A Concrete Example of Symplectic Duality among K-3 Surface*

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Abstract. An explicit example of symplectic duality among two particular K-3 surfaces is given. The example was considered by Iliev and Ranestad. Here, by using projective and computer algebra methods, it is proved that the two surfaces are in fact dominated by a 3-fold.

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1. Introduction

In recent years, K-3 surfaces and the spaces of moduli of vector bundles over them were investigated in great detail (see the fundamental [11]). A lot of results were obtained by considering abstract surfaces and sophisticated methods. However K-3 surfaces can be also approached by using projective geometry.

In [10] the authors, among many other things, consider two particular types of K-3 surfaces: S', a generic section of the Lagrangian Grassmannian $\Sigma \subset \mathbb{P}^9(\mathbb{C})$ by a codimension 4 linear space, and S, the symplectic dual of S', which is a smooth quartic surface in $\mathbb{P}^3(\mathbb{C})$. These two surfaces are related each other by the following result (see Theorem 3.4.8 of [10]): let $M_{S'}(2, L, 6)$ be the moduli space (modulo isomorphisms) of rank 2 stable vector bundles $\mathcal{E}$ over S' such that the Chern classes of $\mathcal{E}$ are: $c_1(\mathcal{E}) = L$ and $c_2(\mathcal{E}) = 6$, where L is the hyperplane section of S'; then $S \simeq M_{S'}(2, L, 6)$. The proof of this result is largely based upon the

projective construction of S and S', which is a particular case of a more general construction of varieties and their symplectic duals (see [10] p. 395).

The aim of this paper is twofold. Firstly we want to give explicit projective constructions for S and S', secondly we want to show that these K-3 surfaces are in fact the target surfaces of two suitable surjective morphisms from a threefold Z. The key point is that S can be considered as the base surface of one of the two types of smooth conic bundles in $\mathbb{P}^5(\mathbb{C})$ (see [3]). These conic bundles can be explicitly constructed by using standard computer algebra techniques and their description given in [7]. Moreover the construction can be performed in such a way that it can be explicitly related to some incidence varieties defined in [10]. We want to stress that here "explicitly" means that we are able to write down polynomial equations. By using the above constructions, a little of Mori's theory and some properties of the syzygies of matrices we can prove the existence and the properties of Z.

The paper is organized as follows. In Section 3 we get explicit equations for any smooth conic bundle X of degree 12 in $\mathbb{P}^5(\mathbb{C})$ classified in [3] (see Proposition 1). In Section 4, after recalling many results contained in [10], we relate the previous construction with the incidence relations I (see Proposition 6) and J (see Proposition 7) defined in [10] and with the surfaces S and S' (see Proposition 8), moreover we get a result about X (see Proposition 9) useful in the sequel. In Section 5 we prove the existence of Z, of the morphisms over S and S' and we describe these morphisms (see Proposition 11 and 12), moreover we get a map from S to $M_{S'}(2, L, 4)$ which is similar to the (more general) map considered by Proposition 3.4.1 of [10], (see Proposition 13 and 14). In Section 6 we give a specialization to $\mathbb{P}^3(\mathbb{C})$ of the construction in [10].

2. Notation

$\mathbb{P}^r$: r-dimensional projective space on $\mathbb{C}$

M_t: transpose of the matrix M

W^*:   dual of the $\mathbb{C}$ vector space W

K_Υ: canonical divisor of the smooth variety Υ

X: 3-dimensional, degree 12, conic bundle in $\mathbb{P}^5$ considered in [3]

H: hyperplane divisor of X

$p : X \to S$:   the natural projection of X over a smooth K-3 surface S

V: 6-dimensional $\mathbb{C}$ vector space such that $\mathbb{P}^5 = \mathbb{P}^5(V)$

$\Psi_{|D|}$:   rational map, induced by some complete linear system $|D|$, from some $\mathbb{P}^r$ to some $\mathbb{P}^{r'}$

Ψ_P: for any point $P \in \mathbb{P}^r \backslash (base\ locus\ of\ |D|)$, it is the closure in $\mathbb{P}^r$ of the fibre of $\Psi_{|D|}$ which is contained in $\mathbb{P}^r \backslash (base\ locus\ of\ |D|)$

$Sm(\Upsilon)$: set of smooth points of a scheme Υ

Υ_{red}: reduced scheme of a scheme Υ

$Sec_d(\Upsilon)$: embedded variety of d-secant lines for the smooth subvariety $\Upsilon \subset \mathbb{P}^r$

$A^i(\Upsilon)$, $A_i(\Upsilon)$: cycles of codimension i (resp. dimension i) in Υ modulo numerical equivalence

$c_i(\mathcal{E})$: i-th Chern class of the vector bundle $\mathcal{E}$

$\mathbb{P}(\mathcal{E})$: projectivization of the vector bundle $\mathcal{E}$.

3. B-O-S-S conic bundle and its explicit equations

In [3] the authors classify all conic bundles in $\mathbb{P}^5 = \mathbb{P}^5(V)$. They proved that there are only two such varieties, one of degree 9 and one of degree 12 (see [3] p. 70). Here we consider the second one, let us call it X.

Theorem 1. *Let V be a 6-dimensional vector space on $\mathbb{C}$, let $\mathbb{P}^5$ be $\mathbb{P}^5(V)$. Let us fix a non degenerate symplectic 2-form $\omega \in \bigwedge^2 V$. Then:*

i) *there is an exact sequence of vector bundles over $\mathbb{P}^5$: $0 \rightarrow \Omega^4_{\mathbb{P}^5}(5) \rightarrow \Omega^2_{\mathbb{P}^5}(3) \rightarrow E \rightarrow 0$, where the injective map $\Omega^4_{\mathbb{P}^5}(5) \rightarrow \Omega^2_{\mathbb{P}^5}(3)$ is induced by ω and the cokernel is the rank 5 vector bundle E over $\mathbb{P}^5$, generated by global sections, with $c_1(E) = 5$, $h^0(E) = 14$, defined by Horrocks in [9];*

ii) *the degeneracy locus of 4 generic sections of E is a smooth, degree 12, 3-fold X in $\mathbb{P}^5$ whose ideal $\mathcal{I}_X$ has a standard presentation: $0 \rightarrow \mathcal{O}^{\oplus 4}_{\mathbb{P}^5} \rightarrow E \rightarrow \mathcal{I}_X(5) \rightarrow 0$;*

iii) *if we call K and H, respectively, the canonical bundle of X and the hyperplane divisor, the rational map associated to the linear system $|K_X + H|$ is a morphism $p := \Psi_{|K_X+H|} : X \rightarrow S$, where S is a smooth quartic K-3 surface in $\mathbb{P}^3$, and every fibre is a smooth plane conic.*

Proof. For proving i) see [7], Proposition 1.2, where E is called $\mathcal{B}(1)$. For proving ii) and iii) see [3] p. 84 and 86. $\qquad \square$

Corollary 1. *From the previous exact sequences defining E and $\mathcal{I}_X$ it is possible to get the following resolution for $\mathcal{I}_X$: $0 \rightarrow O^{\oplus 4}_{\mathbb{P}^5} \oplus \Omega^4_{\mathbb{P}^5}(5) \rightarrow \Omega^2_{\mathbb{P}^5}(3) \rightarrow \mathcal{I}_X(5) \rightarrow 0$ by means of the exterior powers of the cotangent bundle $\Omega^1_{\mathbb{P}^5}$. The resolution can be used, by a computer algebra system, to get an explicit system of generators for $\mathcal{I}_X$, it turns that $\mathcal{I}_X$ is generated by 10 quintics according to the tables at p. 87 of [3].*

Proof. For the first part see [3] p. 86, for the second part you can apply the standard mapping cone technique (see for instance [5]). $\qquad \square$

Remark 1. It is well known that all symplectic, non degenerate, 2-form $\omega \in \bigwedge^2 V$ are projectively equivalent (see for instance [3], proof of Proposition 5.9), so that, up to the choice of the coordinate system in $\mathbb{P}^5$, the variety X is determined only by the choice of the 4 generic sections of E.

In the sequel we will prove some statements by using concrete calculations, so that it will be useful to have a standard way to compute the above 10 quintics from any 4 generic sections in $H^0(E)$. To this aim we will give an alternative method to get the quintics.

Let us assume that $V = < e_1, e_2, \ldots, e_6 >$ and $V^* = < a, b, c, d, e, f >$. Let us fix once for all a coordinate system in $\mathbb{P}^5$ such that the generic point has coordinates $(a : b : c : d : e : f)$, $\omega = e_1 \wedge e_4 + e_2 \wedge e_5 + e_3 \wedge e_6$ and $\omega^* = a \wedge d + b \wedge e + c \wedge f$. In this way the symplectic, non degenerate, 2-form $\omega \in \bigwedge^2 V$ defines a contraction $\neg \omega : \bigwedge^5 V^* \to \bigwedge^3 V^*$, and it can be extended to a morphism of vector bundles $\bigwedge^5 V^* \otimes \mathcal{O}_{\mathbb{P}^5} \to \bigwedge^3 V^* \otimes \mathcal{O}_{\mathbb{P}^5}$, giving rise to an injective map $\varphi : \Omega^4_{\mathbb{P}^5}(5) \to \Omega^2_{\mathbb{P}^5}(3)$ whose cokernel is $E = \mathcal{B}(1)$, as in [7] Proposition 1.2.

Let us look at the following diagram (p. 131 of [7]), where the vertical columns are given by the usual Koszul maps:

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & \varphi & \uparrow \\
\Omega^4_{\mathbb{P}^5}(5) & \to & \Omega^2_{\mathbb{P}^5}(3) \\
\uparrow & & \uparrow \\
\mathcal{O}_{\mathbb{P}^5}^{\oplus 6} = \bigwedge^5 V^* \otimes \mathcal{O}_{\mathbb{P}^5} & \to & \mathcal{O}_{\mathbb{P}^5}^{\oplus 20} = (\bigwedge^3 V^*) \otimes \mathcal{O}_{\mathbb{P}^5} \\
\uparrow & & \uparrow \\
\mathcal{O}_{\mathbb{P}^5}(-1) = (\bigwedge^6 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \to & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 15} = (\bigwedge^4 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \\
\uparrow & & \uparrow \\
0 & & \ldots
\end{array}
$$

As in [7] we have:

$$
H^0(E) \simeq Hom(\mathcal{O}_{\mathbb{P}^5}, E) \simeq \bigwedge^3 V^* / \{Im(\neg \omega : \bigwedge^5 V^* \to \bigwedge^3 V^*)\} \simeq \\
\ker(\neg \omega : \bigwedge^3 V^* \to V^*),
$$

so that we can identify $H^0(E)$ with a suitable 14 dimensional subspace of $\bigwedge^3 V^*$. This subspace is defined by 6 linear relations $\Re$ in $\bigwedge^3 V^*$ depending upon $\neg \omega$. Moreover we have the following diagram:

$$
\begin{array}{ccccccc}
0 & & & & & & 0 \\
\downarrow & & & & & & \downarrow \\
Im \otimes \mathcal{O}_{\mathbb{P}^5} & & & & & & Im^* \otimes \mathcal{O}_{\mathbb{P}^5}(2) \\
\downarrow & & & & & & \downarrow \\
(\bigwedge^3 V^*) \otimes \mathcal{O}_{\mathbb{P}^5} & \to & (\bigwedge^2 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(1) & \simeq & (\bigwedge V^2) \otimes \mathcal{O}_{\mathbb{P}^5}(1) & \to & (\bigwedge^3 V) \otimes \mathcal{O}_{\mathbb{P}^5}(2) \\
\downarrow & & & & & & \downarrow \\
H^0(E) \otimes \mathcal{O}_{\mathbb{P}^5} & & & \to & & & H^0(E)^* \otimes \mathcal{O}_{\mathbb{P}^5}(2) \\
\downarrow & & & & & & \downarrow \\
0 & & & & & & 0
\end{array}
$$

where $Im := Im(\neg \omega : \bigwedge^5 V^* \to \bigwedge^3 V^*)$ and the horizontal map is the composition of the usual Koszul maps and the natural isomorphism $(\bigwedge^2 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(1) \simeq (\bigwedge^2 V) \otimes \mathcal{O}_{\mathbb{P}^5}(1)$ induced by ω.

