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A Geometric Approach to the Trifocal Tensor

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Abstract In geometric computer vision the trifocal tensors are $3 \times 3 \times 3$ tensors T by whose means three different camera views of the same scene are related to each other. In this paper we find two different sets of *constraints*, in the entries of T, that must be satisfied by trifocal tensors. The first set gives exactly the (closure of the) trifocal locus, i. e. all trifocal tensors, but it is very big. The second set, although not complete and still very big, has the property that it is possible to extract from it a set of only eight equations that are generically complete, i.e. for a generic choice of T, they suffice to decide whether T is indeed trifocal. Note that 8 is the codimension of the trifocal locus in its ambient space.

Keywords Multiple view geometry · Trifocal tensor · Constraints

1 Introduction

The *trifocal tensor* is the mathematical object by whose means three different camera views of the same scene are related to each other—for a detailed account of the relevant theory, see [2], ch. 14 and 15, whose notations we follow.

An open problem is finding the *constraints* for the trifocal tensors, which can be formulated as follows: given a $3 \times 3 \times 3$ tensor $T = (t^i_{jk})_{i,j,k=1,2,3}$, find all equations, in the coefficients t^i_{jk}, that must be satisfied by T to be a trifocal tensor, for a suitable configuration of three cameras.

The straightforward solution, namely eliminating the parameters in the relations (6), section 4 below, is computationally intractable, due to the many variables involved, all of them in relations of second degree. So that, to get the first set of *constraints*, we use a little of group theory. We show that the set of trifocal tensors is invariant under a suitable action of a group (corresponding to a change of frames in the planes of cameras, then we use some classical results describing the orbits of this action to find the constraints.

Unfortunately, it turns out that the constraints so found are "too many", in fancier language this means that the (closure of the) set of trifocal tensors is not a complete intersection. Thus, by using elementary geometric arguments, we also determine another set of constraints having the "right number" of elements, namely eight, the codimension of the trifocal locus in its ambient space. This second set of constraints is not complete, but it is *generically* complete, i.e. for a generic choice of a $3 \times 3 \times 3$ tensor T, it suffices to decide whether T is indeed trifocal.

The problem of finding constraints for trifocal tensors has been tackled by several authors, cf. [3], [4], [5], main differences with our results being as follows.

The constraints of Papadopoulo and Faugeras [4] do not give a complete characterization of trifocal tensors, but, just like the ones we find in section 4, a *generically* complete one, in the sense described above; although of low degrees (degrees three and six) their set of constraints contains twelve polynomials, more than the the possible minimum, eight, corresponding to the codimension of the set of all trifocal tensors.

Ressl's constraints, see his PhD thesis [5], are similarly generically complete, and have the "correct" number, i.e. eight constraints; our set of constraints has the advantage of lower degrees, namely degrees three, five and

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six, as opposed to Ressl's degrees three, five and eight. Heyden's paper [3] is different in its scope, as it covers the whole field of multiple view constraints; in particular, concerning trifocal tensors, the constraints found there are similar to those in [4], and the same observations apply, i. e. they are more than the codimension of the trifocal tensor locus.
In conclusion, the main difference between our paper and other known result is twofold: in the first place, we exhibit a new generically complete set of eight constraints that arises from elementary geometric constructions and whose polynomials are of lower degree; besides, we give also another set of constraints (see theorem 23) that is complete–i.e. characterizes completely the closure of the set of all trifocal tensors–even though it consists of 36 polynomials, (of degrees three, nine and twelve) instead of the eight polynomials of the generically complete set of constraints.

The organization of the paper is as follows.
In section 2 we recall the definition of the trifocal tensor, we give the definition of the *trifocal locus* Θ, the smallest algebraic variety containing all trifocal tensors, and we give our main theorem getting a complete set of constraints for Θ in a very simple way. In section 3 we define a rational *trifocal map* such that Θ is the closure of the image of this map; we define an equivariant group action for the trifocal map, and we show that Θ is (the closure of) an orbit of this action. Then we use this fact to recover the same set of constraints with a more abstract, but shorter, proof. At the end of this Section we also relate Θ to a suitable Segre product in Algebraic Geometry to give another proof that the trifocal locus has dimension 18 (see [2] p. 358).
In section 4 we derive another set of constraints starting with the relations (6) giving the coefficients of a trifocal tensor in terms of the entries of the matrices representing the three cameras. Then we extract, from this larger set, a subset of only eight constraints, by using arguments of linear algebra. Although not complete, this set of eight constraints is *generically complete*, i. e. the locus defined by it contains Θ as a component of maximal dimension. Besides being much simpler, this set seems also better suited for applications. At the end of this section we also give a complete description of all irreducible components of the larger set of constraints. This description is useful to extract other sets of eight constraints in case other sets are needed.
Section 5 is devoted to the conclusions.
In the Appendix we have collected some mathematical definitions and results used in the paper.

2 A complete set of constraints for the Trifocal tensor

We denote by $\mathbb{K}$ a field of characteristic zero; for our purposes, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$; also, we denote $V := \mathbb{K}^3, W := \mathbb{K}^4$.

Definition 21 *A* camera *is a projection C from a point P, the center of the camera,*

$$C : \mathbb{P}^3 = \mathbb{P}(W) \dashrightarrow \mathbb{P}^2 = \mathbb{P}(V).$$

A camera C comes from a (surjective) linear map, still denoted by C, $C : W \to V$, hence $C \in V \otimes W^*$; since the linear map is determined up to a nonzero factor, C is actually an element of $\mathbb{P}(V \otimes W^*)$. Choosing coordinates, C is represented by a 3×4 matrix of rank 3, $M \in \mathrm{M}_{3\times 4}(\mathbb{K}), M = (m_i^j), i = 1, 2, 3, j = 1, 2, 3, 4$.
We shall be more interested in the dual map $C^* : \mathbb{P}^{2*} = \mathbb{P}(V^*) \to \mathbb{P}^{3*} = \mathbb{P}(W^*)$, coming now from the dual (injective) linear map $C^* : V^* \to W^*$; of course $C = C^*$ as elements of $V \otimes W^*$. In dual coordinates, C^* is represented by $M^\top$, the transposed of M.
Given three cameras $C^*, C'^*, C''^* : \mathbb{P}(V^*) \to \mathbb{P}(W^*)$, and three lines $l, l', l'' \in \mathbb{P}(V^*)$, we say that they (the lines) are *concurrent* if the planes $\pi = C^*(l), \pi' = C'^*(l'), \pi'' = C''^*(l'') \in \mathbb{P}(W^*)$ intersect along a line $\ell \subset \mathbb{P}(W)$; it is equivalent to: $C(\ell) = l, C'(\ell) = l', C''(\ell) = l''$.
The map

$$T : \mathbb{P}(V^*) \times \mathbb{P}(V^*) \to \mathbb{P}(V^*),$$

is defined when the planes $C'^*(l'), C''^*(l'')$ are different; in this case the condition defining T is: $T(l', l'') = l$ if and only if l, l', l'' are concurrent.
The map T comes from a bilinear map, still denoted by $T : V^* \times V^* \to V^*$; in other words, $T \in V \otimes V \otimes V^*$ is a tensor, called *trifocal tensor*. As in the case of cameras above, trifocal tensors actually belong to $\mathbb{P}(V \otimes V \otimes V^*) = \mathbb{P}^{26}$.
Recall that a *Zariski closed set* is the zero-locus of any finite set of polynomial equations in $\mathbb{P}^{26}$. Our aim is to give a complete set of *constraints* for trifocal tensors in $\mathbb{P}^{26}$ in the following sense: find a minimal (necessarily finite) set of polynomial equations, defining a Zariski closed set, which we will call the *Trifocal Locus* Θ, such that every generic point of Θ is indeed a trifocal tensor. Of course it is not possible to find a finite set of polynomial equations in $\mathbb{P}^{26}$ such that a point is a trifocal tensor if and only if it belongs to the zero-locus of the polynomials, because zero-loci are closed set (in the Zariski topology), while to be a trifocal tensor is an open condition, giving rise to an open set (in the Zariski topology).

To find the constraints we are aiming for, we will use the fact that the group $GL(V)^3$ acts naturally on $V \otimes V \otimes V^*$, the action being–basically–a change of basis in the vector space V (for the definition of a group action, see Appendix, III). This action was thoroughly investigated in [7], whose results we will use.
We now give a brief summary of those results.
Given a tensor $A \in V \otimes V \otimes V$, in a fixed coordinate system A is represented by a $3 \times 3 \times 3$ numerical tensor $A = (a_{ijk})_{i,j,k=1,2,3}$; in the notation of [7], we can identify A with a trilinear form (on V) $F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{ijk} a_{ijk} x_i y_j z_k$, where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{z} = (z_1, z_2, z_3)$ are three sets of three variables each.
The matrix $H_x = (\sum_i a_{ijk} x_i)_{i,j=1,2,3}$, associated to A (or F), is a 3×3 matrix whose entries are linear forms in the variables x_i; define $X(x) := \det(H_x)$, a cubic form in the x_i. Similarly, we consider the matrices $H_y = (\sum_j a_{ijk} y_j)_{i,k=1,2,3}$ and $H_z = (\sum_k a_{ijk} z_k)_{j,k=1,2,3}$, and define the corresponding cubic forms $Y(y) := \det(H_y)$ and $Z(z) := \det(H_z)$.
The main results of [7] are:
(i) the projective classes of the plane cubics, given by the following equations $X(x) = 0, Y(y) = 0, Z(z) = 0$ in $\mathbb{P}^2$, are invariants of the tensor A with respect to the G^3 action, defined in [7], on the trilinear form F ;
(ii) the projective classes of the plane cubics determine the orbit of the tensor A.
To prove our main theorem we will need the following.

