha} = \delta_\beta^\alpha \varphi_0^k$, then in particular

$$0 = B_j^{\beta,k} \gamma^* \theta_{\beta,k}^{0,\alpha}(E_j) = \delta_\beta^\alpha B_j^{\beta,k} \phi_0^k(E_j) = B_j^{\alpha,j} = -h_{jj}^\alpha$$

and we have proved that e is a Darboux frame along J_γ.
Moreover, it is trivial to see that, outside the branch points of J_γ, we have $\gamma_{J_\gamma} = \gamma$, and J_γ is $-$ isotropic, since γ_{J_γ} is holomorphic by assumption.

On the other hand, consider a weakly conformal branched immersion $f : M \to Q_4$ with the property that its Gauss map γ_f can be continuously extended to the branch points, and let e be any Darboux frame along f. If f is $-$ isotropic (outside the branch points), then γ_f is holomorphic, and in this case, with the notations of (4.49), we have

$$\Lambda^0 = k^3 + ik^4, \qquad \Lambda^1 = -\frac{1}{2}\big(L^3 + iL^4\big), \qquad \Lambda^2 = -\frac{i}{2}\big(L^3 + iL^4\big), \qquad \Lambda^3 = 0,$$

so that

$$-2\Lambda^0\Lambda^3 + \sum_k \Lambda^k\Lambda^k = 0$$

and γ_f is a totally isotropic map. Furthermore, $J_{\gamma_f} = f$.
We have therefore proved the following

Theorem 4.17. *Let M be a Riemann surface. There is a bijective correspondence between $-$ isotropic, non totally umbilical, weakly conformal branched immersions $f : M \to Q_4$, whose conformal gauss map can be continuously extended at the branch points, and non constant, holomorphic, totally isotropic maps $\gamma : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ with non constant associated map J_γ. The bijection is realized via the conformal Gauss map.*

4.7 $\mathcal{Q}_2(Q_4)$-valued maps and the conformal Gauss lift

Using an appropriate Grassmann bundle, we can extend the previous result so as to include the totally umbilical surfaces.
Let us consider the product manifold $Q_4 \times \mathcal{Q}_2\big(\mathbb{R}^6\big)$ and define $\mathcal{Q}_2(Q_4)$ as the orbit of the point $([\eta_0], [\varepsilon_3, \varepsilon_4]) \in Q_4 \times \mathcal{Q}_2\big(\mathbb{R}^6\big)$ with respect to the natural left action (defined componentwise) of the group Möb(4). In other words

$$\mathcal{Q}_2(Q_4) = \{([\eta], [s_1, s_2]) \mid \eta = P\eta_0,\ s_1 = P\varepsilon_3,\ s_2 = P\varepsilon_4,\ P \in \text{Möb}(4)\}. \quad (4.53)$$

It is trivial to see that Möb(4) acts transitively on $\mathcal{Q}_2(Q_4)$, the action being given, for $P \in$ Möb(4) and $([\eta], [s_1, s_2]) \in \mathcal{Q}_2(Q_4)$, by

$$P([\eta], [s_1, s_2]) = ([P\eta], [Ps_1, Ps_2]).$$

Let us compute the isotropy subgroup of the point $([\eta_0], [\varepsilon_3, \varepsilon_4])$. If $P \in$ Möb(4) fixes the point $([\eta_0], [\varepsilon_3, \varepsilon_4])$, then in particular it must fix the first component, hence P must be an element of G_0, defined in (1.7), so it is bound to be of the form

$$P = \begin{pmatrix} r^{-1} & {}^t x A & \frac{1}{2}r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix}.$$