It follows that the horizontal map can be represented by a $(14, 14)$ symmetric matrix b_2 whose entries belong to $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$, (see [7] p. 132). As $\mathcal{B}$ is self-dual (see [7], Proposition 1.2), we have that: $E = \mathcal{B}(1) \simeq \mathcal{B}^*(1) = [\mathcal{B}(-1)]^* = [E(-2)]^* = E^*(2)$, so that we also have the following commutative diagram:

$$
\begin{array}{ccc}
H^0(E) \otimes \mathcal{O}_{\mathbb{P}^5} & \rightarrow & H^0(E)^* \otimes \mathcal{O}_{\mathbb{P}^5}(2) \\
\downarrow & & \downarrow \\
E & \simeq & E^*(2) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

hence, by introducing a $\mathbb{C}$-vector space W of dimension 14, we can complete the first diagram in the following way:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \Omega^4_{\mathbb{P}^5}(5) & \rightarrow & \Omega^2_{\mathbb{P}^5}(3) & \rightarrow & E & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \bigwedge^5 V^* \otimes \mathcal{O}_{\mathbb{P}^5} & \rightarrow & (\bigwedge^3 V^*) \otimes \mathcal{O}_{\mathbb{P}^5} & \rightarrow & H^0(E) \otimes \mathcal{O}_{\mathbb{P}^5} & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & (\bigwedge^6 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \rightarrow & (\bigwedge^4 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \rightarrow & W \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & \cdots & & \cdots & &
\end{array}
$$

where the vertical map $W \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \rightarrow H^0(E) \otimes \mathcal{O}_{\mathbb{P}^5}$ is given by a $(14, 14)$ matrix L of linear forms in $\mathbb{P}^5$ whose columns are the syzygies of the columns of b_2 $(b_2 L = 0)$.

Note that, if we consider the mapping cone resolution given by the first diagram, i.e.:

$$
\cdots \rightarrow \bigwedge^5 V^* \otimes \mathcal{O}_{\mathbb{P}^5} \oplus (\bigwedge^4 V^*) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \rightarrow (\bigwedge^3 V^*) \otimes \mathcal{O}_{\mathbb{P}^5} \rightarrow E \rightarrow 0,
$$

if we dualize it:

$$
0 \rightarrow E^* \rightarrow (\bigwedge^3 V) \otimes \mathcal{O}_{\mathbb{P}^5} \rightarrow \bigwedge^5 V \otimes \mathcal{O}_{\mathbb{P}^5} \oplus (\bigwedge^4 V) \otimes \mathcal{O}_{\mathbb{P}^5}(1) \rightarrow \cdots
$$

and if we recall that $E^*(2) \simeq E$, then we have that the sections of E can be described as the syzygies of the $(21, 20)$ matrix of the map $(\bigwedge^3 V) \otimes \mathcal{O}_{\mathbb{P}^5}(2) \rightarrow \bigwedge^5 V \otimes \mathcal{O}_{\mathbb{P}^5}(2) \oplus (\bigwedge^4 V) \otimes \mathcal{O}_{\mathbb{P}^5}(3)$. If you compute such syzygies with a computer algebra system you get a $(20, 14)$ matrix, whose entries belong to $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$, and, by using the above 6 linear relations $\Re$ to eliminate dependent rows, you can get exactly the matrix b_2 of [7].

In the sequel we want to compare our calculations with the results contained in [10], to this aim it is more useful to use a slightly different form of the matrix b_2 of [7]. So that we define the following symmetric matrix $\mathcal{H}$ which is a simple manipulation of rows and columns of b_2, i.e. it corresponds to a different choice for a base in $H^0(E)$:

$$\begin{bmatrix}
0 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & -d^2 & -de & -df & -e^2 & -ef & -f^2 \\
h_1 & 0 & -a^2 & -ab & -ac & -b^2 & -bc & -c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a^2 & 0 & 0 & 0 & f^2 & -ef & e^2 & h_2 & ae & af & 0 & 0 & 0 \\
0 & -ab & 0 & -\frac{1}{2}f^2 & \frac{1}{2}ef & 0 & \frac{1}{2}df & -de & bd & -\frac{1}{2}cf & \frac{1}{2}bf & ae & \frac{1}{2}af & 0 \\
0 & -ac & 0 & \frac{1}{2}ef & -\frac{1}{2}e^2 & -df & \frac{1}{2}de & 0 & cd & \frac{1}{2}ce & -\frac{1}{2}be & 0 & \frac{1}{2}ae & af \\
0 & -b^2 & f^2 & 0 & -df & 0 & 0 & d^2 & 0 & bd & 0 & h_3 & bf & 0 \\
0 & -bc & -ef & \frac{1}{2}df & \frac{1}{2}de & 0 & -\frac{1}{2}d^2 & 0 & 0 & \frac{1}{2}cd & \frac{1}{2}bd & ce & -\frac{1}{2}ad & bf \\
0 & -c^2 & e^2 & -de & 0 & d^2 & 0 & 0 & 0 & 0 & cd & 0 & ce & h_4 \\
-d^2 & 0 & h_2 & bd & cd & 0 & 0 & 0 & 0 & 0 & 0 & c^2 & -bc & b^2 \\
-de & 0 & ae & -\frac{1}{2}cf & \frac{1}{2}ce & bd & \frac{1}{2}cd & 0 & 0 & -\frac{1}{2}c^2 & \frac{1}{2}bc & 0 & \frac{1}{2}ac & -ab \\
-df & 0 & af & \frac{1}{2}bf & -\frac{1}{2}be & 0 & \frac{1}{2}bd & cd & 0 & \frac{1}{2}bc & -\frac{1}{2}b^2 & -ac & \frac{1}{2}ab & 0 \\
-e^2 & 0 & 0 & ae & 0 & h_3 & ce & 0 & c^2 & 0 & -ac & 0 & 0 & a^2 \\
-ef & 0 & 0 & \frac{1}{2}af & \frac{1}{2}ae & bf & -\frac{1}{2}ad & ce & -bc & \frac{1}{2}ac & \frac{1}{2}ab & 0 & -\frac{1}{2}a^2 & 0 \\
-f^2 & 0 & 0 & 0 & af & 0 & bf & h_4 & b^2 & -ab & 0 & a^2 & 0 & 0
\end{bmatrix}$$

where $h_1 = ad+be+cf$, $h_2 = ad-be-cf$, $h_3 = -ad+be-cf$, $h_4 = -ad-be+cf$.

In this case, the matrix $\mathcal{L}$ of the syzygies for the columns of $\mathcal{H}$ is the following:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a \\
f & e & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & -b & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & -c & 0 & a & d & e \\
0 & 0 & 0 & -c & 0 & -b & 0 & 0 & a & b & d & 0 & 0 & f \\
0 & 0 & 0 & 0 & 0 & 0 & -c & a & 0 & 0 & 0 & 0 & e & 0 \\
0 & 0 & 0 & 0 & -c & a & b & 0 & 0 & 0 & e & 0 & f & 0 \\
0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\
0 & 0 & a & 0 & e & 0 & -f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & b & -e & -d & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
a & 0 & c & 0 & 0 & e & d & -f & 0 & -e & 0 & 0 & 0 & 0 \\
0 & b & 0 & d & 0 & 0 & 0 & 0 & -f & 0 & 0 & 0 & 0 & 0 \\
b & c & 0 & 0 & 0 & -d & 0 & 0 & e & 0 & 0 & f & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & -e & 0 & 0
\end{bmatrix}.$$

Now we can prove the

Proposition 1. *Let us fix systems of coordinates in $V, \mathbb{P}^5, H^0(E)$ and a 2-form ω as above, (hence $\mathcal{H}$ and $\mathcal{L}$), then any generic $(14,4)$ matrix M of constants corresponds to a generic choice of 4 elements in $H^0(E)$ and it is possible to get a system of generators (10 quintics) for the ideal $\mathcal{I}_X$ of the corresponding conic bundle X in $\mathbb{P}^5$ depending only on M.*

Proof. Let us pick a $(14,4)$ matrix M of constants and let us consider the exact sequence of 1: $0 \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \to E \to \mathcal{I}_X(5) \to 0$ and the following diagram:

$$\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} & \to & & & E & \to & \mathcal{I}_X(5) & \to & 0 \\
& & \uparrow & & & & \uparrow & & \\
: \quad 0 & \to & \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} & \to & \mathcal{O}_{\mathbb{P}^5}^{\oplus 14} \simeq H^0(E) \otimes \mathcal{O}_{\mathbb{P}^5} & & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & \to & \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 14} \simeq W \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & & \\
& & & & \uparrow & & \\
& & & & \ldots & &
\end{array}$$

where the injective map $0 \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 14}$ is given by M and the vertical map $\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 14} \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 14}$ is given by $\mathcal{L}$. In this way we have chosen 4 elements of $H^0(E)$ and we have got a free resolution for $\mathcal{I}_X(5)$ of this type:

$$\cdots \to \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 14} \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 14} \to \mathcal{I}_X(5) \to 0$$

where the map $\mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 14} \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 14}$ is given by $[\mathcal{L}|M]$. In this case $H^0(\mathcal{I}_X(5))$ is given by the syzygies of the columns of the $(18,14)$ matrix $[\mathcal{L}|M]_t$. By any computer algebra system, as Macaulay, you can get 14 quintics as generators for $\mathcal{I}_X$. The 14 quintics are in fact linearly dependent over $\mathbb{C}$, so that the ideal is generated only by 10 quintics among them. $\square$

4. The basic construction

In [10] the authors describe a very complicated construction by starting with a 6-dimensional $\mathbb{C}$ vector space V and a symplectic, non degenerate, 2-form $\omega \in \bigwedge^2 V$, as in Section 3. They consider the 6-dimensional Lagrangian Grassmannian $\Sigma \subset \mathbb{P}^{13}$ of isotropic (with respect to ω) projective planes in $\mathbb{P}^5 = \mathbb{P}(V)$. Σ is one of the six symmetric Legendre varieties (see table 2 of [12]) and Σ is linked with the rank 5 vector bundle E over $\mathbb{P}^5$ considered in Section 3, because the fibre ($\simeq \mathbb{P}^4$) of $\mathbb{P}(E)$ over any point $P \in \mathbb{P}^5$ can be identified with the span of the smooth 3-dimensional quadric $\mathcal{Q}_P$ in Σ parametrizing the isotropic planes passing through P.

Let S' be a surface section of Σ with a generic $\mathbb{P}^9$. It is well known that S' is a smooth K-3 surface whose curve section is a canonical embedded curve of genus 9 and degree 16 in $\mathbb{P}^8$. One of the most important results contained in [10] (Theorem 3.4.8) is the proof of the existence of an isomorphism among the moduli space $M_{S'}(2, L, 6) := \{$rank 2 vector bundles $\mathcal{E}$ over S', having $c_1(\mathcal{E}) = L$ (hyperplane section) and $c_2(\mathcal{E}) = 6\}$ with a smooth K-3 surface $S \subset \mathbb{P}^3$, which is the intersection of the degree 4 dual hypersurface of Σ in $\mathbb{P}^{13*}$ with $\mathbb{P}^{9*}$. It turns out that S is also the base of a degree 12, smooth BOSS conic bundle $X \subset \mathbb{P}^5$ (see Proposition 2.4.2 of [10]) determined by the surface section of Σ.

Here we want to use the algebraic machinery introduced in Section 3 to give explicit descriptions of some incidence relations considered in [10].