Lemma 22 *A (non zero) homogeneous, real or complex coefficients, degree* 3*, equation in two variables* $(t:s)$ *of the following type:* $at^3 + bt^2s + cts^2 + ds^3$ *has at least a multiple root if and only if* $b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2 = 0$.

Proof: Let f be the polynomial $at^3 + bt^2 + ct + d$.
Let us assume that $a \neq 0$. Then our equation has at least a multiple root if and only if $R(f, f') = 0$ where R is the usual resultant of polynomials. It is easy to see that, in this case, $R(f, f') = 0$ if and only if $b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2 = 0$.
Let us assume $a = 0$ and let g be the polynomial $bt^2 + ct + d$. Then our equation has at least a multiple root if and only if $R(g, g') = 0$. It is easy to see that, if $b \neq 0$, then $R(g, g') = 0$ if and only if $b^2c^2 - 4b^3d = 0$. If $b = 0$ the relation is identically satisfied, but in this case, our equation has the multiple root $(1 : 0)$. $\square$

We can now prove our main theorem. The proof consists in translating the results of [7] in the setting of trifocal tensors.

Theorem 23 *A complete set of constraints for the trifocal locus* Θ *is given by 10 equations of degree three, 20 equations of degree nine and 6 equations of degree twelve on the* 27 *entries of a generic* $3 \times 3 \times 3$ *tensor.*

Proof: We will follow the conventions of [6], [7].
To give a description of the trifocal locus as the closure of a suitable orbit of $V \otimes V \otimes V^*$ we choose a generic trifocal tensor and we identify the corresponding orbit, showing that the $GL(V)^3$–action introduced earlier is the same as the one considered in [6], [7]. Then we degenerate trifocal tensors to detect points in the closure of the orbit not corresponding to trifocal tensors.
We fix once and for all a coordinate system (x, y, z, u) in $\mathbb{P}^3$, such that the center of the first camera is $(0, 0, 0, 1)$ and $u = 0$ is the plane at the infinity, as in [2]. Then we can assume that the matrices of the three cameras have the following form:

$$M = [C|\mathbf{0}],\ M' = [A|\mathbf{a}_4],\ M'' = [B|\mathbf{b}_4]$$

where C, A, B are 3×3 matrices.
As we are dealing with a trifocal tensor we assume that:

(I) C, A, B are non singular matrices

(II) the centers of the cameras are distinct and in general position in $\mathbb{P}^3$.

Choose three coordinate system on $\mathbb{P}(V^*)$, denoted by $\mathbf{w}, \mathbf{x}, \mathbf{z}$, thinking of them as the (dual) coordinates in the image planes of the three cameras, i.e. the coordinates of the generic lines l, l', l'' in the three cameras are $\mathbf{w}, \mathbf{x}, \mathbf{z}$ respectively; in this Section we denote by $\mathbb{P}(V^*)$, $\mathbb{P}(V^*)'$, $\mathbb{P}(V^*)''$ the (dual) image planes of the three cameras under consideration. Let $\mathbf{y}$ be the coordinates of the generic element of $\mathbb{P}(V)$.
According to [2], p. 357, three lines l, l', l'' are linked by the trifocal relation if and only if the 4×3 matrix $N := \begin{bmatrix} C^\top\mathbf{w} & A^\top\mathbf{x} & B^\top\mathbf{z} \\ \mathbf{0} & \mathbf{a}_4^\top\mathbf{x} & \mathbf{b}_4^\top\mathbf{z} \end{bmatrix}$ do not have maximal rank.
It means that there are $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that $\gamma\mathbf{n}_1 = \alpha\mathbf{n}_2 + \beta\mathbf{n}_3$
where $\mathbf{n}_i$ are the columns of N.
By assumption (II) we know that $\mathbf{a}_4$ and $\mathbf{b}_4$ are not $\mathbf{0}$, so that we can write the previous relation in the following form, for a suitable γ (possibly 0):
$\gamma\mathbf{n}_1 = (\mathbf{b}_4^\top\mathbf{z})\mathbf{n}_2 - (\mathbf{a}_4^\top\mathbf{x})\mathbf{n}_3$ i.e.
$\gamma C^\top\mathbf{w} = (\mathbf{b}_4^\top\mathbf{z})A^\top\mathbf{x} - (\mathbf{a}_4^\top\mathbf{x})B^\top\mathbf{z} = (\mathbf{z}^\top\mathbf{b}_4)A^\top\mathbf{x} - (\mathbf{x}^\top\mathbf{a}_4)B^\top\mathbf{z}$
so that the i–th coordinate of $\gamma C^\top\mathbf{w}$, $i = 1, 2, 3$, is given by $[\gamma C^\top\mathbf{w}]_i = \mathbf{x}^\top T^i\mathbf{z}$ with $T^i = \mathbf{a}_i\mathbf{b}_4^\top - \mathbf{a}_4\mathbf{b}_i^\top$ (see [2], p. 357).
Now, if $C = I_3$, $\gamma \neq 0$, for any pair of lines $l' \longleftrightarrow \mathbf{x}$ and $l'' \longleftrightarrow \mathbf{z}$ the third corresponding line in $\mathbb{P}(V)$ has equation (up to a factor γ) $y_1(\mathbf{x}^\top T^1\mathbf{z}) + y_2(\mathbf{x}^\top T^2\mathbf{z}) + y_3(\mathbf{x}^\top T^3\mathbf{z}) = 0$. i.e.

$$\mathbf{x}^\top[y_1T^1 + y_2T^2 + y_3T^3]\mathbf{z} = 0. \tag{1}$$

If $C \neq I_3$, $\gamma \neq 0$, the third corresponding line has equation, up to a factor γ:

$$\mathbf{x}^\top[y_1'T^1 + y_2'T^2 + y_3'T^3]\mathbf{z} = 0, \tag{2}$$

where $\mathbf{y}' = C^{-1}\mathbf{y}$. Note that (1) and (2) include also the case $\gamma = 0$, so that they are equivalent to the trifocal relation.
The conclusion is that the natural action of any $(g_1, g_2, g_3) \in GL(V)^3$ on the three cameras, i.e. the multiplication on the left of the three matrices M, M', M'' by three non singular $(3,3)$ matrices, is exactly the action of $GL(V)^3$ on the trilinear form (1) considered by [7]: g_1 is a linear transformation on $\mathbb{P}(V)$, i.e. on $\mathbf{y}$ coordinates, g_2 is a linear transformation on $\mathbb{P}(V^*)'$, i.e. on $\mathbf{x}$ coordinates, and g_3 is a linear transformation on $\mathbb{P}(V^*)''$, i.e. on $\mathbf{z}$ coordinates.
Any trilinear form of type (1) is determined by a triple $[T^1, T^2, T^3]$, and viceversa such a triple, up to multiplication by a non zero constant, fixes a trilinear form as (1). On the other hand, we have seen that a trifocal tensor, which is defined up to a non zero constant, determines a suitable triple $[T^1, T^2, T^3]$ up to a non zero constant, i.e. a unique trilinear form as (1). From now on we identify trifocal tensors and trilinear forms, both denoted by a triple $[T^1, T^2, T^3]$.
Let us choose any trifocal tensor (recall assumptions (I) and (II)) and let us choose a suitable $(g_1, g_2, g_3) \in GL(V)^3$ such that the original trifocal tensor is transformed in another one for which $C = A = B = I_3$. In this case we have: $\mathbf{a}_4 = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$ and $\mathbf{b}_4 = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$ for some numbers b, c, d, f, g, h,
the center of the second camera is $(-b, -c, -d, 1)$, the center of the third camera is $(-f, -g, -h, 1)$ and the matrix $\begin{bmatrix} 0 & 0 & 0 & 1 \\ -b & -c & -d & 1 \\ -f & -g & -h & 1 \end{bmatrix}$ has maximal rank.
If we compute the corresponding trilinear form we get:

$$T^1 = \begin{bmatrix} f+b & g & h \\ c & 0 & 0 \\ d & 0 & 0 \end{bmatrix},$$

$$T^2 = \begin{bmatrix} 0 & b & 0 \\ f & g+c & h \\ 0 & d & 0 \end{bmatrix},$$

$$T^3 = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ f & g & h+d \end{bmatrix}.$$