But, for $P[\varepsilon_3]$ to belong to $[\varepsilon_3, \varepsilon_4]$, we must have $x^3 = 0$, $A^1_3 = A^2_3 = 0$ and analogously, imposing $P[\varepsilon_4] \in [\varepsilon_3, \varepsilon_4]$, we deduce $x^4 = 0$ and $A^1_4 = A^2_4 = 0$. Putting these conditions together we find that $P \in G_D$. Since in turn any element of G_D fixes $([\eta_0], [\varepsilon_3, \varepsilon_4])$, we can conclude that the isotropy subgroup is exactly G_D. Hence $\mathcal{Q}_2(Q_4) \simeq \text{Möb}(4)/G_D$ is realized as a homogeneous space with projection

$$\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$$

given by

$$\bar{\pi} : P \mapsto ([P\eta_0], [P\varepsilon_3, P\varepsilon_4]),$$

that is, $\bar{\pi} = \pi \times \widehat{\pi}$. Also, we will denote by $\breve{\pi} : \mathcal{Q}_2(Q_4) \to Q_4$ the canonical projection

$$\breve{\pi} : ([\eta], [s_1, s_2]) \mapsto [\eta].$$

Observe that $\mathcal{Q}_2(Q_4)$ has a natural integrable complex structure defined as follows: let ξ be a local section of the bundle $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$; then we declare the forms

$$\begin{aligned} \sigma^{-1} &= \xi^*\Phi^1_0 + i\xi^*\Phi^2_0, \\ \sigma^0 &= \xi^*\Phi^0_3 + i\xi^*\Phi^0_4, \\ \sigma^k &= \xi^*\Phi^k_3 + i\xi^*\Phi^k_4, \\ \sigma^3 &= \xi^*\Phi^3_0 + i\xi^*\Phi^4_0 \end{aligned} \tag{4.54}$$

a local basis of the space of the forms of type $(1,0)$ over $\mathcal{Q}_2(Q_4)$. In order to do this, first we need to check that the ideal they generate is differential. Setting, for the sake of simplicity, $\varphi = \xi^*\Phi$ and using the structure equations, we have

$$\begin{aligned} d\sigma^{-1} &= -\,\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) - \varphi^1_3 \wedge \varphi^3_0 - \varphi^1_4 \wedge \varphi^4_0 - i\varphi^2_3 \wedge \varphi^3_0 - i\varphi^2_4 \wedge \varphi^4_0 = \\ &= -\,\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) + i\sigma^1 \wedge \varphi^4_0 + i\sigma^2 \wedge \varphi^3_0 + \sigma^3 \wedge (\varphi^1_3 + \varphi^2_4) \end{aligned}$$

and analogously for the differentials of the other forms. Lastly, one can easily check that the space generated by these forms is well defined, i.e., it is independent of the choice of the section ξ.

Proposition 4.18. *The fibers of $\breve{\pi} : \mathcal{Q}_2(Q_4) \to Q_4$ are integral submanifolds of the (invariantly defined) Pfaffian system*

$$\begin{cases} \sigma^{-1} = 0 \\ \sigma^3 = 0. \end{cases} \tag{4.55}$$

Proof. Since $\mathcal{Q}_2(Q_4) \subset Q_4 \times \mathcal{Q}_2(\mathbb{R}^6)$, for $([\eta], [s_1, s_2]) \in \mathcal{Q}_2(Q_4)$, we have

$$T_{([\eta],[s_1,s_2])}\mathcal{Q}_2(Q_4) \subset T_{[\eta]}Q_4 \times T_{[s_1,s_2]}\mathcal{Q}_2(\mathbb{R}^6).$$

Thus, we can regard a tangent vector of $\mathcal{Q}_2(Q_4)$ as a pair (X, V) with $X \in T_{[\eta]}Q_4$ and $V \in T_{[s_1,s_2]}\mathcal{Q}_2(\mathbb{R}^6)$. Now $\breve{\pi}$ is the projection on the first component, so

$$\breve{\pi}_{*([\eta],[s_1,s_2])}(X, V) = X$$

and

$$\ker \breve{\pi}_{*([\eta],[s_1,s_2])} = \{(0, V) \in T_{([\eta],[s_1,s_2])}\mathcal{Q}_2(Q_4)\}$$