Let us consider the construction of [10]. Let us fix systems of coordinates in V, $\mathbb{P}^5$, $H^0(E)$ and a 2-form $\omega \in \bigwedge^2 V$, (hence $\mathcal{H}$ and $\mathcal{L}$), as in Section 3. The form ω^* gives rise to a correlation $L_\omega : V \to V^*$, $v \to \omega^*(v, \_) \in V^*$, which is an isomorphism. This isomorphism induces an isomorphism: $L_\omega : \bigwedge^3 V \to \bigwedge^3 V^*$ denoted in the same way. Moreover ω^* gives rise to a contraction $\neg \omega^* : \bigwedge^3 V \to V$, so that we have a linear subspace $V_{14} \subset \bigwedge^3 V$ which is the kernel of $\neg \omega^*$. Note that $V_{14}^* = L_\omega(V_{14}) = \{w^* \in \bigwedge^3 V^* | w^* \wedge \omega^* = 0\} \subset \bigwedge^3 V^*$. If we consider $V = U_0 \oplus U_1$ for fixed 3-dimensional subspaces U_0, U_1, we can decompose

$$\bigwedge^3 V = \bigwedge^3 U_0 \oplus \bigwedge^2 U_0 \otimes U_1 \oplus U_0 \otimes \bigwedge^2 U_1 \oplus \bigwedge^3 U_1.$$

As $\bigwedge^2 U_1 \simeq U_1^* \simeq U_0$ (where the second isomorphism is given by L_ω^{-1}) we have that every (non zero) element w^* of $\bigwedge^3 V^*$ gives rise, by restriction, to a bilinear

map $U_0 \times U_0 \rightarrow \mathbb{C}$ which is symmetric if and only if $w^* \in V_{14}^*$ (see [10] p. 387). I. e. we can associate a conic to any point of $\mathbb{P}(V_{14}^*)$.

For convenience of the reader in the next four propositions we recall some of the results of [10]:

Proposition 2. *The symplectic group* $Sp(3, \mathbb{C})$ *acts naturally on* $\mathbb{P}^{13} := \mathbb{P}(V_{14})$ *and the stratification of the orbits is given by the following subvarieties of* $\mathbb{P}^{13}$:

1) F, *a smooth hypersurfaces of degree* 4 *in* $\mathbb{P}^{13}$;
2) $\Omega := Sing(F)$, *a singular variety of dimension* 9 *and degree* 21; *the ideal of* Ω *is generated by* 14 *forms of degree* 3: *the partial derivatives of the polynomial of* F;
3) $\Sigma \subset \mathbb{P}^{13}$, *the Lagrangian Grassmannian of isotropic (with respect to* ω^*) *projective planes in* $\mathbb{P}^5 := \mathbb{P}(V)$; Σ *is* $Sing(\Omega)$, *it has dimension* 6 *and degree* 16; *the ideal of* Σ *is generated by* 21 *forms of degree* 2.

Moreover: Σ *is a linear section of the usual Grassmannian* $G(3,6)$ *in* $\mathbb{P}(\bigwedge^3 V)$. F *is the tangential variety of* Σ, *i.e. the union of the tangent spaces at points of* Σ, *and it is isomorphic with the dual hypersurface of* Σ. *In the dual space* $\mathbb{P}^{13*} := \mathbb{P}(V_{14}^*)$ *we have dual varieties* $F^* \supset \Omega^* \supset \Sigma^*$ *with the same properties, in particular* F^* *is the dual variety of* Σ *and it is a smooth degree* 4 *hypersurface.*

Proof.   See Section 2.3 of [10].   □

Proposition 3. *For any point* $w^* \in \mathbb{P}^{13*}$, $w^* \in F^* \backslash \Omega^*$ *the associated conic (see above) is a smooth conic, hence if we cut* F^* *with a generic linear subspace* $A \simeq \mathbb{P}^3$, *so that the intersection with* Ω^* *is empty, we get a smooth quartic surface* S *which is the base of a conic bundle* X. *These conic bundles are exactly those considered in* [3].

Proof.   For the first part see Proposition 2.3.3 of [10], for the second part see Section 2.4 of [10], see also [7].   □

Proposition 4. Σ *is a* $VOADP$, *i.e. a variety with one apparent double point. More precisely: for any generic point* $w \in \mathbb{P}^{13}$ *there is only one secant line to* Σ *passing through* w; *if* $w \in \mathbb{P}^{13} \backslash F$ *we have exactly a secant line at two distinct points, if* $w \in F \backslash \Omega$ *we have exactly only one tangent line, if* $w \in \Omega \backslash \Sigma$ *the secant lines span a* $\mathbb{P}^4$ *and the entry locus of* w *(the set of points of* Σ *which are the intersections between* Σ *and the secant lines passing through* w) *is a* 3-*dimensional smooth hyperquadric* $\Sigma \cap \mathbb{P}^4$.

Proof.   See Theorem 2.3.2 and Proposition 2.5.1 of [10].   □

Proposition 5. *For any point* $P \in \mathbb{P}^5 = \mathbb{P}(V)$ *the isotropic planes passing through* P *are parametrized a* 3-*dimensional quadric* $\mathcal{Q}_P \subset \Sigma$. *The* 4-*dimensional linear span* $\mathbb{P}_P^4$ *of* $\mathcal{Q}_P$ *is* $\mathbb{P}(E_P)$, *the projectivization of the fibre of* E *over* P.

Proof.   For the first part see Lemma 2.4.1 of [10], for the second part see [10] p. 390–391.   □

Now we are ready to get some consequences of our machinery. Let us consider the following incidence variety (called I_P at p. 390 of [10]):

$$I := \{(w, P) \in \mathbb{P}^{13} \times \mathbb{P}^5 |\ w \in \mathbb{P}^4_P\}.$$

Proposition 6. *Let us fix coordinates $(x_1 : x_2 : \cdots : x_{14})$ in $\mathbb{P}^{13}$ and let us consider the $(1, 14)$ matrix $\mathcal{L}_{\underline{x}} := [x_1 x_2 \ldots x_{14}]$, and the $(1, 14)$ matrix $C\mathcal{L}_{\underline{x}} := [x_1\ x_2\ x_3\ 2x_4\ 2x_5\ x_6\ 2x_7\ x_8\ x_9\ 2x_{10}\ 2x_{11}\ x_{12}\ 2x_{13}\ x_{14}]$. Then:*

i) *a set of equations for I is given by $\mathcal{L}_{\underline{x}}\mathcal{L} = 0$, where $\mathcal{L}$ is the matrix defined in Section 3,*

ii) *a set of equations for Σ is given by: $C\mathcal{L}_{\underline{x}}\mathcal{H}[C\mathcal{L}_{\underline{x}}]_t = 0$, identically with respect to $(a : b : c : d : e : f)$, where $\mathcal{H}$ is the matrix defined in Section 3.*

Proof. Let us recall that $\mathbb{P}^{13} = \mathbb{P}(V_{14})$ with $V_{14} = \ker(\neg\omega^* : \bigwedge^3 V \to V)$, so that $H^0(\mathbb{P}^{13}, \mathcal{O}_{\mathbb{P}^{13}}(1))$ can be identified with $H^0(E) = \ker(\neg\omega : \bigwedge^3 V^* \to V^*) = V_{14}^*$. Let us recall that, for any point $P \in \mathbb{P}^5$, the 14 columns of the matrix $\mathcal{H}(P)$, where $\mathcal{H}$ is the matrix defined in Section 3, can be considered as the coordinates of 14 points of $\mathbb{P}^{13}$ and the linear span of these 14 points is $\mathbb{P}^4_P$. As $\mathcal{H}\mathcal{L} = \mathcal{L}_t\mathcal{H} = 0$ we have that the linear span of the 14 rows of $[\mathcal{L}(P)]_t$ is $(\mathbb{P}^4_P)^*$ so that $\mathbb{P}^4_P = \{\underline{x} \in \mathbb{P}^{13} | [\mathcal{L}(P)]_t(\mathcal{L}_{\underline{x}})_t = 0\} = \{\underline{x} \in \mathbb{P}^{13} | \mathcal{L}_{\underline{x}}\mathcal{L}(P) = 0\}$. Letting P to vary in $\mathbb{P}^5$ we have that a set of equations for I is given by $\mathcal{L}_{\underline{x}}\mathcal{L} = 0$, so that i) is proved.

To prove ii), let us recall matrix $\mathcal{H}$. The columns of $\mathcal{H}$ correspond to a set of generators for $H^0(E) \subset \bigwedge^3 V^*$ by considering the 6 relations $\Re$. If we choose a base for $\bigwedge^3 V^*$ as in [10] $< u, z, x_{ij}, y_{ij}, i, j = 1, 2, 3 >$ the relations $\Re$ are given by: $x_{ij} = x_{ji}$ and $y_{ij} = y_{ji}$. If we adjoin 6 suitable columns to $\mathcal{H}$ we have a $(14, 20)$ matrix $\mathcal{H}'$ such that from $\mathcal{H}'$ we can get a $(21, 20)$ matrix $\overline{\mathcal{H}'}$ of linear forms giving a morphism $\bigwedge^3 V^* \otimes \mathcal{O}_{\mathbb{P}^{19}} \to S^2(V^*) \otimes \mathcal{O}_{\mathbb{P}^{19}}$ of vector bundles on $\mathbb{P}^{19}$ (see [7] p. 133 where the trick is used for b_2 in $\mathbb{P}^{13}$). If we fix coordinates $(u_1 : \cdots : u_{20})$ in $\mathbb{P}^{19}$, to determine $\overline{\mathcal{H}'}$ we have to consider the product $\mathcal{H}[u_1 \ldots u_{20}]_t$ then we have to consider the coefficients of the 21 quadratic forms generating $S^2(V^*)$. Now it is easy to see that a set of generators for the usual Grassmannian $G(3, 6) \subset \mathbb{P}^{19}$ is given by $[u_1 \ldots u_{20}]\mathcal{H}'[u_1 \ldots u_{20}]_t = 0$, identically with respect to $(a : b : c : d : e : f)$, so that a set of generators for Σ is given by imposing the 6 relations $\Re$ in $\mathbb{P}^{19}$. Alternatively we can use directly $\mathcal{H}$, getting from it a $(21, 14)$ matrix of linear forms $\overline{\mathcal{H}}$ giving a morphism $H^0(E) \otimes \mathcal{O}_{\mathbb{P}^{13}} \to S^2(V^*) \otimes \mathcal{O}_{\mathbb{P}^{13}}$ of vector bundles on $\mathbb{P}^{13}$ (see [7] p. 133). However, in this case, we have to use the different matrix $C\mathcal{L}_{\underline{x}}$ to satisfy relations $\Re$. Note also that the columns of $\overline{\mathcal{H}}$ have only one syzygy (as you can prove by any computer algebra system): a $(14, 1)$ vector given by the 14 cubic forms defining Ω (see [7] p. 133). These forms are the gradient of a polynomial defining F, which is the tangent developable for Σ (see [7], Proposition 1.3).

By using $\mathcal{H}$ in this way, we get a set of quadratic forms generating the same ideal I_Σ considered by [10] if we translate their coordinates in the following way:

$$(u : z : x_{11} : x_{12} : x_{13} : x_{22} : x_{23} : x_{33} : y_{11} : y_{12} : y_{13} : y_{22} : y_{23} : y_{33}) \equiv$$
$$\equiv (x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : x_{10} : x_{11} : x_{12} : x_{14}).$$

$\square$

From Proposition 6 we have the following:

Corollary 2. *Let I be the incidence variety defined above. The projection of I over $\mathbb{P}^5$ is surjective, the fibres are all linear of dimension 4 and $\dim(I) = 9$. The projection of I over $\mathbb{P}^{13}$ is surjective over the 9-dimensional, degree 21, variety Ω; the generic fibre is a point over $\Omega \backslash \Sigma$ and a plane over Σ.*

Proof.   By definition and by Proposition 5 we know that $I = \mathbb{P}(E)$, hence we have the first part of Corollary 2. By Proposition 6 we know set of generators for the ideals of Σ, Ω and F in $\mathbb{P}^{13}$. By projecting I in $\mathbb{P}^{13}$ (by a computer algebra system as Macaulay for instance), we get exactly a set of generators for Ω and you can verify directly the description of the generic fibres. $\square$

Now let us consider an other incidence variety, introduced in [10] (p. 394):

$$J := \{(P, w^*) \in \mathbb{P}(V) \times \mathbb{P}(V_{14}^*) | \ \mathbb{P}_P^4 \subset \mathbb{P}_{w^*}^{12}\}$$

where $\mathbb{P}_{w^*}^{12}$ is the hyperplane of $\mathbb{P}(V_{14})$ corresponding to w^*.