To find (in the list of [7], p. 689) the orbit in which the trilinear form sits, we consider the three following plane cubics: $\det(y_1T^1 + y_2T^2 + y_3T^3) = 0, \det(z_1A^1 + z_2A^2 + z_3A^3) = 0, \det(x_1B^1 + x_2B^2 + x_3B^3) = 0$
where the 3×3 matrices A^i are given by the first, second, third columns of the matrices T^i and the matrices B^i are given by the first, second, third rows of the matrices T^i respectively. A straightforward computation shows that $\det(y_1T^1 + y_2T^2 + y_3T^3) \equiv 0, \det(z_1A^1 + z_2A^2 + z_3A^3) = (fz_1 + gz_2 + hz_3)^2[(f+b)z_1 + (g+c)z_2 + (h+d)z_3] = 0, \det(x_1B^1 + x_2B^2 + x_3B^3) = (bx_1 + cx_2 + dx_3)^2[(f+b)x_1 + (g+c)x_2 + (h+d)x_3] = 0$
i.e. one cubic is identically zero and the other two are reducible, both being the union of a double line and another line. The table of [7], p. 689 shows that this is sufficient to identify the orbit. Let us call it O. Again by [7], we can also pick a very simple representative of this orbit, namely

$$T^1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now let us consider the closure of the orbit. If we drop assumption (II), then the matrix $\begin{bmatrix} b & c & d \\ f & g & h \end{bmatrix}$ has rank 1 (or 0), hence the cubics $\det(z_1A^1 + z_2A^2 + z_3A^3) = 0$ and $\det(x_1B^1 + x_2B^2 + x_3B^3) = 0$ are reducible, both of them as a triple line (or are identically zero). The corresponding trilinear forms (when non zero) belong to other two orbits of [7]'s list, which are characterized by the following type of $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ cubics: *(triple line, zero, triple line)* and *(zero, zero, zero)*.
(By *zero* we mean that the corresponding cubic is identically zero.)
If we drop assumption (I), it is easy to see that the corresponding trilinear form is identically zero or it belongs to the orbit *(zero, zero, zero)*. In any case the trifocal locus, according to our definition, is the closure of the orbit O, so to get a set of equations defining it we can use the characterization of this orbit given by [7].
In other words, a triple $[T^1, T^2, T^3]$ (considered up to a non zero common factor) belongs to the trifocal locus (be careful: this does not mean that it corresponds to a trifocal tensor) if and only if

$$\det(y_1T^1 + y_2T^2 + y_3T^3) \equiv 0 \tag{3}$$

and

$$\begin{aligned} \det(z_1A^1 + z_2A^2 + z_3A^3) = 0 \\ \det(x_1B^1 + x_2B^2 + x_3B^3) = 0 \end{aligned} \tag{4}$$

are reducible cubics, both the union of a double line and another line.
Given a triple $[T^1, T^2, T^3]$ as above, i.e. a point in $\mathbb{P}^{26}$, (3) translates into 10 equations of third degree.
To satisfy (4), we require firstly, that both cubics be

actually three concurrent lines, and then that two of those lines be the same.
The condition that a cubic be three concurrent lines is equivalent to ask to its Hessian being identically zero, and this gives 10 equations of degree 9.
Now, a plane cubic, that is the union of three concurrent lines, actually contains a double line if and only if its intersection with each of the three coordinate axes is not three distinct points. Note that we need to control all the axes because one, or even two, of them might be a component of the cubic (or contain the triple point). Thanks to lemma 22, this condition is equivalent to three equations of degree 4 in the coefficients of the cubic, hence of degree 12 in the entries of $[T^1, T^2, T^3]$. Since both cubics must be of this type, there are then 20 equations of degree 9 and six of degree 12.
We have thus proved that a complete set of equations cutting out the trifocal locus in $\mathbb{P}^{26}$ is given by 10 equations of degree $3, 20$ of degree 9 and 6 of degree 12. $\square$

Remark 24 We give an interpretation, in our setting, of some properties of the orbit O, defined in the previous proof. According to the results in [7], we know that, in this case, there are:
only two matrices $y_1T^1 + y_2T^2 + y_3T^3$ having rank 1;
infinitely many matrices $z_1A^1 + z_2A^2 + z_3A^3$ having rank 1;
infinitely many matrices $x_1B^1 + x_2B^2 + x_3B^3$ having rank 1.
The first two matrices are those for which $(y_1, y_2, y_3) = (b, c, d)$ and $(y_1, y_2, y_3) = (f, g, h)$; the two last sets are given by matrices satisfying, respectively: $fz_1 + gz_2 + hz_3 = 0$ and $bx_1 + cx_2 + dx_3 = 0$.
Recalling that any trilinear form in this orbit is a trifocal tensor, it is easy to see that the previous statements have to be true: the line passing through the center of the first and the second camera has the following point at the infinity (see [2]) $(b, c, d, 0)$, hence $bx_1 + cx_2 + dx_3 = 0$ is precisely the pencil of planes in $\mathbb{P}^3$ passing through this line, given in the coordinates of $\mathbb{P}(V^*)'$. On the other hand the line passing through the center of the first and the third camera has the following point at the infinity (see [2]) $(f, g, h, 0)$, thus $fz_1 + gz_2 + hz_3 = 0$ is precisely the pencil of planes in $\mathbb{P}^3$ passing through this line, given in the coordinates of $\mathbb{P}(V^*)''$. Moreover the first two matrices with rank 1 correspond to the intersections of these lines with the plane $u = 0$, which is the equation of $\mathbb{P}(V)$ embedded in $\mathbb{P}^3$, having chosen the center of the first camera as $(0, 0, 0, 1)$.

3 The Trifocal Map

In this section we give an alternate proof of theorem 23 by defining a suitable map, the trifocal map, and showing that Θ is the closure of the image of this map; thus the theorem follows immediately form the results of [7]. The definition of the trifocal map requires some preliminaries though.
We start by choosing coordinates in V, T is a $3 \times 3 \times 3$ tensor with coefficients in $\mathbb{K}$, i.e. $T = (t^i_{jk}), i, j, k = 1, 2, 3$. Since the choice of coordinates in V (and in W) determines the matrices $M = (m^h_g)$,$M' = (m'^q_p)$,$M'' = (m''^s_r)$, $g, p, r = 1, 2, 3, h, q, s = 1, 2, 3, 4$, (whose transposed matrices represent the cameras C^*, C'^*, C''^* respectively), we want to express the coefficients t as functions of the m, m', m''.
Let us fix coordinates in V and W. Let $l = \tilde{\mathbf{a}}, l' = \tilde{\mathbf{b}}, l'' = \tilde{\mathbf{c}} \in \mathbb{K}^3$, then the corresponding planes are $C^*(l) = \tilde{a}_1\mathbf{m}_1^\top + \tilde{a}_2\mathbf{m}_2^\top + \tilde{a}_3\mathbf{m}_3^\top$, $C'^*(l') = \mathbf{b} = M'^\top\tilde{\mathbf{b}}$, $C''^*(l'') = \mathbf{c} = M''^\top\tilde{\mathbf{c}} \in \mathbb{K}^4$—recall that M, M', M'' are 3×4 matrices and especially $M = \begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\\ \mathbf{m}_3\end{pmatrix}$. Now $T(\tilde{\mathbf{b}}, \tilde{\mathbf{c}}) = \tilde{\mathbf{a}}$ if and only if the planes $C^*(l), C'^*(l'), C''^*(l'')$ intersect along a line, if and only if $M^\top\tilde{\mathbf{a}}, \mathbf{b}, \mathbf{c}$ are linearly dependent, if and only if $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ are solutions of the 4×5 homogeneous linear system

$$\tilde{a}_1\mathbf{m}_1 + \tilde{a}_2\mathbf{m}_2 + \tilde{a}_3\mathbf{m}_3 + \lambda\mathbf{b} + \mu\mathbf{c} = \mathbf{0}.$$

These solutions are given by the determinants of the 4×4 minors, especially $\tilde{a}_i = \epsilon(irs)\det(\mathbf{m}_r\ \mathbf{m}_s\ \mathbf{b}\ \mathbf{c})$, where $\{i, r, s\} = \{1, 2, 3\}$ and $\epsilon(irs) = \pm 1$ is the sign of the permutation $\begin{pmatrix}1 & 2 & 3\\ i & r & s\end{pmatrix}$(see Appendix, I). Choosing as $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}$ the canonical basis vectors of $\mathbb{K}^3$, we get the entries of the tensor T. We can collect the previous arguments in the following proposition.