We want to prove that

$$\begin{cases} \sigma^{-1}_{([\eta],[s_1,s_2])}(0, V) = 0 \\ \sigma^3_{([\eta],[s_1,s_2])}(0, V) = 0, \end{cases}$$

or equivalently that, if ξ is a local section of $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$, then

$$\xi^* \Phi^A_{0\ ([\eta],[s_1,s_2])}(0,V) = 0.$$

To this end we set $g = \xi([\eta],[s_1,s_2])$ and compute

$$\begin{aligned}\xi^* \Phi^A_{0\ ([\eta],[s_1,s_2])}(0,V) =& \Phi^A_{0\ g}(\xi_{*([\eta],[s_1,s_2])}(0,V)) = \big(\Phi_g(\xi_{*([\eta],[s_1,s_2])}(0,V))\big)^A_0 = \\ =& (g^{-1})^A_b \big(\xi_{*([\eta],[s_1,s_2])}(0,V)\big)^b_0,\end{aligned}$$

where in the last equality we used the definition of the Maurer-Cartan form for classical groups:

$$\Phi_P(X) = P^{-1}X.$$

Now take $([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ in the domain of ξ, set $\widetilde{g} = \xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ and observe that

$$\bar{\pi}(\xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])) = \bar{\pi}(\widetilde{g}) = ([\widetilde{g}\eta_0],[\widetilde{g}\varepsilon_3,\widetilde{g}\varepsilon_4])$$

and, since $\bar{\pi} \circ \xi = id$,

$$([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]) = (\bar{\pi} \circ \xi)([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]) = ([\widetilde{g}\eta_0],[\widetilde{g}\varepsilon_3,\widetilde{g}\varepsilon_4]).$$

In particular we have that $[\widetilde{\eta}] = [\widetilde{g}\eta_0]$ and

$$[\widetilde{g}\eta_0] = [\widetilde{g}_0] = [(\xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]))_0] = [\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])],$$

that is, the projective class of the vector $\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ coincides with that of $\widetilde{\eta}$. In other words, calling

$$p : \mathbb{R}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{R}}$$

the canonical projection, we find that $p(\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])) = p(\widetilde{\eta})$. Hence $p \circ \xi_0 = \check{\pi}$ and

$$(p \circ \xi_0)_{*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V) = \check{\pi}_{*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V) = 0,$$

that is

$$p_{*\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V) = 0.$$

Thus $\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V) \in \ker p_{*\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}$, implying

$$\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V) = \lambda \xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$$

for some $\lambda \in \mathbb{R}$. Therefore

$$\big(\xi_{*([\eta],[s_1,s_2])}(0,V)\big)^b_0 = \lambda(\xi([\eta],[s_1,s_2]))^b_0 = \lambda g^b_0.$$

So eventually,

$$\xi^* \Phi^A_{0\ ([\eta],[s_1,s_2])}(0,V) = \lambda (g^{-1})^A_b g^b_0 = \lambda \delta^A_0 = 0.$$

□

Let us consider the canonical projection $c : \mathcal{Q}_2(Q_4) \to \mathcal{Q}_2(\mathbb{R}^6)$ defined by

$$c([\eta],[s_1,s_2]) = [s_1,s_2],$$

which makes the following diagram commutative

$$\begin{array}{ccc} & \text{Möb}(4) & \\ {\scriptstyle \bar{\pi}} \swarrow & & \searrow {\scriptstyle \widehat{\pi}} \\ \mathcal{Q}_2(Q_4) & \xrightarrow{\quad c \quad} & \mathcal{Q}_2\big(\mathbb{R}^6\big) \end{array}$$

that is, $\widehat{\pi} = c \circ \bar{\pi}$.

Proposition 4.19. *The map $c : \mathcal{Q}_2(Q_4) \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ defined above is holomorphic.*

Proof. Fix $p_0 = ([\eta], [s_1, s_2]) \in \mathcal{Q}_2(Q_4)$ and consider ξ a local section of the bundle $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$, defined on a neighbourhood of p_0 and ς a local section of the bundle $\widehat{\pi} : \text{Möb}(4) \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ defined on a neighbourhood of $[s_1, s_2]$. We have to show that $c^*\zeta^0$, $c^*\zeta^k$ and $c^*\zeta^3$, defined as in (3.11), are forms of type $(1, 0)$.
Set $g_0 = \xi(p_0)$. As in the proof of Theorem 4.3, we can assume that the section ς satisfies $\varsigma(\widehat{\pi}(g_0)) = g_0$, and

$$(\varsigma \circ \widehat{\pi})^*(\Phi^0_\alpha)_{g_0} = (\Phi^0_\alpha)_{g_0}.$$