Proposition 7. *Let us fix coordinates $(y_1 : y_2 : \cdots : y_{14})$ in $\mathbb{P}^{13*}$ and let us consider the $(1, 14)$ matrix $\mathcal{L}_{\underline{y}} := [y_1 y_2 \ldots y_{14}]$. Then a set of equations for J is given by $\mathcal{L}_{\underline{y}} \mathcal{H} = 0$, where $\mathcal{H}$ is the matrix defined in Section 3.*

Proof.   From Proposition 6 we know that, for any $P \in \mathbb{P}^5$, $\mathcal{H}(P)[\mathcal{L}_{\underline{x}}]_t$ are the coordinates in $\mathbb{P}^{13}$ of the generic point of $\mathbb{P}_P^4$. Hence the condition: $\{\mathcal{L}_{\underline{y}} \mathcal{H}(P)[\mathcal{L}_{\underline{x}}]_t = 0$ identically with respect to $x_1, x_2, \ldots, x_{14}\}$ implies that $\mathbb{P}_P^4 \subset \mathbb{P}_{w^*}^{12}$ for the fixed hyperplane $w^* = (y_1 : y_2 : \cdots : y_{14})$, so that J is defined by the conditions $\mathcal{L}_{\underline{y}} \mathcal{H} = 0$. $\square$

Corollary 3. *Let J be the incidence variety defined above. The projection of J over $\mathbb{P}^5$ is surjective, all the fibres are linear spaces of dimension 8 and $\dim(J) = 13$. The projection of J over $\mathbb{P}^{13*}$ is surjective over the degree 4, hypersurface $F^* \simeq F$; the generic fibre is a smooth conic.*

Proof.   Obviously, for any $P \in \mathbb{P}^5$, the fibre over P is given by the hyperplanes containing $\mathbb{P}_P^4$. On the other hand by a computer algebra system (as Macaulay for instance) a direct calculation gives F^* and you can verify directly the description of the generic fibres. The transformation $\{y_i = 2x_i \ i = 4, 5, 7, 10, 11, 13, \ y_i = x_i \ i \neq 4, 5, 7, 10, 11, 13\}$ proves that $F \simeq F^*$. $\square$

Remark 2. By Proposition 3 the fibres of J over the points $w^* \in F^* \backslash \Omega^*$ are the smooth conics which are fibres of the conic bundles considered in [3]. To get one of these conic bundles you have to cut F^* with a generic $\mathbb{P}^3 \subset \mathbb{P}^{13*}$.

Corollary 4. *Let I and J be the above incidence varieties. Let Π be an isotropic plane in $\mathbb{P}^5$. Let w_Π be the corresponding point in Σ and let w_Π^* be $L_\omega(w_\Pi)$. Then Π gives rise to:*

   - *a dimension 6, degree 13, genus 6, variety $\Pi_I \subset \Omega$ which is a cone, of vertex w_Π, over the intersection of Σ with the hyperplane corresponding to w_Π^* and a cubic hypersurface;*

- *a dimension* 10, *degree* 12, *genus* 19, *variety* $\Pi_J \subset F^*$ *which is the complete intersection of* F^*, *its polar variety with respect* w_Π^*, *and the hyperplane corresponding to* w_Π.

Proof.   As Σ is a homogeneous variety it suffices to prove Corollary 4 for one point of Σ, i.e. for one isotropic plane in $\mathbb{P}^5$. Then the proof is simply an application of direct calculation by using Propositions 6 and 7. Note that $\Pi_I = \mathbb{P}(E_{|\Pi})$, so that its numerical character also follows from Proposition 2.5.4 of [10]. $\square$

Proposition 8. *Let us choose two generic linear dual spaces:* $A \subset \mathbb{P}(V_{14}^*) = \mathbb{P}^{13*}$, $B \subset \mathbb{P}(V_{14}) = \mathbb{P}^{13}$, *of dimension* 3 *and* 9, *respectively. Let* M *be the* (14, 4) *matrix of constants defining* A. *Let us consider* $S := A \cap F^*$ *and* $S' := B \cap \Sigma$. *Then* S *and* S' *are two smooth* K-3 *surfaces whose Picard group is generated by the hyperplane divisor and* S *is the* $Sp(3)$-*dual of* S', *isomorphic to* $M_{S'}(2, L, 6)$. *If we construct* X *as in Section* 3 *by using* M, *we get that the base surface is exactly* S.

Proof.   Obviously S is a smooth K-3 surface with $Pic(S) \simeq \mathbb{Z}$. The same is true for S' thanks to Proposition 2.5.9 of [10]. The isomorphism $S \simeq M_{S'}(2, L, 6)$ (where L is the hyperplane divisor) follows from Theorem 3.4.8 of [10].

Now, let us define a conic bundle X as in Section 3 starting from M. Let us recall that M corresponds to a choice of 4 generic sections of E and that X is the degeneracy locus of such sections in $\mathbb{P}^5$. Let $a \equiv (x : y : z : u)$ the generic point in A and let $J_X := \{(P, a) \in \mathbb{P}^5 \times B| \ \mathbb{P}_P^4 \subset \mathbb{P}_a^{12}\} = \{(P, a) \in \mathbb{P}^5 \times A| \ [x \ y \ z \ u]M_t\mathcal{H}(P) = 0\} = \{(P, a) \in \mathbb{P}^5 \times A| \ \mathcal{H}(P)M[x \ y \ z \ u]_t = 0\}$ as $\mathcal{H}$ is symmetric. Then the projection of $J_X \to \mathbb{P}^5$ is the degeneracy locus of the 4 sections corresponding to M, i.e. X, and the projection of J_X into A is a smooth quartic surface which is the base of the conic bundle X, according to remark 2. It is easy to prove, by direct calculation, that this quartic surface in $A \simeq \mathbb{P}^3$ is exactly S. $\square$

Remark 3. As B is generic, $\mathbb{P}_P^4 \cap B$ is a point for generic $P \in \mathbb{P}^5$. It can be shown that $X = \{P \in \mathbb{P}^5| \ \dim(\mathbb{P}_P^4 \cap B) \geq 1\}$.

Proposition 9. *Let* X *be a conic bundle as in Section* 3 *and let* C *be any fibre of* p, *spanning a plane* $< C > \subset \mathbb{P}^5$. *Then, for any* C, $< C > \cap X$ *is a reducible scheme given by* C *and by a* 0-*dimensional scheme of length* 6. *For generic* C, *it is given by* C *and* 6 *distinct points which are the intersections of* 4 *distinct lines in general position on* $< C >$.

Proof.   First of all we have that $< C > \cap X$ is given by C and a 0-dimensional scheme, otherwise we would have at least a point x on X such that two different fibres of X would pass through x and this is not possible. Now, let Ξ be a smooth hyperplane section of X containing C. There is the following exact sequence of normal bundles: $0 \to \mathcal{N}_{C|\Xi} \to \mathcal{N}_{C|X} \to (\mathcal{N}_{\Xi|X})_{|C} \to 0$. We have $\mathcal{N}_{C|X} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{N}_{C|\Xi} = \mathcal{O}_{\mathbb{P}^1}(\nu)$ where $\nu = C^2$ in $H^*(\Xi)$. Moreover $\mathcal{N}_{\Xi|X} \simeq H_{|\Xi}$ as $\Xi \equiv H$ (hyperplane section of X) in $Pic(X)$, so that $\deg[c_1(\mathcal{N}_{\Xi|X})_{|C}] = H^2C = 2$, hence $\nu = -2$. Let Ξ' be another smooth hyperplane section of X containing C.

$\Xi' \in |H_{|\Xi} - C|$, hence $H_{|\Xi} - C$ is an effective divisor on Ξ and $h^0(\Xi, H_{|\Xi} - C) > 0$ as we can choose Ξ' in a web of hyperplanes containing $< C >$. It follows that the length of the 0-dimensional above scheme is: $(H_{|\Xi} - C)^2 = (H_{|\Xi})^2 - 2H_{|\Xi}C + C^2 = H^2\Xi - 2H^2C + \nu = 12 - 4 - 2 = 6$. For a generic fibre the points are all distinct and, by using the set of equations for J of Proposition 7, a direct calculation shows that they are the intersections of 4 distinct lines in general position on $< C >$. $\square$

To conclude Section 4 we give some facts, that can be proved by direct calculation, and whose importance will be evident in the sequel. Let us consider the restriction to S' of the projection of I. For any generic point of S' the fibre is a plane, projecting as a plane in $\mathbb{P}(V)$, (remember that the generic fibre is a point only for points of $\Omega \setminus \Sigma$ and here we are on Σ). If we cut X with this plane we get a smooth plane quintic. Moreover if we project into A such plane quintics, by using J, we get singular curves which are complete intersection of S and cubic surfaces with 4 singular points. These cubic surfaces belong to the linear system of cubic surfaces cut on A by the linear system of $\mathbb{P}(V_{14}^*)$ given by the 14 partial derivatives of F^*. Vice versa, if we project into B a smooth conic, which is a fibre of X, we get a hyperplane section of S'.

Remark 4. If you have any smooth quartic surface S in $\mathbb{P}^3$ it is natural to ask if S can be the base of a conic bundle like X. The answer is yes if you can choose coordinates in $\mathbb{P}^3$ such that S is a linear section of F^*, fixed by the [10] construction, depending on the choices of coordinates in V and ω.

5. The concrete linkage translating the symplectic duality

Around the construction in [10] there are at least two interesting birational maps. The first one is well known: the 14 partial derivatives of the equation of F give an involutory birational map $\gamma : \mathbb{P}(V_{14}) \to \mathbb{P}(V_{14}^*)$, γ^{-1} is given by the 14 partial derivatives of the polynomial of F^*. Obviously the base locus of γ is Ω and the exceptional divisor is F (see [7], p. 131).

   Here we want to consider another interesting birational map.

   First of all we need to study the normal bundle $\mathcal{N}_{X|\mathbb{P}^5}$.

Proposition 10. *Let $\mathcal{N}_{X|\mathbb{P}^5}$ be the normal bundle of X in $\mathbb{P}^5$. Let C be a generic fibre of X and recall that $Pic(X) = < H, K >$. Let h be the class of a hyperplane in $\mathbb{P}^5$ in such a way that $H = h_{|X}$. Let $\Delta := \mathbb{P}(\mathcal{N}_{X|\mathbb{P}^5}^*)$, let τ be the tautological divisor of Δ and let $\pi : \Delta \to X$ be the natural map. Then:*
   i) $c_1(\mathcal{N}_{X|\mathbb{P}^5}) = 6H + K$, $c_2(\mathcal{N}_{X|\mathbb{P}^5}) = 12H^2$.
   ii) $(\mathcal{N}_{X|\mathbb{P}^5})_{|C} = \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$.
   iii) $\pi^*H^4 = 0$; $\pi^*H^3\tau = 12$; $\pi^*H^2\tau^2 = -76$; $\pi^*H\tau^3 = 324$; $\tau^4 = -996$.