Proposition 31 *The entries of the trifocal tensor T, relative to the matrices*
$M = \begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\\ \mathbf{m}_3\end{pmatrix}, M' = \begin{pmatrix}\mathbf{m}'_1\\ \mathbf{m}'_2\\ \mathbf{m}'_3\end{pmatrix}, M'' = \begin{pmatrix}\mathbf{m}''_1\\ \mathbf{m}''_2\\ \mathbf{m}''_3\end{pmatrix}$, *are*

$$t^i_{jk} = \epsilon(irs)\det\begin{pmatrix}\mathbf{m}_r\\ \mathbf{m}_s\\ \mathbf{m}'_j\\ \mathbf{m}''_k\end{pmatrix} \quad i, j, k = 1, 2, 3. \tag{5}$$

In a more intrinsic fashion, to give the previous correspondence we can define the *trifocal map*

$$\mathcal{T} : (V \otimes W^*)^3 \to V \otimes V \otimes V^*.$$

To describe $\mathcal{T}$ let us define:
1) the "diagonal" map $\delta : V \otimes W^* \to (V \otimes W^*)^{\otimes 2}$ such

that $\delta(z) = z \otimes z$;
2) the natural multiplication map $\mu : (V \otimes W^*)^{\otimes 2} \to \wedge^2 V \otimes \wedge^2 W^*$, defined on decomposable tensors by $\mu((v \otimes w) \otimes (\bar{v} \otimes \bar{w})) := v \wedge \bar{v} \otimes w \wedge \bar{w}$ and then extended by linearity (for a definition of wedge product, see Appendix, II);
3) the composition $\sigma := \mu \circ \delta : V \otimes W^* \to V^* \otimes \wedge^2 W^*$ (since $\dim V = 3$, there is a canonical identification, up to a nonzero factor $\wedge^2 V \simeq V^*$);
4) the natural multiplication map $\hat{\mu} : V^* \otimes \wedge^2 W^* \times (V \otimes W^*)^2 \to (V \otimes V \otimes V^*) \otimes \wedge^4 W^*$, given by $\hat{\mu}(v \otimes w_1 \wedge w_2, \bar{v} \otimes \bar{w}, \tilde{v} \otimes \tilde{w}) := (\bar{v} \otimes \tilde{v} \otimes v) \otimes (w_1 \wedge w_2 \wedge \bar{w} \wedge \tilde{w})$, extended by linearity. Since $\dim W = 4$, by using again the canonical identification $\wedge^4 W^* \simeq \mathbb{K}$, we get in fact $\hat{\mu} : V^* \otimes \wedge^2 W^* \times (V \otimes W^*)^2 \to V \otimes V \otimes V^*$.
Now we can describe $\mathcal{T}$ with the following proposition.

Proposition 32 *For any* $(C^*, C'^*, C''^*) \in (V \otimes W^*)^3$ *we have:*
$\mathcal{T}(C^*, C'^*, C''^*) = \hat{\mu}(\sigma(C^*), C'^*, C''^*)$.

Proof: Choose lines $l', l'' \in \mathbb{P}^2$, i.e. $l', l'' \in \mathbb{P}(V^*)$, then $C'^*(l'), C''^*(l'') \in \mathbb{P}(W^*)$ are planes whose intersection is the line of $\mathbb{P}^3$ represented by $C'^*(l') \wedge C''^*(l'') \in \wedge^2 W^*$. The projection $C : W \to V$ induces the map $\wedge^2 C : \wedge^2 W \to \wedge^2 V$ i.e. $\wedge^2 C \in \wedge^2 V \otimes \wedge^2 W^*$. This map, restricted to the respective Grassmannians, (varieties parametrizing lines in projective spaces) is the projection from lines of $\mathbb{P}^3$ to lines of $\mathbb{P}^2$. By using the canonical identification $\wedge^2 V \simeq V^*$, it is easy to see that $\wedge^2 C = (\sigma(C^*))$. Thus, the image of the lines l' and l'' (with respect to the trifocal map T determined by C, C', C'') is the image of the line $C'^*(l') \wedge C''^*(l'')$ with respect to the map $\wedge^2 C$, hence, again with the identification $\wedge^4 W^* \simeq \mathbb{K}$, the map T is represented by the tensor $\hat{\mu}(\sigma(C^*), C'^*, C''^*)$. □

The trifocal map induces a rational map, still referred to as trifocal map,

$$\mathfrak{T} : (\mathbb{P}(V \otimes W^*))^3 \to \mathbb{P}(V \otimes V \otimes V^*).$$

Note that the *trifocal locus* Θ is nothing else than the closure (in the Zariski topology) of the image of the trifocal map, i.e.

$$\Theta = \overline{\mathrm{Im}\mathfrak{T}}.$$

As before, the problem of finding conditions for being a trifocal tensor is thus to determine the ideal of Θ, i.e. to find the equations of the trifocal locus in $\mathbb{P}^{26}$. As before, the key point is to use group actions
The map $\mathfrak{T}$ is invariant under the natural action of the group $\Gamma := GL(W^*)$ on $\mathbb{P}(V \otimes W^*)$; it is also covariant under the action of the group $G^3 := GL(V)^3$ in the following sense: an element of G^3 acts on $(V \otimes W^*)^3$ in the natural way, while it acts on $V \otimes V \otimes V^*$ as follows. The dual action of G on V^* is, up to a constant factor, given by $u \wedge v \mapsto u^g \wedge v^g$, via the natural identification $\wedge^2 V = V^*$ (recall that $\dim V = 3$). Choosing coordinates in V, and the corresponding dual coordinates in V^*, an element $g \in GL(V)$ is represented by a 3×3 matrix, say $g = (\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3)$ (column vectors). An element of $GL(V^*)$, denoted by g^*, is represented by the matrix $g^* = (\mathbf{g}_2 \wedge \mathbf{g}_3\ \ \mathbf{g}_3 \wedge \mathbf{g}_1\ \ \mathbf{g}_1 \wedge \mathbf{g}_2)$ with respect to the dual coordinates in V^*.
Using this representation and (5), it is straightforward, albeit quite tedious, to verify that the map $\mathfrak{T}$ is indeed G^3–covariant.
These remarks and the following theorem will allow us to use the results about G^3–action (on $V^{\otimes 3}$), contained in [6], [7].

Theorem 33 *The trifocal locus* Θ *is (the closure of) an orbit of the* $GL(V)^3$*–action (on* $V \otimes V \otimes V^*$*) previously considered.*

Proof: Let M_1, M_2, M_3 be the matrices denoted by M, M', M'' in proposition 31. Any generic $T \in \Theta$ is of type $T = \mathfrak{T}(M_1, M_2, M_3)$, with $M_i \in V \otimes W^*$. Let $\mathbf{g} = (g_1, g_2, g_3)$ be any element in G^3. As the action is covariant, $T^{\mathbf{g}} = \mathfrak{T}(M_1^{g_1}, M_2^{g_2}, M_3^{g_3})$, hence $T^{\mathbf{g}}$ is still a trifocal tensor; in other words, $\mathrm{Im}\mathfrak{T}$ is G^3-invariant.
Conversely, given $M_i \in V \otimes W^*$, $i = 1, 2, 3$, representing three cameras in general position, their centers are three points $P_i = \ker M_i$ in $\mathbb{P}^3 = \mathbb{P}(W)$ in general position—i.e. not collinear. Thus, there is $\sigma \in \Gamma$ such that $P_1^\sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, P_2^\sigma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, P_3^\sigma = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, hence $N_1 := M_1^\sigma$ is a matrix whose fourth column is zero and $N_j := M_j^\sigma, j = 2, 3$, are matrices whose j-th and fourth columns are equal. Since $\mathrm{rk} N_i = \mathrm{rk} M_i = 3$, there are $g_i \in G$ such that $D_i := N_i^{g_i}$, $i = 1, 2, 3$ is of the following type:

$$D_1 = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \end{pmatrix}, D_2 = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 1 \\ 0\ 0\ 1\ 0 \end{pmatrix}, D_3 = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 1 \end{pmatrix}.$$

As the trifocal map $\mathfrak{T}$ is Γ-invariant and G^3-covariant, it follows that, for the generic trifocal tensor $T \in \Theta$, $T = \mathfrak{T}(M_1, M_2, M_3) = \mathfrak{T}(N_1, N_2, N_3) = \mathfrak{T}(D_1^{h_1}, D_2^{h_2}, D_3^{h_3}) = \mathfrak{T}(D_1, D_2, D_3)^{(h_1, h_2, h_3)} = \Delta^{(h_1, h_2, h_3)}$, where $h_i = g_i^{-1}$ and $\Delta := \mathfrak{T}(D_1, D_2, D_3)$.
We have shown that, on a dense open subset of Θ, the action of G^3 is transitive, so the proof is complete. □

The previous proof also shows that the set of trifocal tensors is the orbit the tensor $\Delta := \mathfrak{T}(D_1, D_2, D_3)$, with the D_i defined above. $\Delta = (d^i_{jk})$ can be explicitly computed using (5)—see also (7) below. Writing

$\varDelta^i := (d^i_{jk})_{jk},\ i = 1, 2, 3$, we have:

$$\varDelta^1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \varDelta^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \varDelta^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

As we have seen in Section 2 (see the proof of theorem 23), the action of G^3 on the trilinear form F considered in [7], corresponds to the action of G^3 on the tensor T considered in theorem 33. Then we can detect the orbit of $\varTheta$ by constructing the trilinear form associated to a particular element of $\mathrm{Im}\mathfrak{T}$ and by using the classification of [7]. For instance, if we consider the tensor $\varDelta = (d_{ijk})$—we now write $d_{ijk} = d^i_{jk}$—we get $X(x) \equiv 0, Y(y) = y_3^2(y_3 - y_2), Z(z) = z_2^2(z_3 - z_2)$.