Then, observing that $c = \widehat{\pi} \circ \xi$, we have that

$$\big(c^*\zeta^0\big)_{p_0} = \big(\xi^*\widehat{\pi}^*\zeta^0\big)_{p_0} = \xi^*\big(\widehat{\pi}^*\varsigma^*(\Phi^0_3 + i\Phi^0_4)\big)_{g_0} = \xi^*\Phi^0_{3g_0} + i\xi^*\Phi^0_{4g_0} = \sigma^0_{p_0},$$

and analogously for $c^*\zeta^k$ and $c^*\zeta^3$. □

Definition 4.6. *Let $f : M \to Q_4$ be an immersed oriented surface. The* ***conformal Gauss lift*** *$\Gamma_f : M \to \mathcal{Q}_2(Q_4)$ is defined as*

$$\Gamma_f = f \times \gamma_f,$$

that is, given $p \in M$ and e any Darboux frame along f, defined on a neighbourhood of p,

$$\Gamma_f = \bar{\pi} \circ e;$$

in other words,

$$\Gamma_f : p \mapsto ([e_0]_p, [e_3, e_4]_p).$$

We are now ready to state the generalization of Theorem 4.17.

Theorem 4.20. *Let M be a Riemann surface. There is a bijective correspondence between $-$ isotropic, weakly conformal branched immersions $f : M \to Q_4$ whose conformal Gauss map can be continuously extended at the branch points, and holomorphic maps $\Gamma : M \to \mathcal{Q}_2(Q_4)$, solutions of the Pfaffian system*

$$\begin{cases} \sigma^3 = 0 \\ \sigma^2 - i\sigma^1 = 0 \end{cases}$$

but not of $\sigma^{-1} = 0$. The bijection is realized via the conformal Gauss lift Γ_f.

Proof. Let $f : M \to Q_4$ be as in the statement of the theorem. Then, in order to show that the conformal Gauss lift Γ_f is holomorphic, we proceed as for the conformal Gauss map γ_f in the proof of Theorem 4.3. Let us fix $p_0 \in M$ such that it is not a branch point for f and choose a Darboux frame e along f defined on a neighbourhood U of p_0 and a section ξ of the bundle $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$ defined in a neighbourhood of $\Gamma_f(p_0)$. We set $e(p_0) = g_0$; then since $\bar{\pi} \circ (\xi \circ \bar{\pi}) = \bar{\pi}$, there must exist a function $K : \bar{\pi}^{-1}(U) \to G_D$ such that, for every $g \in \bar{\pi}^{-1}(U)$

$$\xi(\bar{\pi}(g)) = gK(g)$$

and

$$(\xi \circ \bar{\pi})^* \Phi_g = K(g)^{-1} g^{-1} dg K(g) + K(g)^{-1} dK_g$$

In particular we have

$$\begin{aligned}
(\xi \circ \bar{\pi})^* \Phi^k_{0\,g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^k_0 \\
(\xi \circ \bar{\pi})^* \Phi^0_{\alpha\,g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^0_\alpha \\
(\xi \circ \bar{\pi})^* \Phi^k_{\alpha\,g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^k_\alpha \\
(\xi \circ \bar{\pi})^* \Phi^\alpha_{0\,g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^\alpha_0,
\end{aligned}$$

because $K^{-1}dK$ is valued in the Lie algebra of the group G_D. Replacing, if necessary, the section ξ with $\xi K(g_0)^{-1}$, we can assume that

$$\xi(\bar{\pi}(g_0)) = g_0$$

and hence

$$\begin{aligned}
(\xi \circ \bar{\pi})^* \Phi^k_{0\,g_0} &= \Phi^k_{0\,g_0} \\
(\xi \circ \bar{\pi})^* \Phi^0_{\alpha\,g_0} &= \Phi^0_{\alpha\,g_0} \\
(\xi \circ \bar{\pi})^* \Phi^k_{\alpha\,g_0} &= \Phi^k_{\alpha\,g_0} \\
(\xi \circ \bar{\pi})^* \Phi^\alpha_{0\,g_0} &= \Phi^\alpha_{0\,g_0}.
\end{aligned}$$