Proof.   i) Let us consider the exact sequence: $0 \to T_X \to (T_{\mathbb{P}^5})_{|X} \to \mathcal{N}_{X|\mathbb{P}^5} \to 0$. As $c_1(T_X) = -K$ and $c_1(T_{\mathbb{P}^5}) = -6h$, we have $c_1(\mathcal{N}_{X|\mathbb{P}^5}) = 6H + K$. As $cod_{\mathbb{P}^5}(X) = 2$ we have: $c_2(\mathcal{N}_{X|\mathbb{P}^5}) = X_{|X} = (\deg(X))h_{|X}^2 = 12H^2$.

ii) As $C \subset X \subset \mathbb{P}^5$, we have the following exact sequence of normal bundles:

$$0 \to \mathcal{N}_{C|X} \to \mathcal{N}_{C|\mathbb{P}^5} \to (\mathcal{N}_{X|\mathbb{P}^5})_{|C} \to 0. \qquad (*)$$

As $C \simeq \mathbb{P}^1$, $(\mathcal{N}_{X|\mathbb{P}^5})_{|C} = \mathcal{O}_C(\alpha) \oplus \mathcal{O}_C(\beta)$ for some integers α and β, moreover C is a complete intersection in $\mathbb{P}^5$ so that $\mathcal{N}_{C|\mathbb{P}^5} = \mathcal{O}_C(1)^{\oplus 3} \oplus \mathcal{O}_C(2)$. Now, let $\widetilde{S}$ be the pull-back, by $p : X \to S$, of a generic smooth hyperplane section of S such that $\widetilde{S} \supset C$. $\widetilde{S}$ is a smooth ruled surface in $\mathbb{P}^5$ and $C \subset \widetilde{S} \subset X$, so that we have $0 \to \mathcal{N}_{C|\widetilde{S}} \to \mathcal{N}_{C|X} \to (\mathcal{N}_{\widetilde{S}|X})_{|C} \to 0$. $\mathcal{N}_{C|\widetilde{S}} = \mathcal{O}_C$ because C is a fiber of $p_{|\widetilde{S}}$. $\mathcal{N}_{\widetilde{S}|X} = \widetilde{S}_{|\widetilde{S}} \equiv 4C$ (recall that $\deg(S) = 4$ and that C is a generic fibre of p), hence $(\mathcal{N}_{\widetilde{S}|X})_{|C} = \mathcal{O}_C$. Therefore $\mathcal{N}_{C|X} = \mathcal{O}_C \oplus \mathcal{O}_C$. Now we can write $(*)$ in the following way:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(2\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(2\beta) \to 0.$$

Obviously $2\alpha + 2\beta = 10$ and, as $(*)$ does not split, $2\alpha \geq 4$ and $2\beta \geq 4$, so we are done.

iii) From i) we have: $c_1(\mathcal{N}^*_{X|\mathbb{P}^5}) = -6H - K$, $c_2(\mathcal{N}^*_{X|\mathbb{P}^5}) = 12H^2$, so that the Wu-Chern relation for τ is: $\tau^2 = -\pi^*(6H + K)\tau - 12\pi^* H^2$. Then iii) follows from the facts: $\dim(X) = 3$, $H^3 = K^3 = 12$, $HK^2 = -12$, $H^2 K = 4$ (see [3] p. 87) and from the previous relation. $\qquad \square$

Proposition 11. *Let us fix a couple of linear spaces A and B as in Section 4. In this way we have a conic bundle X as above. Let $\sigma : W \to \mathbb{P}^5$ be the blow up of $\mathbb{P}^5$ along X. Let h be the class of a hyperplane in $\mathbb{P}^5$, let $\Delta = \mathbb{P}(\mathcal{N}^*_{X|\mathbb{P}^5})$ be the exceptional divisor for σ and let $\pi : \Delta \to X$ the natural map. W is a smooth 5-dimensional variety such that $Pic(X) = <\sigma^* h, \Delta>$. Let $\Psi : \mathbb{P}^5 \to \mathbb{P}^9$ be the rational map defined by the 10 quintics defining X and let $\Phi : W \to \mathbb{P}^9$ be the morphism induced by Ψ. Then:*

   i) *it is possible to identify $\mathbb{P}^9 \simeq B$ and Ψ is a birational map among $\mathbb{P}^5$ and the 5-dimensional intersection $\Phi(W) = B \cap \Omega$. The exceptional loci of Ψ are, respectively, X and $S' = Sing(B \cap \Omega)$. Φ is the map associated to the linear system $|5\sigma^* h - \Delta|$;*

   ii) *there is a smooth divisor $Y \subset W$ such that $Y = \mathbb{P}(\mathcal{U}_{|S'})$, where $\mathcal{U}$ is the universal bundle over Σ, and the natural map $\psi : Y \to S'$ coincides with $\Phi_{|Y}$;*

   iii) *W is a smooth Fano variety and its Mori's cone is generated by two extremal rays corresponding to Φ and σ;*

   iv) *on $Z := Y \cap \Delta$ two maps are defined:*

   - *$p \circ \pi_{|Z} : Z \to X \to S$ is such that the generic fibre is the double covering of a plane conic, (branched at 12 distinct points)*

   - *$\Phi_{|Z} : Z \to S'$ is such that the generic fibre is a smooth plane quintic.*

Proof. i) From A and B we can get 14 quintics in $\mathbb{P}^5$ (not linearly independent) among which we can choose 10 quintics to generate $\mathcal{I}_X$ as in Section 4 and to define Ψ and Φ. By a computer algebra system, as Macaulay for instance, it is easy to prove that the matrix of the first syzygies of the 14 quintics is a $(14, 14)$ matrix $R = [R_0 | R_1]$ where R_0 is a $(14, 4)$ matrix of constants and R_1 is a $(14, 10)$ matrix of linear forms having rank 9. More precisely, for any $P \in \mathbb{P}^5$, $rank[R_1(P)] < 9$ if and only if $P \in X$. Moreover it is possible to identify the target space of Ψ' (the rational map induced by the 14 quintics) with $\mathbb{P}(V_{14}) = \mathbb{P}^{13}$ in such a way that:
- $(x_1 : x_2 : \cdots : x_{14})$ are the coordinates of the generic point $w \in \mathbb{P}^{13}$,
- $w \in B$ if and only if $[x_1 \ x_2 \ ... \ x_{14}]R_0 = 0$,
- B is the target space of Ψ and Φ.

Now, let us consider $\Psi : \mathbb{P}^5 \to B$. As R_1 is a matrix of linear forms we have that X satisfy condition K_5 (see [14]), then Ψ is an embedding out of $Sec_5(X)$ and the closure of any positive dimensional fibre of Ψ is a linear space in $\mathbb{P}^5$ intersecting X along a hypersurface of degree 5 (see [2] Proposition 1 and [14]), moreover the graph G_Ψ of Ψ in $\mathbb{P}^5 \times B$ is given by (see [1] Proposition 3):

$$[x_1 \ x_2 \ ... \ x_{14}]R_1 = [x_1 x_2 ... x_{14}]R_0 = 0.$$

By direct calculation now it can be shown that:
- G_Ψ is the restriction of I to $\mathbb{P}^5 \times B$, hence $\Phi(W) = B \cap \Omega$,
- the fibre of Ψ over a point $w \in B \cap \Omega$ is a plane in $\mathbb{P}^5$, intersecting X along a plane quintic, if and only if $w \in S'$, otherwise the fibre is a point.

To get i) now it suffices to recall that $S' = B \cap \Sigma$ and $\Sigma = Sing(\Omega)$. Obviously $\Phi = \Phi_{|5\sigma^*\text{h}-\Delta|}$, note that, by using Proposition 10 iii) and by recalling that $\sigma^*\text{h}_{|\Delta} = \pi^*H$, $\Delta_{|\Delta} = -\tau$, we have that $(5\sigma^*\text{h} - \Delta)^5 = 21 = \deg(\Omega)$.

ii) By using G_Ψ in $\mathbb{P}^5 \times B$, by direct calculation, it can be shown that the union of all positive dimensional fibres of Ψ is a hypersurface $\mathcal{Y}$ of degree 8, singular along X, so that there exists a positive integer x such that linear system $|8\sigma^*\text{h} - x\Delta| \neq \emptyset$ and $(5\sigma^*\text{h} - \Delta)^4(8\sigma^*\text{h} - x\Delta) = 0$. By using Proposition 10 iii) as above we have $x = 2$.

Now let us consider the restriction $\overline{I}$ of I to $X \times S'$. As we have seen in i), the projection $\psi : \overline{I} \to S'$ is the natural map of the projectivization of a rank 3 vector bundle over S', the fibres of ψ are isotropic planes in $\mathbb{P}^5$, so that $\overline{I} = \mathbb{P}(\mathcal{U}_{|S'})$. By blowing up $\mathbb{P}^5$ at X, as $\overline{I} \subset G_\Psi$ we have that there exists a smooth divisor $Y \simeq \overline{I}$ in W and Y is the only divisor contracted by Φ, so that Y is the only element of $|8\sigma^*\text{h} - 2\Delta|$ and $\psi = \Phi_{|Y}$. Note that, by direct calculation, it is easy to see that for any generic point $P \in X$ the fibre of I over P is the line $\mathbb{P}^4_P \cap B$, the intersection of this line with Σ is: $\mathbb{P}^4_P \cap \Sigma \cap B = \mathcal{Q}_P \cap (\mathbb{P}^4_P \cap B)$ i.e. a couple of points in $\mathcal{Q}_P$. Hence the line $\mathbb{P}^4_P \cap B$ is a secant line for S' and the intersection points with S' give rise to two isotropic planes in $\mathbb{P}^5$ passing through P. These planes cut X along two plane quintics and the blow up of $\mathbb{P}^5$ along X separates the two plane quintics.

iii) Obviously $Pic(W) = < \sigma^*\text{h}, \Delta >$, so that the Picard number of W is 2. Let f be the numerical class of a fibre of π, then we can assume that $A_1(W) \otimes$

$\mathbb{R} \simeq \mathbb{R}^2$ is generated by $\sigma^*\mathrm{h}^4$ and f, so that any 1-cycle in $A_1(W) \otimes \mathbb{R}$ can be written as $\alpha\sigma^*\mathrm{h}^4 + \beta f$ with $\alpha, \beta \in \mathbb{R}$. Note that $K_W f = (-6\sigma^*\mathrm{h} + \Delta)f = (-6\sigma^*\mathrm{h} + \Delta)_{|\Delta}f = (-6\pi^*H - \tau)f = -1$ and $K_W \sigma^*\mathrm{h}^4 = (-6\sigma^*\mathrm{h} + \Delta)\sigma^*\mathrm{h}^4 = -6$, so that $K_W(\alpha\sigma^*\mathrm{h}^4 + \beta f) = -6\alpha - \beta$, hence the polyhedral part of the (two dimensional) Mori's cone is in the half-plane where $\beta \geq -6\alpha$. Due to the existence of 5-secant lines to X in $\mathbb{P}^5$ we have that the 1-cycle $\sigma^*\mathrm{h}^4 - 5f$ is effective and $K_W(\sigma^*\mathrm{h}^4 - 5f) = -1$, moreover the pull back in W of all these lines are contracted by Φ and cover Y, so that $\sigma^*\mathrm{h}^4 - 5f$ generates a rational extremal ray. Another rational extremal ray, corresponding to σ, is generated by f, so Mori's cone is $< [f] \otimes \mathbb{R}^+, [\sigma^*\mathrm{h}^4 - 5f] \otimes \mathbb{R}^+ >$, it coincides with its polyhedral part and it is entirely contained in the half-plane where $\beta \geq -6\alpha$. It follows that W is a smooth Fano variety (see [4], Theorem 1.27).

iv) Let us consider the intersection $Z := Y \cap \Delta$. As $(8\sigma^*\mathrm{h} - 2\Delta)_{|\Delta} = 2\tau + 8\pi^*H$ we have that $Z = 2\tau + 8\pi^*H$ in $Pic(\Delta) = < \pi^*H, \pi^*K, \tau >$. If we define $\rho := \pi_{|Z} : Z \to X$ we have that ρ is a double covering of X. To find the ramification divisor R_ρ we recall that $K_\Delta = (K_W + \Delta)_{|\Delta} = (-6\sigma^*\mathrm{h} + 2\Delta)_{|\Delta} = -6\pi^*H - 2\tau$, $K_Z = \rho^*K + R_\rho = (K_\Delta + Z)_{|Z} = (2\pi^*H)_{|Z}$, hence $R_\rho = -\pi^*K_{|Z} + (2\pi^*H)_{|Z} = \pi^*(2H - K)_{|Z}$. It follows that the branching divisor of ρ is $D_b = 4H - 2K$ in $Pic(X)$. Any conic C, fibre of p, is numerically equivalent to $\frac{1}{4}(H + K)^2$ in $A^2(X)$, so that the fibres of $p \circ \rho$ are double coverings, of a smooth plane conic, branched at $\frac{1}{4}(H + K)^2(4H - 2K) = 12$ points. Note that the Riemann-Hurwitz formula implies that these curves have genus 5. If we project these curves in $\mathbb{P}^9 \simeq B$ by Φ we get curves of degree 16, in fact $\{\pi^*[\frac{1}{4}(H + K)^2]_{|Z}\}(\tau + 5\pi^*H)_{|Z} = \frac{1}{4}\pi^*(H + K)^2(\tau + 5\pi^*H)(2\tau + 8\pi^*H) = 16$.