Looking at the table of [7] p. 689, we see that there is a unique orbit with these invariants—it is the one corresponding to the entry in the second row and last column in that table.

Hence, a numerical tensor A is a trifocal tensor if and only if the associated cubics are of the following types: $X(x)$ is identically zero and both $Y(y)$ and $Z(z)$ are reducible, each one being the union of a double line and another line.

To translate these conditions into equations on the entries of A we recall that the coefficients of $X(x), Y(y), Z(z)$ are in turn homogeneous polynomials of degree three in the entries t^i_{jk} of A, thus $X(x) \equiv 0$ amounts to impose that all these 10 coefficients are zero, i.e. ten cubic equations.

The requirement that $Y(y)$ is the union of a double line and another line is equivalent to the following two conditions:

(i) $Y(y)$ has a triple point (so that the curve is reducible into 3 concurrent lines) and

(ii) given any three non concurrent lines, e.g. the coordinate lines (in $\mathbb{P}^2$), each of them has at least a double intersection with the curve $Y(y)$.

Condition (i) translates into imposing that Hessian of $Y(y)$ be identically zero. The Hessian being in this case a form of degree three (in $\mathbf{y}$), whose coefficients are homogeneous polynomials of degree nine in the t^i_{jk}, condition (i) is eventually equivalent to ten equations of degree nine.

Condition (ii) is equivalent, by lemma 22, to three equations of degree 4 in the coefficients of $Y(y)$, hence of degree 12 in the t^i_{jk}. Clearly the number of equations arising from conditions (i) and (ii) must be doubled, taking into account $Z(z)$.

In this way we have (re)-proved our main theorem in a shorter way and without any calculation.

We now relate briefly the trifocal locus to a particular Segre embedding. In this way we will prove that its dimension is 18 by a different count of parameters with respect to the strategy used in [2] p. 358.

Let X be the Segre embedding of the product $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$. Any point in $\mathbb{P}^8$ can be viewed as a 3×3 matrix and the Segre variety is exactly the locus of matrices having rank ≤ 1. The tangent variety TX is the cubic hypersurface which is the locus of matrices having rank ≤ 2. We can consider any T^i as a point in this $\mathbb{P}^8$ and the set $T = y_1T^1 + y_2T^2 + y_3T^3$ as a plane in this $\mathbb{P}^8$. If $[T^1, T^2, T^3]$ is a trifocal tensor then we know that $T^i = \mathbf{a}_i\mathbf{b}_4^\top - \mathbf{a}_4\mathbf{b}_i^\top$. It means that we have chosen a point $(\mathbf{a}_4, \mathbf{b}_4) \hookrightarrow \mathbf{a}_4\mathbf{b}_4^\top \in \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, three points in the plane $\mathbb{P}^2 \times \mathbf{b}_4 \subset \mathbb{P}^8$ i.e. $(\mathbf{a}_i, \mathbf{b}_4) \hookrightarrow \mathbf{a}_i\mathbf{b}_4^\top$, three points in the plane $\mathbf{a}_4 \times \mathbb{P}^2 \subset \mathbb{P}^8$ i.e. $(\mathbf{a}_4, \mathbf{b}_i) \hookrightarrow \mathbf{a}_4\mathbf{b}_i^\top$, and one point on each line $\langle(\mathbf{a}_i, \mathbf{b}_4), (\mathbf{a}_4, \mathbf{b}_i)\rangle \hookrightarrow \langle\mathbf{a}_i\mathbf{b}_4^\top, \mathbf{a}_4\mathbf{b}_i^\top\rangle \subset \mathbb{P}^8$ i.e. $T^i = \mathbf{a}_i\mathbf{b}_4^\top - \mathbf{a}_4\mathbf{b}_i^\top$. Hence the plane $T = y_1T^1 + y_2T^2 + y_3T^3$ is contained in the tangent space of X at $(\mathbf{a}_4, \mathbf{b}_4)$, (and *a fortiori* in TX) which is the $\mathbb{P}^4 \subset \mathbb{P}^8$ spanned by the two planes $\mathbb{P}^2 \times \mathbf{b}_4$ and $\mathbf{a}_4 \times \mathbb{P}^2$ intersecting only at $(\mathbf{a}_4, \mathbf{b}_4)$. As we know that in T there are only two matrices having rank 1, we have that the plane T intersects X only at two points (one in $\mathbb{P}^2 \times \mathbf{b}_4$ and one in $\mathbf{a}_4 \times \mathbb{P}^2$).

Conversely, a plane in $\mathbb{P}^8$ which is contained in only one tangent $\mathbb{P}^4$ at a point of X and intersecting X only at two points gives rise to a trifocal tensor, so that the trifocal locus is also the (Zariski) closure of (the set given by the union of) these planes.

This description of the trifocal locus can be used to compute its dimension. Recall the choices made to get a trifocal tensor: a point $(\mathbf{a}_4, \mathbf{b}_4) \in \mathbb{P}^2 \times \mathbb{P}^2$, three points on $\mathbf{a}_4 \times \mathbb{P}^2$, three points on $\mathbb{P}^2 \times \mathbf{b}_4$, a point on each line $\langle(\mathbf{a}_i, \mathbf{b}_4), (\mathbf{a}_4, \mathbf{b}_i)\rangle$, hence there are $4 + 3 \times 2 + 3 \times 2 + 3 = 19$ parameters. As the trifocal tensors are defined up to a non zero constant, the dimension of the trifocal locus is then 18 (see [2] p. 358).

4 Other Constraints for the Trifocal Tensor

In this Section, we determine a second set of constraints for trifocal tensors, other than the ones coming from theorem 23, by means of geometric arguments. What we actually do is to give a set of equations defining a subvariety $\varOmega$ in $\mathbb{P}(V \otimes V \otimes V^*) = \mathbb{P}^{26}$, which contains $\varTheta$. Then we describe how $\varTheta$ sits inside $\varOmega$ as an irreducible component of maximal dimension. The payoff is that only eight equations are needed to define "almost all" trifocal tensors—see corollary 44 below.

The map $\mathfrak{T}$ is invariant under the natural $GL(W^*)$–action on $\mathbb{P}(V \otimes W^*)$, hence given any three matrices

M, M', M'', it is possible to choose coordinates (in W^*) such that M has the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

writing $M' = (a_i^j), M'' = (b_i^j), i = 1,2,3, j = 1,2,3,4$, equation (5) becomes ([2])

$$t^i_{jk} = a^i_j b^4_k - a^4_k b^i_j \quad i,j,k = 1,2,3. \tag{6}$$

At this point, finding the equations of Θ is seemingly straightforward, just eliminate the a's and b's in (6). Computationally, this elimination is quite intractable, so we shall find the equations of the trifocal locus by means of geometric arguments.
Slicing the tensor T with respect to the index i, we get three 3×3 matrices $T^i, i = 1,2,3$, explicitly $T^i = (t^i_{jk})_{j,k=1,2,3}$.
Now, a 3×3 matrix is actually an element of $\mathbb{K}^3 \otimes \mathbb{K}^3$, so from (6) follows

$$T^i = \mathbf{a}_i \otimes \mathbf{b}_4 - \mathbf{a}_4 \otimes \mathbf{b}_i, \quad i = 1,2,3. \tag{7}$$

Thinking of $\mathbf{a}$ and $\mathbf{b}$ as column and row vectors respectively, i.e. $\mathbf{a}$ are 3×1 matrices and $\mathbf{b}$ are 1×3 matrices, (7) can be written as a sum of products of matrices in the form (cf. [2] p. 357)

$$T^i = \mathbf{a}_i \mathbf{b}_4 - \mathbf{a}_4 \mathbf{b}_i \quad i = 1,2,3. \tag{8}$$

Lemma 41 *Let H, K be vector spaces, $\dim H = \dim K = 2$, and let $u \in H, v \in K, \chi \in H^*, \omega \in K^*$ be nonzero elements such that $\ker \chi = \langle u \rangle, \ker \omega = \langle v \rangle$.*
A tensor $S \in H \otimes K$ is of type $S = p \otimes v + u \otimes q$, for suitable $p \in H, q \in K$, if and only if $\chi \cdot S \cdot \omega = 0$.