Therefore we can compute

$$\big(\Gamma_f^* \sigma^{-1}\big)_{p_0} = \big((\xi \circ \bar{\pi} \circ e)^* (\Phi^1_0 + i\Phi^2_0)\big)_{p_0} = \big(e^*(\Phi^1_0 + i\Phi^2_0)\big)_{p_0} = \varphi_{p_0} \qquad (4.56)$$

and likewise for σ^k and σ^3. This proves the holomorphicity of Γ_f outside the set of branch points of f. But since f is continuous and by assumption γ_f can be continuously extended to the branch points, then $\Gamma_f = f \times \gamma_f$ is continuous on M, and therefore holomorphic.
The same computation also proves that Γ_f is a solution of the Pfaffian system $\sigma^3 = 0$, $\sigma^2 - i\sigma^1 = 0$, since it is easily verified that

$$\begin{aligned}
\Gamma_f^* \sigma^3 &= 0, \\
\Gamma_f^* \sigma^1 &= -\frac{1}{2}\big(L^3 + iL^4\big)\varphi \\
\Gamma_f^* \sigma^2 &= -\frac{i}{2}\big(L^3 + iL^4\big)\varphi.
\end{aligned}$$

Moreover, (4.56) assures that

$$\Gamma_f^*\sigma^{-1} \neq 0.$$

On the contrary, assume $\Gamma : M \to \mathcal{Q}_2(Q_4)$ is a holomorphic map such that $\Gamma^*\sigma^3 = 0$, $\Gamma^*\sigma^2 = i\Gamma^*\sigma^1$ and $\Gamma^*\sigma^{-1} \neq 0$ and define $f_\Gamma = \check{\pi} \circ \Gamma$. For any local section ξ of $\bar{\pi}$, the map $e = \xi \circ \Gamma$ is a local frame along f_Γ, since

$$\pi \circ e = \pi \circ \xi \circ \Gamma = \check{\pi} \circ \bar{\pi} \circ \xi \circ \Gamma = \check{\pi} \circ \Gamma = f_\Gamma.$$

Moreover, let φ be a local $(1,0)$-form defining the complex structure on M; then, since Γ is holomorphic, there must exist a smooth function $\mu \not\equiv 0$ such that

$$e^*(\Phi_0^1 + i\Phi_0^2) = \Gamma^*\sigma^{-1} = \mu\varphi.$$

As usual, we set $\phi = e^*\Phi$, so that the previous equality becomes $\phi_0^1 + i\phi_0^2 = \mu\varphi$. Differentiating this last equality and using the structure equation we can deduce that

$$d\mu = -i\mu\big(\omega - \phi_2^1 + i\phi_0^0\big) \qquad \text{mod } \varphi,$$

where ω is such that $d\varphi = i\omega\wedge\varphi$. Hence μ is of analytic type, and its zeros must be isolated and of finite order, proving that f_Γ is a weakly conformal branched immersion. In addition, since by assumption $\Gamma^*\sigma^3 = 0$, we know that e is a first order frame along f_Γ. We can prove that e is actually a Darboux frame along f_Γ using

$$\Gamma^*\sigma^2 = i\Gamma^*\sigma^1. \tag{4.57}$$

Indeed, setting as usual $\phi_i^\alpha = h_{ij}^\alpha \phi_0^j$, $h_{ij}^\alpha = h_{ji}^\alpha$,

$$\Gamma^*\sigma^k = e^*(\Phi_3^k + i\Phi_4^k) = -\phi_k^3 - i\phi_k^4 = -(h_{kj}^3 + ih_{kj}^4)\phi_0^j$$