On the other hand, if we project Z in $\mathbb{P}^9 \simeq B$ by Φ, we obviously get a fibration over S' such that any fibre is a plane quintic, the section of the fibre of ψ with Y, in fact, by recalling that $\Phi_{|Z}$ is given by the linear system $|\tau + 5\pi^*H|_Z$, we have: $[(\tau + 5\pi^*H)_{|Z}]^3 = 0$ and $\frac{1}{16}[(\tau + 5\pi^*H)_{|Z}]^2(\pi^*H)_{|Z} = 5$. $\qquad \square$

From Proposition 11 we know that $\rho = \pi_{|Z} : Z \to X$ is a double covering and $Z \subset Y = \mathbb{P}(\mathcal{U}_{|S'})$. The following proposition tells us that ρ is given by the restriction to Z, of a map induced by a linear system in Y.

Proposition 12. *Let $Y = \mathbb{P}(\mathcal{U}_{|S'})$ be the variety introduced by Proposition 11, let T be its tautological bundle and let L be the generator of $Pic(S')$. Then the morphism $\rho = \pi_{|Z}$ is induced by the linear system $|T|$ restricted to Z while none of the divisors $D \in Pic(Y)$ is such that $p \circ \rho$ is given by $|D|_{|Z}$.*

Proof. First of all we want to determine $Pic(Y)$ and the class of Z inside it. Let us recall that $Pic(S') = < L >$, where L is the hyperplane section of S', by [10] Proposition 2.5.9, so that $Pic(Y) = < T, \psi^*L >$ and $Z = \mu T + \lambda\psi^*L$ in $Pic(S')$ for some integers λ and μ. Let F be the numerical class of a (two dimensional) fibre of ψ. Obviously $Z_{|F}T_{|F} = 5$ as the fibres of $\psi_{|Z}$ are plane quintics, hence $5 = (\mu T + \lambda\psi^*L)_{|F}T_{|F} = \mu \; T^2F = \mu$ and $Z = 5T + \lambda\psi^*L$. To find λ we need some information about $\mathcal{U}_{|S'}$.

Let $\Omega(i, j, k)$ the Schubert cycle of the usual Grassmannian $G(3, 6)$ given by planes intersecting linear spaces in $\mathbb{P}^5$, respectively of dimension i, j, k, along

linear spaces of dimension, respectively, $0, 1, 2$. By 2.4 of [10] we know that $\tau_1 := c_1(\mathcal{U}^*) = \Omega(2, 4, 5)_{|\Sigma} = -c_1(\mathcal{U})$ and $\tau_2 := c_2(\mathcal{U}^*) = \Omega(1, 4, 5)_{|\Sigma} = c_2(\mathcal{U})$, moreover $S' = \tau_1^4$ in $A_2(\Sigma)$ because τ_1 is the class of a hyperplane section of Σ, and $\tau_1^2 = 2\tau_2$ in $A_4(\Sigma)$. Hence $c_1(\mathcal{U}_{|S'}) = c_1(\mathcal{U})_{|S'} = -\tau_{1|S'} = -L$ and $c_2(\mathcal{U}_{|S'}) = c_2(\mathcal{U})_{|S'} = \tau_{2|S'} = \tau_2\tau_1^4 = \frac{1}{2}\tau_1^6 = \frac{1}{2}\deg(\Sigma) = 8$. It follows that the Wu-Chern relation for $\mathcal{U}_{|S'}$ is: $T^3 = -\psi^*LT^2 - 8FT$ and $K_Y = -3T - \psi^*L$ (recall that S' is a K-3 surface).

Now let us consider K_Z^3. As $K_Z = (K_Y + Z)_{|Z} = (2T + (\lambda - 1)\psi^*L)_{|Z}$ we have $K_Z^3 = (2T + (\lambda - 1)\psi^*L)^3(5T + \lambda\psi^*L)$. On the other hand, by Proposition 11 iv), we know that $K_Z = 2(\pi^*H_{|Z})$ by considering $Z \subset \Delta$, hence $K_Z^3 = 8(\pi^*H)^3(2\tau + 8\pi^*H) = 16 \cdot 12$ by using Proposition 10. The only integer value for which $(2T + (\lambda - 1)\psi^*L)^3(5T + \lambda\psi^*L) = 16 \cdot 12$ is $\lambda = 1$. The conclusion is that $Z = 5T + \psi^*L$ in $Pic(Y)$.

By Proposition 11 ii), we have that $K_Y = (K_W + Y)_{|Y} = (2\sigma^*\text{h} - \Delta)_{|Y} = (2\sigma^*\text{h})_{|Y} - Z$. By considering K_Y and Z in $Pic(Y)$ as above we have: $-3T - \psi^*L = (2\sigma^*\text{h})_{|Y} - (5T + \psi^*L)$, so that $\sigma^*\text{h}_{|Y} = T$. As ρ is obviously given by $|\sigma^*\text{h}_{|Y}|$ we get the first part of Proposition 12. Note that $(T_{|Z})^3 = T^3Z = 24 = \deg(\rho)\deg(X)$.

To get the second part of the proposition let us suppose, by contradiction, that there exists a divisor $D = aT + b\psi^*L$ such that $p \circ \rho$ is given by $|D|$. Then we would have $(D_{|Z})^3 = D^3Z = (aT + b\psi^*L)^3(5T + \psi^*L) = 0$, but it is easy to see that there are rational values a, b such that $(D_{|Z})^3 = 0$ only if $a = 0$. On the other hand $p \circ \rho$ is given by $|\pi^*(H + K)_{|Z}|$, so that we would have $\pi^*(H + K)_{|Z} = (b\psi^*L)_{|Z}$ for some b, but it is not possible: as $\sigma^*\text{h}_{|Y} = T$ we have $\pi^*(H)_{|Z} = T_{|Z}$, so that we would have: $16 = \pi^*(H + K)^2\pi^*H(2\tau + 8\pi^*H) = \pi^*(H + K)_{|Z}^2\pi^*(H)_{|Z} = (b\psi^*L)_{|Z}^2T_{|Z} = (b\psi^*L)^2T(5T + \psi^*L) = 5 \cdot 16 \cdot b^2$ and this is a contradiction. $\qquad\square$

Next proposition allows to define a map from S to a suitable moduli space of vector bundles over S'.

Proposition 13. *Let C be any conic which is fibre of p as above. Let R be the ruled surface $\mathbb{P}((\mathcal{N}^*_{X|\mathbb{P}^5})_{|C})$ in Δ, then:*

i) *$\Phi(R)$ is a surface, rational scroll, of degree 10 in $B \simeq \mathbb{P}^9$;*

ii) *$\Phi(R)$ intersects S' along a degree 16 curve Γ which is a singular hyperplane section of S' of genus 5;*

iii) *for generic C, Γ has exactly 4 singular double points: the intersections of S' with the curve on $\Phi(R)$ which is the image of the fundamental section of R.*

Proof. i) To prove i) for the generic C we can use direct calculation by using a computer algebra system and the algebraic description of the incidence relation I (see Proposition 6) restricted to $C \times B$. In the general case let us recall that $(\mathcal{N}^*_{X|\mathbb{P}^5})_{|C} = \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-6)$ by Proposition 10 ii), so that, by using Hartshorne notation (see [8], V.2), the invariant of the rational ruled surface R is $e = 2$ and its tautological divisor is $C_0 - 4f$ where C_0 is the numerical class of the fundamental section (isomorphic to C), and f is the numerical class of any fibre (note that no confusion can arise with the numerical class of the fibres of π because they are in

fact fibres of the same morphism). The morphism $\Phi_{|R}$ is given by $|\tau + 5\pi^*H|_{|R}$. Obviously $\tau_{|R}$ is the tautological divisor of R, i.e. $C_0 - 4f$, while $\pi^*H_{|R} = 2f$ because $R = \frac{1}{4}\pi^*(H + K)$ in $A_2(\Delta)$, so that $R\pi^*H = \frac{1}{4}\pi^*(H + K)\pi^*H = 2f$ in Δ. Hence $\Phi_{|R}$ is given by a subsystem of $|C_0 + 6f|$. Such divisors are very ample on R (see [8] p. 380) and the complete linear system embeds R as a smooth surface of degree $(C_0 + 6f)^2 = 10$ in $\mathbb{P}^{11}$, however, in our case, $\Phi(R) \subset \mathbb{P}^9$ so that $\Phi_{|R}$ is given by a proper subsystem of $|C_0 + 6f|$.

ii) To study Γ we recall that $S' = \Phi(Z)$ so that $\Gamma = \Phi(Z \cap R)$. As $Z = 2\tau + 8\pi^*H$ in Δ, we have that Z cuts on R a curve Γ which is linearly equivalent to $2C_0 + 8f$, hence $\deg(\Gamma) = (2C_0 + 8f)(C_0 + 6f) = 16 = \deg(S')$ and $2g(\Gamma) - 2 = (-2C_0 - 4f + 2C_0 + 8f)(2C_0 + 8f) = 8 \Longrightarrow g(\Gamma) = 5$. As $Pic(S') = <L>$ we have that Γ is a hyperplane section of S and, as the sectional genus of S' is 9, we have that Γ must be singular.

iii) Let $\widetilde{R}$ be the numerical class of R in W, let P_C be the numerical class of the pull back of the plane $<C>$ in W, then $P_C = \sigma^*h^3 - \widetilde{R}$. We want to study the restriction of Φ to P_C. As Φ is given by $|5\sigma^*h - \Delta|$ the degree of the image of the surface in $\mathbb{P}^9$ is $(\sigma^*h^3 - \widetilde{R})(5\sigma^*h - \Delta)^2$. It is immediate to see that $(\sigma^*h^3)(5\sigma^*h - \Delta)^2 = 13$ by standard calculation, while $\widetilde{R}(5\sigma^*h - \Delta)^2 = R(5\pi^*H + \tau)^2 = \frac{1}{4}\pi^*(H + K)(5\pi^*H + \tau)^2 = 10$, by restricting the calculation to Δ and by using Proposition 10. Therefore $(\sigma^*h^3 - \widetilde{R})(5\sigma^*h - \Delta)^2 = 3 = \deg[\Phi(P_C)]$.

Let us recall that, for generic C, $<C>$ is a plane intersecting X along C and at 6 distinct points, say $P_1, \ldots, P_6$, which are the intersections of 4 distinct lines $l_1, \ldots, l_4$, (see Proposition 9). The rational map Ψ, restricted to $<C>$, is given by a linear system of plane quintics whose base locus must be $\{C, P_1, \ldots, P_6\}$. By the existence of the lines l_i we have that the moving part $\mathcal{M}$ of the linear system is given by all plane cubics passing through $\{P_1, \ldots, P_6\}$. So that $\Phi(P_C)$ is a rational cubic surface, the image of the morphism, defined by the pull back of $\mathcal{M}$ on the blow up of $<C> \simeq \mathbb{P}^2$ at $\{P_1, \ldots, P_6\}$. Obviously $<\Phi(P_C)> = \mathbb{P}^3$ and $\Phi(P_C)$ is a singular cubic surface having 4 double points where the lines l_i contract. Moreover the image C' of the pull back of C is a rational, degree 6, curve on $\Phi(P_C)$ which is singular exactly at the double points of $\Phi(P_C)$ because C is cut by any l_i at two distinct points of $<C>$, say P_{l_i} and P'_{l_i} (recall that C is generic).

Now let us recall the degree 8 hypersurface $\mathcal{Y}$ in $\mathbb{P}^5$ considered in the proof of Proposition 11 ii). $\mathcal{Y}$ is the union of all 5-secant lines to X and is singular along X. The intersection $\mathcal{Y} \cap <C>$ is a reducible plane octic, whose components are: C (double), $l_1, \ldots, l_4$, because any l_i is in fact a 5-secant line for X. As we have seen above that any l_i is contracted by Φ, we know that any l_i belongs to one, and only one, of the plane quintics whose pull back in W are fibres of $\Phi_{|Y}$. Let us consider P_{l_i}, we have that one of the two plane quintics passing through P_{l_i} is reducible into l_i and a residual plane quartic. The same is true for P'_{l_i}. Therefore the two fibres of Δ over P_{l_i} and P'_{l_i} are sent by Φ into two intersecting lines in $B \simeq \mathbb{P}^9$. The conclusion is that $\Phi(R)$ has 4 double points, the 4 points $C' \cap \Phi(R)$, and they are 4 singular points for Γ because they are also on $S' \cap \Phi(R)$. $\square$

Remark 5. For generic C the proof of Proposition 13 iii), can also be done by direct calculation, by using the algebraic description of the incidence relation I (see Proposition 6) restricted to $C \times B$. However the above proof shows that for any C the corresponding hyperplane section Γ of S' is singular along a length 4 subscheme of S'. Moreover this subscheme is given by 4 distinct double points when $\mathcal{Y} \cap < C >$ is given by 4 distinct lines, other than the double C.