Proof: One implication is obvious: if $S = p \otimes v + u \otimes q$, then $\chi \cdot S \cdot \omega = \chi(p)\omega(v) + \chi(u)\omega(q) = 0$.
Conversely, let $h \in H, k \in K$ be such that $\{h, u\}$ and $\{k, v\}$ are bases of H and K respectively, then S can be written as $S = \kappa h \otimes k + \lambda h \otimes v + \mu u \otimes k + \nu u \otimes v$, so $\kappa \chi(h)\omega(k) = \chi \cdot S \cdot \omega = 0$; it follows that $\kappa = 0$, hence $S = \lambda h \otimes v + \mu u \otimes k + \nu u \otimes v = (\lambda h) \otimes v + u \otimes (\mu k + \nu v)$. □

Lemma 42 *Let T be a generic element in Θ. For T^i as in (7), let R_i and C_i be the subspaces of $\mathbb{K}^3$ generated by the row and column vectors of T^i respectively. Then the following equations are satisfied.*
(i) Three cubic equations:

$$\det T^i = 0, \quad i = 1,2,3. \tag{9}$$

(ii) 54 equations of degree six:
let $\mathbf{t}^i_{p\bullet}$ be the p–th row of T^i, then

$$\det(\mathbf{t}^1_{p_1\bullet} \wedge \mathbf{t}^1_{q_1\bullet} \;\; \mathbf{t}^2_{p_2\bullet} \wedge \mathbf{t}^2_{q_2\bullet} \;\; \mathbf{t}^3_{p_3\bullet} \wedge \mathbf{t}^3_{q_3\bullet}) = 0, \tag{10}$$

for all $1 \le p_i < q_i \le 3,\ i = 1,2,3$; similarly,

$$\det(\mathbf{t}^1_{\bullet r_1} \wedge \mathbf{t}^1_{\bullet s_1} \;\; \mathbf{t}^2_{\bullet r_2} \wedge \mathbf{t}^2_{\bullet s_2} \;\; \mathbf{t}^3_{\bullet r_3} \wedge \mathbf{t}^3_{\bullet s_3}) = 0, \tag{11}$$

where $\mathbf{t}^i_{\bullet r}$ is the r–th column of T^i.
(iii) 108 quintic equations:

$$(\mathbf{t}^j_{\bullet p} \wedge \mathbf{t}^j_{\bullet q}) \cdot T^i \cdot (\mathbf{t}^k_{r\bullet} \wedge \mathbf{t}^k_{s\bullet}) = 0, \tag{12}$$

for any $i,j,k,p,q,r,s = 1,2,3$, with $j \neq i$ $k \neq i$ $p \neq q$ $r \neq s$.

Proof: (i) Since $R_i = \langle \mathbf{b}_i, \mathbf{b}_4 \rangle$, then $\dim R_i \le 2$. Therefore, as $\dim R_i = \dim C_i \le 2$ means that T^i, as a 3×3 matrix, has rank ≤ 2, so $\det T^i = 0$.
(ii) Since $\mathbf{a}_4$ belong to all three R_i, then $\dim R_1 \cap R_2 \cap R_3 \ge 1$—recall that the intersection of three bidimensional subspaces of $\mathbb{K}^3$ in general position is the null space; similarly $\dim C_1 \cap C_2 \cap C_3 \ge 1$. Note that $\dim R_1 \cap R_2 \cap R_3 \ge 1$ implies that the R_i's are linearly dependent—as planes in $\mathbb{K}^3$. Since R_i, as element of the dual space $\mathbb{K}^{3*}$, is represented by $\mathbf{t}^i_{p\bullet} \wedge \mathbf{t}^i_{q\bullet}$, for any choice of $p \neq q$, where $\mathbf{t}^i_{p\bullet}$ is the p–th row of T^i, the linear dependence translates into (10); considering columns, one gets (11).
(iii) The conclusion follows from lemma 41. In fact, fix for instance $i = 1$, $H = R_2$ and $K = C_2$, then $\mathbf{t}^2_{p\bullet} \wedge \mathbf{t}^2_{q\bullet}$ is a linear functional vanishing on $\mathbf{a}_4$, (but not necessarily on $\mathbf{a}_1$); similarly $\mathbf{t}^1_{\bullet r} \wedge \mathbf{t}^1_{\bullet s}$ is a linear functional vanishing on $\mathbf{b}_4$, (but not necessarily on $\mathbf{b}_1$), so we can apply lemma 41, then equations (12) are satisfied for $1 = 1, j = k = 2$, and so on. □

Now we can prove the following

Theorem 43 *Let Ω be the subvariety of $\mathbb{P}^{26}$ defined by equations (9)—(12), then the trifocal locus Θ is an irreducible component, of maximal dimension, of the variety Ω.*

Proof: Θ is irreducible because it is rational. Thanks to lemma 42 we know that an open subset U of Θ is contained in Ω, which is itself a closed set in $\mathbb{P}^{26}$. Then Θ, being the (Zariski) closure of U, is contained in Ω too. By direct calculation (eight random generators are algebraically independent) it is easy to see that $\dim(\Omega)$ is 18 at most, hence Θ is a component of maximal dimension. □

Theorem 43 tells us that Θ is an irreducible component of Ω, of maximal dimension. We will describe all components of Ω, but we postpone this long and somehow wearisome task until the end of this section. However we want to observe that an important consequence of the analysis of the components is that, to describe the trifocal locus Θ, we do not need the full set of three

cubic, 54 sextic and 108 quintic equations given above, equations that cut out the bigger variety Ω anyway. It is possible to do it, at least generically, using just the right number of equations, namely eight, and two open conditions, i.e. inequalities, of degree four. It means that in such a way we obtain a (Zariski) open subset of Θ, i.e. almost all trifocal tensors.
The idea is to take all three cubic equations (9), one each of the sextic equations of (10) and (11), and three quintic equations among (12), one for each of the slices T^i; to these one adds, as inequalities, two suitably chosen 2×2 minors of the matrices $(\mathbf{t}^1_{\bullet p_1} \wedge \mathbf{t}^1_{\bullet q_1} \quad \mathbf{t}^2_{\bullet p_2} \wedge \mathbf{t}^2_{\bullet q_2} \quad \mathbf{t}^3_{\bullet p_3} \wedge \mathbf{t}^3_{\bullet q_3})$ and $(\mathbf{t}^1_{r_1\bullet} \wedge \mathbf{t}^1_{s_1\bullet} \quad \mathbf{t}^2_{r_2\bullet} \wedge \mathbf{t}^2_{s_2\bullet} \quad \mathbf{t}^3_{r_3\bullet} \wedge \mathbf{t}^3_{s_3\bullet})$.
For example, we take, besides (9), the following sextic equations

$$\begin{aligned} &\det(\mathbf{t}^1_{1\bullet} \wedge \mathbf{t}^1_{2\bullet} \quad \mathbf{t}^2_{1\bullet} \wedge \mathbf{t}^2_{2\bullet} \quad \mathbf{t}^3_{1\bullet} \wedge \mathbf{t}^3_{2\bullet}) = 0, \\ &\det(\mathbf{t}^1_{\bullet 1} \wedge \mathbf{t}^1_{\bullet 2} \quad \mathbf{t}^2_{\bullet 1} \wedge \mathbf{t}^2_{\bullet 2} \quad \mathbf{t}^3_{\bullet 1} \wedge \mathbf{t}^3_{\bullet 2}) = 0, \end{aligned} \tag{13}$$

and quintic equations

$$\begin{aligned} &(\mathbf{t}^2_{1\bullet} \wedge \mathbf{t}^2_{2\bullet}) \cdot T^1 \cdot (\mathbf{t}^2_{\bullet 1} \wedge \mathbf{t}^2_{\bullet 2}) = 0, \\ &(\mathbf{t}^1_{1\bullet} \wedge \mathbf{t}^1_{2\bullet}) \cdot T^2 \cdot (\mathbf{t}^3_{\bullet 1} \wedge \mathbf{t}^3_{\bullet 2}) = 0, \\ &(\mathbf{t}^1_{1\bullet} \wedge \mathbf{t}^1_{2\bullet}) \cdot T^3 \cdot (\mathbf{t}^2_{\bullet 1} \wedge \mathbf{t}^2_{\bullet 2}) = 0; \end{aligned} \tag{14}$$

we denote the quartic inequalities by

$$F(t) \neq 0, \quad G(t) \neq 0 \tag{15}$$

where: $F(t)$ is the 2×2 minor of the matrix $(\mathbf{t}^1_{1\bullet} \wedge \mathbf{t}^1_{2\bullet} \quad \mathbf{t}^2_{1\bullet} \wedge \mathbf{t}^2_{2\bullet} \quad \mathbf{t}^3_{1\bullet} \wedge \mathbf{t}^3_{2\bullet})$, induced by first and second rows and first and second columns; $G(t)$ is the 2×2 minor of the matrix $(\mathbf{t}^1_{\bullet 1} \wedge \mathbf{t}^1_{\bullet 2} \quad \mathbf{t}^2_{\bullet 1} \wedge \mathbf{t}^2_{\bullet 2} \quad \mathbf{t}^3_{\bullet 1} \wedge \mathbf{t}^3_{\bullet 2})$, induced by first and second rows and second and third columns.

Corollary 44 *Let I be the ideal generated by the eight polynomials (9), (13) and (14), then it defines the trifocal locus Θ, outside the hypersurface $F(t)G(t) = 0$.*

Proof: If T is a tensor in the locus of I but outside $F(t)G(t) = 0$, i.e. satisfying both inequalities (15), then T satisfies conditions (16) and (17), so we can repeat the argument, already used for $T \in \Omega$ satisfying (16) and (17), and conclude that T is a trifocal tensor indeed.