and equation (4.57) becomes

$$\begin{cases} h_{1j}^3 = h_{2j}^4 \\ h_{2j}^3 = -h_{1j}^4 \end{cases}$$

which gives

$$h_{11}^3 = h_{21}^4 = -h_{22}^3, \qquad h_{11}^4 = -h_{21}^3 = -h_{22}^4.$$

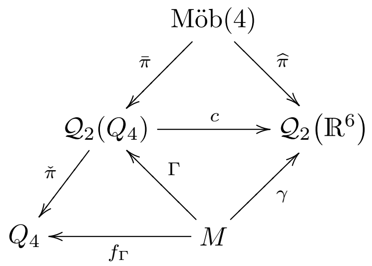
Now since $e = \xi \circ \Gamma$ is a Darboux frame along f_Γ, it makes sense to consider its conformal Gauss map, defined as usual as

$$\gamma_{f_\Gamma} = [e_3, e_4] = \widehat{\pi} \circ e$$

outside the branch points of f_Γ. We want to prove that γ_{f_Γ} can be continuously extended at the branch points, and that the extension is holomorphic. To this end, we define $\gamma : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ as follows

$$\gamma = c \circ \Gamma \tag{4.58}$$

and observe that Proposition 4.19 implies that γ is holomorphic. By the commutativity of the following diagram



we have that, on the open set where γ_{f_Γ} is defined,

$$\gamma_{f_\Gamma} = \widehat{\pi} \circ e = \widehat{\pi} \circ \xi \circ \Gamma = c \circ \bar{\pi} \circ \xi \circ \Gamma = c \circ \Gamma = \gamma.$$

Therefore γ_{f_Γ} is holomorphic, hence f_Γ is $-$ isotropic. Lastly, we obviously have

$$\Gamma_{f_\Gamma} = \bar{\pi} \circ e = \bar{\pi} \circ \xi \circ \Gamma = \Gamma$$

and

$$f_{\Gamma_f} = \breve{\pi} \circ \Gamma_f = \breve{\pi} \circ \bar{\pi} \circ e = \pi \circ e = f,$$

so the claim is proved. $\square$

Bibliography

[1] Beardon, A. F., *The Geometry of Discrete Groups*, Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983.

[2] Bers, L., *Theory of pseudoanalytic functions*, Institute of Mathematical Sciences, New York University, New York, 1952.

[3] Bredon, G. E., *Introduction to compact transformation groups*, Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972.

[4] Bryant, R. L., *A duality theorem for Willmore surfaces*, J. Differential Geometry, vol. **20** (1984), pp. 23–53.

[5] Bryant, R. L., *Conformal and minimal immersions of compact surfaces into the* 4*-sphere*, J. Differential Geometry, vol. **17** (1982), pp. 455–473.

[6] Chern, S. S., *On the Minimal Immersions of the Two-sphere in a Space of Constant Curvature*, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), pp. 27–40. Princeton Univ. Press, Princeton, N.J., 1970.

[7] Ejiri, N., *Willmore surfaces with a duality in* $S^N(1)$, Proc. London Math. Soc. (3), vol. **57** (1988), pp. 383–416.

[8] Eschenburg, J., Tribuzy, R., *Conformal mappings of surfaces and Cauchy-Riemann inequalities*, Differential geometry, pp. 149–170, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[9] Greub, W., Halperin, S., Vanston, R., *Connections, curvature, and cohomology Volume I*, Academic Press, 1972.

[10] Jensen, G. R., Rigoli, M., *Twistor and Gauss lifts of surfaces in four-manifolds*, Recent developments in geometry (Los Angeles, CA, 1987) Contemp. Math., 101, Amer. Math. Soc., Providence, RI, 1989 pp. 197–232.

[11] Rigoli, M., *The conformal Gauss map of submanifolds of the Möbius space*, Ann. Global Anal. Geom., vol. **5**, No. 2 (1987), pp. 97–116.

[12] Rigoli, M., Salavessa, I. M. C., *Willmore submanifolds of the Möbius space and a Bernstein-type theorem*, Manuscripta Math., vol. **81** (1993), pp. 203–222.

[13] Schiemangk, C., Sulanke, R., *Submanifolds of the Möbius space*, Math. Nachr., vol. **96** (1980), pp. 165–183.

[14] Sharpe, R. W., *Differential Geometry. Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics, 166. Springer-Verlag, New York, 1997.

[15] Sulanke, R., *Submanifolds of the Möbius space. II. Frenet formulas and curves of constant curvatures*, Math. Nachr., vol. **100** (1981), pp. 235–247.