But this is indeed the case because $[\mathcal{Y} \cap < C >] \backslash C := C_r$ is a plane curve of degree 4 having a 0-scheme of length 6 as singular locus (by Proposition 9) and not containing any component of degree ≥ 2, otherwise on $< C >$ there would be some k-secant line for X with $k \geq 6$ and this is not possible: such line would be in X and this is not possible by Proposition 9.

Proposition 14. *Let S and S' as above. For any point $s \in S$ it is possible to define a rank 2 vector bundle E_s over S' with $c_1(E_s) = L$ and $c_2(E_s) = 4$, i.e. an element of $M_{S'}(2, L, 4)$.*

Proof.   Let us fix s, hence a conic $C := p^{-1}(s)$ in X. By Proposition 13 and Remark 5 from C we get a hyperplane section Γ of S' and a length 4 0-dimensional subscheme U in S' given by 4 distinct points. As $|K_{S'} - \Gamma| = |-L| = \emptyset$ by Theorem 3.13 of [6] we have the existence of a rank 2 vector bundle E'_s and an exact sequence as follows: $0 \to \mathcal{O}_{S'} \to E'_s \to \mathcal{I}_U \otimes \mathcal{O}_{S'}(-L) \to 0$; by tensorizing it with $\mathcal{O}_{S'}(L)$ we get: $0 \to \mathcal{O}_{S'}(L) \to E_s := E'_s \otimes \mathcal{O}_{S'}(L) \to \mathcal{I}_U \to 0$ and now $c_1(E_s) = L$ and $c_2(E_s) = 4$, i.e. $E_s \in M_{S'}(2, L, 4)$. $\square$

6. A specialization of the previous construction

In this section we want to see how to specialize the construction in [10] in the case $n = 4$, (it can be shown that the construction can not be generalized for $n \geq 8$). Let us fix two dual bases: $V = < e_1, e_2, e_3, e_4 > = U_0 \oplus U_1 = < e_1, e_2 > \oplus < e_3, e_4 >$ and $V^* = < x_1, x_2, x_3, x_4 >$. Let us choose $\omega^* \in \bigwedge^2 V^*$ (symplectic) as follows: $\omega^* = x_1 \wedge x_3 + x_2 \wedge x_4$, so that the induced isomorphism $L_\omega : V \to V^*$ is given by: $(e_1, e_2, e_3, e_4) \to (-x_3, -x_4, x_1, x_2)$. Let us put $U_0^\perp := L_\omega(U_0) = < x_3, x_4 >$ and $U_1^\perp := L_\omega(U_1) = < x_1, x_2 >$ so that $V^* = U_1^\perp \oplus U_0^\perp$ and we have: $\bigwedge^2 V^* = \bigwedge^2 U_1^\perp \oplus (U_1^\perp \otimes U_0^\perp) \oplus \bigwedge^2 U_0^\perp$, while $\bigwedge^2 V = \bigwedge^2 U_0 \oplus (U_0 \otimes U_1) \oplus \bigwedge^2 U_1$. Let us choose the standard dual bases for $\bigwedge^2 V$: $< e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >$ and for $\bigwedge^2 V^*$: $< x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4 >$. Let us consider the base for $\bigwedge^2 V$ as coordinates on $\bigwedge^2 V^*$, as in [10], and let us call them (a, b, c, d, e, f) for simplicity. We have that the coordinates in the previous decomposition of $\bigwedge^2 V^*$ can be arranged, as in [10], as follows: $(a, \begin{bmatrix} -d & b \\ -e & c \end{bmatrix}, f)$.

To any element $w^* \in \bigwedge^2 V^*$ we can associate a bilinear form $U_0 \times U_0 \to \mathbb{C}$, as in [10], in the following way: given w^* we have an element of $U_1^\perp \otimes U_0^\perp$, but $U_1^\perp = U_0^*$ and $U_0^\perp \simeq U_0^*$ by using $-(L_\omega)^{-1}$ so that we have an element of $U_0^* \times U_0^*$ and we are done. Analogously, to any element $w \in \bigwedge^2 V$ we can associate a bilinear form $U_0^* \times U_0^* \to \mathbb{C}$, as in [10], in the following way: given w we have an element of

$U_0 \otimes U_1$, but $U_1 \simeq U_0$ so that we have an element of $U_0 \times U_0$ and we are done. The two bilinear forms are represented by the same matrix $\begin{bmatrix} -d & b \\ -e & c \end{bmatrix}$. There is a contraction $\neg\omega^* : \bigwedge^2 V \to \mathbb{C}$, whose kernel V_5 is a 5-dimensional subspace of $\bigwedge^2 V$. The dual space of the kernel, $V_5^* \subset \bigwedge^2 V^*$, is given by those elements $w^* \in \bigwedge^2 V^*$ having coordinates (a, b, c, d, e, f) such that: $b + e = 0$, i.e. the previous $(2,2)$ matrix must be symmetric. In fact $w^* \in V_5^*$ if and only if $\omega^* \wedge w^* = 0$. In this way we can associate a conic in $\mathbb{P}^1$ to any point of $\mathbb{P}(V_5^*)$ (and $\mathbb{P}(V_5)$). If we consider the standard equation $af - be + cd = 0$ of $G(2, 4) \subset \mathbb{P}^5 = \mathbb{P}(\bigwedge^2 V)$, we have that the linear section of $G(2, 4)$ with $\mathbb{P}(V_5)$ is the Lagrangian Grassmannian Σ of the isotropic lines of $\mathbb{P}^3 := \mathbb{P}(V)$, with respect to the chosen symplectic form, i.e. the smooth quadric in $\mathbb{P}^4 = \mathbb{P}(V_5)$ of equation $af + b^2 + cd = 0$.

It is immediate to see that for any point $u \in \Sigma$ (take for instance $(1 : 0 : 0 : 0 : 0)$) the tangent space at u to Σ intersects Σ along a quadric cone of $\mathbb{P}^3$ having u as vertex. If you take a point $w \in T_u(\Sigma)$ and if you consider the conic in $\mathbb{P}^1$ associated to w, as we have seen before, we have that the rank of this conic is $0, 1, 2$ according to the fact that $w = u$, $w \in T_u(\Sigma) \cap \Sigma$, $w \in T_u(\Sigma) \backslash \Sigma$.

For any point $P \in \mathbb{P}^3$ let us consider the symplectic lines, with respect to ω^*, passing through P. They are parametrized by a line $\mathbb{P}_P^1$ contained in $\Sigma \subset G(2, 4)$ (for instance, for $P \equiv (1 : 0 : 0 : 0)$ you have the line $a = b = d = 0$ in $\mathbb{P}^4$). Then we can define two incidence relations as in [10]:

$$I := \{(w, P) \in \mathbb{P}^4 \times \mathbb{P}^3 | w \in \mathbb{P}_P^1\} \qquad J := \{(P, w^*) \in \mathbb{P}^3 \times \mathbb{P}^{4*} | \mathbb{P}_P^1 \subset \mathbb{P}_{w^*}^3\}$$

where $\mathbb{P}_{w^*}^3$ is the hyperplane of $\mathbb{P}^4$ corresponding to w^*. If we consider the projection $I \to \mathbb{P}^3$ we get that I is $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a rank 2 vector bundle over $\mathbb{P}^3$. To recognize $\mathcal{E}$ we can remark that the situation is similar to that one in [10], but, in that case, $\mathcal{E}$ is self-dual, so that the authors use a dual construction with respect to that one described in [7]. Here we have to follow the original construction of [7].

The symplectic form $\omega^* \in \bigwedge^2 V^*$ defines a null correlation bundle $\mathcal{N}$ over $\mathbb{P}^3$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega_{\mathbb{P}^3}^1(1) \to \mathcal{N} \to 0$$

(see [7], p. 130, see also [13] p. 76). The vector bundle $\mathcal{N}$ is self-dual, in fact, by dualizing the previous sequence, we have

$$0 \to \mathcal{N}^* \to T_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

so that $\det(\mathcal{N}^*) = \mathcal{O}_{\mathbb{P}^3}$, hence $\mathcal{N}^* \simeq \mathcal{N}$. The vector bundle $\mathcal{N}$ plays the role of the vector bundle $\mathcal{B}$ in [7], in fact both of them have no sections. As in [7], let us define $\mathcal{E} := \mathcal{N} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$, note that $h^0(\mathbb{P}^3, \mathcal{E}) = 5$, note also that $H^0(\mathbb{P}^3, \mathcal{E}) = Hom(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{N}) \simeq \ker(\neg\omega : \bigwedge^2 V \to \mathbb{C}) = V_5$ as in [7] p. 132, so that $\mathcal{E}$ is the vector bundle we are looking for.

We have the following diagram for $\mathcal{E}$

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^3} & \to & \Omega^1_{\mathbb{P}^3}(2) & \to & \mathcal{E} \to 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{O}_{\mathbb{P}^3} & \to & \mathcal{O}^{\oplus 6}_{\mathbb{P}^3} = (\bigwedge^2 V^*) \otimes \mathcal{O}_{\mathbb{P}^3} & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} = V^* \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & & \\
 & & & & \uparrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^3}(-2) & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}
$$

where the map $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}^{\oplus 6}_{\mathbb{P}^3} = (\bigwedge^2 V^*) \otimes \mathcal{O}_{\mathbb{P}^3}$ is given by the dual of ω and the vertical maps are the usual Koszul maps.

By the mapping cone theory we get a free resolution of $\mathcal{E}$:

$$
\cdots \to \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus 6}_{\mathbb{P}^3} \to \mathcal{E} \to 0.
$$

By dualizing and twisting:

$$
0 \to \mathcal{E}^*(2) \to \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \to \cdots .
$$

Now let us recall that $\mathcal{E} = \mathcal{N}(1)$, so that $\mathcal{E}^*(2) = \mathcal{N}^*(1) \simeq \mathcal{N}(1) = \mathcal{E}$. So that the sections of $\mathcal{E}$ can be identified with the syzygies of the $(5,6)$ matrix representing the map $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$. This matrix is easy to compute because it is the transpose of the matrix representing the mapping cone map $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus 6}_{\mathbb{P}^3}$, however to get a more symmetric matrix we can proceed as in [7] p. 132. The sections of $\mathcal{E}$ can be identified with the columns of the matrix representing the composite map:

$$
\bigwedge^2 V^* \to V^* \simeq V \to \bigwedge^2 V
$$

where the isomorphism is given by L_ω^{-1} and the other maps are the usual Koszul maps. If we choose coordinates $(x : y : z : u)$ in $\mathbb{P}^3$ we have that the first matrix is

$$
\begin{bmatrix}
-u & -z & -y & 0 & 0 & 0 \\
x & 0 & 0 & -z & -u & 0 \\
0 & x & 0 & y & 0 & -u \\
0 & 0 & x & 0 & z & u
\end{bmatrix},
$$

the second matrix is

$$
\begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

and the third matrix is the transpose of the first one. If we make the computation we get the following $(6, 6)$ matrix, analogous to the matrix b of [7]:

$$
\begin{bmatrix}
0 & xy & -x^2 & y^2 & -xy & -(xz+yu) \\
-xy & 0 & -xu & yz & 0 & -zu \\
x^2 & xu & 0 & -(xz-yu) & -xu & -u^2 \\
-y^2 & -yz & (xz-yu) & 0 & yz & z^2 \\
xy & 0 & xu & -yz & 0 & zu \\
(xz+yu) & zu & u^2 & -z^2 & -zu & 0
\end{bmatrix}.
$$