□

Thus Θ is the (Zariski) closure of $V(I)/V(FG)$, hence it is the zero locus of the quotient ideal $J := (I : FG)$ (see [1] p. 193); explicitly

Corollary 45 *The quotient ideal $J := (I : FG)$ cuts exactly the trifocal locus Θ.*

We now proceed to describe all the components of Ω. Our strategy is the following.
Let T be any tensor belonging to Ω. If T satisfies the following two conditions:

$$\operatorname{rk} T^i = 2, \quad i = 1, 2, 3 \tag{16}$$

$$\dim R_1 \cap R_2 \cap R_3 = \dim C_1 \cap C_2 \cap C_3 = 1. \tag{17}$$

then T is in Θ.
If T does not satisfy either of them, we prove that: either T is in Θ (by showing that T can be written as in (7)), or T gives rise to an irreducible component of Ω (other than Θ), whose dimension we determine. We search for these components in two steps: first we drop (16), then we drop (17), while upholding (16).

To begin with, let us interpret (9)–(17) geometrically.
Equations (9) mean that all vector spaces R_i and C_i, subspaces of $\mathbb{K}^3$, have dimension ≤ 2; dropping condition (16) means that $\dim R_i = \dim C_i \leq 1$ for some i.
In equations (10), the terms $\mathbf{t}^i_{p\bullet} \wedge \mathbf{t}^i_{q\bullet}$, via the canonical identification $\mathbb{K}^3 \wedge \mathbb{K}^3 = \mathbb{K}^{3*}$, is a linear function $\mathbf{m}_i$ whose kernel is R_i, in case $\dim R_i = 2$; otherwise $\mathbf{m}_i = \mathbf{0}$. Equations (10) mean that, if all spaces R_i have dimension two, they share a common one–dimensional subspace, i.e. they form a pencil; if some $\mathbf{m}_i = \mathbf{0}$, then (10) are automatically satisfied.
Dropping condition (17), while upholding condition (16), implies that $\dim R_1 \cap R_2 \cap R_3 = 2$, hence the row spaces R_1, R_2, R_3 coincide.
Similar considerations apply to (11); we denote by $\mathbf{n}_i$ the linear function $\mathbf{t}^i_{\bullet r} \wedge \mathbf{t}^i_{\bullet s}$, whose kernel is the column space C_i.
Equations (12) now become $\mathbf{m}_j \cdot T^i \cdot \mathbf{n}_k = 0$; besides being trivial when $\mathbf{n}_j$ or $\mathbf{m}_k$ is zero (i.e. when R_j or C_k is one–dimensional), it is also automatically satisfied if $R_j = R_i$ or $C_k = C_i$.

We show now that a $T \in \Omega$ satisfying (16) and (17) is in Θ.
Since $T \in \Omega$ satisfies (16), there are suitable $\mathbf{t}^i_{\bullet p_i} \wedge \mathbf{t}^i_{\bullet q_i} \neq \mathbf{0}$, representing the subspaces $C_i \subseteq \mathbb{K}^3$; from (10) now follows that $\dim C_1 \cap C_2 \cap C_3 \geq 1$, hence there are $\mathbf{a}_l \in \mathbb{K}^3, l = 1, ..., 4$ such that, for all $i = 1, 2, 3, C_i = \langle \mathbf{a}_i, \mathbf{a}_4 \rangle$; the same argument, applied to the row vectors, shows that, for suitable $\mathbf{b}_l, R_i = \langle \mathbf{b}_i, \mathbf{b}_4 \rangle$ where $R_i \subseteq \mathbb{K}^3$ are the two–dimensional subspaces generated by the columns.
Since (17) is satisfied, then $C_i \neq C_j$ for some j, hence there exists a nonzero $\mathbf{t}^j_{\bullet p} \wedge \mathbf{t}^j_{\bullet q}$, for suitable p, q, and it represents an element of C_i^* whose value is zero on $\mathbf{a}_4$, but not on $\mathbf{a}_i$; similarly, for the columns, there exists a nonzero $\mathbf{t}^k_{r\bullet} \wedge \mathbf{t}^k_{s\bullet}$, which, as an element of R_i^*, vanishes on $\mathbf{b}_4$, but not on $\mathbf{b}_i$. Now, the condition of the lemma 41 is fulfilled because of (12), hence all T^i are of the form $T^i = \mathbf{a}_i \otimes \mathbf{b}_4 - \mathbf{a}_4 \otimes \mathbf{b}_i$, thus $T \in \Theta$.

Now we enumerate all other components of Ω, and we start by dropping condition (16).

(i) Assume that only one of the T^i has rank one; to fix ideas, let rk $T^1 = 1$, rk $T^2 =$ rk $T^3 = 2$, i.e. $\dim R_1 = \dim C_1 = 1$, $\dim R_2 = \dim C_2 = \dim R_3 = \dim C_3 = 2$. A tensor $T = (T^i)$, with $T^i \in C_i \otimes R_i$ satisfying (12), satisfies in particular $\mathbf{m}_i \cdot T^1 \cdot \mathbf{n}_j = 0$, $i, j = 2, 3$, hence $R_1 \subseteq R_2 \cap R_3$ or $C_1 \subseteq C_2 \cap C_3$; we suppose that $R_1 \subseteq R_2 \cap R_3$, the other case being equivalent. There are several subcases to consider.

(i)$_\mathrm{a}$ If $R_2 \neq R_3$ and $C_2 \neq C_3$, let $\mathbf{b}_4 \in R_2 \cap R_3$, then $T^1 = \mathbf{a}_1 \otimes \mathbf{b}_4$, where $C_1 = \langle \mathbf{a}_1 \rangle$—recall that $\dim C_1 = 1$. Since (12) gives also $\mathbf{m}_3 \cdot T^2 \cdot \mathbf{n}_3 = 0$, and $\mathbf{m}_3$, $\mathbf{n}_3$ are the linear functions χ, ω of lemma 41, then T^2 is of type $T^2 = \mathbf{a}_2 \otimes \mathbf{b}_4 - \mathbf{a}_4 \otimes \mathbf{b}_2$, where $\mathbf{a}_4 \in C_2 \cap C_3$, and similarly $T^3 = \mathbf{a}_3 \otimes \mathbf{b}_4 - \mathbf{a}_4 \otimes \mathbf{b}_3$. It means that $T \in \Theta$, so in this case there is no new component.

(i)$_\mathrm{b}$ If $R_2 \neq R_3$ but $C_2 = C_3$, then, as in the previous case, $T^1 = \mathbf{a}_1 \otimes \mathbf{b}_4$. However, since $C_2 = C_3$, $\mathbf{m}_3$ vanishes on C_2, hence lemma 41 does not apply, so $T^2 \in C_2 \otimes R_2$ with no restriction, and analogously T^3 is any element of $C_3 \otimes R_3$. We have then a component, whose parameters are: two each for R_2, R_3; none for $R_1 = R_2 \cap R_3$; two for C_1; two for both $C_2 = C_3$; one for $T^1 \in C_1 \otimes R_1$; four each for $T^i \in C_i \otimes R_i$, $i = 2, 3$. Total is 17 parameters, then we have a component of dimension 16.

(i)$_\mathrm{c}$ If $R_2 = R_3$, since tensors having form $T^i \in C_i \otimes R_i$, $i = 1, 2, 3$ are among those of case (v) below, then no new component arises in this case.

(ii) Assume rk $T^1 =$ rk $T^2 = 1$, rk $T^3 = 2$, i.e. $\dim R_1 = \dim R_2 = \dim C_1 = \dim C_2 = 1$, $\dim R_3 = \dim C_3 = 2$. It follows that $\mathbf{m}_1 = \mathbf{n}_1 = \mathbf{m}_2 = \mathbf{n}_2 = \mathbf{0}$, while $\mathbf{m}_3$ and $\mathbf{n}_3$ are nonzero. A tensor $T = (T^i)$ with $T^i \in C_i \otimes R_i$ satisfies (9)–(11); to satisfy (12) too, i.e. $\mathbf{m}_3 \cdot T^i \cdot \mathbf{n}_3 = 0$, $i = 1, 2$, it must happen that: $R_i \subset R_3$ and $C_j \subset C_3$, with $\{i, j\} = \{1, 2\}$; or $R_i \subset R_3$ $i = 1, 2$; or $C_i \subset C_3$ $i = 1, 2$.

Let us consider the various subcases.

(ii)$_\mathrm{a}$ Let $R_1 \subset R_3$ and $C_2 \subset C_3$, then the tensors $T^i \in C_i \otimes R_i$ have already been accounted for in case (i)$_\mathrm{a}$.