Of course we have to consider the splitting of $\bigwedge^2 V$ into V_5 and its complement. In consequence of the choice of the coordinates in V_5, here we have to ignore the 5th column and the 5th row of b, so that we get the analogous of the matrix b_2 of [7], which is antisymmetric and of rank 2:

$$
b_2 =
\begin{bmatrix}
0 & xy & -x^2 & y^2 & -(xz+yu) \\
-xy & 0 & -xu & yz & -zu \\
x^2 & xu & 0 & -(xz-yu) & -u^2 \\
-y^2 & -yz & (xz-yu) & 0 & z^2 \\
(xz+yu) & zu & u^2 & -z^2 & 0
\end{bmatrix}.
$$

Let us remember that $(a : b : c : d : f)$ are the coordinates in $\mathbb{P}^4 = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{E}))$. Then, for any point $P \equiv (x : y : z : u) \in \mathbb{P}^3$, $b_2 \begin{bmatrix} a & b & c & d & f \end{bmatrix}^\top$ gives the coordinates of the generic point of $\mathbb{P}^1_P$ in $\mathbb{P}^4$. Let us determine, by a computer algebra system as Macaulay, the matrix $\mathcal{L}$ of the syzygies of the columns of b_2:

$$
\mathcal{L} =
\begin{bmatrix}
0 & 0 & z & -u \\
u & z & -y & x \\
-z & 0 & 0 & y \\
0 & u & x & 0 \\
x & y & 0 & 0
\end{bmatrix},
$$

$\mathcal{L}$ is a $(5, 4)$ matrix of linear forms such that $b_2 \mathcal{L} = [0]$. Now, for any point $P \equiv (x : y : z : u) \in \mathbb{P}^3$, to get the equations of the line $\mathbb{P}^1_P$ in $\mathbb{P}^4$ it is sufficient to consider

$$
\begin{bmatrix} a & b & c & d & f \end{bmatrix} \mathcal{L} = [0]. \tag{$**$}
$$

In fact

$$
\begin{aligned}
(b_2 \begin{bmatrix} a & b & c & d & f \end{bmatrix}^\top)^\top \mathcal{L} &= \begin{bmatrix} a & b & c & d & f \end{bmatrix} (b_2)^\top \mathcal{L} \\
&= -\begin{bmatrix} a & b & c & d & f \end{bmatrix} b_2 \mathcal{L} = 0.
\end{aligned}
$$

The equations $(**)$ are:

$$
ub - zc + xf, \quad zb + ud + yf, \quad za - yb + xd, \quad ua - xb - yc
$$

and these are the equations of I, of dimension 4. The projection of I in $\mathbb{P}^3$ is surjective and the fibre over any point $P \in \mathbb{P}^3$ is the line $\mathbb{P}^1_P$ of Σ. It is well known

that $\mathbb{P}^3$ is the Fano variety of a 3-dimensional smooth hyperquadric, in this way we have given an explicit description of the variety.

On the converse, if we project I in $\mathbb{P}^4$ by using a computer algebra system, as Macaulay, we get exactly the Lagrangian Grassmannian Σ: $af + b^2 + cd = 0$. For any point $w \in \Sigma \subset \mathbb{P}^4$ the set of lines passing through w and contained in Σ gives a 2-dimensional quadric cone (having w as vertex and contained in $T_w(\Sigma)$), however the fibre of I over w is a line in $\mathbb{P}^3$, parametrizing the lines of the cone. For instance, if you take the point $w = (0 : 0 : 0 : 0 : 1)$, the cone is: $a = f = b^2 + cd = 0$, and the fibre in $\mathbb{P}^3$ is the line: $x = y = 0$.

Let us choose $(a' : b' : c' : d' : f')$ as coordinates in $\mathbb{P}^{4*}$, then

$$\begin{bmatrix} a' & b' & c' & d' & f' \end{bmatrix} b_2 \begin{bmatrix} a & b & c & d & f \end{bmatrix}^\top = 0$$

implies that the hyperplane corresponding to $(a' : b' : c' : d' : f') \in \mathbb{P}^{4*}$ contains all the points of $\mathbb{P}^1_P$. Hence $\begin{bmatrix} a' & b' & c' & d' & f' \end{bmatrix} b_2 = [0]$ gives the equations for J:

$$\begin{aligned}
-b'xy + c'x^2 - d'y^2 + f'(xz + yu) &= 0 \\
a'xy + c'xu - d'yz + f'zu &= 0 \\
-a'x^2 - b'xu + d'(xz - yu) + f'u^2 &= 0 \\
a'y^2 + b'yz - c'(xz - yu) - f'z^2 &= 0 \\
-a'(xz + yu) - b'zu - c'u^2 + d'z^2 &= 0.
\end{aligned}$$

If we project J in $\mathbb{P}^{4*}$ by using a computer algebra system, as Macaulay, we get that the projection is surjective. In fact, as we have seen above, the set of fibres of $\mathbb{P}(\mathcal{E})$, projected in $\mathbb{P}^4$, gives rise to all lines of Σ, so that every hyperplane w^* of $\mathbb{P}^4$ contains some lines of Σ, hence some fibres of $\mathbb{P}(\mathcal{E})$. For generic w^* the section with Σ consists of a smooth 2-dimensional quadric, so that it contains two distinct rulings of lines of Σ. Hence the generic fibre over w^* is given by two lines of $\mathbb{P}^3$, each one parametrizing one of the two rulings. For instance, if we choose $w^* = (0 : 1 : 0 : 1 : 0)$, i.e. the hyperplane $b + d = 0$, we get the two lines: $y = z - u = 0$ and $z = x + y = 0$. If w^* is tangent to Σ, then the fibre is given by a double line whose points correspond to the lines of the 2-dimensional cone $w^* \cap \Sigma$.

On the converse, if we project J in $\mathbb{P}^3$ we get that the projection is surjective too, but now the fibre over any point $P \in \mathbb{P}^3$ is the plane in $\mathbb{P}^{4*}$ given by the hyperplanes of $\mathbb{P}^4$ containing the line $\mathbb{P}^1_P$.

To complete the analogy with the construction of [10] we should find a variety in $\mathbb{P}^3$ corresponding to the conic bundle in $\mathbb{P}^5$ of [3] which was the locus where 4 generic sections of $\mathcal{E}$ were dependent. Here $rank(\mathcal{E}) = 2$ so that we have only to deal with the locus of degeneracy of two generic sections, i.e. with $c_1(\mathcal{E}) = \det(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^3}(2)$. In fact if we choose two generic sections of $\mathcal{E}$ and we determine the degeneracy locus by a computer algebra system as Macaulay we get a smooth quadric of $\mathbb{P}^3$. For instance, if we consider the random $(5,2)$ matrix $R := \begin{bmatrix} 0 & 4 & 0 & 4 & 0 \\ 2 & 4 & 4 & 0 & 5 \end{bmatrix}^\top$ and we multiply b_2R, we get a $(5,2)$ matrix giving

two random sections for $\mathcal{E}$. Their degeneracy locus is the smooth quadric Q_R: $xy + y^2 - 2xz + 2yz - \frac{5}{2}z^2 + 2xu + 2yu + \frac{5}{2}zu$.

The matrix R determines the plane $P_R : b + d = 2a + 4b + 4c + 5f = 0$ of $\mathbb{P}^4$ cutting Σ along a smooth plane conic $C_R : a = b = e = cd + d^2 - 2cf + 2df - \frac{5}{2}f^2 = 0$. It is easy to see that $Q_R = \{P \in \mathbb{P}^3 | \mathbb{P}^1_P \cap P_R \neq \emptyset\} = \{P \in \mathbb{P}^3 | \mathbb{P}^1_P \cap C_R \neq \emptyset\}$. Let us study the restriction of I to $C_R \times \mathbb{P}^3$ a little: if we fix any point $w \in C_R$ the fibre over w of the restriction of I is the same as the fibre of I: a line l_w parametrizing the lines of Σ passing through w, but obviously $l_w \in Q_R$. For any point $H \in l_w$ there is a line of Q_R (belonging to the other ruling of Q_R with respect to the ruling containing l_w) parametrizing other lines of Σ intersecting C_R. More precisely, if we consider the pencil $\mathbb{P}^1_R$ given by the hyperplanes of $\mathbb{P}^4$ containing the plane P_R we have that there exists a double covering $\sigma : l_w \simeq \mathbb{P}^1 \to \mathbb{P}^1_R$ such that, for any generic hyperplane $\pi \in \mathbb{P}^1_R$, $\sigma^{-1}(\pi)$ is given by two points of l_w for which there pass two lines of Q_R (belonging to the other ruling of Q_R with respect to the ruling containing l_w) parametrizing the lines of Σ, intersecting C_R, belonging to the two rulings of the smooth 2-dimensional quadric $\pi \cap \Sigma$. The two ramification points of σ correspond to the two hyperplanes of $\mathbb{P}^1_R$ which are tangent to Σ, in these cases $\pi \cap \Sigma$ is a 2-dimensional quadric cone and the lines of these cones are parametrized by one line only of Q_R for any cone (the two lines belong to the other ruling of Q_R with respect to the ruling containing l_w). For instance, if we choose $w = (4 : -1 : -1 : 1 : 0)$, l_w is the line: $z - u = x + y + 4u = 0$. If we choose the two points $H = (1 : -1 : 0 : 0)$ and $H' = (-4 : 0 : 1 : 1)$ on l_w we have that for H and H' there passes, respectively, the two lines: $z = x + y = 0$ and: $y = z - u = 0$ of Q_R (other than l_w of course), these lines parametrize the lines of Σ, intersecting C_R, contained in the smooth 2-dimensional quadric $\pi \cap \Sigma$ (a line for each ruling), where π is the hyperplane: $b + d = 0$.

The matrix R determines also a line L_R in $\mathbb{P}^{4*}$ by considering its columns as the coordinates of two points in $\mathbb{P}^{4*}$. L_R is the dual of P_R. If we take $(s : t)$ as coordinates in L_R and we consider the relation: $\begin{bmatrix} s & t \end{bmatrix} R^\top b_2 = [0]$ we get the restriction of J to $Q_R \times L_R$. For any point $w^* \in L_R$ the fibre of the restriction of J is the same as the fibre of J: it is given by two lines in $\mathbb{P}^3$, but obviously the lines belong to Q_R. Moreover, as the hyperplane w^* contains P_R, we know that these two lines belong to the other ruling of Q_R with respect to the ruling containing l_w : when $w^* = (0 : 1 : 1 : 0)$ (i.e. w^* is the hyperplane $b + d = 0$) the two lines are $z = x + y = 0$ and $y = z - u = 0$ as above. On the other hand, for any point $P \in Q_R$ the fibre is given by a single point of L_R. In fact the fibre of J over P is given by the hyperplanes of $\mathbb{P}^4$ containing $\mathbb{P}^1_P$, but here we are looking for the hyperplanes of $\mathbb{P}^4$ containing $\mathbb{P}^1_P$ and P_R. As $\mathbb{P}^1_P$ intersects transversely $C_R = P_R \cap \Sigma$ we find only one hyperplane.

We can summarize up the situation as follows: if we fix a random matrix R as above we get a smooth quadric $Q_R \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, a smooth conic $C_R \subset \Sigma$ and a line $L_R \subset \mathbb{P}^{4*}$ identified with the pencil $\mathbb{P}^1_R$ of hyperplanes of $\mathbb{P}^4$ containing the span of C_R. Q_R is equipped with a couple of maps: $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to C_R$ is the usual projection onto one factor, say the first one, and the fibre over any $w \in C_R$ is the line l_w ; $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to L_R$ is the composition of the usual projection onto

the second factor and a double covering σ, the fibre over the generic $w^* \in L_R$ is a couple of distinct lines of Q_R, belonging to the other ruling of Q_R with respect to the ruling containing l_w. There are exactly two distinct branching points on L_R corresponding to the two hyperplanes of $\mathbb{P}^1_R$ tangent to Σ.

Note that, to complete the analogy with the construction of [10], we should consider only the intersection of L_R with the dual hypersurface of Σ. In this case the intersection is given by the two branching points of σ, so that the fibre over the intersection is a couple of double lines: in fact a (reducible) conic bundle of dimension 1.

A final remark: let us consider $\mathcal{E}(1)$, in this case $c_1[\mathcal{E}(1)] = \mathcal{O}_{\mathbb{P}^3}(4)$, so that the degeneracy locus of two generic sections of $\mathcal{E}(1)$ is a smooth quartic surface in $\mathbb{P}^3$, i.e. a K-3 surface.

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