Indeed, let $R_1 = \langle \mathbf{b}_4 \rangle \subseteq R_3 = \langle \mathbf{b}_3, \mathbf{b}_4 \rangle$ and let $R_2 = \langle \mathbf{b}_2 \rangle$; define $R'_1 := R_1$, $R'_2 := R_1 + R_2 = \langle \mathbf{b}_2, \mathbf{b}_4 \rangle$, $R'_3 := R_3$ and $C'_1 := C_1$, $C'_2 = C'_3 := C_3$; then $T^i \in C_i \otimes R_i$ is also an element of $C'_i \otimes R'_i$ $i = 1, 2, 3$; but the configuration of spaces R'_i and C'_i satisfies the conditions of case (i)$_\mathrm{a}$, so the claim follows.

The upshot is that the tensors of this case all lie in the component already seen in case (i)$_\mathrm{a}$.

(ii)$_\mathrm{b}$ Let $R_1, R_2 \subset R_3$—the case $C_1, C_2 \subset C_3$ is completely similar; arguing as in the previous case, one sees that tensors under consideration now are among those of case (i)$_\mathrm{c}$, and again there is no new component.

(iii) Assume that all rk $T^i = 1$. This means that all spaces R_i and C_i are one-dimensional, so $T^i \in C_i \otimes R_i$ is of type $T^i \in \mathbf{a}_i \otimes \mathbf{b}_i$, with $\mathbf{a}_i \in C_i$ and $\mathbf{b}_i \in R_i$ $i = 1, 2, 3$. Arguing again along the lines of case (ii)$_\mathrm{a}$, we see that such tensors are among those described in case (i)$_\mathrm{b}$, so they belong to that component.

(iv) The case rk $T^i = 0$ for some i gives no new components, because the tensors satisfying this conditon actually belong to the component for which rk $T^i = 1$, all other conditions remaining unchanged. E.g. if rk $T^1 = 0$, rk $T^2 =$ rk $T^3 = 2$, with $R_2 \neq R_3$ and $C_2 = C_3$, it is easy to see that the corresponding tensors are among those described in case (i)$_\mathrm{b}$ above, because they satisfy the additional condition $T^1 = 0$.

We now assume condition (16) and drop condition (17).

(v) Suppose $\dim R_1 \cap R_2 \cap R_3 = 2$, $\dim C_1 \cap C_2 \cap C_3 = 1$. In this case, $T^i \in C_i \otimes R_i$ $i = 1, 2, 3$ do not necessarily satisfy (7), so another component of Ω pops out.

The parameters of this components are: two for the spaces $R_1 = R_2 = R_3$; five for the spaces C_1, C_2, C_3, because they belong to a pencil; four each for T^1, T^2, T^3. Dimension of the component is 18.

Clearly, there is another component of dimension 18, arising from the symmetric case $\dim R_1 \cap R_2 \cap R_3 = 1$, $\dim C_1 \cap C_2 \cap C_3 = 2$.

(vi) Suppose $\dim R_1 \cap R_2 \cap R_3 = \dim C_1 \cap C_2 \cap C_3 = 2$, write $R = R_i$, $C = C_i$, $i = 1, 2, 3$.

Tensors in this case are among those already accounted for in case (v), so no new component arises.

Summing up, Ω has three components of dimension 18, one being Θ, the other two coming from case (v), and six components of dimension 16, arising from case (i)$_\mathrm{b}$.

5 Conclusions

In this paper we produced two new sets of constraints for the trifocal tensors. The first one consists of 36 polynomial equations of degree 3, 6 and 12 defining a closed algebraic variety in $\mathbb{P}^{26}$ which is the closure of the set of trifocal tensors, i. e. the smallest algebraic variety containing this set. To the best of our knowledge, it is the only complete set of constraints existing in literature. Moreover every constraint has a simply interpretation in terms of the geometry of the three involved cameras. The key idea is to consider a suitable action of a group on the set of trifocal tensors and the to apply known results about this action.

The second set of constraint consists of a large collection of polynomial equations, obtained by elementary geometric arguments. In this case, the closure of the set of trifocal tensors is only an irreducible component of the algebraic variety defined by the polynomial constraints in $\mathbb{P}^{26}$, however it is possible to pick (in many different ways) only 8 equations, to get a generically complete set of constraints. This is very similar to other sets of constraints already published, but compared to them, our set is given by the minimal number of equations and they have lower degrees.

Appendix

We collect here the definitions of some less common mathematical concepts used throughout the paper.

I. *Sign of a permutation.*
A *permutation* (of n objects) σ is a bijective function from the set $\{1, 2, 3, \dots n-1, n\}$ of the first n positive integers into itself; we denote $\sigma(k) = \sigma_k$ and $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix}$; the set (actually it is a group) of all $n!$ such permutations is denoted S_n. An *inversion* occurs whenever $\sigma_j > \sigma_k$ for some $j < k$; the minimum number of inversions for a permutation σ is zero, when $\sigma = \begin{pmatrix} 1\,2 \dots n \\ 1\,2 \dots n \end{pmatrix}$, the maximum is $\frac{n(n-1)}{2}$, for $\sigma = \begin{pmatrix} 1 & 2 & \dots n \\ n & n-1 & \dots 1 \end{pmatrix}$; a permutation σ is *even* or *odd* if it contains an even or odd number of inversions. The *sign* $\epsilon(\sigma)$ is $+1$ or -1 if σ is even or odd.
Here we use the signs of the six permutations of $\{1, 2, 3\}$; of them three are even, namely

$$\begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix},$$

and have sign $+1$; the remaining three

$$\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}$$

are odd, with sign -1.

II. *Wedge product*
The *tensor product* $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b}$ is a $n \times n$ matrix $T = (t_{pq})$, with entries $t_{pq} = a_p b_q, p, q = 1, \dots, n$
The *wedge product* $\mathbf{a} \wedge \mathbf{b}$ is defined by

$$\mathbf{a} \wedge \mathbf{b} := \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a},$$

i.e. it is a a $n \times n$ matrix $W = (w_{pq})$, with entries $w_{pq} = a_p b_q - b_p a_q, p, q = 1, \dots, n$; the set of all wedge products $\mathbf{a} \wedge \mathbf{b}, \mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ generates a subspace $\mathbb{K}^n \wedge \mathbb{K}^n$ of $\mathbb{K}^n \otimes \mathbb{K}^n$, having dimension $\frac{n(n-1)}{2}$; notice that, in the case of $\mathbb{R}^3$, the wedge product $\mathbb{R}^3 \wedge \mathbb{R}^3$ has dimension three, so it can be canonically identified with $\mathbb{R}^3$, and, under this identification, $\mathbf{a} \wedge \mathbf{b}$ coincides with the usual cross product $\mathbf{a} \times \mathbf{b}$.
In general, the wedge product of k vectors, $k \leq n$, is defined by

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k := \sum_{\sigma \in S_k} \epsilon(\sigma) \mathbf{a}_{\sigma_1} \otimes \cdots \otimes \mathbf{a}_{\sigma_k}.$$

The set of all k–wedge products generate a subspace $\wedge^k \mathbb{K}^n$ of $\otimes^k \mathbb{K}^n$, having dimension

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

Note that in the case of $\mathbb{R}^n$, the wedge products $\wedge^{n-1}\mathbb{R}^n$ and $\wedge^n \mathbb{R}^n$, having dimension n and one respectively, can be canonically identified with $\mathbb{R}^n$ and $\mathbb{R}$.

III. *Group action*
Let V be a space and let G be a group. We say that G *acts on* V if every $g \in G$ is an isomorphism $g : V \to V$ and they satisfy the following:
(i) $1_G = id_V$, i.e the unit of G is the identity on V, when viewed as isomorphism of V; and
(ii) $(gh)(v) = g(h(v))$, for any $g, h \in G, v \in V$, i.e. the product of elements of G coincides with the composition of functions, when the same elements are viewed as functions of V.
Most important for us is the case when $V = \mathbb{K}^n$ is the space of vectors and $G = GL(V)$ is the group of non-singular $n \times n$ matrices (with entries in $\mathbb{K}$); S act on V via

$$M(\mathbf{v}) := M\mathbf{v}$$

for all $M \in G, \mathbf{a} \in V$, i.e. the action is the usual product of a matrix and a vector.
$GL(V)$ acts also on any tensor product of V, e.g. $M(\mathbf{v} \otimes \mathbf{w}) := M\mathbf{v} \otimes M\mathbf{w}$.
The *orbit* of an element $\mathbf{v}$ of V, denoted $G\mathbf{v}$ or $\mathbf{v}^G$, is the set

$$G\mathbf{v} := \{\mathbf{w} \in V | \exists g \in G : g\mathbf{v} = \mathbf{w}\}.$$

If a group G acts on a space V, then V admits a partition into orbits, i.e. there is a family of orbits whose union is V and pairwise intersections are empty. If there is a topology on V, orbits are not closed, in general, but only locally closed, i.e. every orbit is the intersection of a closed set and an open set.
When $V = \mathbb{K}^n$, as in our case, only the closure of an orbit, and not the orbit itself, can be described as a Zariski closed set, i.e. the zero locus of a finite set of polynomials.

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